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TITLE:

APPLICATION OF THE COMPARISON PRINCIPLE  
TO ANALYSIS OF NONLINEAR SYSTEMS

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Logan, Utah

R. W. Gunderson  
Principal Investigator

Consider the ordinary differential equation

$$\dot{x} = f(t, x) \quad (I)$$

where  $x$  and  $f$  belong to  $R^n$ ,  $t$  is a real scalar and where  $f$  is assumed to be continuous on  $D_r = \{(t, x) : t \geq 0, x < r\}$ ,  $0 < r \leq +\infty$ . Let  $x(t, t_0, x_0)$  denote a solution of (I), such that  $x(t_0, t_0, x_0) = x_0$ . In the following, if  $a, b$  both belong to  $R^n$  and  $a_i \geq b_i$  ( $a_i > b_i$ ) for all  $i = 1, 2, \dots, n$  we shall write  $a \geq b$  ( $a > b$ ). Further, a function  $f(x)$  where  $f$  and  $x$  belong to  $R^n$ , will be said to be of type K in a set S if  $f_i(a) \leq f_i(b)$  ( $i = 1, 2, \dots, n$ ) for any two points  $a, b$  in S with  $a_i = b_i$  and  $a_k \leq b_k$  for all  $i = k$ .

In addition to (I) we will also consider the "comparison" differential equation

$$\dot{r} = w(t, r) \quad (II)$$

where  $w$  and  $r$  belong to  $R^m$  and  $w$  is assumed continuous for all  $(t, r)$  such that  $t \geq 0$  and  $r \in R^m$ . It will also be necessary to require  $w$  to be a function of type K, for each fixed  $t$ , in  $R^m$ .

The following lemma is given relative to (I) and (II) and is of fundamental importance. It comprises the basic idea of the comparison principle.

Lemma. Let  $v(t, x)$  belong to  $R^m$  and be continuous and locally Lipschitzian, with respect to  $x$ , on  $D_r$ . Suppose that

$$v^*(t, x) = \limsup_{h \rightarrow 0^+} h^{-1} \{v[t+h, x+hf(t, x)] - v(t, x)\} \leq w(t, v(t, x))$$

for all  $(t, x) \in D_r$  and also suppose  $v(0, x_0) \leq r_0$ . Let  $r_m(t, 0, r_0)$  denote a maximal solution of (II) and let  $J$  denote its right maximal interval of existence. Then  $v(t, x(t, 0, x_0)) \leq r_m(t, 0, r_0)$  for all  $t \in J$  such that  $(t, x(t, 0, x_0)) \in D_r$ .

The existence of the maximal solution  $r_m(t, 0, r_0)$ , under the conditions placed on  $w$ , was established by Kamke. It can easily be shown that the local

Lipschitz condition assures that  $v^*(t,x)$  coincides with the upper right derivative of  $v(t,x(t,0,x_0))$  for every  $(t,x) \in D_r$ . The lemma then follows immediately from another theorem of Kamke. With respect to the applications, it should be noted that

$$v^*(t,x) = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \cdot f(t,x)$$

in the case  $v(t,x)$  has continuous partial derivatives.

This lemma, or special cases of it, has been used by a number of individuals to obtain sufficient conditions for the existence of solution proportion of classical stability theory. An example of its application can be found in the paper of Appendix I, "On a stability property of Krasovskii", which was published in the International Journal of Non-Linear Mechanics, Vol. 5, pages 507-512. In this paper, sufficient conditions for a type of practical stability, proposed by Krasovskii, were obtained. The results are of particular value to investigations involving the behavior of systems over a finite interval of time.

A different version of the comparison lemma was given in the paper of Appendix II, "A comparison lemma for higher order trajectory derivatives", published in the Proceedings of the American Mathematical Society, Vol. 27, Number 3, in March 1971. Here the basic lemma on comparison theory was used to obtain a comparison lemma which is useful when higher order trajectory derivatives of Liapunov functions are known.

Appendix III, "A stability condition for linear comparison systems", published in the Quarterly of Applied Mathematics, July 1971, presents a result which is useful when applying the comparison principle following the basic approach given in a paper of Bailey. Using a result of this paper, it is possible to provide a very much simplified proof of an important section of Bailey's original paper.

The paper of Appendix IV, "A result on differential inequalities and its application to higher order trajectory derivatives", was published in the SIAM Journal on Analysis, Vol. 4, Number 4, in November 1973. This paper was concerned with the important problem of obtaining a less restrictive condition for comparison systems than that imposed by the type K condition. In addition to obtaining a less restrictive condition, an example was given of its application to higher order trajectory derivatives. The example considerably improved that given in the paper of Appendix II.

The result obtained in Appendix IV required investigation of the adjoint differential equation of the variational equation of the right side of the inequality. The condition itself could be considered as the failure to satisfy a certain two point boundary value problem. Consequently some effort was devoted to study of two point boundary value problems. Papers resulting from this investigation were those of Appendix V, "An existence theorem for boundary value problems", published in the Quarterly of Applied Mathematics, April 1973; Appendix VI, "Conditioning of linear boundary value problems", published in the Swedish Journal BIT, Vol. 12, 1972; Appendix VII, "Application of Liapunov theory to boundary value problems II", published in the Proceedings of the American Mathematical Society, Vol. 37, Number 1, January 1973; Appendix VIII, "Optimization of bounds for boundary sets", presented at the 1974 International Congress of Mathematicians, Vancouver, Canada.

An application of the comparison principle to the analysis of approximate systems was given in the paper of Appendix IX, "Error estimates for approximate dynamic systems", to be published in a 1974 issue of the International Journal of Control. In this paper, two approaches were considered for obtaining estimates on the error between approximate and exact solutions of dynamic systems.

It sometimes occurs that the matrix of a linear comparison system is not of the required type K. In these cases it is sometimes possible to maintain the inequality while replacing the matrix by one whose off diagonal elements are the absolute values of the original matrix. The paper of Appendix X, "Stability of a class of matrices similar to nearly non-negative matrices", to be published in the SIAM Journal on Applied Mathematics in 1975, is concerned with the question of when the stability of the former matrix implies the stability of the latter and whether a simple test for the stability of such matrices exists.

Closely related to stability theory is the topic of bifurcation analysis. Roughly, a system will bifurcate when it loses stability. The papers of Appendices XI and XII are concerned with this problem. Appendix XI, "Sustained small oscillations in nonlinear control systems", submitted for publication to the International Journal on Control concerns some results from the theory of bifurcating solutions which can be used to obtain conditions which allow sustained small oscillations in aircraft-spacecraft dynamics. Appendix XII, "Structurally stable bifurcating systems", submitted to the International Journal of Nonlinear Mechanics is concerned with showing that a system of ordinary differential equations with a vertical bifurcation cannot be structurally stable. The purpose of these investigations was to determine the application of comparison theory to bifurcation analysis. So far, it is not clear.

Finally, the paper of Appendix XIII, "Scalar n-th order comparison systems", submitted to the IEEE Transactions on Automatic Control discusses results from several papers and their application to comparison theorems involving higher order trajectory derivatives.

APPENDIX 1

ON A STABILITY PROPERTY OF KRASOVSKII  
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APPENDIX 11

A COMPARISON LEMMA FOR HIGHER ORDER TRAJECTORY DERIVATIVES  
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A RESULT ON DIFFERENTIAL INEQUALITIES AND ITS APPLICATION TO HIGER  
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APPENDIX V

AN EXISTENCE THEOREM FOR LINEAR BOUNDARY VALUE PROBLEMS  
J H GEORGE AND R W GUNDERSON

APPENDIX VI

CONDITIONING OF LINEAR BOUNDARY VALUE PROBLEMS  
JOHN H GEORGE AND ROBERT W. GUNDERSON

APPENDIX 1X

ERROR ESTIMATES FOR APPROXIMATE DYNAMIC SYSTEMS  
R W GUNDERSON AND J H GEORGE

PAPERS INTENTIONALLY OMITTED

## APPENDIX VII

APPLICATION OF LIAPUNOV THEORY TO  
BOUNDARY VALUE PROBLEMS, II\*

by

J. H. George\*\* and R. J. York<sup>†</sup>

1. Introduction. In [1], George and Sutton formulated a Liapunov theory for the existence and uniqueness of solutions to the following class of boundary value problems;

$$(1.1a) \quad x'' = f(t, x, x')$$

$$(1.1b) \quad x(a) = A, \quad x(b) = B$$

where  $A, B$  are fixed  $n$ -vectors,  $' = d/dt$ ,  $x$  is a  $n$ -vector with a continuous second derivative on  $[a, b]$ , and the  $n$ -vector function  $f$  is defined and continuous on  $D = [a, b] \times R^{2n}$  where  $[a, b]$  is an interval on the real line. The development in [1] replaced certain conditions of Hartman [2], by Liapunov-type results. Fundamental to this approach was the construction of a function  $g$  which is equal to  $f$  on a bounded region.

LEMMA 1.1. (Hartman [2, p. 432]). Suppose  $f$  is defined and continuous on  $D_M = [a, b] \times \{x : \|x\| \leq M\} \times R^n$  and let  $N > 0$  be given. Then for any  $[t_1, t_2] \subset [a, b]$  there exists a continuous function  $g$  bounded on  $D_M$ , such that  $f = g$  on  $D_1 = [t_1, t_2] \times \{x : \|x\| \leq M\} \times \{x' : \|x'\| \leq N\}$ .

By placing Liapunov conditions on

$$(1.2) \quad x'' = g(t, x, x')$$

it was shown in [1] that the solutions of (1.2) and (1.1b) are also solutions of (1.1).

In this paper, a more restrictive class of Liapunov functions than those considered in [1] are introduced. This allows the existence theorems

in [1] to be proved with conditions in terms of  $f$  rather than  $g$  as in [1]. Thus, a principle difficulty in using the results of [1] are removed. Also, the new class of Liapunov functions is still sufficiently large to include Hartman's condition, [2].

2. Liapunov Conditions for existence of solutions. Let a Liapunov function  $V(t, x, x')$  be a continuously differentiable in  $(t, x, x')$ , real valued function. Let

$$\frac{\partial V}{\partial x} = \left( \frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n} \right), \quad \frac{\partial V}{\partial x'} = \left( \frac{\partial V}{\partial x'_1}, \dots, \frac{\partial V}{\partial x'_n} \right)$$

Let  $\langle \cdot, \cdot \rangle$  denote the dot product in  $n$ -dimensional Euclidean space. Then define the derivative of  $V$  with respect to  $t$  along a solution of (1.1a) to be,

$$V'_f = \frac{\partial V}{\partial t} + \left\langle \frac{\partial V}{\partial x}, x' \right\rangle + \left\langle \frac{\partial V}{\partial x'}, f \right\rangle$$

Also let  $x(t)$  belong to the class  $C^2(t_1, t_2]$  if  $x''(t)$  is continuous on  $[t_1, t_2]$ . We will now restate a Lemma of George and Sutton [1] in terms of this Liapunov function.

LEMMA 2.1. (George and Sutton [1]). Suppose there exists a Liapunov function  $U(t, x, x')$  defined on  $D_2 = [t_1, t_2] \times \{x: \|x\| \leq M\} \times \mathbb{R}^n$ . Let  $X(t)$  be the set of functions  $x(t) \in C^2[t_1, t_2]$  satisfying  $\|x(t)\| \leq M$  on  $[t_1, t_2]$ , and also the following properties:

- (i)  $U(t_1, x(t_1), x'(t_1)) = 0$
- (ii)  $U(t, x, x') \geq (t - t_1) \phi(\|x'\|)$ , where  $\phi$  is a positive continuous function defined on  $[0, \infty)$  such that  $\phi(r) \rightarrow \infty$  as  $r \rightarrow \infty$ .
- (iii) there exists a positive constant  $L$  such that

$$U' = \frac{\partial U}{\partial t} + \left\langle \frac{\partial U}{\partial x}, x' \right\rangle + \left\langle \frac{\partial U}{\partial x'}, x'' \right\rangle \leq L \quad \text{in the interior of } D_2.$$



Then there exists a positive constant  $N$  such that  $\|x'(t)\| \leq N$  on  $[t_1, t_2]$ , for every  $x(t)$  belonging to  $X(t)$ .

EXAMPLE. Let

$$U = (t - t_1) \|x'\|^2.$$

Then conditions (i) and (ii) are satisfied. Condition (iii) becomes

$$(2.1) \quad \langle x'(t), x''(t) \rangle \leq \frac{L - \|x'(t)\|^2}{2(t - t_1)} \quad \text{on } (t_1, t_2]$$

From the lemma, there exists an  $N$  such that if (2.1) is satisfied and  $\|x(t)\| \leq M$  then  $\|x'(t)\| \leq N$  on  $[t_1, t_2]$ . Many lemma's similar to Lemma 2.1 could be developed. This was simply an attempt in [1] to show that alternatives existed to the standard Nagumo type theorems [2, p. 428-9].

THEOREM 2.1. Assume there exists a Liapunov function satisfying the hypothesis of Lemma 2.1. Let  $V(t, x, x')$  be another Liapunov function defined on  $D_3 = [t_1, t_2] \times \{x: \|x\| \geq M\} \times \mathbb{R}^n$  and satisfying:

$$(i) \quad \left\langle \frac{\partial V}{\partial x}, f \right\rangle \leq 0 \quad \text{for } \|x\| \geq M, \quad \|x'\| \geq N + 1$$

and

$$(ii) \quad \left\langle \frac{\partial V}{\partial x}, \frac{M}{\|x\|} f(t, \frac{Mx}{\|x\|}, x') \right\rangle \geq \left\langle \frac{\partial V}{\partial x}, f \right\rangle \quad \text{for } \|x\| \geq M \quad \text{and} \quad \|x'\| \leq N$$

Then  $V'_f \geq 0$  in the interior of  $D_3$  implies  $V'_g \geq 0$  in the interior of  $D_3$ .

Proof. Let  $N$  be the constant provided by Lemma 2.1. As in Hartman

$$[2], \text{ let } g(t, x, x') = \begin{cases} 1 - \delta(\|x'\| - N) f(t, x, x') & \text{on } D_M \\ \frac{M}{\|x\|} g(t, \frac{Mx}{\|x\|}, x') & \text{when } \|x\| > M. \end{cases}$$

Here,

$$\delta(s) = \begin{cases} 0 & \text{if } s < 0 \\ s & \text{if } 0 \leq s \leq 1 \\ 1 & \text{if } s > 1 \end{cases}$$

The proof now consists of establishing that  $V'_g \geq 0$  on  $D_m$  with  $\|x\| = M$ .

Then, the argument is extended to  $||x|| > M$ . The proof is motivated by a result of Knoblock [3].

On  $D_{II}$  with  $||x|| = M$ ,

$$(2.1) \quad V'_g = \frac{\partial V}{\partial t} + \langle \frac{\partial V}{\partial x}, x' \rangle + \langle \frac{\partial V}{\partial x'}, f \rangle \tau$$

where  $\tau = 1 - \delta(||x'|| - N) \in [0,1]$ . Since (2.1) is a linear function of  $\tau \in [0,1]$ , if  $V'_g \geq 0$  for  $\tau = 0$  and  $\tau = 1$  then  $V'_g \geq 0$  for all  $\tau \in [0,1]$ . At  $\tau = 1$ ,  $V'_g = V'_f \geq 0$ . At  $\tau = 0$ ,  $||x'|| > N + 1$ , we have from condition (i)

$$V'_g = \frac{\partial V}{\partial t} + \langle \frac{\partial V}{\partial x}, x' \rangle \geq \frac{\partial V}{\partial t} + \langle \frac{\partial V}{\partial x}, x' \rangle + \langle \frac{\partial V}{\partial x'}, f \rangle = V'_f \geq 0$$

If  $||x|| > M$ ,

$$V'_g = \frac{\partial V}{\partial t} + \langle \frac{\partial V}{\partial x}, x' \rangle + \langle \frac{\partial V}{\partial x'}, \frac{M}{||x||} g(t, \frac{Mx}{||x||}, x') \rangle =$$

$$\frac{\partial V}{\partial t} + \langle \frac{\partial V}{\partial x}, x' \rangle + \frac{M\tau}{||x||} \langle \frac{\partial V}{\partial x'}, f(t, \frac{Mx}{||x||}, x') \rangle \geq 0$$

by repeating the argument above for  $\tau = 0$  and  $\tau = 1$ , using condition (ii) when  $\tau = 1$  and condition (i) when  $\tau = 0$ .

Using this result, THEOREM 4.2 in [1] can now be stated as follows:

THEOREM 2.2. Let  $x(t)$  be a solution of (1.2) defined on  $[t_1, t_2]$   $[a, b]$  and suppose that  $||x(t_1)|| \leq M$ ,  $||x(t_2)|| \leq M$ . Let there exist a Liapunov function  $V(t, x, x')$  defined on  $D_3$  such that

- (i)  $V(t, x, x') = 0$  whenever  $||x|| = M$
- (ii)  $V(t, x, x') > 0$  whenever  $||x|| > M$
- (iii)  $\langle \frac{\partial V}{\partial x'}, f \rangle \leq 0$  for  $||x|| \geq M$ ,  $||x'|| \geq N + 1$
- (iv)  $\langle \frac{\partial V}{\partial x'}, \frac{M}{||x||} f(t, \frac{Mx}{||x||}, x') \rangle \geq \langle \frac{\partial V}{\partial x'}, f \rangle$  for  $||x|| > M$ ,  $||x'|| \leq N$
- (v)  $V'_f \geq 0$  in the interior of  $D_3$

Then  $\|x(t)\| \leq M$  for all  $t \in [t_1, t_2]$ .

Proof. From Theorem 2.1, condition (v) can be replaced by  $V'_g \leq 0$  in the interior of  $D_3$ . The proof continues exactly the same as the proof of Theorem 4.2 in [1].

EXAMPLE.  $V = \|x'\|^2 (\|x\|^2 - M^2)$  shows that  $\frac{\partial V}{\partial x}$ , need not be zero. In this case, (iii), (iv) and (v) will impose conditions on  $f$ .

THEOREM 2.3. Suppose the Liapunov function in Lemma 2.1 has the additional properties  $U'_f \leq L$  in the interior of  $D_2$  and

$$(1) \quad \left\langle \frac{\partial u}{\partial x}, f \right\rangle \geq 0 \quad \text{for} \quad \|x\| \leq M, \quad \|x'\| \geq N + 1.$$

Then  $U'_g \leq L$  in the interior of  $D_2$ .

Proof. In a similar fashion to Theorem 2.1,

$$U'_g = \frac{\partial u}{\partial t} + \left\langle \frac{\partial u}{\partial x}, x' \right\rangle + \left\langle \frac{\partial u}{\partial x'}, f \right\rangle \tau$$

At  $\tau = 1$ ,  $U'_g = U'_f \leq 0$ . When  $\tau = 0$ ,  $U'_g = \frac{\partial u}{\partial t} + \left\langle \frac{\partial u}{\partial x}, x' \right\rangle \leq \frac{\partial u}{\partial t} + \left\langle \frac{\partial u}{\partial x}, x' \right\rangle + \left\langle \frac{\partial u}{\partial x'}, f \right\rangle = U'_f \leq L$ , by condition (1).

The results of [1] can now be stated entirely in terms of  $f$ .

THEOREM 2.4. Suppose  $f(t, x, x')$  is defined and continuous on  $[a, b] \times R^{2n}$ . Suppose there exists two Liapunov functions  $V(t, x, x')$  and  $U(t, x, x')$  as described in Theorems 2.2 and 2.3. Then the boundary value problem (1.1) and  $x(t_1) = x_1, x(t_2) = x_2, \|x_1\|, \|x_2\| \leq M$  has at least one solution.

Proof. From Theorem 2.2 and 2.3, we have  $V'_g \geq 0$  in the interior of  $D_3$  and  $U'_g \leq L$  in the interior of  $D_2$ . The proof is now the same as Theorem 4.4 in [1].

EXAMPLE. Hartman [2, p. 433] gives the following condition to insure  $\|x(t)\| \leq M$  on  $[t_1, t_2]$ :

$$\langle x, f \rangle + \|x'\|^2 > 0 \text{ if } \langle x, x' \rangle = 0 \text{ and } \|x\| \geq M$$

By choosing  $V(t, x, x') = \langle x, x \rangle - M^2$ ,  $V'_f = 2\langle x, x' \rangle$ ,  $V''_f = 2[\langle x, f \rangle + \|x'\|^2]$ . Hartman's condition implies  $V$  evaluated along a solution  $x(t)$  of (1.1a) does not have a maximum at any point  $t \in [t_1, t_2]$  when  $\|x(t)\| \geq M$ . This follows since if  $V'_f = 0$ ,  $V''_f > 0$ . Now  $V = 0$  when  $\|x\| = M$  and  $V > 0$  if  $\|x\| > M$  if there existed a solution  $x(t)$  such that  $\|x(t)\| > M$ , then  $V'_f \geq 0$  and all conditions of the theorem are satisfied with the observation that  $\frac{\partial V}{\partial x'} = 0$ .

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## FOOTNOTES

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\*\* University of Wyoming, Visiting Professor at Utah State University, summer 1971.

† Western Kentucky University.

## APPENDIX VIII

## EXAMPLES CONCERNING SOLUTIONS TO BOUNDARY VALUE PROBLEMS\*

Ross Fraker\*\*

1. Consider the boundary value problem given by

$$(1) \quad y'' = f(t, y, y')$$

$$(2) \quad y(0) = A \quad y(1) = B$$

where  $f(t, y, y')$  is continuous on  $[0, 1] \times \mathbb{R}^2$  and  $A$  and  $B$  are real.

A recently developed technique for solving boundary value problems with very general endpoint constraints developed by Bebernes and Wilhelmson [1], [2], [3], did not lend itself directly to boundary value problems (BVP) (1) - (2). Beyond the assumption of lower and upper solutions of equation (1) their principal restriction was a differential boundedness condition imposed on the solutions of equation (1).

By strengthening the differential boundedness to a Nagumo condition Bebernes and Fraker [4] demonstrated a modification of the techniques suitable for solving BVP (1) - (2). The results possible because of the modification have been discussed in different settings by Jackson and Schrader [5] and Jackson and Klaasen [6].

Examples are given showing that the differential boundedness condition is neither sufficient nor necessary for demonstrating the existence of solutions to BVP (1) - (2) using the techniques in [1], [2], or [3].

2. Let  $\alpha(t)$  and  $\beta(t) \in C^2[0, 1]$  be lower and upper solutions, respectively, of equation (1) on  $[0, 1]$  with  $\alpha(t) \leq \beta(t)$  for all  $t \in [0, 1]$ . A solution,  $y(t)$ , of equation (1) is said to be initially differentially bounded

(to the right) with respect to a lower solution - upper solution pair  $(\alpha \leq \beta)$  if there exists a positive  $N(n, t_0)$  such that  $|y'(t)| \leq N(n, t_0)$  for  $t \in [0, t_0]$  whenever  $|y'(0)| \leq n$  and  $\alpha(t) \leq y(t) \leq \beta(t)$  for all  $t \in [0, t_0]$ .

Let  $S_1$  be a compact connected set in the  $(0, y, y')$  - plane such that there exist points  $(0, y_1, y_1')$  and  $(0, y_2, y_2')$  with  $y_1 = \beta(0)$ ,  $y_1' \geq \beta'(0)$ ,  $y_2 = \alpha(0)$ , and  $y_2' \leq \alpha'(0)$ , and let  $S_2$  be a closed connected set in the  $(t_0, y, y')$  - plane such that the set

$$S_2 \cap \{(t_0, y, y') : \alpha(t_0) \leq y \leq \beta(t_0), y' = k \text{ (arbitrary)}\}$$

is non-empty for every  $k$ . If the solutions of equation (1) are initially differentially bounded with respect to some lower solution - upper solution pair  $(\alpha \leq \beta)$  then there exists at least one solution to equation (1) with  $(0, y(0), y'(0)) \in S_1$ ,  $(t_0, y(t_0), y'(t_0)) \in S_2$ , and  $\alpha(t) \leq y(t) \leq \beta(t)$  for all  $t \in [0, t_0]$ ; (see the theorem of [3]), It is clear that this result can not in general be applied to BVP (1) - (2).

3. It is noted in [2] that the properties of the set  $S_1$  are not necessary for the existence of a solution originating between  $\alpha(0)$  and  $\beta(0)$ . It is sufficient only that the solution funnel of a compact connected component of the initial set intersect both the upper solution and the lower solution at least by time  $t_0$ . A Nagumo condition can be used to eliminate the necessity of checking this property. Under both conditions BVP (1) - (2) can be solved. The following example shows that even the weaker differentially boundedness condition is not necessary for the solvability of BVP (1) - (2).

Consider  $y'' = (y')^3$  on  $[0, 1]$ . Let  $\alpha(t) = -1$  and  $\beta(t) = 1$ . In order to solve BVP (1) - (2) for  $\alpha(0) \leq A \leq \beta(0)$  and  $\alpha(1) \leq B \leq \beta(1)$  it is sufficient to take  $S_1 = \{(0, A, y') : |y'| \leq 2\}$  since by the mean value theorem

$$y'(t) \geq y'(0) + t \cdot \min f(t, y, y') \geq 2$$

$$0 \leq t \leq 1$$

$$-1 \leq y \leq 1$$

$$y' \geq 2$$

when  $y'(0) \geq 2$  for example. The connectedness of the funnel cross section at  $t = 1$  yields a solution if

$$S_2 = \{(1, B, y') : |y'| < \infty\}.$$

The example we give now shows that differential boundedness is not sufficient to solve BVP (1) - (2).

Consider  $y'' = -(y')^3$  on  $[0, 1]$ . Let  $\alpha(t) = -1$  and  $\beta(t) = 1$ . Clearly the solutions, when they exist, are differentially bounded. Let  $A = -1$  and  $B = 1$ . To solve the differential equation we consider the auxiliary equation  $z' = -z^3$  to find that

$$y'(t) = (2t + (y'(0))^{-2})^{-1/2}, \quad y'(0) > 0,$$

$$y'(t) = -(2t + (y'(0))^{-2})^{-1/2}, \quad y'(0) < 0, \text{ and}$$

$$y'(t) = 0, \quad y'(0) = 0.$$

Since we want  $y(0) = -1$ ,  $y(1) = 1$ , and  $-1 \leq y(t) \leq 1$  for all  $t \in [0, 1]$  we must take  $y'(0) > 0$  and because of the form of the equation we must require  $y'(0) > 2$ . This leads to the solution

$$y(t) = (2t + (y'(0))^{-2})^{1/2} + c$$

for appropriate constants  $y'(0)$  and  $c$ .

The boundary values lead to the equations

$$-1 = |y'(0)|^{-1} + c \quad \text{and}$$

$$1 = (2 + (y'(0))^{-2})^{1/2} + c.$$



Eliminating  $c$  we find that

$$2 + |y'(0)|^{-1} = (2 + (y'(0))^{-2})^{1/2}$$

or

$$4 + (y'(0))^{-2} + 4|y'(0)| = 2 + (y'(0))^{-2}.$$

Hence we must have

$$|y'(0)| = -\frac{1}{2}$$

which is clearly impossible. Thus no solution to this boundary value problem is possible.

APPENDIX X

STABILITY OF A CLASS OF MATRICES  
SIMILAR TO NEARLY NON-NEGATIVE MATRICES

R. W. Gunderson and E. E. Underwood\*

Abstract. A simple stability condition is obtained for a class of matrices which occur in the application of linear differential inequalities.

1. Introduction. Let  $a, b$  belong to  $R^n$  and write  $a \leq b$  if and only if  $a_i \leq b_i$  for each  $i = 1, 2, \dots, n$ . Let  $A$  be an  $n \times n$  real matrix and  $z(t)$  a vector function satisfying

$$(1) \quad z'(t) \leq Az(t)$$

for  $t \in [t_0, T]$ . Suppose further that

$$z(t_0) \leq x(t_0)$$

where  $x(t)$  is a solution of

$$x' = Ax$$

over the same interval  $[t_0, T]$ . Then it is known ([1], p.137) that

$$z(t) \leq x(t)$$

for all  $t \in [t_0, T]$  if and only if the matrix  $A$  is nearly non-negative; that is,  $a_{ij} \geq 0$  for all  $i \neq j$  and  $i, j = 1, 2, \dots, n$ . Further, it is known that the nearly non-negative matrix  $A$  is stable if and only if its  $k^{th}$  successive principal minor has sign  $(-1)^k$  ([2], p.74).

The above results have a number of interesting applications, ranging from estimating the errors of approximate solutions of differential equations [3] to the stability analysis of composite, or inter-connected,

\*Dept. of Mathematics, Utah State University. The research of the first author was supported by NASA Grant No. NGR-002-016.

systems [4]-[6]. It often occurs, however, that the matrix  $A$  of (1) does not turn out to be of the required type, i.e. nearly non-negative. In these cases it is sometimes possible to maintain the inequality (1) and at the same time replace  $A$  by  $|A|$ , where  $|A|$  is obtained from  $A$  by replacing the off-diagonal elements by their absolute value. Thus,  $|A|$  is nearly non-negative and the preceding results apply.

It is of interest in the applications to know when the spectrum of  $A$  is retained by  $|A|$ . In particular, it is of interest to know when the stability of  $A$  implies the stability of  $|A|$  and whether a simple test for the stability of such matrices exists, such as it does for  $|A|$ .

## 2. A Family of Matrices Similar to Nearly Non-Negative Matrices.

For a given  $n \times n$  matrix  $A$ , draw a line through the first row in which a negative non-diagonal element appears. Denote the number of the row by  $i_1$  and draw a line through the column  $i_1$ . Now draw a line through the next row which has a negative non-diagonal element, not in column  $i_1$ . Denote this row by  $i_2$  and draw a line through column  $i_2$ , and so forth. In this manner the matrix is covered with lines over every negative element. Now suppose that every non-diagonal element covered by a line is non-positive, with the exception of those covered at intersections of lines, which are non-negative. Then it is clear that there exists a permutation matrix  $P$  such that

$$(3) \quad P^T A P = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix}$$

where the square matrices  $A_1$  and  $A_2$  are nearly non-negative and  $B_1$  and  $B_2$

are non-positive. If  $A_1$  is of dimension  $(n-m) \times (n-m)$  and  $A_2$  is  $m \times m$ , then  $A$  will be similar to a nearly non-negative matrix  $K$ , where

$$(4) \quad K = \begin{pmatrix} -I_{m-n} & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} \begin{pmatrix} -I_{m-n} & 0 \\ 0 & I_m \end{pmatrix}.$$

Remark 1. In the applications of comparison theory, (1) occurs as a system of differential inequalities satisfied by the trajectory derivatives of vector Liapunov functions [3]-[6]. Since the component Liapunov functions are almost always positive definite, it is worthwhile to consider (1) under the assumption  $z \geq 0$  and with  $A$  of (1) assumed similar to a nearly non-negative matrix as described above. In such cases, define the vector  $y$  by

$$y = P^T z.$$

Since  $y$  is only a rearranged version of  $z$ , it follows from (1) and the definition of  $P$  that

$$y' \leq P^T A P y = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} y$$

and, since  $y \geq 0$ , finally that

$$y' \leq Ky$$

i.e.

$$z' \leq |A|z$$

where

$$|A| = PKP^T$$

Remark 2. In equilibrium theory of economics, a matrix  $A$  satisfying (4) is called a Moroshima matrix if, in addition, it is irreducible and has all negative diagonal elements [7]. The requirement of irreducibility allows a generalized Perron-Frobenius theorem to hold, from which some of the results of this paper can be deduced. The approach taken here does not require the Perron-Frobenius theorem and, hence eliminates the restriction to irreducible matrices.

### 3. Stability of Matrices Similar to Nearly Non-Negative Matrices.

Following Marcus and Minc [8], let  $Q_{r,n}$  denote the set of all strictly increasing sequences of length  $r$ , formed from the first  $n$  integers. Any submatrix of the  $n \times n$  matrix  $A$  can then be represented by  $A[\alpha|\beta]$  where  $\alpha$  and  $\beta$  both belong to  $Q_{r,n}$  for  $1 \leq r \leq n$  and where  $\alpha$  determines the sequence of rows and  $\beta$  the sequence of columns. Using this notation, an  $n \times n$  matrix  $H$  is said to be Hicksian [7] if

$$\det A[\alpha|\alpha] = (-1)^r |\det A[\alpha|\alpha]|$$

for each  $\alpha \in Q_{r,n}$  and all  $1 \leq r \leq n$ ; that is, if every  $r^{\text{th}}$  order principal minor of the matrix  $A$  has sign  $(-1)^r$ .

Definition. A  $n \times n$  matrix  $P$  will be called a signed permutation matrix if the matrix obtained by replacing the elements of  $P$  by their absolute values is a permutation matrix.

Lemma. Let  $P$  be an  $n \times n$  signed permutation matrix and consider the family of all  $r \times r$  minors

$$\det P[\alpha|\beta]$$

where  $\alpha, \beta \in Q_{r,n}$ . For a given  $\alpha$  there corresponds one sequence  $\beta^*$

such that

$$\det P[\alpha|\beta^*] = \pm 1$$

and

$$\det P[\alpha|\beta] = 0$$

otherwise.

Proof. First observe that

$$PP^T[\alpha|\alpha] = 1$$

and that

$$\det P^T[w|\alpha] = \det P[\alpha|w]$$

for any  $\alpha, w \in Q_{r,n}$ . By the Cauchy-Binet theorem

$$\begin{aligned} \det PP^T[\alpha|\alpha] &= \sum_{w \in Q_{r,n}} \det P[\alpha|w] \det P^T[w|\alpha] \\ &= \sum_{w \in Q_{r,n}} (\det P[\alpha|w])^2 = 1 \end{aligned}$$

It follows that  $\det P[\alpha|w] = \pm 1$  for one and only one  $w$  and is zero otherwise.

Theorem 1. Let  $A$  be Hicksian and  $P$  a signed permutation matrix, both of dimension  $n \times n$ . Then, the matrix

$$H = PAP^T$$

is also Hicksian.

Proof. Again apply the Cauchy-Binet theorem to obtain

$$\begin{aligned} \det PAP^T[\alpha|\alpha] &= \sum_{w \in Q_{r,n}} \det PA[\alpha|w] \det P^T[w|\alpha] \\ &= \sum_{w \in Q_{r,n}} \det P^T[w|\alpha] \sum_{\gamma \in Q_{r,n}} \det P[\alpha|\gamma] \det A[\gamma|w] \\ &= \sum_{w \in Q_{r,n}} \sum_{\gamma \in Q_{r,n}} \det P[\alpha|w] \det P[\alpha|\gamma] \det A[\gamma|w]. \end{aligned}$$

By the lemma,

$$\det P[\alpha|w] \det P[\alpha|\gamma] = 0$$

unless  $w = \gamma = \beta^*$ , and it follows that

$$\det PAP^T[\alpha|\alpha] = \det A[\beta^*|\beta^*].$$

Since  $A$  is Hicksian, so is  $PAP^T$ .

Theorem 2. Suppose there exists a signed permutation matrix  $P$  such that

$$PAP^T = K$$

where  $K$  is nearly non-negative. Then  $|A|$  is stable if and only if  $A$  is Hicksian.

Proof. From the similarity established in the preceding section, it follows that if  $|A|$  is stable then  $K$  is also stable. Since  $K$  is stable its successive principal diagonal minors have sign  $(-1)^k$ , as mentioned in the introduction. By a result of Kotelyanskii ([2], p.71) it follows that  $K$  is Hicksian. Hence by the lemma,  $A$  is Hicksian. If  $A$  is Hicksian, then  $K$  is Hicksian and thus stable, since its successive principal diagonal minors then have sign  $(-1)^k$ . But, if  $K$  is stable,  $|A|$  is stable.

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APPENDIX XI

SUSTAINED SMALL OSCILLATIONS IN NONLINEAR CONTROL SYSTEMS

J. H. George\*  
R. W. Gundersen\*\*  
H. Hahn\*\*\*

Abstract. Some results from the theory of bifurcating solutions are used to obtain conditions which allow sustained small oscillations in aircraft-spacecraft dynamics.

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\*University of Wyoming; Institut für Informationsverarbeitung, Universität Tübingen (Germany), supported as an Alexander von Humboldt awardee.

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\*\*\*Institut für Informationsverarbeitung, Universität Tübingen (Germany)

## 1. Introduction

Consider the system of first order equations

$$\dot{x} = f(x, \mu) \quad (1)$$

where  $\mu$  is a parameter,  $f$  is assumed analytic in  $x$  and  $\mu$  at the origin of  $(x, \mu)$  space and  $f(0, \mu) = 0$ . Suppose there exists a family of solutions

$$\bar{x}(t, \mu) \quad (2)$$

such that to each neighborhood of the origin  $(x, \mu)$  there corresponds at least one value,  $\mu$ , such that  $(\bar{x}(t, \mu), \mu)$  is contained in that neighborhood. In this case, the origin is called a bifurcation point, the family of solutions  $\bar{x}(t, \mu)$  is called a bifurcating branch and the solutions corresponding to fixed values of the parameter  $\mu$  are called bifurcating solutions (Krasnosel'skii 1964).

Example. Consider the scalar equation

$$\dot{x} = x(x - \mu)$$

It can easily be seen that

$$\bar{x}(t, \mu) = \mu$$

is a bifurcating branch of solutions corresponding to a bifurcation branch at the origin of the two-dimensional  $(x, \mu)$  space and that the bifurcating solutions are constant, or steady state solutions.

In the following we shall utilize some results from bifurcation theory to investigate the existence of small amplitude periodic behavior in launch vehicle dynamics. It will be assumed that the nonlinearity exists as a cubic term in the rudder response.

Starting with Poincare, there have been a number of important contributors to the theory. Among the early contributors to the theory of periodic bifurcations were Hopf (1942) and Friedrichs (1965). We shall follow quite closely the approach given in Sattinger (1973).

## 2. Bifurcations in System Theory

In addition to the existence of bifurcating solutions, either steady state or periodic, it is usually necessary in practice to determine their stability properties. The definition of asymptotic stability of steady state solutions is well-known and need not be presented here. However, the definition of orbital stability of periodic solutions is perhaps less well-known and given here for convenience.

Definition. Let  $\gamma$  denote the closed path  $x = p(t)$  in  $x$ -space. The periodic solution  $p(t)$  is said to be orbitally stable if for each  $\epsilon > 0$  there exists a corresponding  $\delta > 0$  such that every solution  $x(t)$  of (33) whose distance from  $\gamma$  is less than  $\delta$  for  $t = 0$  is defined and remains at a distance less than  $\epsilon$  from  $\gamma$  for all  $t \geq 0$ . It is said to be orbitally asymptotically stable (and  $\gamma$  is said to be a limit cycle) if in addition the distance of  $x(t)$  from  $\gamma$  tends to zero as  $t \rightarrow \infty$ .

Consider now the special case of equation (1) given by

$$x' = (A + \mu B)x + N(x) \quad N(0) = 0 \quad (3)$$

where  $A, B$  are real  $n \times n$  matrices and  $N(x)$  is an analytic vector function beginning with non-zero  $k$ -th order terms,  $k \geq 2$ . In addition, assume that the pair  $(A, B)$  has one of the following properties:

Property I. The matrix  $A$  has a simple eigenvalue at the origin and if  $A\phi_0 = A^T\psi_0 = 0$ , then  $\langle B\phi_0, \psi_0 \rangle \neq 0$ , where  $\langle \cdot, \cdot \rangle$  is the dot product of  $\mathbb{R}^n$ . Further, the matrix  $A + \mu B$  has all roots with negative real part when  $\mu < 0$  and has exactly one unstable root for  $\mu > 0$ .

Property II. The matrix  $A$  has a pair of simple, pure imaginary, roots at  $\pm i\omega_0$  and if  $A\xi_0 = i\omega_0, A^T\eta_0 = i\omega_0$  then  $\langle B\xi_0, \eta_0 \rangle \neq 0$ . Further assume  $A + \mu B$  has all its eigenvalues with negative real parts when  $\mu < 0$  and has exactly one unstable complex pair if  $\mu > 0$ .

The following theorem can then be given relative to the above properties:

Theorem. (Hopf (1942), Sattinger (1973), Friedrichs (1965)).

- a) Assume that (3) satisfies Property I and, in addition, assume that  $\langle N(\phi_0), \psi_0 \rangle \neq 0$ . Then the origin is a bifurcation point and the bifurcating branch consists of asymptotically stable steady state solutions.
- b) Assume that (3) satisfies Property II and, in addition, that  $\langle N(\xi_0), \eta_0 \rangle \neq 0$ . Then the origin is a bifurcation point and the bifurcation branch consists of orbitally asymptotically stable periodic solutions.

Remark. The theorem is proven by showing the existence of an analytic one-parameter family of solutions  $(x(\epsilon), \mu(\epsilon))$  satisfying  $x(0) = 0$ ,  $\mu(0) = 0$ . The conditions  $\langle N(\phi_0), \psi_0 \rangle \neq 0$  and  $\langle N(\xi_0), \eta_0 \rangle \neq 0$  then allow the use of the implicit function theorem to solve for  $\epsilon$  as a function of  $\mu$ , providing the bifurcating branch.

Remark. Properties I and II both require that the simple eigenvalues, either real or pure imaginary, pass from the left to the right side of

the complex plane when  $\mu = 0$ . This, of course will, not occur in general, but a simple transformation will place the system in proper form.

Suppose, for example, the crossing over occurs for  $\mu = \mu_0$  in equation (3). Then set

$$\lambda = \mu - \mu_0$$

$$A_1 = A + \mu_0 B$$

$$B_1 = B$$

and the new system

$$x' = (A_1 + \lambda B_1)x + N(x)$$

will satisfy the condition.

### 3. Application to Flexible Body Dynamics

Consider the following system of equations, obtained from the analysis of flexible body dynamics together with rigid body and control system dynamics.

$$\ddot{\phi}_r + c_2 \beta = 0$$

$$\ddot{\phi}_b + 2k\zeta\omega\dot{\phi}_b + \omega^2\phi_b = \omega^2 k\beta$$

$$\beta = f(\sigma) = \sigma - a_2\sigma^3$$

$$\sigma = a_0(\phi_r + \phi_b) + a_1(\dot{\phi}_r + \dot{\phi}_b)$$

where

$\phi_r$  - attitude angle of the rigid body

$\beta$  - rudder (control force) deflection

$c_2$  - control effectiveness coefficient

$a_0, a_1$  - control system gains

$\phi_b$  - attitude angle due to bending dynamics (fixed mode)

$\zeta, \omega$  - damping and natural frequency of bending mode

$k$  - normalization constant

$\sigma$  - control command

$f(\sigma)$  - nonlinear control command

Letting

$$x = \begin{pmatrix} \phi_r \\ \dot{\phi}_r \\ \phi_b \\ \dot{\phi}_b \end{pmatrix}$$

equations (4) can be put into the form

$$x' = (A + \mu B)x + b\sigma^3$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -c_2 a_0 & -c_2 a_1 & -c_2 a_0 & -c_2 a_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega^2 a_0 & \omega^2 a_1 & \omega^2 a_0 & \omega^2 a_1 - 2\zeta\omega \end{pmatrix}$$

$$b = \begin{pmatrix} 0 \\ c_2 a_2 \\ 0 \\ -ka_2 \omega^2 \end{pmatrix}$$

and

$$\mu = +k$$

It can easily be seen that there exists a pure imaginary pair of complex eigenvalues at  $\pm j\omega$ , providing  $c_2 a_1 > 0$ . It remains to enforce the conditions of Property II. To find  $\xi_0$  and  $\eta_0$  it is necessary to solve

$$A\xi_0 = i\omega\xi_0$$

$$A^T\eta_0 = i\omega\eta_0$$

Carrying out the calculation, it turns out that

$$\xi_0 = \begin{pmatrix} b_1 \\ b_2 \\ 1 \\ i\omega \end{pmatrix} \quad \eta_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\omega \end{pmatrix}$$

where  $b_1, b_2$  are complex constants depending on  $a_0, a_1, \omega$  and  $c_2$ . Consequently, the dot product conditions become

$$\langle B\xi_0, \eta_0 \rangle = 2\zeta\omega \neq 0$$

and

$$\langle N(\xi_0), \eta_0 \rangle \neq 0$$

if

$$a_2 \neq 0$$

$$3a_1^2\omega^2 \neq (a_0 b_1 + a_0 + a_1 b_2)^2$$

$$a_0 b_1 + a_0 + a_1 b_2 \neq 0$$

#### 4. Conclusion

In the preceding, we have assumed a cubic nonlinearity in the rudder dynamics and have determined conditions under which a bifurcating branch

of orbitally stable periodic solutions will exist. In the case considered, it was possible to determine rather easily conditions under which the system matrix had a pair of simple, pure imaginary, eigenvalues. In more complicated cases this can still be accomplished by utilizing various linear stability techniques. The D-decomposition method of determining stability regions (Siljak 1969) ought to prove especially useful in this application.

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## APPENDIX XII

STRUCTURALLY STABLE BIFURCATING SYSTEMS

J. H. GEORGE\*, H. HAHN\*\*, R. W. GUNDERSON\*\*\*

## Abstract

It is shown that a system of ordinary differential equations with a vertical bifurcation cannot be structurally stable. Conditions are then derived which can be used to detect the presence of a vertical bifurcation.

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\*University of Wyoming, Institut für Informationsverarbeitung, Universität Tübingen (Germany), supported as an Alexander von Humboldt awardee.  
\*\*Institut für Informationsverarbeitung, Universität Tübingen (Germany)  
\*\*\*Utah State University, this work was supported by the National Aeronautics and Space Administration under Grant No. NGR-45-002-016.

1. Introduction. In the applications it is usually required that a system possesses the property of structural stability. Briefly, this means that small changes in the differential equation result in topologically equivalent solution behavior. In this paper we show that a system which has a vertical bifurcation cannot be structurally stable in a neighborhood of the origin and develop a procedure to determine the existence of such bifurcations.

2. Structural Stability and Vertical Bifurcations.

Consider the system

$$x' = (A + \lambda B)x + N(x) \quad (1)$$

where  $A, B$  are  $n \times n$  matrices and  $N(x)$  is an analytic vector function defined on  $R^n$  and begins with non-zero  $k$ -th order terms,  $k \geq 2$ . We shall assume, for simplicity, that  $N$  is a bilinear form.

Suppose there exists a family of solutions

$$\bar{x}(t, \lambda) \neq 0$$

such that to each neighborhood of the origin of the  $n + 1$  space  $(x, \lambda)$  there corresponds at least one value,  $\lambda$ , such that  $(\bar{x}(t, \lambda), \lambda)$  is contained in that neighborhood. Then the origin is called a bifurcation point; the family of solutions  $\bar{x}(t, \lambda)$  is called a bifurcating branch and the solutions corresponding to fixed values of the parameter are called bifurcating solutions (Krasnosel'skii [1]).

A system will be called structurally stable in a neighborhood of the origin if, for any sufficiently small perturbations, there exists

a homeomorphism that carries each trajectory of the original system to a trajectory of the perturbed system (Hirsch and Smale [2]). A bifurcating branch  $(\bar{x}(t, \lambda), \lambda)$  is structurally stable if for all values of  $\lambda \neq 0$ , sufficiently small, the system obtained by substituting  $x = y - \bar{x}(t, \lambda)$  into (1) is structurally stable in a neighborhood of the origin.

3. Existence of Vertical Bifurcations. Following Sattinger [3,4], assume (i) that  $A$  has a one-dimensional null space,  $Au_0 = 0$ . Since  $A$  is  $n \times n$  matrix it satisfies a Fredholm alternative. That is, the system  $Au = f$  has a solution if and only if  $\langle f, u_0^* \rangle = 0$ , where  $A^T u_0^* = 0$  and  $\langle u_0, u_0^* \rangle = 1$ . Further  $\langle Av, u_0^* \rangle = 0$  for all  $v$  in  $R^n$ .

I.e.  $M_0 = \{u | u \in R^n, \langle u, u_0^* \rangle = 0\}$ . By the above  $A$  is invertible on  $M_0$ . Denote its generalized inverse by  $K_0$  and the projection onto  $M_0$  by  $Q$ .

Finally, assume (ii)  $A + \lambda B$  has its spectrum in the left half plane for  $\lambda < 0$ , has a one dimensional null space when  $\lambda = 0$  and, when  $\lambda > 0$ , this eigenvalue crosses into the right half plane, while all others remain in the left half plane.

The steady state bifurcations of (1) can now be obtained by setting

$$(A + \lambda B)u + N(u, u) = 0 \tag{2}$$

Let  $[y]$  denote the inner product  $\langle y, u_0^* \rangle$  and introduce a parameter  $\epsilon$  into (2) by setting

$$u = \varepsilon(u_0 + \xi), \quad \lambda = \varepsilon\sigma$$

where

$$\xi = Q(u_0 + \xi), \quad [ \xi ] = 0.$$

Equation (2) then takes the form

$$A\xi + \varepsilon\sigma B(u_0 + \xi) + \varepsilon N(u_0 + \xi, u_0 + \xi) = 0. \quad (3)$$

Operating on (3) first by  $KQ$ , then taking the  $[ \cdot ]$  on (3) leads to

$$\xi + \varepsilon\sigma KQB(u_0 + \xi) + \varepsilon KQN(u_0 + \xi, u_0 + \xi) = 0 \quad (4)$$

and

$$\sigma[B \phi_0] + \sigma[B \xi] + [N(u_0 + \xi, u_0 + \xi)] = 0 \quad (5)$$

Equations (4) and (5) are called the Liapanov-Schmidt equations [3,4].

The implicit function theorem can now be used to prove the following:

Theorem 1. (Sattinger [3,4]). Under the assumptions (i) and (ii), and if  $[B u_0] \neq 0$  there exists an analytic one-parameter family  $u = u(\varepsilon)$ ,  $\lambda = \lambda(\varepsilon)$  of solutions of (2) such that  $[u(\varepsilon)] = \varepsilon$  and  $\lambda(0) = 0$ ,  $u(0) = 0$ .

Remark 1. The statement of the theorem and the proof do not rule out the possibility that  $u(\varepsilon) \neq 0$ ,  $\lambda(\varepsilon) \equiv 0$ . If this is the case, the corresponding branch is called a vertical bifurcation.

Example 1. Consider the scalar linear equation

$$x' = \lambda x$$

With each  $\lambda = 0$ , the equation for steady state solutions is satisfied for any value of  $x$ . That is, a vertical bifurcation occurs at the origin, which is a bifurcation point.

Example 2. Consider

$$u' = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_1^2 \end{pmatrix} \quad (6)$$

which also satisfies the conditions of Theorem 1. The steady state solutions are  $u \equiv 0$  when  $\lambda \neq 0$  and  $u_2 = u_1^2 + u_1$  when  $\lambda \equiv 0$  and  $u_1 \neq 0$ .

We next show that a vertical bifurcation cannot be structurally stable.

Theorem 2. Suppose the system

$$x' = F(x) \quad (7)$$

has the property that

(a) there exists  $G_1(x_1)$  such that the solution  $x_1 = 0$  of  $x_1' = F(x_1) + G_1(x_1)$  is asymptotically stable

(b) there exists  $G_2(x_2)$  such that the solution  $x_2 = 0$  of  $x_2' = F(x_2) + G_2(x_2)$  is unstable.

Then the system (7) is not structurally stable. Here  $F, G_1, G_2$  are sufficiently smooth,  $|G_i(x)| < \alpha |x|$ , and  $\alpha$  sufficiently small.

Proof: Suppose the system is structurally stable. It follows that there exists a homeomorphism  $x_2 = Tx_1$ . Since the system of part (a) is asymptotically stable, its solutions form a compact set so that the solutions of part (b) must form a bounded set. However, the solutions of (b) are unstable and hence not bounded.

Corollary. If the system (2) has a vertical bifurcation then it is not structurally stable.

Existence of Vertical Bifurcations. Assume that

$$\xi = \xi_1 \varepsilon + \xi_2 \varepsilon^2 + \dots + \dots$$

$$\sigma = \sigma_0 + \sigma_1 \varepsilon + \dots + \dots$$

and substitute into equation (4) to obtain

$$-\xi_1 = KQN(\phi_0, \phi_0) + \sigma_0 KQB \phi_0$$

$$-\xi_2 = 2KQN(\phi_0, \xi_1) + \sigma_1 KQB \phi_0 + \sigma_0 KQB \xi_1$$

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$$-\xi_n = 2KQN(\phi_0, \xi_{n-1}) + \dots + \begin{cases} 2KQN(\xi_{\frac{n-1}{2}}, \xi_{\frac{n}{2}}), & n \text{ even} \\ KQN(\xi_{\frac{n-1}{2}}, \xi_{\frac{n-1}{2}}), & n \text{ odd} \end{cases} \quad (8)$$

$$+ \sigma_{n-1} [B \phi_0] + \dots + \sigma_0 [B \xi_{n-1}]$$

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and use (5) to obtain the solvability conditions:

$$\sigma_0 [B \phi_0] + [N(\phi_0, \phi_0)] = 0$$

$$\sigma_1 [B \phi_0] + \sigma_0 [B \xi_1] + 2[N(\phi_0, \xi_1)] = 0$$

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$$\sigma_{n-1} [B \phi_0] + \dots + \sigma_0 [B \xi_{n-1}] +$$

$$+ 2[N(\phi_0, \xi_{n-1})] + \dots + \begin{cases} 2[N(\xi_{\frac{n-1}{2}}, \xi_{\frac{n}{2}})], & n \text{ even} \\ [N(\xi_{\frac{n-1}{2}}, \xi_{\frac{n-1}{2}})], & n \text{ odd} \end{cases} = 0 \quad (9)$$

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**Theorem 2.** Suppose  $[N(\phi_0, \phi_0)] = 0$  and  $N(\phi_0, \phi_0)$  is not identically zero and that there exists a basis  $\{x_1, x_2, \dots, x_k\}$ ,  $k \leq n - 1$ , for the space spanned by  $\{\xi_1, \xi_2, \dots, \xi_n, \dots\}$  which satisfies

$$\begin{aligned} [N(\phi_0, x_i)] &= 0 \quad i = 1, 2, \dots, k \\ [N(x_i, x_j)] &= 0 \quad i, j = 1, 2, \dots, k, \quad i < j \end{aligned} \quad (10)$$

Then equation (8) and (9) have the solution  $\xi(\epsilon) \neq 0$ ;  $\sigma(\epsilon) \equiv 0$ . That is, (2) has a vertical bifurcation.

**Proof:** Since  $\{x_1, \dots, x_k\}$  is a basis and since  $N(x)$  is a bilinear form it follows that condition (10) implies for  $i > j$ ,  $i, j = 1, 2, \dots$

$$\begin{aligned} [N(\phi_0, \xi_i)] &= 0 \\ [N(\xi_i, \xi_j)] &= 0 \end{aligned} \quad (11)$$

From (9),  $\sigma(\epsilon) \equiv 0$ . Solving (8) for  $\xi_i$  yields the desired nonzero solution.

**Theorem 3.** There always exists a family of bilinear forms for which the hypothesis of Theorem 2 is satisfied, providing  $[N(\phi_0, \phi_0)] = 0$  and  $N(\phi_0, \phi_0)$  is not identically zero, when  $n \geq 2$ .

**Proof:** From (9)  $\sigma_0 = 0$  and from (8)

$$\xi_1 = -KQN(\phi_0, \phi_0).$$

Let  $\xi_1 = x_1$  and assume

$$[N(\phi_0, x_1)] = 0. \quad (12)$$



Then, from (8)

$$\xi_2 = -2KQN(\phi_0, x_1).$$

If  $x_1, \xi_2$  are linearly independent, let  $\xi_2 = x_2$  and set

$$\begin{aligned} [N(\phi_0, x_2)] &= 0 \\ [N(x_1, x_2)] &= 0. \end{aligned} \tag{13}$$

If they are dependent, continue on with  $\xi_3, \dots, \xi_j \dots$  until determining a vector  $\xi_j$  which is linearly independent. In this manner, it is possible to generate a set of linearly independent vectors  $(x_1, \dots, x_k)$ ,  $k \leq n - 1$ , from a finite set of vectors  $(\xi_1, \dots, \xi_m)$ ,  $m \geq k$ , belonging to  $\{\xi_i\}$ ,  $i = 1, 2, \dots$ . By construction there exist constants  $\alpha_{jm}$ ,  $i = 1, 2, \dots$ ,  $m = 1, 2, \dots, k$  such that

$$\xi_j = \alpha_{j1} x_1 + \dots + \alpha_{jk} x_k$$

for all  $j = 1, 2, \dots$ .

Now the conditions (12), (13), ... can be summarized as

$$\begin{aligned} [N(\phi_0, x_i)] &= 0 \\ [N(x_i, x_j)] &= 0 \end{aligned} \tag{14}$$

for  $i, j = 1, 2, \dots, k$ ,  $i < j$ .

Equation (14) along with  $[N(\phi_0, \phi_0)] = 0$  consists of

$$\frac{1}{2} (k^2 + 3k + 4) \leq \frac{1}{2} (n^2 + n + 2)$$

conditions for  $k \leq n - 1$ . Now  $N(x, x)$  can be written as

$$N(x, x) = (N_1 x, \dots, N_n x)x$$

where  $N_1, \dots, N_n$  are symmetric  $n \times n$  matrices, which yields  $\frac{1}{2} n^2 (n + 1)$  elements which can be chosen arbitrarily to satisfy (13). Since  $\frac{1}{2} n^2 (n + 1) \geq \frac{1}{2} (n^2 + n + 2)$ , equation (13) represents a linear system in the  $\frac{1}{2} n^2 (n + 1)$  arbitrary elements of the  $N_1, \dots, N_n$ , it is clear that conditions can be imposed to assure the existence of the quadratic nonlinear term from (13).

Remark. If conditions are imposed on the solvability condition (9) to insure that  $\sigma(\varepsilon) \neq 0$ , the vertical bifurcation does not occur. Further, using Sattinger's arguments on the exchange of stability, it is possible to prove that the bifurcating branch of solutions now obtained by the implicit function theorem is either asymptotically stable or unstable from the first variation equation of (1). Appealing to a theorem of Hartman [5], we can now conclude that the bifurcating branch of solutions is structurally stable for  $\varepsilon > 0$ ,  $\varepsilon$  sufficiently small.

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APPENDIX XIII

SCALAR n-th ORDER COMPARISON SYSTEMS

R. W. Gunderson\*

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Department of Mathematics, Utah State University, Logan, Utah 84321. This work was supported by the National Aeronautics and Space Administration under Grant NGR-45-002-016.

INTRODUCTION

The theory of differential inequalities has found an important application in the theory of comparison systems (vector Liapunov functions). The theory of composite systems, for example, has been developed extensively since first introduced by Bailey in [1] (see Siljak [2] for an excellent survey). At the same time, there have been several papers dealing with the application of higher order trajectory derivatives in the analysis of dynamical systems [3], [4], [5], and [6]. In this note we shall discuss some recent results on the application of differential inequalities to comparison theorems involving higher order trajectory derivatives.

NOTATION AND DEFINITIONS

Consider the system of first order differential equations

$$x = f(t,x) \tag{1}$$

and the m-th order comparison system

$$u^{(m)}(t) = \omega(t,u,u',\dots,u^{(m-1)}) \tag{2}$$

where  $x, f$  belong to  $R^n$  and  $t, u$  are scalars. Assume  $f$  continuous on  $D_r = \{(t,x) \mid 0 \leq t \leq T < +\infty, \quad x < r\}$  and the right side of (2) continuous on  $[0,T] \times R^m$ . A solution of (1) satisfying the initial condition  $x_0$  at  $t_0$  will be denoted by  $x(t,t_0,x_0)$  and a solution of (2) satisfying  $u^j(t_0) = u_j \ (j = 1,2,\dots,m-1)$  will be denoted by  $u(t,t_0,U_0)$ .

Definition. The scalar function  $g(x)$ ,  $x$  belonging to  $R^n$ , will be said to be of type  $W^*$  on a set  $S \subset R^n$  if  $g(a) \leq g(b)$  for any  $a, b$  in  $S$  such that  $a_n = b_n, \ a_i \leq b_i \ (i=1,2,\dots,n-1)$ .

COMPARISON THEOREMS

The following lemma was given in [5]:

Lemma. Let  $v: D_r \rightarrow R$  and let  $v$  belong to  $C^m$ ,  $f$  belong to  $C^{m-1}$  on  $D_r$ . Let  $\omega$

of equation (2) be of type  $W^*$  in  $S$  for each  $t$ , where  $S = \{t, v(t,x), v'(t,x), \dots, v^{(m-1)}(t,x)\}$  and

$$v^{(j)}(t,x) = \frac{\partial v^{(j-1)}}{\partial t} + \frac{\partial v^{(j-1)}}{\partial x} f(t,x) \quad (3)$$

Suppose

$$v^{(m)}(t,x) \leq \omega(t, v, v', \dots, v^{(m-1)}) \quad (4)$$

for  $(t,x)$  belonging to  $D_r$  and set  $v^j(0, x_0) = u_j$ . Let  $J$  denote the maximal interval of existence of the right maximal solution  $u_m(t, 0, U_0)$ . Then

$$v^{(j)}(t, x(t, 0, x_0)) \leq u_m^{(j)}(t, 0, U_0) \quad (5)$$

for each  $t$  belonging to  $J$   $[0, T]$  and  $j=1, 2, \dots, m-1$ .

Remark. Since the inequality (4) implies the system of first order inequalities

$$\begin{aligned} v_1 &= v_2 \\ v_2 &= v_3 \\ &\vdots \\ v_n &\leq \omega(t, v_1, v_2, \dots, v_m) \end{aligned} \quad (6)$$

it is natural to view higher order derivatives as a means of obtaining vector Liapunov functions. (The fact that  $\omega$  is of type  $W^*$  assures that the right side of (6) satisfies the corresponding monotone property for vector functions). There are several practical difficulties, however, to this approach of constructing vector Liapunov functions. For example, one such difficulty is the requirement that the right side of (4) be of type  $W^*$ . Since the characteristic polynomial of a linear  $n$ -th order comparison equation must then be of the form

$$s^n \pm a_{n-1} s^{n-1} - a_{n-2} s^{n-2} - \dots - a_1 s - a_0 = 0 \quad (a_i \geq 0)$$

to be of type  $W^*$ , it follows that even in the linear case, the comparison equation cannot be stable, except when  $n=1$ .

In order to improve the applicability of scalar  $n$ -th order comparison theorems, we introduce the following special case of a more general theorem whose proof is given in [7],

Theorem 1. Assume

$$x^{(n)} = \omega(t, x, x', \dots, x^{(n-1)}), \quad (7)$$

where  $\omega$  is the right side of (2), holds for  $t$  belonging to some interval  $I$ . Suppose the solution  $z(t, t_0, E)$  of the adjoint equation to (7) is non-negative on the interval  $t_0, t$  for each  $t \geq t_0$ ;  $t_0, t$  belonging to  $I$ , and where  $E = (1, 0, \dots, 0)$ . Then (7) and (2) together imply

$$u(t) \leq x(t, t_0, U_0)$$

for all  $t \geq t_0$ ,  $t$  belonging to  $I$ .

The following is an example of a second order system which is not of type  $W^*$  but which satisfies the conditions of the theorem:

Example. The second order system

$$x'' + 3x' + 2x = 0$$

has the adjoint equation

$$z'' - 3z' + 2z = 0$$

which has the solution

$$z(t) = e^{(t-t_0)}(2 - e^{(t-t_0)})$$

satisfying the conditions

$$z(t_0) = 1$$

$$z'(t_0) = 0$$

Clearly  $z(t) > 0$  for all  $t < t_0$ , so that the conclusion of the theorem holds on any interval  $I$ .

In fact, it is not difficult to show that the conditions of the theorem are satisfied by any second order system with real, unequal, roots.

Using the above theorem it is then possible to prove comparison theorems of the following type [7]:

Theorem 2. Let  $v(x) = x^T Hx$  be a positive definite quadratic form and suppose the trajectory derivatives of  $v$  formed relative to (1) satisfy

$$v^{(m)} + a_{m-1} u^{(m-1)} + \dots + a_0 v \leq 0$$

for  $(t, x)$  belonging to  $D_r$ . Suppose the comparison system

$$u^{(m)} + a_{m-1} u^{(m-1)} + \dots + a_0 u = 0 \quad (8)$$

satisfies the conditions of theorem 1. Then there exists constants

$c, \dots, c_m$  such that

$$x(t, t_0, x_0)^2 \leq \frac{1}{\gamma} \sum_{j=1}^m c_j u_j(t)$$

for  $t \geq t_0$ , where  $\gamma$  is the minimal eigenvalue of  $H$  and the functions  $u_j(t)$  are linearly independent solutions of (8).



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