ATTITUDE DETERMINATION OF A HIGH ALTITUDE BALLOON SYSTEM

PART I

DEVELOPMENT OF THE MATHEMATICAL MODEL

by

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CHAPTER I

INTRODUCTION

1.1 Motivation and Relevance of Report. In April of 1974 the National Aeronautics and Space Administration conducted a high altitude balloon experiment called LACATE (Lower Atmosphere Composition and Temperature Experiment) which employed an infrared radiometer to sense remotely vertical profiles of the concentrations of selected atmospheric trace constituents and temperature. The constituents were measured by inverting infrared radiance profiles of the earth's horizon. The radiometer line of sight was scanned vertically across the horizon at approximately 0.25° per second, requiring 30 seconds to acquire a complete radiance profile. The specifications require that the relative vertical position of the data points making up a profile be known to approximately 30 arc seconds.

The balloon system for accomplishing the mission (see figure 1) consisted of: (1) a 45 million cubic feet balloon, (2) a load-bar containing the balloon control equipment, (3) a package containing additional balloon control electronics and a gondola recovery parachute, and (4) a gondola containing the research payload. Instrumentation to determine altitude consisted of a magnetometer and three orthogonally oriented precision rate gyros, the latter were used to obtain accurate time histories of the roll, pitch and yaw motions of the gondola. The magnetometer and gyros were flown with the research payload and their outputs were telemetered to ground operations

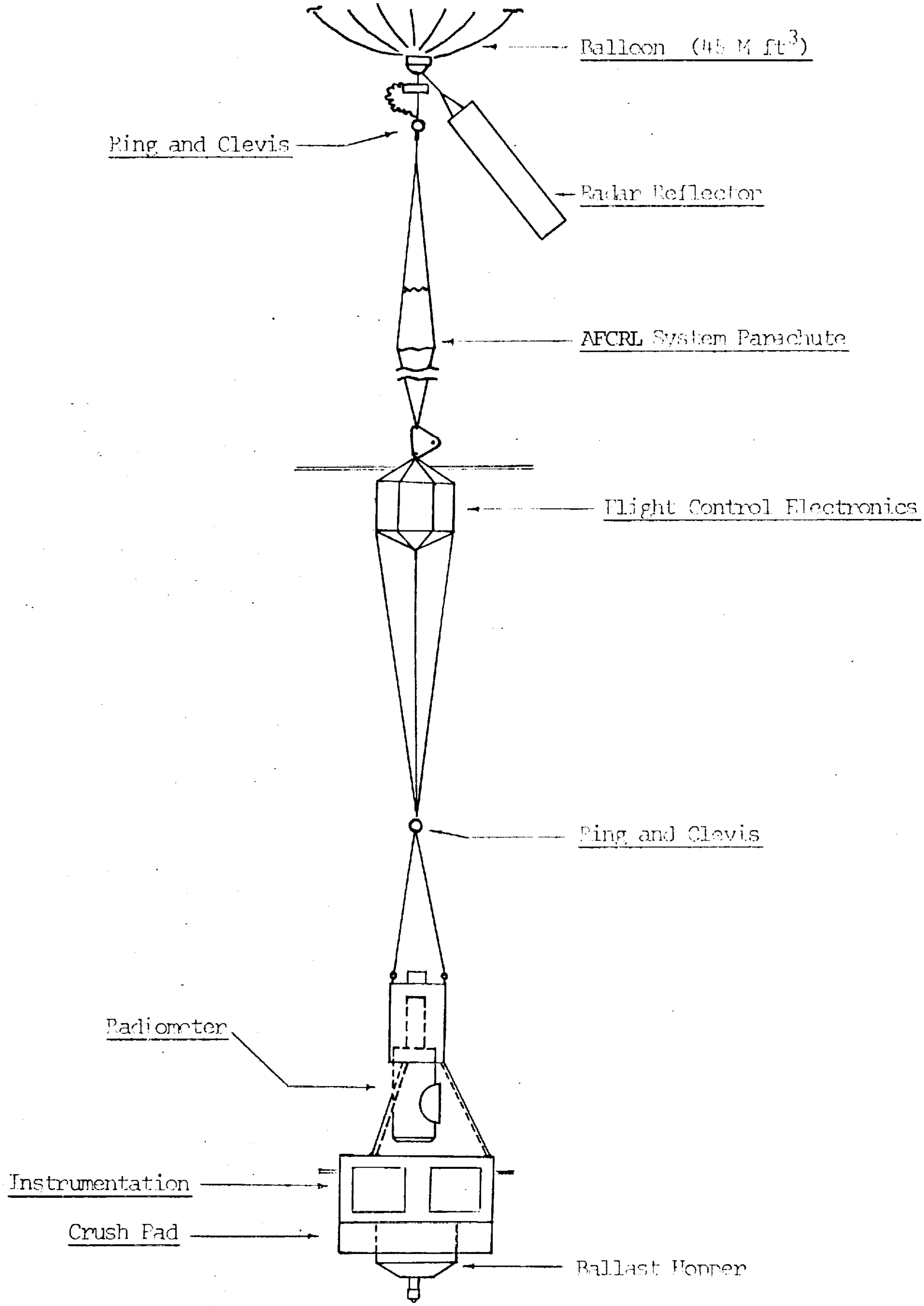


Fig. 1 - Balloon System

for recording and real time data reduction and display.

Figure 2 gives the dimensions and weight of the various subsystems comprising the balloon system. Two kinds of connectors between subsystems were employed: a clevis and ring shown in figure 3a and a triangular plate connection shown in figure 3b.

In order to fix the orientation of the line of sight of the radiometer, it is necessary to be able to determine the configuration of the platform in space, i.e. the attitude of the system. This can be accomplished by simulating the balloon system and using the gyro output in conjunction with a parameter estimation process. This process is described in greater detail in reference 1. The critical problem arising in the simulation of the high altitude balloon system is the development of the mathematical model. This model must enable one to predict the orientation of the platform with sufficient precision such that the position of the data points can be determined within the required values. At the same time the model must be as simple as possible in order to be amenable to simulation.

The actual motion of the balloon system (once it reaches float altitude) is extremely complex and involves various types of oscillations including bounce (vertical oscillation), pendulations (in plane motion) and spin (rotation). In the general case, where the system is subjected to arbitrary input (initial conditions) these oscillations will be coupled. The modeling problem is further complicated by other factors;

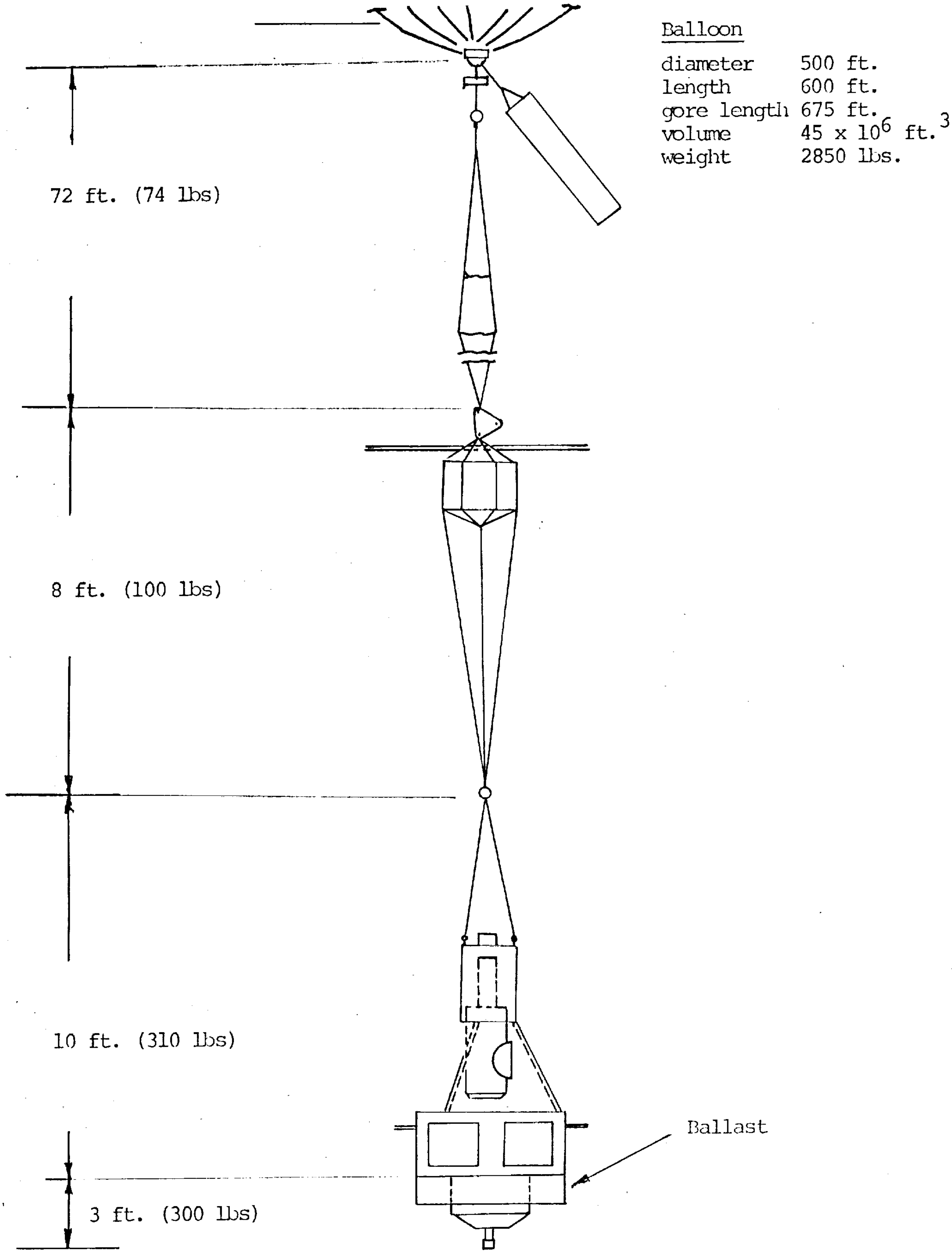


Figure 2. Balloon System(Dimensions and Weights)

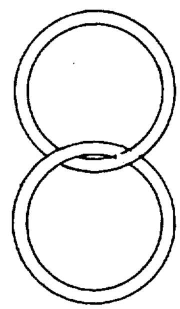


Fig. 3a Ring and Clevis Connector

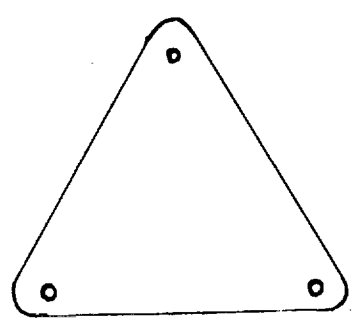


Fig. 3b Triangular Plate Connector

e.g. the presence of damping forces, the significant mass (relative to the payload) and elasticity of the cable, the tendency (due to support conditions) of some of the gondolas to oscillate independently of the main system, and the fact that the balloon itself is actually a distributed parameter system and hence its properties (e.g. moment of inertia) must first be approximated in order to develop a lumped parameter model for the entire system.

1.2 Objectives of Report. The main objective of this report will be to develop a mathematical model for predicting the three dimensional motion of the balloon system. The model will incorporate the various factors discussed previously and includes the effects of bounce, pendulation and spin of each subsystem. Also, some work will be done in analyzing the balloon itself, i.e. determine the best way for treating it as an equivalent lumped parameter system. This will require some investigation of boundary layer effects and the aerodynamic forces acting on the balloon. Finally, various simplified forms of the system mathematical model will be developed based on an "order-of-magnitude" analysis.

1.3 Idealization of System. In general, it is necessary to idealize the physical system before one obtains the mathematical model. For purposes of this study, the following assumptions will be made:

(1) The masses of balloon, subsystems and interconnecting subsystems will be "lumped" at the locations shown in figure 4.

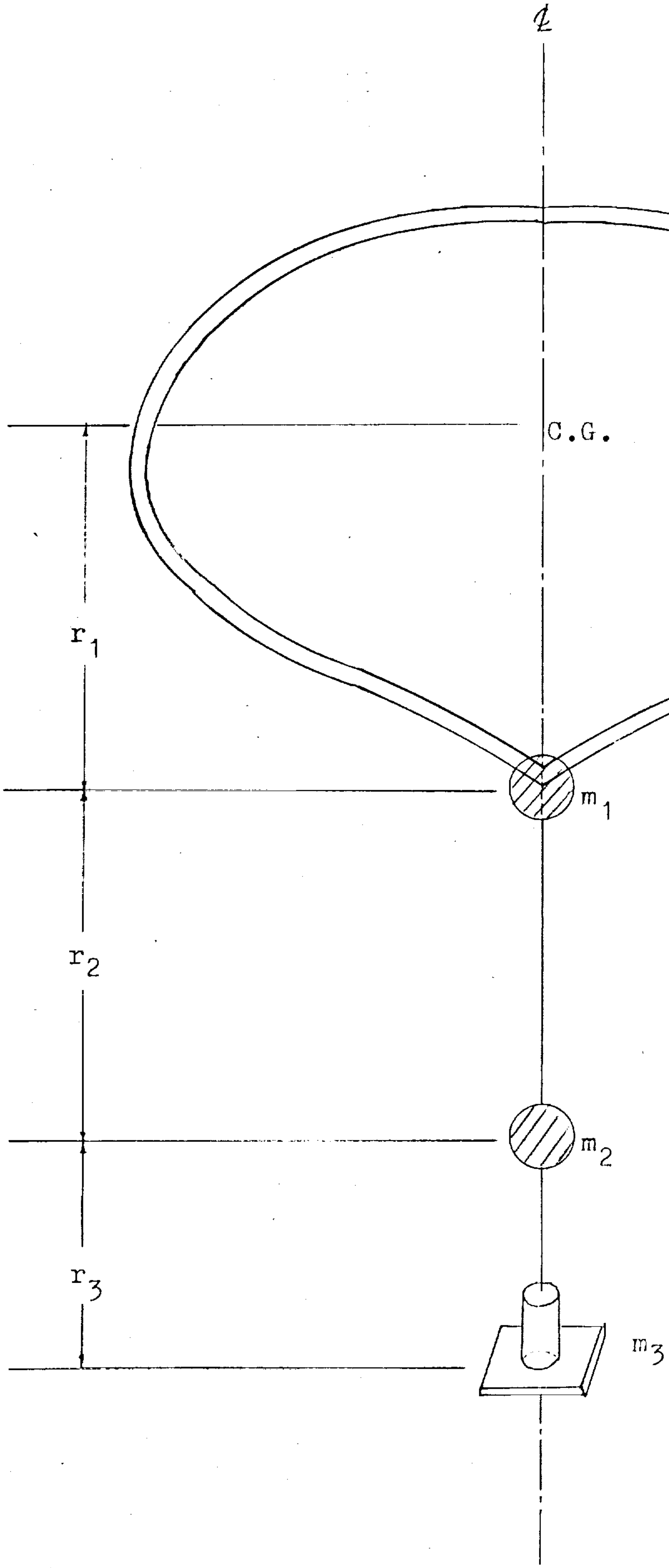


Fig. 4 Idealized Balloon System

(2) The balloon will be treated as an "equivalent" rigid body.

(3) The generalized drag reactions will be computed by assuming that balloon and masses are spherical in shape.

(4) The altitude of the balloon static equilibrium position (float altitude) will be assumed to be a constant during the entire period of observation, i.e. changes in this altitude due to the losses of or changes in the properties of helium will be neglected.

(5) The cables will be treated as though they were inflexible.

Further assumptions are made in this study but these will be discussed in the main body of the thesis.

CHAPTER II

BASIC CONCEPTS

2.1 Equilibrium Principles of Mechanics. The vector (Newton's law) and the variational (Hamilton's principle and Lagrange's Equation) principles of mechanics provide two methods for obtaining mathematical descriptions of the same realm of natural phenomena. The vector theory bases everything on two fundamental vector quantities (momentum and force), while the variational theory bases everything on two scalar quantities (kinetic energy and generalized work function). In the case of free particles, i.e. particles whose motion is not constrained, the two forms of description lead to identical equations. However, for systems with constraints the analytical treatment is simpler and more efficient. With the variational method, the given constraints are considered in a natural way by letting the system move along all the possible paths (in configuration space) in harmony with them. With the vector method, the forces which maintain the constraints must be considered. The vector method does not restrict the nature of a force, while the variational method requires that the acting forces can be derived from a scalar quantity, termed the work function. Forces, which cannot be derived from a generalized work function (e.g. Coulomb friction) must be included via the generalized force concept.

There are two advantages in modeling the balloon system under study by the analytical method.

(a) The energy terms and generalized forces can be computed in a straightforward manner without great difficulty.

(b) The internal reaction forces which exist at the interface of the subsystems do no work during the motion, and hence they can be ignored. This results in a simplification of the form of the model.

2.2 Hamilton's Principle and Lagrange's Equation. Hamilton's principle states that the motion of any mechanical system in configuration space is such that

$$\delta\int_{t_1}^{t_2} (T + W^*)dt = 0 \ , \qquad (2\text{-}1)$$

where:

T = kinetic energy,

$W^* = \sum_{i=1}^{n} Q_j q_j$ = generalized work function,

n = number of degrees of freedom,

$Q_j = \sum_{i=1}^{n} \overline{F}_i \cdot \frac{\partial \overline{r}_i}{\partial q_i}$ = generalized force,

$\overline{F}_i$ = external forces acting on system,

$\overline{r}_i$ = position vector to point where the external force acts,

$\overline{q}_i$ = generalized coordinates, and

N = number of external forces.

From the calculus of variation, the necessary and sufficient conditions (Euler-Lagrange Equations) for equation (2-1) to be satisfied are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i \ \ (i=1\ldots n) \ , \qquad (2\text{-}2)$$

and $\delta q_i(t_1) = \delta q_i(t_2) = 0.$

Moreover, if (a) the generalized forces can be divided into conservative and non-conservative components, i.e.

$$Q_i = Q_i^C + Q_i^N \text{ , where}$$

$$Q_i^C = \partial V/\partial q_i \text{ , and if}$$

(b) the potential energy (V) is not a function of the generalized velocities ($\dot{q}_i$), then equation (2-2) can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i^N \ , \tag{2-3}$$

where the Lagrangian L = T - V.

Equation (2-3) can be written in state-variable form by first defining a generalized momentum (p_i),

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \ . \tag{2-4}$$

Substituting (2-4) into (2-3) yields

$$\dot{p}_i = \frac{\partial L}{\partial q_i} + Q_i^N \ . \tag{2-5}$$

Equations (2-4) and (2-5) can be employed to obtain the 2n state variable equations (in terms of generalized coordinates and generalized momentum) by inverting equation (2-4) to get the generalized velocities in terms of p_i and q_i and then substituting these expressions into equation (2-5).

If (a) the system is conservative ($Q_i^N = 0$),

(b) the generalized forces can be divided into applied forces (Q_i^a) and reaction forces (Q_i^R) such that

(c) the work done by the reaction forces during a virtual displacement is equal to zero, and

(d) the applied forces can be written in the form

$$Q_i^{\,a} = - \frac{\partial V}{\partial q_i} ,$$

then, equations (2-4) and (2-5) yield the Canonical equations of Hamilton; i.e.

$$\dot{p}_i = - \frac{\partial H}{\partial q_i} \tag{2-6}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{2-7}$$

where H, the Hamiltonian, is defined by

$$H(p,q,t) = \sum_i \dot{q}_i p_i - L(q_i,\dot{q}_i,t) \ .$$

2.3 Generalized Coordinates. The generalized coordinates for a given system are those coordinates which are employed to specify the configuration of the system at any instant of time. In any mechanical system there will be as many generalized coordinates as there are degrees of freedom. In the case of the idealized lumped parameter system shown in figure 4, eleven generalized coordinates are required to specify the configuration. These consist of three translational coordinates to locate the mass center of the balloon (relative to a set of axes fixed in the space) and eight Euler angles to specify the orientation of the three pendulum subsystems.

In general, the Euler angles give the orientation of the body coordinate system (x_i''') relative to a fixed system (X_i). If the two systems are initially coincident, a series of three rotations about the body axis is sufficient to allow the body axes to attain any orientation.

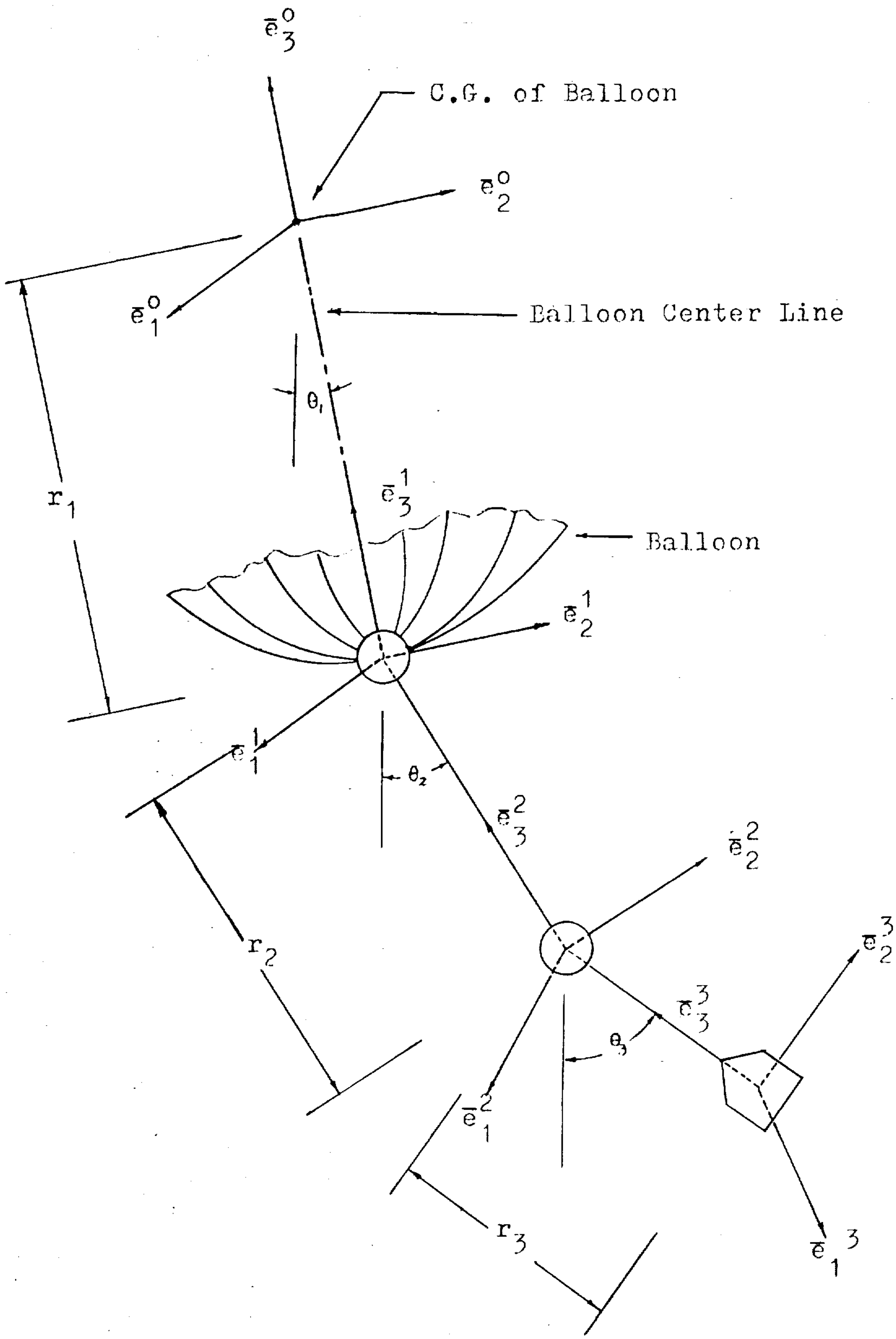


Fig. 5 Balloon System Configuration

Several sets of Euler angles are possible for fixing the orientation of the balloon and subsystems. Three sets were investigated in this work and they are shown in figure 6. The sequence of the three rotations which define the set which was finally employed* in this work is described below.

(a) A positive rotation θ about the X_1 axis resulting in the x_i' body system,

(b) a positive rotation ψ about the x_2' axis resulting in the x_i'' body system, and

(c) a positive rotation ϕ of the body about the x_3'' axis relative to the x_i'' system. Since the final rotation is relative to the x_i'' system, the transformation equation is given as

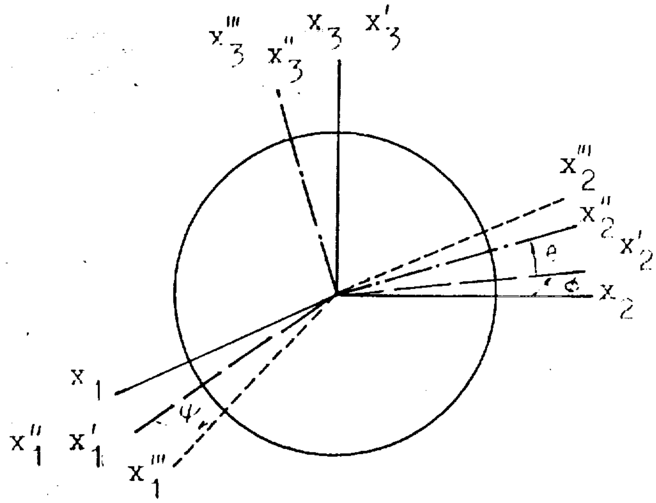
$$\overline{x}'' = A\,\overline{X}\,, \tag{2-8}$$

where

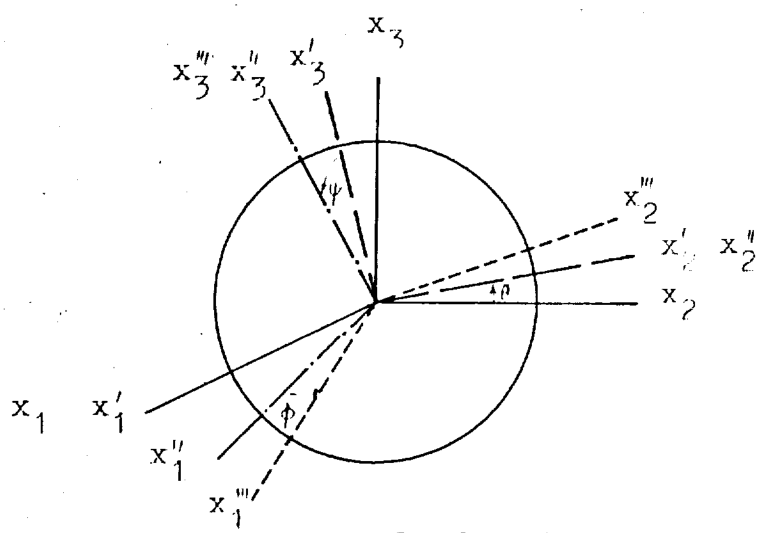
$$A = \begin{pmatrix} c(\psi) & s(\psi)s(\theta) & -s(\psi)c(\theta) \\ 0 & c(\theta) & s(\theta) \\ s(\psi) & -s(\theta)c(\psi) & c(\psi)c(\theta) \end{pmatrix} \tag{2-9}$$

This set is convenient whenever a body has at least one axis of symmetry.

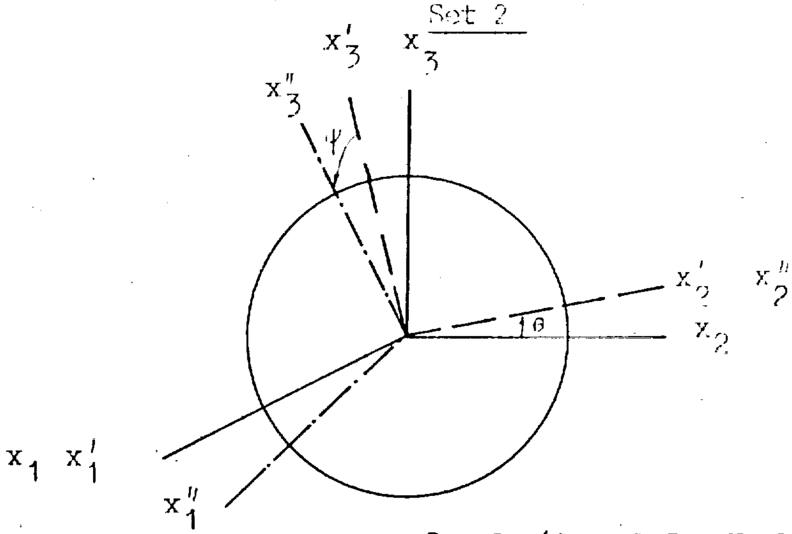
2.4 <u>Aerodynamic Reactions</u>. The exact mathematical model for the balloon system consists of the equations of motion of the solid (balloon fabric and payload) and the fluid-dynamic equations. These equations are coupled through the boundary conditions which must be satisfied at the interface of the solid and fluid media. The resulting mathematical model is extremely complex and consists of a system of



Set 1



Set 2



Set 3 (Actual Set Employed)

Fig. 6 Euler Angles for Balloon Orientation coupled non-linear differential equations which must be solved simultaneously in order to predict the behavior of the system.

For purposes of this work the mathematical model will be simplified by treating the system as a lumped parameter system. In order to accomplish this it is first necessary to develop approximate expressions for the forces and torques which result due to the interactions between the solid system and the fluid media (air-helium). In the lumped parameter model, these forces and torques are then treated as external reactions on the solid system.

The reactions which the fluid media exert on the solid system consist of (a) a "static" lift force due to buoyancy, (b) "dynamic" buoyancy forces due to the acceleration of the fluid approaching the system, (c) viscous drag forces due to the translational motion of a solid body in a fluid medium, (d) viscous drag torques due to the rotation of the balloon in the fluid media, and (e) inertia drag forces and torques due to the unsteady motion of the fluid media relative to the balloon system. The expressions for these reaction forces and torques are presented below.

2.5 <u>Static Buoyancy Force</u>. The net lift force due to the aerostatic pressures is given as

$$F_L = V_H g(\rho_a - \rho_H) \quad , \tag{2-10}$$

where F_L = static lift force in the z direction,

V_H = volume of helium,

ρ_a = density of air, and

ρ_H = density of helium .

If we assume (a) that the mission takes place in the isothermal region of the atmosphere and (b) that helium undergoes an adiabatic process, then the expressions for ρ_a and ρ_H are given as

$$\rho_a = \rho_{oa} \left(e^{-\frac{\rho_{oa}\, g}{p_o} z} \right) , \tag{2-11}$$

and

$$\rho_H = \rho_{oH} \left(e^{-\frac{\rho_{oa}\, g}{p_o} z} \right)^{\frac{1}{n_H}} , \tag{2-12}$$

where ρ_{oa} = density of air at the static equilibrium position (i.e. the float altitude),

ρ_{oH} = density of helium at the static equilibrium position,

p_o = atmospheric pressure at static equilibrium position,

z = elevation measured from the static equilibrium position, and

n_H = ratio of specific heats for helium, (c_p/c_v).

Substitution of (2-11) and (2-12) into equation (2-10) yields the expression for the static lift force (F_L) as

$$F_L = \rho_H V_H\, g \left[\frac{\rho_{oa}}{\rho_{oH}}\, e^{-\frac{\rho_{oa} gz}{p_o}\left(1 - \frac{1}{n_H}\right)} - 1 \right] . \tag{2-13}$$

The presence of a static-buoyancy force results in an external torque (see figure 7) whenever the balloon shape is such that the center of gravity of the shell does not coincide with the center of buoyancy. The magnitude of this torque is given as

$$\bar{N}_L = \bar{d} \times \bar{F}_L , \tag{2-14}$$

where $\bar{N}_L$ = external torque, and

$\bar{d}$ = vector between the center of gravity and center of buoyancy.

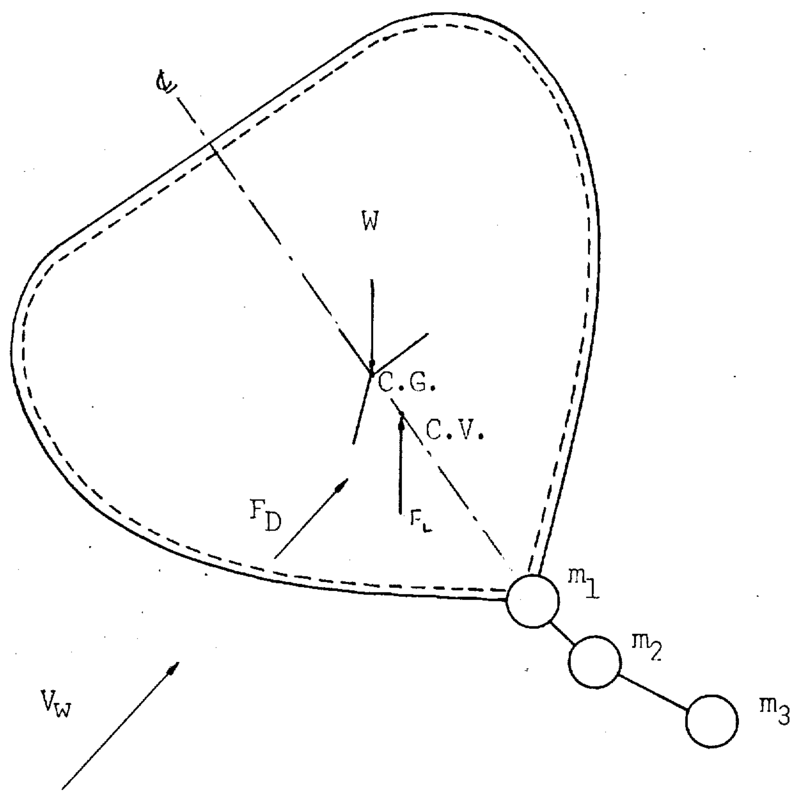


Fig. 7 Location of Static and Dynamic Buoyancy Forces on Idealized Balloon System.

Equation (2-16) indicates that the static lift force is a non-linear function of z. A linear expression for F_L, valid for small z, can be obtained from a truncated Taylor-series expansion:

$$F_L = F_o + \left(\frac{dF_L}{dz}\right)_{z=0} z \ ,$$

where F_o = lift force at the static equilibrium position.

$(F_o = w)$,

w = weight of the solid balloon system.

The equivalent spring constant $K_o = \left.\frac{dF_L}{dz}\right)_{z=0}$ is obtained by differentiating equation (2-13) and considering $\rho_H V_H g$ to be a constant. This development is presented in Appendix A.

2.6 Dynamic Buoyancy Force. The absolute acceleration of the fluid approaching the balloon results in a uniform pressure drop across the balloon in the direction of the acceleration of the fluid. This pressure drop, when integrated over the balloon surface, results in a force which acts on the balloon in the direction of the absolute acceleration of the fluid. The expression for this force, termed dynamic buoyancy force, is given as

$$\overline{F}_B = V_H \rho_a \dot{\overline{V}}_W \tag{2-15}$$

where $\overline{F}_B$ = dynamic buoyancy force, and

$\dot{\overline{V}}_W$ = absolute acceleration of air.

This force acts at the center of buoyancy and results in a torque whenever the center of buoyancy does not coincide with the center of gravity (see figure 7).

2.7 Inertia Drag Force. Whenever a body accelerates in a fluid medium, it is subjected to a resisting force even if the fluid is assumed to be frictionless. This resisting force, termed the inertia drag force, has the same effect as that obtained by increasing the body's mass by an amount equal to that of the fluid carried along with it. The expression for the inertia drag force on a spherical body due to the external flow of air (2) is given as

$$\bar{F}_I = -\frac{1}{2}\rho_a V_H \left(\dot{\bar{V}}_b - \dot{\bar{V}}_W\right) , \tag{2-16}$$

while, for the internal flow of helium, the expression is given as

$$\bar{F}_I = -\rho_H V_H \dot{\bar{V}}_b , \tag{2-17}$$

where $\bar{F}_I$ = inertia drag force, and

$\dot{\bar{V}}_b$ = acceleration of the balloon.

2.8 Viscous Drag Force. The expression for the resisting viscous drag force (3) which acts on a spherical solid due to its translation relative to air is given as

$$\bar{F}_D = -\frac{C_D}{2}\rho_a A \left|\bar{V}_b - \bar{V}_W\right| (\bar{V}_b - \bar{V}_W) , \tag{2-18}$$

where $\bar{F}_D$ = viscous drag force,

C_D = drag coefficient,

A = projected area of the body on a plane normal to the relative velocity,

$\bar{V}_b$ = balloon velocity, and

$\bar{V}_W$ = air velocity.

The magnitude of C_D depends on the Reynolds number.

2.9 Viscous Drag Torque due to the Rotation of a Spherical Balloon in Viscous Fluid Media. The expression for the drag torque on a balloon rotating about a single axis depends on the range of the Reynolds numbers under consideration. For high Reynolds numbers, $Re = \frac{\omega r_o^2 \rho_a}{\mu_a}$, and $||\omega||^2 >> ||\dot{\omega}||$, the expression (4) is given as

$$N_V = -K_1 (\rho\mu)^{1/2} \omega^{1/2} r_o^4 \omega \text{ *,} \tag{2-19}$$

where N_V = viscous drag torque,

μ = viscosity of the fluid,

ω = angular velocity of balloon along the rotation axis,

r_o = mean radius of the balloon,

$\dot{\omega}$ = angular acceleration of the balloon,

$||\omega||$ = amplitude of ω, and

$||\dot{\omega}||$ = amplitude of $\dot{\omega}$.

The above expression is valid for both oscillatory and non-oscillatory motion.

Equation (2-19) can be used to obtain the drag torque due to both internal and external fluids and was developed under the following assumptions:

(1) The boundary layer thickness of the fluid is a constant and is taken to be equal to the boundary layer thickness on an infinite rotating disk. This thickness (5) is given as

$$\delta = 4.5 \sqrt{\frac{\mu}{\rho\omega}} , \tag{2-20}$$

*Refer to Appendix B for development of this expression.

where

δ = boundary layer thickness, and

ω = magnitude of angular velocity of the balloon.

(2) The boundary layer thickness is assumed to be small compared to the mean radius of the balloon, and

(3) The tangential velocity is assumed to vary parabolically across the boundary layer thickness.

For the case of a sphere undergoing oscillatory motion with frequency Ω about a single axis and having low Reynolds number, the expression (6) for the viscous torque due to the relative motion of the inner fluid is given as

$$N_V = -\frac{8}{3}\pi\,\mu_H\,r_o^3\,(\beta_H r_o)\,\omega\ , \qquad (2\text{-}21)$$

where N_V = internal viscous torque due to helium,

μ_H = viscosity of helium,

Ω = frequency of rotational oscillation, and

ν_H = kinematic viscosity of helium.

For the relative motion of the outer fluid, the expression (6) for the torque is given as

$$N_V = -\frac{8}{3}\pi\mu_a r_o^3 \left(\frac{3 + 6\beta_a r_o + 6\beta_a^2 r_o^2 + 2\beta_a^3 r_o^3}{1 + 2\beta_a r_o + 2\beta_a^2 r_o^2}\right)\omega\ , \qquad (2\text{-}22)$$

where N_V = external viscous torque due to air,

μ_a = viscosity of air,

$\beta_a = (\Omega/2\nu_a)^{1/2}$, and

ν_a = kinematic viscosity of air.

2.10 Inertia Drag Torque due to the Unsteady Rotation of a Spherical Balloon in Viscous Fluid Media. The expression for the inertia drag torque on a spherical balloon rotating about a single axis depends on the magnitude of the Reynolds number. For high Reynolds number and $||\omega||^2 >> ||\dot{\omega}||$, the expression is (4)

$$N_I = -K_2 \ (\rho\mu)^{1/2}\omega^{-1/2} r_o^4 \dot{\omega} \ *, \tag{2-23}$$

where N_I = inertia drag torque.

Equation (2-23) is valid only for the case of non-oscillatory motion but can be used to determine the inertia drag torque due to both inner and outer flows.

For low Reynolds number, the expression (6) for the inner flow is given as

$$N_I = -\frac{4}{3}\pi \rho_H r_o^5 (\beta_H r_o)^{-1} \dot{\omega} \ , \tag{2-24}$$

where N_I = internal inertia drag torque.

For the case of outer flow, the expression for the torque is given as

$$N_I = -\frac{8}{3}\pi\rho_a r_o^5 \frac{1 + \beta_a r_o}{1 + 2\beta_a r_o + 2\beta_a^2 r_o^2} \dot{\omega} \ , \tag{2-25}$$

where N_I = external inertia drag torque.

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*See Appendix C for development of this expression.

CHAPTER III

DEVELOPMENT OF STATE VARIABLE EQUATIONS

3.1 Kinematics. In order to obtain the kinetic energy of the balloon system, it is first necessary to develop the kinematic expressions for the velocities (angular and linear) of the balloon and subsystems shown in figure 4. By employing the Eulerian Angles which were described in Chapter II, the angular velocities for the balloon and subsystems can be written as

$$\bar{\omega}^0 = \bar{\omega}^1 = \dot{\theta}_1 c(\psi_1)\ \bar{e}_1^1 + \dot{\psi}_1\ \bar{e}_2^1 + (\dot{\theta}_1 s(\psi_1) + \dot{\phi}_1)\ \bar{e}_3^1 ,$$

$$\bar{\omega}^2 = \dot{\theta}_2 c(\psi_2)\ \bar{e}_1^2 + \dot{\psi}_2\ \bar{e}_2^2 + (\dot{\theta}_2 s(\psi_2) + \dot{\phi}_1)\ \bar{e}_3^2 , \qquad (3\text{-}1)$$

$$\bar{\omega}^3 = \dot{\theta}_3 c(\psi_3)\ \bar{e}_1^3 + \dot{\psi}_3\ \bar{e}_2^3 + (\dot{\theta}_3 s(\psi_3) + \dot{\phi}_3)\ \bar{e}_3^3 ,$$

where $\bar{\omega}^0$ = angular velocity of the balloon,

$\bar{\omega}^i$ = angular velocity of the i^{th} subsystem (i=1,2,3),

θ_i, ψ_i, ϕ_i = Euler angles of rotation, and

$\bar{e}_i^j$ = unit vectors for the j^{th} subsystem (see figure 4)

For purpose of this work, the spin of subsystem 2 is assumed to be identical to the spin of the balloon, i.e. $\dot{\phi}_1 = \dot{\phi}_2$. This assumption is reasonable because of the type of connectors employed (refer figure 3).

The linear motion of the balloon center is referred to an axis which is fixed in space at the balloon static equilibrium position (see figure 8). The velocity expression is given as

$$\bar{v}^0 = \dot{x}\ \bar{i} + \dot{y}\ \bar{j} + \dot{z}\ \bar{k} , \qquad (3\text{-}2)$$

where $\bar{v}^0$ = velocity of balloon center, and

$\dot{x}, \dot{y}, \dot{z}$ = components of the velocity of the balloon center.

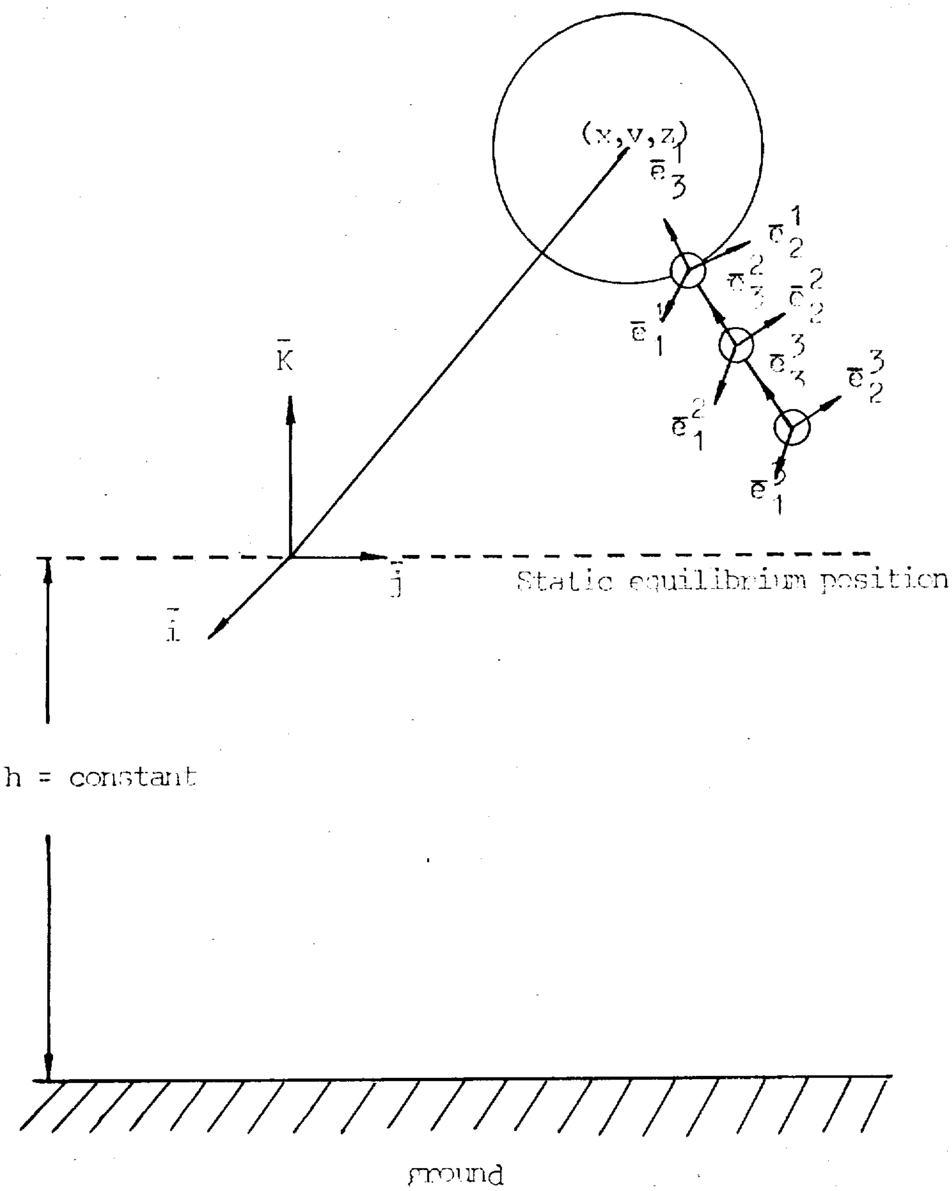


Fig. 8 Coordinate System

The velocity expressions for subsystems are given as

$$\bar{v}^{i} = \bar{v}^{i-1} - r_i \, \bar{\omega}^{i} \times \bar{e}_3^{i} \, , \qquad (i = 1,2,3) \tag{3-3}$$

where $\bar{v}^{i}$ = velocity of the mass center of the i^{th} subsystem,

$\bar{v}^{i-1}$ = velocity of the mass center of the $(i-1)^{th}$ subsystem,

r_i = distance between m_i and m_{i-1}.

The detailed velocity expressions in terms of the Euler angles and their rates are obtained by substituting equation (3-1) and (3-2) into (3-3) and these are presented in Appendix D.

3.2 Kinetic Energy. The general kinetic energy expression for the balloon shell which is undergoing both rotational and translational motion is

$$T^{o} = \frac{m_o}{2} \bar{V}^{o} \cdot \bar{V}^{o} + \frac{I_{o1}}{2} \left[(\omega_1^{o})^2 + (\omega_2^{o})^2 \right] + \frac{I_{o3}}{2} (\omega_3^{o})^2 \, , \tag{3-4}$$

where T^{o} = kinetic energy,

m_o = mass of balloon shell,

I_{o1}, I_{o3} = moments of inertia of the balloon shell relative to the $\bar{e}_i^{o}$ modified body axis, and

ω_i^{o} = components of the balloon angular velocity along $\bar{e}_i^{o}$ modified body axis.

The detailed expression for the kinetic energy is obtained by substituting equations (3-1) and (3-2) into equation (3-4). This expression is presented in Appendix E.

The kinetic energy expressions for the subsystems are obtained by substituting equations (3-1) and (3-3) into the following

$$T^{i} = \frac{m_i}{2} \bar{V}^{i} \cdot \bar{V}^{i} + \frac{I_{i1}}{2} (\omega_1^{i})^2 + \frac{I_{i2}}{2} (\omega_2^{i})^2 + \frac{I_{i3}}{2} (\omega_3^{i})^2 , \tag{3-5}$$

(i=1,2,3)

where T^i = kinetic energy of the i^{th} subsystem,

m_i = mass of the i^{th} subsystem,

I_{ij} = moments of inertia of the i^{th} subsystem relative to $\bar{e}^i_j$ modified body axis, and

ω^i_j = components of angular velocity of the i^{th} subsystem along $\bar{e}^i_j$ direction.

For purposes of this work, the moments of inertia I_{i1} and I_{i2} were neglected in which case equation (3-5) becomes

$$T^i = \frac{m_i}{2} \bar{V}^i \cdot \bar{V}^i + \frac{I_{i3}}{2} (\omega^i_3)^2 \;, \quad (i=1,2,3) \tag{3-6}$$

The detailed expressions for T^i are given in Appendix E . The total kinetic energy (T) of the balloon system is obtained by summing up the expressions in equations (3-4) and (3-6), i.e.

$$T = T^o + \sum_{i=1}^{3} T^i \tag{3-7}$$

3.3 Potential Energy. The expressions for potential energy due to the gravitational forces acting on the subsystems are given as

$$V^i = - m_i \, g \sum_{j=1}^{i} r_j c(\psi_j) c(\theta_j) \;, \qquad i=1,2,3 \tag{3-8}$$

where V^i = potential energy of the i^{th} subsystem with datum through the balloon mass center.

The expression for the potential energy due to the static buoyancy force is given as

$$V_s = \frac{K_o}{2} z^2 \tag{3-9}$$

where V_s = potential energy due to static buoyancy force,

K_o = equivalent spring constant, and

z = elevation measured from balloon static equilibrium position.

The total potential energy (V) for the balloon system is obtained by summing up equations (3-8) and (3-9), i.e.

$$V = \sum_{i=1}^{3} V^i + V_s \tag{3-10}$$

The Lagrangian (L), which is defined in equation (2-3), is obtained by subtracting the total potential energy in equation (3-10) from the total kinetic energy in equation (3-7) and is given as

$$L = T - V = T^o + \sum_{i=1}^{3} T^i - \sum_{i=1}^{3} V^i - V_s . \tag{3-11}$$

3.4 Generalized Forces. The generalized force Q_i, corresponding to the generalized coordinate q_i, can be determined from equation (2-1) or, in the case of torques, by determining the virtual work done by the torque during a virtual displacement δq_i (7). In this manner the generalized forces corresponding to each of the aerodynamic reactions (refer Chapter II) were determined and the resulting non-zero expressions are presented below.

(a) Static Buoyancy Force*

$$Q^1_{\theta 1} = w\ d\ s(\theta_1), \text{ and} \tag{3-12}$$

$$Q^1_{\psi 1} = w\ d\ s(\psi_1).$$

---

*The generalized forces presented here are due (only) to the torque of the static buoyancy force, since the force itself is considered via the potential energy expression (see equation 3-9).

(b) Dynamic Buoyancy Force*

$$Q_x^2 = V_H \rho_a \dot{v}_1 \,,$$

$$Q_y^2 = V_H \rho_a \dot{v}_2 \,, \text{ and} \qquad (3\text{-}13)$$

$$Q_z^2 = V_H \rho_a \dot{v}_3 \,,$$

where $\bar{V}_W = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$

$= \text{wind velocity.}$

(c) Inertia Drag Force

$$Q_x^3 = -0.5\rho_a V_H(\ddot{x} - \dot{v}_1) - \rho_H V_H \ddot{x} \,,$$

$$Q_y^3 = -0.5\rho_a V_H(\ddot{y} - \dot{v}_2) - \rho_H V_H \ddot{y} \,, \text{ and} \qquad (3\text{-}14)$$

$$Q_z^3 = -0.5\rho_a V_H(\ddot{z} - \dot{v}_3) - \rho_H V_H \ddot{z} \,.$$

(d) Viscous Drag Force**

$$Q_x^4 = -\frac{1}{2} C_D A \rho_a [(\dot{x}-v_1)^2 + (\dot{y}-v_2)^2 + (\dot{z}-v_3)^2]^{1/2}(\dot{x}-v_1),$$

$$Q_y^4 = -\frac{1}{2} C_D A \rho_a [(\dot{x}-v_1)^2 + (\dot{y}-v_2)^2 + (\dot{z}-v_3)^2]^{1/2}(\dot{y}-v_2), \ \& \qquad (3\text{-}15)$$

$$Q_z^4 = -\frac{1}{2} C_D A \rho_a [(\dot{x}-v_1)^2 + (\dot{y}-v_2)^2 + (\dot{z}-v_3)^2]^{1/2}(\dot{z}-v_3),$$

where $A = \pi r_o^2$.

(e) Viscous Drag Torque

(1) High Reynolds Numbers

$$Q_{\theta 1}^5 = -3.72\, r_o^4[(\rho_a\mu_a)^{1/2}+(\rho_H\mu_H)^{1/2}](\dot{\theta}_1^2+\dot{\psi}_1^2+\dot{\phi}_1^2)^{1/4}(\dot{\theta}_1+\dot{\phi}_1 s(\dot{\psi}_1)),$$

$$Q_{\psi 1}^5 = -3.72\, r_o^4[(\rho_a\mu_a)^{1/2}+(\rho_H\mu_H)^{1/2}](\dot{\theta}_1^2+\dot{\psi}_1^2+\dot{\phi}_1^2)^{1/4}(\dot{\psi}_1), \ \& \qquad (3\text{-}16)$$

$$Q_{\phi 1}^5 = -3.72\, r_o^4[(\rho_a\mu_a)^{1/2}+(\rho_H\mu_H)^{1/2}](\dot{\theta}_1^2+\dot{\psi}_1^2+\dot{\phi}_1^2)^{1/4}(\dot{\theta}_1 s(\dot{\psi}_1)+\dot{\phi}_1).$$

---

* The torque due to the dynamic buoyancy force was neglected in this study.

**The translational drag forces acting on the subsystems m_i are neglected in this study.

(2) Low Reynolds Numbers

$$Q^5_{\theta 1} = \left[-\frac{8}{3}\pi\mu_a r_o^3 \left(\frac{3+6\beta_a r_o + 6\beta_a^2 r_o^2 + 2\beta_a^3 r_o^3}{1+2\beta_a r_o + 2\beta_a^2 r_o^2} \right) - \frac{8}{3}\pi\mu_H r_o^3 (\beta_H r_o) \right]$$

$$(\dot{\theta}_1 + \dot{\phi}_1 s(\psi_1)) \ ,$$

$$Q^5_{\psi 1} = \left[-\frac{8}{3}\pi\mu_a r_o^3 \left(\frac{3+6\beta_a r_o + 6\beta_a^2 r_o^2 + 2\beta_a^3 r_o^3}{1+2\beta_a r_o + 2\beta_a^2 r_o^2} \right) - \frac{8}{3}\pi\mu_H r_o^3 (\beta_H r_o) \right]$$

$$(\dot{\psi}_1), \text{ and} \qquad (3\text{-}17)$$

$$Q^5_{\phi 1} = \left[-\frac{8}{3}\pi\mu_a r_o^3 \left(\frac{3+6\beta_a r_o + 6\beta_a^2 r_o^2 + 2\beta_a^3 r_o^3}{1+2\beta_a r_o + 2\beta_a^2 r_o^2} \right) - \frac{8}{3}\pi\mu_H r_o^3 (\beta_H r_o) \right]$$

$$(\dot{\theta}_1 s(\psi_1) + \dot{\phi}_1) .$$

(f) <u>Inertia Drag Torque</u>

(1) $\omega >> \Omega$

$$Q^6_{\theta 1} = -6.28[(\rho_a\mu_a)^{\frac{1}{2}} + (\rho_H\mu_H)^{\frac{1}{2}}](\dot{\theta}_1^2 + \dot{\psi}_1^2 + \dot{\phi}_1^2)^{-\frac{1}{4}}$$

$$r_o^4(\ddot{\theta}_1 + \dot{\psi}_1\dot{\phi}_1 c(\psi_1) + \ddot{\phi}_1 s(\psi_1)) \ ,$$

$$Q^6_{\psi 1} = -6.28[(\rho_a\mu_a)^{\frac{1}{2}} + (\rho_H\mu_H)^{\frac{1}{2}}](\theta_1^2 + \psi_1^2 + \phi_1^2)^{-\frac{1}{4}}$$

$$r_o^4(\ddot{\psi}_1 - \dot{\theta}_1\dot{\phi}_1 c(\psi_1)) \ , \text{ and} \qquad (3\text{-}18)$$

$$Q^6_{\phi 1} = -6.28[(\rho_a\mu_a)^{\frac{1}{2}} + (\rho_H\mu_H)^{\frac{1}{2}}](\dot{\theta}_1^2 + \dot{\psi}_1^2 + \dot{\phi}_1^2)^{-\frac{1}{4}}$$

$$r_o^4(\ddot{\theta}_1 s(\psi_1) + \dot{\theta}_1\dot{\psi}_1 c(\psi_1) + \ddot{\phi}_1) \ .$$

(2) $\Omega >> \omega$

$$Q^6_{\theta 1} = \left[-\frac{8}{3}\pi\rho_a r_o^{\,5}\left(\frac{1+\beta_a r_o}{1+2\beta_a r_o+2\beta_a^{\,2} r_o^{\,2}}\right) - \frac{4}{3}\pi\rho_H r_o^{\,5}(\beta_H r_o)^{-1}\right]$$

$$(\ddot{\theta}_1+\dot{\psi}_1\dot{\phi}_1 c(\psi_1)+\ddot{\phi}_1 s(\psi_1)) \ ,$$

$$Q^6_{\psi 1} = \left[-\frac{8}{3}\pi\rho_a r_o^{\,5}\left(\frac{1+\beta_a r_o}{1+2\beta_a r_o+2\beta_a^{\,2} r_o^{\,2}}\right) - \frac{4}{3}\pi\rho_H r_o^{\,5}(\beta_H r_o)^{-1}\right]$$

$$(\ddot{\psi}_1-\dot{\theta}_1\dot{\phi}_1 c(\psi_1)) \ , \text{ and} \qquad (3\text{-}19)$$

$$Q^6_{\phi 1} = \left[-\frac{8}{3}\pi\rho_a r_o^{\,5}\left(\frac{1+\beta_a r_o}{1+2\beta_a r_o+2\beta_a^{\,2} r_o^{\,2}}\right) - \frac{4}{3}\pi\rho_H r_o^{\,5}(\beta_H r_o)^{-1}\right]$$

$$(\ddot{\theta}_1 s(\psi_1)+\dot{\theta}_1\dot{\psi}_1 c(\psi_1)+\ddot{\phi}_1) \ .$$

3.5 Generalized Momenta. The generalized momenta corresponding to each generalized coordinate is given below

$$P_x = \frac{\partial T}{\partial \dot{x}} ,$$

$$P_y = \frac{\partial T}{\partial \dot{y}} ,$$

$$P_z = \frac{\partial T}{\partial \dot{z}} ,$$

$$P_{\theta 1} = \frac{\partial T}{\partial \dot{\theta}_1} ,$$

$$P_{\theta 2} = \frac{\partial T}{\partial \dot{\theta}_2} ,$$

$$P_{\theta 3} = \frac{\partial T}{\partial \dot{\theta}_3} , \qquad (3\text{-}20)$$

$$P_{\psi 1} = \frac{\partial T}{\partial \dot{\psi}_1} ,$$

$$P_{\psi 2} = \frac{\partial T}{\partial \dot{\psi}_2} ,$$

$$P_{\psi 3} = \frac{\partial T}{\partial \dot{\psi}_3} ,$$

$$P_{\phi 1} = \frac{\partial T}{\partial \dot{\phi}_1} , \text{ and}$$

$$P_{\phi 3} = \frac{\partial T}{\partial \dot{\phi}_3} .$$

The expressions for the generalized momenta are obtained by substituting the expression of kinetic energy in equation (3-7) into equation (3-20). The resulting detailed expressions are presented in Appendix F.

3.6 State Variable Equations. The state variable equations are obtained by first inverting equation (3-20) and solving for the generalized velocity $\dot{q}_i$ and employing these equations in conjunction with the following

$$\dot{P}_x = \frac{\partial L}{\partial x} + \sum_{i=1}^{6} Q_x^i ,$$

$$\dot{P}_y = \frac{\partial L}{\partial y} + \sum_{i=1}^{6} Q_y^i ,$$

$$\dot{P}_z = \frac{\partial L}{\partial z} + \sum_{i=1}^{6} Q_z^i ,$$

$$\dot{P}_{\theta 1} = \frac{\partial L}{\partial \theta_1} + \sum_{i=1}^{6} Q_{\theta 1}^i ,$$

$$\dot{P}_{\theta 2} = \frac{\partial L}{\partial \theta_2} + \sum_{i=1}^{6} Q_{\theta 2}^i ,$$

$$\dot{P}_{\theta 3} = \frac{\partial L}{\partial \theta_3} + \sum_{i=1}^{6} Q_{\theta 3}^i , \qquad (3\text{-}21)$$

$$\dot{P}_{\psi 1} = \frac{\partial L}{\partial \psi_1} + \sum_{i=1}^{6} Q_{\psi 1}^i ,$$

$$\dot{P}_{\psi 2} = \frac{\partial L}{\partial \psi_2} + \sum_{i=1}^{6} Q_{\psi 2}^i ,$$

$$\dot{P}_{\psi 3} = \frac{\partial L}{\partial \psi_3} + \sum_{i=1}^{6} Q_{\psi 3}^i ,$$

$$\dot{P}_{\phi 1} = \frac{\partial L}{\partial \phi_1} + \sum_{i=1}^{6} Q_{\phi 1}^i , \text{ and}$$

$$\dot{P}_{\phi 3} = \frac{\partial L}{\partial \phi_3} + \sum_{i=1}^{6} Q^i_{\phi 3} .$$

The detailed expressions for equation (3-21) are presented in Appendix G.

CHAPTER IV

RESULTS AND CONCLUSIONS

4.1 Linearized Mathematical Model. The general form of the mathematical model can be obtained by direct expansion of the state variable equations which are presented in Appendix G (i.e. equation 3-20 and 3-21). It is clear that this expansion would be quite tedious and, one expects the form of the resulting math model to be extremely complex thus requiring a numerical integration method for solution. In this work, the form of the mathematical model was simplified by neglecting all non-linear terms, i.e. by dropping those terms which involve products of the state variables (e.g. $q_i q_j$, $q_i \dot{q}_j$, etc.). This approximation is justified in view of the fact that actual observations indicate that the displacements and velocities of the balloon systems are small, i.e. $|q_i|$, $|\dot{q}_i| << 1$. The result of this is a linear model which admits a closed form solution thus simplifying the simulation process. Two forms of the linearized model were obtained in this study and these are discussed below.

4.2 First Approximation of Mathematical Model. The first approximation of the mathematical model was obtained by neglecting:

(a) viscous drag forces and torques,

(b) inertia drag torques,

(c) the relative spin of the subsystems, i.e. $\dot{\phi}_1=\dot{\phi}_2=\dot{\phi}_3$,

(d) the acceleration of the wind, and

(e) all nonlinear terms.

Under these assumptions equation 3-20 and 3-21 can be written

$$\bar{P}_1 = B_1 \dot{\bar{q}}_1 ,$$
$$\bar{P}_2 = B_2 \dot{\bar{q}}_2 , \qquad (4-1)$$
$$\bar{P}_3 = B_3 \dot{\bar{q}}_3$$

$$\dot{\bar{P}}_1 = C_1 \bar{q}_1 + D_1 \ddot{\bar{q}}_1 ,$$
$$\dot{\bar{P}}_2 = C_2 \bar{q}_2 + D_2 \ddot{\bar{q}}_2 , \text{ and} \qquad (4-2)$$
$$\dot{\bar{P}}_3 = C_3 \bar{q}_3 + D_3 \ddot{\bar{q}}_3 ,$$

where

$$\bar{P}_1 = \begin{Bmatrix} P_x \\ P_{\psi 1} \\ P_{\psi 2} \\ P_{\psi 3} \end{Bmatrix} , \quad \bar{P}_2 = \begin{Bmatrix} P_y \\ P_{\theta 1} \\ P_{\theta 2} \\ P_{\theta 3} \end{Bmatrix} , \quad \bar{P}_3 = \begin{Bmatrix} P_z \\ P_{\phi 1} \end{Bmatrix} ,$$

$$\bar{q}_1 = \begin{Bmatrix} x \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} , \quad \bar{q}_2 = \begin{Bmatrix} y \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} , \quad \bar{q}_3 = \begin{Bmatrix} z \\ \phi_1 \end{Bmatrix} .$$

and the elements of the B_i, C_i and D_i matrices (i=1,2,3) are defined in Appendix H

The state variable form of the model is obtained by first inverting equation (4-1), i.e.

$$\dot{\bar{q}}_1 = \beta_1 \bar{P}_1 ,$$
$$\dot{\bar{q}}_2 = \beta_2 \bar{P}_2 , \qquad (4-3)$$
$$\dot{\bar{q}}_3 = \beta_3 \bar{P}_3 ,$$

where $\beta_i = B_i^{-1}$ (i=1,2,3)

and the β_i matrices are defined in Appendix I. Differentiation of equation (4-3) and substitution into equation (4-2) yields

$$\dot{\overline{P}}_1 = A_1 \overline{q}_1 ,$$
$$\dot{\overline{P}}_2 = A_2 \overline{q}_2 , \qquad (4-4)$$
$$\dot{\overline{P}}_3 = A_3 \overline{q}_3 ,$$

where $A_i = (I - D_i\beta_i)^{-1}C_i$. (i=1,2,3) (4-5)

The elements of the A_i matrices are presented in Appendix J . Equations (4-3) and (4-4) comprise the state variable form of the math model. These equations can be written in second order form by differentiating equation (4-3) with respect to time and employing equation (4-4) to yield

$$\ddot{\overline{q}}_1 = \alpha_1 \overline{q}_1 ,$$
$$\ddot{\overline{q}}_2 = \alpha_2 \overline{q}_2 , \text{ and} \qquad (4-6)$$
$$\ddot{\overline{q}}_3 = \alpha_3 \overline{q}_3 ,$$

where $\alpha_i = \beta_i A_i$. (i=1,2,3) (4-7)

The elements of the matrices α_i are defined in Appendix K.

4.3 Second Approximation of Mathematical Model. The second approximation of the mathematical model was obtained by retaining the dynamic buoyancy, viscous drag and inertia drag reactions along with the effects of wind velocity and acceleration. However, as in the case of the first approximation, all non-linear terms were dropped and the relative spin of the subsystems was neglected under these assumptions, the state variable equations are given as

$$\dot{\overline{P}}_1 = \gamma_1\overline{q}_1\delta + {}_1\overline{P}_1 + \overline{F}_1 \ ,$$

$$\dot{\overline{P}}_2 = \gamma_2\overline{q}_2\delta + {}_2\overline{P}_2 + \overline{F}_2 \ , \text{ and} \tag{4-8}$$

$$\dot{\overline{P}}_3 = \gamma_3\overline{q}_3\delta + {}_3\overline{P}_3 + \overline{F}_3 \ .$$

$$\dot{\overline{q}}_1 = \beta_1\overline{P}_1 \ ,$$

$$\dot{\overline{q}}_2 = \beta_2\overline{P}_2 \ , \text{ and} \tag{4-9}$$

$$\dot{\overline{q}}_3 = \beta_3\overline{P}_3 \ .$$

where $\overline{P}_i$, $\overline{q}_i$ and β_i are defined as in equation (4-1), γ_i and δ_i are matrices defined in Appendix L and $\overline{F}_i$ is defined in Appendix M.

4.4 Recommendation for Future Study. Before employing the mathematical models (refer sections 4.2 and 4.3) in the attitude determination process, the models should be evaluated in order to determine the contribution of various terms on the system response and attitude. For example, the importance of retaining the inertia drag torque in the system model can be determined by: (a) solving the mathematical model given in equations (4-3) and (4-4), (b) modifying the form of this model to include the effect of $\overline{N}_I$ (see equations (2-23 - 2-25)), and (c) resolving the modified model. By comparing the results, e.g. natural frequencies, angular velocity, etc., one can determine whether this term is important. In the same manner one can determine whether the viscous drag reactions, wind velocity and wind acceleration are important. Finally the effect of the non-linear terms can be investigated by developing a third approximation of the mathematical model which retains first and second order terms and employing the same process. This work will be conducted in the near future.

Appendix A

Development of Equivalent Spring Constant

The net lift force acting on the balloon is

$$F_L = \rho_H V_H g \left(\frac{\rho_a}{\rho_H} - 1 \right) . \tag{A-1}$$

If (a) the mission takes place in the isothermal region of the atmosphere, and

(b) the helium undergoes an adiabatic process;

then

$$\rho_a = \rho_{oa} \left(e^{-\frac{\rho_{oa} g}{P_o} Z} \right) , \text{ and} \tag{A-2}$$

$$\rho_H = \rho_{oH} \left(e^{-\frac{\rho_{oa} g}{P_o} Z} \right)^{\frac{1}{n_H}} \tag{A-3}$$

Substitution of equation (A-2) and (A-3) into equation (A-1), gives

$$F_L = \rho_H V_H g \left[\frac{\rho_{oa}}{\rho_{oH}} \; e^{-\frac{\rho_{oa} g Z}{P_o} \left(1 - \frac{1}{n_H} \right)} - 1 \right] . \tag{A-4}$$

The equivalent spring constant is obtained by differentiating equation (A-4) with respect to Z, then setting Z=0 and considering $\rho_H V_H g$ to be a constant, i.e.

$$K_o = - \left. \frac{dF_L}{dZ} \right)_{Z=0} = \frac{\rho_H V_H g^2 \rho_{oa}^2}{\rho_{oH} P_o} \left(1 - \frac{1}{n_H} \right) . \tag{A-5}$$

Appendix B

Development of Viscous Drag Torques due to Rotation of a Spherical Balloon

(a) Velocity Distribution of Fluid Across the Boundary Layer Thickness

assume (1) parabolic velocity distribution(see figure)

(2) a,b,c are determined by the boundary conditions,

$$V_\phi = r_o \omega \sin\theta \qquad \text{at } y=0$$

$$V_\phi = 0 \qquad \text{at } y=\delta$$

$$\frac{dV_\phi}{dy} = 0 \qquad \text{at } y=\delta$$

$$V_\phi = r_o \omega \sin\theta - \frac{2 r_o \omega \sin\theta}{\delta} y + \frac{r_o \omega \sin\theta}{\delta^2} y^2 . \qquad (B-1)$$

(b) Shear Stress on the Surface of the Balloon

$$\tau_{r\phi}\Big)_{r=r_o} = -\mu \left(\frac{1}{r \sin\theta} \frac{\partial V_r}{\partial \phi} + r \frac{\partial}{\partial r}\left(\frac{V_\phi}{r}\right)\right)_{r=r_o} \quad , \quad \delta << r_o$$

$$= -\mu \left(r \frac{\partial}{\partial r}\left(\frac{V_\phi}{r}\right)\right)_{r=r_o} \cong -\mu \left(\frac{\partial V_\phi}{\partial r}\right)_{r=r_o} = -\mu \left.\frac{\partial V_\phi}{\partial y}\right)_{y=0} \qquad (B-2)$$

Substitution equation (B-1) into (B-2) yields

$$\tau_{r\phi}\Big)_{r=r_o} = -\mu b = \frac{2 \mu r_o \omega \sin\theta}{\delta} \qquad (B-3)$$

(c) Viscous Drag Torque

$$N_D = \int_0^\pi (\tau_{r\phi})_{r=r_o} (2\pi r_o \sin\theta)(r_o \sin\theta)(r_o d\theta)$$

Substitution for $(\tau_{r\phi})_{r=r_o}$, we obtain

Appendix C

Development of Inertia Drag Torque due to Unsteady Rotation of A Spherical Balloon in A Viscous Fluid Media

Average momentum of fluid per unit mass (see figure 9)

$$M_\phi = \frac{1}{\delta} \int_0^\delta V_\phi \, dy = \frac{1}{3} r_o \omega \sin\theta$$

Angular momentum (H_ϕ) of the fluid

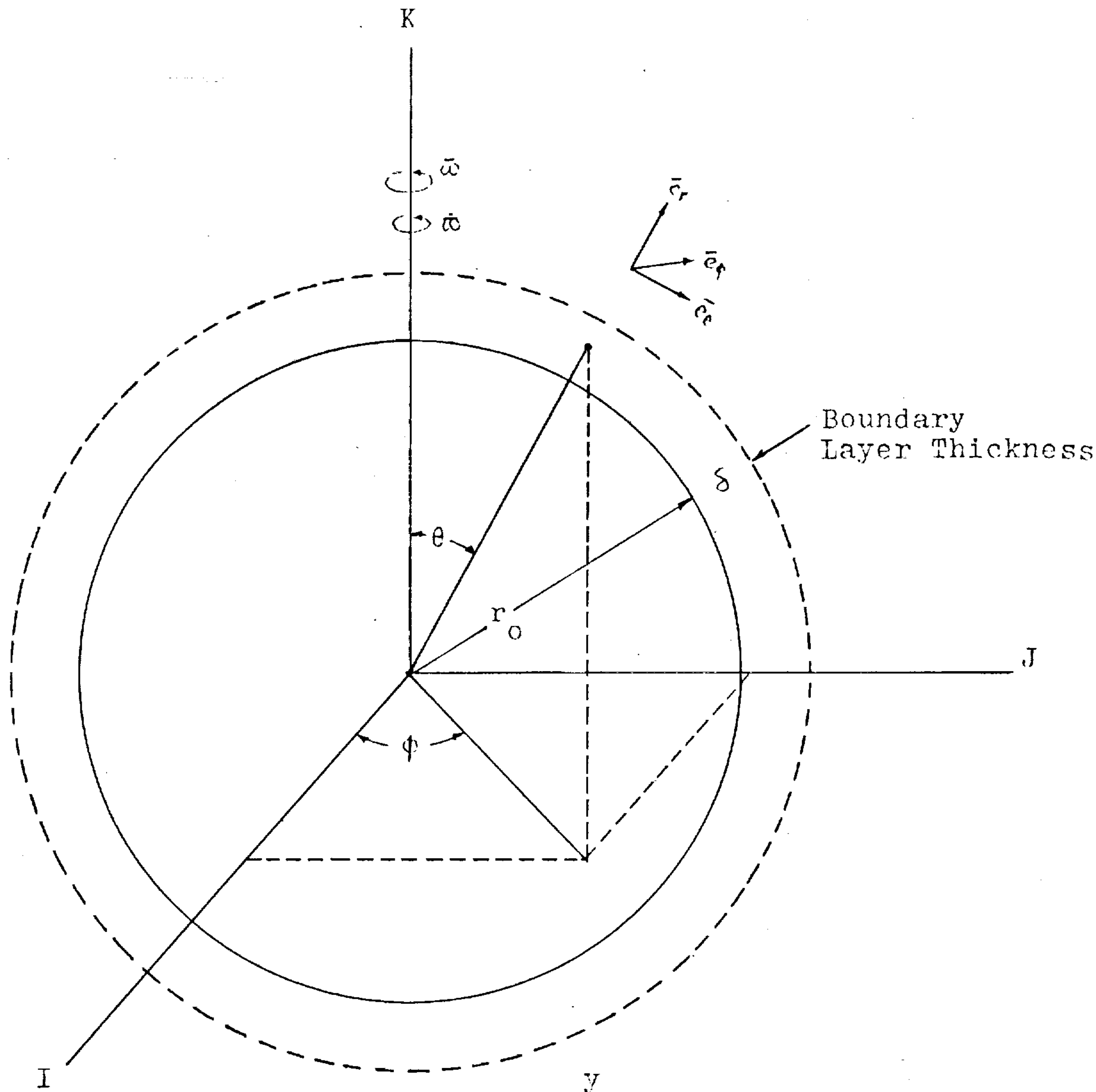
$$H_\phi = \int_c^\pi (\text{average momentum per unit mass}) \ (\text{arm}) \ dm$$

$$= \int_0^\pi \frac{1}{3} r_c \omega \sin\theta \, (r_o \sin\theta) \, \rho \, (2\pi r_o \sin\theta) \, r_o d\theta \, (4.5 \ \mu^{1/2} / (\rho\omega)^{1/2}$$

$$= 12.56 \ (\rho\mu)^{1/2} \omega^{1/2} r_o^4$$

Inertia Drag Torque (N_I) is given as

$$N_I = \frac{dH_\phi}{dt} = 6.28 \ (\rho\mu)^{1/2} r_c^4 \ \omega^{-1/2} \dot{\omega}$$



$$\delta = 4.5\sqrt{\frac{\mu}{\rho\omega}}$$

$$\tau_{r\phi} = \frac{2\mu r_0 \omega \sin\theta}{\delta} \quad at\ r = r_0$$

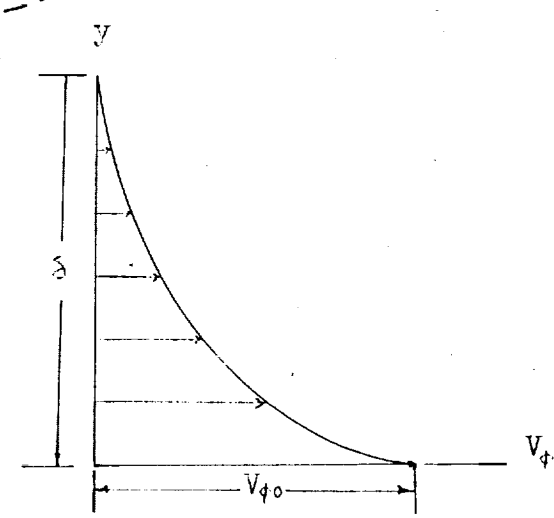


Fig. 9 Fluid Effect on Rotating Balloon

Appendix D

Velocity Expressions for Subsystems

Mass m_1

$$\bar{V}_1 = (\dot{x} - r_1 c(\psi_1)\dot{\psi}_1)\,\bar{i} + (\dot{y} + r_1 c(\psi_1)c(\theta_1)\dot{\theta}_1 - r_1 s(\psi_1)s(\theta_1)\dot{\psi}_1)\,\bar{j} + (\dot{z} + r_1 c(\psi_1)s(\theta_1)\dot{\theta}_1 + r_1 s(\psi_1)c(\theta_1)\dot{\psi}_1)\,\bar{k} \qquad (D\text{-}1)$$

Mass m_2

$$\bar{V}_2 = (\dot{x} - \sum_{i=1}^{2} r_i c(\psi_i)\dot{\psi}_i)\,\bar{i} + (\dot{y} + \sum_{i=1}^{2} r_i c(\psi_i)c(\theta_i)\dot{\theta}_i - \sum_{i=1}^{2} r_i s(\psi_i)s(\theta_i)\dot{\psi}_i)\,\bar{j} + (\dot{z} + \sum_{i=1}^{2} r_i c(\psi_i)s(\theta_i)\dot{\theta}_i + \sum_{i=1}^{2} r_i s(\psi_i)c(\theta_i)\dot{\psi}_i)\,\bar{k} \qquad (D\text{-}2)$$

Mass m_3

$$\bar{V}_3 = (\dot{x} - \sum_{i=1}^{3} r_i c(\psi_i)\dot{\psi}_i)\,\bar{i} + (\dot{y} + \sum_{i=1}^{3} r_i c(\psi_i)c(\theta_i)\dot{\theta}_i - \sum_{i=1}^{3} r_i s(\psi_i)s(\theta_i)\dot{\psi}_i)\,\bar{j} + (\dot{z} + \sum_{i=1}^{3} r_i c(\psi_i)s(\theta_i)\dot{\theta}_i + \sum_{i=1}^{3} r_i s(\psi_i)c(\theta_i)\dot{\psi}_i)\,\bar{k} \qquad (D\text{-}3)$$

Appendix E

Expressions for Kinetic Energy of the Balloon and Subsystems

(a) For Balloon m_0

$$T^0 = \frac{m_0}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{I_{01}}{2}(\dot{\theta}_1^2 c^2(\psi_1) + \dot{\psi}_1^2) + \frac{I_{03}}{2}(\dot{\theta}_1^2 s^2(\psi_1) + \dot{\phi}_1^2 + 2s(\psi_1)\dot{\theta}_1\dot{\phi}_1) \qquad (E-1)$$

(b) For m_1

$$T^1 = \frac{m_1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + r_1^2\dot{\psi}_1^2 + r_1^2 c^2(\psi_1)\dot{\theta}_1^2 - 2r_1 c(\psi_1)\dot{x}\dot{\psi}_1 + 2r_1 c(\psi_1)c(\theta_1)\dot{y}\dot{\theta}_1 - 2r_1 s(\psi_1)s(\theta_1)\dot{y}\dot{\psi}_1 + 2r_1 c(\psi_1)s(\theta_1)\dot{z}\dot{\theta}_1 + 2r_1 s(\psi_1)c(\theta_1)\dot{z}\dot{\psi}_1) + \frac{I_{13}}{2}(\dot{\theta}_1^2 s^2(\psi_1) + \dot{\phi}_1^2 + 2s(\psi_1)\dot{\theta}_1\dot{\phi}_1) \qquad (E-2)$$

(c) For m_2

$$T^2 = \frac{m_2}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + \sum_{i=1}^{2} r_i^2\dot{\psi}_i^2 + 2r_1 c(\psi_1) r_2 s(\psi_2) s(\theta_1-\theta_2)\dot{\theta}_1\dot{\psi}_2 + 2r_1 s(\psi_1) r_2 s(\psi_2) c(\theta_1-\theta_2)\dot{\psi}_1\dot{\psi}_2 + 2r_1 c(\psi_1) r_2 c(\psi_2) c(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 - 2r_1 s(\psi_1) r_2 c(\psi_2) s(\theta_1-\theta_2)\dot{\psi}_1\dot{\theta}_2 + \sum_{i=1}^{2} r_i^2 c^2(\psi_i)\dot{\theta}_i^2 - 2\dot{x}\sum_{i=1}^{2}\dot{\psi}_i c(\psi_i) r_i + 2r_1 c(\psi_1) r_2 c(\psi_2)\dot{\psi}_1\dot{\psi}_2 + 2r_1 c(\psi_1)c(\theta_1)\dot{y}\dot{\theta}_1 - 2r_1 s(\psi_1)s(\theta_1)\dot{y}\dot{\psi}_1 + 2r_2 c(\psi_2)c(\theta_2)\dot{y}\dot{\theta}_2 - 2r_2 s(\psi_2)s(\theta_2)\dot{y}\dot{\psi}_2 + 2r_1 c(\psi_1)s(\theta_1)\dot{z}\dot{\theta}_1 + 2r_1 s(\psi_1)c(\theta_1)\dot{z}\dot{\psi}_1 + 2r_2 c(\psi_2)s(\theta_2)\dot{z}\dot{\theta}_2 + 2r_2 s(\psi_2)c(\theta_2)\dot{z}\dot{\psi}_2 + \frac{I_{23}}{2}(\dot{\theta}_2^2 s^2(\psi_2) + \dot{\phi}_1^2 + 2\dot{\phi}_1\dot{\theta}_2 s(\psi_2))) \qquad (E-3)$$

(d) For m_3

$$
\begin{aligned}
T^3 = \frac{m_3}{2}\Big(& \dot{x}^2+\dot{y}^2+\dot{z}^2+\sum_{i=1}^{3} r_i^2\dot{\psi}_i^2 + 2r_1 c(\psi_1) r_2 s(\psi_2) s(\theta_1-\theta_2)\,\dot{\theta}_1\dot{\psi}_2 \\
& + 2r_1 s(\psi_1) r_2 s(\psi_2) c(\theta_1-\theta_2)\,\dot{\psi}_1\dot{\psi}_2 + 2r_1 c(\psi_1) r_2 c(\psi_2) c(\theta_1-\theta_2)\,\dot{\theta}_1\dot{\theta}_2 \\
& - 2r_1 s(\psi_1) r_2 c(\psi_2) s(\theta_1-\theta_2)\,\dot{\psi}_1\dot{\theta}_2 + \sum_{i=1}^{3} r_i^2 c^2(\psi_i)\dot{\theta}_i^2 - 2\dot{x}\sum_{i=1}^{3} r_i c(\psi_i)\dot{\psi}_i \\
& + 2r_1 c(\psi_1) r_2 c(\psi_2)\,\dot{\psi}_1\dot{\psi}_2 + 2\dot{y}\sum_{i=1}^{3} r_i\dot{\theta}_i c(\psi_i) c(\theta_i) - 2\dot{y}\sum_{i=1}^{3} r_i\dot{\psi}_i s(\psi_i) s(\theta_i) \\
& + 2\dot{z}\sum_{i=1}^{3} r_i\dot{\theta}_i c(\psi_i) s(\theta_i) + 2\dot{z}\sum_{i=1}^{3} r_i\dot{\psi}_i s(\psi_i) c(\theta_i) \\
& + 2r_3\dot{\psi}_3 s(\psi_3)\sum_{i=1}^{2} r_i\dot{\theta}_i c(\psi_i) s(\theta_i-\theta_3) + 2r_3\dot{\psi}_3 s(\psi_3)\sum_{i=1}^{2} r_i\dot{\psi}_i s(\psi_i) c(\theta_i-\theta_3) \\
& + 2r_3\dot{\theta}_3 c(\psi_3)\sum_{i=1}^{2} r_i\dot{\theta}_i c(\psi_i) c(\theta_i-\theta_3) - 2r_3\dot{\theta}_3 c(\psi_3)\sum_{i=1}^{2} r_i\dot{\psi}_i s(\psi_i) s(\theta_i-\theta_3) \\
& + 2r_3\dot{\psi}_3 c(\psi_3)\sum_{i=1}^{2} r_i\dot{\psi}_i c(\psi_i) + \frac{I_{33}}{2}\Big(\dot{\theta}_3^2 s^2(\psi_3) + \dot{\phi}_3^2 \\
& + 2\dot{\theta}_3 s(\psi_3)\dot{\phi}_3\Big)
\end{aligned}
\qquad (E-4)
$$

Appendix F

Development of Expressions for Generalized Momenta

$$P_x = \sum_{i=0}^{3} m_i \dot{x} - \sum_{i=1}^{3} m_i r_1 c(\psi_1)\dot{\psi}_1 - \sum_{i=2}^{3} m_i r_2 c(\psi_2)\dot{\psi}_2 - m_3 r_3 c(\psi_3)\dot{\psi}_3 \quad (F-1)$$

$$P_y = \sum_{i=0}^{3} m_i \dot{y} + \sum_{i=1}^{3} m_i r_1 c(\psi_1) c(\theta_1)\dot{\theta}_1 + \sum_{i=2}^{3} m_i r_2 c(\psi_2) c(\theta_2)\dot{\theta}_2 \quad (F-2)$$
$$+ m_3 r_3 c(\psi_3) c(\theta_3)\dot{\theta}_3 - \sum_{i=1}^{3} m_i r_1 s(\psi_1) s(\theta_1)\dot{\psi}_1 - \sum_{i=2}^{3} m_i r_2 s(\psi_2) s(\theta_2)\dot{\psi}_2$$
$$- m_3 r_3 s(\psi_3) s(\theta_3)\dot{\psi}_3$$

$$P_z = \sum_{i=0}^{3} m_i \dot{z} + \sum_{i=1}^{3} m_i r_1 c(\psi_1) s(\theta_1)\dot{\theta}_1 + \sum_{i=2}^{3} m_i r_2 c(\psi_2) s(\theta_2)\dot{\theta}_2 \quad (F-3)$$
$$+ m_3 r_3 c(\psi_3) s(\theta_3)\dot{\theta}_3 + \sum_{i=1}^{3} m_i r_1 s(\psi_1) c(\theta_1)\dot{\psi}_1 + \sum_{i=2}^{3} m_i r_2 s(\psi_2) c(\theta_2)\dot{\psi}_2$$
$$+ m_3 r_3 s(\psi_3) c(\theta_3)\dot{\psi}_3$$

$$P_{\theta_1} = \left(I_{c1} c^2(\psi_1) + I_{13} s^2(\psi_1) + I_{c3} s^2(\psi_1) + \sum_{i=1}^{3} m_i r_1^2 c^2(\psi_1) \right)\dot{\theta}_1 \quad (F-4)$$
$$+ \sum_{i=2}^{3} m_i r_1 c(\psi_1) r_2 c(\psi_2) c(\theta_1-\theta_2)\dot{\theta}_2 + m_3 r_3 c(\psi_3) r_1 c(\psi_1) c(\theta_1-\theta_3)\dot{\theta}_3$$
$$+ \left(I_{c3} s(\psi_1) + I_{13} s(\psi_1) \right)\dot{\phi}_1 + \sum_{i=1}^{3} m_i r_1 c(\psi_1) c(\theta_1)\dot{y} + \sum_{i=1}^{3} m_i r_1 c(\psi_1) s(\theta_1)\dot{z}$$
$$+ \sum_{i=2}^{3} m_i r_1 c(\psi_1) r_2 s(\psi_2) s(\theta_1-\theta_2)\dot{\psi}_2 + m_3 r_3 s(\psi_3) r_1 c(\psi_1) s(\theta_1-\theta_3)\dot{\psi}_3$$

$$P_{\theta_2} = \sum_{i=2}^{3} m_i r_2 c(\psi_2) c(\theta_2)\dot{y} + \sum_{i=2}^{3} m_i r_2 c(\psi_2) s(\theta_2)\dot{z} \quad (F-5)$$
$$+ \left(\sum_{i=2}^{3} m_i r_1 c(\psi_1) r_2 c(\psi_2) c(\theta_1-\theta_2)\right)\dot{\theta}_1 + \left(I_{23} s^2(\psi_2) + \sum_{i=2}^{3} m_i r_2^2 c^2(\psi_2) \right)\dot{\theta}_2$$
$$+ m_3 r_3 c(\psi_3) r_2 c(\psi_2) c(\theta_2-\theta_3)\dot{\theta}_3 - \sum_{i=2}^{3} m_i r_1 s(\psi_1) r_2 c(\psi_2) s(\theta_1-\theta_2)\dot{\psi}_1$$
$$+ m_3 r_3 s(\psi_3) r_2 c(\psi_2) s(\theta_2-\theta_3)\dot{\psi}_3 + I_{23}\dot{\phi}_1 s(\psi_2)$$

$$P_{\theta_3} = m_3 r_3 c(\psi_3) c(\theta_3)\dot{y} + m_3 r_3 c(\psi_3) s(\theta_3)\dot{z} + \sum_{i=1}^{2} m_3 r_3 c(\psi_3) r_i c(\psi_i) c(\theta_i-\theta_3)\dot{\theta}_i \quad (F-6)$$
$$+ \left(I_{33} s^2(\psi_3) + m_3 r_3^2 c^2(\psi_3) \right)\dot{\theta}_3 + I_{33} s(\psi_3)\dot{\phi}_3$$
$$- \sum_{i=1}^{2} m_3 r_3 c(\psi_3) r_i s(\psi_i) s(\theta_i-\theta_3)\dot{\psi}_i$$

$$P_{\psi_1} = -\sum_{i=1}^{3} m_i r_1 C(\psi_1)\,\dot{x} - \sum_{i=1}^{3} m_i r_1 S(\psi_1) S(\theta_1)\,\dot{y} + \sum_{i=1}^{3} m_i r_1 S(\psi_1) C(\theta_1)\,\dot{z} \qquad (F-7)$$

$$+ \left(I_1 + \sum_{i=1}^{3} m_i r_1^2\right)\dot{\psi}_1 + \left(\sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 C(\psi_2) + \sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 S(\psi_2) C(\theta_1-\theta_2)\right)\dot{\psi}_2$$

$$+ \left(m_3 r_3 S(\psi_3) r_1 S(\psi_1) C(\theta_1-\theta_3) + m_3 r_3 C(\psi_3) r_1 C(\psi_1)\right)\dot{\psi}_3$$

$$- \sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 C(\psi_2) S(\theta_1-\theta_2)\,\dot{\theta}_2 - m_3 r_3 C(\psi_3) r_1 S(\psi_1) S(\theta_1-\theta_3)\,\dot{\theta}_3$$

$$P_{\psi_2} = -\sum_{i=2}^{3} m_i r_2 C(\psi_2)\,\dot{x} - \sum_{i=2}^{3} m_i r_2 S(\psi_2) S(\theta_2)\,\dot{y} + \sum_{i=2}^{3} m_i r_2 S(\psi_2) C(\theta_2)\,\dot{z} \qquad (F-8)$$

$$+ \sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 S(\psi_2) S(\theta_1-\theta_2)\,\dot{\theta}_1 - m_3 r_3 C(\psi_3) r_2 S(\psi_2) S(\theta_2-\theta_3)\,\dot{\theta}_3$$

$$+ \left(\sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 S(\psi_2) C(\theta_1-\theta_2) + \sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 C(\psi_2)\right)\dot{\psi}_1$$

$$+ \sum_{i=2}^{3} m_i r_2^2\,\dot{\psi}_2 + \left(m_3 r_3 S(\psi_3) r_2 S(\psi_2) C(\theta_2-\theta_3) + m_3 r_3 C(\psi_3) r_2 C(\psi_2)\right)\dot{\psi}_3$$

$$P_{\psi_3} = -m_3 r_3 C(\psi_3)\,\dot{x} - m_3 r_3 S(\psi_3) S(\theta_3)\,\dot{y} + m_3 r_3 S(\psi_3) C(\theta_3)\,\dot{z} \qquad (F-9)$$

$$+ m_3 r_3 S(\psi_3) r_1 C(\psi_1) S(\theta_1-\theta_3)\,\dot{\theta}_1 + m_3 r_3 S(\psi_3) r_2 C(\psi_2) S(\theta_2-\theta_3)\,\dot{\theta}_2$$

$$+ \left(m_3 r_3 S(\psi_3) r_1 S(\psi_1) C(\theta_1-\theta_3) + m_3 r_3 C(\psi_3) r_1 C(\psi_1)\right)\dot{\psi}_1$$

$$+ \left(m_3 r_3 S(\psi_3) r_2 S(\psi_2) C(\theta_2-\theta_3) + m_3 r_3 C(\psi_3) r_2 C(\psi_2)\right)\dot{\psi}_2$$

$$+ m_3 r_3^2\,\dot{\psi}_3$$

$$P_{\phi_1} = \left(I_{c3} S(\psi_1) + I_{13} S(\psi_1)\right)\dot{\theta}_1 + \left(I_{c3} + I_{13} + I_{23}\right)\dot{\phi}_1 \qquad (F-10)$$

$$+ I_{23} S(\psi_2)\,\dot{\theta}_2$$

$$P_{\phi_3} = I_{33} S(\psi_3)\,\dot{\theta}_3 + I_{33}\,\dot{\phi}_3 \qquad (F-11)$$

Appendix G

Development of Expressions for State Variable Equations

$$\dot{P}_x = \rho_a V_H \dot{v}_1 - \tfrac{1}{2}\rho_a V_H(\ddot{x} - \dot{v}_1) - \tfrac{1}{2} C_D A \rho_a\left[(\dot{x}-v_1)^2 + (\dot{y}-v_2)^2 + (\dot{z}-v_3)^2\right]^{\frac{1}{2}}(\dot{x}-v_1) - \rho_H V_H \ddot{x} \qquad (F\text{-}12)$$

$$\dot{P}_y = \rho_a V_H \dot{v}_2 - \tfrac{1}{2}\rho_a V_H(\ddot{y} - \dot{v}_2) - \tfrac{1}{2} C_D A \rho_a\left[(\dot{x}-v_1)^2 + (\dot{y}-v_2)^2 + (\dot{z}-v_3)^2\right]^{\frac{1}{2}}(\dot{y}-v_2) - \rho_H V_H \ddot{y} \qquad (F\text{-}13)$$

$$\dot{P}_z = \rho_a V_H \dot{v}_3 - K_z \rho_a V_H(\ddot{z} - \dot{v}_3) - \tfrac{1}{2} C_D A \rho_a\left[(\dot{x}-v_1)^2 + (\dot{y}-v_2)^2 + (\dot{z}-v_3)^2\right]^{\frac{1}{2}}(\dot{z}-v_3) - K_c z - \rho_H V_H \ddot{z} \qquad (F\text{-}14)$$

$$\begin{aligned}
\dot{P}_{\theta_1} = & -\sum_{i=1}^{3} m_i r_1 c(\psi_1) s(\theta_1)\dot{y}\dot{\theta}_1 - \sum_{i=1}^{3} m_i r_1 s(\psi_1) c(\theta_1)\dot{y}\dot{\psi}_1 + \sum_{i=1}^{3} m_i r_1 c(\psi_1) c(\theta_1)\dot{z}\dot{\theta}_1 \qquad (F\text{-}15)\\
& - \sum_{i=1}^{3} m_i r_1 s(\psi_1) s(\theta_1)\dot{z}\dot{\psi}_1 + \sum_{i=2}^{3} m_i r_1 c(\psi_1) r_2 s(\psi_2) c(\theta_1-\theta_2)\dot{\theta}_1\dot{\psi}_2 \\
& - \sum_{i=2}^{3} m_i r_1 s(\psi_1) r_2 s(\psi_2) s(\theta_1-\theta_2)\dot{\psi}_1\dot{\psi}_2 - \sum_{i=2}^{3} m_i r_1 c(\psi_1) r_2 c(\psi_2) s(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 \\
& - \sum_{i=2}^{3} m_i r_1 s(\psi_1) r_2 c(\psi_2) c(\theta_1-\theta_2)\dot{\psi}_1\dot{\theta}_2 + m_3 r_3 s(\psi_3) r_1 c(\psi_1) c(\theta_1-\theta_3)\dot{\theta}_1\dot{\psi}_3 \\
& - m_3 r_3 s(\psi_3) r_1 s(\psi_1) s(\theta_1-\theta_3)\dot{\psi}_1\dot{\psi}_3 - m_3 r_3 c(\psi_3) r_1 c(\psi_1) s(\theta_1-\theta_3)\dot{\theta}_1\dot{\theta}_3 \\
& - m_3 r_3 c(\psi_3) r_1 s(\psi_1) c(\theta_1-\theta_3)\dot{\psi}_1\dot{\theta}_3 - \sum_{i=1}^{3} m_i g r_1 c(\psi_1) s(\theta_1) + W d\, s(\theta_1) \\
& - K_1 r_0^4 (\dot{\theta}_1^2 + \dot{\psi}_1^2 + \dot{\phi}_1^2)^{\frac{1}{4}}\left[(\rho_a \mu_a)^{\frac{1}{2}} + (\rho_H \mu_H)^{\frac{1}{2}}\right](\dot{\theta}_1 + \dot{\phi}_1 s(\psi_1)) \\
& - K_2 \left[(\rho_a \mu_a)^{\frac{1}{2}} + (\rho_H \mu_H)^{\frac{1}{2}}\right] r_0^4 (\dot{\theta}_1^2 + \dot{\psi}_1^2 + \dot{\phi}_1^2)^{-\frac{1}{4}}(\ddot{\theta}_1 + \dot{\psi}_1\dot{\phi}_1 c(\psi_1) + \ddot{\phi}_1 s(\psi_1)) \\
& - \left[\frac{8}{3}\pi \mu_a r_0^3 \left(\frac{3 + 6\beta_a r_0 + 6\beta_a^2 r_0^2 + 2\beta_a^3 r_0^3}{1 + 2\beta_a r_0 + 2\beta_a^2 r_0^2}\right) + \frac{8}{3}\pi \mu_H r_0^3 (\beta_H r_0)\right](\dot{\theta}_1 + \dot{\phi}_1 s(\psi_1)) \\
& + \left[-\frac{8}{3}\pi \rho_a r_0^5 \left(\frac{1 + \beta_a r_0}{1 + 2\beta_a r_0 + 2\beta_a^2 r_0^2}\right) \frac{4}{3}\pi \rho_H r_0^5 (\beta_H r_0)^{-1}\right] \\
& (\ddot{\theta}_1 + \dot{\psi}_1\dot{\phi}_1 c(\psi_1) + \ddot{\phi}_1 s(\psi_1))
\end{aligned}$$

$$\dot{P}_{\theta_2} = -\sum_{i=2}^{3} m_i r_2 C(\psi_2) S(\theta_2) \dot{y}\, \dot{\theta}_2 - \sum_{i=2}^{3} m_i r_2 S(\psi_2) C(\theta_2) \dot{y}\, \dot{\psi}_2 - \sum_{i=2}^{3} m_i r_2 S(\psi_2) S(\theta_2) \dot{z}\, \dot{\psi}_2$$
$$+ \sum_{i=2}^{3} m_i r_2 C(\psi_2) C(\theta_2) \dot{z}\, \dot{\theta}_2 - \sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 S(\psi_2) C(\theta_1-\theta_2) \dot{\theta}_1 \dot{\psi}_2$$
$$+ \sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 S(\psi_2) S(\theta_1-\theta_2) \dot{\psi}_1 \dot{\psi}_2 + \sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 C(\psi_2) S(\theta_1-\theta_2) \dot{\theta}_1 \dot{\theta}_2$$
$$+ \sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 C(\psi_2) C(\theta_1-\theta_2) \dot{\psi}_1 \dot{\theta}_2 - \sum_{i=2}^{3} m_i g\, r_2 C(\psi_2) S(\theta_2)$$
$$+ m_3 r_3 S(\psi_3) r_2 C(\psi_2) C(\theta_2-\theta_3) \dot{\theta}_2 \dot{\psi}_3 - m_3 r_3 S(\psi_3) r_2 S(\psi_2) S(\theta_2-\theta_3) \dot{\psi}_2 \dot{\psi}_3$$
$$- m_3 r_3 C(\psi_3) r_2 C(\psi_2) S(\theta_2-\theta_3) \dot{\theta}_2 \dot{\theta}_3 - m_3 r_3 C(\psi_3) r_2 S(\psi_2) C(\theta_2-\theta_3) \dot{\psi}_2 \dot{\theta}_3$$

$$\dot{P}_{\theta_3} = -m_3 r_3 S(\psi_3) r_1 C(\psi_1) C(\theta_1-\theta_3) \dot{\theta}_1 \dot{\psi}_3 + m_3 r_3 S(\psi_3) r_1 S(\psi_1) S(\theta_1-\theta_3) \dot{\psi}_1 \dot{\psi}_3$$
$$- m_3 r_3 S(\psi_3) r_2 C(\psi_2) C(\theta_2-\theta_3) \dot{\theta}_2 \dot{\psi}_3 + m_3 r_3 S(\psi_3) r_2 S(\psi_2) S(\theta_2-\theta_3) \dot{\psi}_2 \dot{\psi}_3$$
$$+ m_3 r_3 C(\psi_3) r_1 C(\psi_1) S(\theta_1-\theta_3) \dot{\theta}_1 \dot{\theta}_3 + m_3 r_3 C(\psi_3) r_1 S(\psi_1) C(\theta_1-\theta_3) \dot{\psi}_1 \dot{\theta}_3$$
$$+ m_3 r_3 C(\psi_3) r_2 C(\psi_2) S(\theta_2-\theta_3) \dot{\theta}_2 \dot{\theta}_3 + m_3 r_3 C(\psi_3) r_2 S(\psi_2) C(\theta_2-\theta_3) \dot{\psi}_2 \dot{\theta}_3$$
$$- m_3 r_3 C(\psi_3) S(\theta_3) \dot{y}\, \dot{\theta}_3 - m_3 r_3 S(\psi_3) C(\theta_3) \dot{y}\, \dot{\psi}_3 + m_3 r_3 C(\psi_3) C(\theta_3) \dot{z}\, \dot{\theta}_3$$
$$- m_3 r_3 S(\psi_3) S(\theta_3) \dot{z}\, \dot{\psi}_3 - m_3 g\, r_3 C(\psi_3) S(\theta_3).$$

$$\dot{P}_{\psi_1} = -\sum_{i=1}^{3} m_i g\, r_1 S(\psi_1) C(\theta_1) + \big(-I_{c1} C(\psi_1) S(\psi_1) + I_{c3} S(\psi_1) C(\psi_1) + I_{13} S(\psi_1) C(\psi_1)$$
$$- \sum_{i=1}^{3} m_i r_1^2 C(\psi_1) S(\psi_1) \big) \dot{\theta}_1^2 + \sum_{i=1}^{3} m_i r_1 S(\psi_1) \dot{x}\, \dot{\psi}_1 - \sum_{i=1}^{3} m_i r_1 C(\psi_1) S(\theta_1) \dot{y}\, \dot{\psi}_1$$
$$- \sum_{i=1}^{3} m_i r_1 S(\psi_1) S(\theta_1) \dot{z}\, \dot{\theta}_1 + \sum_{i=1}^{3} m_i r_1 C(\psi_1) C(\theta_1) \dot{z}\, \dot{\psi}_1 - \sum_{i=1}^{3} m_i r_1 S(\psi_1) C(\theta_1) \dot{y}\, \dot{\theta}_1$$
$$+ \sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 S(\psi_2) C(\theta_1-\theta_2) \dot{\psi}_1 \dot{\psi}_2 - \sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 S(\psi_2) S(\theta_1-\theta_2) \dot{\theta}_1 \dot{\psi}_2$$
$$+ \big(I_{c3} C(\psi_1) + I_{13} C(\psi_1) \big) \dot{\theta}_1 \dot{\phi}_1 - \sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 C(\psi_2) S(\theta_1-\theta_2) \dot{\psi}_1 \dot{\theta}_2$$
$$- \sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 C(\psi_2) C(\theta_1-\theta_2) \dot{\theta}_1 \dot{\theta}_2 - \sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 C(\psi_2) \dot{\psi}_1 \dot{\psi}_2$$
$$- m_3 r_3 S(\psi_3) r_1 S(\psi_1) S(\theta_1-\theta_3) \dot{\theta}_1 \dot{\psi}_3 + m_3 r_3 S(\psi_3) r_1 C(\psi_1) C(\theta_1-\theta_3) \dot{\psi}_1 \dot{\psi}_3$$
$$- m_3 r_3 C(\psi_3) r_1 S(\psi_1) C(\theta_1-\theta_3) \dot{\theta}_1 \dot{\theta}_3 - m_3 r_3 C(\psi_3) r_1 C(\psi_1) S(\theta_1-\theta_3) \dot{\psi}_1 \dot{\theta}_3$$

$$- m_3 r_3 C(\psi_3) r_1 S(\psi_1) \dot{\psi}_1 \dot{\psi}_3 + W d S(\psi_1)$$

$$- K_1 r_0^4 (\dot{\theta}_1^2 + \dot{\psi}_1^2 + \dot{\phi}_1^2)^{\frac{1}{4}} [(\rho_a \mu_a)^{\frac{1}{2}} + (\rho_H \mu_H)^{\frac{1}{2}}] \dot{\psi}_1$$

$$- K_2 [(\rho_a \mu_a)^{\frac{1}{2}} + (\rho_H \mu_H)^{\frac{1}{2}}] r_0^4 (\dot{\theta}_1^2 + \dot{\psi}_1^2 + \dot{\phi}_1^2)^{-\frac{1}{4}} (\ddot{\psi}_1 - \dot{\theta}_1 \dot{\phi}_1 C(\psi_1))$$

$$- \left[\frac{8}{3} \pi \mu_a r_0^3 \left(\frac{3 + 6\beta_a r_0 + 6\beta_a^2 r_0^2 + 2\beta_a^3 r_0^3}{1 + 2\beta_a r_0 + 2\beta_a^2 r_0^2} \right) + \frac{8}{3} \pi \mu_H r_0^3 (\beta_H r_0) \right] \dot{\psi}_1$$

$$- \left[\frac{8}{3} \pi \rho_a r_0^5 \left(\frac{1 + \beta_a r_0}{1 + 2\beta_a r_0 + 2\beta_a^2 r_0^2} \right) + \frac{4}{3} \pi \rho_H r_0^5 (\beta_H r_0)^{-1} \right]$$

$$(\ddot{\psi}_1 - \dot{\theta}_1 \dot{\phi}_1 C(\psi_1))$$

$$\dot{P}_{\psi_2} = -\sum_{i=2}^{3} m_i g r_2 S(\psi_2) C(\theta_2) + \left(I_{23} S(\psi_2) C(\psi_2) - \sum_{i=2}^{3} m_i r_2^2 C(\psi_2) S(\psi_2) \right) \dot{\theta}_2^2$$

$$+ \sum_{i=2}^{3} m_i r_2 S(\psi_2) \dot{x} \dot{\psi}_2 - \sum_{i=2}^{3} m_i r_2 C(\psi_2) S(\theta_2) \dot{y} \dot{\psi}_2 + \sum_{i=2}^{3} m_i r_2 C(\psi_2) C(\theta_2) \dot{z} \dot{\psi}_2$$

$$+ \sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 C(\psi_2) C(\theta_1 - \theta_2) \dot{\psi}_1 \dot{\psi}_2 - \sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 S(\psi_2) \dot{\psi}_1 \dot{\psi}_2$$

$$- \sum_{i=2}^{3} m_i r_2 S(\psi_2) C(\theta_2) \dot{y} \dot{\theta}_2 - \sum_{i=2}^{3} m_i r_2 S(\psi_2) S(\theta_2) \dot{z} \dot{\theta}_2$$

$$- \sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 S(\psi_2) C(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + \sum_{i=2}^{3} m_i r_1 C(\psi_1) r_2 C(\psi_2) S(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\psi}_2$$

$$+ \sum_{i=2}^{3} m_i r_1 S(\psi_1) r_2 S(\psi_2) S(\theta_1 - \theta_2) \dot{\psi}_1 \dot{\theta}_2 - m_3 r_3 S(\psi_3) r_2 S(\psi_2) S(\theta_2 - \theta_3) \dot{\theta}_2 \dot{\psi}_3$$

$$+ m_3 r_3 S(\psi_3) r_2 C(\psi_2) C(\theta_2 - \theta_3) \dot{\psi}_2 \dot{\psi}_3 - m_3 r_3 C(\psi_3) r_2 S(\psi_2) C(\theta_2 - \theta_3) \dot{\theta}_2 \dot{\theta}_3$$

$$- m_3 r_3 C(\psi_3) r_2 C(\psi_2) S(\theta_2 - \theta_3) \dot{\psi}_2 \dot{\theta}_3 - m_3 r_3 C(\psi_3) r_2 S(\psi_2) \dot{\psi}_2 \dot{\psi}_3$$

$$+ I_{23} C(\psi_2) \dot{\phi}_1 \dot{\theta}_2$$

$$\dot{P}_{\psi_3} = (I_{33} S(\psi_3) C(\psi_3) - m_3 r_3^2 C(\psi_3) S(\psi_3)) \dot{\theta}_3^2 - \sum_{i=1}^{2} m_3 r_3 S(\psi_3) r_i C(\psi_i) C(\theta_i - \theta_3) \dot{\theta}_i \dot{\theta}_3$$

$$+ \sum_{i=1}^{2} m_3 r_3 C(\psi_3) r_i S(\psi_i) C(\theta_i - \theta_3) \dot{\psi}_i \dot{\psi}_3 + \sum_{i=1}^{2} m_3 r_3 C(\psi_3) r_i C(\psi_i) S(\theta_i - \theta_3) \dot{\theta}_i \dot{\psi}_3$$

$$+ \sum_{i=1}^{2} m_3 r_3 S(\psi_3) r_i S(\psi_i) S(\theta_i - \theta_3) \dot{\psi}_i \dot{\theta}_3 + m_3 r_3 S(\psi_3) \dot{x} \dot{\psi}_3$$

$$- m_3 r_3 S(\psi_3) r_1 C(\psi_1) \dot{\psi}_1 \dot{\psi}_3 - m_3 r_3 S(\psi_3) r_2 C(\psi_2) \dot{\psi}_2 \dot{\psi}_3 - m_3 r_3 S(\psi_3) C(\theta_3) \dot{y} \dot{\theta}_3$$
$$- m_3 r_3 C(\psi_3) S(\theta_3) \dot{y} \dot{\psi}_3 - m_3 r_3 S(\psi_3) S(\theta_3) \dot{z} \dot{\theta}_3 + m_3 r_3 C(\psi_3) C(\theta_3) \dot{z} \dot{\psi}_3$$
$$+ I_{33} C(\psi_3) \dot{\theta}_3 \dot{\phi}_3 - m_3 g r_3 S(\psi_3) C(\theta_3) .$$

$$\dot{P}_{\phi_1} = -K_1 r_0^4 \left(\dot{\theta}_1^2 + \dot{\psi}_1^2 + \dot{\phi}_1^2 \right)^{\frac{1}{4}} \left[(\rho_a \mu_a)^{\frac{1}{2}} + (\rho_H \mu_H)^{\frac{1}{2}} \right] \left(\dot{\theta}_1 S(\psi_1) + \dot{\phi}_1 \right)$$
$$- K_2 \left[(\rho_a \mu_a)^{\frac{1}{2}} + (\rho_H \mu_H)^{\frac{1}{2}} \right] r_0^4 \left(\dot{\theta}_1^2 + \dot{\psi}_1^2 + \dot{\phi}_1^2 \right)^{-\frac{1}{4}} \left(\ddot{\theta}_1 + \dot{\psi}_1 \dot{\phi}_1 C(\psi_1) + \ddot{\phi}_1 S(\psi_1) \right)$$

$$- \left[\frac{8}{3} \pi \mu_a r_0^3 \left(\frac{3 + 6\beta_a r_0 + 6\beta_a^2 r_0^2 + 2\beta_a^3 r_0^3}{1 + 2\beta_a r_0 + 2\beta_a^2 r_0^2} \right) + \frac{8}{3} \pi \mu_H r_0^3 (\beta_H r_0) \right]$$

$$\left(\dot{\theta}_1 S(\psi_1) + \dot{\phi}_1 \right) - \left[\frac{8}{3} \pi \rho_a r_0^5 \left(\frac{1 + \beta_a r_0}{1 + 2\beta_a r_0 + 2\beta_a^2 r_0^2} \right) \right.$$
$$\left. + \frac{4}{3} \pi \rho_H r_0^5 (\beta_H r_0)^{-1} \right] \left(\ddot{\theta}_1 S(\psi_1) + \dot{\theta}_1 \dot{\psi}_1 C(\psi_1) + \ddot{\phi}_1 \right)$$

$$\dot{P}_{\phi_3} = 0 .$$

Appendix H

Expressions for the Matrices B_i, C_i and D_i

$$B_1 = \begin{bmatrix} \sum_{i=0}^{3} m_i & -r_1 \sum_{i=1}^{3} m_i & -r_2 \sum_{i=2}^{3} m_i & -m_3 r_3 \\ -r_1 \sum_{i=1}^{3} m_i & I_{01} + r_1^2 \sum_{i=1}^{3} m_i & r_1 r_2 \sum_{i=2}^{3} m_i & m_3 r_1 r_3 \\ -r_2 \sum_{i=2}^{3} m_i & r_1 r_2 \sum_{i=2}^{3} m_i & r_2^2 \sum_{i=2}^{3} m_i & m_3 r_2 r_3 \\ -m_3 r_3 & m_3 r_1 r_3 & m_3 r_2 r_3 & m_3 r_3^2 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \sum_{i=0}^{3} m_i & r_1 \sum_{i=1}^{3} m_i & r_2 \sum_{i=2}^{3} m_i & m_3 r_3 \\ r_1 \sum_{i=1}^{3} m_i & I_{01} + r_1^2 \sum_{i=1}^{3} m_i & r_1 r_2 \sum_{i=2}^{3} m_i & m_3 r_1 r_3 \\ r_2 \sum_{i=2}^{3} m_i & r_1 r_2 \sum_{i=2}^{3} m_i & r_2^2 \sum_{i=2}^{3} m_i & m_3 r_2 r_3 \\ m_3 r_3 & m_3 r_1 r_3 & m_3 r_2 r_3 & m_3 r_3^2 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \sum_{i=0}^{3} m_1 & 0 \\ 0 & \sum_{i=0}^{3} I_{i3} \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Wd-\sum_{i=1}^{3} m_i g r_1 & 0 & 0 \\ 0 & 0 & -\sum_{i=2}^{3} m_i g r_2 & 0 \\ 0 & 0 & 0 & -m_3 g r_3 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Wd-\sum_{i=1}^{3} m_i g r_1 & 0 & 0 \\ 0 & 0 & -\sum_{i=2}^{3} m_i g r_2 & 0 \\ 0 & 0 & 0 & -m_3 g r_3 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} -K_0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -K_x \rho_a V_H - \rho_H V_H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} -K_y \rho_a V_H - \rho_H V_H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_3 = \begin{bmatrix} -K_z \rho_a V_H - \rho_H V_H & 0 \\ & \\ 0 & 0 \end{bmatrix}$$

Appendix I

Expressions for the Matrices β_i

$$\beta_1 = \begin{bmatrix} \beta^1_{11} & \beta^1_{12} & \beta^1_{13} & 0 \\ \beta^1_{21} & \beta^1_{22} & \beta^1_{23} & 0 \\ \beta^1_{31} & \beta^1_{32} & \beta^1_{33} & \beta^1_{34} \\ \beta^1_{41} & \beta^1_{42} & \beta^1_{43} & \beta^1_{44} \end{bmatrix}$$

$$\beta_2 = \begin{bmatrix} \beta^2_{11} & \beta^2_{12} & \beta^2_{13} & 0 \\ \beta^2_{21} & \beta^2_{22} & \beta^2_{23} & 0 \\ \beta^2_{31} & \beta^2_{32} & \beta^2_{33} & \beta^2_{34} \\ \beta^2_{41} & \beta^2_{42} & \beta^2_{43} & \beta^2_{44} \end{bmatrix}$$

$$\beta_3 = \begin{bmatrix} \beta^3_{11} & 0 \\ 0 & \beta^3_{22} \end{bmatrix}$$

$$\beta'_{11} = \frac{I_{c1} r_2 + m_1 r_1^2 r_2}{m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2}$$

$$\beta'_{12} = \frac{m_1 r_1 r_2}{m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2}$$

$$\beta'_{13} = \frac{I_{c1}}{m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2}$$

$$\beta'_{21} = \frac{r_1}{I_{c1}} - \frac{m_0 r_1 (I_{c1} + m_1 r_1^2)}{I_{c1}(m_0 I_{c1} + m_1 I_{c1} + m_0 m_1 r_1^2)}$$

$$\beta'_{22} = \frac{1}{I_{c1}} - \frac{m_0 m_1 r_1^2}{I_{c1}(m_0 I_{c1} + m_1 I_{c1} + m_0 m_1 r_1^2)}$$

$$\beta'_{23} = -\frac{m_0 r_1}{m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2}$$

$$\beta'_{31} = \frac{(I_{c1} + m_1 r_1^2)\sum_{i=0}^{2} m_i}{m_2 (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)} - \frac{r_1^2 (m_0 + m_1)}{I_{c1} m_2 r_2} + \frac{(m_1 r_1 + m_0 r_1) m_0 r_1 (I_{c1} + m_1 r_1^2)}{m_2 I_{c1} (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)} - \frac{1}{m_2 r_2}$$

$$\beta'_{32} = \frac{m_1 r_1 \sum_{i=0}^{2} m_i}{m_2 (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)} - \frac{m_1 r_1 + m_0 r_1}{m_2 r_2 I_{c1}} + \frac{m_0 r_1 (m_1 r_1 + m_0 r_1) m_1 r_1}{m_2 I_{c1} (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)}$$

$$\beta'_{33} = \frac{\sum_{i=0}^{2} m_i I_{c1}}{m_2 r_2 (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)} + \frac{m_0 r_1 (m_1 r_1 + m_0 r_1)}{m_2 r_2 (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)}$$

$$\beta'_{34} = -\frac{1}{m_2 r_2 r_3}$$

$$\beta'_{41} = \frac{I_{c1} + m_1 r_1^2}{r_3 (m_0 I_{c1} + m_1 I_{c1} + m_0 m_1 r_1^2)} - \left(\frac{r_1}{r_3}\right)\left(\frac{r_1}{I_{c1}} - \frac{m_0 r_1 (I_{c1} + m_1 r_1^2)}{I_{c1}(m_0 I_{c1} + m_1 I_{c1} + m_0 m_1 r_1^2)}\right) - \left(\frac{r_2}{r_3}\right)$$

$$\left(\frac{\sum_{i=0}^{2} m_i (I_{c1} + m_1 r_1^2)}{m_2 (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)} - \frac{r_1 (m_1 r_1 + m_0 r_1)}{I_{c1} m_2 r_2} + \frac{(m_1 r_1 + m_0 r_1) m_0 r_1 (I_{c1} + m_1 r_1^2)}{m_2 I_{c1} (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)} - \frac{1}{m_2 r_2} \right)$$

$$\beta^1_{42} = \frac{m_1 r_1}{r_3(m_0 I_{c1} + m_1 I_{c1} + m_0 m_1 r_1^2)} - \frac{r_1}{r_3}\left(\frac{1}{I_{c1}} - \frac{m_0 r_1^2 m_1}{I_{c1}(m_0 I_{c1} + m_1 I_{c1} + m_0 m_1 r_1^2)}\right)$$

$$- \frac{r_2}{r_3}\left(\frac{m_1 r_1 (m_0 + m_1 + m_2)}{m_2 (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)} - \frac{m_2 r_1 + m_1 r_1}{m_2 r_2 I_{c1}} + \frac{m_0 r_1 (m_1 r_1 + m_2 r_1) m_1 r_1}{m_2 I_{c1} (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)}\right)$$

$$\beta^1_{43} = \frac{I_{c1}}{r_3 (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)} + \frac{r_1}{r_3}\,\frac{m_0 r_1}{m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2}$$

$$- \frac{r_2}{r_3}\left(\frac{\sum_{i=0}^{2} m_i I_{c1}}{m_2 r_2 (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)} + \frac{(m_1 r_1 + m_2 r_1)(m_0 r_1)}{m_2 r_2 (m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2)}\right)$$

$$\beta^1_{44} = \frac{1}{m_3 r_3^2} + \frac{1}{m_2 r_3^2}$$

$\beta^2_{11} = \beta^1_{11}$

$\beta^2_{12} = -\beta^1_{12}$

$\beta^2_{13} = -\beta^1_{13}$

$\beta^2_{21} = -\beta^1_{21}$

$\beta^2_{22} = \beta^1_{22}$

$\beta^2_{23} = \beta^1_{23}$

$\beta^2_{31} = -\beta^1_{31}$

$\beta^2_{32} = \beta^1_{32}$

$\beta^2_{33} = \beta^1_{33}$

$\beta^2_{34} = \beta^1_{34}$

$$\beta^{2}_{41} = -\beta^{1}_{41}$$

$$\beta^{2}_{42} = \beta^{1}_{42}$$

$$\beta^{2}_{43} = \beta^{1}_{43}$$

$$\beta^{2}_{44} = \beta^{1}_{44}$$

$$\beta^{3}_{11} = \frac{1}{\sum_{i=0}^{3} m_i}$$

$$\beta^{3}_{22} = \frac{1}{\sum_{i=0}^{3} I_{i3}}$$

Appendix J

Expressions for the Matrices A_i

$$A_1 = \begin{bmatrix} o & a^1_{12} & a^1_{13} & o \\ o & a^1_{22} & o & o \\ o & o & a^1_{33} & o \\ o & o & o & a^1_{44} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} o & a^2_{12} & a^2_{13} & o \\ o & a^2_{22} & o & o \\ o & o & a^2_{33} & o \\ o & o & a^2_{44} & o \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a^3_{11} & o \\ o & o \end{bmatrix}$$

where

$$a^1_{12} = \frac{(K_x \rho_a + \rho_H) V_H m_1 r_1 r_2 \left(\sum_{i=1}^{3} m_i g r_1 - W d\right)}{(m_c r_2 + m_1 r_2) I_{c1} + m_c m_1 r_1^2 r_2 + (K_x \rho_a + \rho_H) V_H (I_{c1} r_2 + m_1 r_1^2 r_2)}$$

$$a^1_{13} = \frac{(K_x \rho_a + \rho_H) V_H I_{c1} \sum_{i=2}^{3} m_i g r_2}{(m_c r_2 + m_1 r_2) I_{c1} + m_c m_1 r_1^2 r_2 + (K_x \rho_a + \rho_H) V_H (I_1 r_2 + m_1 r_1^2 r_2)}$$

$$a^1_{22} = W d - \sum_{i=1}^{3} m_i g r_1$$

$$a^1_{33} = -\sum_{i=2}^{3} m_i g r_2$$

$$a^1_{44} = -m_3 g r_3$$

$$a^2_{12} = -\frac{(K_y \rho_a + \rho_H) V_H m_1 r_1 r_2 \left(\sum_{i=1}^{3} m_i g r_1 - W d\right)}{(m_c r_2 + m_1 r_2) I_{c1} + m_c m_1 r_1^2 r_2 + (K_y \rho_a + \rho_H) V_H (I_1 r_2 + m_1 r_1^2 r_2)}$$

$$a^2_{13} = -\frac{(K_y \rho_a + \rho_H) V_H I_{c1} \sum_{i=2}^{3} m_i g r_2}{(m_c r_2 + m_1 r_2) I_{c1} + m_c m_1 r_1^2 r_2 + (K_y \rho_a + \rho_H) V_H (r_2 I_{c1} + m_1 r_1^2 r_2)}$$

$$a^2_{22} = \left(W d - g r_1 \sum_{i=1}^{3} m_i\right)$$

$$a^2_{33} = -\sum_{i=2}^{3} m_i g r_2$$

$$a^2_{44} = -m_3 g r_3$$

$$a^3_{11} = \frac{(K_z \rho_a + \rho_H) V_H K_c}{\sum_{i=0}^{3} m_i + (K_z \rho_a + \rho_H) V_H} - K_c$$

Appendix K

Expressions for the Matrices α_i

$$\alpha_1 = \begin{bmatrix} 0 & \beta^1_{11}a^1_{12} + \beta^1_{12}a^1_{22} & \beta^1_{11}a^1_{13} + \beta^1_{13}a^1_{33} & 0 \\ 0 & \beta^1_{21}a^1_{12} + \beta^1_{22}a^1_{22} & \beta^1_{21}a^1_{13} + \beta^1_{23}a^1_{33} & 0 \\ 0 & \beta^1_{31}a^1_{12} + \beta^1_{32}a^1_{22} & \beta^1_{31}a^1_{13} + \beta^1_{33}a^1_{33} & \beta^1_{34}a^1_{44} \\ 0 & \beta^1_{41}a^1_{12} + \beta^1_{42}a^1_{22} & \beta^1_{41}a^1_{13} + \beta^1_{43}a^1_{33} & \beta^1_{44}a^1_{44} \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} 0 & \beta^2_{11}a^2_{12} + \beta^2_{12}a^2_{22} & \beta^2_{11}a^2_{13} + \beta^2_{13}a^2_{33} & 0 \\ 0 & \beta^2_{21}a^2_{12} + \beta^2_{22}a^2_{22} & \beta^2_{21}a^2_{13} + \beta^2_{23}a^2_{33} & 0 \\ 0 & \beta^2_{31}a^2_{12} + \beta^2_{32}a^2_{22} & \beta^2_{31}a^2_{13} + \beta^2_{33}a^2_{33} & \beta^2_{34}a^2_{44} \\ 0 & \beta^2_{41}a^2_{12} + \beta^2_{42}a^2_{22} & \beta^2_{41}a^2_{13} + \beta^2_{43}a^2_{33} & \beta^2_{44}a^2_{44} \end{bmatrix}$$

$$\alpha_3 = \begin{bmatrix} \beta^3_{11}a^3_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

Appendix L

Expressions for the Matrices γ_i , δ_i

$$\dot{P}_x = -(K_x \rho_a + \rho_H) V_H \ddot{x} - C_x \dot{x} + (\rho_a + K_x \rho_a) V_H \ddot{v}_1 + C_x \dot{v}_1$$

$$= m_{11} \ddot{x} + m_{12} \dot{x} + m_{13} \ddot{v}_1 + m_{14} \dot{v}_1$$

$$\dot{P}_{\psi 1} = -C_4 \ddot{\psi}_1 - C_3 \dot{\psi}_1 + (Wd - \sum_{i=1}^{3} m_i g r_i) \psi_1 = m_{21} \ddot{\psi}_1 + m_{22} \dot{\psi}_1 + m_{23} \psi_1$$

$$\dot{P}_{\psi 2} = -(m_2 + m_3) g r_2 \psi_2 = m_{31} \psi_2$$

$$\dot{P}_{\psi 3} = -m_3 g r_3 \psi_3 = m_{41} \psi_3$$

$$\dot{x} = \frac{(I_{c1} r_2 + m_1 r_1^2 r_2) P_x + m_1 r_1 r_2 P_{\psi 1} + I_{c1} P_{\psi 2}}{m_0 r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2}$$

$$= a_{11} P_x + a_{12} P_{\psi 1} + a_{13} P_{\psi 2}$$

$$\dot{\psi}_1 = a_{21} P_x + a_{22} P_{\psi 1} + a_{23} P_{\psi 2}$$

where $a_{21} = \beta'_{21}$,

$a_{22} = \beta'_{22}$

$a_{23} = \beta'_{23}$

define
$$t_1 = \frac{(a_{12} m_{21})(a_{21} m_{12} a_{11} + a_{22} m_{22} a_{14}) + (a_{12} m_{22} a_{21} + a_{11}^2 m_{12})(1 - a_{22} m_{21})}{(1 - a_{11} m_{11})(1 - a_{22} m_{21}) - a_{12} m_{21} a_{21} m_{11}}$$

$$t_2 = \frac{a_{12} m_{21}(a_{21} m_{12} a_{12} + a_{22}^2 m_{22}) + (a_{12} m_{22} a_{22} + a_{11} m_{12} a_{12})(1 - a_{22} m_{21})}{(1 - a_{11} m_{11})(1 - a_{22} m_{21}) - a_{12} m_{21} a_{21} m_{11}}$$

$$t_3 = \frac{a_{12} m_{21}(a_{21} m_{12} a_{13} + a_{22} m_{22} a_{23}) + (a_{12} m_{22} a_{13} + a_{11} m_{12} a_{13})(1 - a_{22} m_{21})}{(1 - a_{11} m_{11})(1 - a_{22} m_{21}) - a_{12} m_{21} a_{21} m_{11}}$$

$$t_4 = \frac{a_{12} m_{21} a_{22} m_{23} + a_{12} m_{23}(1 - a_{22} m_{21})}{(1 - a_{11} m_{11})(1 - a_{22} m_{21}) - a_{12} m_{21} a_{21} m_{11}}$$

$$t_5 = \frac{a_{12} m_{21} a_{23} m_{31} + a_{13} m_{31}(1 - a_{22} m_{21})}{(1 - a_{11} m_{11})(1 - a_{22} m_{21}) - a_{12} m_{21} a_{21} m_{21}}$$

$$t_6 = \frac{a_{12} n_{21} a_{21} n_{13} + a_{11} n_{13} (1 - a_{22} n_{21})}{(1 - a_{11} n_{11})(1 - a_{22} n_{21}) - a_{12} n_{21} a_{21} n_{11}}$$

$$t_7 = \frac{a_{12} n_{21} a_{21} n_{14} + a_{11} n_{14} (1 - a_{22} n_{21})}{(1 - a_{11} n_{11})(1 - a_{22} n_{21}) - a_{12} n_{21} a_{21} n_{11}}$$

$$\dot{P}_y = -(K_y f_\alpha + f_H) v_H \ddot{y} - C_y \dot{y} + (f_\alpha + K_y f_\alpha) \dot{v}_z + C_y v_z$$

$$= n_{11} \ddot{y} + n_{12} \dot{y} + n_{13} \dot{v}_z + n_{14} v_z$$

$$\dot{P}_{\theta 1} = -C_2 \ddot{\theta}_1 - C_1 \dot{\theta}_1 + (w d - \sum_{i=1}^{3} m_i g r_i) \theta_1 = n_{21} \ddot{\theta}_1 + n_{22} \dot{\theta}_1 + n_{23} \theta_1$$

$$\dot{P}_{\theta 2} = -(m_2 + m_3) g r_2 \theta_2 = n_{31} \theta_2$$

$$\dot{P}_{\theta 3} = -m_3 g r_3 \theta_3 = n_{41} \theta_3$$

$$\dot{y} = \frac{(r_2 I_{c1} + m_1 r_1^2 r_2) P_y - m_1 r_1 r_2 P_{\theta 1} - I_{c1} P_{\theta 2}}{m_c r_2 I_{c1} + m_1 r_2 I_{c1} + m_0 m_1 r_1^2 r_2}$$

$$= b_{11} P_y + b_{12} P_{\theta 1} + b_{13} P_{\theta 2}$$

$$\dot{\theta}_1 = b_{21} P_y + b_{22} P_{\theta 1} + b_{23} P_{\theta 2}$$

where $b_{21} = \beta_{21}^2$

$b_{22} = \beta_{22}^2$

$b_{23} = \beta_{23}^2$

define

$$l_1 = \frac{b_{12} n_{21} (b_{21} n_{12} b_{11} + b_{22} n_{22} b_{21}) + (b_{12} n_{22} b_{21} + b_{11} n_{12} b_{11})(1 - b_{22} n_{21})}{(1 - b_{11} n_{11})(1 - b_{22} n_{21}) - b_{12} n_{21} b_{21} n_{11}}$$

$$l_2 = \frac{b_{12} n_{21} (b_{21} b_{12} n_{12} + b_{22} n_{22} b_{22}) + (b_{12} n_{22} b_{22} + b_{11} n_{12} b_{12})(1 - b_{22} n_{21})}{(1 - b_{11} n_{11})(1 - b_{22} n_{21}) - b_{12} n_{21} b_{21} n_{11}}$$

$$l_3 = \frac{b_{12} n_{21} (b_{21} n_{12} b_{13} + b_{22} n_{22} b_{23}) + (b_{21} n_{22} b_{23} + b_{11} b_{13} n_{12})(1 - b_{22} n_{21})}{(1 - b_{11} n_{11})(1 - b_{22} n_{21}) - b_{12} n_{21} b_{21} n_{11}}$$

$$l_4 = \frac{b_{12} n_{21} b_{22} n_{23} + b_{12} n_{23} (1 - b_{22} n_{21})}{(1 - b_{11} n_{11})(1 - b_{22} n_{21}) - b_{12} n_{21} b_{21} n_{11}}$$

$$l_5 = \frac{b_{12} n_{21} b_{23} n_{31} + b_{13} n_{31} (1 - b_{22} n_{21})}{(1 - b_{11} n_{11})(1 - b_{22} n_{21}) - b_{12} n_{21} b_{21} n_{11}}$$

$$l_6 = \frac{b_{12} n_{21} b_{21} n_{13} + b_{11} n_{13} (1 - b_{22} n_{21})}{(1 - b_{11} n_{11})(1 - b_{22} n_{21}) - b_{12} n_{21} b_{21} n_{11}}$$

$$l_7 = \frac{b_{12} n_{21} b_{21} n_{14} + b_{11} n_{14} (1 - b_{22} n_{21})}{(1 - b_{11} n_{11})(1 - b_{22} n_{21}) - b_{12} n_{21} b_{21} n_{11}}$$

$$\gamma_1 = \begin{bmatrix} 0 & \gamma_{12}^1 & \gamma_{13}^1 & 0 \\ 0 & \gamma_{22}^1 & \gamma_{23}^1 & 0 \\ 0 & 0 & \gamma_{33}^1 & 0 \\ 0 & 0 & 0 & \gamma_{44}^1 \end{bmatrix}$$

$$\gamma_2 = \begin{bmatrix} 0 & \gamma_{12}^2 & \gamma_{13}^2 & 0 \\ 0 & \gamma_{22}^2 & \gamma_{23}^2 & 0 \\ 0 & 0 & \gamma_{33}^2 & 0 \\ 0 & 0 & 0 & \gamma_{44}^2 \end{bmatrix}$$

$$\gamma_3 = \begin{bmatrix} \gamma_{11}^3 & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$\gamma_{12}^1 = m_{11} t_4$$

$$\gamma_{13}^1 = m_{11} t_5$$

$$\gamma_{22}^1 = m_{21} \left(\frac{a_{21} m_{11} t_4}{1 - a_{22} m_{21}} + \frac{a_{22} m_{23}}{1 - a_{22} m_{21}} \right) + m_{23}$$

$$\gamma_{23}^1 = m_{21} \left(\frac{a_{21} m_{11} t_5}{1 - a_{22} m_{21}} + \frac{a_{23} m_{31}}{1 - a_{22} m_{21}} \right)$$

$$\gamma_{33}^1 = m_{31}$$

$$\gamma_{44}^1 = m_{41}$$

$$\gamma_{12}^2 = n_{11} l_4$$

$$\gamma_{13}^2 = n_{11} l_5$$

$$\gamma_{22}^2 = n_{21} \left(\frac{b_{21} n_{11} l_4}{1 - b_{22} n_{21}} + \frac{b_{22} n_{23}}{1 - b_{22} n_{21}} \right) + n_{23}$$

$$\gamma_{23}^2 = n_{21} \left(\frac{b_{21} n_{11} l_5}{1 - b_{22} n_{21}} + \frac{b_{23} n_{31}}{1 - b_{22} n_{21}} \right)$$

$$\gamma_{33}^2 = n_{31}$$

$$\gamma_{44}^2 = n_{41}$$

$$\gamma_{11}^3 = \frac{V_H (K_z f_a + f_H) K_0}{\sum_{i=0}^{3} m_i + V_H (f_a K_z + f_H)} - K_0$$

$$\delta_1 = \begin{bmatrix} \delta^1_{11} & \delta^1_{12} & \delta^1_{13} & 0 \\ \delta^1_{21} & \delta^1_{22} & \delta^1_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\delta_2 = \begin{bmatrix} \delta^2_{11} & \delta^2_{12} & \delta^2_{13} & 0 \\ \delta^2_{21} & \delta^2_{22} & \delta^2_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\delta_3 = \begin{bmatrix} \delta^3_{11} & 0 \\ 0 & \delta^3_{22} \end{bmatrix}$$

$$\delta^1_{11} = m_{11} t_1 + m_{12} a_{11}$$

$$\delta^1_{12} = m_{11} t_2 + m_{12} a_{12}$$

$$\delta^1_{13} = m_{11} t_3 + m_{12} a_{13}$$

$$\delta^1_{21} = m_{21} \left(\frac{a_{21} m_{11} t_1}{1 - a_{22} m_{21}} + \frac{a_{21} m_{12} a_{11} + a_{22} m_{22} a_{21}}{1 - a_{22} m_{21}} \right) + m_{22} a_{21}$$

$$\delta^1_{22} = m_{21} \left(\frac{a_{21} m_{11} t_2}{1 - a_{22} m_{21}} + \frac{a_{21} m_{12} a_{12} + a_{22}^2 m_{22}}{1 - a_{22} m_{21}} \right) + m_{22} a_{22}$$

$$\delta^1_{23} = m_{21} \left(\frac{a_{21} m_{11} t_3}{1 - a_{22} m_{21}} + \frac{a_{21} m_{12} a_{13} + a_{22} m_{22} a_{23}}{1 - a_{22} m_{21}} \right) + m_{22} a_{23}$$

$$\delta^2_{11} = n_{11} l_1 + n_{12} b_{11}$$

$$\delta^2_{12} = n_{11} l_2 + n_{12} b_{12}$$

$$\delta^2_{13} = n_{11} l_3 + n_{12} b_{13}$$

$$\delta^2_{21} = n_{21} \left(\frac{b_{21} n_{11} l_1}{1 - b_{22} n_{21}} + \frac{b_{21} n_{12} b_{11} + b_{22} n_{22} b_{21}}{1 - b_{22} n_{21}} \right) + n_{22} b_{21}$$

$$\delta^2_{22} = n_{21} \left(\frac{b_{21} n_{11} l_2}{1 - b_{22} n_{21}} + \frac{b_{21} n_{12} b_{12} + b_{22} n_{22} b_{22}}{1 - b_{22} n_{21}} \right) + n_{22} b_{22}$$

$$\delta^2_{23} = n_{21} \left(\frac{b_{21} n_{11} l_3}{1 - b_{22} n_{21}} + \frac{b_{21} n_{12} b_{13} + b_{22} n_{22} b_{23}}{1 - b_{22} n_{21}} \right) + n_{22} b_{23}$$

$$\delta^3_{11} = \frac{V_{11} (k_z \rho_a + \rho_{11}) C_z}{\sum_{i=0}^{3} m_i \left(\sum_{i=0}^{3} m_i + V_{11} (\rho_a K_z + \rho_{11}) \right)} - \frac{C_z}{\sum_{i=0}^{3} m_i}$$

$$\delta^3_{22} = \frac{-C_5}{\sum_{i=0}^{3} I_{i3}} + \frac{c_5 c_6}{\left(\sum_{i=0}^{3} I_{i3} + c_6 \right) \left(I_{03} + I_{13} + I_{23} + I_{33} \right)}$$

Appendix M
Expressions for the $\overline{F}_i$

$$f_1^1 = (m_{11} t_6 + m_{13})\, v_1 + (m_{11} t_7 + m_{14})\, v_1$$

$$f_2^1 = m_{21} \left(\frac{a_{21} m_{11} t_6}{1 - a_{22} m_{21}} + \frac{a_{21} m_{13}}{1 - a_{22} m_{21}} \right) v_1 + m_{21} \left(\frac{a_{21} m_{11} t_7}{1 - a_{22} m_{21}} + \frac{a_{21} m_{14}}{1 - a_{22} m_{21}} \right) v_1$$

$$f_1^2 = (n_{11} l_6 + n_{13})\, v_2 + (n_{11} l_7 + n_{14})\, v_2$$

$$f_2^2 = n_{21} \left(\frac{b_{21} n_{11} l_6}{1 - b_{22} n_{21}} + \frac{b_{21} n_{13}}{1 - b_{22} n_{21}} \right) v_2 + n_{21} \left(\frac{b_{21} n_{11} l_7}{1 - b_{22} n_{21}} + \frac{b_{21} n_{14}}{1 - b_{22} n_{21}} \right) v_1$$

$$f_1^3 = \left(\frac{-V_H^2 (K_2 \rho_a + \rho_H)(\rho_a + K_2 \rho_a)}{\sum_{i=0}^{3} m_i + V_H (\rho_a K_2 + \rho_H)} + (\rho_a V_H + K_2 \rho_a V_H) \right) v_3 + \left(- \frac{V_H (K_2 \rho_a + \rho_H) C_2}{\sum_{i=0}^{3} m_i + V_H (\rho_a K_2 + \rho_H)} + C_2 \right) v_3$$

$$\overline{F}_1 = \begin{bmatrix} f_1^1 \\ f_2^1 \\ 0 \\ 0 \end{bmatrix} \qquad \overline{F}_2 = \begin{bmatrix} f_1^2 \\ f_2^2 \\ 0 \\ 0 \end{bmatrix} \qquad \overline{F}_3 = \begin{bmatrix} f_1^3 \\ 0 \end{bmatrix}$$

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