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THE ITERATIVE SOLUTION OF THE PROBLEM OF ORBIT DETERMINATION USING CHEBYSHEV SERIES

FINAL TECHNICAL REPORT

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1. INTRODUCTION

A method of orbit determination is investigated which employs Picard iteration and Chebyshev series. The method is applied to the problem of determining the orbit of an earth satellite from range and range-rate observations contaminated by noise. The method is shown to be readily applicable and to possess linear convergence.

2. MATHEMATICAL ANALYSIS OF THE PROBLEM

2.1 FORMULATION OF THE PROBLEM

Consider the mathematical model of a dynamical system subject to observation given by

$$\dot{x} = f(x, t) \tag{1}$$

$$y = g(x, t) \tag{2}$$

where

x denotes an n-dimensional state vector;

f, an n-vector function of the state;

y, an m-vector of observations;

g, an m-vector function of state variables; and

t, the time.

The dot denotes differentiation with respect to time.

Let the observations be made at discrete time points t_i (i=0,1,2,. . . N) in the interval (t_0, t_N). Let these observations be denoted by

$$\hat{y}(t_i),\ i=0,1,2,. . . N \tag{3}$$

In the above model, the functions f and g are nonlinear in general and the observations are contaminated by measurement errors. The problem of state estimation can now be stated as follows:

Find the solution x(t) of the differential system (1) such that

$$U = \sum_{i=0}^{N} \left|\left| g(x(t_i), t_i) - \hat{y}(t_i) \right|\right|^2 \tag{4}$$

is a minimum. This, in effect, is to find from the possible solutions of the system (1), the solution that satisfies the least square criterion with the given observations. If the system has a unique solution for a given set of initial conditions, then the problem is to determine the set of initial conditions, c, for which the solution minimizes the function U.

2.2 PRESENT METHOD OF SOLUTION

The present work comprises a study of Picard iteration [1] as a method for the solution of the estimation problem. Picard iteration has been used to solve the initial value problem [2] and the two point boundary value problem [3, 4]. In this case the integration constants are determined such that the solution satisfies the boundary conditions at each iteration. The convergence criteria have been established for this process. The present work is a natural extension of this method to the estimation problem. The technique and the method of implementation are discussed in the following sections.

2.3 PICARD ITERATION

Picard iteration successively approximates the solution of the differential equation (1) using the following

$$\dot{x}_i = f(x_{i-1}, t) \tag{5}$$

where i denotes the iteration number and where the initial (guessed) solution or zeroth approximation is denoted by $x_o(t)$. This procedure reduces the solution of the differential equation to a sequence of simple integrations. It is also seen that at the end of each iteration the integration constants c are to be determined. In the case of boundary value problems, the constants are chosen such that the given boundary conditions are satisfied by the new approximate solution.

For the present problem, the constants are determined so that the solution $x_i(t)$ minimizes the function U given by Equation 4. With these new values for the constants and the corresponding new approximate solution, the iteration process continues to determine the next approximation $x_{i+1}(t)$. It is expected that the iteration procedure converges linearly because of its similarity to the classical Picard iteration.

2.4 MINIMIZATION OF THE RESIDUALS

The minimization of the function U (or the sum of the squares of the residuals) can be carried out using any of the conventional schemes to solve the function minimization problem. For instance, the method of steepest descent could be used for this purpose [5]. On the other hand, at the extremum, the function satisfies the following

$$\frac{\partial U}{\partial c} = 0 = \sum_{i=0}^{N} 2 \left(\frac{\partial x_k^T}{\partial c} \frac{\partial q^T}{\partial x_k} \right) [g(x_k, t_i) - \hat{y}(t_i)] \qquad (6)$$

where

$\frac{\partial U}{\partial c}$ is n x 1 matrix

$\frac{\partial x_k^T}{\partial c}$ is an n x n matrix

This is a necessary condition for the extremum of the function and gives n algebraic equations for the n unknowns in c. These equations are in general nonlinear in c. However, if the observations are linear functions of the state, it is evident from Equation (6) that this equation set is linear in c (as x is a linear function of c).

The problem of determining c now reduces to the problem of solving the set of nonlinear algebraic equations (6). These equations can be solved using Newton's method for finding the zeros of a function [5, 6]. The derivative matrix for $L = \nabla U$ can be written as follows:

$$\frac{\partial L^T}{\partial c} = \Sigma \left\{ \frac{\partial x_k^T}{\partial c} \frac{\partial g^T}{\partial x_k} \left[\frac{\partial x_k^T}{\partial c} \frac{\partial g^T}{\partial x_k} \right]^T + (g - \hat{y}(t_i))^T \frac{\partial}{\partial c} \left[\frac{\partial x_k^T}{\partial c} \frac{\partial g^T}{\partial x_k} \right] \right\} \quad (7)$$

This matrix can be used to compute the new value of c using the following:

$$c_{j+1} = c_j - \left[\frac{\partial L}{\partial c} \right]^{-1} L_j \quad \text{where} \quad (8)$$

$\frac{\partial L}{\partial c}$ is an n x n matrix and

L_j corresponds to c_j.

2.5 POLYNOMIAL REPRESENTATION

In order to implement Picard iteration, a method of evaluating the integral (5) must be chosen. The initial guessed solution, $x_o(t)$, and the subsequent iterates, $x_k(t)$, can be approximated by a polynomial of k-th degree. A polynomial representation, although other forms are possible, has the advantage that it can be integrated easily.

For a given degree k, the polynomial $P_k(t)$ of best approximation to the function $\Phi(t)$ defined on the interval (-1, 1) minimizes the norm

$$\max_{\tau} [\Phi(\tau) - P_k(\tau)] \quad (9)$$

Polynomials which minimize this norm are said to satisfy the minimax principle [7]. For many functions, a truncated series of Chebyshev polynomials is very close to the polynomial of best approximation [7].

For this study, a series of Chebyshev polynomials is used to represent the function $x_k(t)$. The Chebyshev polynomial of k-th degree is given by.

$$T_k(\tau) = \cos\ (k \arccos\ (\tau))\ ;\ -1 \leq \tau \leq +1 \qquad (10)$$

These polynomials can be generated from the recurrence relations:

$$\begin{aligned} T_{k+1}(\tau) &= 2\ \tau\ T_k(\tau) - T_{k-1}(\tau) \\ T_o(\tau) &= 1; \qquad T_1(\tau) = \tau \end{aligned} \qquad (11)$$

Picard iteration can easily be implemented using the orthogonality properties given in Reference 12. The interval (t_o, t_f) is mapped onto the uniform interval (1, -1) by the transformation

$$\tau = 1 - 2t/(t_f - t_o) \qquad (12)$$

To take advantage of the orthogonality properties, the functions are evaluated at the special points given by

$$\tau_i = \cos(i\ \pi/K) \qquad (13)$$

where K is degree of polynomial representation.

The forcing function $f(x_i, t)$ as given in Equation 5 is represented by

$$f(x_i,\ t) = \sum_{j=0}^{K} b_j\ T_j\ (\tau) \qquad (14)$$

The constants b_j can be evaluated by knowing the function values of f at the special points τ_j. The integration of these Equations (14) can easily be done using the integration relations [7] obtaining the polynomial representation for $x_{i+1}(\tau)$.

3. APPLICATION TO ORBIT DETERMINATION PROBLEM

The technique discussed above is employed to solve the problem of orbit determination with range and range-rate observations from earth-bound tracking stations. The model considered and the results are discussed in the following sections. All computations were performed on an IBM 360/65.

3.1 EQUATIONS OF MOTION

The equations of motion are formulated in an inertial reference frame. This reference frame has its origin at the center of the earth. The X-Y plane coincides with the equatorial plane of the earth with the X-axis pointing to the first point of Aries. The Z-axis is in the direction of the north pole and the Y-axis forms a right-handed triad with the other two axes (see Figure 1).

The equations of motion of a satellite, considered to be a mass point moving in the central gravity field of the earth, in the inertial system of co-ordinates are given by [8]

$$\ddot{X} = -\mu X/R^3 \tag{15}$$

$$\ddot{Y} = -\mu Y/R^3 \tag{16}$$

$$\ddot{Z} = -\mu Z/R^3 \tag{17}$$

where

$R = (X^2 + Y^2 + Z^2)^{1/2}$

μ = gravitational constant (3.986×10^5 km^3/sec^2)

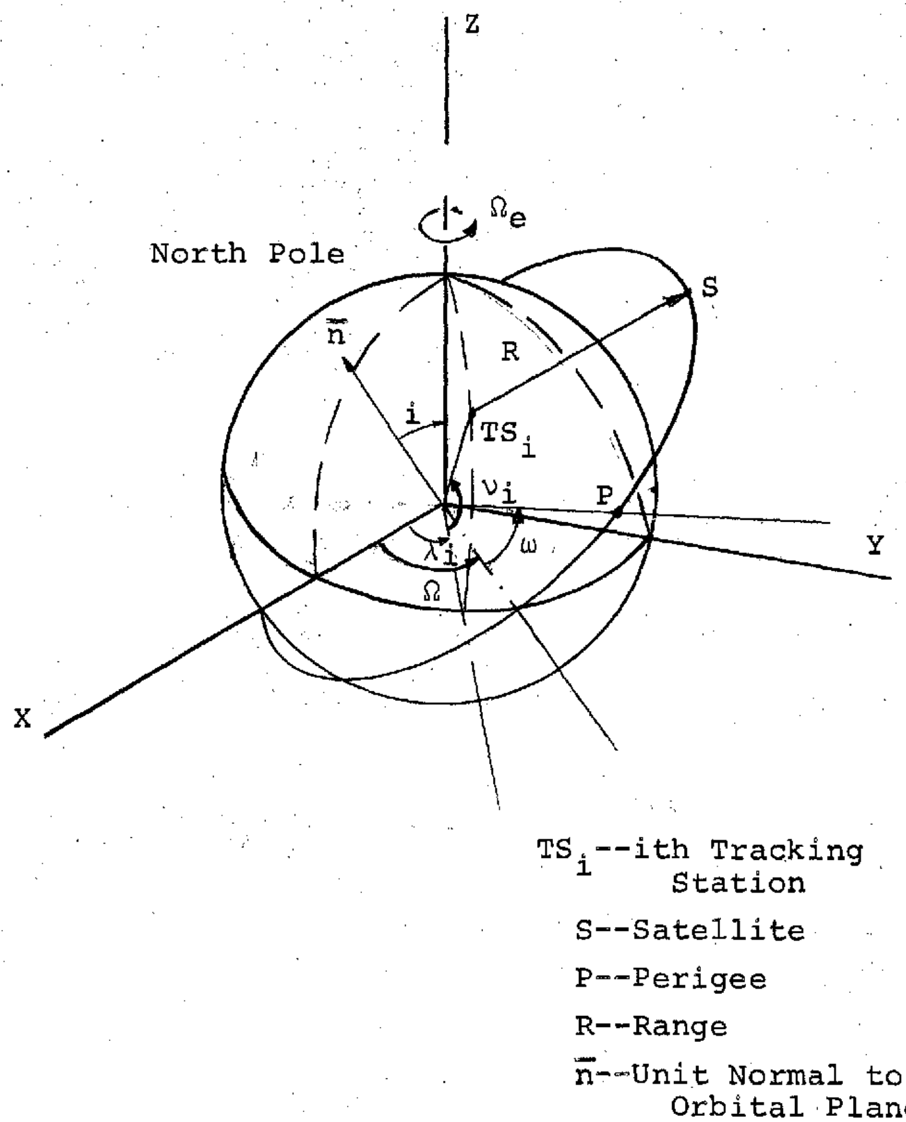

Figure 1. Relative positions of satellite and tracking station.

3.2 OBSERVATION MODEL

The earth is modeled as a sphere rotating with an angular velocity of 360 deg/day about the Z-axis. The location of the tracking station is specified by giving the latitude and the longitude of the station measured with respect to the inertial co-ordinate system (XYZ) as shown in Figure 1 at the instant t_o.

Two types of observations are made by each station: (1) the range (the distance between the tracking station and the satellite) and (2) the range-rate (the time rate of the range at the time of observation) [9, 10]. In a practical case the range and range-rate are obtained by the radar network. These observations are denoted by

$$\hat{y}(t_i) = \begin{pmatrix} \hat{R}_i^{(j)} \\ \hat{\dot{R}}_i^{(j)} \end{pmatrix} \qquad \begin{matrix} j = 1, 2, \ldots S \\ i = 0, 1, 2, \ldots N \end{matrix} \tag{18}$$

Here, the superscript j refers to the station number with the total number of stations considered being S.

The functional form of the observation vector g is yet to be given. Let $X^{(j)}(t)$, $Y^{(j)}(t)$ and $Z^{(j)}(t)$ be the co-ordinates of the j-th tracking station at time t. Then the observation vector as a function of the state variables can immediately be written as

$$g(t) = \begin{pmatrix} R^{(j)}(t) \\ \dot{R}^{(j)}(t) \end{pmatrix} \tag{19}$$

where

$$R^{(j)}(t) = \{[X(t) - X^{(j)}(t)]^2 + [Y(t) - Y^{(j)}(t)]^2 + [Z(t) - Z^{(j)}(t)]^2\}^{1/2} \tag{20}$$

$$\dot{R}^{(j)}(t) = \frac{1}{R^{(j)}} \{ [X(t) - X^{(j)}(t)] [\dot{X}(t) - \dot{X}^{(j)}(t)] + [Y(t) - Y^{(j)}(t)] [\dot{Y}(t) - \dot{Y}^{(j)}(t)] + [Z(t) - Z^{(j)}(t)] [\dot{Z}(t) - \dot{Z}^{(j)}(t)] \} \quad (21)$$

From the above it is seen that both the observations and the equations of motion are nonlinear.

3.3 DETAILS OF NUMERICAL SOLUTION

The Equations 15 to 17 representing the system dynamics are second-order nonlinear differential equations. These can be written as a set of six first order nonlinear equations, one for each of the state variables (viz, X, Y, Z, $\dot{X}$, $\dot{Y}$, $\dot{Z}$) in the form given by Equation 1. The state variables are represented by a series of Chebyshev polynomials with 16 terms, the number of terms being chosen based on the recommendations of the earlier studies [1, 11] on the representation of the orbits using Chebyshev polynomials. The computation of the forces is performed only at the selected points as given by Equations 12 and 13 in order to maintain the orthogonality property of the polynomials. The integration of the second-order system gives rise to two vector constants of integration (each with three elements). Let these six constants be represented by c. The function to be minimized for the problem is given by

$$U = \sum_{j=1}^{S} \sum_{i=0}^{N} \{ [R^{(j)}(t_i) - \hat{R}_i^{(j)}]^2 + [\dot{R}^{(j)}(t_i) - \hat{\dot{R}}_i^{(j)}]^2 \} \quad (22)$$

The function given by Equation 22 is minimized with respect to c by finding the solution of the system of equations given by

$$\nabla U = \frac{\partial U}{\partial c} = 0 \quad (23)$$

Newton's method is used to find the zeros of ∇U. No major difficulties are encountered in implementing this scheme. It is observed that U decreases as the solution is approached.

3.4 NUMERICAL RESULTS AND DISCUSSIONS

To demonstrate the validity of the method and the scheme of implementation two examples are considered. The first one is the determination of the state of a satellite moving in a circular orbit. The orbital elements are given in Table I. The observations are made in the interval (0, 1513.1) seconds. This corresponds to one-fourth of a revolution in the orbit. The tracking station data (viz, the location and the number of observations) are also presented in Table I. It should be noted that the observations are not simultaneous except for the first. This does not in any way constrain the applicability of this method for a general set of observations. The observations for both examples presented here are simulated on the computer using a random number generator. At the i^{th} iteration the error is

$$E = [(X_i - X)^2 - (Y_i - Y)^2 + (Z_i - Z)^2]^{1/2} \quad (24)$$

The initial guessed solution for this problem is in error by about 72 km throughout the interval considered. In Figure 2, the error as a function of time is plotted for the first two iterations. From this figure it can be seen that although the initial guess is far from the solution, within two iterations the error is reduced to less than 5 km. To have a clear pictorial representation of the convergence of the process, the logarithm of the absolute error is presented in Figure 3, as a function of the iteration number. The error at selected time points is plotted in order to

TABLE I

STATION AND ORBIT DATA (EXAMPLE I)

Parameters	Station Number 1	2	3
Latitude (deg)	18.0	12.0	10.0
Longitude (deg)	0.0	28.0	14.0
Number of observations	10	20	30
Interval between observations (sec)	168.0	79.0	52.0
Semi-major axis (km)		$a = 7178.145$	
Eccentricity		$e = 0.0$	
Inclination (deg)		$i = 20$	
Longitude of ascending node		$\Omega = 0.0$	
Argument of perigee		$\omega = 0.0$	

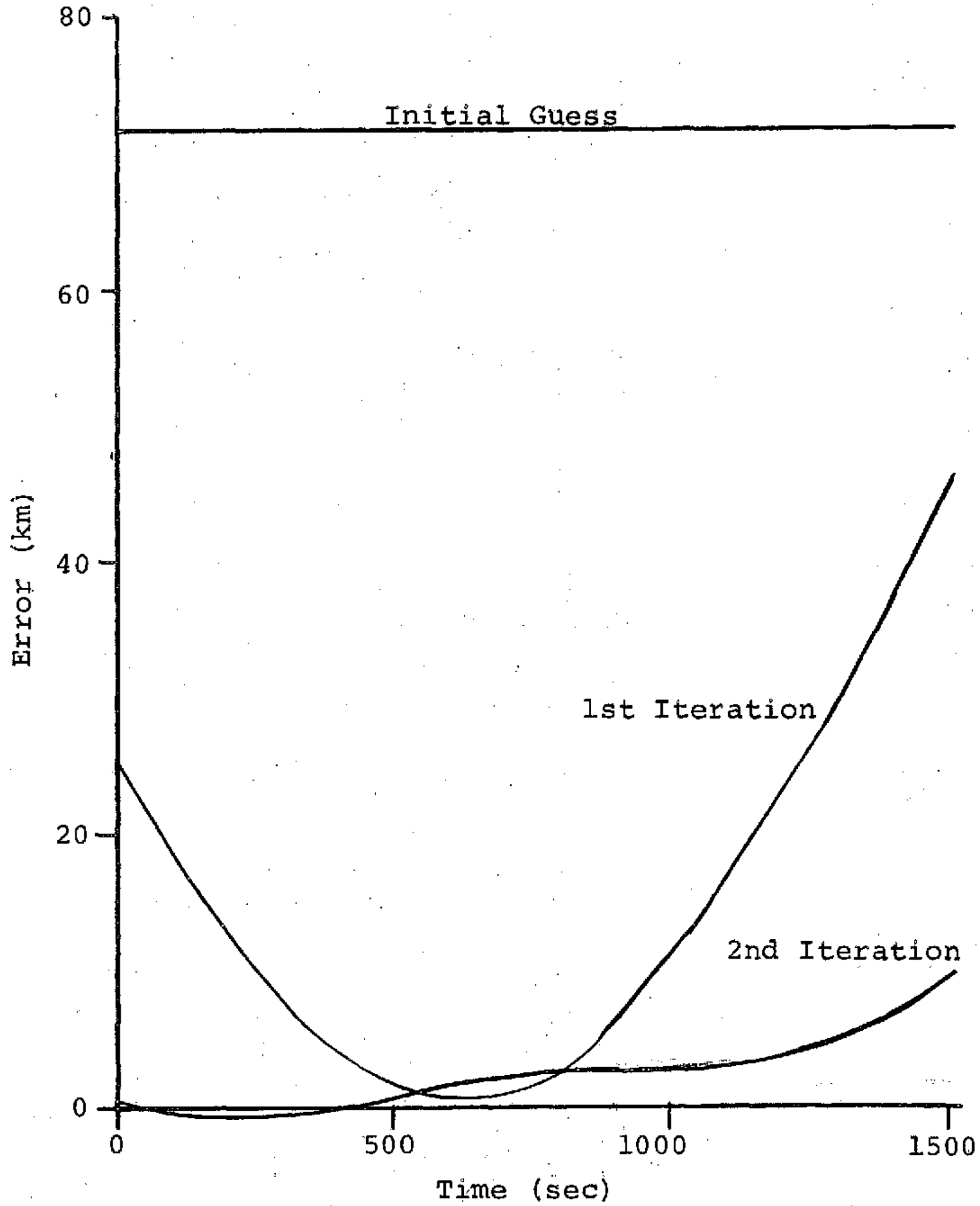

Figure 2. Error as a function of time for different iterations (circular orbit).

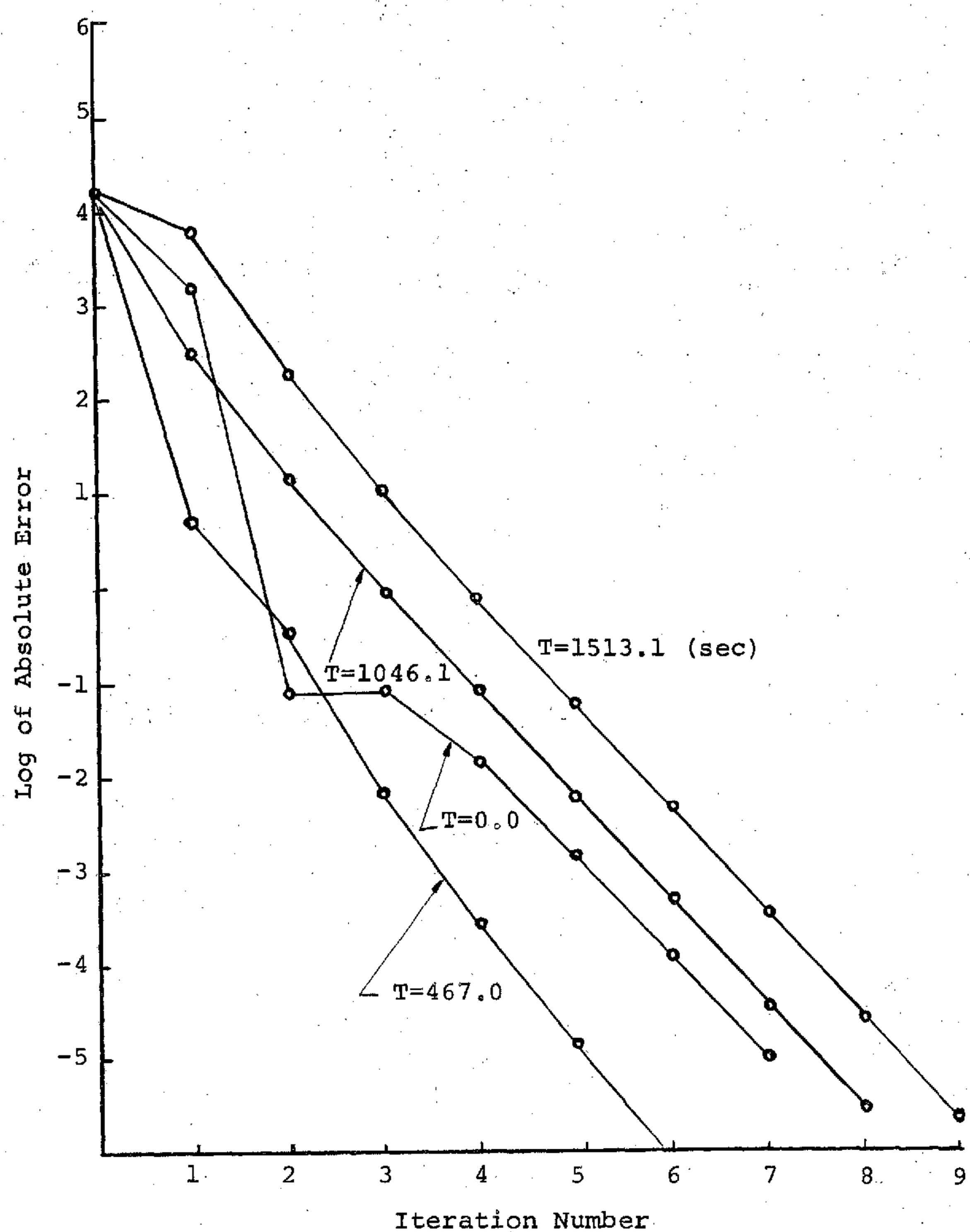

Figure 3. Convergence of the method for a circular orbit.

depict the convergence of the over-all solution. It is apparent from this that the error reduction is linear as the solution approaches the true solution.

The convergence criterion is that the error be less than a meter. The iteration is terminated when this criterion is satisfied. For this problem, the convergence of the process is attained at the end of nine iterations. The time for the computation is 21.6 seconds.

The second problem considered is the case of a satellite moving in an elliptic orbit. The orbit has an eccentricity of 0.0557 with a perigee height of 400 km. Here, the arc considered for the solution is from $t_o = 0$ (perigee) to $t_f = 1513.1$ seconds. The circular orbit of the first example and this eccentric orbit have the same semi-major axes, and therefore, the same orbital periods. This example allows a study of the effect, if any, of eccentricity on the method of solution. The tracking stations and the times of observation are unaltered from those of the first problem. The tracking station data and the orbit data are presented in Table II.

Figure 4 contains the graphical representation of the errors in the solution for the first two iterations. The error of about 70 km for the guessed solution is reduced to about 10 km in two iterations. The logarithm of the absolute error is plotted in Figure 5 as a function of the iteration number. Here again, it is evident that the convergence is linear. The process converges, for this problem, in 11 iterations to a solution which is in error by about a meter.

In summary, it is seen that the method is applicable to the problem of orbit determination and that the convergence is linear as expected. The comparison of the two problems considered here is presented in Table III. It is seen that the elliptic orbit takes more iterations.

TABLE II

STATION AND ORBIT DATA (EXAMPLE II)

Parameters	Station Number 1	2	3
Latitude (deg)	18.0	12.0	10.0
Longitude (deg)	0.0	28.0	14.0
Number of observations	10	20	30
Interval between observations (sec)	168.0	79.0	52.0
Semi-major axis (km)		a = 7178.145	
Eccentricity		e = 0.0557	
Inclination (deg)		i = 20	
Longitude of ascending node		Ω = 0.0	
Argument of perigee		ω = 0.0	

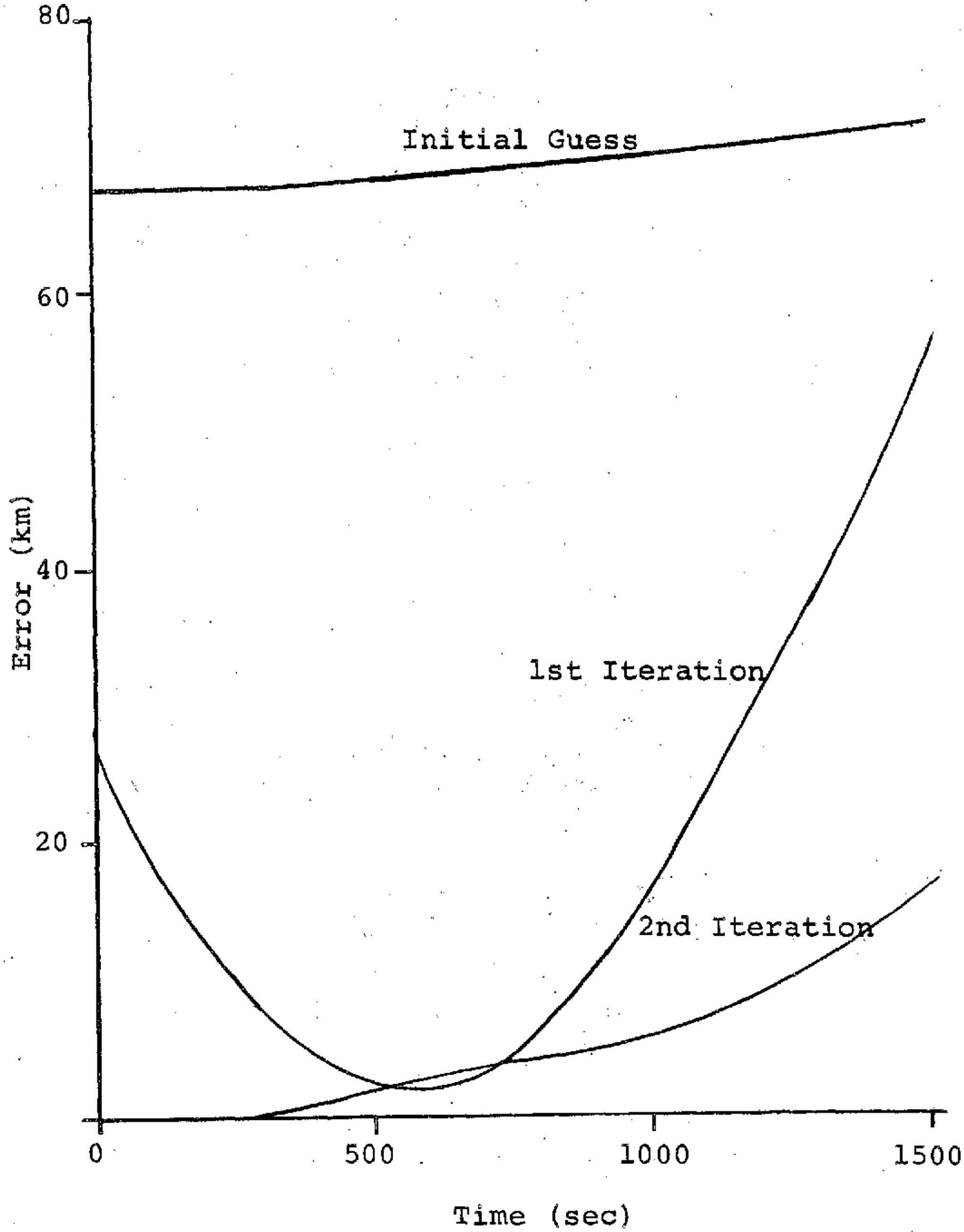

Figure 4. Error as a function of time for different iterations (elliptic orbit).

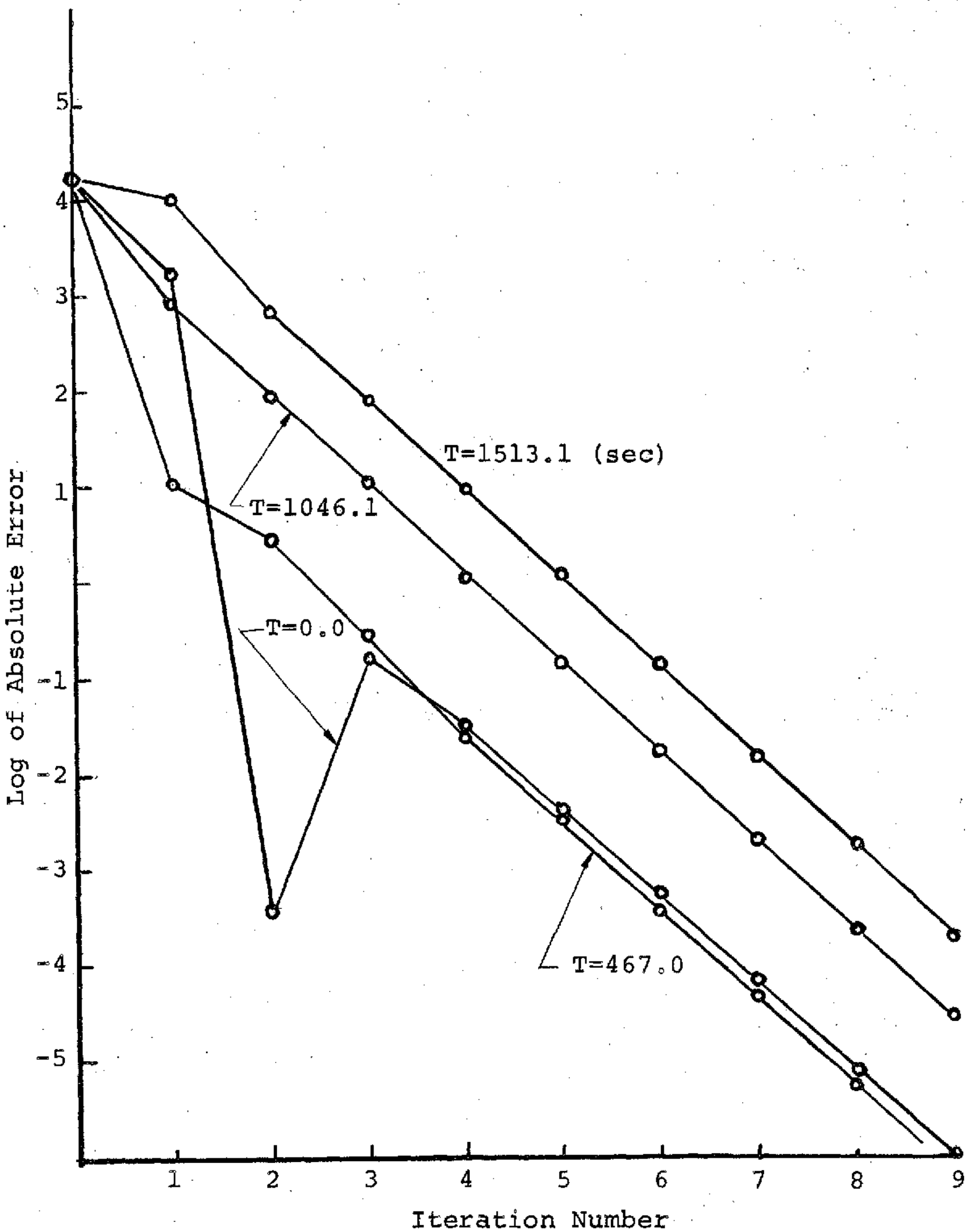

Figure 5. Convergence of the method for an elliptic orbit.

TABLE III

COMPARISON OF THE TWO EXAMPLES

Parameter	Example I (Circular Orbit)	Example II (Elliptic Orbit)
Arc length in time (sec)	1513.1	1513.1
Number of observations	50	50
Eccentricity	0.0	0.0557
Initial error in state variables	1.0%	1.0%
Number of iterations for convergence	9	11
Computer time (sec)	21.6	28.9

4. CONCLUSIONS

An iterative scheme using Picard iteration has been investigated for the solution of the state estimation problem with discrete observations. The applicability of this scheme to practical problems has been demonstrated [12] using as an example the problem of orbit determination of an earth satellite with range and range-rate observations from earth-bound tracking stations.

Unlike some of the more commonly used methods, this method does not require the formulation or the solution of the linear perturbation equations. From the examples considered, it is seen the method has linear convergence.

This scheme can be extended easily to a general problem or orbit determination (including the effects of drag, oblateness of the earth, gravity fields of other bodies, etc.). However, further work is necessary to study the convergence of Chebyshev series, and the number of terms needed to represent the solution over a given interval.

A sequential algorithm for estimating the orbit using Chebyshev series is presently being investigated for use in a real time orbit determination environment.

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