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EFFECT OF RESONANCE-OBLATENESS COUPLING ON A SATELLITE ORBIT

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ABSTRACT

Second order effects of the coupling between geopotential resonance and oblateness on a satellite orbit are calculated. These effects arise from the interaction of resonance with the secular changes of the orbit's node, perigee and mean anomaly. They have the same period and phase as first order resonance perturbations. But their amplitudes are proportional to the square of the period and dominate the first order effects as the orbit becomes commensurate. A striking example of this coupling is seen in the 18 day resonance variation of the node of the orbit of the first earth resources technology satellite. Analysis of this one arc second (31m) variation yields a strong 14th order constraint to the geopotential field for odd degree terms. This constraint (lumped coefficient) is poorly predicted by current models.

INTRODUCTION

The analytic perturbations of a Kepler orbit by the geopoetntial have been extensively discussed in the literature [i.e.; Kaula, 1966]. The zeroth order perturbations are the integration of the usual non-linear Lagrange planetary equations:

$$\dot{\mathbf{E}} = \mathbf{f}(\mathbf{E}, C_{\ell,m}, S_{\ell,m}), \tag{1}$$

with constant right hand side (except for the steady progression of the mean anomaly). In Equation (1), **E** is a vector of six Kepler elements. The $C_{\ell m}$ and $S_{\ell,m}$ are the usual geopotential coefficients (but fully normalized; Kaula (1966), p. 7) defined from the representation of the potential as:

$$V_e = \frac{\mu}{r}\left[1 + \sum_{\ell=2}^{\infty}\sum_{m=0}^{\ell} (a_e/r)^{\ell} P_{\ell m}(\sin\phi)\,[C_{\ell m}\cos m\lambda + S_{\ell m}\sin m\lambda]\right],$$

where r, ϕ, λ are the satellite's distance from the earth's center of mass, geocentric latitude and longitude; μ is the Gaussian gravitational constant, a_e is the earth's mean equatorial radius and the $P_{\ell,m}$ are the usual (but fully normalized) associated Legendre polynomials. At this initial level of approximation the well known secular effects on the node (Ω), perigee (ω) and mean anomaly (M) due to oblateness ($C_{2,0}$) dominate since this term is about one thousand times larger than the others. These zeroth order secular perturbations are:

$$\dot{\Omega}_0 = \frac{3\mu^{1/2} C_{2,0} \cdot \sqrt{5}\, a_e^2 a^{-7/2} \cos I}{2(1 - e^2)^2}$$

$$\dot{\omega}_0 = \frac{3\mu^{1/2} C_{2,0} \cdot \sqrt{5}\, a_e^2 a^{-7/2} (1 - 5 \cos^2 I)}{4(1 - e^2)^2} \tag{2}$$

$$\dot{M}_0 = \frac{- 3\mu^{1/2} C_{2,0} \cdot \sqrt{5}\, a_e^2 a^{-7/2}}{4(1 - e^2)^{3/2}} [3 \cos^2 I - 1],$$

where a, e and I are the orbit's semimajor axis, eccentricity and inclination. Zeroth order perturbations of short period (less than an orbit revolution) also exist, again dominated by the $C_{2,0}$ term. However, these are usually calculated with respect to the secularly precessing ellipse defined by Equation (2) and are thus first order perturbations.

First order variations; integrations of Equation (1) incorporating the zeroth order effects (of the precessing ellipse) in their right hand sides, are also well known. The best example is the "odd" zonal (m = 0) oscillation of long period in e, i, ω, Ω and M with frequency $\dot{\omega}$.

In general, the first order perturbations have frequencies:

$$\dot{\psi} = (\ell - 2p)\, \dot{\omega}_0 + (\ell - 2p + q)\, \dot{M} + m(\dot{\Omega}_0 + \dot{\theta}_e), \tag{3}$$

where p and q are additional indices related to the representation of the geopotential in terms of the Kepler elements, $\dot{\theta}_e$ is the rotation rate of the earth and $\dot{M} = n + \dot{M}_0$, where $n = \mu^{1/2}\ a^{-3/2}$. In particular, longitude harmonic ($m \neq 0$) resonance occurs when $\dot{\psi} \approx 0$.

As is well known, the first order perturbations are of order $J_{\ell,m}(n/\dot{\psi})$; $J_{\ell m} = (C^2_{\ell m} + S^2_{\ell m})^{1/2}$. But of course, first order resonance perturbations are not adequate as $\dot{\psi} \to 0$ (i.e., Gedeon, 1969). Even for so called "shallow resonance," where the character of the motion is still sinusoidal, a second order perturbation in the mean anomaly, of order $J_{\ell m}(n/\dot{\psi})^2$, dominates the first order variation.

This particular second order perturbation arises from the interaction (coupling) of the first order resonance variation of the semimajor axis with the two body mean motion. Following Kaula (1966), the element rates are expanded in a Taylor series with respect to the element perturbations:

$$\dot{E} = f_0 + \Sigma \frac{\partial f_0}{\partial E} \Delta E + \cdots \tag{5}$$

For the semimajor axis, the first order variation, of order $J_{\ell m}(n/\dot{\psi})$, for a particular term (ℓ, m, p, q) is:

$$\Delta_1 a = (n/\dot{\psi})\, a (a_e/a)^{\ell}\, 2F_{\ell m p}(I)\, G_{\ell pq}(e)(\ell - 2p + q)\, S_{\ell m p q} \tag{6}$$

where F and G are inclination and eccentricity functions and;

$$S_{\ell mpq} = \begin{array}{c} \begin{bmatrix} C_{\ell m} \\ \\ -S_{\ell m} \end{bmatrix} \end{array}\!\!\begin{array}{l} \ell-m \text{ even} \\ \\ \ell-m \text{ odd} \end{array} \cos\psi + \begin{array}{c} \begin{bmatrix} S_{\ell m} \\ \\ C_{\ell m} \end{bmatrix} \end{array}\!\!\begin{array}{l} \ell-m \text{ even} \\ \\ \ell-m \text{ odd} \end{array} \sin\psi .$$

[The characteristic longitude is $\psi = (\ell - 2p)\omega + (\ell - 2p + q)\, M + m(\Omega - \theta_e)$].

Writing Equation (5) for the variation of the mean anomaly;

$$\dot{M} = n + \frac{\partial n}{\partial a} \Delta_1 a, \tag{7}$$

the second term on the right gives the interaction with the linear perturbation of Equation (6):

$$\Delta_2 \dot{M} = (-3n/2a)\, \Delta_1 a.$$

The integration of this "second order" effect with respect to ψ [letting $dt = d\psi/\dot{\psi}$] yields:

$$\Delta_2 M = -3(a_e/a)^{\ell} (n/\dot{\psi})^2 \, FG(\ell - 2p + q)\, \overline{S} = (\Delta M)_2\, \overline{S} \tag{8}$$

where

$$\overline{S} = \int S d\psi.$$

The second order (resonance) interaction perturbations to be discussed in this paper are entirely analogous to $\Delta_2 M$ except that they are of order $J_{2,0}\;\; J_{\ell,m}(n/\dot{\psi})^2$. Though $J_{2,0} \simeq 10^{-3}$; $\dot{\psi} << 1$ in resonance and these effects will dominate the linear perturbations of order $J_{\ell m}(n/\dot{\psi})$ as $\dot{\psi} \to 0$.

ANALYSIS

The resonance perturbations of the node, perigee and mean anomaly are written as a sum of direct (linear) and indirect effects:

$$\Delta(\Omega,\, \omega,\, M) = \Delta_1(\Omega,\, \omega,\, M)_{\mathrm{DIRECT}} + \int dt \left[\Sigma \frac{\partial \dot{\Omega}_0,\, \dot{\omega}_0,\, \dot{M}_0}{\partial a,\, e,\, I} \Delta_1 a,\, e,\, I \right]$$

$$+ \Delta_2 M(\text{TWO BODY INTERACTION}) \tag{9}$$

Using the zeroth order secular rates of Equation (2);

$$\frac{\partial \dot{\Omega}_0}{\partial a} = -2\mu^{1/2} C_{2,0} \cdot \sqrt{5} \cos I\, a_e^2 a^{-9/2}/4(1-e^2)^2,$$

$$\frac{\partial \dot{\Omega}_0}{\partial e} = 6\mu^{1/2} C_{2,0} \cdot \sqrt{5}\, e \cos I a_e^2\, a^{-7/2}/(1-e^2)^3,$$

$$\frac{\partial \dot{\Omega}_0}{\partial I} = \frac{-3\mu^{1/2} C_{2,0} \cdot \sqrt{5} \sin I\, a_e^2 a^{-7/2}}{2(1-e^2)^2},$$

$$\frac{\partial \dot{\omega}_0}{\partial a} = -21\mu^{1/2} C_{2,0} \cdot \sqrt{5}\,(1-5\cos^2 I)\, a_e^2 a^{-9/2}/8(1-e^2)^2,$$

$$\frac{\partial \dot{\omega}_0}{\partial e} = 3\mu^{1/2} C_{2,0} \cdot \sqrt{5}\, e(1-5\cos^2 I)\, a_e^2\, a^{-7/2}/(1-e^2)^3,$$

$$\frac{\partial \dot{\omega}_0}{\partial I} = 15\mu^{1/2} C_{2,0} \cdot \sqrt{5} \sin I \cos I\, a_e^2 a^{-7/2}/2(1-e^2)^2,$$

$$\frac{\partial \dot{M}_0}{\partial a} = 21\mu^{1/2} C_{2,0} \cdot \sqrt{5}\,(3\cos^2 I - 1)\, a_e^2\, a^{-9/2}/8(1-e^2)^{3/2},$$

$$\frac{\partial \dot{M}_0}{\partial e} = -9\mu^{1/2} C_{2,0} \cdot \sqrt{5}\, e(3\cos^2 I - 1)\, a_e^2\, a^{-9/2}/4(1-e^2)^{5/2},$$

and

$$\frac{\partial \dot{M}_0}{\partial I} = 9\mu^{1/2} C_{2,0} \cdot \sqrt{5} \sin I \cos I\, a_e^2 a^{-7/2}/2(1-e^2)^{3/2}. \tag{10}$$

The first order (resonance) perturbation Δ_1 a has already been given in Equation (6). The other direct resonance variations are:

$$\Delta_1 e = \Delta_1 a\{(1 - e^2)^{1/2} [(1 - e^2)^{1/2} - (\ell - 2p)]\}/2ae$$

$$\Delta_1 I = \Delta_1 a\{(\ell - 2p) \cos I - m\}/2a(1 - e^2)^{1/2} \sin I \tag{11}$$

Recall that resonance occurs when $\dot{M} \doteq \dot{\theta}_e\, m/k$, or a rational fraction of the earth's rotation rate. Here, $k = (\ell - 2p + q) = 1, 2, 3, \ldots$ In these circumstances, $\dot{\psi} \approx 0$. A resonant family is characterized by m and k having all the q indices as species (or members). A specification of m and q however is sufficient to denote a particular species (or member) when the orbit's mean motion is known. It is noted that the strongest resonances usually occur in the families for k = 1 when q = 0 since the G functions are of order $e^{|q|}$.

To show the relative dominance of the direct and interactive terms, recall [from Kaula, 1966 p. 40] [that the direct effect in the node is given as:

$$\Delta_1 \Omega = \frac{(n/\dot{\psi})\,(a_e/a)^{\ell}\,\partial F/\partial I\; G\bar{S}}{(1 - e^2)^{1/2} \sin I} = (\Delta\Omega)_1\,\bar{S}, \tag{12}$$

where

$$\bar{S} = \int S(\psi)\, d\psi$$

The indirect, resonance-oblateness coupled perturbation is evaluated from Equations (6), (9), (10), and (11). For the integral, the change of independent variable $d\psi = \dot{\psi}\, dt$ is made and all parameters except ψ are assumed constant. Then

$$\Delta_2\Omega = \frac{\bar{S}(n/\dot{\psi})C_{2,0}\cdot\sqrt{5}}{(1-e^2)^2\,(a/a_e)^2}\,\{2(n/\dot{\psi})\,FGk/(a/a_e)^{\ell}\}\left\{\frac{-21\cos I}{4}\right.$$

$$\left.+\frac{3\cos I}{(1-e^2)}\,[(1-e^2)^{1/2}\,[(1-e^2)^{1/2}-(k-q)]]-\frac{3[(k-q)\cos I-M]}{4(1-e^2)^{1/2}}\right\}\cdot \quad (13)$$

Combining (12) and (13);

$$\Delta\Omega(\mathrm{TOTAL}) = \bar{S}\,\frac{(n/\dot{\psi})}{(a/a_e)^{\ell}}\,GF\left\{\frac{(\partial F/\partial I)/F}{(1-e^2)^{1/2}\,\sin I}\right.$$

$$+\frac{2(n/\dot{\psi})\,C_{2,0}\cdot\sqrt{5}\,k}{(1-e^2)^2\,(a/a_e)^2}\left[\frac{-21\cos I}{4}+\frac{3\cos I}{(1-e^2)}\,(1-e^2)^{1/2}\,[(1-e^2)^{1/2}\right.$$

$$\left.\left.-(k-q)]-\frac{3}{4}\,\frac{[(k-q)\cos I-m]}{(1-e^2)^{1/2}}\right]\right\}. \quad (14)$$

Consider a resonant member of the family $k = 1$, $m = 15$ ($\dot{M} = 15$ revolutions/day, or $n \sim 5300$ °/day). The dominant resonant effects are generally due to the lower degree terms where $\ell - m$ is small. Their sinusoidal inclination functions have characteristic wavelengths of $2\pi/\ell - m + 1$ (Allan, 1973). Typically then, in about one radian the change in F is equal to F, or $\partial F/\partial I = 0(F)$. Better, for a sinusoidal function F with wavelength of λ, the value of $\partial F/\partial I$ (rms) is $2\pi/\lambda$ times the value of F (rms). (A more exact "average" is derived in Appendix A). Thus for a close satellite orbit the order of the direct term will be about 4 (taking sin I into consideration) while that of the indirect term will be $(n/\dot{\psi}) \times 10^{-3} \times 10$ or $55^{\circ/d}/\dot{\psi}^{\circ/d}$. In other words, at a resonant period of about 25 days, the two effects are about the same for the average close orbit. This "shallow resonance" period is somewhat longer than the majority of the useful geodetic orbits. The typical geodetic orbit has a dominant resonance period of about 4 days. But even with indirect effects 1/6 of the direct, these interaction terms cannot be ignored.

Returning to the calculation of the second order oblateness-resonance effects in terms of the first order perturbations $(\Delta_1 a, \Delta_1 e, \Delta_1 I)$ from Equations (9), (10) and (11); these are combined with the direct effects to give the total resonant variations as;

$$\Delta\Omega = \left\{(\Delta\Omega)_1 + \frac{(n/\dot{\psi})\, C_{2,0}\cdot\sqrt{5}}{(a/a_e)^2\,(1-e^2)^2}\left[\frac{-21}{4}\cos I(\Delta a)_1/a + \frac{6e\cos I(\Delta e)_1}{(1-e^2)} - \frac{3}{2}\sin I(\Delta I)_1\right]\right\}\overline{S}$$

$$\Delta\omega = \left\{(\Delta\omega)_1 + \frac{(n/\dot{\psi})\, C_{2,0}\cdot\sqrt{5}}{(a/a_e)^2\,(1-e^2)^2}\left[\frac{-21}{8}(1-5\cos^2 I)\,(\Delta a)_1/a + \frac{3e(1-5\cos^2 I)}{(1-e^2)}(\Delta e)_1 + \frac{15\sin I\cos I}{2}(\Delta I)_1\right]\right\}\overline{S},$$

and

$$\Delta M = \left\{(\Delta M)_1 + (\Delta M)_2 + \frac{(n/\dot{\psi})\, C_{2,0}\cdot\sqrt{5}}{(a/a_e)^2\,(1-e^2)^{3/2}}\left[\frac{21(3\cos^2 I-1)\,(\Delta a)_1/a}{8} - \frac{9e(3\cos^2 I-1)\,(\Delta e)_1}{4(1-e^2)} + \frac{9\sin I\cos I(\Delta I)_1}{2}\right]\right\}\overline{S}, \quad (15)$$

where:

$(\Delta\omega)_1 = \Delta_1\omega/\bar{S}$, $(\Delta M)_1 = \Delta_1 M/\bar{S}$, $(\Delta a)_1 = \Delta_1 a/S$

$(\Delta e)_1 = \Delta_1 e/S$, $(\Delta I)_1 = \Delta_1 I/S$,

and the Δ_1 variations are the linear resonance effects given by Kaula, 1966, p. 40.

An important point to note from Equation (15) is that both the phase and the frequency of the coupling effects are identical to those of first order. They will therefore be ordinarily indistinguishable from the linear resonance perturbations on a single orbit. But incorrect resonant coefficients will result in gravitational determinations from many orbits which have significant coupling and employ only linear effects. Generally, the error in the mean anomaly and argument of perigee is not serious. This is because the in-plane resonance variation is usually dominated by the stronger second order resonance coupling with the two body mean motion which have always been included in analyses (i.e., Gaposchkin and Lambeck, 1970). However, the resonance out-of-plane from the variation of the node may have serious errors if only linear analysis is used as in Gaposchkin and Lambeck, 1970 and Gaposchkin, 1973. Table 1 presents these maximum linear and total variations of the node for the 21 satellite orbits (excluding ERTS) used in the Smithsonian Standard Earth's (see also Gaposchkin, 1966). Almost half show more than 10% change when the indirect effects are included.

DISCUSSION AND RESULTS

The analysis here is a straightforward application of perturbation theory. In fact, Kaula 1966, p. 49 has worked out two such second order variation; one is a long period variation of the node from odd zonal-J_2 interaction. These effects are entirely analogous to those here. Born (1974) has worked out coupling effects of order $(J_{\ell m} n/\dot{\psi})^2$ from the interaction of periodic resonant terms with themselves. This was done for the analytic theory of the Mariner 9 Mars orbiter ($n \simeq$ 2 revs/day) which had a shallow resonant period of 18 days. For the close earth satellite these resonant-resonant interactions are of the order of 10^{-3} less than the resonant-oblateness ones. For Mars, the longitude gravity field is more than an order of magnitude stronger than on earth and the high Mariner 9 orbit further diminishes the obleteness effects. As a result, the resonant-resonant effects are roughly comparable to the resonant-oblateness ones. More surprising, even though the Mars oblateness is almost twice earth's, the indirect effect on the Mariner 9 node is only about 2% of the direct. This is due to the height of the orbit and a significant cancellation of indirect terms.

For example, I use the following parameters for the Mariner 9 orbit about Mars:

$a = 12631$ km($\sim$2 rev's/day), $e = 0.62$, $I = 64.8°$, $360°/\dot{\psi} = 18.3$ days

with respect to the term (2,2,1,1) $10^6\ C_{2,0} = -1960/\sqrt{5}$, $10^6\ J_{2,2} = 96.7$. Then, Equation (12) gives the amplitude of the direct perturbation of the node as:

$$|\Delta_1 \Omega| = .02580°.$$

This agrees well with Born (1974). On the other hand the indirect effect from Equation (13) is calculated as:

$$|\Delta_2 \Omega| = .00051° \tag{17}$$

This amplitude is somewhat greater than the amplitude of the periodic part of the resonant-resonant interaction (Born, 1974). It also appears to be sufficient to explain the remaining discrepancy of the Mariner 9 theory (for the node) with numerical integration.

ERTS RESULT FOR VARIATION OF THE NODE

The ERTS 1 spacecraft was intensively tracked by Unified S-Band range and range rate stations (worldwide) in 1972 and 1973. From this tracking history, precise, almost daily osculating elements have been obtained (Arthur Fuchs, Goddard Space Flight Center). These have been converted to mean elements by numerically averaging minute Keplerian ephemerides over each day of tracking, with respect to a precessing ellipse. Before averaging, the osculating vectors were first reduced of most short and medium period terms analytically (Douglas; Marsh and Mullins, 1972).

Figure 1 (circled points) shows a residual variation of the mean node in a 50 day period in late 1972. These residuals are the "observed" mean nodes minus calculated values from a trajectory computed with all the long term effects of the zonal geopotential, the sun and moon (including earth tides), radiation pressure and atmospheric drag (Wagner, Douglas and Williamson, 1974). No resonant geopotential effects were included on this trajectory.

The ERTS orbit's ground trace repeats in 18 days (by design) which is seen to be the period of this variation. The amplitude is only about 30 m but well defined. The ERTS orbit parameters are; n $\simeq$ 14 rev's day, e = .001 and I = 99.1°. It is natural, therefore, to ascribe this variation to a resonance with geopotential terms of order 14. In particular, from Equation (3), $360°/\dot{\psi}$ = 18.0 days, when $\dot{\psi} = (\dot{\omega} + \dot{M}) + 14(\dot{\Omega} + \dot{\theta}_e)$. This characteristic rate holds for all odd degree resonant terms (ℓ, m, p, q) of order 14 where q = 0. By almost two orders of magnitude this family has the strongest effect on the node. (See Table 2.)

C. A. Wagner and S. M. Klosko, unpublished, have analysed the variation of the semi-major axis, inclination, node and eccentricity of the ERTS orbit by the same semi-numerical trajectory program which calculated the residuals in Figure 1. In this analysis only a pair of two odd and even degree coefficients of 14th order was used, with a single 28th order pair. They found the pair of two "lumped" odd degree, 14th order coefficients which best fit the mean element data to be:

$$\begin{aligned} 10^9 (C, S)_{19,14} &= -23.1 \pm 0.7, \quad -97.5 \pm 0.7 \\ 10^9 (C, S)_{21,14} &= 37.0 \pm 0.8, \quad 138.4 \pm 0.8 \end{aligned} \tag{18}$$

Using these values in Equation (12) (with the sum of the two terms taken) or scaling to the values in Table 2, the analytic variation of the node considering only first order effects is:

$$\Delta\Omega_1'' = -0.347 \sin\psi - 0.149 \cos\psi \tag{19}$$

The corresponding variation including the 2nd order effects of oblateness-resonant interaction is (from Equation (15)):

$$\Delta\Omega'' = -0.696 \sin\psi - 0.238 \cos\psi . \tag{20}$$

The longitude ψ in the 50-day period shown in Figure 1 is nearly linear in time:

$$\psi^\circ \doteq -11.7 + 18(\mathrm{MJD} - 41622.33). \tag{21}$$

The two variations are plotted in Figure 1. Clearly, the second order variation is more faithful to the data.

However, it should be apparent that any fixed biases in the trackers such as station coordinate and constant calibration errors can simulate this resonant effect (i.e., Allan, 1973, p. 223). Such tracking depends on the orbit-station configuration and would repeat in the same period as the ground track, which is the resonant period. It is difficult to determine the likelihood or magnitude of such an effect over a complex of tracking stations and operating times. However, with regard to station position error (center of mass coordinates), the ERTS trackers should be known to better than 10m individually. Any residual bias in the node should be considerably less than this in orbits determined from all the stations. It should be added that such a bias would affect all geodetic results from resonant satellites, but especially those with effects less than 10m. A considerable amount of the shallow resonant information in current satellite determined fields is only available at a level below 10m (i.e. Table 1 and Lerch et al., 1974).

Finally, it is instructive to write out the full resonant constraint for the nodal variation of the ERTS orbit as defined by Equation (20). The lumped coefficients of this constraint are merely the coefficients of the sine and cosine terms in ψ expanded as a sum of all the relevant 14th order harmonics. The "observed" lumped coefficients can then be compared to calculated values from more complete models to judge the accuracy of those models. In addition, the constraint can be used as an equation (set) to improve the geopotential. For this purpose I have used the (q = 0, second order effects) values in Table 2 divided by $\sqrt{2}\ 10^{-5}/\ell^2$ and normalized to the maximum, and find a lumped coefficient equivalent to Equation (20) to be:

$$
\begin{aligned}
10^9 (C, S)^{\Omega}_{14,\,0} = (-11.2 \pm 1.5,\ -32.8 \pm 1.5) &= (C, S)_{15,\,14} - .292(C, S)_{17,\,14} \\
&- .803(C, S)_{19,\,14} - .803(C, S)_{21,\,14} - .565(C, S)_{23,\,14} \\
&- .286(C, S)_{25,\,14} - .054(C, S)_{27,\,14} + .100(C, S)_{29,\,14} + \cdots \qquad (22)
\end{aligned}
$$

The solutions of this equation are plotted in Figure 2. A number of remarks on this figure are in order:

1. The observed value is in fair agreement with values calculated from recent comprehensive fields determined without use of the ERTS orbit.
2. All the fields displayed use surface gravity and satellite tracking data with the exception of GEM 1 which uses only satellite optical data (Smith, Lerch and Wagner, 1973). This aspect does not seem too significant except that the "best" result is achieved for the WGS 72 field (Department of Defense, classified, 1974) which combines

considerably more data than the others. The GEM 6 model (Lerch, et al., 1974) contains considerably more satellite data than GEM 1, as well as surface gravity information, yet shows almost no improvement in predicting the ERTS nodal variation. The gravitational parameters of WGS-72 are determined from extensive Doppler laser and optical tracking on about 30 satellite orbits combined with surface gravity and astrogeodetic data. The field is complete to (19, 19) with resonant and zonal terms as high as 28th degree. The SAO SE 3 (Gaposchkin, 1973) though it uses more satellite (especially laser) and surface data than SAO SE 2 (Gaposchkin and Lambeck, 1970) gives only a marginally improved ERTS result.

3. While taken together the fields show considerable variability (up to 32% in amplitude) from the ERTS result, the variation is much smaller compared to the amplitude estimated on the basis of Kaula's rule. This estimation is $\sqrt{2}$ times the root sum of the squares of the terms in (22) with $(C, S)_{\ell,m} = 10^{-5}/\ell^2$. The relevant terms of WGS-72 for example are (on average) 74% of the rule, while the predicted amplitude with WGS-72 is only 34% of Kaula's rule. The variability of the amplitudes for all the fields is only 14% with respect to the rule.

Finally, it should be emphasized again that these second order effects, proportional to $(n/\dot{\psi})^2$, are particularly important for the variations in deep resonance (commensurability) where $\dot{\psi} \to 0$. For example, they do not appear to have been considered by King Hele (1974) in his analysis of the node and perigee of the orbit of Cosmos 72 during its resonance pass with 15th order geopotential terms in 1972. The character of the perturbation through commensurability for the node and perigee remains to be investigated. These variables probably do not librate since the acceleration of ψ is under the control of the stronger resonance in the mean motion.

SUMMARY

The interaction of secular oblateness effects with long periodic resonance on satellite orbits is calculated. The result is to add a perturbation (of the same phase and period) of order $C_{2,0}\ J_{\ell,m}\ (n/\dot{\psi})^2$ to the direct (first order) resonance effect on the node, perigee and mean anomaly. The direct effect is of order $J_{\ell m}\ (n/\dot{\psi})$. Where $\dot{\psi}$ is small (shallow and deep resonance) the interaction term is significant or dominates the resonant motion of the node and perigee. These effects have not been considered in previous analytic orbit analyses of geodetic satellites. For a number of shallow resonant earth orbits this failure results in significant error with respect to the cross track information from the node. The effects are especially large on the node of the ERTS orbit (18 day period) for which the first order theory would underestimate the amplitude (30m) by a factor of 2.

Results from the ERTS orbit yield a geopotential constraint which will be useful in refining current models for 14th order terms to about the 29th degree.

APPENDIX

INCLINATION FUNCTION RATIOS

The dominance of direct or indirect terms in the resonance of the node, perigee and mean anomaly depends critically on the ratio $(\partial F/\partial I)/F$ of the derivative of the inclination function to its value. For medium to high inclination satellites the resonance is generally dominated by the terms of a species where $\ell - m$ is small and $q = 0$ since they are of the lowest degree and do not contain the eccentricity as a factor (Allan, 1965; Allan, 1973).

It is instructive (and simplest) to work out the root mean square (rms) value of this ratio for these dominant functions where $\ell = m$, $q = 0$ and $k = 1$; the odd order resonances. Allan (1973) gives the normalized inclination function for these as:

$$F = \frac{N_{mm}(2m)\,!}{2^m \left[\frac{1}{2}(m-1)\right]!\left[\frac{1}{2}(m+1)\right]!}\, c^{m+1} s^{m-1} \tag{A1}$$

where

$$N_{mm}^2 = 2(2m+1)/(2m)\,!,$$

and

$$c = \cos I/2, \quad s = \sin I/2$$

The full inclination function for the direct term in (14) is:

$$F' = \frac{(\partial F/\partial I)/F}{\sin I}.$$

Expanded, this is simply:

$$F' = \frac{1}{\sin I}\left[-\frac{1}{2}(m+1)\,c^m s^m + \frac{1}{2}(m-1)\,s^{m-2}c^{m+2}\right]\Big/ c^{m+1}s^{m-1}$$

$$= \frac{1}{\sin I}\left[(m+1)\,c^{-1}s + \frac{1}{2}(m-1)\,cs^{-1}\right] = \frac{1}{\sin I\,cs}\left[-\frac{1}{2}(m+1)\,s^2\right.$$

$$\left.+\frac{1}{2}(m-1)\,c^2\right] = \frac{1}{2\sin I\,cs}\left[m(c^2-s^2)-(c^2+s^2)\right]$$

$$= [m\cos I - 1]/\sin^2 I$$

Except for $m = 1$, F' is singular at $I = 0, 180°$. But since both $(\partial F/\partial I)/\sin I$ and F are zero at $I = 0, 180°$ for $m > 2$, both direct and indirect effects are negligible for the close circular orbit equatorial satellite.

Therefore only the range of $30° \rightarrow 150°$ will be taken for the rms computation.

The indefinite integral:

$$\int (F')^2\,dI = -\frac{m\cot^3 I}{3} - 2m\left[-\frac{\cot^2 I}{3\sin^3 I} - \frac{1}{3\sin I}\right]$$

$$\frac{-\cos I}{3\sin^3 I} - \frac{2\cot I}{3}$$

Therefore:

$$\text{rms } F'(30° \rightarrow 150°) = \frac{3}{2\pi}\int_{\pi/6}^{5\pi/6} (F')^2\,dI$$

$$= \left[\frac{3}{2\pi}(3.46\,m + 6.93)\right]^{1/2} \qquad \text{(A2)}$$

For the 13 rev/day orbit, Equation (A2) yields:

$$\text{rms } F'(30° \rightarrow 150°) = 4.98.$$

This compares to the value 4 estimated in the text from a cruder sinusoidal model for F.

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Table 1

Resonance Perturbations of the Node with Second Order Effects

Effects Due to $J_{\ell m} = \sqrt{2}\, 10^{-5}/\ell^2$

Satellite Orbit	Inclination (deg's)	Apogee Ht. (km)	Eccentricity	Maximum Resonance Perturbation Term: meters		Resonant	
				First Order Only	With Second Order	Period (days)	Order
Agena (1964-1A)	69.9	926	.0010	10.6	10.8	5.0	14
Anna 1B (1962-60A)	50.1	1184	.0082	5.7	5.3	4.8	13
BEB (1964-64A)	79.7	1075	.0135	5.9	5.4	3.0	14
BE-C (1965-32A)	41.2	1322	.0257	5.1	4.8	5.6	13
Courrier 1B (1960-13A)	28.3	1211	.0161	1.4	1.4	3.8	13
DI-C (1967-11A)	40.0	1354	.0532	2.8	2.9	2.5	14
DI-D (1967-14A)	39.5	1890	.0848	8.7	9.2	8.4	13
Echo Rocket (1960-9B)	47.21	1682	.0118	65.3	59.6	11.9	12
GEOS 1 (1965-89A)	59.4	2277	.0719	6.2	7.2	7.0	12
GEOS 2 (1968-2A)	105.8	1588	.033	5.9	4.4	5.7	13
GRS (1963-26A)	49.8	1294	.0598	20.6	25.3	10.7	14
Injun	66.8	995	.0079	7.8	7.6	3.8	14
Midas	95.8	3728	.0112	0.9	0.8	3.0	9
OGO 2	87.4	1515	.0752	14.0	14.3	3.8	14
Oscar 7	89.7	1199	.0224	4.7	4.9	2.2	14
Secor 5	69.2	2420	.0793	3.2	3.2	3.4	12
Transit	66.8	1000	.0076	7.2	7.0	3.5	14
Vanguard 2	32.9	3282	.1641	3.3	3.2	2.7	11
SBN-2	90.0	1128	.0058	2.8	3.1	2.4	13
Vanguard 3	33.3	3748	.1901	24.9	42.2	12.2	11
Telstar	44.8	5640	.2429	22.9	21.8	14.9	9
ERTS-1	99.1	908	.0080	10.2	40.2	18.0	14

Table 2

Resonant Effects on the Node of the ERTS Orbit

Unit = 10^{-4} Degrees

Amplitude of Effects Due to $(C, S)_{\ell,m} = 10^{-5}/\ell^2$

(ℓ, m) q = 0	First Order Effect Only	With Second Order Effect	(ℓ, m) q = -1	With Second Order Effect	(ℓ, m) q = +1	With Second Order Effect
15,14	-0.72	-3.71	14,14	.00	14,14	-.02
17,14	1.91	.85	16,14	.02	16,14	.03
19,14	2.05	1.86	18,14	.00	18,14	.03
21,14	1.41	1.52	20,14	-.01	20,14	.02
23,14	.72	.90				
25,14	.24	.38				
27,14	-.03	.06				
29,14	-.15	-.10				
31,14	-.18	-.15				
28,28	.05					
30,28	.05					
32,28	-.03					
34,28	-.05					
36,28	-.04					

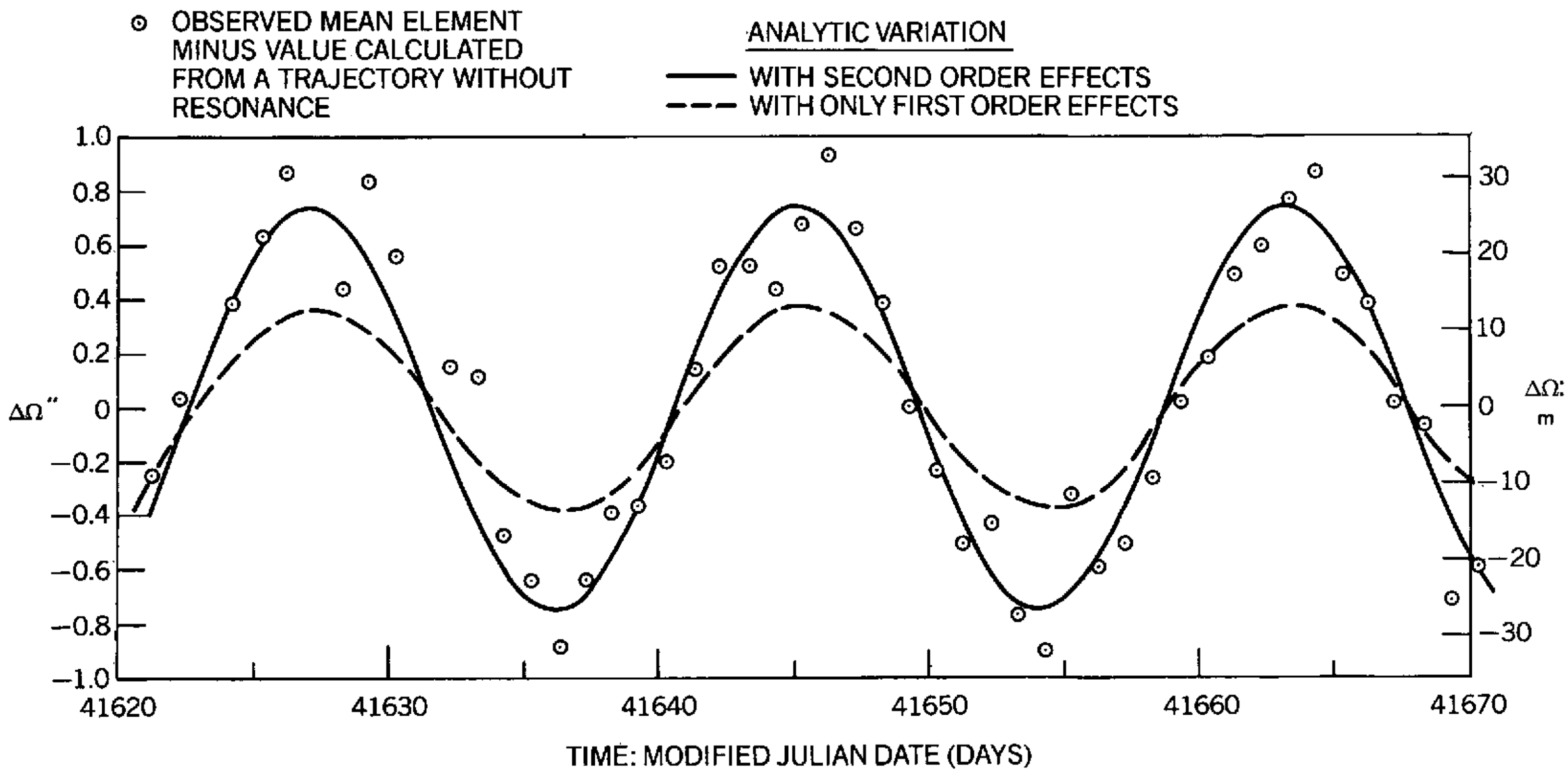

Figure 1. Resonant variation of the node of the ERTS orbit.

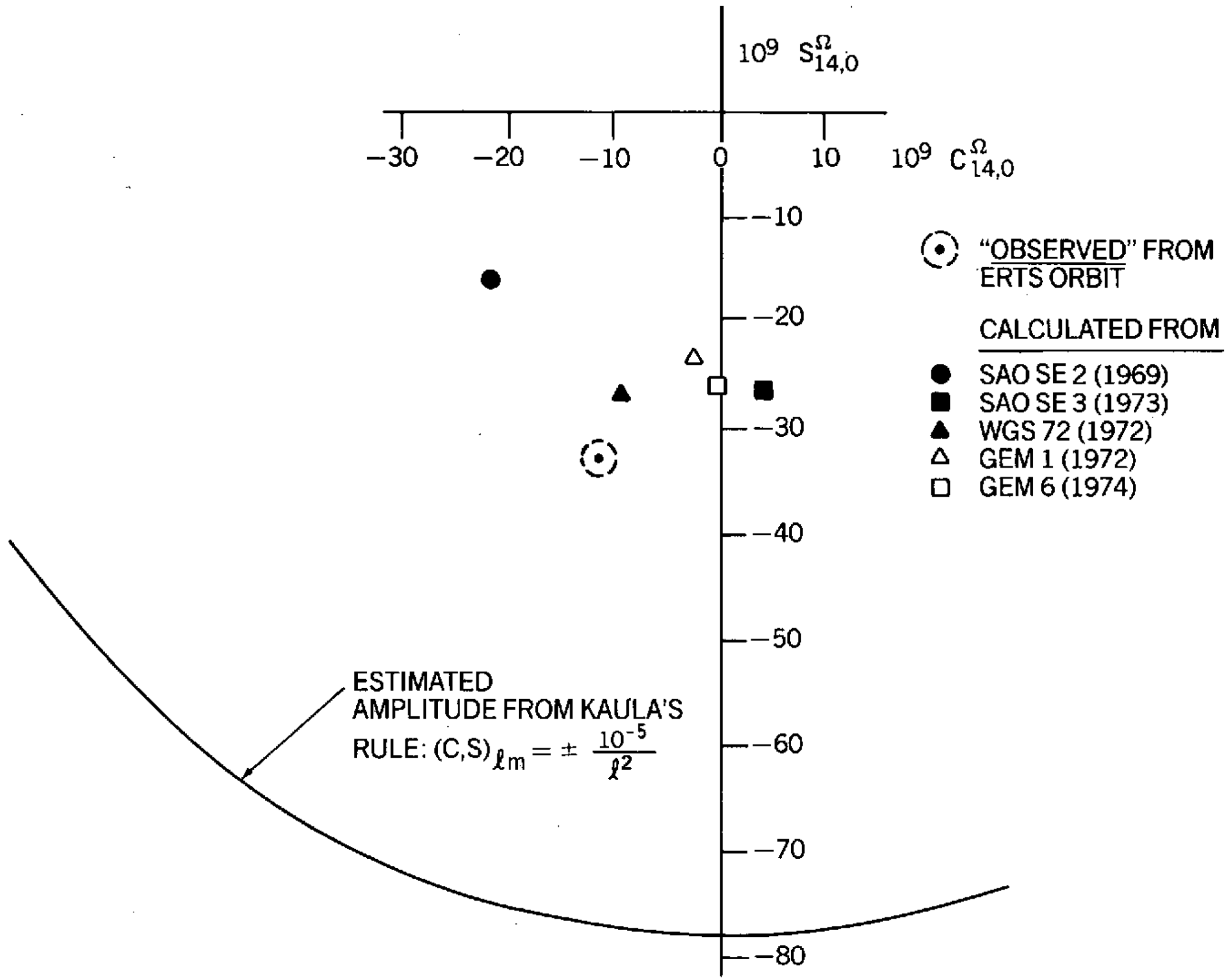

Figure 2. Lumped coefficient from resonance in node for ERTS.