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UNCERTAINTIES IN DERIVED TEMPERATURE-HEIGHT PROFILES

R. A. MINZNER

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ABSTRACT

Nomographs have been developed for relating uncertainty in temperature T (or in the corresponding scale height H) to uncertainty in the observed height profiles of both pressure p and density ρ. The relative uncertainty $\delta T/T$ is seen to depend not only upon the relative uncertainties $\delta p/p$ or $\delta\rho/\rho$, and to a small extent upon the value of T or H, but primarily upon the sampling-height increment Δh, the height increment between successive observations of p or ρ. For a fixed value of $\delta p/p$, the value of $\delta T/T$ varies inversely with Δh. For $\Delta h = \sqrt{2} \cdot H$, which is of the order of 10 km, $\delta T/T = \delta p/p$. For $\Delta h = 0.01$ km, $\delta T/T$ is about 1000 $\delta p/p$, while for $\Delta h = 35$ km, $\delta T/T$ is about 0.3 $\delta p/p$. In the case of T derived from density-height data, this inverse relationship between $\delta T/T$ and Δh applies only for large values of Δh, i.e., for $\Delta h > 35$ km. For $\Delta h < 1$ km, $\delta T/T \simeq \delta\rho/\rho$, independent of the size of Δh. No limit exists in the fineness of usable height resolution of T which may be derived from densities, while a fine height resolution in pressure-height data leads to temperatures with unacceptably large uncertainties.

CONTENTS

ILLUSTRATIONS

SYMBOLS

Basic Quantities and Coefficients		Equation or Figure
A	The dimensionless coefficient indicating the factor by which $(\delta p/p)$ is amplified to yield $(\delta T/T)$	(20)
G	The standard geopotential gravitational constant $9.80665\ m^2 sec^{-2}\ (m')^{-1}$	(1)
H	Scale height in units of geopotential height, m' (or km')	(1)
h	Geopotential height in units of m' (nearly equal to geometric height, but accounting for height variation of the acceleration of gravity: i.e., $Gdh = g(z)dz$	(1)
k	A dimensional constant $0.0341632\ K/m'$ resulting from the combining of the three constants GM/R into a single value	(4)
M	mean molecular weight of air $28.9644\ kg\ kmol^{-1}$	(1)
p	atmospheric pressure Nm^{-2}	(1)
R	Universal gas constant 8.31432×10^3 joules $K^{-1}\ kmol^{-1}$	(1)
S	The value of a particular series of terms involving density ratios	(30)
T	Atmospheric Temperature K	(1)
x	A general designation for a function of a number of generalized variables	(14)

Basic Quantities and Coefficients (Cont'd.)		Equation or Figure
y	A general designation for the set of variables of which x is a function	(14)
α	A subscripted dimensionless coefficient with a value near unity	(23)
ρ	atmospheric density kg m^{-3}	(7)

Single Subscripts

1	Refers to a particular height h_1 so that p_1, T_1, ρ_1, and h_1 all refer to the same point in a given equation	(4), (11)
2	Similar to 1	(4)
i	Refers to an individual member of a set of related variables having the general designation y_i	(14)
i	Also refers to a variable in an isothermal atmosphere as ρ_i in the graph of (ρ_i/ρ_r) versus h in Figure 1	
j	A general designation for an integer which may vary between 2 and q, and which is simultaneously associated with a geopotential height as h_j and with the related density as ρ_j	(21)
q	A general designation for an integer which may have any positive value and which is associated with the lowest density-height data point (i.e., h_q, ρ_q) involved in a particular evaluation of an integral to determine the value of the related temperature T_q.	(11)

	Single Subscripts (Cont'd.)	Equation or Figure
q	Also used as S_q where q implies the number of terms in the series S	(30)
r	Designates a specific reference value for the basic quantity of which it is a subscript, i. e., p_r and ρ_r	Figure 1

Double Subscripts

1, 2	Refers to a particular height layer as between h_2 and h_1, where h_1 and h_2 are ordered in such a way that the quantities within the subscripted parentheses have particular signed significance, i. e., Δh is always positive while ΔH is positive only when dT/dh is positive in the related layer	(4) (10a), (23) and (24)
j-1, j	Similar to 1, 2	(25)
j, j+1	Similar to 1, 2	(26)
q-1, q	Similar to 1, 2	(27)
q, max	designating maximum value of S_q, as in Figure 6.	
q, min	designating minimum value of S_q, as in Figure 6.	(36)

Operators and Functions

	Overbar as in $\overline{T}$ or $\overline{H}$ indicates the mean value of T or H for the related layer	(4)
$\sqrt{\ }$	Square root as $\sqrt{2}$	(20)
$\int$	Integral as $\int \rho(h)dh$	(11)

Operators and Functions (Cont'd.)		Equation or Figure
∂	Partial differential as in ∂x	(14)
d	Differential as in dh	(1)
ℓn	Natural logarithm as $\ell n p$ and $\ell n \rho$	(1) and (7)
Δ	An increment as in Δh and ΔH	(4) and (10a)
δ	An increment or random uncertainty as in δy_i, δp_1, $\delta \Delta h$, δT_1, $\delta \rho_j$, δT_q, and $\delta \rho_q$	(14), (15) and (21)
Σ	A summation of a set of terms	(12), (14)

AN ANALYSIS OF THE ERRORS ASSOCIATED WITH THE DETERMINATION OF ATMOSPHERIC TEMPERATURE FROM ATMOSPHERIC PRESSURE AND DENSITY DATA

I. INTRODUCTION

Relationships between pressure, temperature, and height in the earth's atmosphere are well known, and for many years have been the basis for height determination in balloon-radiosonde flights making observations of pressure and temperature in meteorological probings of the lowest 30 kilometers of the earth's atmosphere. When radar-tracked rocket vehicles extended the potential atmospheric probing capabilities to heights up to 100 kilometers, into regions where pressure is still measurable, but where existing technology does not allow for immersion sensing of temperature, these same mathematical relationships were used to extract temperature from the measured pressures and radar-derived rocket-height data. The unsmoothed temperature-height values from many of these soundings represented a very jagged height profile, with the degree of jaggedness apparently increasing as the height increment between successive data points decreased. Investigators usually have been unable to determine how much of this jaggedness represents real temperature variability, and how much is attributable to measurement error.

The hydrostatic equation and the equation of state lead to another set of height relationships: those between atmospheric density, temperature, and height, such that temperature may also be computed from density-height data. In cases where such computations have been made, particularly in the height region of 30 to 100 km, the resulting temperature-height profiles appeared to be less jagged than those derived from pressure-height data with comparable height resolution.

The apparent difference in the jaggedness of density-derived temperature-height profiles from those associated with pressure-height data suggest that the height increments of the pressure and density data do affect the uncertainty in the derived temperatures, and that the influence of height increments in relation to density data may be different from that in relation to pressure data. Obviously, the error analyses presented in this paper, involving both the pressure-temperature-height relationship, and the density-temperature-height relationship, are needed. The error analyses confirm the fact that important differences do exist between these two sets of relationships, particularly in regard to the influence of the height increment on the propagation of measurement uncertainties into the temperature-height profile. These differences strongly favor the use of density-height data over pressure-height data.

II. FUNDAMENTAL CONSIDERATIONS

A. Pressure-Height Relationships

The equation of state, when combined with the differential form of the hydrostatic equation to eliminate density ρ, yields an expression frequently referred to as the hypsometric equation:

$$\frac{d\ell np}{dh} = \frac{-GM}{RT} = \frac{-1}{H} \qquad (1)$$

where p is atmospheric pressure,

h is a measure of the height above sea level, in geopotential meters m′

T is absolute temperature of the atmosphere at h,

R is the universal gas constant,

M is the mean molecular weight of air,

G is a constant when h is expressed in geopotential, and

H is the scale height in geopotential units.

Solving equation (1) for T and H, respectively, yields

$$T = \frac{GM}{R} \cdot \frac{-dh}{d\ell np} \qquad (2)$$

and

$$H = \frac{RT}{GM} = \frac{-dh}{d\ell np} \qquad (3)$$

When values of ℓnp versus h are known from numerical data rather than from an analytical function, it is convenient to replace the expression $-dh/d\ell np$, which applies to a specific height, with a numerical approximation $(h_2 - h_1)/(\ell np_1 - \ell np_2)$.

In this approximation, p_1 is the pressure at h_1, and p_2 is the pressure at h_2. The approximation therefore represents the mean value of the reciprocal of the derivative over the height interval (h_2-h_1). When the point value of the derivative is replaced by the numerical approximation, in both equations (2) and (3), the related values of T and H no longer apply to a single height, but rather become the mean values $\overline{T}$ and $\overline{H}$, respectively, associated with the height interval (h_2-h_1). Thus, we have

$$\overline{T} = \frac{GM}{R} \cdot \frac{h_2 - h_1}{\ell np_1 - \ell np_2} = k \cdot \frac{(\Delta h)_{1,2}}{\ell np_1 - \ell np_2} \tag{4}$$

where $k = GM/R = .034163\ ^\circ K/m'$, and $(\Delta h)_{1,2} = h_2 - h_1$, and where, from the relationship between T and H implicity in equation (1), we see that

$$\overline{H} = \frac{h_2 - h_1}{\ell np_1 - \ell np_2} = \frac{(\Delta h)_{1,2}}{\ell np_1 - \ell np_2} \tag{5}$$

From equation (5), we obtain the following expression which will be of special importance in the uncertainty analysis in Section III:

$$\frac{1}{\ell np_1 - \ell np_2} = \left(\frac{\overline{H}}{\Delta h}\right)_{1,2} \tag{6}$$

The interrelationships between ℓnp, $d\ell np/dh$, T, and H, all as a function of height as expressed by equations (1), (2) and (3), are shown pictorally by four of the six height-related graphs of Figure 1. This figure, which depicts the properties of a portion of the 1974 U. S. Standard Atmosphere (in preparation), was developed from a preliminary set of abbreviated tables representing a portion of this revised standard atmosphere (Kantor and Cole 1973). Geopotential

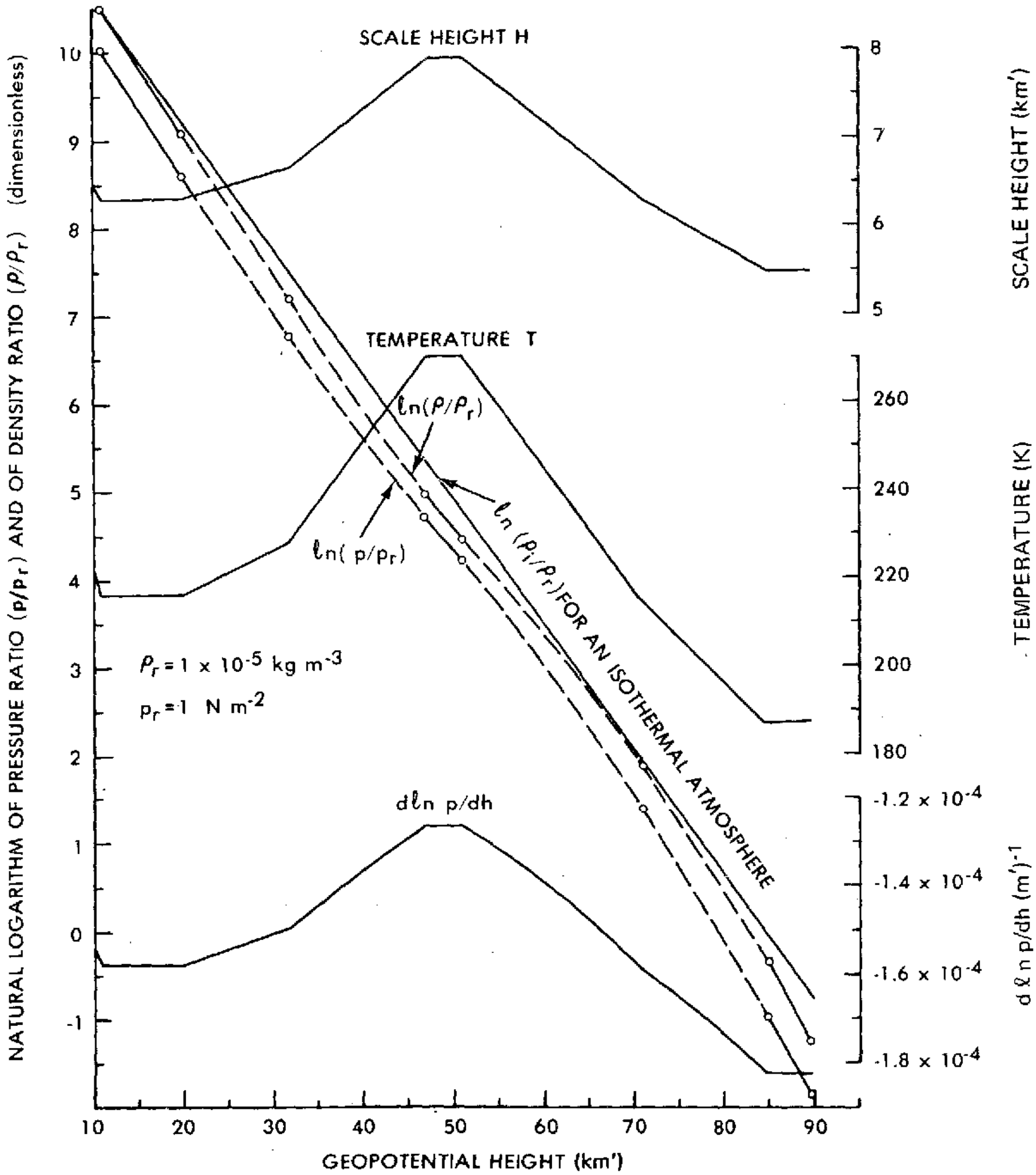

Figure 1. The U.S. Standard Atmosphere as Represented by Graphs of Five Atmospheric Properties versus Height, i.e., Temperature, Scale Height, Natural Log of Density Ratio, Natural Log of Pressure Ratio, and Derivative of Natural Log of Pressure Ratio with Respect to Height, Plus a Sixth Property for Comparison, i.e., Natural Log of Density Ratio versus Height for an Isothermal Atmosphere

height is scaled linearly along the abscissa of Figure 1. Atmospheric pressure, which is plotted in the form of the natural logarithm of the ratio (p/p_r), has its ordinate scale at the left of the figure, and is depicted by the lower of the three lines diagonally crossing the entire figure from upper left to lower right. In this presentation of pressure, p_r in the ratio (p/p_r) has been made equal to one Newton per square meter (i.e., $1\ N\ m^{-2}$), so that the numerical values of (p/p_r) are those of p in $N\ m^{-2}$, while the scale of $\ell n(p/p_r)$ at the left of the figure implies no dimensions.

The graph of $d\ell np/dh$, which quantity is negative and inversely proportional to T, shows that the slope of $\ell n(p/p_r)$ versus h, or equivalently the slope of ℓnp versus h, is constant only in height regions over which T is invariant. These portions of the graph of $d\ell np/dh$ versus height corresponding to constant T (i.e., the three height intervals 11 to 20 km′, 47 to 57 km′, and 84.852 to 89.716 km′) consist of horizontal straight-line segments, while the remainder of that graph consists of 5 curved segments. The scale height, which is directly proportional to T, is seen to remain within the range of 5.45 and 7.95 km′, with a median value of 6.5 km′.

B. Density-Height Relationship 1

When the differential form of the hydrostatic equation is combined with the differential form of the equation of state to eliminate the differential of pressure, we obtain the following relationship between temperature and the density-height function:

$$T = \frac{-dh}{d\ell n\rho}\left[\frac{GM}{R} + \frac{dT}{dh}\right] \tag{7}$$

In the absence of an analytical expression for $\rho(h)$, a numerical approximation for $dh/d\ell np$ leads to a mean value of temperature $\bar{T}$ for the height interval $h_2 - h_1$:

$$\bar{T} = \frac{h_2 - h_1}{\ell n\rho_1 - \ell n\rho_2}\left[\frac{GM}{R} + \frac{T_2 - T_1}{h_2 - h_1}\right] \tag{8}$$

In terms of scale height, this expression becomes

$$\bar{H} = \frac{h_2 - h_1}{\ell n\rho_1 - \ell n\rho_2}\left[1 + \frac{H_2 - H_1}{h_2 - h_1}\right] \tag{9}$$

It is important to note that while equation (8) expresses the mean temperature for the layer h_1 to h_2, the expression involves the average gradient of T with respect to h within that layer. A similar situation prevails for the expression of the mean scale height in equation (9), where $\bar{H} = (H_2 + H_1)/2$. Since prior knowledge of the value of neither $(T_2 - T_1)/(h_2 - h_1)$ nor $(H_2 - H_1)/(h_2 - h_1)$ is generally available, equations (8) and (9) are not of themselves useful relationships for temperature-height determination. One version of the following expression derived from equation (9), however, will be of special importance in an uncertainty analysis in Section IV. This expression is

$$\frac{1}{\ell n\rho_1 - \ell n\rho_2} = \frac{(H_2 + H_1)/2}{(h_2 - h_1) + (H_2 - H_1)} \tag{10}$$

Equation (10) is written for the case in which $h_2 > h_1$, as when data are being analyzed from lower to greater heights, such that $\rho_1 > \rho_2$. If the equation is to apply to data being analyzed from greater to lower heights, as is the preferred case for the integral form of the density-height function, involving the normal atmosphere, h_1 is greater than h_2 such that $\rho_2 > \rho_1$, and equation (10) must be rewritten as

$$\frac{1}{\ell n\rho_2 - \ell n\rho_1} = \frac{(H_1 + H_2)/2}{(h_1 - h_2) + (H_1 - H_2)} = \left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{1,2} \tag{10a}$$

In this expression Δh is always positive, and ΔH is positive in regions of positive temperature gradients, zero in regions of zero temperature gradients, and negative in regions of negative temperature gradients.

C. Density-Height Relationship 2

The integral of the hydrostatic equation, when combined with the equation of state to eliminate the pressure, yields the following directly useful integral expression relating temperature to density:

$$T_q = \frac{\rho_1}{\rho_q} \cdot T_1 + \frac{GM}{R} \cdot \frac{1}{\rho_q} \cdot \int_{h_1}^{h_q} \rho(h)\, dh \tag{11}$$

The temperature T_q is not the mean value of temperature for some layer, as in the case of $\bar{T}$ in equation (4), rather, it is the temperature at the specific height h_q.

It has been demonstrated (Minzner, et al. 1964 and 1966) that the properties of equation (11), when applied to a helium atmosphere, differ markedly from those observed when the equation is applied to an argon atmosphere. With an air atmosphere having a mean molecular weight of about 29, the properties of this equation are similar to those found when the equation is applied to an argon atmosphere in which the integration optimumly proceeds from the greatest to the lowest altitude of the density-height data. In this situation, T_1, ρ_1, and h_1 are each associated with the greatest height of the data set, while T_q, ρ_q, and h_q are each associated with the running value of h. This is the height for which the value of T_q is being computed, a height which varies progressively from h_1 to the lowest height of the data set as the calculation of the profile proceeds.

Because $\rho(h)$ is usually known numerically rather than analytically, it is convenient to replace the integral of equation (11) with an appropriate series approximation, one of which is governed by the logarithmic trapezoidal rule (Smith 1965, Minzner, et al. 1965). With the use of this approximation, equation (11) may be rewritten as

$$T_q = \frac{\rho_1}{\rho_q} T_1 + \frac{GM}{R} \cdot \frac{1}{\rho_q} \cdot \sum_{j=2}^{q} \frac{(h_{j-1} - h_j)(\rho_j - \rho_{j-1})}{\ell n \rho_j - \ell n \rho_{j-1}} \tag{12}$$

The logarithmic-trapezoidal-rule approximation is a particularly suitable one for the integration of atmospheric density with respect to height, because, in a graph of measured values of $\ell n \rho$ versus h, the straight-line segments between successive data points exactly represent the conditions of the logarithmic trapezoidal rule, and also very closely represent the conditions of the real atmosphere. The latter is evident from the nearly straight-line segments of an idealized version of the real atmosphere shown in the graph of $\ell n(\rho/\rho_r)$ versus height in Figure 1. For this graph, which represents the densities of the 1974 U.S. Standard

Atmosphere, the value of ρ_r was chosen to be 1×10^{-5} kg m^{-3}, so that the single ordinate scale at the left of the figure applies to both $\ell n(p/p_r)$ and $\ell n(\rho/\rho_r)$. The shape of the curve $\ell n(\rho/\rho_r)$ versus h is identical to the shape which the curve $\ell n\rho$ versus h would have, since the values of $\ell n(\rho/\rho_r)$ vs. h and of $\ell n\rho$ vs. h are off-set by the constant difference $\ell n\rho_r$ at all heights.

Figure 1 also contains a graph of $\ell n(\rho_i/\rho_r)$ in the form of a continuous straight line immediately above the graph of $\ell n(\rho/\rho_r)$. This single straight line diagonally across the entire figure is characteristic of the density of an isothermal atmosphere extending upward from a height of 7.625 km where the standard-atmosphere density is 5.8281×10^{-1} kg m^{-3}, and where the temperature and the corresponding scale height are 238.587 K and 7,000 km$'$, respectively. For this specialized atmosphere, the logarithmic-trapezoidal rule represents a series which exactly duplicates the integral of equation (11). The small deviations of the slopes of $\ell n(\rho/\rho_r)$ versus height from those of $\ell n(\rho_i/\rho_r)$ versus height show the small influence of the variation of atmospheric temperature from the fixed value, 238.587 K, (in the height region of 10 to 90 km$'$) on the general shape of the curves of $\ell n(\rho/\rho_r)$ and $\ell n\rho$ versus h. (The influence of temperature variation upon the curves of $\ell n(p/p_r)$ and ℓnp versus h is similarly small.)

In the evaluation of T_q by equation (11), or by any appropriate approximation of that equation as exemplified by equation (12), a knowledge of the initial temperature T_1 is of importance only for the upper regions of the profile. For $h_q = h_1$, the value of $(\rho_1/\rho_q)T_1$ is exactly T_1, since $\rho_1 = \rho_q$ and the integral term is, of course, zero. As the value of h_q decreases from h_1, i.e., as $(h_1 - h_q)$ increases, the value of the relative contribution of the density-ratio term decreases, while that of the integral term correspondingly builds up. When $(h_1 - h_q)$ increases from zero, first by one scale height and then by three scale heights, the relative contribution of the density-ratio term to T_q decreases from 100% to about 37%, and then to about 3%, because of the large decrease in the value of (ρ_1/ρ_q) over these height regions. Simultaneously, the value of the integral term grows correspondingly. At heights of more than three scale heights below h_1, the value of T_q is determined almost completely by the integral term alone. Figure 2 shows the relative contributions made to T_q by the density-ratio term, and by the integral term of equation (11) as a function of the range of the limits of integration $(h_1 - h_q)$ expressed in units of scale height.

The graphs in Figure 2 are based on the density-height profile for an isothermal atmosphere (T = 238.587 K) for which $\ell n\rho$ is a linear function of height as previously shown in the graph of $\ell n(\rho_i/\rho_r)$ in Figure 1. Only small variations from the values of the two terms of equation (12) depicted in Figure 2 would be seen for calculations based on a real, variable-temperature atmosphere.

The elimination of T between equation (12) and equation (3) yields the following expression for scale height:

$$H_q = \frac{\rho_1}{\rho_q} \cdot H_1 + \frac{1}{\rho_q} \cdot \sum_{j=2}^{q} \frac{(h_{j-1} - h_j)(\rho_j - \rho_{j-1})}{\ell n\rho_j - \ell n\rho_{j-1}} \qquad (13)$$

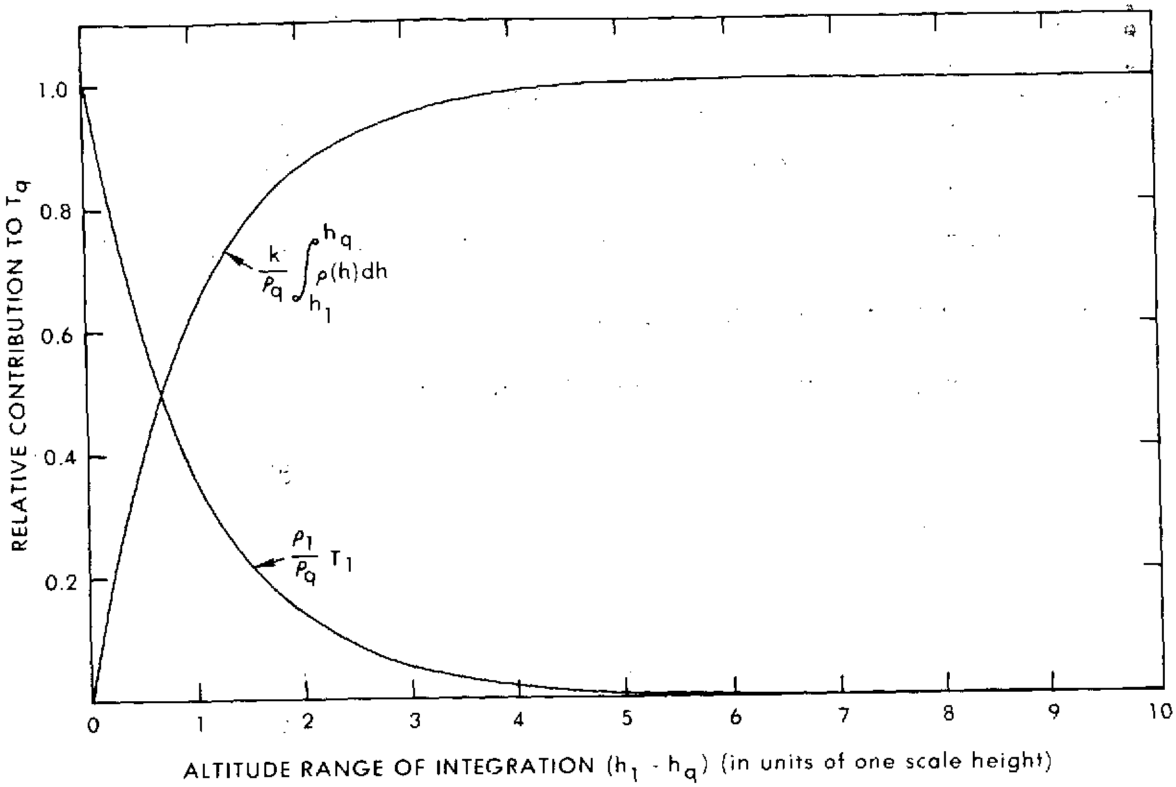

Figure 2. Relative Contribution Made to the Calculated Temperature by the Density-Ratio Term and by the Integral Term of Equation (11) as a Function of the Size of the Altitude Range of Integration, for a 238 K Isothermal Atmosphere

The relative contribution of each of the two terms of this equation to H_q, as a function of the range of integration (h_1 - h_q), follows exactly the same pattern shown in Figure 2 with regard to T_q.

Atmospheric soundings of both pressure and density have for many years served as the basis for the determination of the height profiles of temperature and scale height. From pressures, these profiles are obtained through equations (4) and (5) respectively, while from densities, they are obtained through one or another version of equations (12) and (13), respectively. It is obvious that the uncertainties of these derived temperatures and scale heights are a function of the uncertainty in the measured quantities, pressure or density. Somewhat less obvious is the fact that the height interval between successive observations of pressure or density strongly influences the propagation of the observational uncertainties into the computed temperatures and scale heights. A rigorous error analysis of both of these pairs of equations demonstrates this situation. The pressure-related equations are analyzed in Section III, while the density-related equations are analyzed in Section IV.

III. UNCERTAINTY OF TEMPERATURES DEDUCED FROM PRESSURES

The error-analysis method employed is the first-order Gaussian method, wherein each variable y_i entering into the expression of a particular function of these variables $x(y_i)$ is assumed to have an observational uncertainty δy_i which meets the conditions of a Gaussian or normal distribution about $\bar{y}_i$, where $\bar{y}_i$ is the mean of a set of individual observations of the ith variable, or the true value of the ith variable. Thus, if the value of x is determined from the functional expression $x(y_i)$, the value of δx, the implicit uncertainty in x, is given by

$$(\delta x)^2 = \sum_i \left(\frac{\partial x}{\partial y_i} \cdot \delta y_i \right)^2 \tag{14}$$

Applying this relationship to the variables of equation (4) yields

$$\delta \bar{T} = \left[\left(\frac{\partial \bar{T}}{\partial p_1} \cdot \delta p_1 \right)^2 + \left(\frac{\partial \bar{T}}{\partial p_2} \cdot \delta p_2 \right)^2 + \left(\frac{\partial \bar{T}}{\partial \Delta h} \cdot \delta \Delta h \right)^2 \right]^{1/2} \tag{15}$$

It may be shown that the partial derivative of $\bar{T}$ with respect to any one of the independent variables y_i in equation (4) has the general form

$$\frac{\partial \bar{T}}{\partial y_i} \cdot \delta y_i = \frac{-k\Delta h}{(\ell n p_1 - \ell n p_2)^2} \cdot \frac{\delta y_i}{y_i} \tag{16}$$

where Δh is associated with $(h_2 - h_1)$ without benefit of subscripts. Applying this generalized expression to equation (15) yields

$$\delta \bar{T} = \frac{+k\Delta h}{(\ell n p_1 - \ell n p_2)^2}\left[\left(\frac{\delta p_1}{p_1}\right)^2 + \left(\frac{\delta p_2}{p_2}\right)^2 + \left(\frac{\delta \Delta h}{\Delta h}\right)^2\right]^{1/2} \qquad (17)$$

where δp_1 and δp_2 are the pressure uncertainties of two consecutive pressure-height values, and $\delta\Delta h$ is the uncertainty of the height interval Δh between the corresponding two pressure-height values.

Dividing each side of equation (17) by the appropriate side of equation (4) yields the relative uncertainty

$$\frac{\delta \bar{T}}{\bar{T}} = \frac{1}{\ell n p_1 - \ell n p_2}\left[\left(\frac{\delta p_1}{p_1}\right)^2 + \left(\frac{\delta p_2}{p_2}\right)^2 + \left(\frac{\delta \Delta h}{\Delta h}\right)^2\right]^{1/2} \qquad (18)$$

It is convenient to assume that all the uncertainty of a pressure-height point is in the pressure, and that the height increment is exact. In this case, the values of both δp_1 and δp_2 are correspondingly increased over their actual values, and the term involving $\delta\Delta h$ in equation 18 vanishes. It is further assumed that the relative uncertainty in measured values of p is the same at h_1 and h_2, i.e., $\delta p_1/p_1 = \delta p_2/p_2 = \delta p/p$. These two assumptions permit equation (18) to be reduced to the simpler form,

$$\frac{\delta \bar{T}}{\bar{T}} = \frac{\sqrt{2}}{\ell n p_1 - \ell n p_2} \cdot \frac{\delta p}{p} \qquad (19)$$

From equation (6) it is apparent that $1/(\ell n p_1 - \ell n p_2)$ in equation (19) may be replaced by $(\bar{H}/\Delta h)_{1,2}$ where $\bar{H}$ is the mean scale height for the layer

$h_2 - h_1 = (\Delta h)_{1,2}$. Thus, equation (19) may be rewritten as

$$\frac{\delta\bar{T}}{\bar{T}} = \sqrt{2}\left(\frac{\bar{H}}{\Delta h}\right)_{1,2} \cdot \frac{\delta p}{p} = A_{1,2} \cdot \left(\frac{\delta p}{p}\right) \qquad (20)$$

This equation indicates that $(\delta\bar{T}/\bar{T})$, the relative uncertainty in $\bar{T}$ the mean temperature of a layer $(\Delta h)_{1,2}$ (i.e., the layer bounded by pressures p_1 and p_2), is equal to the relative uncertainty in the pressure measurements $(\delta p/p)$ times a coefficient $A_{1,2}$. This coefficient is seen to depend upon only two quantities, the mean temperature $\bar{T}$ of the layer $h_2 - h_1$ ($\bar{T}$ being implicit in $\bar{H}$), and $(1/\Delta h)$, the reciprocal of the thickness of the pressure-sampling interval $(h_2 - h_1)$. Since the maximum value of $\bar{T}$ in the earth's atmosphere below $h = 100\,\text{km}'$ is less than twice its minimum value, and since Δh can, in principle, be made to vary over many orders of magnitude, it is apparent that Δh is the dominant factor in determining the propagation of $(\delta p/p)$ into $(\delta\bar{T}/\bar{T})$.

A graph of the coefficient $A_{1,2}$ as a function of Δh for three specific values of $\bar{T}$, 169.10 K, 241.57 K, and 314.04 K, is given in Figure 3. The first and third of these temperatures were selected in part because they are close to the lowest and highest atmospheric temperatures normally observed at heights below 120 km, while the second is the mean of the extremes. In addition, these three values were specifically selected to correspond, respectively, to a particular set of three scale heights, i.e., $(7/\sqrt{2})$, $(10/\sqrt{2})$, and $(13/\sqrt{2})$ km$'$. In each of these three cases, the value of $A_{1,2}$ is unity when Δh (equal to $\sqrt{2} \cdot \bar{H}$) is a particular integer multiple of one geopotential kilometer, i.e., 7, 10, and 13 km$'$, respectively.

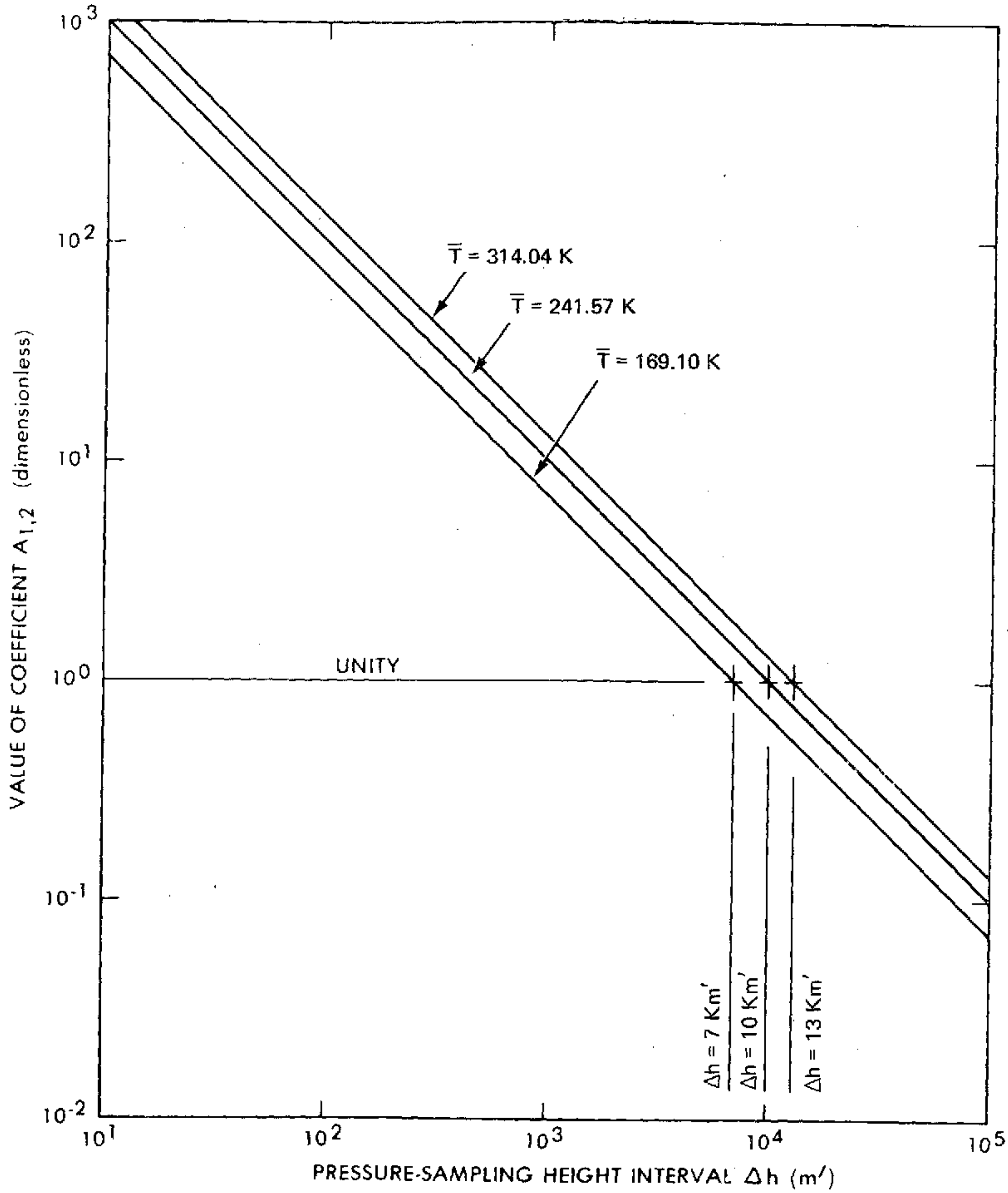

Figure 3. Value of the Coefficient "$A_{1,2}$" in Equation (20) as a Function of Pressure-Sampling Height Interval for Each of Three Atmospheric Temperatures Representing Approximately the Maximum, Median, and Minimum Values, Respectively, for Heights Below 100 Geopotential Kilometers

It is interesting to note that, for the entire range of normally observed temperatures at heights below 100 km′, the pressure-sampling interval Δh corresponding to unity for the coefficient $A_{1,2}$ varies between the limited range of 7 to 13 km′. The pressure-sampling height interval, however, may actually vary over several orders of magnitude depending upon the design of the measuring system. The value of the coefficient $A_{1,2}$ could correspondingly vary over several orders of magnitude depending upon the choice of Δh.

Concentrating on the median value of $\breve{T}$, and allowing the pressure-sampling interval Δh to decrease, first from 10^4 m′ to 10^3 m′, and then to 10^2 m′, causes the value of $A_{1,2}$ to increase first from 1 to 10, and then to 100, respectively, such that the uncertainty in the value of $\bar{T}$ as expressed by equation (20) increases by identical factors for a fixed uncertainty in the pressure. Conversely, as Δh is increased from 10^4 m′ to 10^5 m′, the value of $A_{1,2}$ decreases from unity to 0.1, such that the uncertainty in a related mean value of T decreases correspondingly.

Figure 3 shows that when mean temperatures are deduced for successive layers from pressure observations having a fixed uncertainty, finer height resolution in the resulting temperature-height profile is obtained at the expense of temperature accuracy, while increased temperature accuracy is achieved at the expense of height resolution. Since the ultimate accuracy of pressures obtained from any high-altitude balloon-borne or rocket-borne pressure-sensing device is limited by considerations of various perturbing phenomena such as outgassing, boundary layer, and shock wave, the value of $\delta p/p$ has a practical lower bound. Thus, $(\delta\bar{T}/\bar{T}) \cdot (\Delta h/\sqrt{2} \cdot \bar{H})$ which is equal to $\delta p/p$ similarly has a practical lower bound, and for this minimum value of $\delta p/p$, $\delta\bar{T}/\bar{T}$ is governed by the value of the ratio $(\Delta h/\bar{H})$.

Figure 3 shows the effect of variations of both T and Δh on the value of the coefficient $A_{1,2} = (\sqrt{2} \cdot \bar{H}/\Delta h)$. By expressing Δh in multiples of one scale height, the number of variables in equation (20) is effectively reduced by one, and values of $\delta\bar{T}/\bar{T}$ can be plotted as a function of $\Delta h = n \cdot \bar{H}$ (where n is the value along the abscissa) for any particular value of $\delta p/p$. Figure 4 presents such a graph for each of nine values of pressure uncertainty.

In Figure 4, the straight-line graph for 1% uncertainty in p intersects the coordinate for 1% uncertainty in $\bar{T}$ at the abscissa coordinate value of 1.414. This same straight-line graph for 1% uncertainty in p intersects the ordinate values of 10%, 100%, and 1000% at abscissa values of Δh equal to 0.1414, 0.01414, and 0.001414, respectively. This series of four successively decreasing values of Δh, each being one tenth of the preceeding one, yields a geometric series of four increasing values of $\delta\bar{T}/\bar{T}$, each being ten times the preceeding one. A similar situation prevails for each of the eight other values of $\delta p/p$, for which lines have been plotted. The value of $\delta\bar{T}/\bar{T}$ is obviously varied over orders of magnitude by comparable variations of the value of the ratio $\bar{H}/\Delta h$. For height increments with a value of $\sqrt{2}$ times one scale

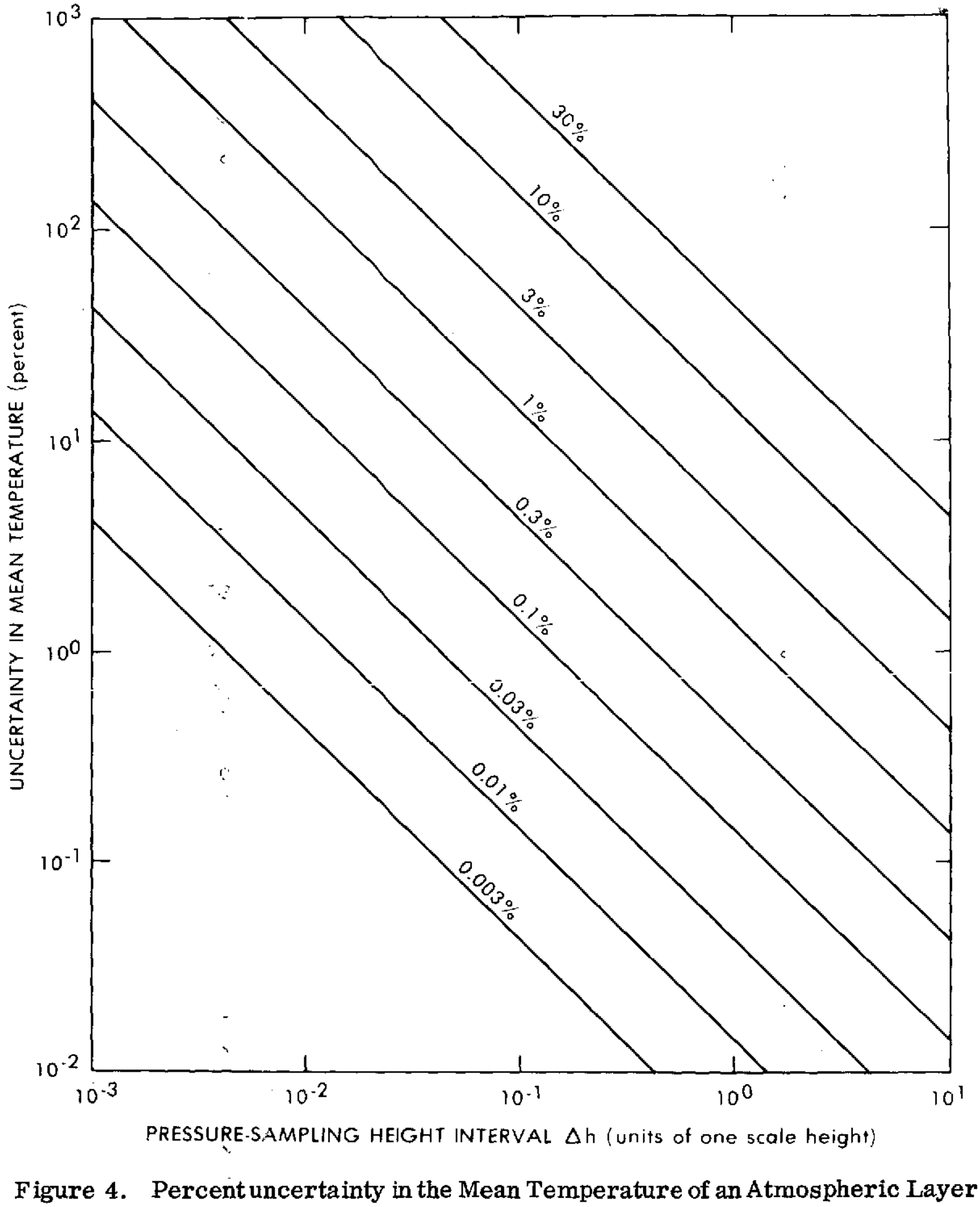

Figure 4. Percent uncertainty in the Mean Temperature of an Atmospheric Layer (for any one of nine percent uncertainties in the pressure-height data) as a function of the Pressure-Sampling Height Interval (i.e., the thickness of that Layer) when that Mean Temperature is Deduced from the Atmospheric Pressure at the Upper and Lower Boundaries of that Layer, and When the Mean Temperature is At or Near 241.57 K.

height, relative temperature uncertainties are seen to be equal to relative pressure uncertainties for all nine values plotted.

In order to show simultaneously the small additional influence produced by the allowable range in the value of $\bar{T}$ as it appears both in $\delta \bar{T}/\bar{T}$ as well as intrinsically in H, the single-line graph for each value of $\delta p/p$ in Figure 4 is expanded into a band in Figure 5 where the pressure-sampling height is expressed in meters as in Figure 3.

The lower left-hand edge of each band corresponds to T = 169.10 K, while the upper right-hand edge of each band corresponds to T = 314.04 K with other points across each band corresponding linearly to intermediate temperatures. Thus, Figure 5 serves as a nomograph from which one may estimate the percent uncertainty in particular atmospheric temperatures computed from pressure-height data measured with a specified uncertainty over particular height increments.

Three sample applications of this nomograph are cited: In the first it is desired to determine the maximum pressure-gauge uncertainty allowable to achieve a 1% uncertainty in a mean temperature of about 240 K for a layer thickness of 100 m′. We look for the intersection of the 100 m′ abscissa value with the 1% ordinate value, and find that it lies near the 242 K value of the 0.01% pressure-uncertainty band. Thus, pressures would have to be measured with an uncertainty no greater than 0.01% to achieve the desired results. Since it is

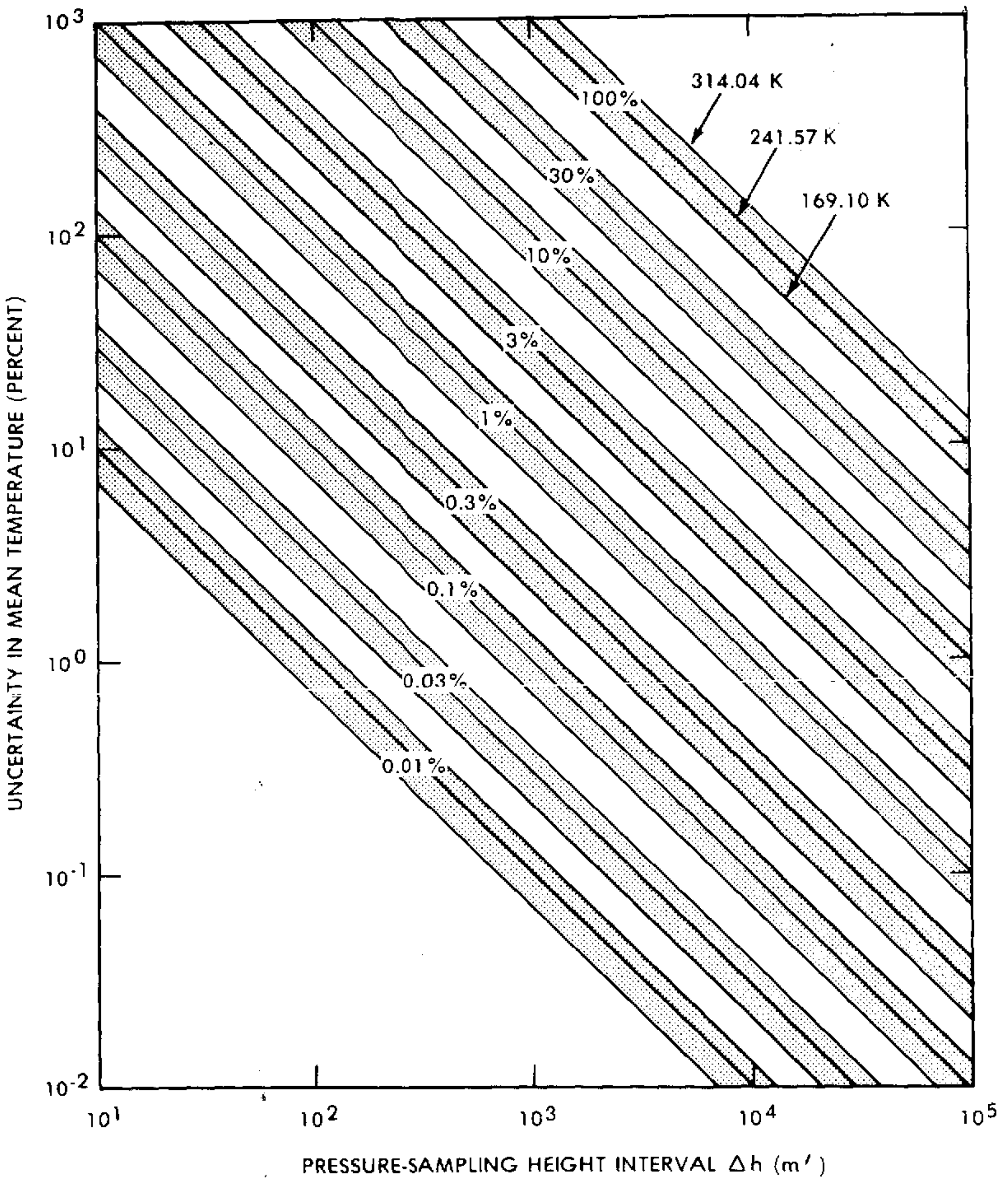

Figure 5. Nomograph for Estimating Relative Uncertainty in Temperatures (Calculated from Pressure-Height Data) as a Function of the Pressure-Height Sampling Interval, and as a Function of the Value of the Temperature, for Each of Nine Values of Relative Uncertainty in the Pressure-Height Observations

essentially impossible to achieve such a small uncertainty in any rocket or balloon measurements of pressure, this combination of height resolution and temperature uncertainty is essentially impossible to achieve from pressure-height data.

In the second example, we assume a pressure-gauge uncertainty of 3 percent, and a mean temperature of 180 K, and seek the temperature uncertainty associated with particular sampling-height increments. If the pressure sampling-height interval is 3 kilometers, this ordinate value is seen to intersect the appropriate region of the band for 3 percent pressure uncertainty at a value corresponding to a temperature uncertainty on the ordinate scale of about 7.4%, or about 13.3 K. A doubling of the layer thickness to 6 km would halve the temperature uncertainty.

The third example involves the inverse problem of estimating the uncertainty in the computed pressure differential associated with an assumed isothermal layer of fixed thickness. If the layer has a thickness of 3 km, as may be the situation in the grenade experiment (Nordberg and Smith 1964), and if there is an uncertainty of about 2% or 5 K in an assumed mean temperature of 250 K, the graph shows that the combined uncertainty of the two boundary pressures lies between 0.3% and 1.0% or about 0.6% on a logarithmic scale. For a layer thickness of 1 km, the boundary-pressure uncertainties would increase by a factor of 3. If the assumed isothermal layers were used to generate density-height profiles rather than pressure-height profiles, the uncertainties in the densities would be somewhat larger than in the pressures because of the required vector addition of the temperature uncertainty and the pressure uncertainty to obtain the density uncertainty.

IV. UNCERTAINTY IN TEMPERATURES DEDUCED FROM DENSITIES

Applying the first-order Gaussian method, equation (14), to the determination of the uncertainty in temperatures deduced from density-height data through equation (11) leads to

$$\delta T_q = \left[\left(\frac{\partial T_q}{\partial T_1}\cdot \delta T_1\right)^2 + \left(\frac{\partial T_q}{\partial \rho_1}\cdot \delta\rho_1\right)^2 + \sum_{j=2}^{q-1}\left(\frac{\partial T_q}{\partial \rho_j}\cdot \delta\rho_j\right)^2 + \left(\frac{\partial T_q}{\partial \rho_q}\cdot \delta\rho_q\right)^2\right]^{1/2} \quad (21)$$

provided that we assume all the uncertainty in a density-height point to be concentrated in the density. Because scale height combines the temperature with several constants, its use facilitates the expansion and analysis of an equation like equation (21). Consequently, it is convenient to rewrite that equation as

$$\delta H_q = \left[\left(\frac{\partial H_q}{\partial H_1}\cdot \delta H_1\right)^2 + \left(\frac{\partial H_q}{\partial \rho_1}\cdot \delta\rho_1\right)^2 + \sum_{j=2}^{q-1}\left(\frac{\partial H_q}{\partial \rho_j}\cdot \delta\rho_j\right)^2 + \left(\frac{\partial H_q}{\partial \rho_q}\cdot \delta\rho_q\right)^2\right]^{1/2} \quad (22)$$

Both equations (21) and (22) involve (q + 1) terms, where q is the number of density-height data points. The first term is associated with the uncertainty of the temperature or the scale height at h_1, while each of the q additional terms deals with the uncertainty of one of the successive q density-height data points, respectively. Three of the (q + 1) terms have a unique format. These are the first, the second, and the last terms; those involving δT_1 (or δH_1), $\delta\rho_1$, and $\delta\rho_q$, respectively. The remaining terms, however, the third through the qth term involving $\delta\rho_2$ through $\delta\rho_{q-1}$, respectively, have a common format. Thus the sum of these common-format terms may be expressed as the summation of a general term. Applying the operators indicated in equation (22), dividing both sides of the resulting equation by H_q, and introducing equation (10a) yields the following equation:

$$\left(\frac{\delta H_q}{H_q}\right)^2 = \left(\frac{H_1}{H_q}\cdot\frac{\rho_1}{\rho_q}\right)^2\left(\frac{\delta H_1}{H_1}\right)^2 + \left\{\frac{H_1}{H_q}\cdot\frac{\rho_1}{\rho_q} - \alpha_{1,2}\left[\frac{\rho_1}{\rho_q} - \left(\frac{\rho_2-\rho_1}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h+\Delta H}\right)_{1,2}\right]\right\}^2\left(\frac{\delta\rho_1}{\rho_1}\right)^2 +$$
$$\sum_{j=2}^{q-1}\left\{\alpha_{j-1,j}\left[\frac{\rho_j}{\rho_q} - \left(\frac{\rho_j-\rho_{j-1}}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h+\Delta H}\right)_{j-1,j}\right] - \alpha_{j,j+1}\left[\frac{\rho_j}{\rho_q} - \left(\frac{\rho_{j+1}-\rho_j}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h+\Delta H}\right)_{j,j+1}\right]\right\}^2\left(\frac{\delta\rho_j}{\rho_j}\right)^2 \quad (23)$$
$$+\left\{\alpha_{q-1,q}\left[\frac{\rho_q}{\rho_q} - \left(\frac{\rho_q-\rho_{q-1}}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h+\Delta H}\right)_{q-1,q}\right] - \frac{H_1}{H_q}\cdot\frac{\rho_1}{\rho_q} - \sum_{j=2}^{q}\alpha_{j-1,j}\left(\frac{\rho_j-\rho_{j-1}}{\rho_q}\right)\right\}^2\left(\frac{\delta\rho_q}{\rho_q}\right)^2$$

where

$$\alpha_{1,2} = \left(\frac{\bar{H}}{H_q}\cdot\frac{\Delta h}{\Delta h+\Delta H}\right)_{1,2} \quad (24)$$

$$\alpha_{j-1,j} = \left(\frac{\bar{H}}{H_q}\cdot\frac{\Delta h}{\Delta h+\Delta H}\right)_{j-1,j} \quad (25)$$

$$\alpha_{j,j+1} = \left(\frac{\bar{H}}{H_q}\cdot\frac{\Delta h}{\Delta h+\Delta H}\right)_{j,j+1} \quad (26)$$

$$\alpha_{q-1,q} = \left(\frac{\bar{H}}{H_q}\cdot\frac{\Delta h}{\Delta h+\Delta H}\right)_{q-1,q} \quad (27)$$

Equation (23), which is developed in Appendix A, is very much more complicated than the comparable uncertainty expression in terms of pressure-height data as represented by equation (18). The considerable difference in the complexity between the two equations stems largely from the fact that equation (18) involves the uncertainty of only two pressure-height data points and the mean temperature for the corresponding height layer, while equation (23) involves the uncertainty of many consecutive density-height data points, each of which contributes at least a small amount to the value and uncertainty of the temperature or scale height at height h_q. It is emphasized that the quantities T_q or H_q in equations (11) through (13), and again in equations (21) through (23), represent values for specific heights, and not mean values for specific layers as in the case of the pressure-related equations.

Each element of equations (23) through (27) is associated either with a specific height or with a specific layer. Elements having a single subscript, e.g., H_1, ρ_1, ρ_j, ρ_{j-1}, ρ_{j+1}, ρ_{q-1}, ρ_q, and H_q signify the value of the particular quantity at heights h_1, h_j, h_{j-1}, h_{j+1}, h_{q-1}, and h_q, respectively. Quantities with a double subscript, e.g., $\alpha_{1,2}$, $\alpha_{j-1,j}$, etc., represent a quantity associated with particular layers, i.e., the layers bounded by h_1 and h_2, or by h_{j-1} and h_j, etc., respectively. Thus, $\alpha_{1,2}$, as expressed by equation (24), represents an algebraic expression of four different quantities three of which, namely $\bar{H}$, Δh, and ΔH, are associated with the layer h_1 to h_2. In equation (24), $\bar{H}$ is the mean scale height for the layer $\Delta h = h_1 - h_2$, while ΔH represents the change in scale height within that layer, i.e., $H_1 - H_2$, such that only in a region where the gradient of H or T with respect to height is positive will ΔH be positive. The quantity H_q is, of course, the value of H at height h_q. Similarly, $\alpha_{j-1,j}$, $\alpha_{j,j+1}$, and $\alpha_{q-1,q}$, as defined by equations (25), (26), and (27), respectively, each represent quantities associated with the particular appropriate layers. Equation (23) also includes the doubly subscripted quantities

$$\left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{1,2}, \left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{j-1,j}, \left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{j,j+1}, \left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{q-1,q}$$

each of which represents one factor of the right-hand side of equations (24), (25), (26), and (27), respectively. The influence of non-zero gradients in the temperature-height profile is impressed upon $\delta H_q/H_q$ through all the double subscripted quantities in equation (23).

In order to more clearly see the influence of sampling-height interval alone on δH_q, it is convenient to assume an isothermal atmosphere, and to correspondingly simplify equation (23). In this case, scale height would not vary, but would remain fixed at a value H over the entire region of integration, and the following relationships would apply: $H_1 = H_q = H$, $\bar{H} = H$, and ΔH is zero in all layers. Under these conditions, equations (24), (25), (26), and (27) become unity, as does the ratio H_1/H_q, and equation (23) may be rewritten as

$$\left(\frac{\delta H_q}{H_q}\right)^2 = \left[\left(\frac{\rho_1}{\rho_q}\right)\left(\frac{\delta H_1}{H_1}\right)\right]^2 + \left[\left(\frac{\rho_2 - \rho_1}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h}\right)_{1,2} \cdot \left(\frac{\delta\rho_1}{\rho_1}\right)\right]^2$$
$$+ \sum_{j=2}^{q-1}\left\{\left[\left(\frac{\rho_{j+1} - \rho_j}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h}\right)_{j,j+1} - \left(\frac{\rho_j - \rho_{j-1}}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h}\right)_{j-1,j}\right]\left(\frac{\delta\rho_j}{\rho_j}\right)\right\}^2 \tag{28}$$
$$+ \left\{\left[\left(1 - \frac{\rho_1}{\rho_q}\right) - \sum_{j=1}^{q}\left(\frac{\rho_j - \rho_{j-1}}{\rho_q}\right) - \left(\frac{\rho_q - \rho_{q-1}}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h}\right)_{q-1,q}\right]\left(\frac{\delta\rho_q}{\rho_q}\right)\right\}^2$$

It can be shown that

$$\sum_{j=1}^{q}\left(\frac{\rho_j - \rho_{j-1}}{\rho_q}\right) = \left(1 - \frac{\rho_1}{\rho_q}\right), \tag{29}$$

such that these two terms in the coefficient of $(\delta\rho_q/\rho_q)$ in equation (28) cancel each other. Then, if we impose the additional condition that the height increments between successive density-height data points are constant over the entire region of integration, this condition plus equation (29) permit the further simplification of equation (28) to

$$\left(\frac{\delta H_q}{H_q}\right)^2 = \left(\frac{\rho_1}{\rho_q}\cdot\frac{\delta H_1}{H_1}\right)^2 + \left(\frac{\bar{H}}{\Delta h}\right)^2\left[\left(\frac{\rho_2-\rho_1}{\rho_q}\cdot\frac{\delta\rho_1}{\rho_1}\right)^2 + \sum_{j=2}^{q-1}\left(\frac{\rho_{j+1}-2\rho_j+\rho_{j-1}}{\rho_q}\cdot\frac{\delta\rho_j}{\rho_j}\right)^2 + \left(\frac{\rho_q-\rho_{q-1}}{\rho_q}\cdot\frac{\delta\rho_q}{\rho_q}\right)^2\right] \tag{30}$$

If we impose still another restriction, i.e., that the relative uncertainty of the density data has the constant value $\delta\rho/\rho$ for all heights within the range of integration, $\delta\rho_1/\rho_1 = \delta\rho_j/\rho_j = \delta\rho_q/\rho_q = \delta\rho/\rho$, and we may rewrite equation (30) as

$$\left(\frac{\delta H_q}{H_q}\right)^2 = \left(\frac{\rho_1}{\rho_q}\right)^2\left(\frac{\delta H_1}{H_1}\right)^2 + \left(\frac{\bar{H}}{\Delta h}\right)^2\left[\left(\frac{\rho_2-\rho_1}{\rho_q}\right)^2 + \sum_{j=2}^{q-1}\left(\frac{\rho_{j+1}-2\rho_j+\rho_{j-1}}{\rho_q}\right)^2 + \left(\frac{\rho_q-\rho_{q-1}}{\rho_q}\right)^2\right]\left(\frac{\delta\rho}{\rho}\right)^2 \tag{31}$$

The restriction permitting this simplification, while somewhat unrealistic from the point of view of any measuring system, is acceptable because, it will be shown below, the relative uncertainty of only ρ_q enters significantly into the value of $\delta H_q/H_q$. It is convenient to represent the sum of the series of the squares of the several density ratios in the coefficient of $(\delta\rho/\rho)^2$ in equation (31) by $(S_q)^2$, such that

$$S_q = \left[\left(\frac{\rho_2-\rho_1}{\rho_q}\right)^2 + \sum_{j=2}^{q-1}\left(\frac{\rho_{j+1}-2\rho_j+\rho_{j-1}}{\rho_q}\right)^2 + \left(\frac{\rho_q-\rho_{q-1}}{\rho_q}\right)^2\right]^{1/2} \tag{32}$$

with this simplification equation (31) may be rewritten as

$$\left(\frac{\delta H_q}{H_q}\right)^2 = \left[\left(\frac{\rho_1}{\rho_q}\right)\left(\frac{\delta H_1}{H_1}\right)\right]^2 + \left[S_q\cdot\left(\frac{\bar{H}}{\Delta h}\right)\left(\frac{\delta\rho}{\rho}\right)\right]^2 \tag{33}$$

Finally, recalling from Figure 2 that the ratio (ρ_1/ρ_q) causes the contribution of H_1 to H_q in equation (13) to become negligible when the integration has proceeded downward from h_1 by more than about three scale heights, we recognize that the contribution of $\delta H_1/H_1$ to $\delta H_q/H_q$ in equation (33) must similarly be negligible for h_q sufficiently below h_1. Thus, for this condition, equation (33) may be approximated by

$$\frac{\delta H_q}{H_q} \simeq S_q \cdot \left(\frac{\overline{H}}{\Delta h}\right)\left(\frac{\delta \rho}{\rho}\right) \tag{34}$$

Also, since $\delta H_q/H_q$ is identically equal to $\delta T_q/T_q$, it is convenient at this point to return from scale-height notation to temperature notation, and equation (34) is rewritten as

$$\frac{\delta T_q}{T_q} \simeq S_q \cdot \left(\frac{\overline{H}}{\Delta h}\right)\left(\frac{\delta \rho}{\rho}\right) \tag{35}$$

This equation represents the relative uncertainty in T_q, the temperature at height h_q, as deduced from a set of density-height data measured with a constant relative uncertainty $(\delta\rho/\rho)$ over the entire range of a height region h_1 down to h_q, where $(h_1 - h_q)$ is equal to at least three scale heights, in a portion of a planetary atmosphere which is isothermal, at some value associated with $\overline{H}$. These conditions might apply to the earth's atmosphere at heights above about 400 km. Since equation (35) excludes several of the variables included in the most general form of the uncertainty expression as given by equation (23), this simpler version is more readily examined for the influence which variations in the sampling-height interval Δh alone have upon $\delta T_q/T_q$.

Equation (35) for $\delta T_q/T_q$, in terms of the relative uncertainty in density, is very similar to equation (20) for $\delta\bar{T}/\bar{T}$ in terms of the relative uncertainty in pressure, except that S_q in equation (35) takes the place of $\sqrt{2}$ in equation (20). Therein lies a great difference, because the quantity S_q is a dimensionless function of three variables, $\bar{H}$, Δh, and $(h_1 - h_q)$, and has a value which varies between $\sqrt{2}$ and zero, as Δh varies from large values to zero. For $\Delta h > 5\,H$, the value of S_q approaches $\sqrt{2}$ asymptotically as Δh approaches infinity, independently of the value of $\bar{H}$ or $(h_1 - h_q)$. For $\Delta h < 0.2\,H$, the value of S_q decreases linearly with decreasing values of Δh, for any fixed value of $\bar{H}$ or $(h_1 - h_q)$.

The influence of variations of both Δh and $(h_1 - h_q)$ upon S_q, and implicitly upon $\delta T_q/T_q$, is seen in Figure 6. Here, for a fixed value of $\bar{H} = (10/\sqrt{2})$ km, a band of values of S_q is plotted as a function of Δh on a fully logarithmic scale, with ordinate values of S_q given on the right-hand side of the figure. The upper limit of the band is designated $S_{q,max}$, and is plotted as a line of long dashes, while the lower limit of the band is designated $S_{q,min}$, and is plotted as a solid line. The range of the values of S_q within the band, at any particular value of Δh, represents the influence of variations of $(h_1 - h_q)$ at that value of Δh, with the value of S_q increasing as $(h_1 - h_q)$ decreases. The smallest possible value of $(h_1 - h_q)$, at any particular value of Δh, is of course Δh, and hence $S_{q,max}$ depicts the locus of the values of S_q associated with the condition that $(h_1 - h_q) = \Delta h$. Thus, this upper limit of S_q represents the value of equation (32) when that equation involves only the first two density-altitude points of a data set, i.e., the value of S_q for the case when $\delta T_q/T_q$ is being computed for the height h_q associated with the

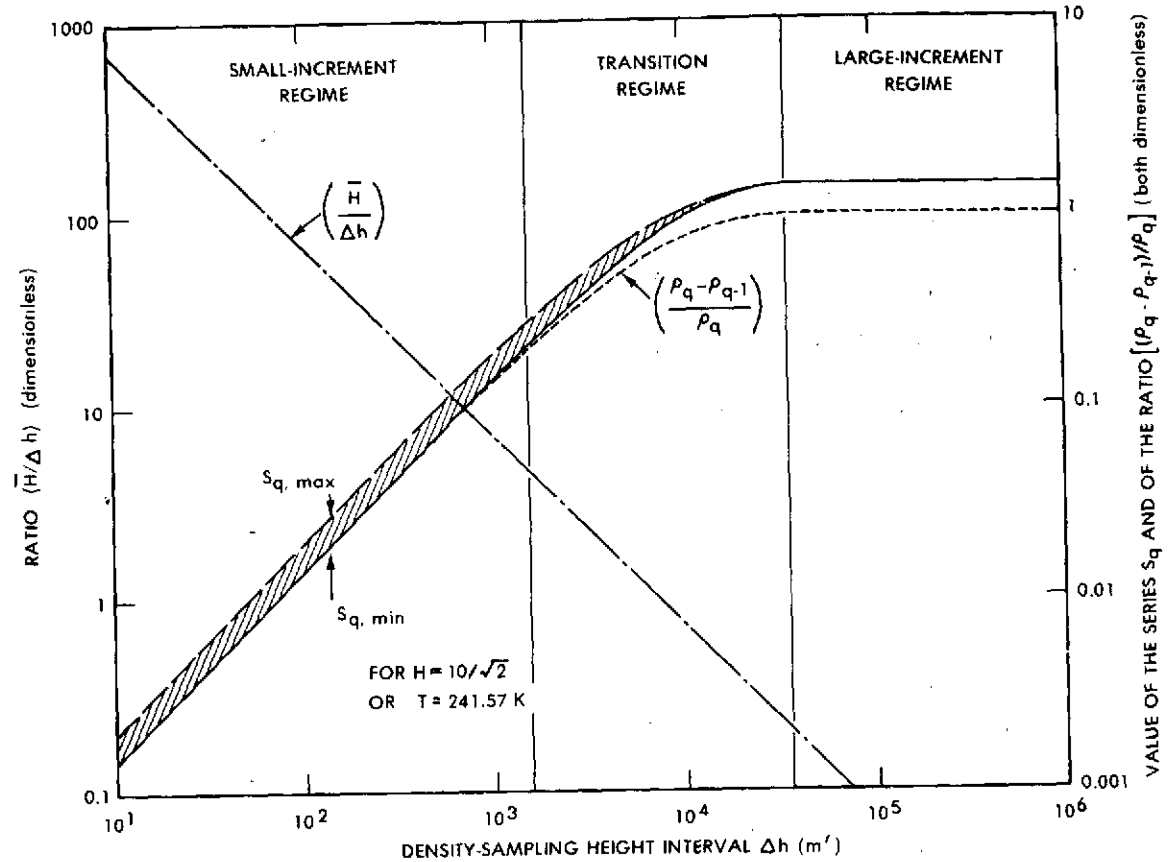

Figure 6. Graphs of Three Quantities as a Function of Density-Sampling Height Interval Δh, i.e., (1) the Limited Band of Values of a Particular Series of Terms Defined by S_q, and Related to Δh, (2) the Value of the Last Term of that Series, and (3) the Value of the Ratio of Mean Scale Height to Δh

second-highest density-height data point in the sounding. For this situation, the right-hand side of equation (32) reduces to $[\sqrt{2}(\rho_q - \rho_1)/\rho_q]$. As the height for which one computes $\delta T_q/T_q$ is lowered from h_1, such that $(h_1 - h_q)$ becomes increasingly large, the value of S_q slowly decreases, and asymptotically approaches a lower limit $S_{q,min}$ when $(h_1 - h_q)$ is greater than three scale heights.

It is convenient to define the ratio of $S_{q,max}$ to $S_{q,min}$ as R, and to examine the variation of R with respect to variations in Δh. As long as Δh is smaller than about 0.2 H, R retains a nearly constant value (a value which is seen graphically to be approximately equal to $\sqrt{2}$), and both $S_{q,max}$ and $S_{q,min}$ are seen to make an angle of about +45° with respect to the abscissa. Because of the feature of a nearly constant slope of +1 for both $S_{q,max}$ and $S_{q,min}$ in the region where $\Delta h < 0.2\,H$, this region is hereafter designated as the small-increment regime. As Δh exceeds 0.2 H, and increases toward 5 H, the slopes of both $S_{q,max}$ and $S_{q,min}$ gradually decrease toward zero. However, the slope of $S_{q,max}$ decreases more rapidly than that of $S_{q,min}$ so that $S_{q,max}$ approaches $S_{q,min}$ at a common value $\sqrt{2}$, while the value of the ratio R decreases from $\sqrt{2}$ toward unity. For $\Delta h > 5\,H$, the value of R is essentially constant at unity, and the slope of the line common to both $S_{q,max}$ and $S_{q,min}$ is essentially zero. The feature of an essentially zero slope for this common line representing S_q versus Δh, specifies a region which, because of the related large value of Δh (i.e., $\Delta h > 5\,H$) is hereafter referred to as the large-increment regime. The region between the small-increment regime and the large-increment regime, i.e., the region for which $0.2\,H < \Delta h < 5\,H$, is hereafter referred to as the transition regime.

In the large-increment regime, where $S_{q,max}$ and $S_{q,min}$ are essentially identical, it is apparent that S_q is essentially independent of $(h_1 - h_q)$. In the small-increment regime, the concern for the influence of $(h_1 - h_q)$ upon S_q remains only for values of $(h_1 - h_q)$ less than about 3 H. Because equation (35) is based upon the restriction that $(h_1 - h_q)$ is greater than 3 H, and because S_q approaches $S_{q,min}$ for such values of $(h_1 - h_q)$ in all three regimes, it is immediately apparent that the general factor S_q in equation (35) should be replaced by the specific value $S_{q,min}$. It is somewhat less apparent that $\delta\rho/\rho$, the general expression for relative uncertainty in equation (35), aught to be replaced by $\delta\rho_q/\rho_q$, the specific relative uncertainty for h_q, particularly when equation (35) is applied to the small-increment regime. The reasons for the latter replacement stem from the facts that the following dual relationship exists for the dual conditions $(h_1 - h_q) > 3$ H, and $\Delta h < 0.2$ H:

(1) The value of the series S_q, which becomes essentially $S_{q,min}$ for $(h_1 - h_q) > 3$ H, simultaneously approaches the value of $(\rho_q - \rho_{q-1})/\rho_q$, the last term in equation (32), defining S_q. This near equality between $S_{q,min}$ and $(\rho_q - \rho_{q-1})/\rho_q$ in the small-increment regime is demonstrated graphically in Figure 6.

(2) The term $(\rho_q - \rho_{q-1})/\rho_q$ is properly associated with the specific uncertainty $\delta\rho_q/\rho_q$ rather than with $\delta\rho/\rho$. This situation is evident from the fact that the quantity $(\rho_q - \rho_{q-1})/\rho_q$ is a coefficient of $\delta\rho_q/\rho_q$ in equation (30), which, except for an isothermal condition, is a general expression equally applicable to both $\delta H_q/H_q$ and $\delta T_q/T_q$. Thus, at least for the small-increment regime, equation (35) may be rewritten as

$$\frac{\delta T_q}{T_q} \simeq S_{q,min} \cdot \left(\frac{\bar{H}}{\Delta h}\right)\left(\frac{\delta\rho_q}{\rho_q}\right) \tag{36}$$

Even in the large-increment regime, where the value of $S_{q,min}$ approaches $[\sqrt{2}(\rho_q - \rho_{q-1})/\rho_q]$, the value of $S_{q,min}$ is still dominated by $(\rho_q - \rho_{q-1})/\rho_q$, (which is equal to unity in this regime), and therefore, $S_{q,min}$ should still be associated with $\delta\rho_q/\rho_q$.

Because $(\rho_q - \rho_{q-1})/\rho_q$ approaches unity in the large-increment regime, while $S_{q,min}$ approaches $\sqrt{2}$, it is reasonable to write the following special form of equation (36) for that regime:

$$\frac{\delta T_q}{T_q} \simeq \sqrt{2} \cdot \left(\frac{\bar{H}}{\Delta h}\right)\left(\frac{\delta \rho_q}{\rho_q}\right) \tag{37}$$

This equation is seen to be analogous to equation (20) involving uncertainties in pressure-height data.

In addition to the graphs of $S_{q,max}$, $S_{q,min}$, and $(\rho_q - \rho_{q-1})/\rho_q$ versus Δh, Figure 6 also contains a graph of the function $(\bar{H}/\Delta h)$ versus Δh, for $\bar{H} = (10/\sqrt{2})$ km, or $\bar{T} = 241.57$ K. This function, which obviously varies inversely with Δh, is seen to have a constant negative 45° slope for a fixed value of H. When this function is multiplied by $S_{q,min}$, a quantity which according to Figure 6 is seen to have a +45° slope in the small-increment regime, the product for the small-increment regime is essentially a constant (independent of Δh) with a value near unity, for all values of $\bar{H}$ or $\bar{T}$. This situation is depicted in Figure 7 where the value of the product $[S_{q,min} \cdot (\bar{H}/\Delta h)]$ versus Δh is plotted in all three regimes for each of three values of $\bar{H}$, $(7/\sqrt{2})$, $(10/\sqrt{2})$, and $(13/\sqrt{2})$ km', consistent with the mean temperatures 169.10, 241.57, and 314.04 K, respectively. Because of the characteristics of this product, equation (35) as applied

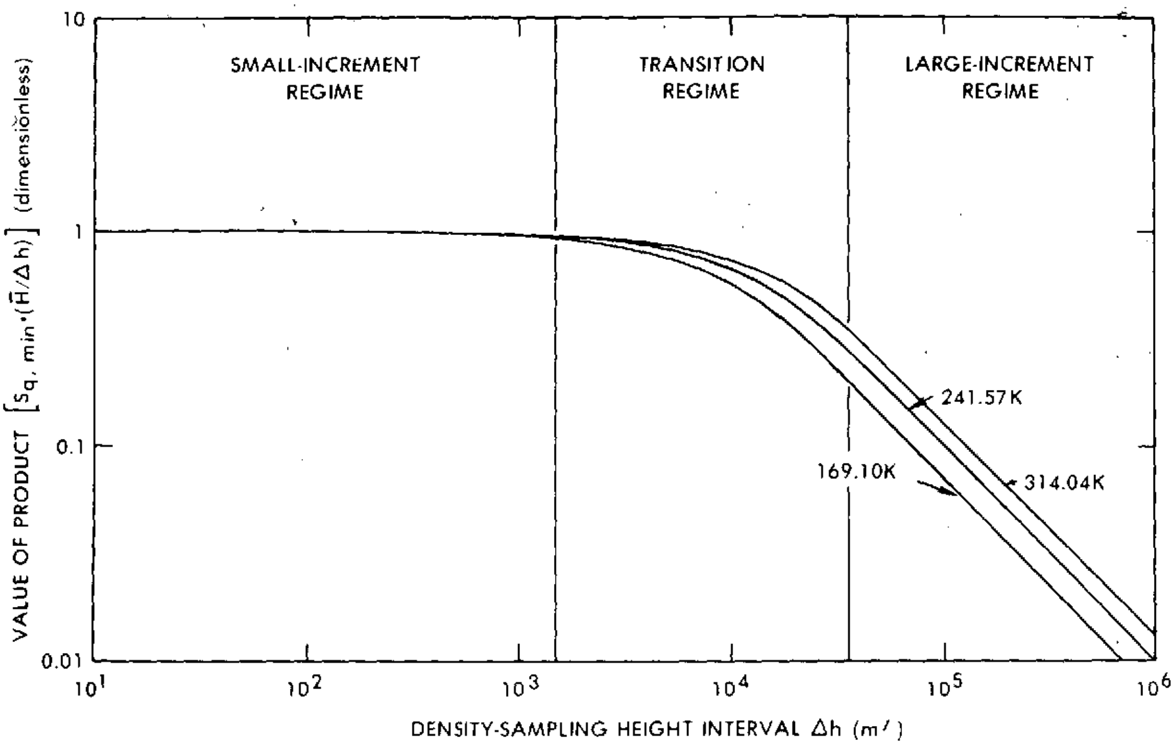

Figure 7. Value of the Product of Two Factors (Comprising the Coefficient of the Uncertainty of Observed Densities) in Equation (36) as a Function of Density-Sampling Height Interval, for Each of Three Values of Mean Scale Height Consistent with Three Specified Temperatures

to the small-increment regime may be replaced by

$$\frac{\delta T_q}{T_q} \simeq \frac{\delta \rho_q}{\rho_q} \tag{38}$$

Thus, three different expressions represent the value of $\delta T_q/T_q$ as a function of uncertainty in density-height data, equation (38) in the small-increment regime, equation (37) in the large-increment regime, and equation (36) in the transition regime.

The application of the data from Figure 7, as representative of the three equations noted, to each of nine particular values of uncertainty in density-height data, yielded the nomograph shown in Figure 8. This figure shows the percent uncertainty in the temperature, derived from density-height data through equation (12), as a function of density-sampling height interval for a band of normal atmospheric temperatures, 169 K to 314 K, and for a wide range of density-height uncertainty, 0.01% to 100%. This nomograph can serve as the basis for estimating the relationship between the uncertainty in each value of a set of density-height data to each point of the related Temperature-height profile, or vice versa, for the region of the earth's atmosphere below about 120km.

The examples of hypothetical use of the pressure-data nomograph, discussed in Section III, could be readily transposed to apply to this density-data nomograph. In particular, Figure 8 shows that, in order to have no more than a 1% uncertainty in a 245 K temperature, for a temperature height-profile with a height resolution of 100 meters, the relative uncertainty in the density observations need not be smaller than 1%. This value is 100 times larger than the 0.01% uncertainty required of pressure data to meet the same conditions, and represents a very significant relaxation of the measurement requirements.

The estimates deduced from the use of this nomograph are theoretically accurate, as long as the data to which the nomograph is applied comply with the three restrictions imposed during the preceeding development. It will be seen, however, that one of these restrictions may be essentially eliminated, while a second has

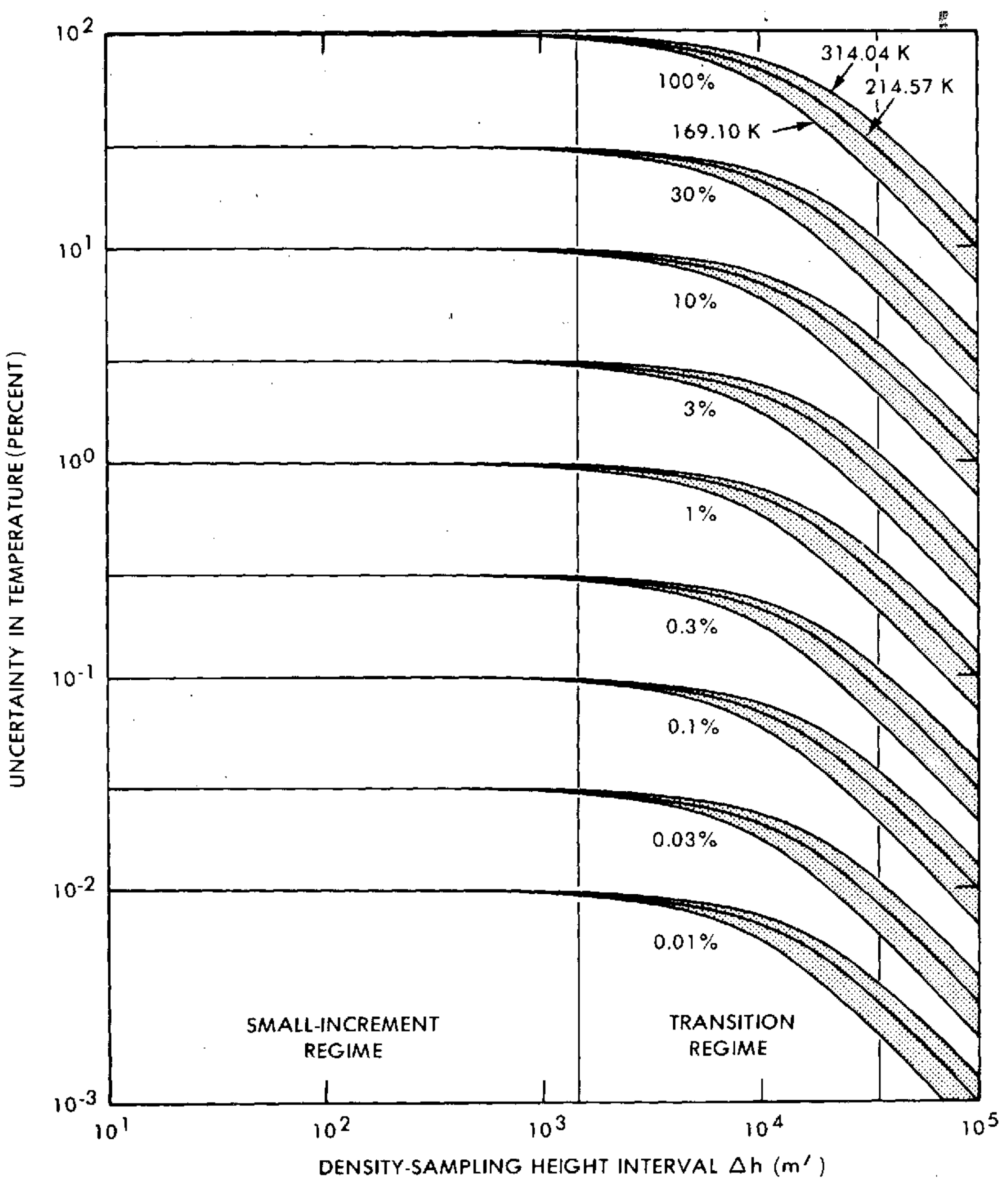

Figure 8. Nomograph for Estimating Relative Uncertainty in Temperatures (Calculated from Density-Height Data) as a Function of the Density-Height Sampling Interval, and as a Function of the Value of the Temperature, for Each of Nine Values of Relative Uncertainty in the Density-Height Observations

only a small influence. One may recall that these restrictions are as follows:

1. The height h_q associated with T_q is more than 3 scale heights below h_1 the greatest height of the sounding, i.e., $(h_1 - h_q) > 3$ H.

2. The relative uncertainty of the density-height data is constant over the entire height range of the sounding.

3. The atmosphere is isothermal over the height range of the sounding.

The first of these restrictions must be retained unless one has some independent means for determining T_1 or H_1 and its uncertainty as required by equation (33). Only for $(h_1 - h_q) > 3$ H will the ratio ρ_1/ρ_q be sufficiently small to permit the associated term to be neglected in equations (34) through (38).

The second restriction is not a very significant one since it has been shown that the constant factor $(\delta\rho/\rho)$, used as the general expression for density uncertainty in equation (35), is dominated by the particular value $(\delta\rho_q/\rho_q)$ associated with the density-height data at h_q. Thus, the need for $(\delta\rho/\rho)$ to be constant over the entire height range of a particular sounding can be relaxed without significantly affecting the validity of any of equations (36), (37), or (38).

The third restriction appears to be philosophically important because the earth's atmosphere below 120 km is certainly not isothermal. Actually, however, the existance of the various non-zero gradients in the temperature profile has little effect on $\delta T_q/T_q$. This situation is due to the fact that $\delta T_q/T_q$ depends primarily on the conditions between the data points (ρ_{q-1}, h_{q-1}) and (ρ_q, h_q), and hardly at all upon the condition between other pairs of data points. Even a non-zero temperature gradient between the two specified data points has only a small effect, and this can be readily accounted for. This is accomplished by reintroducing into equation (36) certain temperature-gradient-dependent coefficients which are associated with $\delta\rho_q/\rho_q$ in the general version of $\delta H_q/H_q$ (or equivalently $\delta T_q/T_q$), as expressed by equation (23).

To begin with, $(\bar{H}/\Delta h)$ in equation (36) is replaced by its original form $\bar{H}/(\Delta h + \Delta H)$ as used in equation (23). Then, $S_{q,min}$, which has been shown to be almost exactly equal to $[(\rho_q - \rho_{q-1})/\rho_q]$ for isothermal conditions in the small-increment regime, is replaced by the product of this ratio times its associated coefficient $\alpha_{q-1,q}$ from equation (23). Thus, for an atmosphere with varying temperature or scale height, we may rewrite equation (38) for the small-increment regime as

$$\frac{\delta T_q}{T_q} \simeq \alpha_{q-1,q}\left(\frac{\rho_q - \rho_{q-1}}{\rho_q}\right)\left[\left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{q-1,q}\right]\left(\frac{\delta\rho_q}{\rho_q}\right) \tag{39}$$

Using the value of α_{q-1}, as defined by equation (27), and replacing $\bar{H}$ by $(2H_q + \Delta H)/2$, where $\Delta H = (H_{q-1} - H_q)$, and remembering that $\Delta h = (h_{q-1} - h_q)$, we have an expression for $\delta T_q/T_q$ in terms of H_{q-1} and ρ_{q-1} at height h_{q-1}, and in terms of H_q, ρ_q, and $\delta\rho_q$ at height h_q:

$$\frac{\delta T_q}{T_q} \simeq \left[\left(\frac{(2H_q + \Delta H)^2}{4H_q} \cdot \frac{\Delta h}{(\Delta h + \Delta H)^2}\right)_{q-1,q}\right]\left(\frac{\rho_q - \rho_{q-1}}{\rho_q}\right)\left(\frac{\delta\rho_q}{\rho_q}\right) \tag{40}$$

It is interesting to examine an example of the use of equation (40) in terms of a realistic set of data, applicable to the small-increment regime, as taken from the U.S. Standard Atmosphere 1962. Choosing the height interval, $\Delta h = 1\,km'$, between $h_{q-1} = 66$ km' and $h_q = 65$ km', in a region of negative temperature gradient, $\Delta T/\Delta h = -4$ K/km', we find ρ_{q-1} and ρ_q to be 1.3482×10^{-4} and 1.5331×10^{-4} kg m^{-3} respectively, $H_q = 7.0709$ km', and $\Delta H = -0.1173$ km'. The substitution of these data into equation (40) leads to a value of 1.076 for the coefficient of $\delta\rho_q/\rho_q$, a value involving only a second-order difference from the unity coefficient of equation (38); i.e., $\delta T_q/T_q$, in this case, would be 1.076 times $\delta\rho_q/\rho_q$, instead of 1.000 times $\delta\rho_q/\rho_q$ for the isothermal case. Thus, the assumption of isothermality, both in the development and use of equations 36, 37, and 38, is seen to introduce only a second-order error, and accounting for this error is unnecessary in most uncertainty determinations.

V. CONCLUSIONS: A COMPARISON OF THE DENSITY-DATA NOMOGRAPH WITH THAT FOR PRESSURE DATA

The nomograph relating $\delta T/T$ to sampling-height interval of density data for various temperatures and density uncertainties depicted in Figure 8 is based on the same range of values of temperature and sampling-height interval used in the nomograph of Figure 5, which relates $\delta\overline{T}/\overline{T}$ to the sampling-height interval of pressure data. In addition, the nine assumed values of $\delta\rho/\rho$ in Figure 8 are identical to the nine assumed values of $\delta p/p$ in Figure 5. Thus, the two figures are exactly comparable, with the only difference being that Figure 8 is for density data, while Figure 5 is for pressure data. A comparison of these two figures shows that, at least from the point of view of height resolution and uncertainty of derived temperatures, density-height data are to be preferred over pressure-height data.

In the large-increment regime, i.e., $\Delta h > 5\overline{H}$, the nomograph of Figure 8 is essentially identical to that of Figure 5. In this regime, both figures show that the uncertainty in the derived temperature is decreased as the height resolution of the related temperature profile is decreased, i.e., as the sampling-height interval is increased. Even the smaller sampling-height intervals within this regime are so large, however, i.e., $\Delta h \simeq 5\overline{H}$, or about 30 km', that the height resolution makes this regime essentially useless for temperature-height determinations at heights below 120 km'.

In the small-increment regime as well as in the transition regime, Figure 5 and 8 are quite different from each other. In the small-increment regime ($\Delta h <$ $0.2H$), the uncertainty in the temperatures derived from density-height data is seen to be independent of both Δh and T (or H), as long as $(h_1 - h_q) > 3H$. In this regime, $\delta T_q/T_q$ derived from densities is dependent only upon the uncertainty of the density-height value associated with the related height h_q. It is apparent that uncertainty considerations place no limits on the usable fineness of the height resolution of the density-height data. This situation is in contrast to that associated with pressure-height data in Figure 5 where $\delta \bar{T}/\bar{T}$ is seen to depend upon $\bar{T}$ and Δh, as well as upon $\delta p/p$, and where $\delta \bar{T}/\bar{T}$ becomes prohibitively large for reasonable values of $\delta p/p$ when Δh becomes smaller than about 1 km.

In the transition regime, Figure 8 shows less difference from Figure 5 than in the small-increment regime. In this transition regime, the characteristics of the function determining $\delta T/T$ from density-height data vary between those of the small-increment regime and those of the large-increment regime, such that $\delta T/T$ decreases slightly as Δh varies from about $0.2\bar{H}$ to about $5\bar{H}$. In going from the small-increment regime to the transition regime, however, the increased coarseness of the height resolution of the related temperature-height profile would more than offset the correspondingly small decrease in temperature uncertainty, particularly since the minimum values of Δh in this regime are already of the order of 1 km'. Even in this regime, however, the density-height data yield smaller values of $\delta T/T$ than are obtained from pressure-height data.

In general, a comparison of Figure 8 with Figure 5 shows that the density-height data are far more desirable than pressure-height data at least from the point of view of the size of the uncertainty and of the height resolution of the derived temperature-height profile.

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APPENDIX A

DEVELOPMENT OF THE GENERAL EXPRESSION FOR $(\delta T_q/T_q)^2$ AND FOR $(\delta H_q/H_q)^2$

It is well known that temperature may be deduced from density-height data by means of the following relationship:

$$T_q = \frac{\rho_1}{\rho_q} \cdot T_1 + \frac{GM}{R} \cdot \frac{1}{\rho_q} \cdot \int_{h_1}^{h_q} \rho(h)\, dh \qquad \text{(A-1)}$$

where

$\rho(h)$ is the function relating atmospheric density to geopotential height,

h_1 is the geopotential height of the upper limit of the region of integration, in geopotential meters (m′),

h_q is the geopotential height of the lower limit of the region of integration, in geopotential meters (m′),

G is the geopotential gravity constant 9.80665 $m^2 sec^{-2} (m')^{-1}$,

M is the mean molecular weight of the air 28.9644 kg $(kmol)^{-1}$,

R is the universal gas constant 8.31432×10^3 joules $K^{-1}\ kmol^{-1}$,

ρ_1 is the atmospheric density at h_1,

T_1 is the atmospheric temperature at h_1,

ρ_q is the atmospheric density at h_q, and

T_q is the sought-after atmospheric temperature at h_q.

Because of the defined relationship between temperature and scale height H, i.e., $H = TR/GM$, it is convenient to rewrite equation (A-1) as

$$H_q = \frac{\rho_1}{\rho_q} \cdot H_1 + \frac{1}{\rho_q} \cdot \int_{h_1}^{h_q} \rho\,(h)\,dh \tag{A-2}$$

where

H_1 is the scale height at h_1 and

H_q is the scale height at h_q.

Since $\rho(h)$ is generally not known as an analytical function, for which one might find a perfect integral, but rather is known as a set of numerical values of density versus geometric or geopotential height, it is convenient to replace the integral term in equation (A-2) by a series approximation. One possible form of such an approximation is a series of terms each representing the area of successive trapezoids under the graph of the natural logarithm of density versus h. When the data, ρ versus h, is plotted on a semilogarithmic scale, i.e., $\ell n\rho$ versus h, with successive data points connected by straight-line segments, as is frequently the situation for closely spaced density-height data, the series of logarithmic trapezoids exactly fits the area under the graph. In using this approximation to the integral equation, (A-2) may be rewritten as

$$H_q = \frac{\rho_1}{\rho_q} \cdot H_1 + \frac{1}{\rho_q} \cdot \sum_{j=2}^{q} \frac{(h_{j-1} - h_j)(\rho_j - \rho_{j-1})}{\ell n\rho_j - \ell n\rho_{j-1}} \tag{A-3}$$

The validity of the approximation of equation (A-2) by equation (A-3) improves as the height increment between successive density values decreases.

The uncertainty in the computed value of H_q is based on a function involving the partial derivative of H_q with respect to each of the independent variables. These include H_1 and the appropriate number of density-height data pairs; i.e.; h_1, ρ_1; h_j, ρ_j (for j = 2 to q - 1); as well as h_q, ρ_q. With the assumption that the uncertainty in each data pair is entirely in the density value, we are interested in the partial derivatives of H_q, as expressed by equation (A-3), with respect to only the following variables: H_1, ρ_1, ρ_j (for j = 2 to q - 1), and ρ_q, having the general designation y_i. These partial derivatives $\partial H_q / \partial y_i$ multiplied by the corresponding uncertainty δy_i are given in the following equations:

The product of δH_1 times the partial derivative of H_q with respect to H_1 is simply

$$\frac{\partial H_q}{\partial H_1} \cdot \delta H_1 = \left\{ \frac{\rho_1}{\rho_q} \cdot \frac{H_1}{1} \right\} \frac{\delta H_1}{H_1} \tag{A-4}$$

The product of $\delta\rho_1$ times the partial derivative of H_q with respect to ρ_1 is

$$\frac{\partial H_q}{\partial \rho_1} \cdot \delta\rho_1 = \left\{ \frac{\rho_1}{\rho_q} \cdot \frac{H_1}{1} - \left(\frac{\Delta h}{\ell n\rho_2 - \ell n\rho_1} \right) \left[\frac{\rho_1}{\rho_q} - \left(\frac{\rho_2 - \rho_1}{\rho_q} \right) \left(\frac{1}{\ell n\rho_2 - \ell n\rho_1} \right) \right] \right\} \frac{\delta\rho_1}{\rho_1} \tag{A-5}$$

It can be shown, however, that

$$\frac{1}{\ell n\rho_2 - \ell n\rho_1} = \left(\frac{\bar{H}}{\Delta h + \Delta H} \right)_{1,2} \tag{A-6}$$

where the double subscript "1, 2" on the right-hand member of equation (A-6) indicates that $\Delta h = (h_1 - h_2)$, $\Delta H = (H_1 - H_2)$, and $\bar{H} = (H_1 + H_2)/2$. When h_1 is greater than h_2, the left-hand side of equation (A-6) is positive, Δh is positive, and ΔH has the sign of the temperature gradient $\partial T / \partial h$ in the region

h_1 to h_2. The same equation, with other consecutive digits, as for example, 2, 3 or 3, 4, etc., indicates these same relationships for the corresponding height intervals.

Equation (A-6) combined with equation (A-5) yields

$$\frac{\partial H_q}{\partial \rho_1} \cdot \delta\rho_1 = \left\{ \frac{\rho_1}{\rho_q} \cdot \frac{H_1}{1} - \left[\left(\frac{\bar{H}}{1} \cdot \frac{\Delta h}{\Delta h + \Delta H} \right)_{1,2} \right] \left[\frac{\rho_1}{\rho_q} - \left(\frac{\rho_2 - \rho_1}{\rho_q} \right) \left(\frac{\bar{H}}{\Delta h + \Delta H} \right)_{1,2} \right] \right\} \frac{\delta\rho_1}{\rho_1} \qquad \text{(A-7)}$$

The product of $\delta\rho_2$ times the derivative of H_q with respect to ρ_2 is a quantity which, when modified by the introduction of equation (A-6), with each of two different, but appropriate pairs of digits, becomes

$$\begin{aligned} \frac{\partial H_q}{\partial \rho_2} \cdot \delta\rho_2 = & \left\{ \left[\left(\frac{\bar{H}}{1} \cdot \frac{\Delta h}{\Delta h + \Delta H} \right)_{1,2} \right] \left[\frac{\rho_2}{\rho_q} - \left(\frac{\rho_2 - \rho_1}{\rho_q} \right) \left(\frac{\bar{H}}{\Delta h + \Delta H} \right)_{1,2} \right] + \right. \\ & \left. - \left[\left(\frac{\bar{H}}{1} \cdot \frac{\Delta h}{\Delta h + \Delta H} \right)_{2,3} \right] \left[\frac{\rho_2}{\rho_q} - \left(\frac{\rho_3 - \rho_2}{\rho_q} \right) \left(\frac{\bar{H}}{\Delta h + \Delta H} \right)_{2,3} \right] \right\} \frac{\delta\rho_2}{\rho_2} \end{aligned} \qquad \text{(A-8)}$$

In this equation the subscript "1, 2" on each of two factors has the same significance as it has on these same two factors in equation (A-7), while the subscript "2, 3" on each of two other factors signifies that the quantities $\bar{H}$, ΔH, and Δh within each of these factors are associated with the geopotential height increment h_2 to h_3. Thus, for these two factors, $\Delta h = h_2 - h_3$, $\bar{H} = (H_2 + H_3)/2$, and $\Delta H = H_2 - H_3$. Equation (A-8) obviously involves two height increments, the two which are separated by the height h_2.

The partial derivative of H_q with respect to each of ρ_3, ρ_4, ..., ρ_{q-2}, and ρ_{q-1}, when multiplied by $\delta\rho_j$ (where j is successively 3, 4, ..., q-2, and q-1) and, when further modified by the appropriate introduction of equation (A-6), is identical to equation (A-8) except for the successive incrementing of the subscripts. Thus, a convenient form of the product of $\delta\rho_{q-1}$, times the particular expression for the partial derivative of H_q with respect to ρ_{q-1} is

$$\frac{\partial H_q}{\partial H_{q-1}} \cdot \delta\rho_{q-1} = \left\{ \left[\left(\frac{\bar{H}}{I} \cdot \frac{\Delta h}{\Delta h + \Delta H} \right)_{q-2,q-1} \right] \left[\frac{\rho_{q-1}}{\rho_q} - \left(\frac{\rho_{q-1} - \rho_{q-2}}{\rho_q} \right) \left(\frac{\bar{H}}{\Delta h + \Delta H} \right)_{q-2,q-1} \right] + \right.$$
$$\left. - \left[\left(\frac{\bar{H}}{I} \cdot \frac{\Delta h}{\Delta h + \Delta H} \right)_{q-1,q} \right] \left[\frac{\rho_{q-1}}{\rho_q} - \left(\frac{\rho_q - \rho_{q-1}}{\rho_q} \right) \left(\frac{\bar{H}}{\Delta h + \Delta H} \right)_{q-1,q} \right] \right\} \frac{\delta\rho_{q-1}}{\rho_{q-1}} \qquad \text{(A-9)}$$

In equation (A-9) the subscript "q-2, q-1" on each of two factors signifies that the quantities $\bar{H}$, ΔH, and Δh within each of those two factors are associated with the geopotential-height increment h_{q-2} to h_{q-1}, so that $\Delta h = h_{q-2} - h_{q-1}$, $\bar{H} = (H_{q-2} + H_{q-1})/2$, and $\Delta H = H_{q-2} - H_{q-1}$. Similarly, in that equation, the subscript "q-1, q" on each of two other factors signifies that the quantities $\bar{H}$, ΔH, and Δh within each of these two factors are associated with the geopotential-height increment h_{q-1} to h_q, so that $\Delta h = h_{q-1} - h_q$, $\bar{H} = (H_{q-1} + H_q)/2$, and $\Delta H = H_{q-1} - H_q$. Again this expression involves two height increments, in this instance the two separated by the height h_{q-1}.

Because of the common form of the partial derivatives of H_q with respect to ρ_2, ρ_3, ..., ρ_{q-2}, and ρ_{q-1}, it is desirable to write a general version of equations (A-8) and (A-9) to express the product of $\delta\rho_j$ times the partial derivative of H_q with respect to ρ_j, where j is understood to have values ranging from 2 to q-1.

This general equation is

$$\frac{\partial H_q}{\partial \rho_j} \cdot \delta\rho_j = \left\{ \left[\left(\frac{\bar{H}}{1} \cdot \frac{\Delta h}{\Delta h + \Delta H} \right)_{j-1,j} \right] \left[\frac{\rho_j}{\rho_q} - \left(\frac{\rho_j - \rho_{j-1}}{\rho_q} \right) \left(\frac{\bar{H}}{\Delta h + \Delta H} \right)_{j-1,j} \right] + \right.$$

$$\left. - \left[\left(\frac{\bar{H}}{1} \cdot \frac{\Delta h}{\Delta h + \Delta H} \right)_{j,j+1} \right] \left[\frac{\rho_j}{\rho_q} - \left(\frac{\rho_{j+1} - \rho_j}{\rho_q} \right) \left(\frac{\bar{H}}{\Delta h + \Delta H} \right)_{j,j+1} \right] \right\} \frac{\delta\rho_j}{\rho_j} \tag{A-10}$$

Finally, the product of $\delta\rho_q$ times the partial derivative of H_q with respect to ρ_q when modified by the introduction of equation (A-6), is

$$\frac{\partial H_q}{\partial \rho_q} \cdot \delta\rho_q = \left\{ \left[\left(\frac{\bar{H}}{1} \cdot \frac{\Delta h}{\Delta h + \Delta H} \right)_{q-1,q} \right] \left[\frac{\rho_q}{\rho_q} - \left(\frac{\rho_q - \rho_{q-1}}{\rho_q} \right) \left(\frac{\bar{H}}{\Delta h + \Delta H} \right)_{q-1,q} \right] + \right.$$

$$\left. - \left(\frac{H_1}{1} \cdot \frac{\rho_1}{\rho_q} \right) - \sum_{j=1}^{q} \left[\left(\frac{\bar{H}}{1} \cdot \frac{\Delta h}{\Delta h + \Delta H} \right)_{j-1,j} \right] \left(\frac{\rho_j - \rho_{j-1}}{\rho_q} \right) \right\} \frac{\delta\rho_q}{\rho_q} \tag{A-11}$$

In this equation the subscript "q-1, q" on each of two factors has the same significance as in equation (A-9), while the subscript "j-1, j" on another factor in the general term signifies that the quantities $\bar{H}$, ΔH, and Δh within that term are associated with the general geopotential height increment h_{j-1} to h_j. Thus, for this factor, $\Delta h = h_{j-1} - h_j$, $\bar{H} = (H_{j-1} + H_j)/2$, and $\Delta H = H_{j-1} - H_j$.

It is evident that one part of this equation deals with the data associated with the single height increment $\Delta h = (h_{q-1} - h_q)$ between the lowest two density-height values involved in the calculation of H_q, while another part of the equation deals with all the height increments between h_1 and h_q, and their associated density data.

Each of equations (A-4), (A-7), (A-10), and (A-11) is directly involved in the Gausian expression for relative uncertainty $\delta H_q/H_q$ which follows:

$$\frac{\delta H_q}{H_q} = \frac{1}{H_q}\left[\left(\frac{\partial H_q}{\partial H_1}\cdot \delta H_1\right)^2 + \left(\frac{\partial H_q}{\partial \rho_1}\cdot \delta \rho_1\right)^2 + \sum_{j=2}^{q-1}\left(\frac{\partial H_q}{\partial \rho_j}\cdot \delta \rho_j\right)^2 + \left(\frac{\partial H_q}{\partial \rho_q}\cdot \delta \rho_q\right)^2\right]^{1/2} \qquad \text{(A-12)}$$

The first, second, and fourth terms within the brackets of this equation are seen to represent exactly the squares of equation (A-4), (A-7), and (A-11), respectively. The third term represents the squares of equation (A-10) evaluated for j ranging from 2 to q-1. The bracketed portion of the right-hand side of equation (A-12) is seen to be multiplied by the reciprocal of H_q, thereby implying that each term of equations (A-4), (A-7), (A-10), and (A-11) must ultimately be multiplied by $1/H_q$, either before or after these equations are squared and summed to equal $(\delta H_q/H_q)^2$. It is convenient in this case to do the multiplication before squaring and summing. Equations (A-7), (A-10), and (A-11), each contain the doubly subscripted factor

$$\left(\frac{\bar{H}}{1}\cdot\frac{\Delta h}{\Delta h + \Delta H}\right)$$

in at least one term. It is convenient therefore to accomplish this multiplication operation in the appropriate terms by introducing H_q into the denominator of this doubly subscripted factor, thereby converting this factor into a nondimensional coefficient with a value close to unity. Then, in order to put the resulting coefficient into proper prospective with respect to the remainder of the equation, and also to conserve space in the uncertainty expression being developed, it is convenient to define each of these modified factors by a specific symbol. Thus, from equation (A-7) the modified factor is defined as the coefficient $\alpha_{1,2}$, i.e.,

$$\alpha_{1,2} = \left(\frac{\bar{H}}{H_q} \cdot \frac{\Delta h}{\Delta h + \Delta H}\right)_{1,2} \tag{A-13}$$

while the two modified factors involved in equation (A-11) are defined as

$$\alpha_{j-1,j} = \left(\frac{\bar{H}}{H_q} \cdot \frac{\Delta h}{\Delta h + \Delta H}\right)_{j-1,j} \tag{A-14}$$

and

$$\alpha_{j,j+1} = \left(\frac{\bar{H}}{H_q} \cdot \frac{\Delta h}{\Delta h + \Delta H}\right)_{j,j+1} \tag{A-15}$$

The modified factor from equation (A-10) is defined as the coefficient $\alpha_{q-1,q}$, i.e.,

$$\alpha_{q-1,q} = \left(\frac{\bar{H}}{H_q} \cdot \frac{\Delta h}{\Delta h + \Delta H}\right)_{q-1,q} \tag{A-16}$$

Substituting equations (A-4), (A-7), (A-10), and (A-11), respectively, into the successive terms on the right-hand side of equation (A-12), dividing each of these equations by H_q (i.e., replacing each of the "1's" in the denominators of these equations by H_q), and simultaneously replacing the resulting modified doubly subscripted factors in these equations by the equivalent coefficient forms defined in equations (A-13) through (A-16), leads to the following expression for $(\delta H_q/H_q)^2$:

$$\begin{aligned}
\left(\frac{\delta H_q}{H_q}\right)^2 = {} & \left(\frac{H_1}{H_q} \cdot \frac{\rho_1}{\rho_q} \cdot \frac{\delta H_1}{H_1}\right)^2 + \left\{\frac{H_1}{H_q} \cdot \frac{\rho_1}{\rho_q} - \alpha_{1,2}\left[\frac{\rho_1}{\rho_q} - \left(\frac{\rho_2 - \rho_1}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{1,2}\right]\right\}^2 \left(\frac{\delta\rho_1}{\rho_1}\right)^2 + \\
& \sum_{j=2}^{q-1}\left\{\alpha_{j-1,j}\left[\frac{\rho_j}{\rho_q} - \left(\frac{\rho_j - \rho_{j-1}}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{j-1,j}\right] - \alpha_{j,j+1}\left[\frac{\rho_j}{\rho_q} - \left(\frac{\rho_{j+1} - \rho_j}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{j,j+1}\right]\right\}^2 \left(\frac{\delta\rho_j}{\rho_j}\right)^2 \\
& + \left\{\alpha_{q-1,q}\left[\frac{\rho_q}{\rho_q} - \left(\frac{\rho_q - \rho_{q-1}}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h + \Delta H}\right)_{q-1,q}\right] - \frac{H_1}{H_q} \cdot \frac{\rho_1}{\rho_q} - \sum_{j=2}^{q} \alpha_{j-1,j}\left(\frac{\rho_j - \rho_{j-1}}{\rho_q}\right)\right\}^2 \left(\frac{\delta\rho_q}{\rho_q}\right)^2
\end{aligned} \tag{A-17}$$

Because $\delta H_q/H_q$ is identically equal to $\delta T_q/T_q$, equation (A-17) may be rewritten as

$$\begin{aligned}
\left(\frac{\delta T_q}{T_q}\right)^2 = {} & \left(\frac{H_1}{H_q}\cdot\frac{\rho_1}{\rho_q}\right)^2\left(\frac{\delta H_1}{H_1}\right)^2 + \left\{\frac{H_1}{H_q}\cdot\frac{\rho_1}{\rho_q} - \alpha_{1,2}\left[\frac{\rho_1}{\rho_q} - \left(\frac{\rho_2-\rho_1}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h+\Delta H}\right)_{1,2}\right]\right\}^2\left(\frac{\delta\rho_1}{\rho_1}\right)^2 + \\
& \sum_{j=2}^{q-1}\left\{\alpha_{j-1,j}\left[\frac{\rho_j}{\rho_q} - \left(\frac{\rho_j-\rho_{j-1}}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h+\Delta H}\right)_{j-1,j}\right] - \alpha_{j,j+1}\left[\frac{\rho_j}{\rho_q} - \left(\frac{\rho_{j+1}-\rho_j}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h+\Delta H}\right)_{j,j+1}\right]\right\}^2\left(\frac{\delta\rho_j}{\rho_j}\right)^2 \\
& + \left\{\alpha_{q-1,q}\left[\frac{\rho_q}{\rho_q} - \left(\frac{\rho_q-\rho_{q-1}}{\rho_q}\right)\left(\frac{\bar{H}}{\Delta h+\Delta H}\right)_{q-1,q}\right] - \frac{H_1}{H_q}\cdot\frac{\rho_1}{\rho_q} - \sum_{j=2}^{q}\alpha_{j-1,j}\left(\frac{\rho_j-\rho_{j-1}}{\rho_q}\right)\right\}^2\left(\frac{\delta\rho_q}{\rho_q}\right)^2
\end{aligned} \qquad \text{(A-18)}$$