FINITE DIFFERENCE METHODS FOR TRANSIENT SIGNAL PROPAGATION IN STRATIFIED DISPERSIVE MEDIA

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ABSTRACT

Explicit difference equations are presented for the solution of a signal of arbitrary waveform propagating in an ohmic dielectric, a cold plasma, a Debye model dielectric, and a Lorentz model dielectric. These difference equations are derived from the governing time-dependent integro-differential equations for the electric fields by a finite difference method. A special difference equation is derived for the grid point at the boundary of two different media. Employing this difference equation, transient signal propagation in an inhomogeneous media can be solved provided that the medium is approximated in a step-wise fashion. The solutions are generated simply by marching on in time. By appropriate choice of the time and space intervals, numerical stability and convergence are always obtained. Numerous examples are given to demonstrate the wide range of applicability of the difference solution. These include: the transmission and reflection of an electromagnetic pulse normally incident on a multilayered ohmic dielectric; a step-modulated sine wave propagating in a dispersive media, a problem originally considered by Sommerfeld and Brillouin; the reflection of a short gaussian pulse normally incident on an inhomogeneous lossy cold plasma with a longitudinal d.c. magnetic field, and many others. It is concluded that while the classical transform methods will remain useful in certain cases, with the development of the finite difference methods described in this dissertation, an extensive class of problems of transient signal propagating in stratified dispersive media can be effectively solved by numerical methods.

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The propagation of a transient signal in dispersive media is an important and interesting problem in electrical science. Take, for example, the historically famous problem considered by Sommerfeld[1]. In this work he showed that the front of a signal propagates exactly with the speed of light in free space regardless of what the group velocity might be. Thus it supported, at a controversial time[2], one of the fundamental postulates in Einstein's special theory of relativity. More recently, there has been an interest in the remote sensing of the thickness of ice layers over water, and in the sensing of subsurface geological structure using short pulse techniques[3]. Other applications are the distortion of the waveform of a signal reflected from the ionosphere[4], and the measurement of the electrical properties of dielectrics utilizing time-domain reflectometry[5].

To analyze these transient problems, Fourier and Laplace transform methods are usually employed. Ahtough these standard methods apply to many cases, their failure to accommodate others has warranted additional and new methods of solution. In the case of cold plasmas, for instance, Bowhill[6] proposed the multiple-scattering technique, and Field[7] suggested the method of characteristics. This dissertation presents some finite difference methods for the solution of transient signal propagation problems in stratified dispersive media.

The numerical solution of partial differential equations has its root deep in the past. In 1928, Courant, Friedrichs and Lewy[8] published a celebrated paper in which they proposed originally the idea of solving the wave equation, the diffusion equation and the Laplace equation by algebraic operations. In particular, when they replaced the wave equation

(1) $$\frac{\partial^2 E}{\partial t^2} - c^2 \frac{\partial^2 E}{\partial z^2} = 0$$

by the difference equation

$$\frac{E(z,t+\Delta t)-2E(z,t)+E(z,t-\Delta t)}{\partial t^2} - c^2 \frac{E(z+\Delta z,t)-2E(z,t)+E(z-\Delta z,t)}{\partial z^2} = 0,$$

they found that the numerical solution always converges if the grid ratio $c\Delta t/\Delta z$ is less than or equal to one. Furthermore, if the equality sign is chosen, the resulting difference equation

(2) $E(z,t+\Delta t) = - E(z,t-\Delta t) + E(z+\Delta z,t) + E(z-\Delta z,t)$

generates exactly the analytic solution[9]. More than 30 years later, in 1970 Chiu[10] carried this remarkable solution through an interface boundary for the first time. He derived an appropriate difference equation for this case and showed that the numerical method again produces an exact solution.

We shall extend these techniques to the integro-differential equation of the type:

(3) $$\frac{\partial^2 E}{\partial t^2} - c^2 \frac{\partial^2 E}{\partial z^2} + a \frac{\partial E}{\partial t} + bE(z,t) = \int_0^t K(t-\beta)E(z,\beta)d\beta.$$

As will be seen this type of equation, or various special cases of it, arises in a wide variety of problems when a plane electromagnetic wave propagates in linear dispersive media.

In Chapter II we shall discuss in detail the numerical solution of (3) in a homogeneous medium. A procedure will then be presented to derive the appropriate difference equation for a stratified medium. As an example the wave equation is considered here. By approximating the inhomogeneous dielectric medium with a large number of thin layers, it will be shown how one of the most challenging boundary-value problems in electromagnetic field theory, the reflection and transmission of a plane electromagnetic wave from an inhomogeneous dielectric slab[11], can be solved as an initial-value problem - by utilizing the step-modulated sinusoidal wave as the incident signal and marching the time domain solution long enough to allow the steady-state to be established.

In Chapter III a difference equation for a stratified lossy dielectric medium is first derived. This difference equation is then utilized to obtain the waveform of a unit-step signal propagating in a homogeneous lossy dielectric medium, the transmission of a gaussian pulse from the air into the earth, and the reflected waveform of a sine-squared pulse from a three-layered lossy dielectric. In all cases normal incidence is assumed.

In Chapter IV a numerical solution to Sommerfeld's "About the propagation of light in dispersive media"[1] is investigated. A difference equation is derived from the governing integro-differential equation for the electric field. Employing the resulting difference equation, we have been able to observe the real-time propagation process vividly on a high-speed digital computer having a CRT display facility. The dynamic behavior of the precursor of a signal thus can be seen. We shall reexamine this historical problem and discuss some of our findings.

Chapter V deals with transient signal propagation in a stratified cold plasma in which a constant magnetic field parallel to the direction of propagation can be assumed. The coupled integro-differential equations for the electric fields are first derived from Maxwell's equations and the equation of motion for the electrons. They are next transformed into an explicit difference equation; the solution is then obtained simply by marching on in time. By approximating the plasma medium with a large number of thin layers, it is shown that the time history of a signal of arbitrary waveform in an inhomogeneous, anisotropic (due to the presence of a constant magnetic field), lossy, cold plasma can be easily obtained. Again, only the case of normal incidence is considered.

In Chapter VI we calculate the reflected waveforms of a unit-step signal normally incident from air to water directly as well as with an ice layer on it. Here the water is considered as a Debye dielectric[12]. We first derive the governing time-dependent integro-differential equation for the electric field propagating in this type of dispersive medium, and then deduce the corresponding difference equation. The solution is then generated again by marching on in time.

Finally in Chapter VII we summarize the results obtained, and discuss briefly the application to nonlinear dispersive media as developed elsewhere[13,14], a subject of great interest in recent years[15].

It is concluded that while the classical Fourier and Laplace transform methods will remain useful in certain cases, with the development of the finite difference methods described here as well as the availability of a high-speed digital computer, an extensive class of problems of transient signal propagation in stratified dispersive media can be effectively solved by numerical methods.

CHAPTER II
DIFFERENCE METHODS FOR AN INTEGRO-DIFFERENTIAL EQUATION

Although the numerical solution of partial differential equations by finite difference methods is well-established[16,17,18], little is known of its application to the initial-value problem of the telegraph and related equations[19]. In this chapter we shall discuss the numerical methods for the integro-differential equation (3) in detail. We shall start from the wave equation, then add one lower order term after another. In this manner, the lossy wave equation, the telegraph equation, and the integro-differential equation itself are treated. We shall also present a procedure to derive the difference equation for a grid point at the boundary of two different media, in order to be able to treat propagation in inhomogeneous media.

A. The Wave Equation

As was mentioned earlier, if we replace the wave equation

$$\frac{\partial^2 E}{\partial t^2} - c^2 \frac{\partial^2 E}{\partial z^2} = 0$$

by the difference equation

$$\frac{E(z,t+\Delta t)-2E(z,t)+E(z,t-\Delta t)}{\Delta t^2} - c^2 \frac{E(z+\Delta z,t)-2E(z,t)+E(z-\Delta z,t)}{\Delta z^2}$$

or

$$(4) \qquad E(z,t+\Delta t) = -E(z,t-\Delta t) + 2\left[1 - \left(\frac{c\Delta t}{\Delta z}\right)^2\right]E(z,t) +$$

$$+ \left(\frac{c\Delta t}{\Delta z}\right)^2 [E(z+\Delta z,t) + E(z-\Delta z,t)]$$

then the numerical solution always converges if the stability criterion

$$(5) \qquad \frac{c\Delta t}{\Delta z} \leq 1$$

is satisfied. Furthermore, if the equality sign is chosen, i.e.,

(6) $\Delta z = c\Delta t,$

then the resulting difference equation

(7) $E(z,t+\Delta t) = - E(z,t-\Delta t) + E(z+\Delta z,t) + E(z-\Delta z,t)$

generates exactly the same values as the analytic solution. This is to say that as far as the numerical values of the solution are concerned, one could not distinguish whether they were either obtained from solving the intial value problem of the wave equation or obtained from (7) by marching on in time. This fact has long been known by numerical analysts[9,17].

In order to visualize this remarkable property, let us consider the Taylor series expansions

$$\frac{\partial^2 E}{\partial z^2} = \frac{E(z+\Delta z,t)-2E(z,t)+E(z-\Delta z,t)}{\Delta z^2} - \frac{\partial^4 E}{\partial z^4}\frac{\Delta z^2}{12} \cdots .$$

A similar expression can be obtained for the time variable t. Now if we approximate the second partial derivatives by only the first term on the right-hand side, obviously a truncation error results. Therefore the replacement of the wave equation by (4), which is now written in the notation $E_i^n = E(i\Delta z, n\Delta t)$ as

(8) $$E_i^{n+1} = -E_i^{n-1} + 2\left[1 - \left(\frac{c\Delta t}{\Delta z}\right)^2\right] E_i^n + \left(\frac{c\Delta t}{\Delta z}\right)^2 (E_{i+1}^n + E_{i-1}^n),$$

introduce a truncation error

(9) $$T = \frac{1}{12}\left(\frac{\partial^4 E}{\partial z^4}\Delta z^2 - \frac{1}{c^4}\frac{\partial^4 E}{\partial t^4}(c\Delta t)^2\right) + \text{higher order terms}.$$

Equation (8) tells us that if the initial data at the time steps n= 0,1 are given, then the fields at subsequent times can be obtained recursively. However the result will be correct only if a proper value of $c\Delta t/\Delta z$ is used.

A procedure to determine the permitted value of this ratio, or the stability criterion as it is called in numerical analysis, is described by Hildebrand[17, pp. 235]. Let the solution of (8) be the form

(10) $E_i^m = \beta^n e^{j\alpha i}$

where α, β are constants, with α real and $j = \sqrt{-1}$. Since the number of time steps is arbitrary, the index n can be increased without bound. Therefore the magnitude of β in (10) must be less than one for a finite solution. This condition is now used to determine the stability criterion.

Substituting (10) into (8) yields the secular equation

$$(11) \qquad \beta^2 - 2\left[1 - 2\left(\frac{c\Delta t}{\Delta z}\right)^2 \sin^2 \frac{\alpha}{2}\right]\beta + 1 = 0.$$

Since

$$\sum_{i=1}^{2} \beta_i = 2\left[1 - 2\left(\frac{c\Delta t}{\Delta z}\right)^2 \sin^2 \frac{\alpha}{2}\right]$$

where β_i, i=1,2 are the roots of (11), the requirement of $|\beta_i| \leq 1$, i=1,2 implies

$$\left|1 - 2\left(\frac{c\Delta t}{\Delta z}\right)^2 \sin^2 \frac{\alpha}{2}\right| \leq 1 .$$

This inequality is true if and only if the absolute value inside the bracket is less than one,

$$\left|\left(\frac{c\Delta t}{\Delta z}\right)^2 \sin^2 \frac{\alpha}{2}\right| \leq 1.$$

Since α is real, the value inside the absolute sign is always positive,

$$\frac{c\Delta t}{\Delta z} \sin \frac{\alpha}{2} \leq 1.$$

This inequality is true if and only if

$$(12) \qquad \frac{c\Delta t}{\Delta z} \leq 1$$

for all real α. This is the well-known stability criterion for the numerical solution of the wave equation by the finite difference method[16].

If the equality sign in (12) is chosen, three remarkable properties are seen:

1. The time increment Δt is the largest one permitted,
    thus allowing a problem to be solved with least number
    of time steps.

2. Calculations at each step are reduced to a minimum because
    (7) is the simplest version of (4).

3. But above all, exact solution is obtained from the numerical
    method because the truncation error in (9) has now vanished.

It is indeed remarkable all these sought-after properties in a
numerical solution of differential equations happen at the same time.

We have therefore identified the scheme $\Delta z = c\Delta t$ optimum in every
sense for the numerical solution of the wave equation by the finite
difference method. This scheme seems to force the difference equation
to march along the characteristics of the wave equation. Since an
initial disturbance propagates along the characteristics of the wave
equation without changing form, it is not surprising that the dif-
ference equation produces an exact solution.

The above conjecture can be verified by showing the difference
equation satisfying the general solution of the wave equation. To
do so, let the general solution of the wave equation be the form

$$E(z,t) = f_1(z-ct) + f_2(z+ct).$$

We compute and verify that

$$-E(z,t-\Delta t) + E(z+\Delta z,t) + E(z-\Delta z,t)$$

$$= -[f_1(z-c(t-\Delta t))+ f_2(z+c(t-\Delta t))] +$$

$$+ [f_1(z+\Delta z-ct)+f_2(z+\Delta z+ct)] + [f_1(z-\Delta z-ct)+f_2(z-\Delta z+ct)]$$

$$= f_1(z-ct-\Delta z)+ f_2(z+ct+\Delta z)$$

$$= f_1[z-c(t+\Delta t)]+ f_2[z+c(t+\Delta t)]$$

$$= E(z,t+\Delta t)$$

$$Q.E.D.$$

In these manipulations the relation $\Delta z=c\Delta t$ has been used.

We now proceed to extend these results to a number of more
general situations.

B. The Lossy Wave Equation

The numerical solution to the lossy wave equation

(13) $\qquad \dfrac{\partial^2 E}{\partial t^2} - c^2 \dfrac{\partial^2 E}{\partial z^2} + a \dfrac{\partial E}{\partial t} = 0$

is also known[17, pp. 311]. While the first two terms are approximated, as usual, by (8), the first derivative is approximated by the central finite difference

(14) $\qquad \dfrac{\partial E}{\partial t} \approx \dfrac{E(z,t+\Delta t)-E(z,t-\Delta t)}{2\Delta t} \sim \dfrac{E_i^{n+1} + E_i^{n-1}}{2\Delta t}$

for a better accuracy. Equation (13) becomes

(15) $\qquad E_i^{n+1} = \dfrac{1}{1 + \frac{a\Delta t}{2}} \left[- \left(1 - \dfrac{a\Delta t}{2}\right) E_i^{n-1} + 2 \left[1 - \left(\dfrac{c\Delta t}{\Delta z}\right)^2 \right] E_i^{n} + \right.$

$\qquad\qquad \left. + \left(\dfrac{c\Delta t}{\Delta z}\right)^2 (E_{i+1}^{n} + E_{i-1}^{n}) \right]$.

Its stability criterion can be determined by following the same steps as (10)-(12) to obtain

(16) $\qquad \beta^2 - 2 \; \dfrac{1-2\left(\dfrac{c\Delta t}{\Delta z}\right)^2 \sin^2 \frac{\alpha}{2}}{1 + \frac{a\Delta t}{2}} + \dfrac{1 - \frac{a\Delta t}{2}}{1 + \frac{a\Delta t}{2}} = 0$

The requirement for $|\beta_i| \leq 1$, $i=1,2$ leads to the stability criterion for the lossy wave equation

(17) $\qquad \dfrac{c\Delta t}{\Delta z} \leq \sqrt{1 + \dfrac{a\Delta t}{4}}$.

Since we shall consider only $a \geq 0$, (17) is always satisfied if the scheme $\Delta z = c\Delta t$ is selected. Consequently (15) is simplified to

(18) $\qquad E_i^{n+1} = \dfrac{1}{1 + \frac{a\Delta t}{2}} \left[- \left(1 - \dfrac{a\Delta t}{2}\right) E_i^{n-1} + E_{i+1}^{n} + E_{i-1}^{n} \right]$

which is an appropriate difference equation for the numerical solution to the lossy wave equation by the finite difference method.

C. The Klein-Gordon Equation

The addition of another kind of lower order term to the wave equation yields the Klein-Gordon equation[20]

$$(19) \qquad \frac{\partial^2 E}{\partial t^2} - c^2 \frac{\partial^2 E}{\partial z^2} + bE = 0.$$

Although a numerical solution to this equation has been discussed[19], it has never been carried out according to the best knowledge of the author. Nevertheless, at the first glance the difference solution for this case seems quite straight-forward. One simply adds the lower order term to (8) and obtains

$$(20) \qquad E_i^{n+1} = - E_i^{n-1} + 2 \left[1 - \left(\frac{c\Delta t}{\Delta z} \right)^2 \right] E_i^n + \left(\frac{c\Delta t}{\Delta z} \right)^2 (E_{i+1}^n + E_{i-1}^n) - b\Delta t^2 E_i^n.$$

However, the stability criterion for this case is found to be

$$(21) \qquad \frac{c\Delta t}{\Delta z} \leq \sqrt{1 - \frac{1}{4} b\Delta t^2}.$$

Hence the scheme $\Delta z = c\Delta t$ can no longer be chosen if $b>0$. This is the case for cold plasmas but not for the Debye dielectrics as will be seen.

One may of course select a scheme such that (21) is satisfied and proceed to obtain a numerical solution. In so doing not only have all the remarkable properties of the difference solution for the wave equation been lost, but as will be seen later, the scheme $\Delta z = c\Delta t$ is found to play an essential part in the practical aspects of the numerical solution. There is another reason to reject the above idea however. It was conjectured previously that the scheme $\Delta z = c\Delta t$ is not just a numerical convenience, but it also coincides with the characteristic theory of the hyperbolic partial differential equation, in that this scheme forces the difference equation to march on the characteristics of the wave equation. Since the presence of a lower order term in the wave equation does not affect its characteristics[21], it is anticipated that the optimum scheme for the wave equation should hold true for the Klein-Gordon equation. It turns out this is correct.

For the time being, let us accept the fact that if we approximate the electric field in the lower order term by three consecutive times such as

$$(22) \qquad E_i^n = \frac{1}{2}\left[E_i^n + \frac{1}{2} E_i^{n+1} + \frac{1}{2} E_i^{n-1} \right] ,$$

then it can be shown that the stability condition for the resulting difference equation

$$(23) \quad E_i^{n+1} = - E_i^{n-1} + \frac{1}{1 + \frac{1}{4} b\Delta t^2}\left[2\left(1 - \left(\frac{c\Delta t}{\Delta z}\right)^2 \right)E_i^n + \left(\frac{c\Delta t}{\Delta z}\right)^2 (E_{i+1}^n + E_{i-1}^n) - \frac{1}{2} b\Delta t^2 E_i^n \right] ,$$

agrees exactly with that of the wave equation. Consequently the optimum scheme $\Delta z = c\Delta t$ can be selected again! Equation (23) then simplifies to

$$(24) \qquad E_i^{n+1} = - E_i^{n-1} + \frac{1}{1 + \frac{1}{4} b\Delta t^2}\left[E_{i+1}^n + E_{i-1}^n - \frac{1}{2} b\Delta t^2 E_i^n \right]$$

which is an appropriate difference equation for the Klein-Gordon equation. We did not find out the scheme (22) by chance however. In the next section we shall demonstrate how it was derived.

D. The Telegraph Equation

Combining the lower order terms in the two previous cases, we have the telegraph equation

$$(25) \qquad \frac{\partial^2 E}{\partial t^2} - c^2 \frac{\partial^2 E}{\partial z^2} + a \frac{\partial E}{\partial t} + bE = 0.$$

When $a = GL + RC$, $b = RG$, and $c^2 = 1/LC$, (25) is the familiar transmission line equation in which R, G, L, C are the resistance, conductance, inductance, and capacitance per unit length of the transmission line. Our goal here is to show that the electric field in the last term should be approximated by three consecutive times as suggested in (22). The approach will be to find a numerical solution which is consistent for both the second order partial differential equation and the first order equations from which it was derived.

To deduce this result, let us return to the second order wave equation

$$\frac{\partial^2 E}{\partial t^2} - c^2 \frac{\partial^2 E}{\partial z^2} = 0.$$

It has been shown that if the scheme

(26) $\Delta z = c\Delta t$

is selected, the resulting difference equation

(27) $E_i^{n+1} = - E_i^{n-1} + E_{i+1}^n + E_{i-1}^n$

generates exactly analytic solution. This must be true if the corresponding first order system of equations, the Maxwell's equations

$$\varepsilon_0 \frac{\partial E_x}{\partial t} + \frac{\partial H_y}{\partial z} = 0,$$

(28)

$$\mu_0 \frac{\partial H_y}{\partial t} + \frac{\partial E_x}{\partial z} = 0,$$

are used as the starting point for the same problem. Therefore, if we approximate (28) by the difference equations[16, pp. 262]

$$\varepsilon_0 \frac{E_i^{n+1} - E_i^n}{\Delta t} + \frac{H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2}}{\Delta z} = 0,$$

(29)

$$\mu_0 \frac{H_{i+1/2}^{n+3/2} - H_{i+1/2}^{n+1/2}}{\Delta t} + \frac{E_{i+1}^{n+1} - E_i^{n+1}}{\Delta z} = 0$$

or

$$E_i^{n+1} = E_i^n - \frac{\Delta t}{\varepsilon_0 \Delta z} (H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2}),$$

(30)

$$H_{i+1/2}^{n+3/2} = H_{i+1/2}^{n+1/2} - \frac{\Delta t}{\mu_0 \Delta z} (E_{i+1}^{n+1} - E_i^{n+1}).$$

we should be able to obtain (27) by eliminating the magnetic field in (30). Indeed, we substitute the second difference equation in (30) to the first difference equation and obtain

$$E_i^{n+1} = E_i^n - \frac{\Delta t}{\varepsilon_0 \Delta z} \left\{ \left[H_{i+1/2}^{n-1/2} - \frac{\Delta t}{\mu_0 \Delta z} (E_{i+1}^n - E_i^n) \right] - \left[H_{i-1/2}^{n-1/2} - \frac{\Delta t}{\mu_0 \Delta z} (E_i^n - E_{i-1}^n) \right] \right\}$$

$$= E_i^n - \frac{\Delta t}{\varepsilon_0 \Delta z} (H_{i+1/2}^{n-1/2} - H_{i-1/2}^{n-1/2}) + (E_{i+1}^n - 2E_{i+1}^n + E_{i-1}^n)$$

$$= E_i^n + (E_i^n - E_i^{n-1}) + (E_{i+1}^n - 2E_i^n + E_{i-1}^n)$$

$$= - E_i^{n+1} + E_{i+1}^n + E_{i-1}^n$$

which is exactly identical to (27).

Now we can do the same thing to the lossy wave equation

$$(31) \qquad \frac{\partial^2 E}{\partial t^2} - c^2 \frac{\partial^2 E}{\partial z^2} + \frac{\sigma}{\varepsilon_0} \frac{\partial E}{\partial t} = 0.$$

The difference equation for this lossy wave equation has been given in (18). It is

$$(32) \qquad E_i^{n+1} = \frac{1}{1 + \frac{\sigma \Delta t}{2\varepsilon_0}} \left[- \left(1 - \frac{\sigma \Delta t}{2\varepsilon_0} \right) E_i^{n-1} + E_{i+1}^n + E_{i-1}^n \right].$$

The corresponding Maxwell's equations

$$\varepsilon_0 \frac{\partial E_x}{\partial t} + \frac{\partial H_y}{\partial z} + \sigma E_x = 0,$$

$$(33)$$

$$\mu_0 \frac{\partial H_y}{\partial t} + \frac{\partial E_x}{\partial z} = 0$$

can be written in difference equation form

(34)

$$\varepsilon_0 \frac{E_i^{n+1} - E_i^n}{\Delta t} + \frac{H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2}}{\Delta z} + \frac{\sigma}{2}(E_i^{n+1} + E_i^n) = 0,$$

$$\mu_0 \frac{H_{i+1/2}^{n+3/2} - H_{i+1/2}^{n+1/2}}{\Delta t} + \frac{E_{i+1}^{n+1} - E_i^{n+1}}{\Delta z} = 0$$

or

(35)

$$E_i^{n+1} = \frac{1}{1 + \frac{\sigma \Delta t}{2\varepsilon_0}} \left[\left(1 - \frac{\sigma \Delta t}{2\varepsilon_0}\right) E_i^n - \frac{\Delta t}{\varepsilon_0 \Delta z} \left(H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2}\right) \right],$$

$$H_{i+1/2}^{n+3/2} = H_{i+1/2}^{n+1/2} - \frac{\Delta t}{\mu_0 \Delta z} \left(E_{i+1}^{n+1} - E_i^{n+1}\right).$$

Notice that in (34), the electric field in the lossy term has been approximated by two consecutive times. It is desired to show that (32) can be deduced from (35) by eliminating the magnetic field.

    We first compute

(36)

$$\frac{\Delta t}{\varepsilon_0 \Delta z} \left(H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2}\right) = \frac{\Delta t}{\varepsilon_0 \Delta z} \left[H_{i+1/2}^{n-1/2} - \frac{\Delta t}{\mu_0 \Delta z}(E_{i+1}^n - E_i^n) \right]$$

$$- \left[H_{i-1/2}^{n-1/2} - \frac{\Delta t}{\mu_0 \Delta z}(E_i^n - E_{i-1}^n) \right]$$

$$= \frac{\Delta t}{\varepsilon_0 \Delta z}\left(H_{i+1/2}^{n-1/2} - H_{i-1/2}^{n-1/2}\right) - (E_{i+1}^n - 2E_i^n + E_{i-1}^n)$$

$$= \left[-\left(1 + \frac{\sigma \Delta t}{2\varepsilon_0}\right) E_i^n + \left(1 - \frac{\sigma \Delta t}{2\varepsilon_0}\right) E_i^{n-1} \right]$$

$$- (E_{i+1}^n - 2E_i^n + E_{i-1}^n)$$

and substitute it to the last bracket in the first difference equation in (35) to obtain (32). The important feature in (34) is the approximation made in the electric field in the lossy term. Thus, for the transmission line equations

$$C \frac{\partial v}{\partial t} + \frac{\partial i}{\partial z} + Gv = 0,$$

(37)

$$L \frac{\partial i}{\partial t} + \frac{\partial v}{\partial z} + Ri = 0,$$

we can approximate the voltage v in Gv and the current i in Ri by
its average values at two consecutive times as

$$C\,\frac{v_j^{n+1}-v_j^n}{\Delta t} + \frac{i_{j+1/2}^{n+1/2} - i_{j-1/2}^{n+1/2}}{\Delta z} + G\,\frac{1}{2}\,(v_j^{n+1} + v_j^n) = 0,$$

$$L\,\frac{i_{j+1/2}^{n+3/2}-i_{j+1/2}^{n+1/2}}{\Delta t} + \frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta z} + R\,\frac{1}{2}\,(i_{j+1/2}^{n+3/2} + i_{j+1/2}^{n+1/2}) = 0$$

or

$$\left(1 + \frac{G\Delta t}{2C}\right)v_j^{n+1} = \left(1 - \frac{G\Delta t}{2C}\right)v_j^n - \frac{\Delta t}{C\Delta z}\,(i_{j+1/2}^{n+1/2}-i_{j-1/2}^{n+1/2}) = 0,$$

(38)

$$\left(1 + \frac{R\Delta t}{2L}\right)i_{j+1/2}^{n+3/2} = \left(1 - \frac{R\Delta t}{2L}\right)i_{j+1/2}^{n+1/2} - \frac{\Delta t}{L\Delta z}\,(v_{j+1}^{n+1} - v_j^{n+1}).$$

Now we eliminate the current i in (38) to obtain a difference equation
for the voltage v by computing first the term

$$\frac{\Delta t}{C\Delta z}\,(i_{j+1/2}^{n+1/2} - i_{j-1/2}^{n+1/2}) = -\frac{1}{\left(1+\frac{R\Delta t}{2L}\right)}\left\{\left(1 - \frac{R\Delta t}{2L}\right)\left[\left(1 + \frac{G\Delta t}{2C}\right)v_j^n -\right.\right.$$

$$\left.\left. - \left(1 - \frac{G\Delta t}{2C}\right)v_j^{n-1}\right] + (v_{j+1}^n - 2v_j^n + v_{j-1}^n)\right\}$$

and substituting it into the first difference equation in (38) to obtain
a difference equation for the voltage

$$(39)\quad \left[1 + \frac{(GL+RC)\Delta t}{2LC} + \frac{RG(\Delta t)^2}{4LC}\right]v_j^{n+1} = -\left[1 - \frac{(GL+RC)\Delta t}{2LC} + \frac{RG(\Delta t)^2}{4LC}\right]v_j^{n-1}$$

$$+ (v_{j+1}^n + v_{j-1}^n) - \frac{RG(\Delta t)^2}{4LC}\,v_j^n$$

in which the scheme

$$(40)\qquad \Delta z = \frac{1}{\sqrt{LC}}\,\Delta t$$

has been used.

It can be verified easily that if we replace the second order telegraph equation

$$(41) \qquad LC \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial z^2} + (GL+RC) \frac{\partial v}{\partial t} + RG \quad v = 0$$

by the difference equation

$$(42) \quad LC \frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2} - \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta z^2} + (GL+RC) \frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t} +$$

$$+ \frac{RG}{2} (v_j^n + \frac{1}{2} v_j^{n+1} + \frac{1}{2} v_j^{n-1}) = 0$$

and use the scheme (40), then (39) will result.

It is seen that the voltage v in the last term RGv in (41) must be approximated by its value at three consecutive times in order that the difference equations for the first order (transmission line) and the second order (telegraph) partial differential equations are compatible. It was from here that we have identified the scheme for the Klein-Gordon equation as presented in (22).

E. The Source Term

We come finally to consider the effect of adding an integral

$$S(z,t) = \int_0^t K(t-\beta) E(z,\beta) d\beta$$

to the wave equation to produce the integro-differential equation of Eq. (3). It is always possible to reduce the integral to recursive form by splitting it into two parts

$$S(z,t) = \int_0^{t-\Delta t} K(t-\beta) E(z,\beta) d\beta + \int_{t-\Delta t}^t K(t-\beta) E(z,\beta) d\beta$$

and integrating the last integral from $t-\Delta t$ to t by the trapezoidal rule of integration. The result is given by

$$(43) \qquad S(z,t) = \int_0^{t-\Delta t} K[(t-\Delta t)-\beta+\Delta t] E(z,\beta) d\beta +$$

$$+ \frac{\Delta t}{2} [K(0)E(z,t)+K(\Delta t)E(z,t-\Delta t)].$$

Here the first term can always be written in terms of the functional $S(z,t-\Delta t)$.

So far, we have examined the numerical solution of (1) in a homogeneous medium by finite difference methods in detail. By recognizing the optimum scheme $\Delta z = c\Delta t$ for the wave equation, we have concluded that this scheme can be chosen for the lossy wave equation, the Klein-Gordon equation and the Telegraph equation. The corresponding difference equations for each of these cases have been given explicitly in (18), (24), and (39). The stability for each of these difference equations has also been analyzed. On the other hand, we have not been able to analyze the stability of a difference equation resulting from an integro-differential equation. Stability analysis in this case is not available at the present time. Under such circumstances, one has recourse to intuition and numerical evidence. Fortunately, in all our problems considered later, we have never experienced numerical instability of any kind.

The developments up to this point are strictly valid for homogeneous media. Their extensions to stratified media are now in order. In the next section we shall consider the wave equation and show how the propagation of an electromagnetic wave in an inhomogeneous dielectric medium can be solved by the difference method. The other applications will be followed up in subsequent chapters where some interesting transient problems in dispersive media will be discussed.

F. Difference Equation for a Stratified Dielectric Medium

To extend previous techniques to a stratified medium, it is worthwhile to recall again the remarkable property of the scheme

$$(44) \qquad \Delta z = c\Delta t$$

for the numerical solution of the wave equation by the finite difference method, which is that the resulting difference equation

$$(45) \qquad E_i^{n+1} = -E_i^{n-1} + E_{i+1}^n + E_{i-1}^n$$

generates an exact solution. Although this fact has been known for a long time[9,17] it was believed to be of little value for practical purposes because the analytic solution for the initial-value problem of the wave equation is very simple[22]. In 1970 Chiu[10] in the study of stress wave propagation in an elastic bar with discontinuities, made an important contribution by carrying this remarkable difference solution through an interface boundary for the first time. We shall see this can also be done for a multitude of interface boundaries.

The generalization of the scheme $\Delta z = c\Delta t$ to a problem involving an interface boundary is not difficult to visualize. Let us consider this situation as shown in Fig. 1. If we subdivided each region such that

(46)

$$\Delta z_1 = c_1 \Delta t, \quad \left(c_1 = \frac{1}{\sqrt{\mu_0 \varepsilon_1}} \right)$$

$$\Delta z_2 = c_2 \Delta t, \quad \left(c_2 = \frac{1}{\sqrt{\mu_0 \varepsilon_2}} \right)$$

the difference equation (45) then applies to all interior grid points except the interface grid point. In order to complete the solution, an appropriate difference equation for this interface grid point must be found. This can be accomplished by considering the following mathematical procedure.
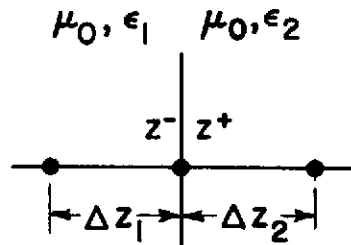
Fig. 1--Interface between two dielectric media.

Referring to Fig. 1, we see that the wave equation is defined everywhere within each region, hence it is valid in the vicinity of the interface boundary. Therefore, the wave equation at z^+ and z^- respectively are

(47)

$$\frac{\partial^2 E(z^+,t)}{\partial t^2} - c_2^2 \frac{\partial^2 E(z^+,t)}{\partial z^2} = 0,$$

$$\frac{\partial^2 E(z^-,t)}{\partial t^2} - c_1^2 \frac{\partial^2 E(z^-,t)}{\partial z^2} = 0.$$

At the interface boundary, the boundary conditions on the continuity of the tangential electric and magnetic fields, lead to the condition

$$E(z^+,t) = E(z^-,t),$$

(48)

$$\frac{\partial E(z^+,t)}{\partial z} = \frac{\partial E(z^-,t)}{\partial z},$$

the second of these being a consequence of the fact that the time derivative of the magnetic field is also continuous.

To derive a difference equation from the mathematical problem as posed in (47) and (48), we expand the fields at $z^++\Delta z_2$ and $z^--\Delta z_1$ respectively by the Taylor series

$$E(z^--\Delta z_1,t) = E(z^-,t) - \frac{\partial E(z^-,t)}{\partial z}\Delta z_1 + \frac{\partial^2 E(z^-,t)}{\partial z^2}\frac{\Delta z_1^2}{2}$$

(49)

$$E(z^++\Delta z_2,t) = E(z^+,t) + \frac{\partial E(z^+,t)}{\partial z}\Delta z_2 + \frac{\partial^2 E(z^+,t)}{\partial z^2}\frac{\Delta z_2^2}{2}$$

in which the higher order terms have been neglected. The second partial derivatives with respect to z in (49) are first substituted by (47) to obtain

$$E(z^--\Delta z_1,t) = E(z^-,t) - \frac{\partial E(z^-,t)}{\partial z}\Delta z_1 + \frac{\partial^2 E(z^-,t)}{\partial t^2}\frac{\Delta z_1^2}{2c_1^2},$$

(50)

$$E(z^++\Delta z_1,t) = E(z^+,t) + \frac{\partial E(z^+,t)}{\partial z}\Delta z_2 + \frac{\partial^2 E(z^+,t)}{\partial t^2}\frac{\Delta z_2^2}{2c_2^2}.$$

We eliminate the first derivatives by the use of the boundary conditions in (48) and then replace the second partial derivatives with respect to time t by the central finite differences

$$\frac{\partial^2 E(z^\pm,t)}{\partial t^2} = \frac{E(z^\pm,t+\Delta t)-2E(z^\pm,t)+E(z^\pm,t-\Delta t)}{\Delta t^2}.$$

Doing all these steps, we finally obtain a difference equation for the interface grid point

(51) $E(z^+,t+\Delta t) = - E(z^+,t-\Delta t) + \dfrac{2}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}\; [\sqrt{\varepsilon_1}\,E(z^--\Delta z_1,t) + \sqrt{\varepsilon_2}\,E(z^++\Delta z_2,t)].$

in which the condition (46) has been employed.

Since (51) is derived from a canonical problem, it applies to any interface boundary and hence a multiplicity of plane boundaries as well. Furthermore, it reduces to (45) when the dielectric medium at both sides of an interface grid point i is the same. Consequently, (51) can be used to obtain the time history of a signal in an inhomogeneous dielectric medium provided that the medium is approximated in a step-wise fashion.

To be explicit, the general difference equation for a multilayered lossless dielectric medium is given by

(52) $E_i^{n+1} = -E_i^{n-1} + \dfrac{2}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}\;(\sqrt{\varepsilon_1}\,E_{i-1}^{n} + \sqrt{\varepsilon_2}\,E_{i+1}^{n})$

where $\varepsilon_1, \varepsilon_2$ are the permittivities of the medium immediately to the left and to the right of the grid point i. Note that (52) must be used in conjunction with the modeling

(53) $\Delta z_i = \dfrac{1}{\sqrt{\mu_0 \varepsilon_i}}\; \Delta t$

for the thin layers. The subscript i denotes the ith thin layer and Δt is arbitrary. Because of the constraint in (53), exact modeling for the slab is possible only when Δt is extremely small. In practice, high precision is seldom required, and the approximation (53) produces results of adequate accuracy.

G. A Numerical Example -- Reflection of a Step-Modulated Sine Wave from an Inhomogeneous Slab

We illustrate the validity of (52) by the example shown in Fig. 2. Here a step-modulated sine wave is normally incident on an inhomogeneous dielectric slab with the profile

(54) $\varepsilon(z) = \dfrac{1}{2}\,(\varepsilon_1+\varepsilon_2) + \dfrac{1}{2}\,(\varepsilon_1-\varepsilon_2)\cos\dfrac{\pi z}{d}\;.$

The slab is first divided into thin layers according to (53), (52) is then used to produce the reflected waveform of the incident signal.

$$\epsilon(z) = \tfrac{1}{2}(\epsilon_1 + \epsilon_2) + \tfrac{1}{2}(\epsilon_1 - \epsilon_2)\cos\frac{\pi z}{d}$$
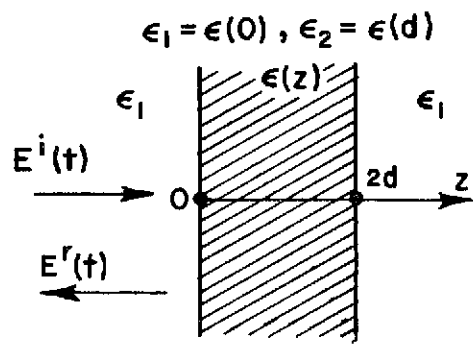
(a)
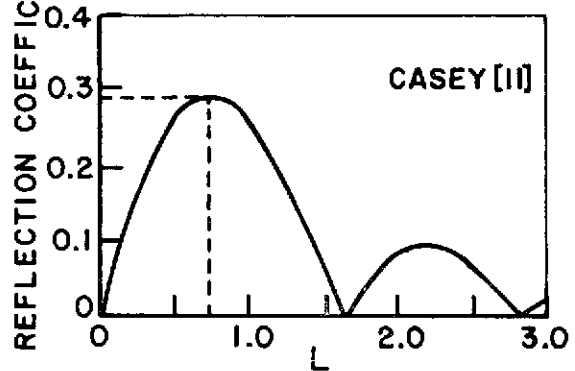
(b)
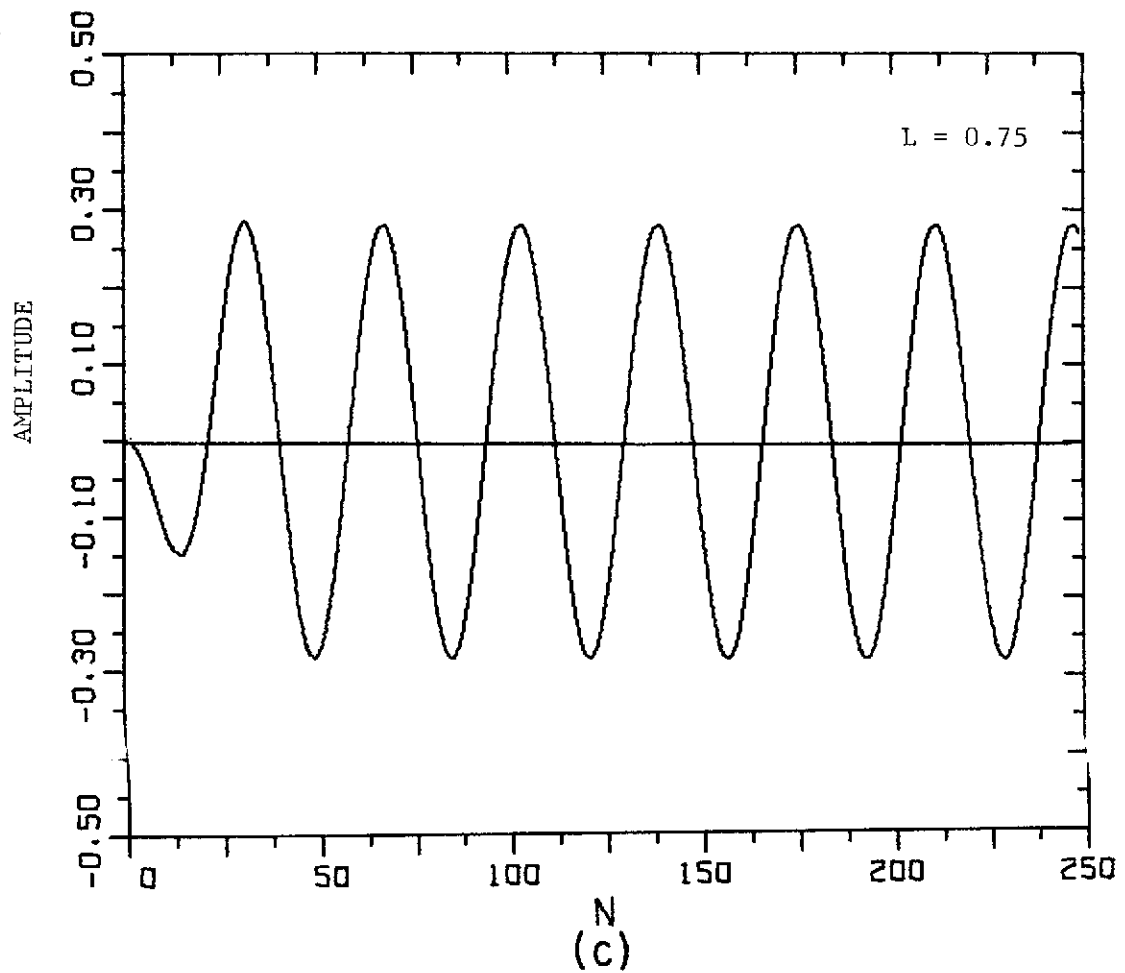
(c)

Fig. 2--Reflected CW wave from an inhomogeneous
dielectric slab versus N, $t=N\Delta t$, $\omega\Delta t=2\pi/36$.

$$L = \frac{2k_a d}{\pi}, \quad k_a^2 = \tfrac{1}{2}\omega^2\mu_0(\epsilon_1+\epsilon_2), \quad \frac{\epsilon_2}{\epsilon_1} = 2.$$

Figure 2c shows the real-time reflected waveform as observed at the air-dielectric interface boundary for the case of a normalized slab thickness L=0.75 versus time step N. The steady-state value of the reflection coefficient obtained by Casey[11] is shown in Fig. 2b. We see that the amplitude of the time domain solution at late time does agree with the frequency domain solution. According to Casey's results, there shall be no reflection of electromagnetic energy at steady-state when L=1.650 and 2.825. Indeed, Fig. 3 confirms this result. By varying the frequency of the incident signal and repeating the calculations for each case, the magnitude of the steady-state reflection coefficienct can be obtained. By comparing the steady-state waveform of the time domain solution with a reference signal, phase information can also be available.

It is interesting to see that the difficult boundary-value problem of plane electromagnetic wave propagation in an inhomogeneous dielectric medium[23,24] can be solved in this fashion. In a single calculation, the reflected field, the transmitted field, and even the fields in the medium are all available during both the transient period and at steady-state. However, only the case of normal incidence is considered here.

H. Some Practical Aspects of a Difference Solution

In this section we shall discuss various aspects of the numerical solution. They include the simulation of the infinite boundary, the moving time window calculation technique, proper treatment of initial conditions, numerical modelling, etc.

To illustrate these techniques we consider the problem of a short pulse incident on an inhomogeneous dielectric slab of finite thickness (see Fig. 4). This slab is divided into thin layers such that

$$(55) \qquad \Delta z_i = \frac{\Delta t}{\sqrt{\mu_0 \epsilon_i}}$$

where $\Delta z_i, \epsilon_i$; i=2, 3, 4, $\cdots$ are determined from $\epsilon(z)$.

A technique to accomplish this is to determine a temporary thickness

$$\Delta z_2' = \frac{\Delta t}{\sqrt{\mu_0 \epsilon(0)}}$$

and obtain the permittivity of layer 2 by

(a)

(b)
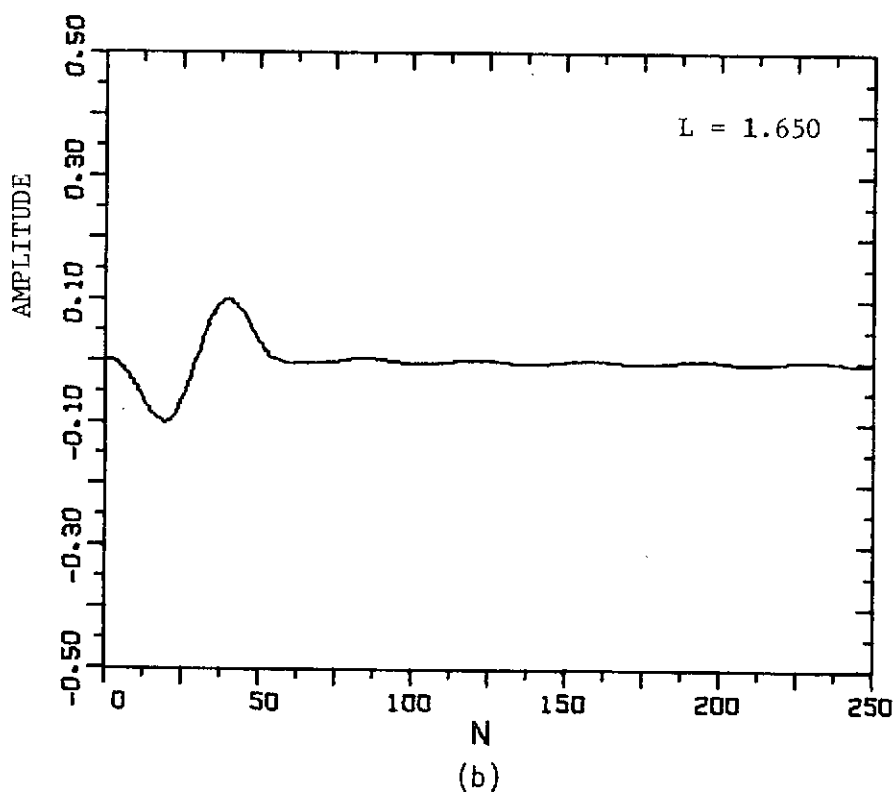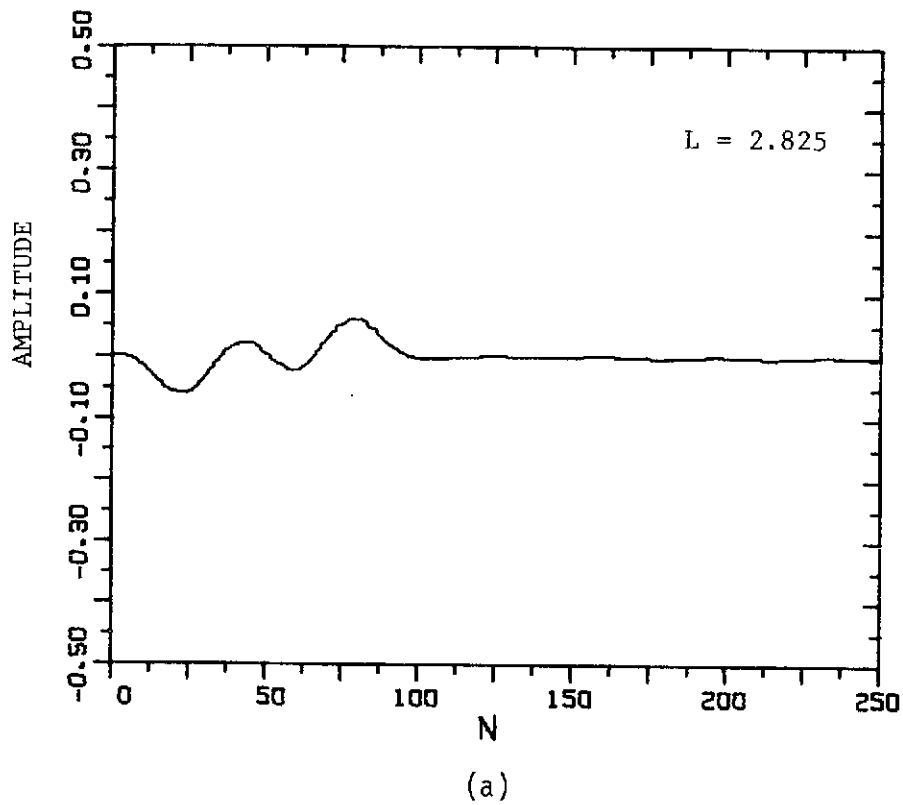
Fig. 3--Reflected CW wave from an inhomogeneous
dielectric slab versus N, t=NΔt. $\varepsilon_2/\varepsilon_1$=2.0.

(56) $\varepsilon_2 = \frac{1}{2} [\varepsilon(0) + \varepsilon(\Delta z_2')]$.

The actual thickness of layer 2 is then determined

$$\Delta z_2 = \frac{\Delta t}{\sqrt{\mu_0 \varepsilon_2}} \quad .$$

The above process is then repeated (considering $z=\Delta z_2$ as the initial guess for $\Delta z_3'$) to determine ε_3, Δz_3; ε_4, Δz_4 $\cdots$ and so on. In general, it is impossible to model the slab exactly unless t is extremely small. However such high precision is usually not required and the problem can always be solved approximately. Once the permittivity in each layer is determined, (52) can be used to generate the fields at all grid points.

Suppose the generator at z < 0 (see Fig. 4) is turned on at n= 0. As the time step marches on, the signal arrives at the air/dielectric boundary; reflection and transmission occur. The reflectes wave will eventually reach the send-end boundary at z<0. Physically this reflected signal could propagate through the boundary to the left and never enter the picture again. However, this is not the case numerically. The zero value of the short pulse at later times acts like a perfectly conducting boundary. So the reflected signal from the slab will appear to be re-reflected from the sending-end back toward the slab. Here we see the numerical solution generating a non-physical signal. This fact applies to the far-end boundary as well. Unless some kind of numerical technique is incorporated at both boundaries, a large number of grid points is needed to model the problem so as to prevent these non-physical signals from coming back to the observation point. Consequently the usefulness of the difference method will be severely limited.

For the far-end boundary, it turns out that these problems can be avoided in a simple way by terminating the infinite boundary at a point just outside the slab. The value at this grid point, rather than being computed from the difference equation, is assigned by

(57) $E_{IEND}^{n+1} = E_{IEND-1}^{n}$.

This follows from the fact that a signal at the point E_{IEND-1}^{n} will propagate without distortion to E_{IEND}^{n+1} in free space. Hence (57) simulates the infinite boundary exactly in this case. A similar technique exists for sending-end boundary. If we put the generator at i=1 and the air/dielectric interface at i=2, then when we compute the fields at d (see Fig. 4) from (52)

(58) $d = -b + \dfrac{2}{\sqrt{\varepsilon_0} + \sqrt{\varepsilon_2}} (\sqrt{\varepsilon_0}\, E_{i-1}^{n} + \sqrt{\varepsilon_2}\, e)$
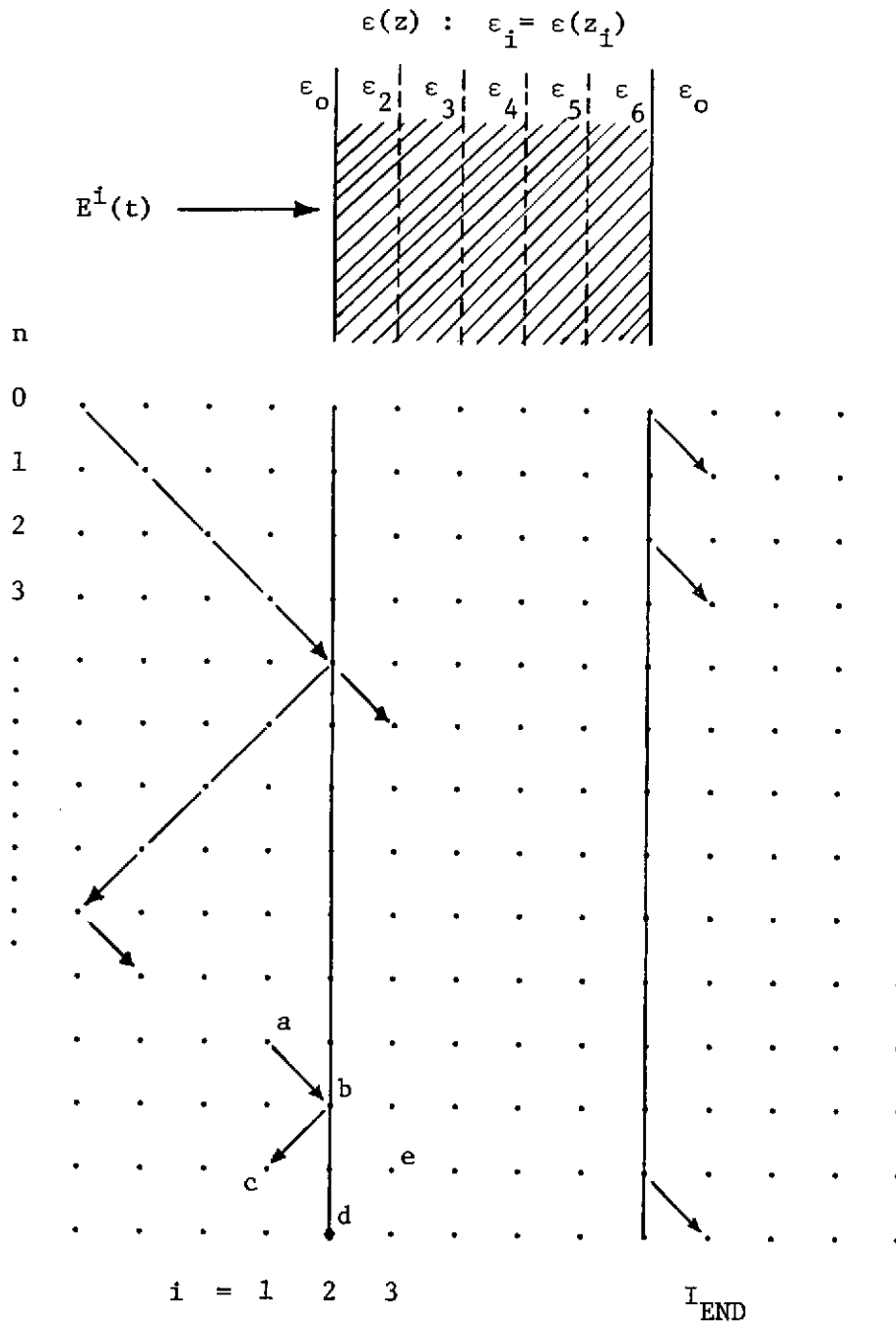
Fig. 4--A signal normally incident on an inhomogeneous dielectric slab in space-time grid points. $z = i\Delta z$, $t = n\Delta t$.

It is important to interpret E_{i-1}^n correctly. Physically, the value at each grid point represents the total field at that point. Therefore, the quantity E_{i-1}^n has two components: an incident field to d generated at c, and a reflected field from b. Thus,

$$E_{i-1}^n = \text{incident field} + \text{reflected field from b,}$$

(59)

$$= c + (b-a)$$

Once the value of d is determined, the reflected wave can be picked up at i=2 by

(60) $$E^r(n\ t) = d-c$$

In this way all the unnecessary calculation in the free space has been eliminated and generation of a non-physical signal has also been avoided. It is worthwhile to mention that all these techniques are possible because the scheme $\Delta z = c\Delta t$ has been selected in the difference solution.

When the far-end boundary is other than free space, one can always terminate the infinite boundary at the point

(61) $$I_{END} = I_{obs} + \frac{(N-I_{obs})}{2} + 1$$ (I_{obs}=observation point, N=number of time steps to be calculated)

without affecting the numerical solution. When the initial field is impressed at z=0, the solution is straightforward and no special technique is required at the sending end.

The above discussion concentrates on the initial conditions prescribed in time at z=0 or normally incident from z 0. When the initial conditions are prescribed in space, i.e.,

(62)
$$E(z,0) = f(z)$$

$$\frac{\partial E(z,0)}{\partial t} = g(z)$$

then it is important to model the information at the time step Δt correctly. Carelessness here may sometimes lead to unexpected results. One example is shown in Fig. 5. Here f(z) is assumed to be a gaussian short pulse (see N=0) and g(z) is zero. It appears therefore that (62) for this case can be replaced by
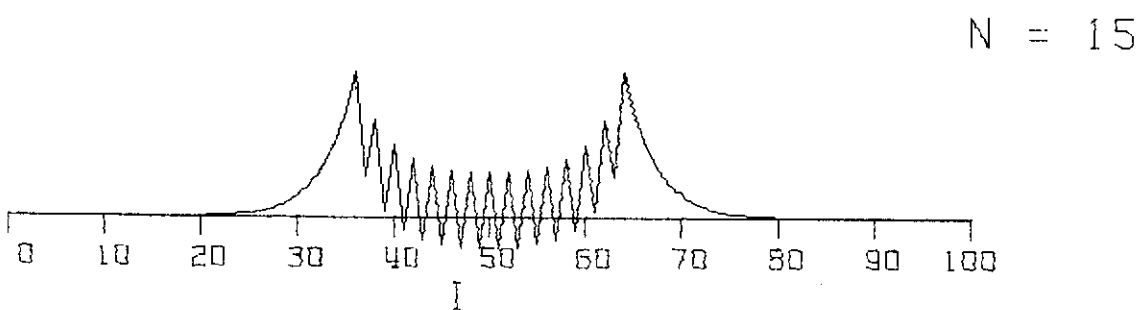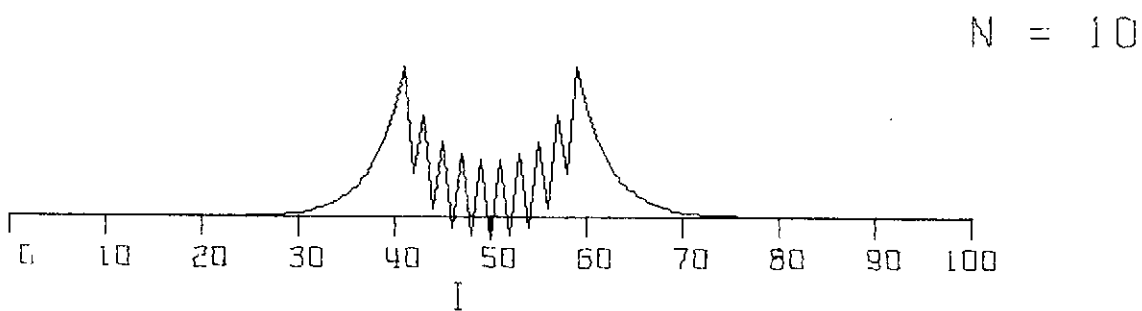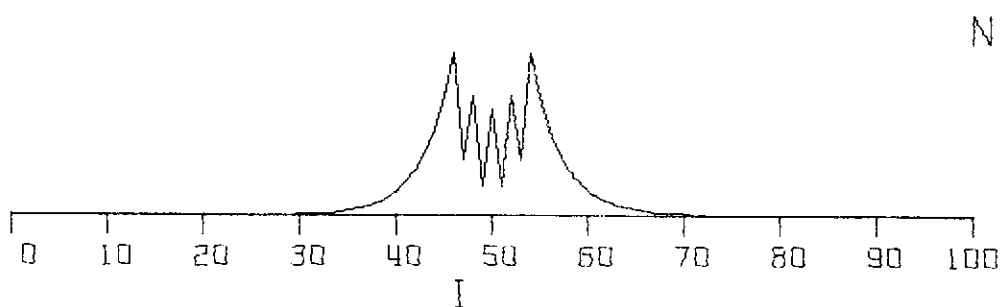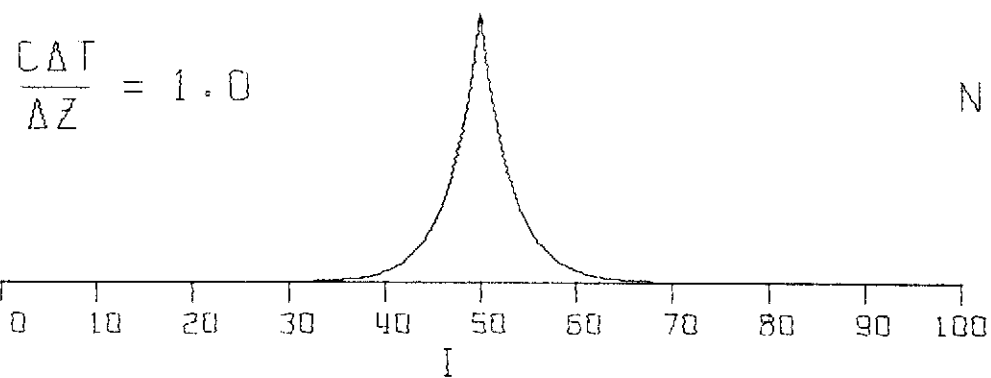
Fig. 5--Spatial variation of a Gaussian pulse,
with the initial data $E_I^1 = E_I^0$, t=NΔt, z=IΔz.

$$E_i^0 = f_i,$$

(63) $E_i^1 = E_i^0$ (i.e., $\frac{\partial E(z,0)}{\Delta t} = \frac{E(z,\Delta t) - E(z,0)}{\Delta t}$)

However the numerical solution is quite strange as can be seen in the figure. One might attribute the numerical instability to the numerical solution itself, i.e., an unstable difference solution. Since we know that the numerical solution should produce an exact solution in this case as mentioned previously, there must be some other source of error. The truth is that the initial data E_i^1 in (63) is incorrect, because it implies the field at Δt is exactly identical to the fields at t =0. This is not physical. Consequently, a correct solution should not be expected.

The correct modelling for this case is to employ the central finite difference[17, pp. 231]

(64) $\frac{\partial E(z,0)}{\partial t} = \frac{E(z,-\Delta t)-E(z,\Delta t)}{2\Delta t} = 0$

or

$$E(z,-\Delta t) = E(z,\Delta t).$$

Employing the difference equation (7) to Δt

$$E(z,\Delta t) = - E(z,-\Delta t) + E(z+\Delta z,0) + E(z-\Delta z,0),$$

we can eliminate $E(z,-\Delta t)$ to obtain

(65) $E(z,\Delta t) = \frac{1}{2} [E(z+\Delta z,0) + E(z-\Delta z,0)]$

which is the corrected initial data at Δt for this case. In general, one should always ask, if the initial waveform is given at t=0, what would it be at Δt physically? By recourse to this kind of physical reasoning, unexpected solutions may be avoided.

Finally it is always desirable and possible to incorporate the moving time window calculation technique to minimize unnecessary calculations in a difference solution. This can be carried out from a knowledge of the domain of dependence for the observation point.

A computer program which is designed for Casey's problem[11] is presented in Appendix A (Computer Program 1). Here the techniques discussed above are worked out in detail.

CHAPTER III
A DIFFERENCE EQUATION FOR STRATIFIED LOSSY DIELECTRICS

In this chapter we shall extend the techniques developed earlier to the case of a stratified lossy dielectric medium. A difference equation appropriate for an interface grid point is derived, as done for the wave equation in the preceding chapter. The resulting difference equation is employed to solve three transient problems which were considered by other authors using standard transform methods. They include: a unit-step signal propagating in a homogeneous lossy dielectric, the transmission of a normally incident gaussian pulse from air to a conducting earth; and the reflected waveform of a sine-squared pulse from a three-layered lossy medium. In all cases normal incidence is assumed.

It has been known for some time that the reflection of short pulses from stratified media offers a diagnostic tool in geophysics [25]. For example, Sivaprasad and Stotz[26] recently in a theoretical investigation concluded that the detection of water layers in dry earth at depths up to approximtely 10 meters is feasible. From another view point, the shielding property of the earth for electro-magnetic pulses has been of concern[27].

It will be shown that the solution of these transient problems can be easily obtained from a simple difference equation developed in this chapter.

A. A Difference Equation

It has been suggested in Section II.B that the lossy wave equation in a homogeneous medium[22]

$$
(66) \qquad \mu\varepsilon \frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial z^2} + \mu\sigma \frac{\partial E}{\partial t} = 0
$$

can be approximated by the difference equation

$$
(67) \qquad \mu\varepsilon \frac{E_i^{n+1}-2E_i^n+E_i^{n-1}}{\Delta t^2} - \frac{E_{i+1}^n-2E_i^n+E_{i-1}^n}{\Delta z^2} + \mu\sigma \frac{E_i^{n+1}-E_i^{n-1}}{2\Delta t} = 0
$$

When the scheme

$$
(68) \qquad \Delta z = c\Delta t
$$

is selected, (67) takes the simple form

$$(69) \qquad E_i^{n+1} = \frac{1}{1 + \frac{\sigma \Delta t}{2\varepsilon}} \left[-\left(1 - \frac{\sigma \Delta t}{2\varepsilon}\right) E_i^{n-1} + E_{i+1}^n + E_{i-1}^n \right]$$

which is then the difference equation for a __homogeneous__ medium.

To extend it to the case of a stratified medium, let us consider an interface boundary between two lossy dielectric media as shown in Fig. 6. Within each thin layer, the lossy wave equation is valid everywhere, hence it is true in the vicinity of the interface boundary. We write (66) at z^-, z^+ respectively

$$(70) \qquad \begin{aligned} \mu_1 \varepsilon_1 \frac{\partial^2 E(z^-,t)}{\partial t^2} - \frac{\partial^2 E(z^-,t)}{\partial z^2} + \mu_1 \sigma_1 \frac{\partial E(z^-,t)}{\partial t} &= 0, \\[2mm] \mu_2 \varepsilon_2 \frac{\partial^2 E(z^+,t)}{\partial t^2} - \frac{\partial^2 E(z^+,t)}{\partial z^2} + \mu_2 \sigma_2 \frac{\partial E(z^+,t)}{\partial t} &= 0. \end{aligned}$$
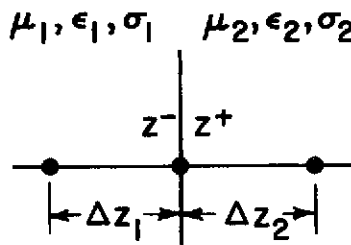
Fig. 6--Interface between two lossy dielectric media.

At the interface the boundary conditions are

$$(71) \qquad \begin{aligned} E(z^+,t) &= E(z^-,t), \\[2mm] \frac{1}{\mu_2} \frac{\partial E(z^+,t)}{\partial z} &= \frac{1}{\mu_1} \frac{\partial E(z^-,t)}{\partial z} \quad . \end{aligned}$$

To derive a difference equation for the mathematical problem as posed in (70) and (71), we expand the fields at $z^+ + \Delta z_2$ and $z^- - \Delta z_1$ respectively by the Taylor series

$$E(z^{+}+\Delta z_2) = E(z^{+},t) + \frac{\partial E(z^{+},t)}{\partial z} \Delta z_2 + \frac{\partial^2 E(z^{+},t)}{\partial z^2} \frac{(\Delta z_2)^2}{2}$$

(72)

$$E(z^{-}-\Delta z_1) = E(z^{-},t) - \frac{\partial E(z^{-},t)}{\partial z} \Delta z_1 + \frac{\partial^2 E(z^{-},t)}{\partial z^2} \frac{(\Delta z_1)^2}{2} \ .$$

The second partial derivatives with respect to z in the above expressions are first replaced by (70), the first partial derivatives with respect to z are eliminated by the boundary conditions in (71), and finally the second and first partial derivatives with respect to t are approximated by

(73) $\qquad \dfrac{\partial^2 E(z^{\pm},t)}{\partial t^2} = \dfrac{E(z^{\pm},t+\Delta t)-2E(z^{\pm},t)+E(z^{\pm},t-\Delta t)}{\Delta t^2}$,

(74) $\qquad \dfrac{\partial E(z^{\pm},t)}{\partial t} = \dfrac{E(z^{\pm},t+\Delta t) - E(z^{\pm},t-\Delta t)}{2\Delta t}$.

Doing all these steps, it can be shown that the resulting difference equation is given by

(75) $\qquad E_i^{n+1} = \dfrac{1}{\alpha} (-\beta\ E_i^{n-1} + \gamma E_{i+1}^n + E_{i-1}^n)$

where

$$\gamma = \frac{\mu_1 \sqrt{\mu_2 \epsilon_2}}{\mu_2 \sqrt{\mu_1 \epsilon_1}} \ ,$$

$$\alpha = \frac{1}{2} \left(1 + \frac{\sigma_1 \Delta t}{2\epsilon_1}\right) + \frac{\gamma}{2} \left(1 + \frac{\sigma_2 \Delta t}{2\epsilon_2}\right)$$

$$\beta = \frac{1}{2} \left(1 - \frac{\sigma_1 \Delta t}{2\epsilon_1}\right) + \frac{\gamma}{2} \left(1 - \frac{\sigma_2 \Delta t}{2\epsilon_2}\right)$$

and

$$\Delta z_1 = \frac{1}{\sqrt{\mu_1 \epsilon_1}} \Delta t,$$

$$\Delta z_2 = \frac{1}{\sqrt{\mu_2 \varepsilon_2}} \Delta t \; .$$

Note that μ_1, ε_1, σ_1 and μ_2, ε_2, σ_2 are the constant values of the permeability, permittivity, conductivity of the thin layers to the left and to the right of the interface grid point i respectively.

We see that (75) reduces to (69) when the electrical properties of the layers at both sides of an interface boundary are the same. It also simplifies to (52) when the inhomogeneous dielectric is loss-less, consequently, (75) can be used to obtain the transient response of a signal in an inhomogeneous lossy dielectric medium provided that the medium is approximated by many thin layers according to

$$(76) \qquad \Delta z_i = \frac{1}{\sqrt{\mu_i \varepsilon_i}} \Delta t$$

where the subscript i denotes the ith layer. In practice it is recommended that Δt be chosen such that $\sigma \Delta t / 2\varepsilon < 0.1$ for accurate results.

B. Numerical Examples

Three numerical examples are presented to illustrate the applications of the difference equations in (75).

(1) Unit-step signal propagating in a homogeneous lossy medium

We assume a unit-step signal is impressed at z=0, and it is desired to receive the signal at z=0.3, 0.6, and 1.05 meter in a medium with $\varepsilon = 16\varepsilon_0$ and $\sigma = 0.02$ mhos/meter. With discrete time interval $\Delta t = 1$ nanosecond and $\Delta z = 0.075$ meter, the waveform received at the corresponding observation point I = 5, 9, 15 are shown in Fig. 7. These results agree with those computed by Fuller and Wait[28] by analytical transform techniques.

(2) Transmission of a gaussian pulse from the air into the earth

This problem was originally considered by King and Harrison[27] in order to determine the shielding properties of the earth. They solved the problem by Fourier transform method. Here we obtain the time history of the pulse directly in the time domain. This approach, perhaps, is more efficient because a long excursion in the frequency domain has been avoided. Figure 8 shows the waveforms of the trans-mitted pulse at depths 0, 10, 25 meters below the surface of the earth
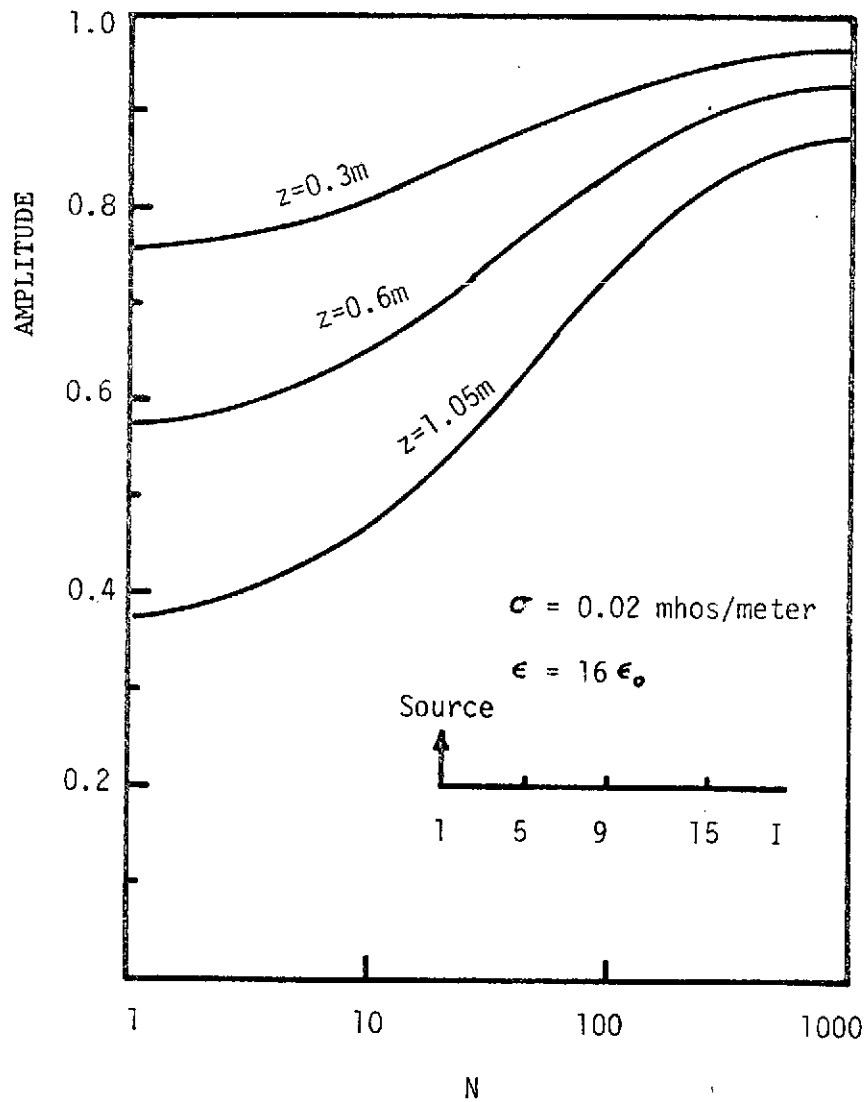
Fig. 7--Step function response for several spatial
observation points. t=NΔt, z=IΔz; Δt=1.0
nanosecond, Δz=0.075 meters.
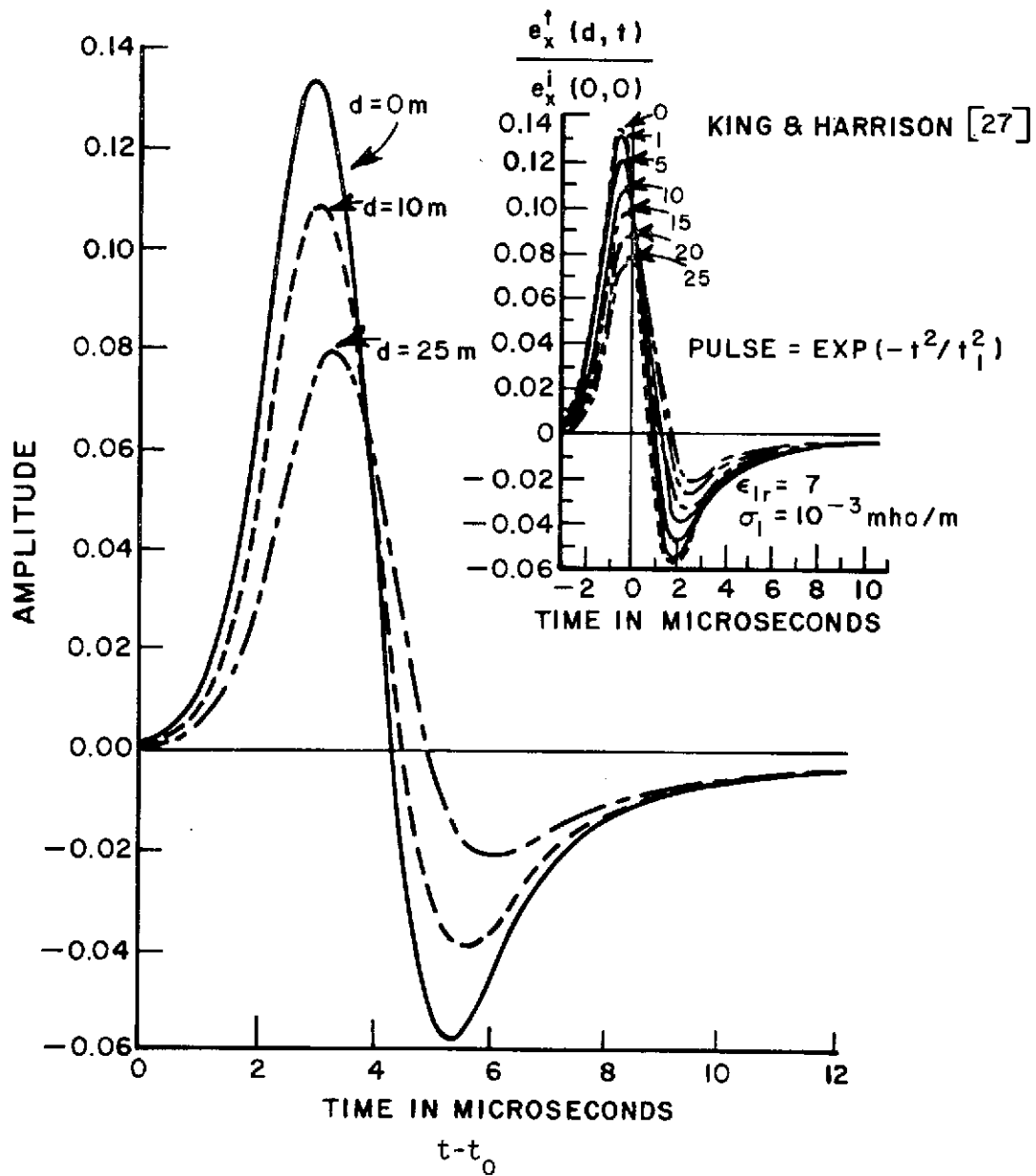
Fig. 8--Transmitted part of a Gaussian pulse received at
depth d in dry earth. t_0 is the signal front
arrival time; $t_1 = 1.4142$ microsecond.

for the case of dry earth ($\varepsilon = 7\varepsilon_0$, $\sigma = 0.001$ mhos/m). The results obtained by King and Harrison are also shown. It is seen that the comparison of the results obtained by the two approaches is excellent.

(3) Reflection of a sine-squared pulse from a three-layered lossy dielectric medium

We finally come to consider a three-layered stratified medium. Here a sine-squared pulse is normally incident on a layered earth ($\varepsilon = 7\varepsilon_0$, $\sigma = 0.001$ mhos/m) of thickness d with water ($\varepsilon = 81\varepsilon_0$, $\sigma = 0.001$) as the bed rock, and the reflected waveform of the diagnostic pulse is of interest. Figure 9 shows the results as obtained from a difference solution for several thicknesses of the earth layer. Our results again compare favorably with those obtained by Sivaprasad and Stotz[26] using the Fourier transform methods. It is interesting to mention in passing that the difference solution used about 1 second in computer CPU (central processing unit) time while the frequency domain approach requires about 20 seconds. This is not surprising because the latter requires responses at many frequencies to construct the time domain results, while the former involves only repetitive additions and multiplications, which are particularly suited for a digital computer.

In summary, an explicit difference equation for application to a stratified lossy dielectric medium has been presented. Its validity has been demonstrated by three examples. A computer program which is designed for the last example is presented in Computer Program 2 in Appendix A; little change is needed to produce the numerical r.sults for the other two examples.

One final comment is that the electrical properties of dielectrics are always frequency-dependent. The propagation of a short pulse in a medium with constant values of permittivity, permeability and conductivity is thus a convenient rather than a realistic model. Nevertheless, when the electrical properties of the medium do not vary significantly over the frequency spectrum of the pulse, then this model is a good approximation to the physical problem, which is often mathematically intractable by transform methods. One can also employ the difference equation to obtain the steady-state response for an inhomogeneous lossy dielectric medium by marching on in time long enough to allow the steady-state to be established. This has been illustrated in an example in the previous chapter.
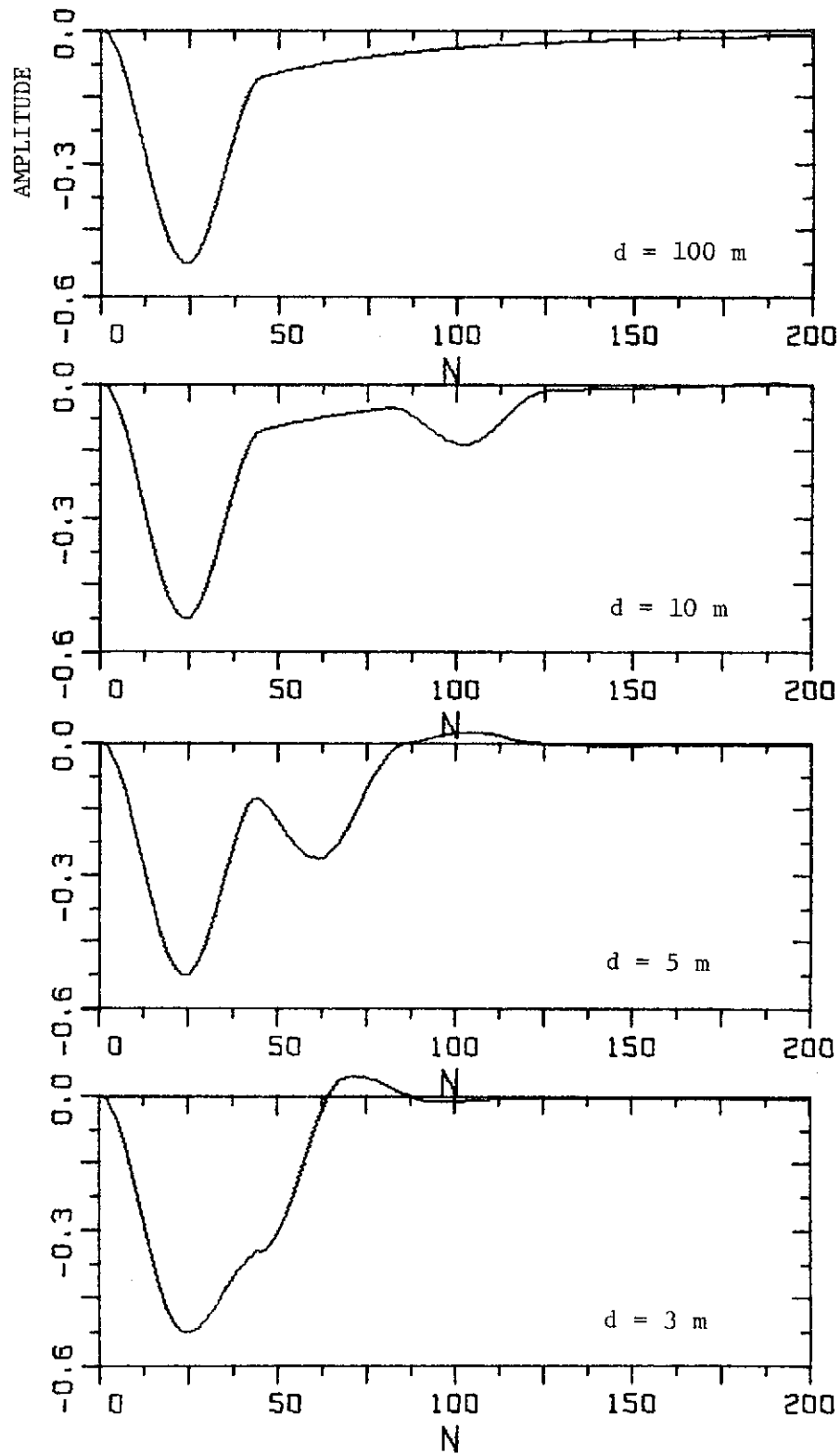
Fig. 9--Reflected waveform of a sine-square pulse normally
incident on earth of thickness d with water as
the bed rock. t=NΔt, Δt=2.2 nanoseconds.

CHAPTER IV
A NUMERICAL SOLUTION TO SOMMERFELD'S "ABOUT THE PROPAGATION OF LIGHT IN DISPERSIVE MEDIA"

A. Introduction

One of the famous problems considered by Sommerfeld is a step-modulated sinusoidal signal propagating in a dispersive medium[1]. When Einstein in 1905 postulated that a signal can never be transmitted with a velocity greater than c, the speed of light in free space, a controversy was raised among the physicists at that time[2] because according to Stratton[22], it was then generally believed that the group velocity was necessarily equivalent to the velocity of energy propagation; but it had also been known that in the absorption region the group velocity can be greater than c. This apparent contradiction was clarified by Sommerfeld who in 1914 used an elegant mathematical analysis to show that the front of the signal propagates exactly with c regardless of what group the velocity might be. Thus at a controversial time, this result conformed with one of the fundamental postulates in Einstein's special theory of relativity. Subsequently, Brillouin[29] extended the analysis and gave a complete picture of the waveform of the precursor. He concluded that the main signal may be considered as arriving with the appropriate group velocity in the normally dispersive region, but that the group velocity loses its significance in the absorption region.

In this chapter we shall present an extremely simple difference equation for the solution of this historically famous problem. The difference equation is derived from an integro-differential equation for the electric field by a finite difference method. The solution is then generated in a recursive manner in time. Employing the difference equation, we have been able to simulate the propagation process in real-time on a high-speed digital computer equipped with a CRT display facility. The dynamic behavior of the signal in a dispersive medium thus can be seen. In fact, we have observed vividly the formation of the precursor as the signal traverses the medium. We shall describe some of these observations and discuss briefly the numerical method. Extensive results will be presented to give a clear picture of the whole shape of the signal.

B. An Integro-differential Equation for the Electric Field

Consider an x-polarized plane electromagnetic wave propagating in the z direction in a dispersive medium. The governing equations for the propagation process are the Maxwell's equations[30]

$$\varepsilon_o \frac{\partial E_x}{\partial t} + \frac{\partial H_y}{\partial z} + Ne \frac{\partial s}{\partial t} = 0,$$

(77)

$$\mu_o \frac{\partial H_y}{\partial t} + \frac{\partial E_x}{\partial z} = 0$$

and the equation of motion for the electrons

$$(78) \qquad \frac{d^2 s}{dt^2} + \nu \frac{ds}{dt} + \omega_o^2 s = \frac{e}{m} E_x$$

where s represents the displacements of the electrons from their equilibrium positions. Here ν, ω_0 are the electron collision frequency and the electron characteristic frequency respectively.

Assuming the electrons are initially undisturbed, i.e., s, ds/dt = 0 at t = 0, the solution of (78) is given by [47]

$$(79) \qquad s(t) = \frac{e}{m} \frac{1}{h} \int_0^t E_x(z,\beta)\sin h(t-\beta) \exp[-\rho(t-\beta)]d\beta$$

where $h = \sqrt{\omega_0^2 - \rho^2}$ and $\rho = \nu/2$. Eliminating the magnetic field H_y in (77), the use of (79) yields an integro-differential equation for the electric field

$$(80) \qquad \frac{\partial^2 E_x}{\partial t^2} - c^2 \frac{\partial^2 E_x}{\partial z^2} + \omega_p^2 E_x(z,t) = 2\omega_p^2 \rho \int_0^t a(t-\beta)E_x(z,\beta)d\beta +$$

$$+ \omega_p^2 \frac{(h^2-\rho^2)}{h} \int_0^t b(t-\beta)E_x(z,\beta)d\beta$$

where

$$(81) \qquad a(t-\beta) = \cos h(t-\beta) \exp[-\rho(t-\beta)],$$

$$b(t-\beta) = \sin h(t-\beta) \exp[-\rho(t-\beta)]$$

and $c = 1/\sqrt{\mu_0 \varepsilon_0}$, $\omega_p^2 = Ne^2/m\varepsilon_0$, c is the speed of light in free space, ωp is the plasma frequency. The subscript x in E_x will be dropped hereafter.

C. The Difference Equation

The numerical solution for this type of integro-differential equations has been discussed in Chapter II and is briefly treated here. The numerical method essentially replaces the second partial derivatives by the central finite differences

$$\frac{\partial^2 E}{\partial z^2} = \frac{E(z+\Delta z,t)-2E(z,t)+E(z-\Delta z,t)}{\Delta z^2} \quad,$$

(82)

$$\frac{\partial^2 E}{\partial t^2} = \frac{E(z,t+\Delta t) - 2E(z,t) + E(z,t-\Delta t)}{\Delta t^2} \quad.$$

To ensure a stable solution, it has been demonstrated in Section II.C that the electric field in the third term in (89) has to be approximated by the scheme

$$(83) \qquad E(z,t) = \frac{1}{2} [E(z,t) + \frac{1}{2} E(z,t+\Delta t) + \frac{1}{2} E(z,t-\Delta t)].$$

Finally, the integrals in the right hand side of (80) can be written as

$$A(z,t) = \frac{1}{\Delta t} \left(\int_0^{t-\Delta t} + \int_{t-\Delta t}^{t} \right) \cos h(t-\beta) \exp[-\rho(t-\beta)] E(z,\beta)d\beta,$$

$$B(z,t) = \frac{1}{\Delta t} \left(\int_0^{t-\Delta t} + \int_{t-\Delta t}^{t} \right) \sin h(t-\beta) \exp[-\rho(t-\beta)] E(z,\beta)d\beta \quad.$$

Evaluating the integration from $t-\Delta t$ to t by the trapezoidal rule of integration yields the recursive relations

$$(84) \qquad \begin{aligned} A(z,t) &= g[KA(z,t-\Delta t)+SB(z,t-\Delta t)] + \frac{1}{2} S\, E(z,t-\Delta t), \\ B(z,t) &= g[KB(z,t-\Delta t)-SA(z,t-\Delta t)] + \frac{1}{2} g\, KE(z,t-\Delta t) + \frac{1}{2} E(z,t) \end{aligned}$$

where g=exp[-ρΔt], K=cos(hΔt), S=sin(hΔt). Combining the results of (82)-(84) and employing the notation $E_i^n = E(z,t)_{z=iΔz}^{t=nΔt}$, (80) is readily transformed into the difference equations

(85) $\qquad E_i^{n+1} = -E_i^{n-1} + \dfrac{1}{1+\frac{1}{4}\tau^2} [E_{i+1}^n + E_{i-1}^n - \dfrac{\tau^2}{2} E_i^n + k_1 A_i^n + k_2 B_i^n]$

where

$\qquad A_i^n = g[KA_i^{n-1} + SB_i^{n-1}] + \dfrac{1}{2} SE_i^{n-1}$,

$\qquad B_i^n = g[KB_i^{n-1} - SA_i^{n-1}] + \dfrac{1}{2} gKE_i^{n-1} + \dfrac{1}{2} E_i^n$

and

$\qquad \tau = \omega_p \Delta t, \quad k_1 = 2\tau^2(\rho\Delta t), \quad k_2 = \dfrac{1}{(h\Delta t)} \tau^2 [(h\Delta t)^2 - (\rho\Delta t)^2]$.

Note that the scheme

(86) $\qquad z = c\Delta t, \quad c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$

has been used. In practice, it is recommended that the time increment Δt be chosen such that $\omega_p \Delta t \leq 0.1$.

When initial conditions are given, (85) can be used to obtain the time history of a signal of arbitrary waveform in the dispersive medium by marching on in time.

D. Numerical Results

It is convenient to summarize Sommerfeld and Brillouin's results by considering a simulated experiment which we shall do shortly. A sinusoidal signal generator is located at Z=0; a detector of infinite sensitivity is set up a distance Z from the origin in the dispersive medium to detect the signal. For convenience we normalize the space and time variables by $Z = \omega_p z/c$ and $T = \omega_p t$. At time T=0 the generator is turned on. Then according to Sommerfeld and Brillouin[2],

1. After a period of time T=Z (i.e., t=z/c), the front of the signal should be detected. The amplitude of the signal received immediately after is very small and can be described by

$$(87) \qquad f(Z,T) = \frac{\omega}{\omega_p}\sqrt{\frac{2(T-Z)}{Z}} \; J_1(\sqrt{2Z(T-Z)}) \; u(T-Z)$$

This is the first precursor and is valid only for small value of 2(T-Z)/Z. u(T-Z) is the unit-step function.

2. At time $T=Z\sqrt{1 + (\omega_p/\omega_0)^2}$ the second precursor arrives.

3. At time $T_g = Zc/v_g$ (i.e., $t = z/v_g$, v_g is the group velocity) the detector should detect an intensity equal to 1/4 the final intensity of the signal, which is true in the normally dispersive region.

We shall attempt to verify these theoretical results by performing a numerical simulated experiment by the use of the difference equation.

Instead of using a single detector, we set up four detectors at Z=0, 20, 30, and 40 respectively. The dispersive medium is assumed to be lossless and specifically $\omega_0/\omega_p = 1$. Later, loss will be included.

Figure 10 shows the waveforms of the signal received at four detectors for the case $\omega=0.2\omega_0$. It is seen that the signal is not distorted. This can be easily explained by considering the dispersion relation (assuming $e^{j\omega t}$)

$$(88) \qquad k = \frac{\omega}{c}\sqrt{1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2}} \sim \sqrt{2}\,\frac{\omega}{c} \quad \text{(for } \omega_p/\omega_0 = 1 \text{ and } \omega/\omega_0 = 0.2)$$

which indicates that the medium is essentially non-dispersive for low frequency signals. The signal velocity thus can be defined clearly and is a little less than 0.7c. We can verify this velocity "experimentally" (in a simulation sense) by measuring the time T for the first peak of the signal and define without ambiguity the signal velocity

$$v_s = \frac{Z}{T - T_0} \; c$$

where T_0 is the time when the generator reaches its first peak. Thus we have in this case

Fig. 10--Received waveform at z of a step-modulated
sinusoidal signal in a dispersive medium.
$\omega = 0.2\omega_0$, $\omega_0 = \omega_p$, T=0.1N.

$$v_s(Z=20) = \frac{20}{37.2 - 7.8} c = 0.680c,$$

$$v_s(Z=30) = \frac{30}{51.7 - 7.8} = 0.683c,$$

$$v_s(Z=40) = \frac{40}{66 - 7.8} = 0.687c$$

which agree with the theoretical result. In addition, one may also notice a minor disturbance in front of the main signal which is actually the precursor. It begins exactly at T=Z and thus confirms that the signal front propagates exactly with the speed of light in free space. We shall prove this later.

Perhaps, a more clear picture of the precursor can be seen by increasing the frequency to $\omega=0.5\omega_0$ as shown in Fig. 11. Here a larger amplitude of the precursor is obtained as predicted by (87), but the signal no longer propagates without distortion. The leading portion of the signal loses its amplitude as it is traveling in the medium. However the waveform received at late times is undistorted. A little arrow has also been added on the figure to indicate where the second precursor should occur as claimed by Brillouin. However it cannot be identified as distinctively as the first precursor.

The actual propagation process in space for this case is shown in Fig. 12. As the signal front is traveling to the right in the medium, its leading edge breaks up into little pulses. The deeper the signal travels, the more the numbers of pulses are formed, and the weaker is the amplitude on the leading peaks of the main signal. If a detector is located very far from the generator, one should receive more pulses in the precursor. Figure 13 shows this for a detector at Z=100.

We now return to continue the experiment and increase the frequency to $\omega = \omega_0$. The received signals are shown in Fig. 14. Here the waveform is distorted to such a degree that it has nothing in common with the original signal as it was launched (i.e., at Z=0). Consequently, there will not be a suitable definition for the velocity of propagation in this case. But one thing is clear: the front of the signal always arrives with a speed equal to c.
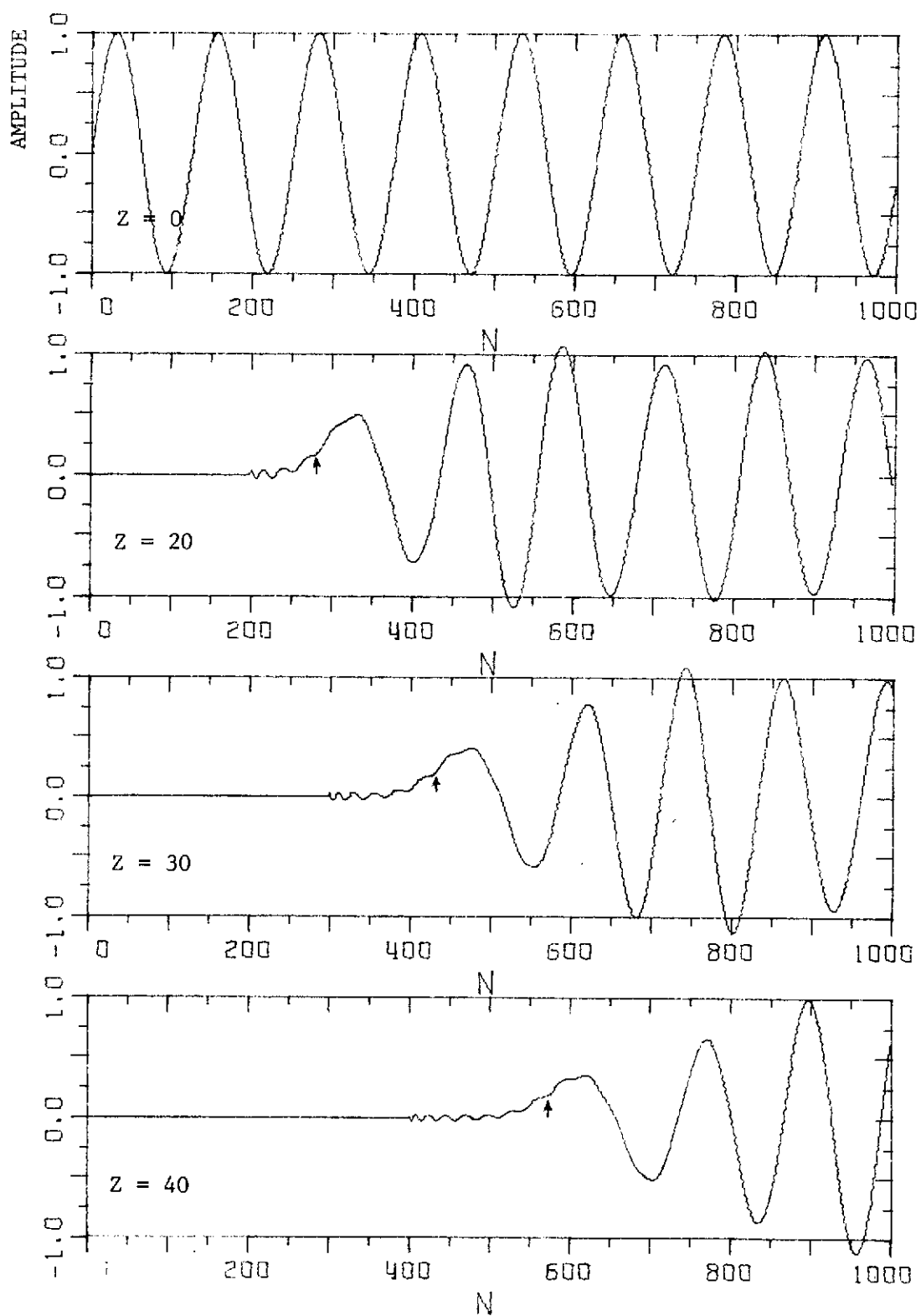
Fig. 11--Received waveform at Z of a step-modulated signal in a
dispersive medium. $\omega=.5\omega_0$, $\omega_0=\omega_p$, T=0.1N. Arrow
indicates normal position of the second precursor.

T = 5

T = 25

Z

0   20   40

T = 10

T = 30

T = 15

T = 35

T = 20

T = 40

T = 25

T = 45
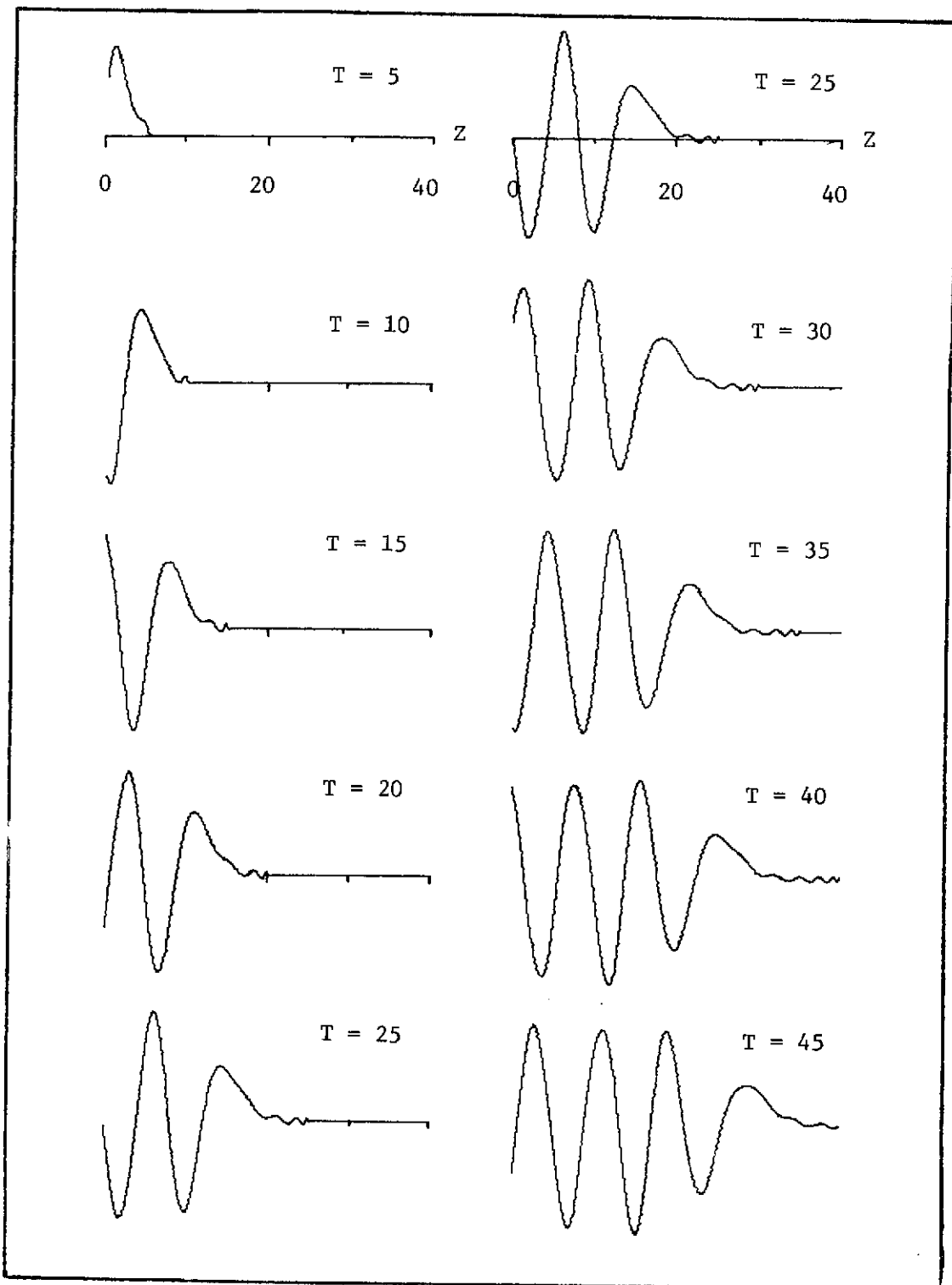
Fig. 12--Spatial variation of a step-modulated sinusoidal signal
in a dispersive medium at selected times.
$\omega = 0.5\omega_0$, $\omega_0 = \omega_p$.

Fig. 13--Received waveform at Z=100 of a step-modulated
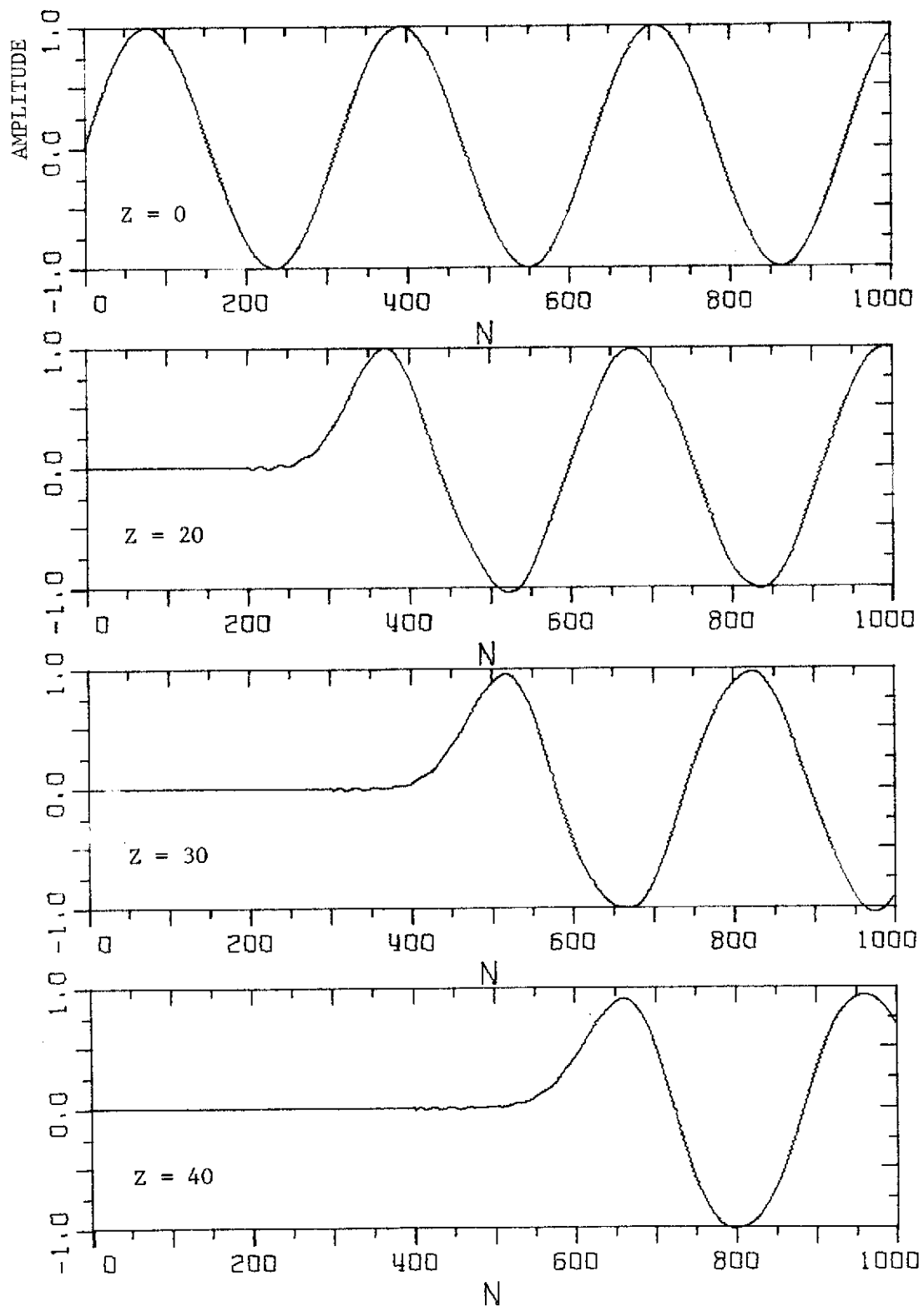sinusoidal signal in a dispersive medium.
$\omega_0 = \omega_p$, T = 0.1N.

Fig. 14--Received waveform of a step-modulated sinusoidal
signal in a dispersive medium, $\omega=\omega_o$. $\omega_o=\omega_p$, T=0.1N.

We further increase the frequency to $\omega=1.5\omega_0$. The waveforms received at each detector are shown in Fig. 15. In this case the signal is building up gradually and there is no clear distinction between the precursor and the main signal. One useful characterization, the build-up time $t_g = z/v_g$, can be defined however. According to Brillouin, it should occur at a time when the detector detects an intensity equal to 1/4 the final intensity. Notice that this also applies to previous cases, when the signal is built up gradually, such as the case $\omega=0.5\omega_0$ in Fig. 13.

Finally, we consider a specific loss $\rho/\omega_0=0.1$ in the medium and repeat the experiment. The received signals are shown in Figs. 16-18 for $\omega/\omega_0 = 0.2$, 0.5, 1.0 respectively. These results indicate that the presence of loss simply attenuates the amplitude of the signal. This is particularly apparent in the case $\omega=0.5\omega_0$ by comparing Fig. 17 and Fig. 11.

We present here a simple proof, using the difference equation, of the fact that the front of a signal always propagates with velocity c and that the front also preserves its amplitude in the course of propagation. Let us consider the space-time grid pattern as shown in Fig. 19. Here the rectangular net is constructed such that $\Delta z=c\Delta t$ ($z=i\Delta z$, $t=n\Delta t$). The signal generator is located in the first column $i=0$. Now we apply the difference equation (85) to all grid points along the diagonal line AB and determine

$$(89) \qquad E_i^n = \frac{1}{\left[1 + \dfrac{(\omega_p\Delta t)^2}{4}\right]^n} E_o^o \qquad (n=1,2,3,\cdots; \; i=n)$$

which describes the course of propagation for the signal front E_o^o. Equation (89) can be approximated by

$$E_i^n = [1 - \tfrac{1}{4} n(\omega_p\Delta t)^2] E_o^o,$$

$$= (1 - \tfrac{1}{4} t \, \omega_p^2\Delta t) E_o^o$$

where t is the time of flight for the signal front from the source (Z=0) to the observation point and hence is finite (actually $t = z/c$ as we shall see). We then have in the limit $\Delta t \to 0$ that

$$(90) \qquad E_i^n = E_o^o \qquad (n=1,2,3,\cdots; \; i=n).$$

Suppose n=1, we have $E_1^1 = E_0^0$. The interpretation is that the signal front E_0^0, after spending Δt seconds, has travelled a distance Δz meters to E_1^0 without losing its amplitude. Hence the velocity of propagation for the front is

$$(91) \qquad v_{s.f.} = \frac{\Delta z}{\Delta t} = c$$

Thus the signal front not only propagates with c, but also preserves its amplitude in the course of propagation. For example, if an ideal unit-step signal of zero risetime is suddenly applied at z=0, then at any point in a dispersive medium one should detect an electric field of 1 volt/m arriving exactly with the speed of light in free space. In this connection it is found that the statements in some books such as "··· so that the process starts always from zero amplitude."[22, pp. 337] and "··· However, the amplitude of the first impulse is zero···"[31] is true only when $E_0^0 = 0$. This is certainly the case for a step-modulated sinusoidal signal considered by Sommerfeld.

E. Summary

In summary, we have found "experimentally" that the front of the signal indeed always propagates with the speed of light in free space. In addition, the front of a signal preserves its amplitude in the course of propagation. The physical reason for this is that, for the Lorentz model dielectric, the relative complex dielectric constant tends to unity for high frequencies. As the signal travels through the medium, its leading edge breaks up into little pulses. These pulses are picked up first by the detector and are known as the precursor. When $\omega \ll \omega_0$, its amplitude is small and the signal propagates essentially without distortion. As ω increases, so does the amplitude of the precursor. But now the leading portion of the main signal suffers a decrease in amplitude as it traverses the medium. When $\omega \sim \omega_0$, the signal is severely distorted. Hence the concept of velocity of propagation is not useful here even though the front always travels at speed c. When $\omega \gg \omega_0$, the signal received at a detector is built up gradually and there is no clear distinction between the precursor and the main signal. A build-up time can be defined however and it is related to the group velocity. The presence of loss in the medium simply attenuates the strength of the signal.

A computer program is presented (Computer Program 3) in Appendix A which can be used to generate the results in this chapter.
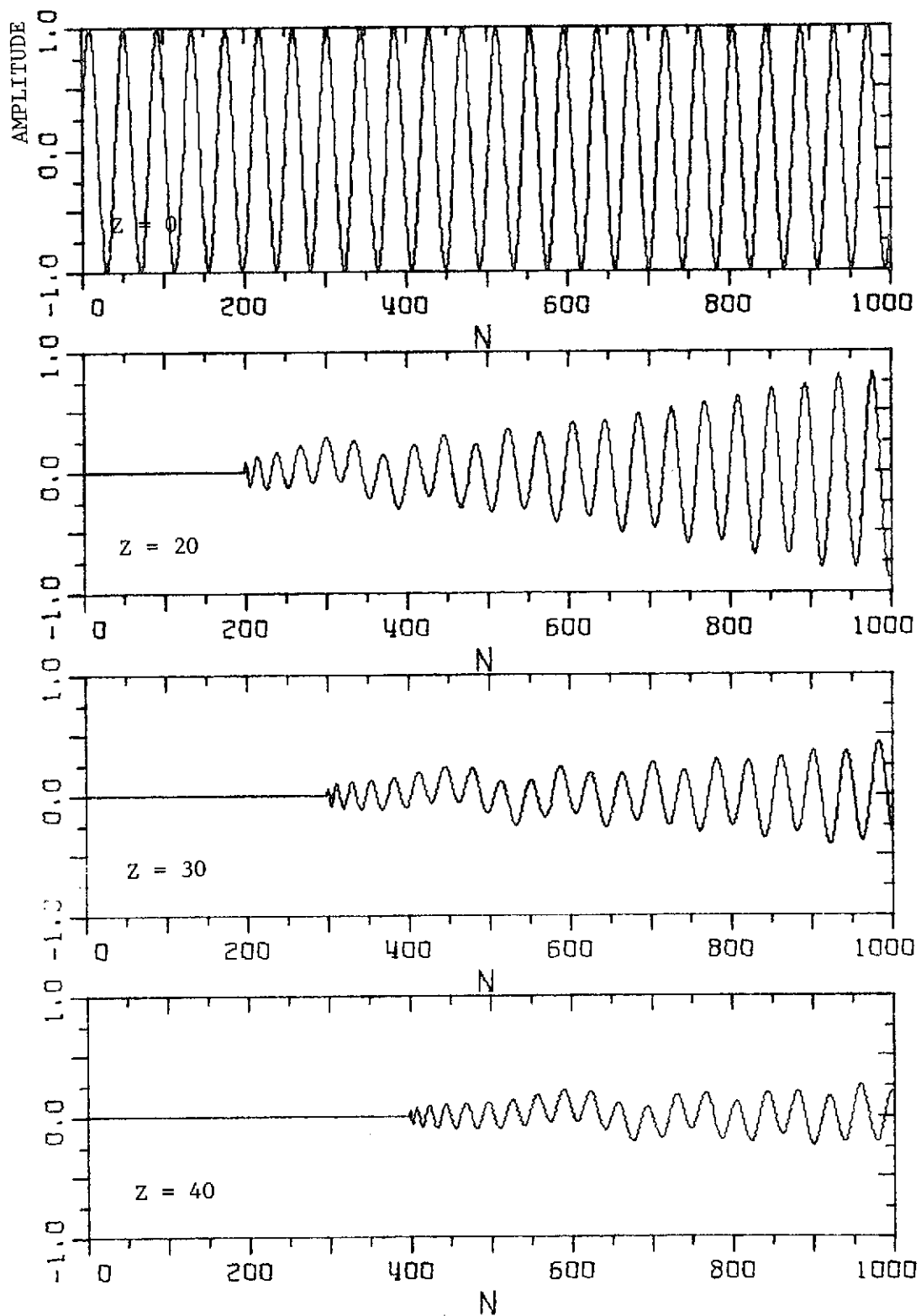
Fig. 15--Received waveform at Z of a step-modulated sinusoidal
signal in a dispersive medium. $\omega=1.5\omega_0$, T=0.1N.
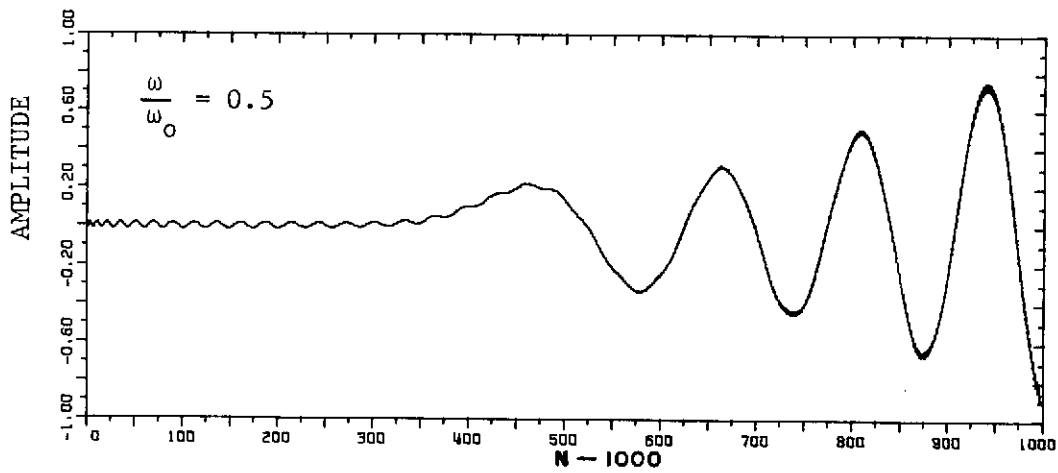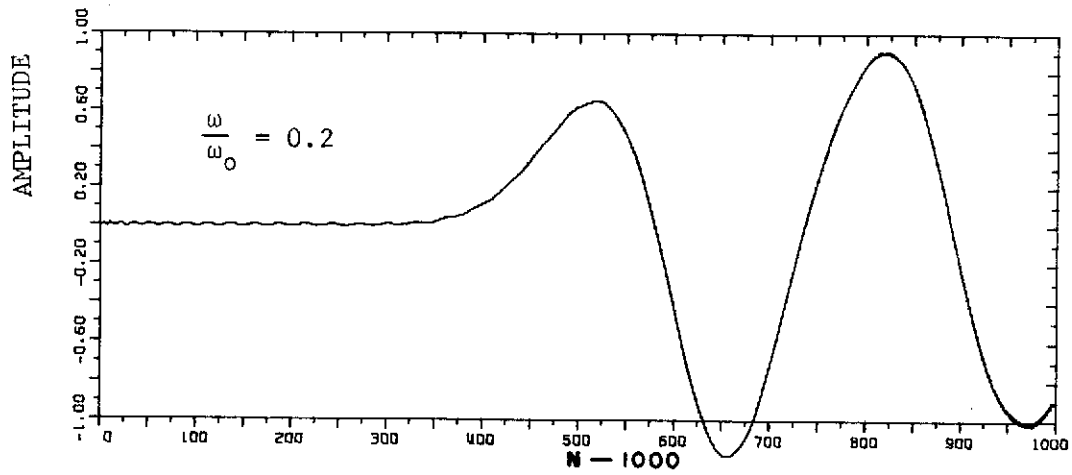
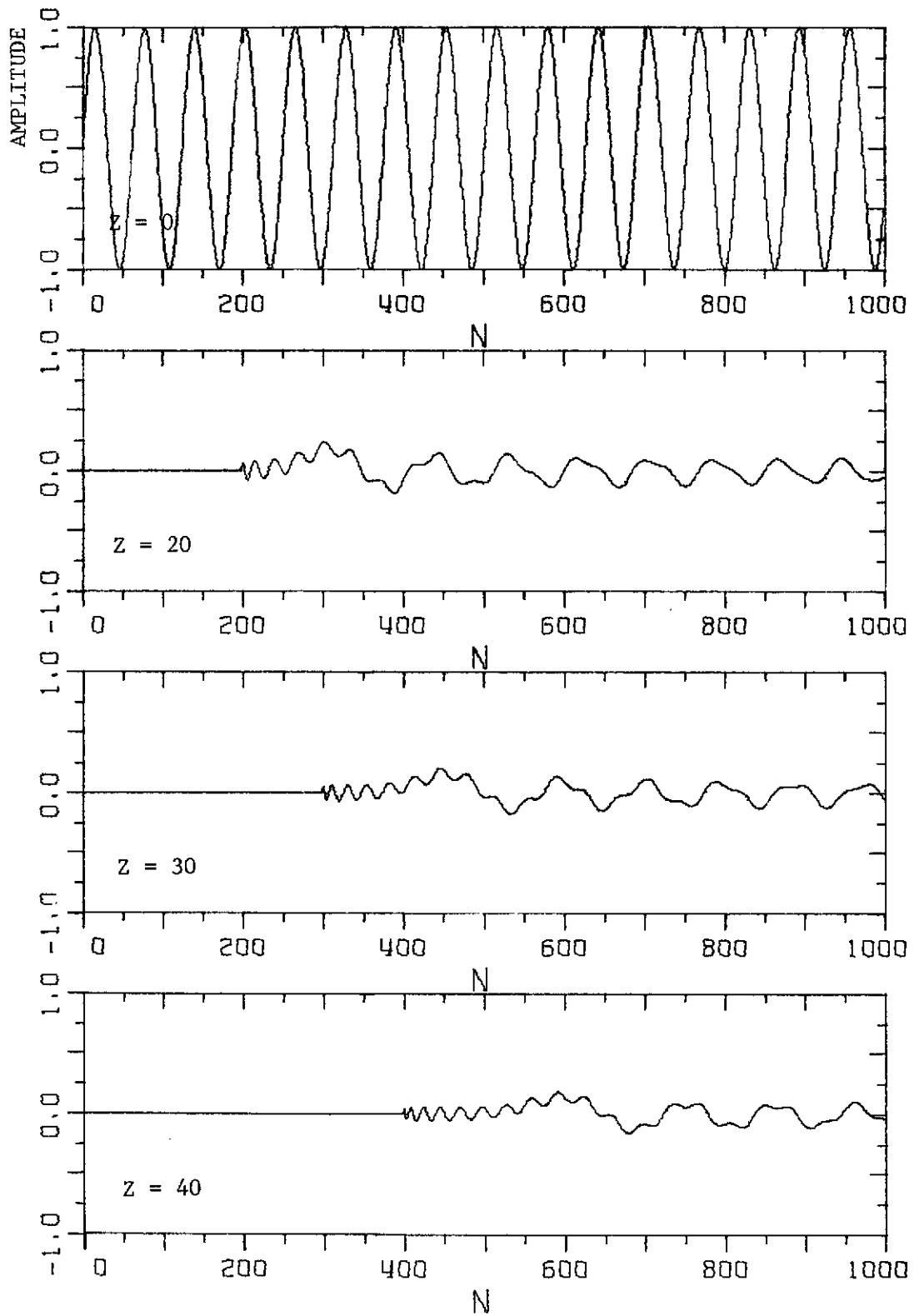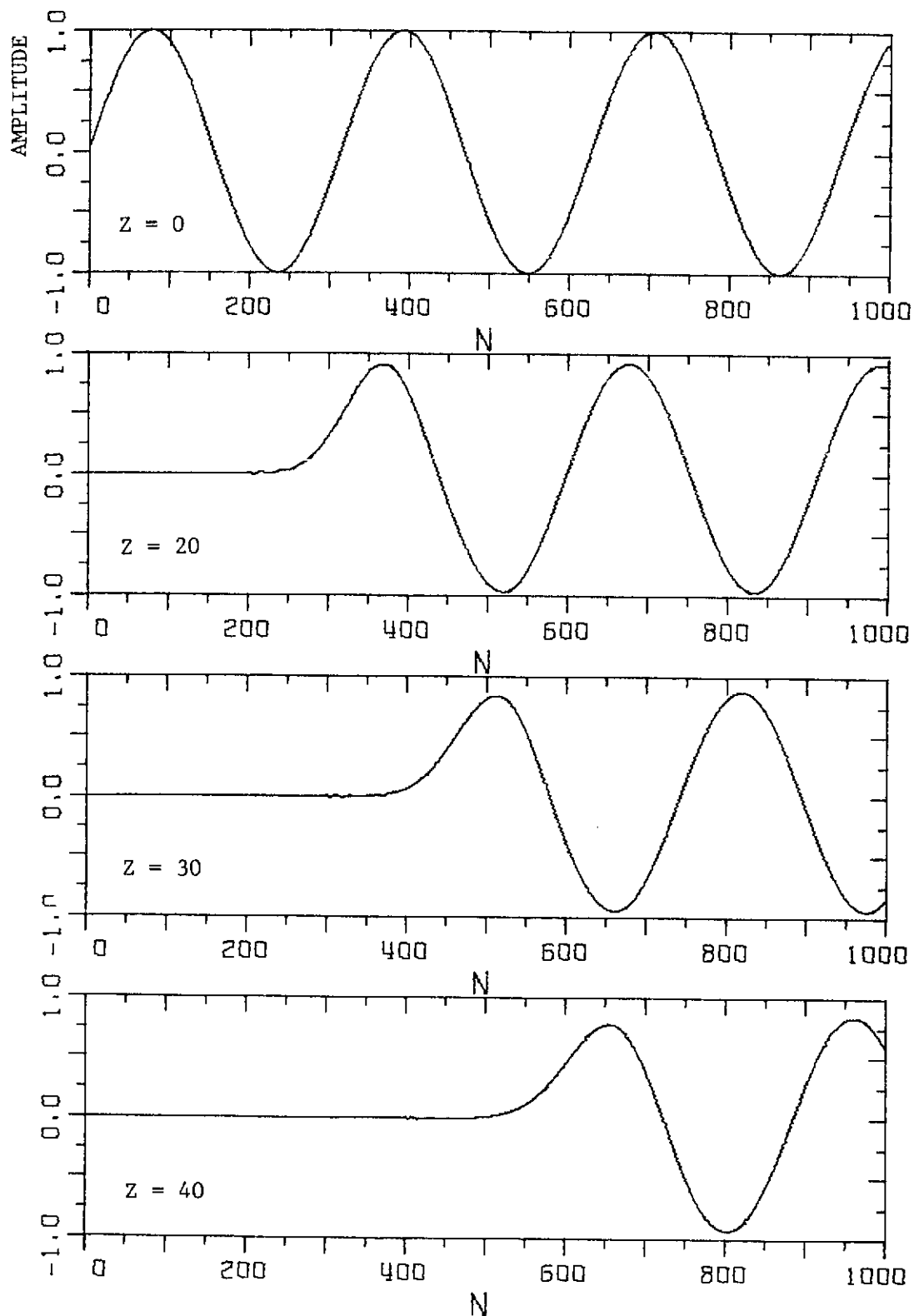Fig. 16--Received waveform at Z of a step-modulated
sinusoidal signal in a dispersive medium.
$\omega=0.2\omega_0$, $\rho/\omega_0=0.1$.
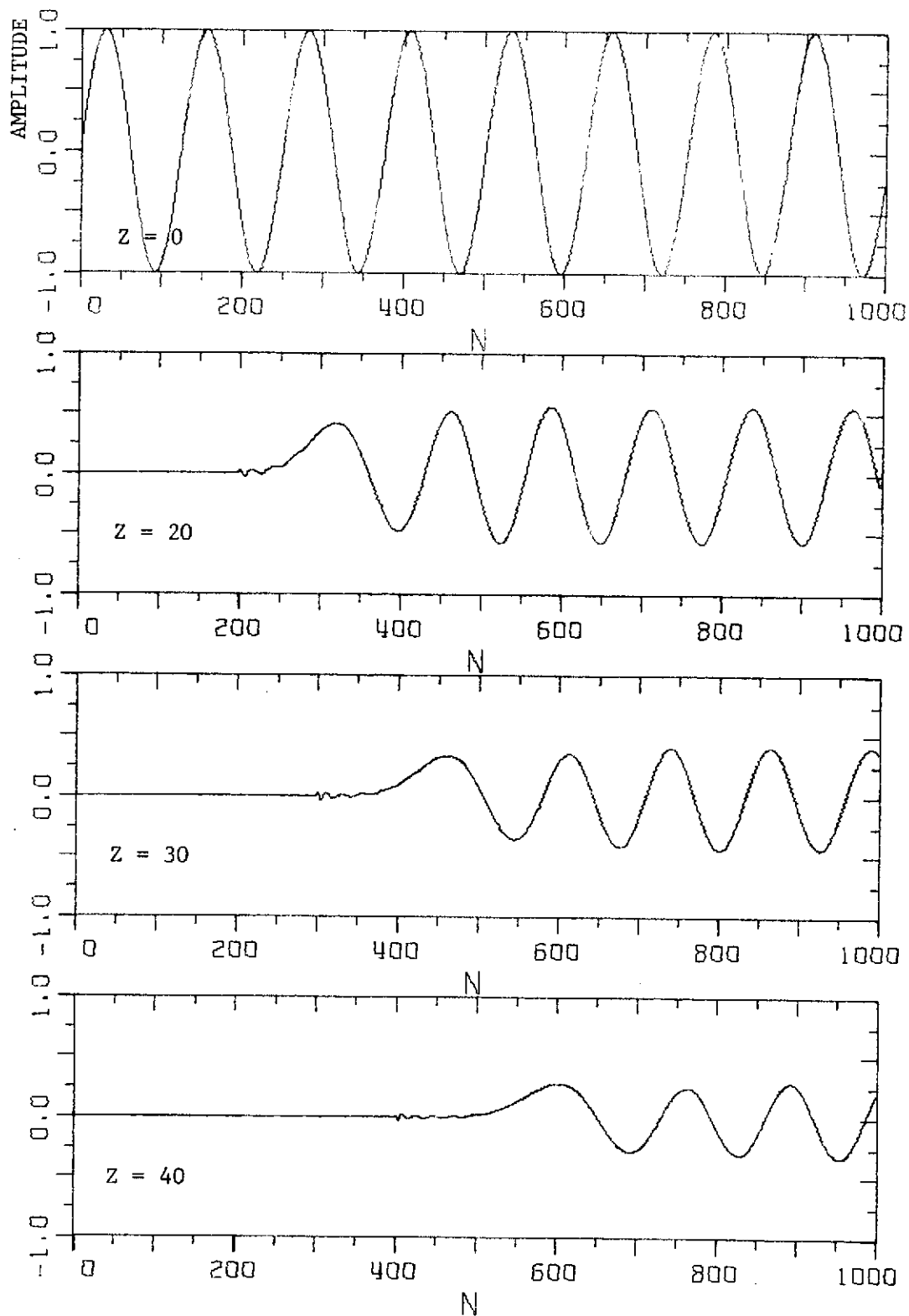
Fig. 17--Received waveform at Z of a step-modulates signal
in a dispersive medium. $\rho/\omega_0 = 0.1$, $\omega=0.5\omega_0$.
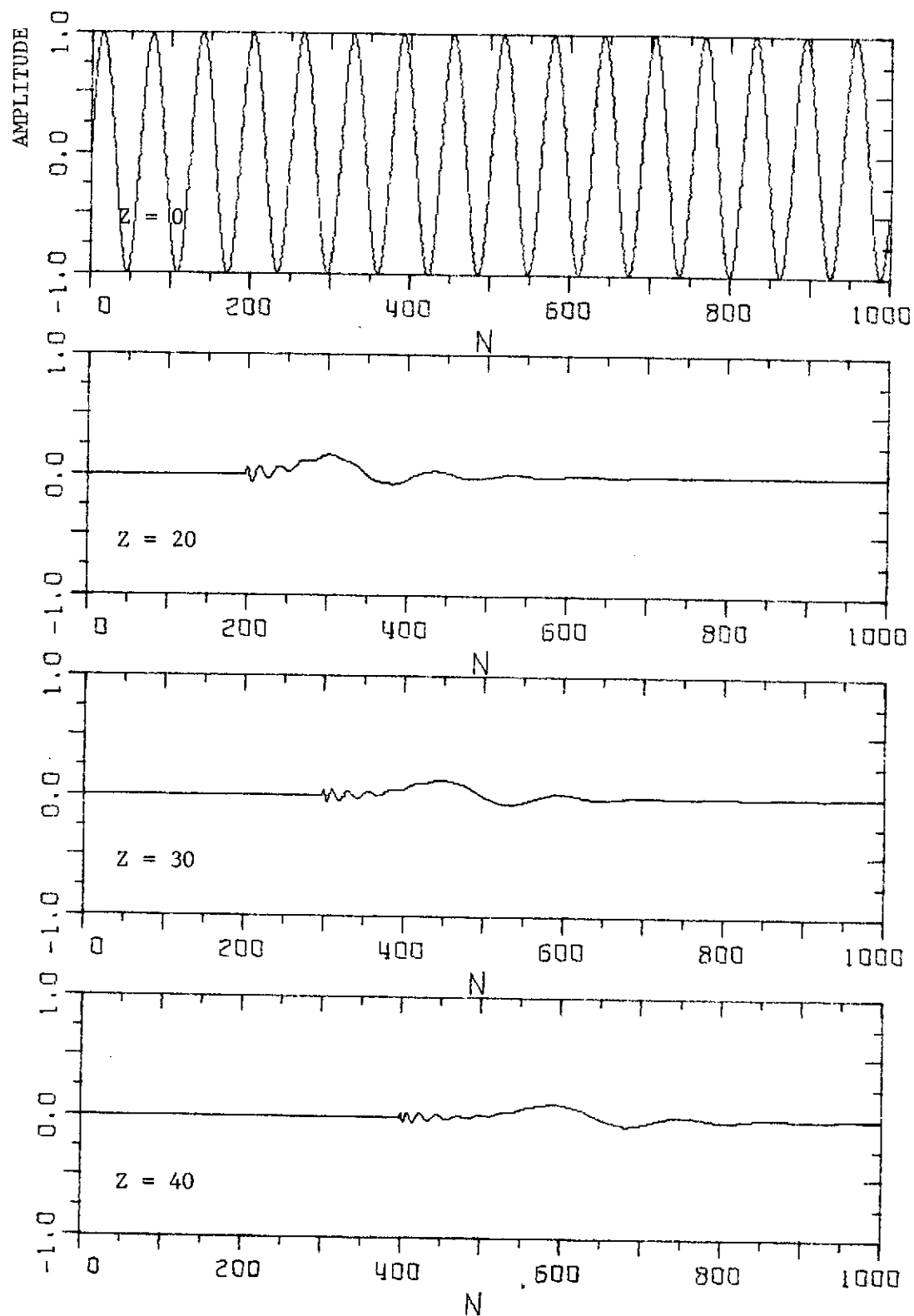
Fig. 18--Received waveform of a step-modulated sinusoidal
signal in a dispersive medium. $\rho/\omega_0 = 0.1$, $\omega = \omega_0$
(For expanded scale, see next page)
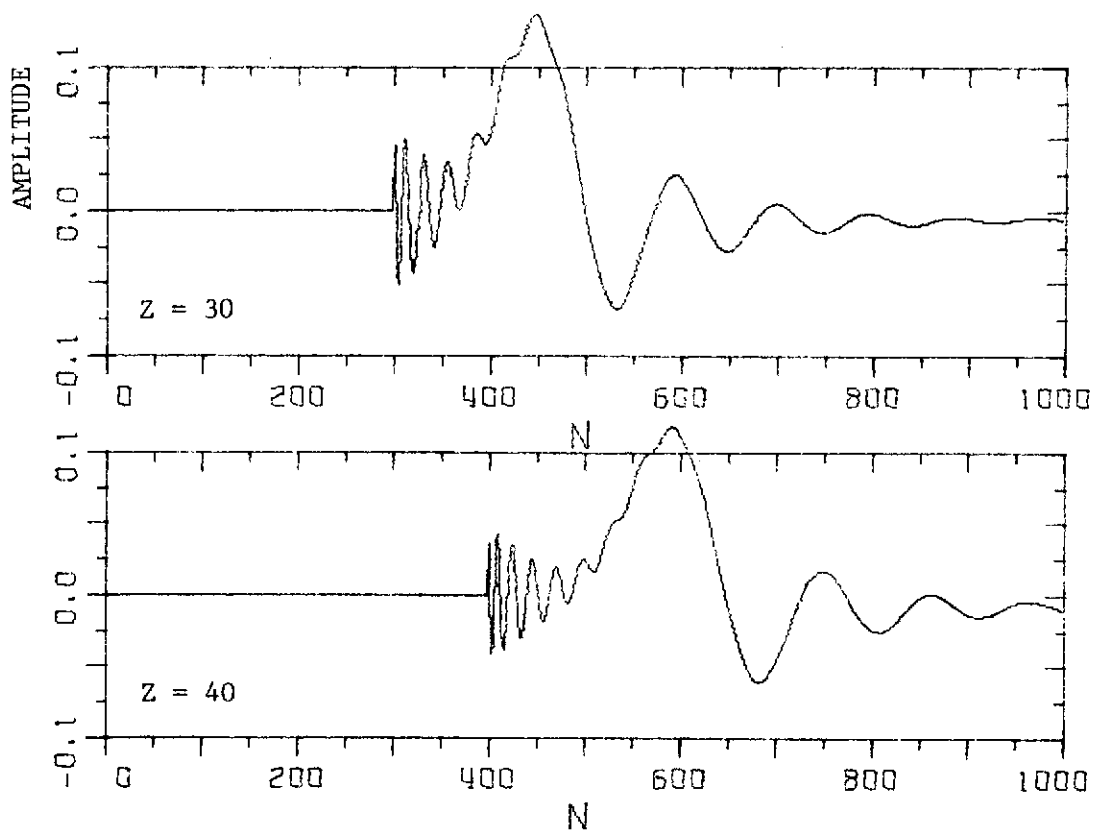
Note: Expanded scale for Fig. 18
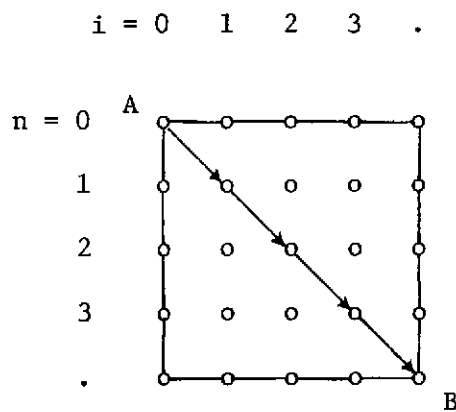
Fig. 19--Signal front at A propagating
to B, $z=i\Delta z$, $t=n\Delta t$.

CHAPTER IV
TRANSIENT SIGNAL PROPAGATION IN COLD STRATIFIED PLASMAS

A. Introduction

The propagation of a transient signal in cold plasmas has been investigated extensively in recent years[e.g., 4, 32, 33, 34]. To analyze this class of problems, Fourier and Laplace transform methods are usually employed. Although these standard methods apply to a number of cases, their failure to accommodate others has warranted additional and new methods of solution. For instance, Bowhill[6] suggested the multiple-scattering technique while Field[7] proposed the method of characteristics.

This chapter extends the previous one to a stratified plasma where a constant magnetic field parallel to the direction of propagation is also assumed. By approximating the plasma medium with a large number of thin layers, it is shown that the time history of a signal of arbitrary waveform in an inhomogeneous, anisotropic (because of the constant magnetic field), lossy, cold plasma can be easily obtained from difference equations by marching on in time. Only the case of normal incidence is considered.

In Section B the coupled integro-differential equations for the electric fields are first derived from the Maxwell's equations together with the equation of motion for the electrons. These equations are next transformed into difference equations in Section C. Numerical techniques and examples are presented to demonstrate the versatile applicability of the resulting difference equations.

B. Coupled Integro-differential Equations for the Electric Fields

Consider a plane electromagnetic wave propagation in the z direction in a homogeneous plasma in which a constant magnetic field B_0 parallel to the direction of propagation, is assumed. The governing equations for this problem have been derived from Maxwell's equations by Stratton[22, pp. 328] and can be written in complex form

$$i\varepsilon_0 \frac{\partial E}{\partial t} + \frac{\partial H}{\partial z} + iNev = 0,$$

(92)

$$i\mu_0 \frac{\partial H}{\partial t} - \frac{\partial E}{\partial z} = 0$$

with the corresponding equation of motion for the electrons

(93) $\dfrac{dv}{dt} + (\nu + i\omega_c)v = \dfrac{e}{m} E$

where $E = E_x + iE_y$, $H = H_x + iH_y$, $v = dx/dt + i\,dy/dt$, and $\omega_c = eB_0/m$. Here N, e, m, ν, ω_c are the electron density, the charge of an electron, the mass of an electron, the electron collision frequency, and the electron cyclotron frequency respectively.

Assuming the electrons are initially rest, the solution of (93) is given by

(94) $v(z,t) = \dfrac{e}{m} \displaystyle\int_0^t E(z,\beta)\exp[-(\nu+i\omega_c)(t-\beta)]d\beta.$

Eliminating the magnetic field H in (92), the use of (94) yields the integro-differential equation

(95) $\dfrac{\partial^2 E}{\partial t^2} - c^2 \dfrac{\partial^2 E}{\partial z^2} + \omega_p^2 E(z,t) = \omega_p^2(\nu+i\omega_c)\displaystyle\int_0^t E(z,\beta)\exp[-(\nu+i\omega_c)(t-\beta)]d\beta$

where $c = 1/\sqrt{\mu_0\varepsilon_0}$, $\omega_p^2 = Ne^2/m\varepsilon_0$ and c is the speed of light in free space, ω_p is the plasma frequency. Equation (95) is further decomposed into real and imaginary parts to obtain the coupled time-dependent integro-differential equations for the electric fields

(96a) $\dfrac{\partial^2 E_x}{\partial t^2} - c^2 \dfrac{\partial^2 E_x}{\partial z^2} + \omega_p^2 E_x(z,t) = \omega_p^2 \displaystyle\int_0^t a(t-\beta)E_x(z,\beta)d\beta$

$\qquad\qquad\qquad\qquad\qquad - \omega_p^2 \displaystyle\int_0^t b(t-\beta)E_y(z,\beta)d\beta,$

(96b) $\dfrac{\partial^2 E_y}{\partial t^2} - c^2 \dfrac{\partial^2 E_y}{\partial z^2} + \omega_p^2 E_y(z,t) = \omega_p^2 \displaystyle\int_0^t a(t-\beta)E_y(z,\beta)d\beta$

$\qquad\qquad\qquad\qquad\qquad + \omega_p^2 \displaystyle\int_0^t b(t-\beta)E_x(z,\beta)d\beta$

where

$$a(t-\beta)=[\nu\cos\omega_c(t-\beta)+\omega_c\sin\omega_c(t-\beta)]\exp[-\nu(t-\beta)],$$

$$b(t-\beta)=[\omega_c\cos\omega_c(t-\beta)-\nu\sin\omega_c(t-\beta)]\exp[-\nu(t-\beta)].$$

Here we can see clearly the coupling of the electric field components which occur in the last terms in (96).

To visualize this coupling, let us consider an x-polarized plane electromagnetic wave in free space normally incident on a plasma medium. Before the signal front reaches the air/plasma interface, the electric fields are, of course, governed by the wave equation, i.e., by setting $\omega_p=0$ in (96). Since there is no E_y component in the incident wave, E_y is identically zero in this period of time. At the instant E_x enters the plasma, an E_y field is set up by the last term in (96b). This induced electric field then propagates both into the plasma and back to the free space. Therefore, the reflected wave has an E_y component. It is clear that the last terms in (96) are responsible for converting the polarization of the incident signal. When the kernal $b(t-\beta)$ is zero, i.e., there exists no constant magnetic field, then (96) becomes

$$(97) \qquad \frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial z^2} + \omega_p^2 E = \omega_p^2 \nu \int_0^t \exp[-\nu(t-\beta)E(z,\beta)d\beta$$

where E can be x or y polarized and the subscript is dropped. Here, the polarization would be preserved since the equations are now u. coupled. One can also obtain this integro-differential equation from (80) by setting the characteristic frequency $\omega_0=0$.

C. Difference Equations for the Electric Fields

To transform (96) and (97) into difference equations, the plasma medium is first divided into a number of thin layers of equal thickness

$$(98) \qquad \Delta z = c\Delta t$$

in which the time increment Δt is arbitrary but small; its choice will be clear later. We then attach to each interface boundary a grid point as shown in Fig. 20.

(1) Homogeneous plasma

A procedure to derive the difference equations for this case has been described in Chapter II. The numerical method essentially replaces the second partial derivatives by the central finite differences

$$\frac{\partial^2 E_\xi}{\partial z^2} = \frac{E_\xi(z+\Delta z,t) - 2E_\xi(z,t) + E_\xi(z,t-\Delta t)}{\Delta z^2}$$

(99)

$$\frac{\partial^2 E_\xi}{\partial t^2} = \frac{E_\xi(z,t+\Delta t) - 2E_\xi(z,t) + E_\xi(z,t-\Delta t)}{\Delta t^2}$$

where ξ stands for both x and y. To ensure a stable numerical solution the electric fields in the third term in (96) are approximated by three consecutive times

(100) $\quad E_\xi(z,t) = \frac{1}{2} [E_\xi(z,t) + \frac{1}{2} E_\xi(z,t+\Delta t) + \frac{1}{2} E_\xi(z,t-\Delta t)].$

Finally, the integrals in the right-hand sides of (96), (97) are written as

$$A_\xi(z,t) = \left(\int_0^{t-\Delta t} + \int_{t-\Delta t}^t \right) a(t-\beta) E_\xi(z,\beta) d\beta,$$

$$B_\xi(z,t) = \left(\int_0^{t-\Delta t} + \int_{t-\Delta t}^t \right) b(t-\beta) E_\xi(z,\beta) d\beta,$$

$$D(z,t) = \frac{1}{\Delta t} \left(\int_0^{t-\Delta t} + \int_{t-\Delta t}^t \right) \exp[-\nu(t-\beta)] E(z,\beta) d\beta.$$

Evaluating the integration from t-Δt to t by the trapezoidal rule of integration yields the recursive relations

$$A_{\xi i}^n = g(KA_{\xi i}^{n-1} + SB_{\xi i}^{n-1}) + \frac{1}{2} [\alpha E_{\xi i}^n + (\theta S + \alpha K)gE_{\xi i}^{n-1}],$$

$$(101) \quad B_{\xi i}^n = g(KB_{\xi i}^{n-1} - SA_{\xi i}^{n-1}) + \frac{1}{2} [\theta E_{\xi i}^n + (\theta K - \alpha S)gE_{\xi i}^{n-1}]$$

$$D_i^n = gD_i^{n-1} + \frac{1}{2} [E_i^n + gE_i^{n-1}],$$

where $K = \cos\theta$, $S = \sin\theta$, $g = \exp(-\alpha)$, $\alpha = \nu\Delta t$, $\theta = \omega_c\Delta t$ and the notation $E_{\xi i}^n = E_\xi(z,t)_{\substack{t=n\Delta t \\ z=i\Delta z}}$ has been used.

Combining the results of (98)-(101), Eqs. (96) are readily transformed into the explicit difference equations

$$E_{xi}^{n+1} = - E_{xi}^{n-1} + \eta[E_{x\ i+1}^n + E_{x\ i-1}^n - \frac{\tau^2}{2} E_{xi}^n + \tau^2(A_{xi}^n - B_{yi}^n)],$$

$$(102)$$

$$E_{yi}^{n+1} = - E_{yi}^{n-1} + \eta[E_{y\ i+1}^n + E_{y\ i-1}^n - \frac{\tau^2}{2} E_{yi}^n + \tau^2(A_{yi}^n + B_{xi}^n)]$$

where $\tau = \omega_p\Delta t$, $\eta = \dfrac{1}{1 + \dfrac{\tau^2}{4}}$

and similarly (97) is transformed to the difference equation

$$(103) \quad E_i^{n+1} = -E_i^{n-1} + \eta[E_{i+1}^n + E_{i-1}^n - \frac{\tau^2}{2} E_i^n + \alpha\tau^2 D_i^n].$$

(2) Stratified Medium

When a grid point is located at the interface between two different plasmas, or the air/plasma interface, strictly speaking Eq. (102) no longer applies. Not only is the finite difference in (99) for the z variable no longer valid, but even the integro-differential equations themselves are undefined here. Fortunately, appropriate difference equations can be derived.

Without loss of generality let us consider the interface between layer 1 and 2 in Fig. 20. Within each thin layer the integro-differential equations are continuous everywhere, hence

they are valid in the vicinity of the interface boundaary. Thus we write (96) at z^+, z^- respectively

$$\frac{\partial^2 E_\xi(z^+,t)}{\partial t^2} - c^2 \frac{\partial^2 E_\xi(z^+,t)}{\partial z^2} + \omega_{p_2}^2 E_\xi(z^+,t) = \omega_{p_2}^2 F_2[\nu_2, E(z^+,t)],$$

(104)

$$\frac{\partial^2 E_\xi(z^-,t)}{\partial t^2} - c^2 \frac{\partial^2 E_\xi(z^-,t)}{\partial z^2} + \omega_{p_1}^2 E_\xi(z^-,t) = \omega_{p_1}^2 F_1[\nu_1, E_\xi(z^-,t)]$$

where the subscripts 1 and 2 denote the thin layers to the left and to the right of an interface boundary respectively. Each of the above expressions actually stands for two equations; the functionals F_1, F_2 are understood as representing the source term in the right-hand sides of (96).
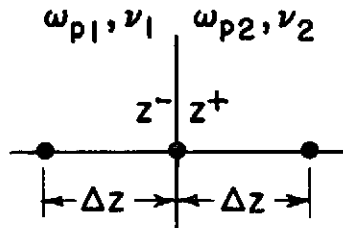
Fig. 20--Interface between two lossy cold plasmas.

At the interface the boundary conditions are

$$E_\xi(z^+,t) = E_\xi(z^-,t),$$

(105)

$$\frac{\partial E_\xi(z^+,t)}{\partial z} = \frac{\partial E_\xi(z^-,t)}{\partial z} \quad .$$

To derive the difference equations for the mathematical problem as posed in (104) and (105), we expand the fields at the points $z^{+}+\Delta z$ and $z^{-}-\Delta z$ by the Taylor series expansions

$$E_{\xi}(z^{+}+\Delta z,t) = E_{\xi}(z^{+},t) + \frac{\partial E_{\xi}(z^{+},t)}{\partial z}\Delta z + \frac{\partial^{2}E_{\xi}(z^{+},t)}{\partial z^{2}}\frac{\Delta z^{2}}{2},$$

(106)

$$E_{\xi}(z^{-}-\Delta z,t) = E_{\xi}(z^{-},t) - \frac{\partial E_{\xi}(z^{-},t)}{\partial z}\Delta z + \frac{\partial^{2}E_{\xi}(z^{-},t)}{\partial z^{2}}\frac{\Delta z^{2}}{2}$$

in which the higher order terms have been neglected. The second partial derivatives in these two expressions are first substituted by (104). The first derivatives are next eliminated by the use of the boundary conditions in (105). Finally we replace the second partial derivatives with respect to t in the resulting equation by the finite difference in (99). Performing all these steps, we obtain the difference equation

(107) $$E_{\xi}(z^{+},t+\Delta t) = -E_{\xi}(z^{+},t-\Delta t)+E_{\xi}(z^{+}+\Delta z,t)+E_{\xi}(z^{-}-\Delta z,t)$$

$$+ \frac{1}{2}(\tau_{1}^{2} + \tau_{2}^{2})\ E_{\xi}^{*}(z^{+},t)$$

$$+ \frac{1}{2}\{\tau_{1}^{2}F_{1}[\nu_{1},E_{\xi}(z^{-},t)]+\tau_{2}^{2}F_{2}[\nu_{2},E_{\xi}(z^{+},t)]\}$$

where $\tau_{1}=\omega_{p_{1}}\Delta t$, $\tau_{2}=\omega_{p_{2}}\Delta t$. A "*" is added to indicate that this term must be approximated by the scheme in (100) for numerical stability considerations.

The above results indicate that for an interface grid point one simply evaluates the lower order terms (i.e., the plasma and the source terms) on both sides of the interface and then take the average value of them. Therefore, (107) can be written explicitly as

$$E_{xi}^{n+1} = -E_{xi}^{n-1} + \eta[E_{xi+1}^{n}+ E_{x\ i-1}^{n}-\gamma E_{xi}^{n} + \frac{1}{2}\sum_{j=1}^{2}\tau_{j}^{2}(A_{xji}^{n}-B_{yji}^{n})],$$

(108)

$$E_{yi}^{n+1} = -E_{yi}^{n-1}+ \eta[E_{y\ i+1}^{n}+E_{y\ i-1}^{n}-\gamma E_{yi}^{n} + \frac{1}{2}\sum_{j=1}^{2}\tau_{j}^{2}(A_{yji}^{n}+B_{xji}^{n})]$$

where

$$A^n_{\xi j i} = g_j(KA^{n-1}_{\xi j i} + SB^{n-1}_{\xi j i}) + \frac{1}{2}[\alpha_j E^n_{\xi i} + (\theta S + \alpha_j K)g_j E^{n-1}_{\xi i}],$$

$$B^n_{\xi j i} = g_j(KB^{n-1}_{\xi j i} - SA^{n-1}_{\xi j i}) + \frac{1}{2}[\theta E^n_{\xi i} + (\theta K - \alpha_j S)g_j E^{n-1}_{\xi i}]$$

and

$$\tau_1 = \omega_{p_1}\Delta t, \quad \tau_2 = \omega_{p_2}\Delta t, \quad \alpha_j = \nu_j \Delta t, \quad g_j = \exp(-\alpha_j), \quad \theta = \omega_c \Delta t,$$

$$K = \cos\theta, S = \sin\theta, \quad \gamma = \frac{1}{4}(\tau_1^2 + \tau_2^2), \quad \eta = \frac{1}{1 + \frac{1}{2}\gamma}.$$

Similarly, we can write down the difference equations for an isotropic lossy plasma from (97)

$$(109) \quad E^{n+1}_i = -E^{n-1}_i + \eta[E^n_{i+1} + E^n_{i-1} - \gamma E^n_i + \frac{1}{2}\sum_{j=1}^{2}\alpha_j \tau_j^2 D^n_j].$$

It is seen that (108) reduces to (102) when the plasmas at both sides of a grid point is the same. Furthermore, since (108) is derived from a canonical problem of an interface boundary, it applies to other interface and hence a multitude of boundaries as well. Consequently, (108) can be used to obtain the time history of a signal in an inhomogeneous plasma provided the medium is approximated in a step-wise fashion.

D. Some Practical Aspects of the Difference Solution

To illustrate some pertinent techniques for the application of (108), let us consider the space-time grid pattern as shown in Fig. 21. Here the space is divided into thin layers of thickness Δz. The time is quantized to an increment Δt (recall that $\Delta z = c\Delta t$). The field at a grid point is denoted by E^n_i. Distance and time are defined as $z = (i-1)\Delta z$, $t = n\Delta t$.

For the time being we assume the incident signal is impressed at $z=0$ ($i=1$). It is desired to obtain the transient response up to a time $T = \omega_p t$ at the observation point $Z = \omega_p z/c$. For simplicity we consider that the plasma is homogeneous, lossless and isotropic; although the discussions that follow apply to a general situation as well.
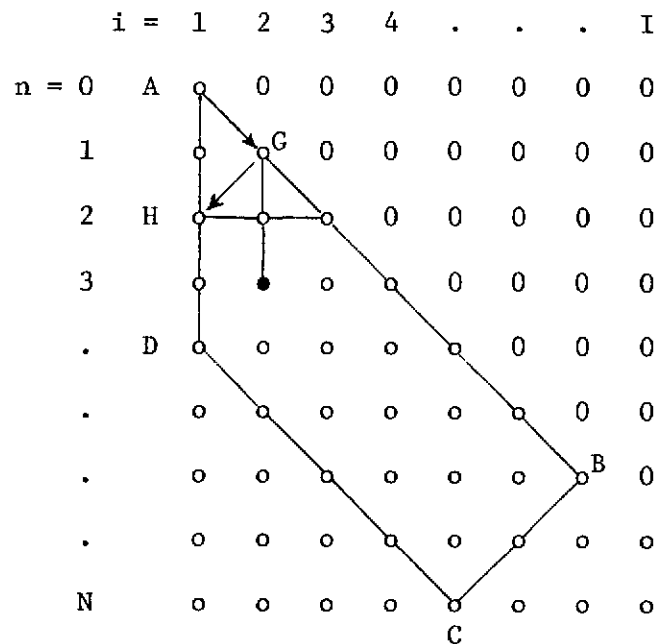
Fig. 21--Space-time grid point pattern,
z=(i-1)Δz, t=nΔt.

The difference equation for this case is the simplest one and is given by

$$E_i^n = -E_i^{n-2} + \frac{1}{1 + \frac{1}{4}\tau^2} (E_{i+1}^{n-1} + E_{i-1}^{n-1} - \frac{1}{2}\tau^2 E_i^{n-1})$$

(110)

$$(i=2,3,4,\cdots,I-1; \; n=2,3,4,\cdots,N)$$

where E may be x or y polarized. At the column i=1 we have the initial condition

(111) $E_1^n = f(n\Delta t) \; (n=0,1,2,\cdots,N)$

where f(t) is the source signal impressed at z=0. Since we have assumed the plasma is initially undisturbed, we have at n=0,1

$$E_i^0 = 0 \quad (i \geq 2),$$

$$E_i^1 = 0 \quad (i \geq 3).$$

Now the field at E_2^1 is no longer zero because the signal front at E_1^0 has reached this point. We can however apply (110) and determine

$$(112) \qquad E_2^1 = \frac{1}{1 + \frac{\tau^2}{4}} E_1^0 .$$

Here the fact $E_2^{-1}=0$ has been used. This follows from causality of the fields.

We finally select an appropriate value of τ, and employ (110) to generate the solution up to the time step

$$(113) \qquad N = \frac{T}{\tau} .$$

At each step the response is simply picked up at the observation point

$$(114) \qquad i_{obs} = \frac{Z}{\Delta z} = \frac{Z}{\tau} + 1 .$$

The far-end boundary can be ended at

$$(115) \qquad I = i_{obs} + \frac{1}{2} (N-i_{obs}) + 1$$

without affecting the numerical solution. This follows from the domain of dependence for the observation point. For this very same reason, only those grid points inside and on the trapezoid ABCD in Fig. 21 need to be calculated.

The condition in (114) is good for an infinite plasma medium. When the plasma is bounded by a perfectly-conducting boundary, one can simply set

$$(116) \qquad E_I^n = 0 \quad (n=0,1,2,\cdots N)$$

at the end point of the conducting boundary.

When the plasma slab is bounded by the free space, then the assignment

$$(117) \qquad E_I^n = E_{I-1}^{n-1}$$

would simulate the infinite free space.

There remains an important case for this class of problems; that for which the incident signal, rather than being impressed at z=0, is incident from z<0 in the air. It is then numerically efficient to assume the signal front arrives at z=0 (i=1) at the instant t=0. This is equivalent to assigning again the initial fields to column i=1, although a different physical problem is encountered here.

Now the air/plasma interface can be conveniently chosen at i=2. While (110) remains valid for all interior points in the plasma, the difference equation for i=2 is

$$(118) \qquad E_2^n = -E_2^{n-2} + \frac{1}{1 + \frac{1}{8}\tau^2} \left(E_3^{n-1} + S - \frac{1}{4}\tau^2 E_2^{n-1} \right)$$

where S is literally equal to E_1^{n-1}, but has a different physical significance in this case. Physically the quantity S means the total field at a point where it is in the air. Thus it has two components: the incident field and the reflected field. Refer to Fig. 21 and suppose E_2^n (say n=3) is being evaluated; then the total field in the air is given by

$$(119) \qquad S = \text{Primary field H + Reflected field from G}$$

$$= E_1^{n-1} + (E_2^{n-2} - E_1^{n-3}).$$

Here E_1^{n-1} is the incident field as discussed earlier, E_2^{n-2} is the total field at G and E_1^{n-3} is the incident field to G from A.

Hence the <u>reflected</u> field from G is equal to the difference be-
tween these two quantities. The reflected signal is then simply
picked up at i=2 according to

$$(120) \qquad E^r(n\Delta t) = E_2^n - E_1^{n-1} \quad (1 \leq n \leq N)$$

The problem is then completely solved.

We now want to comment on the selection of the time step size
Δt. In practice, it is recommended that Δt be chosen such that

$$(121) \qquad (\omega_p \Delta t, \ \omega_c \Delta t, \ \nu \Delta t) \leq 0.1$$

whenever possible. The reason for the choice (121) is solely due
to numerical stability and accuracy considerations. One should recall
that we have obtained the recursive relations for the integrals by
trapezoidal rule of integration as in (101). When the step size is
too large, it is not clear whether the error introduced at one step
will jeopardize the stability of the solution at later times. Under
such circumstances, we have had recourse to intuition and numerical
evidence, since we have not derived a stability condition for this
type of equation.

E.   Numerical Examples

The validity of the resulting difference equation is demonstrated
by several examples selected from the literature.

The first example is based on the work of McIntosh and
El-Khany[35] in which a group velocity concept was used to synthesize
a "chirp pulse", and the transform methods were applied to obtain the
actual "compressed" pulse. Starting with their signal waveform,
we have calculated the waveform received at Z=15 in a homogeneous
isotropic plasma for two cases $\nu/\omega p = 0, 0.1$ (see Fig. 22). Although
only the lossless case was considered in [35], we see that pulse
enhancement still occurs in a lossy plasma. More recently, pulse
compression in bounded dispersive media has also been demonstrated
by transform methods[36]. With the aid of the difference equation,
the response of the signal constructed from a number of simple models
can be easily computed in various plasma conditions. This practice may
lead to computer-aided design procedure for pulse compression studies.
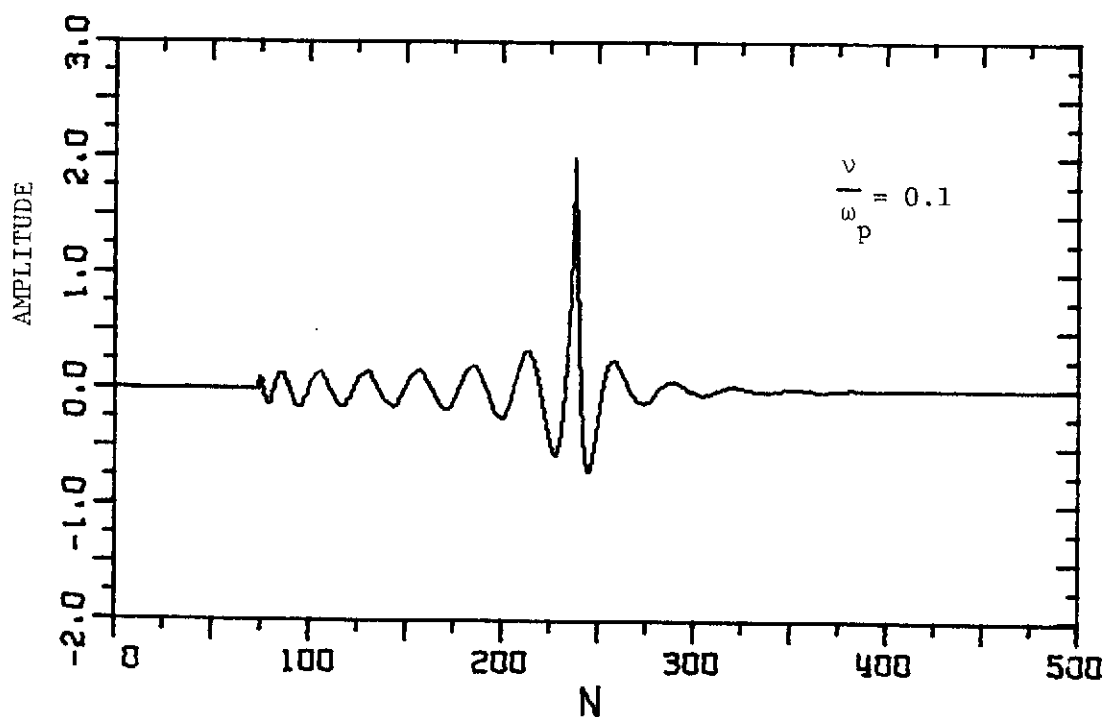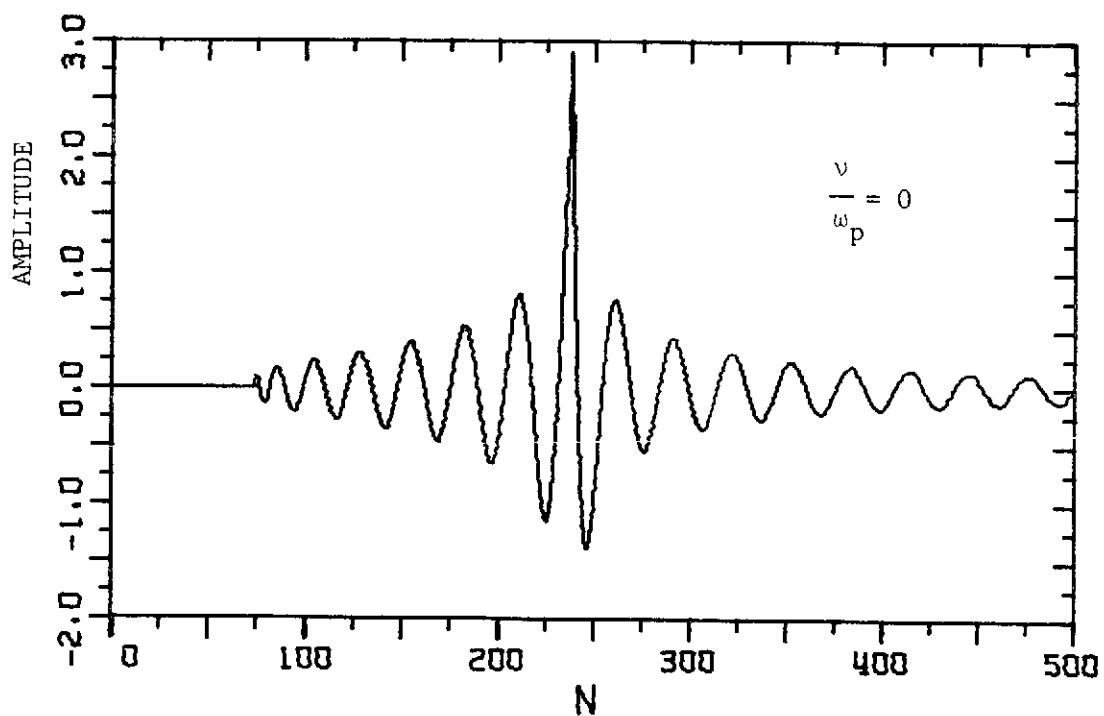
Fig. 22--Pulse enhancement at a receiver
Z=15, T=N$\omega_p \Delta$t, $\omega_p \Delta$t=0.2

The next example compares the difference equation solutions for the reflection of a unit-step signal with results obtained by Wait[37] (for the lossless case only) by transform methods. Figure 23 shows the reflection of a unit-step signal normally incident on a homogeneous plasma half-space. The curves are presented for ν/ω_p = 0.0, 0.1, 0.5 and 1.0. In an accompaning table the difference solution for two different time step sizes (= 0.1, 0.01) is compared with the analytic solution[37] for the lossless plasma case. The convergence of the numerical solution is clearly indicated. As was mentioned earlier, if the electric field in the plasma term is not weighted at three consecutive time steps as in (100), then the difference solution is unstable. This solution is given in parenthesis, and is unrecognizable after T > 5.

The third example is to calculate the waveforms of a short gaussian pulse reflected from an inhomogeneous plasma having a linear electron density profile ($\omega_p^2=2\times10^{10}z$) and constant collision frequency with a vertical magnetic field. This is shown in Fig. 24. This problem was originally considered by Hill and Wait[38] using transform methods, and numerical procedures were used to obtain the results. The difference solution are obtained by the use of (108) by marching on in time. Comparing Fig. 24 with the corresponding results in [38], we conclude that the accuracy of the difference solution is indeed excellent.

The fourth example (Fig. 25) shows the reflection of a unit-step signal normally incident on a lossless plasma slab for 4 different normalized slab thickness Z = 30, 5, 2, 1. Notice that the first case is essentially the half-space case. Wait[37] has obtained an analytic expression for this problem in the form of an infinite series involving Bessel functions. In the presentations of his curves, there are still noticeable oscillations in the reflected waveform at T > 10 for the cases Z = 1, 2. However, our difference solution does not exhibit these oscillations. Antonucci[45] had constructed an artificial TEM line to simulate the plasma and also noted that his result did not agree with Wait in these cases because his results showed no oscillations at late times.

The fifth example shows the response at Z = 100 of a step carrier signal at Z = 0 in a homogeneous lossless cold plasma. Notice that the signal builds up gradually and oscillates about the steady state values at late times. This phenomenon is predicted analytically[32] and confirmed experimentally[48,49] in related problems.

For our final example we show the spatial variation of a step carrier at z=0 for three frequencies ω/ω_p = 1, 1.2, 1.4 propagating in a lossless plasma at several normalized times (see Fig. 27). When $\omega = \omega_p$ (and $\omega<\omega_p$), we see that the waveform is strongly attenuated as

it propagates to the right. But the wavefront of the signal always propagates with the speed of light. Other velocities become difficult to define. When $\omega > \omega_p$, the signal builds up gradually depending on the frequency of the propagating signal.

The above results can be obtained from two computer programs presented in Appendix A. Computer Program 4 calculates the propagation and reflection of a unit-step, step-modulated sinusoidal signal, and phase-varying rectangular pulse[35] for a homogeneous, lossy, half-space or slab geometry. Computer Program 5 calculates the reflected waveforms of a unit-step, step-modulated sinusoidal wave, and a short gaussian pulse from an inhomogeneous plasma[38] or homogeneous plasma half-space with a vertical magnetic field.

$\omega_p t$	Analytic[37]	$\omega_p \Delta t = 0.01$	$\omega_p \Delta t = 0.1$
1	-0.1199	-0.1188(-0.1201)	-0.1094(-0.1218)
2	-0.4233	-0.4215(-0.4249)	-0.4061(-0.4394)
3	-0.7740	-0.7723(-0.7798)	-0.7578(-0.8303)
5	-1.1310	-1.1309(-1.1658)	-1.1302(-1.4657)
8	-0.9413	-0.9415(-1.3492)	-0.9429(-4.8505)
10	-0.9913	-0.9910(-3.2625)	-0.9884(-22.914)
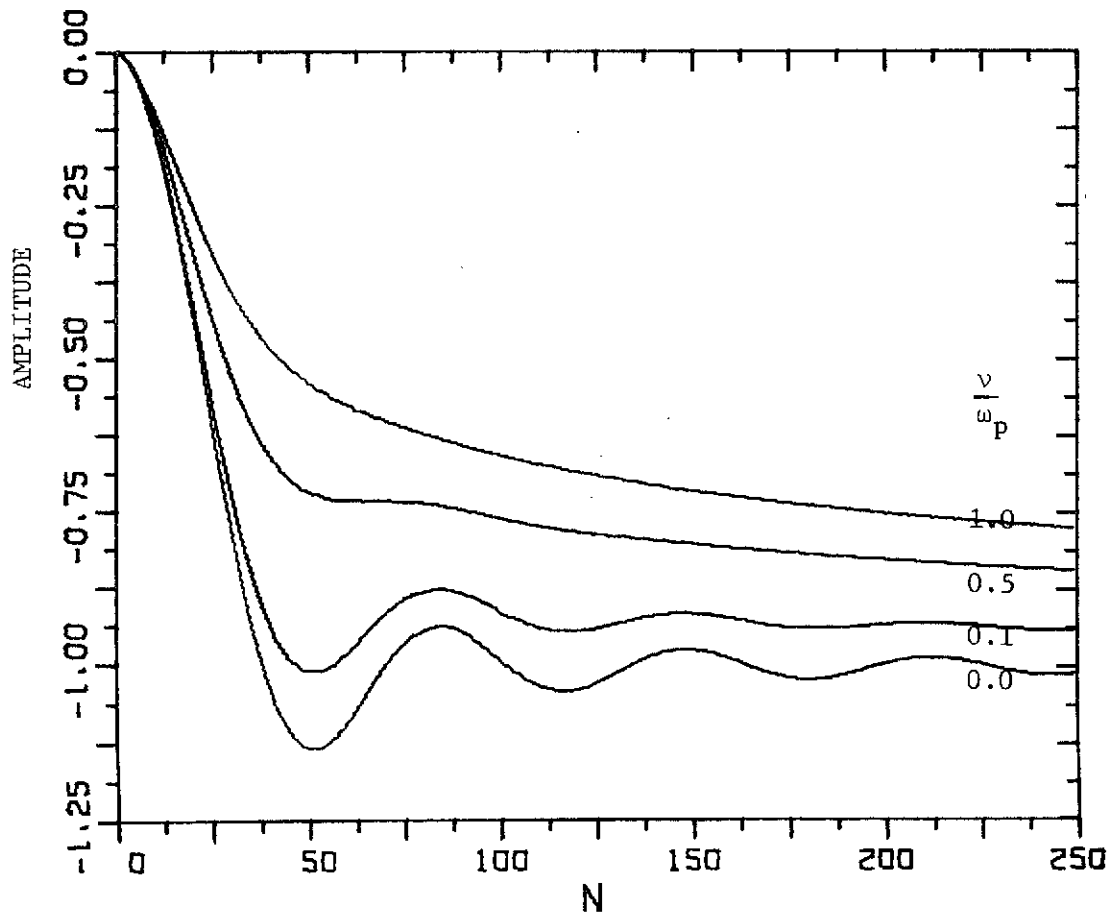
Fig. 23--Reflection of a unit-step signal from an isotropic
plasma half-space versus N. $T = N\omega_p \Delta t$, $\omega_p \Delta t = 0.1$.
(Entries in parenthesis show the case in which
E_x is not approximated by 3 successive values)
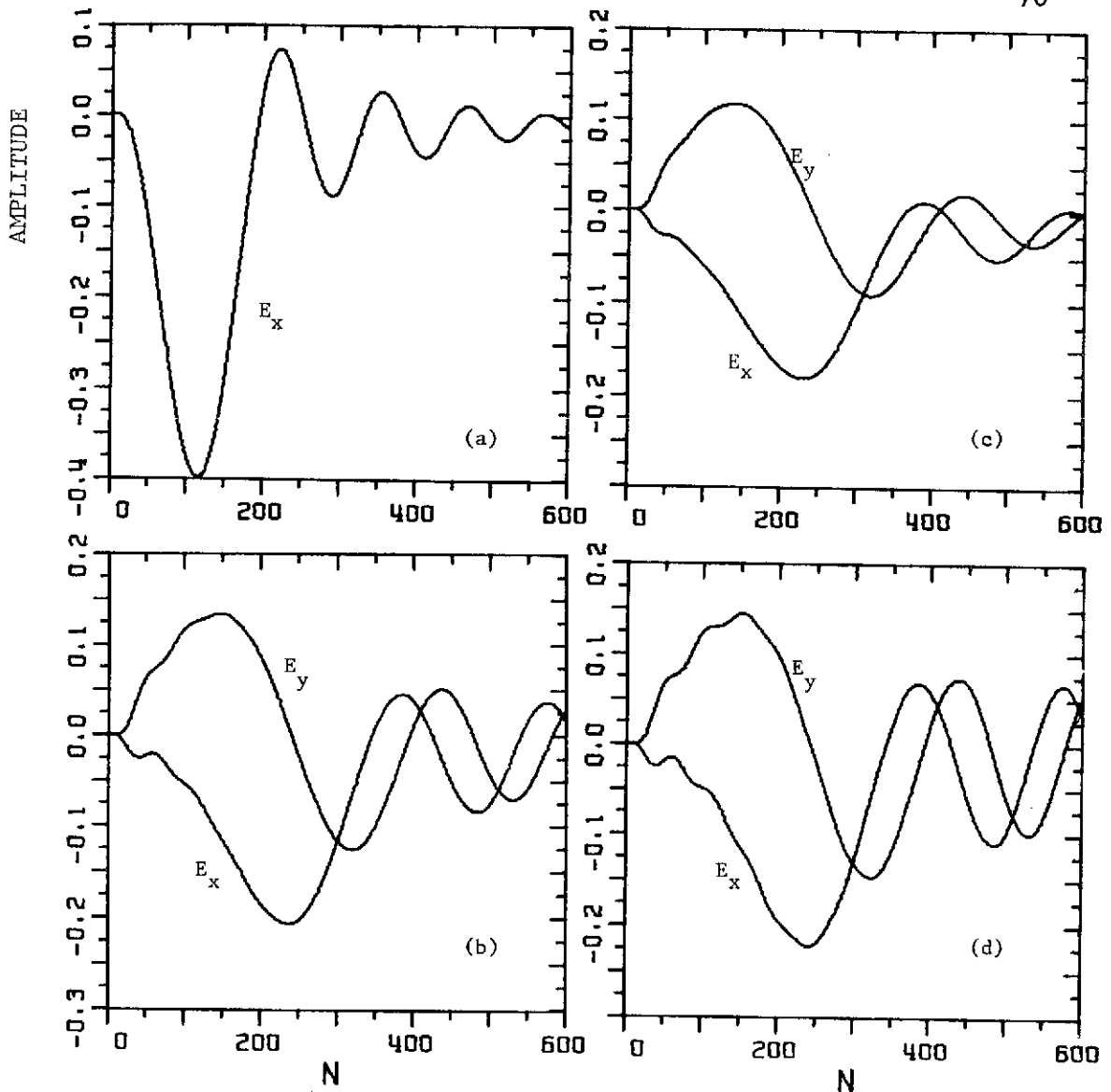
Fig. 24--Reflected waveforms of a gaussian short pulse normally incident on an inhomogeneous plasma having a linear electron density profile with a vertical magnetic field versus N, t=NΔt, Δt=17.4 nanoseconds.

(a) $\nu=10^6$ sec^{-1}, and $\omega_c=7\times10^6$ for

(b) $\nu=10^6$ sec^{-1},

(c) $\nu=2\times10^6$,

(d) $\nu=0.5\times10^6 \cdot E_y^{inc}=0, E_x^{inc}= \dfrac{1}{\sqrt{2\pi(0.02k)^2}} \exp[-(N-11)^2/2k^2]$

$(k=4, \ 0\leq N\leq 22)$.
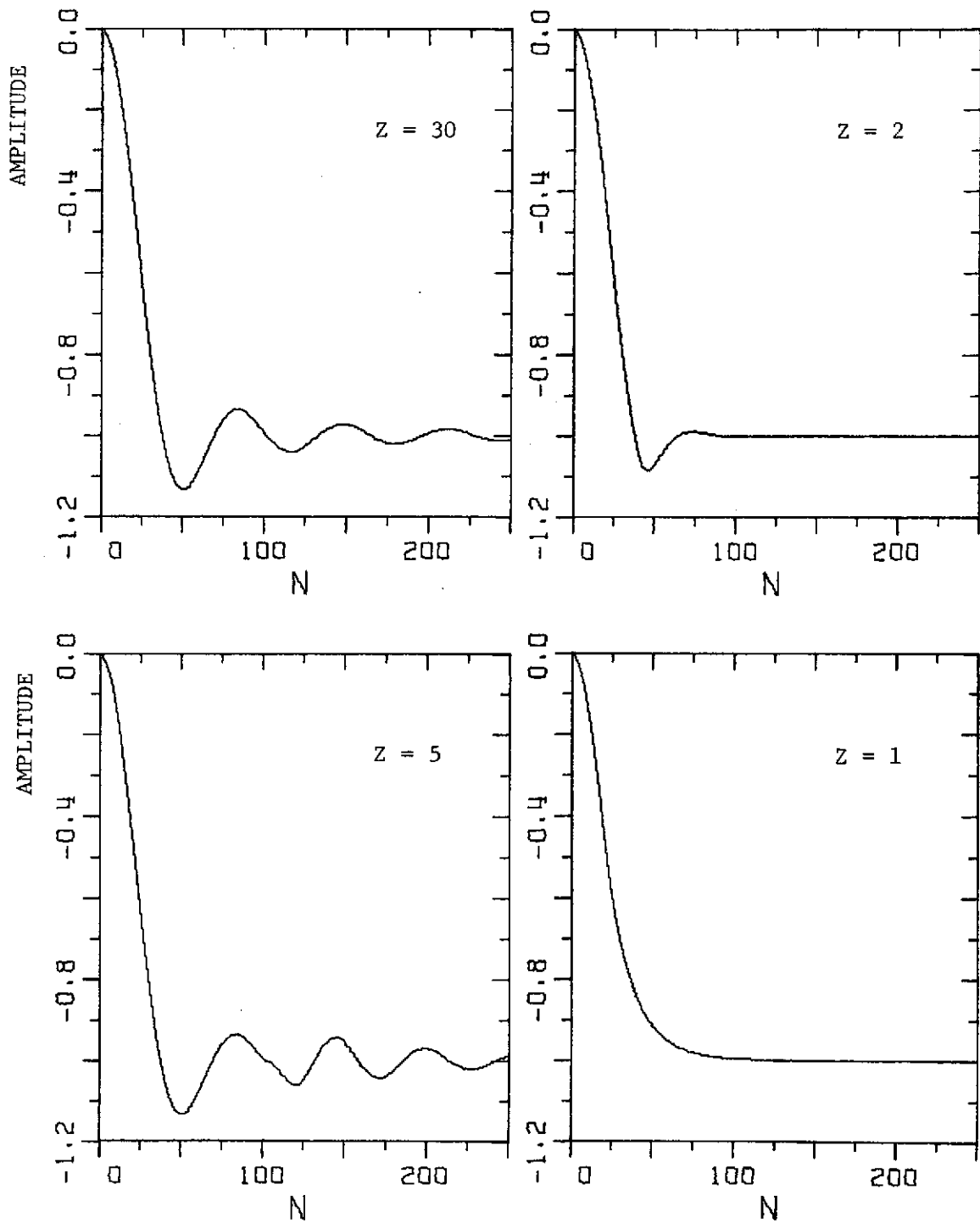
Fig. 25--Reflected waveform of a unit-step signal normally
incident on a homogeneous cold plasma slab of thickness Z.
T=0.1N
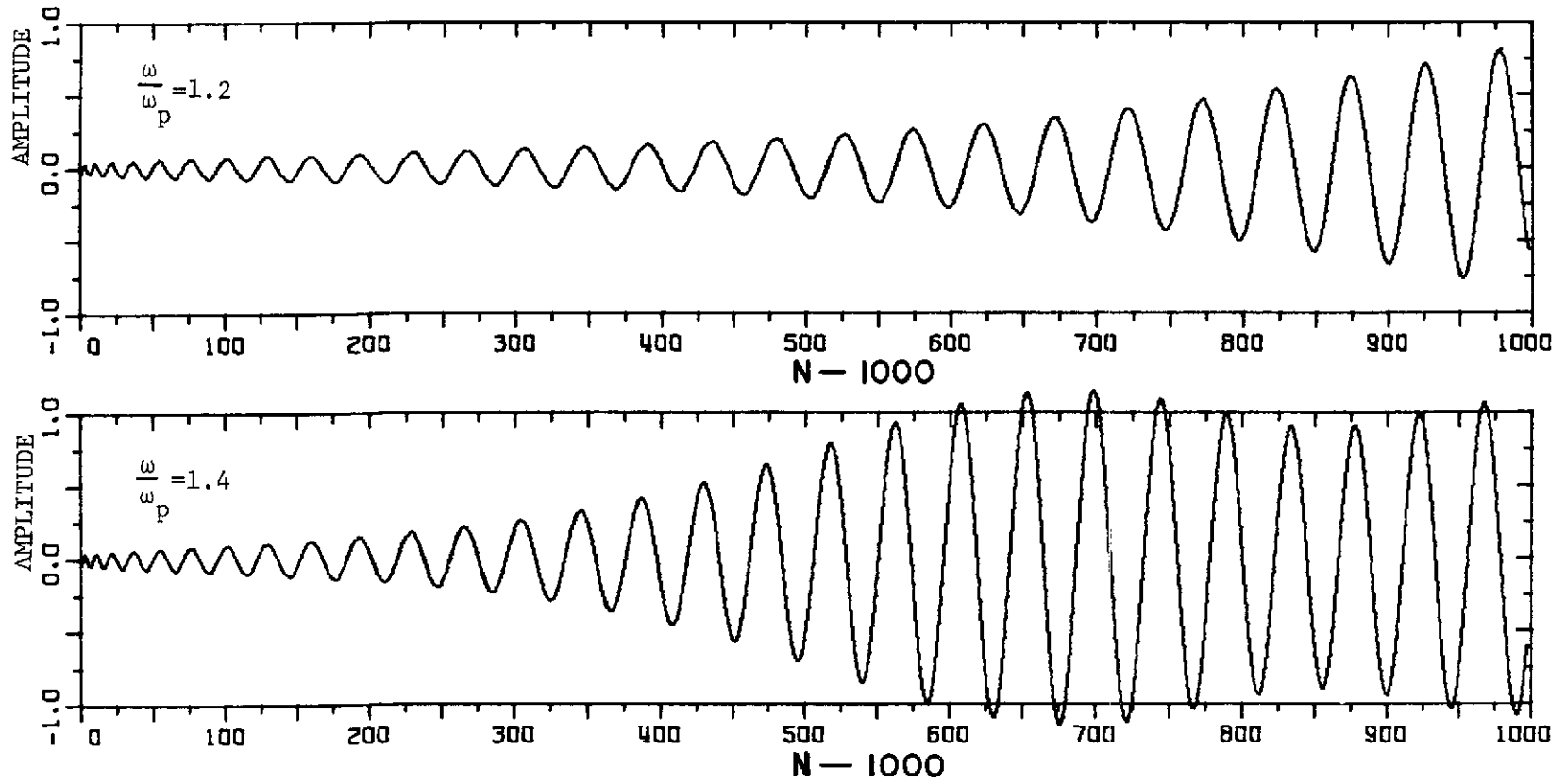
Fig. 26--Received waveform at Z=100 for a step-modulated sinusoidal signal in a lossless isotropic plasma, T=0.1N.
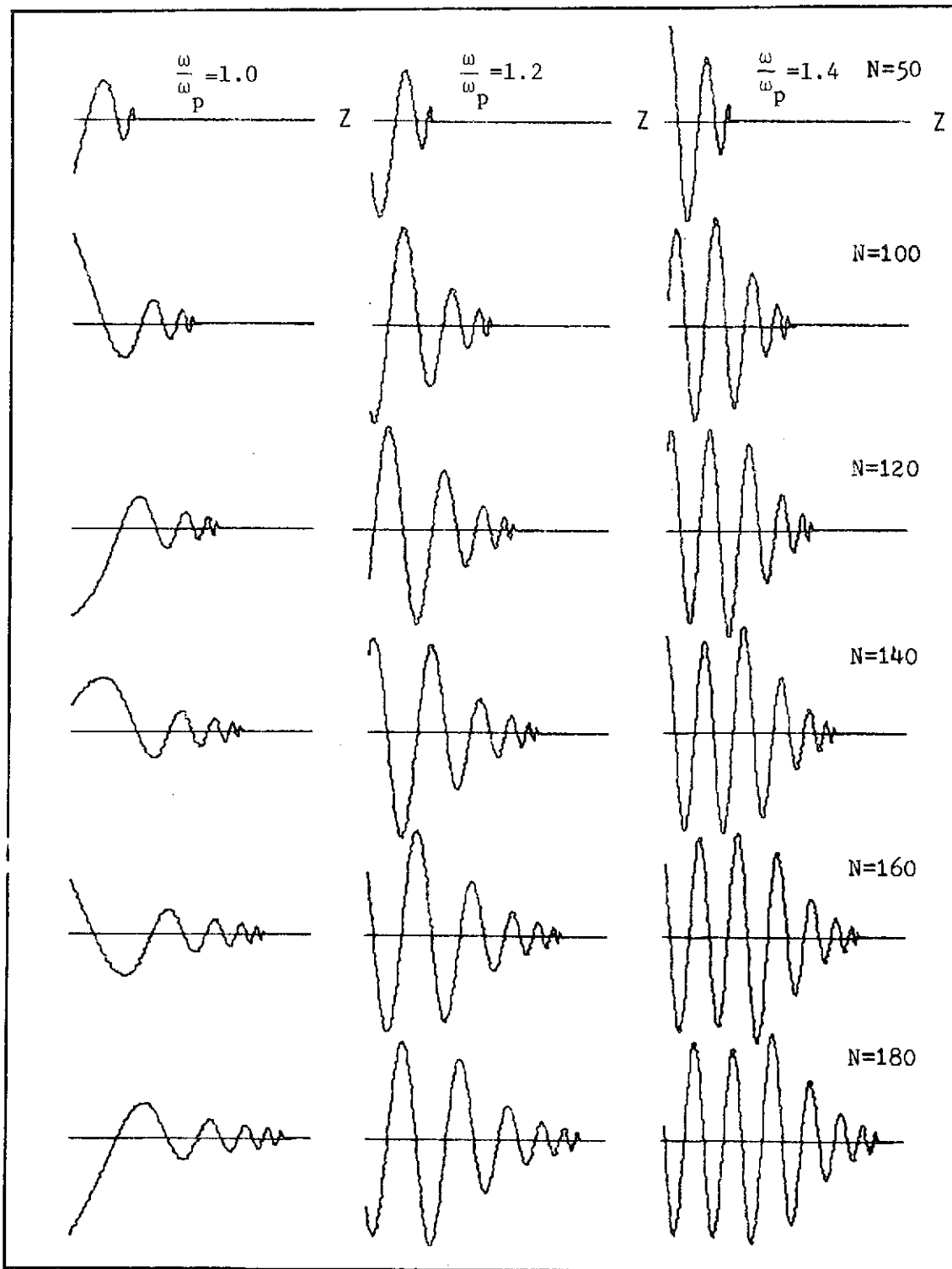
Fig. 27--Spatial variation of a sinusoidal signal in a
lossless plasma at several normalized times.
$T=N\omega_p\Delta t$, $\omega_p\Delta t=0.2$

CHAPTER VI
UNIT-STEP SIGNAL REFLECTED FROM A DEBYE DIELECTRIC

A. Introduction

Fellner-Feldegg[40,41] and others[42] have recently demonstrated quantitatively the feasibility of determining the dielectric constant of dielectrics over a wide band of frequencies by utilizing time-domain reflectometry, i.e., by measuring the reflection of a unit-step signal from a sample of the dielectric terminating a coaxial transmission line. In order to investigate the possibility of a direct determination of the low frequency permittivity, the high frequency permittivity, and the relaxation time spectrum of a dielectric in the time domain, theoretical calculations of the reflected waveform of a signal must be made. This has been done in the past using Laplace transform method[43,44]. In this chapter it is shown that this problem can also be solved effectively by the finite difference method. An explicit difference equation is derived from the governing time-dependent integro-differential equation for the electric field in a Debye dielectric with a single relaxation time. Employing this difference equation the time history of a signal of arbitrary waveform can be generated by marching on in time. Thus the response of a nonideal unit-step signal with finite rise-time can be computed. This may give a better comparison wih the measurements, because in practice a signal always has a finite rise-time. As an example of the treatment of the Debye dielectric case, we shall also calculate the reflected waveform of a unit-step signal incident on an ice layer on water. Here the water is assumed to be a Debye dielectric.

B. An Integro-Differential Equation for the Electric Field

Consider the Maxwell's equations

$$(122) \qquad \frac{\partial D_x}{\partial t} + \frac{\partial H_y}{\partial z} = 0,$$

$$\mu_o \frac{\partial H_y}{\partial t} + \frac{\partial E_x}{\partial z} = 0$$

which is equivalent to

$$(123) \qquad \frac{\partial^2 D_x}{\partial t^2} - \frac{1}{\mu_o} \frac{\partial^2 E_x}{\partial z^2} = 0 .$$

For a Debye dielectric with a single relaxation time, we have[12]

(124) $\quad D_x(z,\omega) = \varepsilon^*(\omega)E_x(z,\omega)$

where

(125) $\quad \varepsilon^*(\omega) = \varepsilon_\infty + \dfrac{\varepsilon_\ell - \varepsilon_\infty}{1 + j\omega\tau_0}$

and $\varepsilon_\ell, \varepsilon_\infty$ are the low and high frequency permittivities, τ_0 is the relaxation time. Then (124) and (125) are transformed into the time domain

$$D_x(z,t) = \int_{-\infty}^{\infty} \varepsilon(t-\beta)E_x(z,\beta)d\beta \ ,$$

$$\varepsilon(t) = \varepsilon_\infty\delta(t) + \frac{\varepsilon_\ell - \varepsilon_\infty}{\tau_0} \exp\left(-\frac{t}{\tau_0}\right) u(t)$$

and thus,

(126) $\quad D_x(z,t) = \varepsilon_\infty E_x(z,t) + \dfrac{\varepsilon_\ell-\varepsilon_\infty}{\tau_0} \displaystyle\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\beta)}{\tau_0}\right) E_x(z,\beta)u(t-\beta)d\beta.$

Here the lower integration limit can be changed to 0 because of causality of the fields, and the upper integration limit can be changed to t by omitting the unit-step function in the integrand. We differentiate (126) twice with respect to t and obtain from (123)

(127) $\quad \dfrac{\partial^2 E_x}{\partial t^2} - c_\infty^2 \dfrac{\partial^2 E_x}{\partial z^2} + \left(\dfrac{\sigma_0}{\varepsilon_\infty} + a\omega_0\right)\dfrac{\partial E_x}{\partial t} - a\omega_0^2 E_x(z,t)$

$$= -a\omega_0^3 \int_0^t \exp[-\omega_0(t-\beta)]E_x(z,\beta)d\beta$$

where $\quad c_\infty = \dfrac{1}{\sqrt{\mu_0\varepsilon_\infty}}$, $a = \dfrac{\varepsilon_\ell - \varepsilon_\infty}{\varepsilon_\infty}$, and $\omega_0 = \dfrac{1}{\tau_0}$.

For completeness the static conductivity σ_0 of the Debye dielectric (as in the case of sea water) is included. (127) governs the electric field component of a plane electromagnetic wave in a Debye dispersive medium.

C. The Difference Equation

In this section we shall derive the difference equation from which one can obtain the reflected waveforms of a signal when it is suddenly applied to the air dielectric interface. The geometry of the problem is shown in Fig. 28. The Debye dielectric is first approximated by thin layers of thickness

$$(128) \qquad \Delta z_2 = \frac{1}{\sqrt{\mu_0 \epsilon_\infty}} \, \Delta t$$

where Δt is arbitrary but small. There are two difference equations needed: one for the interior point in the dielectric, and one for the interface point. However no calculation is needed for the points in the air, as discussed in Section II.H, for the signal incident from $z < 0$.

Employing the notation $E_i^n = E(z,t)_{\substack{z=i\Delta z \\ t=n\Delta t}}$, the difference equation

for interior points can be written directly from (127)

$$(129) \qquad \frac{E_i^{n-1} - 2E_i^n + E_i^{n-1}}{\Delta t^2} - c_\infty^2 \frac{E_{i+1}^n - 2E_i^n + E_{i-1}^n}{\Delta z_2^2} +$$

$$+ \left(\frac{\sigma_0}{\epsilon_\infty} + a\omega_0 \right) \frac{E_i^{n+1} - E_i^{n-1}}{2\Delta t} - a\omega_0^2 E_i^n = -a\omega_0^3 \Delta t \, S_i^n .$$

Here S_i^n denotes $S(z,t)_{\substack{z=i\Delta z \\ t=n\Delta t}}$ which is defined as

$$S(z,t) = \frac{1}{\Delta t} \int_0^t \exp[-\omega_0(t-\beta)]E(z,\beta)d\beta = \frac{1}{\Delta t} \left(\int_0^{t-\Delta t} + \int_{t-\Delta t}^t \right)$$

$$\cdot \exp[-\omega_0(t-\beta)]E(z,\beta)d\beta$$

Thus,

$$(130) \qquad S_i^n = gS_i^{n-1} + \frac{1}{2} (E_i^n + gE_i^{n-1}), \quad g = \exp(-\omega_0\Delta t).$$

Employing the scheme (128), Eq. (129) can be simplified to

$$(131) \quad \left(1 + \frac{\sigma \Delta t}{2\varepsilon_\infty}\right) E_i^{n+1} = -\left(1 - \frac{\sigma \Delta t}{2\varepsilon_\infty}\right) E_i^{n-1} + E_{i+1}^n + E_{i-1}^n +$$

$$+ a(\omega_0 \Delta t)^2 E_i^n - a(\omega_0 \Delta t)^3 S_i^n$$

where $\sigma = \sigma_0 + a\omega_0 \varepsilon_\infty$. Equation (131) applies to all grid points in the Debye dielectric.

The difference equation for the interface point can be derived by the same procedure as used several times previously. Refer to Fig. 28. In the vicinity of the air/dielectric interface we have

$$(132) \quad \frac{\partial^2 E(z^-,t)}{\partial t^2} - c^2 \frac{\partial^2 E(z^-,t)}{\partial z^2} = 0 \quad \left(c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right),$$

$$\frac{\partial^2 E(z^+,t)}{\partial t^2} - c_\infty^2 \frac{\partial^2 E(z^+,t)}{\partial z^2} + \frac{\sigma}{\varepsilon_\infty} \frac{\partial E(z^+,t)}{\partial t} - a\omega_0^2 E(z^+,t) + F(z^+,t) = 0$$

where

$$F(z^+,t) = a\omega_0^3 \int_0^t e^{-\omega_0(t-\beta)} E(z^+,\beta) d\beta$$
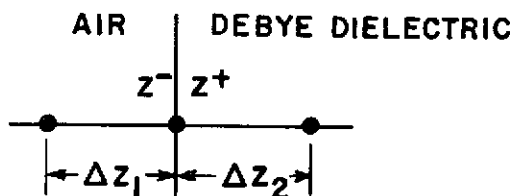
Fig. 28--Interface between air-Debye dielectric.

The boundary conditions at the interface are

$$E(z^+,t) = E(z^-,t),$$

(133)

$$\frac{\partial E(z^+,t)}{\partial z} = \frac{\partial E(z^-,t)}{\partial z} \quad .$$

To derive the difference equation from (132) and (133), we approximate the air and dielectric by thin layers with different thickness in each region such that

$$\Delta z_1 = c \, \Delta t,$$

(134)

$$\Delta z_2 = c_\infty \Delta t.$$

We then expand the field at $z^+ + \Delta z_2$ and $z^- - \Delta z_1$ by the Taylor's series

$$E(z^+ + \Delta z_2, t) = E(z^+, t) + \frac{\partial E(z^+,t)}{\partial z} \Delta z_2 + \frac{\partial^2 E(z^+,t)}{\partial z^2} \frac{\Delta z_2^2}{2} \quad ,$$

(135)

$$E(z^- - \Delta z_1, t) = E(z^-, t) - \frac{\partial E(z^-,t)}{\partial z} \Delta z_1 + \frac{\partial^2 E(z^-,t)}{\partial z^2} \frac{\Delta z_1^2}{2} \quad .$$

The second partial derivatives in the above expressions are first substituted by (132), the first partial derivatives are eliminated by the use of the boundary conditions in (133) and second partial derivatives with respect to t are approximated by the central finite differences

$$\frac{\partial^2 E(z^\pm,t)}{\partial t^2} = \frac{E(z^\pm,t+\Delta t)-2E(z^\pm,t) + E(z^\pm,t-\Delta t)}{\Delta t^2} \quad .$$

Completing all these steps we obtain the difference equation for the interface point

(136) $$\alpha E_i^{n+1} = - \beta E_i^{n-1} + E_{i+1}^n + \sqrt{\frac{\epsilon_0}{\epsilon_\infty}} \, E_{i-1}^n + \frac{1}{2} \, [a(\omega_0 \Delta t)^2 E_i^n - a(\omega_0 \Delta t)^3 S_i^n]$$

where

$$\alpha = \frac{1}{2}\left(1 + \sqrt{\frac{\varepsilon_0}{\varepsilon_\infty}} + \frac{\sigma\Delta t}{2\varepsilon_\infty}\right) \; ,$$

$$\beta = \frac{1}{2}\left(1 + \sqrt{\frac{\varepsilon_0}{\varepsilon_\infty}} - \frac{\sigma\Delta t}{2\varepsilon_\infty}\right) \; .$$

Since (136) will be used only on the air/dielectric interface at the point i=2, we can write it specifically

$$(137) \qquad \alpha E_2^{n+1} \doteq -\beta E_2^{n-1} + E_3^n + \sqrt{\frac{\varepsilon_0}{\varepsilon_\infty}}\, E_1^n + \frac{1}{2}\,[a(\omega_0\Delta t)^2 E_2^n - a(\omega_0\Delta t)^3 S_2^n].$$

Here the quantity E_1^n represents the __total__ field for a point in the air. Thus, for the unit-step signal

$$\text{Total field } E_1^n = \text{Incident field} + \text{Reflected field}$$

$$= 1.0 + (E_2^{n-1} - 1.0)$$

$$\therefore \quad E_1^n = E_2^{n-1} \, .$$

The complete algorithm for the solution of a unit-step signal reflected from a Debye dielectric is then given by

n=0
$$E_i^n = \begin{array}{ll} 1.0 & (i=1) \\ 0.0 & (i>1) \end{array}$$

n=1
$$E_i^n = \begin{cases} 1.0 & (i=1) \\ \dfrac{1}{\alpha}\sqrt{\dfrac{\varepsilon_0}{\varepsilon_\infty}}\; E_1^{n-1} & (i=2) \\ 0.0 & (i>2), \end{cases}$$

(138)

n=2,3,4···
$$E_i^n = \begin{cases} E_2^{n-1} & (i=1) \\ (137) & (i=2) \\ (131) & (i>2) \; . \end{cases}$$

D. Numerical Results

Figure 29 shows the reflected waveform of a unit-step signal from Debye dielectric with $\varepsilon_\ell=80$, $\varepsilon_\infty=8$, and $\sigma_0\tau_0/\varepsilon_0=0$, 11.3, 113, 1130. These results agree excellently with those obtained by employing the Laplace transform method[44]. Note that the amplitude of the waveform for early and late times corresponds to the reflection coefficient for high and low frequencies respectively for the lossless case[44].

We next consider an ice layer on water as shown in Fig. 30. The thickness of the ice is 30 cm and its permittivity is $\varepsilon_1=3$. The water is assumed to be a Debye dielectric with $\varepsilon_\ell=4.5$, $\varepsilon_\infty=81$ and τ_0 is the relaxation time. A unit-step signal is suddenly applied to the air/ice interface; after a time of

$$T_o = \frac{0.30}{3\times10^8 \frac{1}{\sqrt{3}}} = 1.73 \times 10^{-9} \text{ seconds}$$

the front of the transmitted signal reaches the ice/water interface; transmission and reflection occurs, the reflected signal takes another T_0 seconds to return to the air/ice boundary. Thus in the period $0<t<2T_0$ the waveform received in the air is solely due to the reflection from the air/ice interface. Now the reflected signal from the ice/water will be reflected again by the air/ice boundary, but this signal will not arrive the ice/water interface until $t=3T_0$. Thus we may neglect the multiple reflections in the ice by terminating the transient response at $3T_0$. The result of such a calculation is shown in Fig. 30. This problem could not be solved practically without the techniques discussed in Section II.H because then a large number of grid points (about 30000) is needed to model the ice. This is because of the exceedingly small relaxation time of the water relative to the time for propagation through the ice.

A computer program for this problem is given in Computer Program 6 in Appendix A. In this program the static conductivity σ_0 is also included. The result in an earlier example (Fig. 29) can also be obtained from this program by setting the permittivity of the second layer, i.e., the ice, equal to 1.
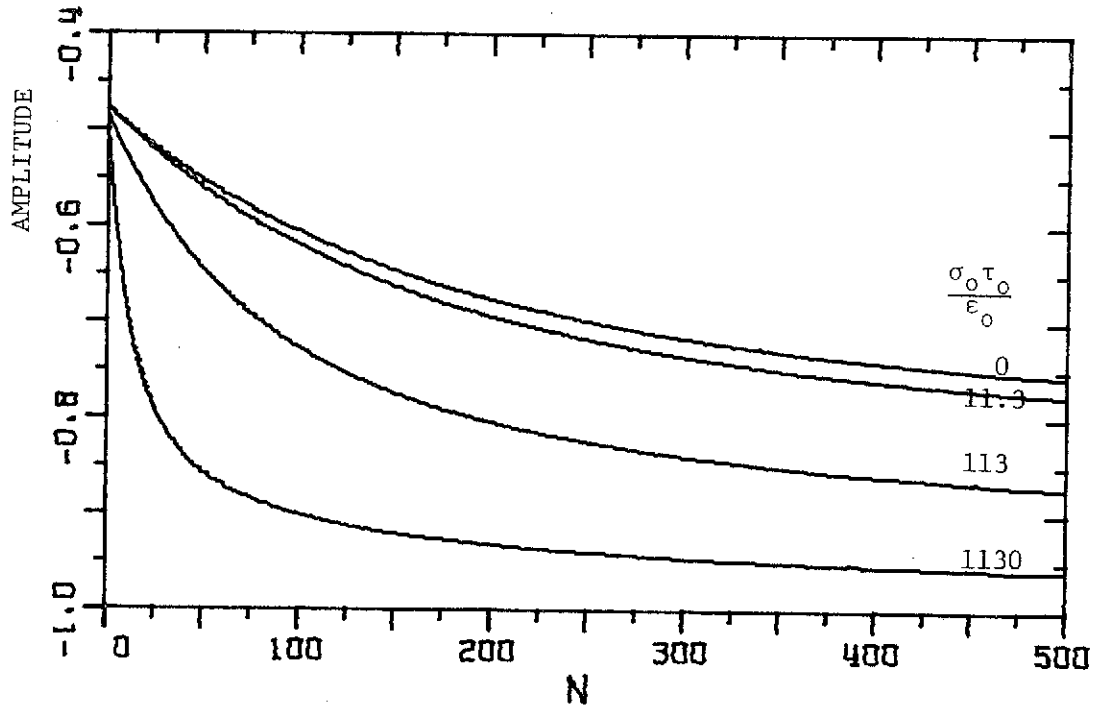
Fig. 29--Reflected waveform of a unit-step signal normally
incident on a Debye dielectric, $\varepsilon_\ell=80$, $\varepsilon_\infty=8$,
including static conductivity σ_0.
$$\frac{t}{\tau_0} = N\frac{\Delta t}{\tau_0}, \quad \frac{\Delta t}{\tau_0} = 0.001.$$
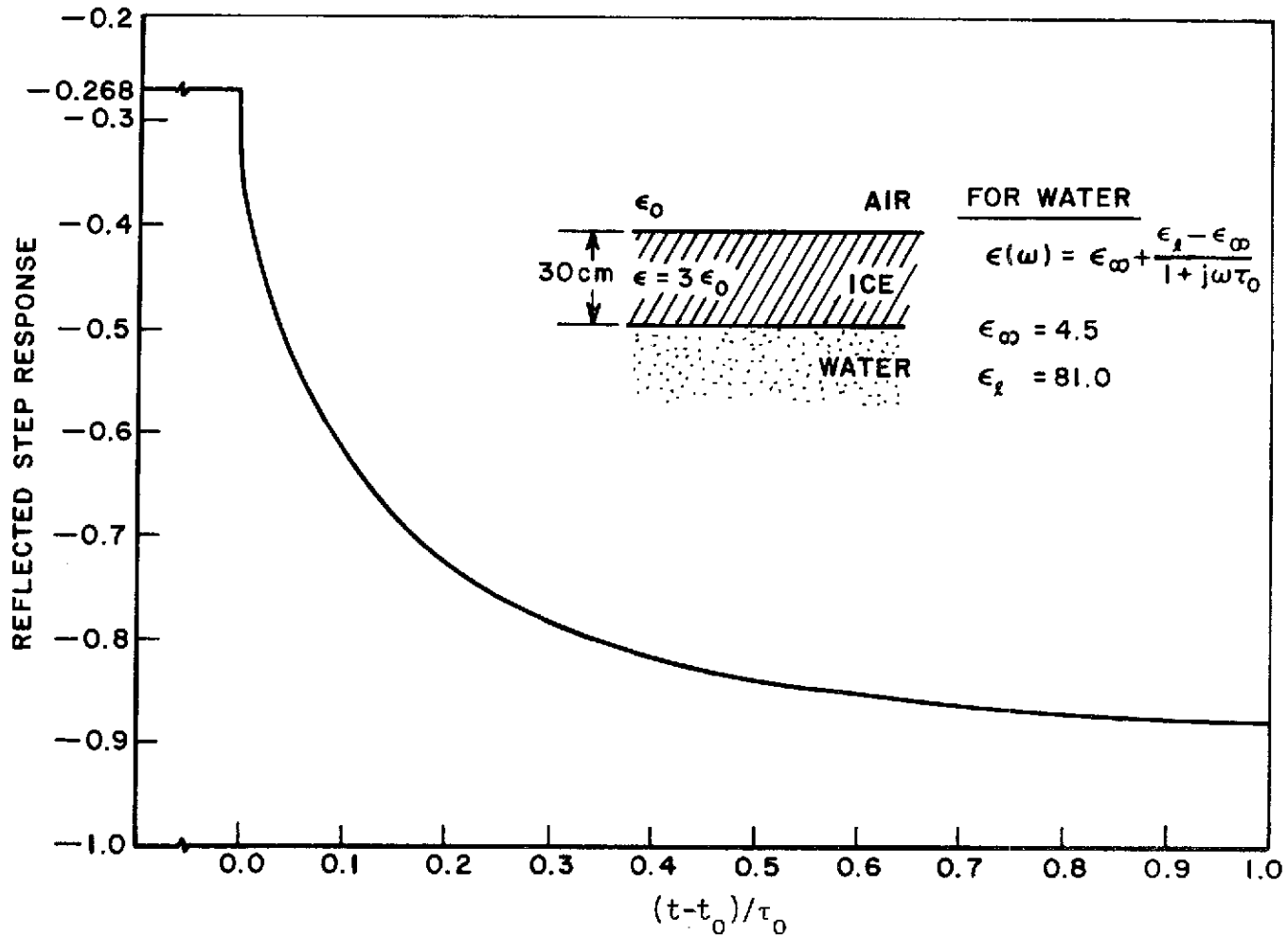
Fig. 30--Reflected waveform of a unit-step signal normally incident on an ice layer of thickness 30 cm on water. t_0=3.46 nanoseconds.

CHAPTER VII
DISCUSSION AND CONCLUSIONS

In the previous chapters we have shown how finite difference methods can be applied to linear dispersive waves. But they are even more powerful for nonlinear waves, which unlike the linear case, cannot be handled by transform methods. One of the nonlinear dispersive waves which caught the attention of many investigators in recent years is the Korteweg-deVries equation[13, 46]

$$(139) \qquad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \delta^2 \frac{\partial^3 u}{\partial x^3} = 0 \; .$$

This equation was originally suggested for shallow water waves (1895), but it has been found to describe the long-time evolution of small, but finite amplitude nonlinear dispersive waves, such as the magneto-hydrodynamic waves and ion-acoustic waves in cold plasmas.

In 1965, Zabusky and Kruskal[13] proposed the difference equation for (139)

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\Delta t} + \frac{1}{3} (u_{i+1}^n+u_i^n+u_{i-1}^n) \frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x} + \delta^2 \frac{1}{2\Delta x^3}$$

$$\cdot (u_{i+2}^n-2u_{i+1}^n+2u_{i-1}^n-u_{i-2}^n) = 0$$

which can be written as

$$(140) \qquad u_i^{n+1} = u_i^{n-1} - \frac{1}{3} \frac{\Delta t}{\Delta x} (u_{i+1}^n+u_{i-1}^n)(u_{i+1}^n+u_i^n+u_{i-1}^n)$$

$$- \frac{\delta^2 \Delta t}{\Delta x^3} (u_{i+2}^n - 2u_{i+1}^n + 2u_{i-1}^n - u_{i-2}^n) \; .$$

Given the initial conditions, (140) can be used to calculate the evolution of the initial waveform. Such a calculation for the case of the periodic initial condition

$$(141) \qquad u(x, t=0) = \cos \pi x$$

is shown in Fig. 31, given by Zabusky and Kruskal. We see that the initial waveform breaks up into eight pulses eventually. After that time these pulses, now called solitons, begin to move without changing their shapes; and the soliton with large amplitude moves faster than the one with smaller amplitude. Thus, after a certain period of time, all these solitons interact and overlap in such a way that the initial waveform reoccurs. This remarkable phenomenon was first discovered in 1965 in the difference solution discussed above. This phenomenon has also been confirmed experimentally and found in several physical nonlinear systems. Because of this remarkable property, great efforts have been spent to devise analytical methods to predict the performance of these nonlinear systems. All these fascinating developments in the last ten years are summarized in a recent article by Scott[15]. Although we have not considered any non-linear problem here, it can be said that the finite difference methods are a powerful tool in the analysis of linear and nonlinear dispersive waves.
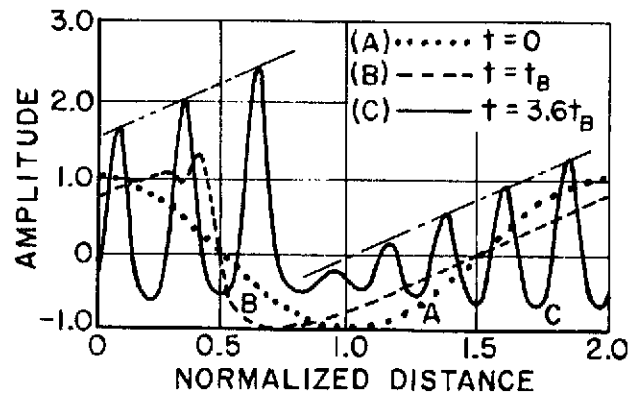
Fig. 31--Solutions of the Korteweg-deV.ries equation, δ=0.022, at three different times. $\left(t_B=1/\pi\right)$

The main thrust of this dissertation is in the development of some finite difference methods for linear dispersive waves. To this end we have obtained simple difference equations for the solution of transient signal propagation in several kinds of stratified dispersive media. We demonstrated the numerical stability and accuracy of the difference equations, and described a number of practical techniques to greatly simplify the calculation procedure. The speed and efficiency with which time domain solutions may now be obtained using these methods lead us to expect many significant applications in the future.

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APPENDIX A
COMPUTER PROGRAMS

Six computer programs which produce most of the results in this dissertation are collected here.

1. Computer Program 1 calculates the reflected waveform of a unit-step sinusoidal wave normally incident on a specific inhomogeneous dielectric slab in [11]. Input data include:

   EPSR = 0.5 or 2.0; ε_r in Casey's paper $(\varepsilon_r = \varepsilon_2/\varepsilon_1)$
   NSTEP= number of time steps to be calculated
   XL = normalized thickness L of the slab
   D = 1.0 if a transition layer,
       2.0 if a dielectric duct.

2. Computer Program 2 calculates the reflection of a sine-squared pulse from a three-layered lossy dielectric medium[26]. Input data are:

   Z = thickness of the second layer
   NSTEP = number of time steps to be calculated
   DT = time step size (e.g., $\Delta t = 2.5 \times 10^{-9}$ sec.)

3. Computer Program 3 generates the results in Chapter IV - Sommerfeld and Brillouin's transient problem. Input data are

   NSTEP = number of time steps to be calculated
   WPDT = $\omega_p \Delta t$
   WOWP = ω_0/ω_p
   PWO = ρ/ω_0
   IOBSER= $Z_1/\omega_p\Delta t + 1$
   IOB2 = $Z_2/\omega_p\Delta t + 1$
   WWP = ω/ω_p.
   ($Z_2 < Z_1$ where Z_1, Z_2 are spatial observation points)

4. Computer Program 4 calculates the propagation and reflection of a transient unit-step, step carrier, and a chirp pulse in a homogeneous lossy plasma half-space or slab geometry. The input data are

   ICASE = 1 for propagation, 2 for reflection
   GWP = ν/ω_p
   NSTEP = number of time steps to be calculated
   WPDT = $\omega_p \Delta t$

Z   = $\omega_p z/c$. In the case of propagation (ICASE=1) it is the observation point. In the case of reflection (ICASE=2) it represents the thickness of the plasma slab.

WWP = ω/ω_p, frequency of the step carrier if applicable

IP   = 1 unit-step
         2 step carrier
         3 chirp pulse.

5.   Computer Program 5 computes the reflected waveform of a short gaussian pulse, a unit-step and a step carrier normally incident on an inhomogeneous lossy plasma (specifically the linear electron density profile[38]) or a homogeneous cold plasma half-space in which a longitudinal dc magnetic field is present.   The input data are

WPDT1 = $\omega_p \Delta t$ if a homogeneous plasma, or a normalized factor for the incident short gaussian pulse.

WCDT = $\omega_c \Delta t$

GDT   = $\nu \Delta t$

WWP   = ω/ω_p

NSTEP = number of time steps to be calculated

IP   = 1 unit-step
         2 step carrier
         3 gaussian short pulse

DT   = 0 if a homogeneous half-space, or the time step size Δt otherwise.

6.   Computer Program 6 computes the reflected waveform of a unit-step signal normally incident to an ice layer on water in which water assumes a Debye model. The input data are

NSTEP1 = number of time steps to be calculated

WODT   = $\omega_0 \Delta t$ ($\tau_0 = 1/\omega_0$ is the relaxation time)

EPS1   = dielectric constant of the second layer (ice in this case)

EPS20 = ε_ℓ low frequency dielectric constant

EPS2N = ε_∞ high frequency dielectric constant

CONDO = normalized low frequency conductivity of the Debye dielectric
           $\sigma_0 \tau_0 / \varepsilon_0$.

```
1 C        ********** COMPUTER PROGRAM 1 **********
2 C        REFLECTED WAVE FROM AN INHOMOGENEOUS DIELECTRIC SLAB
3 C        REFERENCE: CASEY, IEEE G-AP, MAY 1972, PP. 368-374
4          DIMENSION F0(500),F1(500),F2(500),EPS(500),DZ(500),
5        1          WAFORM(500)
6          DATA PI,PI2,Z0,FOLD,DZ(1)/3.14159,6.2832,0.0,0.0,0.0/
7          READ(8,-) EPSR, NSTEP, XL, D
8          WRITE(6,9) NSTEP, EPSR, XL, D
9      9   FORMAT(' NSTEP=',I4/,' EPSR =',F4.2/,'   L =',F5.3/,
10       1' D(TRANSITION LAYER,D=1.0; DUCT,D=2.0)=',F3.1//)
11 C        PROFILE OF THE INHOMOGENEOUS DIELECTRIC
12         FDT=1.0/36.0
13         AA=4.0*FDT/XL
14         DEL=(EPSR-1.0)/(EPSR+1.0)
15         EPS(1)=1.0-DEL
16         DO 10 I=2, 500
17         EPS0=1.0-DEL*COS(PI*Z0)
18         DELZD=AA/SQRT(EPS0)
19         EPS1=1.0-DEL*COS(PI*(Z0+DELZD))
20         EPS(I)=0.5*(EPS0+EPS1)
21         DZ(I)=AA/SQRT(EPS(I))
22         Z0=Z0+DZ(I)
23     10  IF(Z0.GT.D) GO TO 11
24     11  IEND1=I
25         IEND=IEND1+1
26         WRITE(6,16) (EPS(I),I=1,IEND1)
27         DO 12 I=1, IEND1
28     12  EPS(I)=SQRT(EPS(I))
29 C        CW SOURCE AT I=1; AIR/DIELECTRIC INTERFACE AT I=2
30         DO 13 I=1,IEND
31         F0(I)=0.0
32     13  F1(I)=0.0
33         F1(1)=SIN(PI2*FDT)
34         F1(2)=2.0*EPS(1)*F0(1)/(EPS(1)+EPS(2))
35         WAFORM(1)=F1(2) - F0(1)
36         DO 15 N=2, NSTEP
37         F2(1)=SIN(N*PI2*FDT)
38         IF(N.LT.IEND1) MOVING=N+1
39         DO 14 I=2, MOVING
40         TOTALF=F1(I-1)
41         IF(I.EQ.2) TOTALF=F1(1)+(F0(2)-FOLD)
42         T12=2.0/(EPS(I-1)+EPS(I))
43     14  F2(I)=-F0(I)+T12*(EPS(I-1)*TOTALF+EPS(I)*F1(I+1))
44         F2(IEND)=F1(IEND1)
45         WAFORM(N)=F2(2)-F1(1)
46         FOLD=F0(1)
47         DO 15 I=1, IEND
48         F0(I)=F1(I)
49     15  F1(I)=F2(I)
50     16  FORMAT(' DIELECTRIC PROFILE'/,100(10F6.3/)/)
51         WRITE(6,17) (WAFORM(N),N=1,NSTEP)
52     17  FORMAT(' CW REFLECTED WAVEFORM'/,100(10F6.3/))
53         END
```

```
 1 C      *********** COMPUTER PROGRAM 2 ***********
 2 C      DIRECT TIME DOMAIN SOLUTION OF EMP REFLECTED FROM
 3 C      A THREE-LAYERED LOSSY DIELECTRIC MEDIUM.
 4 C      REFERENCE: SIVAPRASAD & STOTZ, IEEE GEOSC. ELECTRONICS,
 5 C              PP. 161-164, JULY 1973.
 6        DIMENSION F0(500), F1(500), F2(500), WAFORM(500)
 7        DATA EPS1,EPS2,EPS3/1.0, 7.0, 81.0/
 8        DATA COND1, COND2, COND3/0.0, 0.001, 0.001/
 9        DATA PI, TAU, EPS0/3.141592, 0.1E-6, 8.854E-12/
10        READ(8,-) Z, NSTEP, DT
11        IEND=2 + NSTEP/2
12        IEND1=IEND-1
13        DZ=DT*(3.0E+8/SQRT(EPS2))
14        ID=Z/DZ
15        I2=2+ID
16        Q1=COND1*DT/(2.0*EPS0*EPS1)
17        Q2=COND2*DT/(2.0*EPS0*EPS2)
18        Q3=COND3*DT/(2.0*EPS0*EPS3)
19        C3=SQRT(EPS2/EPS1)
20        C0=0.5*(C3*(1.0+Q2)+(1.0+Q1))
21        C1=0.5*(C3*(1.0-Q2)+(1.0-Q1))
22        D0=1.0+Q2
23        D1=1.0-Q2
24        E3=SQRT(EPS3/EPS2)
25        E0=0.5*(E3*(1.0+Q3)+(1.0+Q2))
26        E1=0.5*(E3*(1.0-Q3)+(1.0-Q2))
27        G0=(1.0+Q3)
28        G1=(1.0-Q3)
29        DO 10 I=2, IEND
30        F0(I)=0.0
31    10  F1(I)=0.0
32 C      A SINE-SQUARE PULSE IS GENERATED AT I=1
33        F0(1)=0.0
34        F1(1)=(SIN(DT*PI/TAU))**2
35        F1(2)=F0(1)/C0
36        WAFORM(1)=F1(2)-F0(1)
37        EOLD=0.0
38        DO 100 N=2, NSTEP
39        T=N*DT
40        IF(T .LE. TAU) F2(1)=(SIN(T*PI/TAU))**2
41        IF(T .GT. TAU) F2(1)=0.0
42        TOTALF=F1(1) + (F0(2)-EOLD)
43 C      I=2 IS THE AIR/DIELECTRIC INTERFACE
44        F2(2)=(-C1*F0(2)+C3*F1(3)+TOTALF)/C0
45        IF(N .LT. IEND1) MOVING=N+1
46        IF(N .GT. IEND1) MOVING=MOVING-1
47 C      I2 IS THE (2ND/3RD) LAYERS' INTERFACE
48        DO 20 I=3, MOVING
49        IF(I .LT. I2) F2(I)=(-D1*F0(I)+F1(I-1)+F1(I+1))/D0
50        IF(I .EQ. I2) F2(I)=(-E1*F0(I)+F1(I-1)+E3*F1(I+1))/E0
51        IF(I .GT. I2) F2(I)=(-G1*F0(I)+F1(I-1)+F1(I+1))/G0
52    20  CONTINUE
```

```
53 C      PICK UP THE REFLECTED WAVEFORM & STORE PREVIOUS FIELDS
54        WAFORM(N)=F2(2)-F1(1)
55        EOLD=F0(1)
56        DO 30 I=1, MOVING
57        F0(I)=F1(I)
58   30   F1(I)=F2(I)
59   100  CONTINUE
60        Z=ID*DZ
61        WRITE(6,31) NSTEP,Z,DT,(WAFORM(N),N=1,NSTEP)
62   31   FORMAT(' NSTEP=',I4/,'   Z  =',F7.4/,'  DT  =',E10.3//,
63        1' SINE-SQUARED PULSE REFLECTED WAVEFORM'///,50(10F6.3/))
64        END
```

```
 1   NSTEP= 150
 2     Z  = 4.9891
 3    DT  =  .220E -8
 4
 5   SINE-SQUARED PULSE REFLECTED WAVEFORM
 6
 7   0.000 -.002 -.009 -.019 -.034 -.053 -.076 -.101 -.130 -.161
 8   -.193 -.227 -.262 -.296 -.330 -.363 -.394 -.422 -.448 -.471
 9   -.490 -.505 -.516 -.523 -.525 -.522 -.516 -.505 -.490 -.472
10   -.450 -.426 -.399 -.371 -.341 -.311 -.281 -.251 -.223 -.197
11   -.173 -.153 -.138 -.128 -.125 -.127 -.134 -.143 -.154 -.165
12   -.176 -.188 -.199 -.211 -.222 -.232 -.241 -.248 -.255 -.259
13   -.262 -.262 -.261 -.258 -.253 -.246 -.237 -.227 -.214 -.201
14   -.186 -.171 -.155 -.138 -.122 -.105 -.089 -.074 -.060 -.047
15   -.035 -.025 -.016 -.009 -.004 -.001  .001  .002  .004  .006
16    .008  .010  .012  .014  .016  .017  .019  .020  .022  .023
17    .024  .024  .024  .025  .024  .024  .023  .022  .021  .020
18    .018  .017  .015  .013  .011  .009  .008  .006  .004  .003
19    .001 -.000 -.001 -.002 -.003 -.003 -.003 -.003 -.004 -.004
20   -.004 -.005 -.005 -.005 -.005 -.006 -.006 -.006 -.006 -.007
21   -.007 -.007 -.007 -.007 -.007 -.007 -.007 -.007 -.007 -.007
```

```
 1 C       ********** COMPUTER PROGRAM 3 **********
 2 C       A NUMERICAL SOLUTION TO SOMMERFELD'S "ABOUT THE
 3 C       PROPAGATION OF LIGHT IN DISPERSIVE MEDIA"--D. H. LAM
 4         DIMENSION F0(1001),F1(1001),F2(1001),C(1001),S(1001),
 5        1          WAFORM(1000),WAVE20(1000)
 6         DATA III,SS,WAFORM(1),WAVE20(1)/2,0.0,0.0,0.0/
 7         READ(8,-) NSTEP,WPDT,W0WP,WWP,PW0,IOBSER,IOB2
 8         WRITE(6,10) NSTEP,WPDT,W0WP,WWP,PW0
 9     10  FORMAT(' NSTEP=',I4/,' WPDT =',F4.2/,' W0WP =',
10        1F4.2/,' WWP  =',F4.2/,' PW0  =',F4.2//)
11         IEND=NSTEP/2+IOBSER
12         IEND1=IEND-1
13         NN1=NSTEP-IOB2
14         IF(W0WP.EQ.0.0.AND.PW0.EQ.0.0) GO TO 11
15         W0DT=W0WP*WPDT
16         PDT=PW0*W0DT
17         ADT=SQRT(W0DT*W0DT-PDT*PDT)
18         A=WPDT*WPDT*2.0*PDT
19         B=WPDT*WPDT*(ADT*ADT-PDT*PDT)/ADT
20         G=EXP(-PDT)
21         CN=COS(ADT)
22         SN=SIN(ADT)
23         GN=G*CN
24     11  Q2=1.0+0.25*WPDT*WPDT
25         Q3=0.5*WPDT*WPDT
26         DO 12 I=1, IEND
27         C(I)=0.0
28         S(I)=0.0
29         F0(I)=0.0
30         F1(I)=0.0
31     12  F2(I)=0.0
32         WDT=WWP*WPDT
33         F1(1)=SIN(WDT)
34         F1(2)=F0(1)/Q2
35         DO 21 N=2, NSTEP
36         F2(1)=SIN(N*WDT)
37         IF(N .LT. IEND1) MOVING=N+1
38         IF(N .GT. IEND1) MOVING=MOVING-1
39         IF(N .GT. NN1  ) III=III+1
40         DO 20 I=III, MOVING
41         IF(W0WP.EQ.0.0.AND.PW0.EQ.0.0) GO TO 20
42         SOLD=S(I)
43         S(I)=G*(CN*S(I)+SN*C(I)+0.5*SN*F0(I))
44         C(I)=G*(CN*C(I)-SN*SOLD)+0.5*(F1(I)+GN*F0(I))
45         SS=A*C(I)+B*S(I)
46     20  F2(I)=-F0(I)+(F1(I-1)+F1(I+1)-Q3*F1(I)+SS)/Q2
47         WAFORM(N)=F2(IOBSER)
48         WAVE20(N)=F2(IOB2)
49         DO 21 I=1, MOVING
50         F0(I)=F1(I)
51     21  F1(I)=F2(I)
52         WRITE(6,22) IOBSER,(WAFORM(N),N=1,NSTEP)
53     22  FORMAT(' WAFORM SEEN AT IOBSER=',I4/,100(10F6.3/))
54         END
```

```
 1 C      ********* COMPUTER PROGRAM 4 *********
 2 C      THIS PROGRAM CALCULATES THE PROPAGATION AND REFLECTION
 3 C      OF A UNIT- STEP, CW, PHASE-VARIATION PULSE IN AN
 4 C      ISOTROPIC PLASMA HALF-SPACE OR SLAB GEOMETRY.
 5        DIMENSION FO(800),F1(800),F2(800),SUM(800),WAFORM(800)
 6        COMMON WDT, WWP, WPDT, Z, IP
 7        READ(8,-) ICASE, GWP, NSTEP, WPDT, Z, WWP, IP
 8        WDT=WWP*WPDT
 9        ID=(Z+0.00001)/WPDT
10        I2=2+ID
11        IOBSER=Z/WPDT + 1
12        IF(ICASE .EQ. 2) IOBSER=2
13        IEND=IOBSER + (NSTEP-IOBSER)/2 + 1
14        IEND1=IEND-1
15        GDT=GWP*WPDT
16        EXP1=EXP(-GDT)
17        Q4=GDT*WPDT*WPDT
18        Q45=0.5*Q4
19        Q3=0.5*WPDT*WPDT
20        Q35=Q3/2.0
21        Q1=1.0+Q35/2.0
22        Q2=1.0+Q3/2.0
23        WRITE(6,8) GWP, WPDT, IEND, NSTEP, Z
24     8  FORMAT(' GDT/WPDT = ',F7.3,' WPDT    = ',F7.3,/,
25       2' # GRID POINT USED = ',I4,' # TIME STEPS RUN = ',I4,//,
26       3' WP*Z/C = ',F5.1,//)
27        WRITE(6, 88) ICASE, IP, WWP, IOBSER
28    88  FORMAT(' ICASE(PROPAGATION=1; REFLECTION=2) = ',I2,/,
29       1' IP(UNIT-STEP=1; CW WAVE=2) = ',I2,/,
30       2' WWP(FREQUENCY OF CW SOURCE) = ',F5.2,//,
31       3' WAFORM OBSERVED AT GRID POINT = ',I3,//)
32        DO 11 I=2, IEND
33        SUM(I)=0.0
34        FO(I)=0.0
35        F1(I)=0.0
36    11  F2(I)=0.0
37        FO(1)=PULSE(0)
38        F1(1)=PULSE(1)
39        GO TO(12, 13), ICASE
40    13  F1(2)=FO(1)/Q1
41        WAFORM(1)=F1(2)-PULSE(0)
42        EINC=0.0
43        GO TO 14
44    12  F1(2)=FO(1)/Q2
45        WAFORM(1)=F1(IOBSER)
46    14  DO 21 N=2, NSTEP
47        F2(1)=PULSE(N)
48        IF(N .LT. IEND1) MOVING=N+1
49        GO TO(30, 31), ICASE
50    30  IF(N.GT.IEND1) MOVING=MOVING-1
51        DO 27 I=2, MOVING
```

```
52          IF(Q4.GT.0.)  SUM(I)=SUM(I)*EXP1+0.5*(EXP1*F0(I)+F1(I))
53     27   F2(I)=-F0(I)+(F1(I-1)+F1(I+1)-Q3*F1(I)+Q4*SUM(I))/Q2
54          WAFORM(N)=F2(IOBSER)
55          GO TO 32
56  C       I=2 IS THE AIR/PLASMA INTERFACE
57     31   SOURCE=F1(1)+(F0(2)-EINC)
58          I=2
59          IF(Q4.GT.0.0)  SUM(I)=SUM(I)*EXP1+0.5*(EXP1*F0(I)+F1(I))
60          F2(I)=-F0(I)+(SOURCE+F1(I+1)-Q35*F1(I)+Q45*SUM(I))/Q1
61          DO 26 I=3, MOVING
62          IF(Q4.GT.0. .AND. I.LE.I2) SUM(I)=SUM(I)*EXP1+
63        1                 0.5*(EXP1*F0(I)+F1(I))
64          IF(I.LT.I2) F2(I)=-F0(I)+(F1(I-1)+F1(I+1)-Q3*F1(I)
65        1                 +Q4*SUM(I))/Q2
66          IF(I.EQ.I2) F2(I)=-F0(I)+(F1(I-1)+F1(I+1)-Q35*F1(I)
67        1                 +Q45*SUM(I))/Q1
68     26   IF(I.GT.I2) F2(I)=F1(I-1)
69          WAFORM(N)=F2(2)-F1(1)
70          EINC=F0(1)
71     32   CONTINUE
72  C       SIMULATION 5/14/74
73          F2(IEND)=F1(IEND-1)/Q1
74          F0(IEND)=F1(IEND)
75          F1(IEND)=F2(IEND)
76          DO 21 I=1, MOVING
77          F0(I)=F1(I)
78     21   F1(I)=F2(I)
79          WRITE(6,22) (WAFORM(N),N=1,NSTEP)
80     22   FORMAT(100(13F6.3/))
81          END
82          FUNCTION PULSE(N)
83          COMMON WDT, WWP, WPDT, Y, IP
84          IF(IP .EQ. 3) GO TO 100
85          IF(IP .EQ. 1) PULSE=1.0
86          IF(IP .EQ. 2) PULSE=SIN(WDT*N)
87          RETURN
88    100   IF(N.GT.0) GO TO 101
89          P=WWP
90          R=4.0
91          Q=Y/SQRT(1.0-P*P)
92          S1=WPDT/Q
93          TOR=Q-Y*R/SQRT(R*R-P*P)
94          IF(WPDT .NE. 0.0) NOR=1+TOR/WPDT
95    101   PULSE=0.0
96          IF(N .GT. NOR) RETURN
97          SN=N*S1
98          CETA=Q*(P-SQRT(SN*SN-2.0*SN+P*P))
99          PULSE=SIN(CETA)
100         RETURN
101         END
```

```
 1 C     ********** COMPUTER PROGRAM 5 **********
 2 C     .THIS PROGRAM COMPUTES THE REFLECTED WAVEFORM OF A SIGNAL
 3 C     FROM A HOMOGENEOUS AND INHOMOGENEOUS PLASMA HALF-SPACE
 4 C     WHERE A VERTICAL MAGNETIC FIELD IS ASSUMED
 5       DIMENSION EX0(510),EX1(510),EX2(510),SX(510),SY(510),
 6      1          EY0(510),EY1(510),EY2(510),CX(510),CY(510),
 7      2          REFEX(1000),REFEY(1020),WPDT(510)
 8       COMMON IP, WDT, WPDT1
 9       DATA EX0,EX1,EY0,EY1,CX,CY,SX,SY,EINC/510*0.0,510*0.0,
10      1510*0.0,510*0.0,510*0.0,510*0.0,510*0.0,510*0.0,0.0/
11       READ(8,-) WPDT1, WCDT, GDT, WWP, NSTEP, IP, DT
12       WRITE(6,4) DT,WPDT1,WCDT,GDT,WWP,NSTEP,IP
13     4 FORMAT('     DT=',F6.3,' NANOSECONDS'/,' WPDT1=',F6.4/,
14      1' WCDT=',F6.4/,'     GDT=',F6.4/,'     WWP=',F6.4/,
15      2' NSTEP=',I4/,' IP(UNIT-STEP=1,CW WAVE=2,IMPULSE=3)='
16      3,I2//,' REFLECTED WAVEFORMS: EX & EY'/)
17 C     FOR IMPULSE SIGNAL : WPDT1=0.02 WITH 600 TIME STEPS
18       IEND=2+NSTEP/2
19       IEND1=IEND-1
20       DZ=0.3*DT
21 C     LINEAR ELECTRON DENSITY PROFILE
22       DO  5 I=2, IEND
23       Z=(I-1.5)*DZ
24     5 WPDT(I)=0.00014142*DT*SQRT(Z)
25       WPDT(1)=0.0
26       IF(DT.NE.0.0) GO TO 7
27       DO 6 I=1, IEND
28     6 WPDT(I)=WPDT1
29     7 AA=WPDT(1)*WPDT(1)
30       DO 8 I=2, IEND
31       BB=WPDT(I)*WPDT(I)
32       WPDT(I)=0.25*(AA+BB)
33     8 AA=BB
34       C=COS(WCDT)
35       S=SIN(WCDT)
36       GG=EXP(-GDT)
37       GGS=GG*S
38       GGC=GG*C
39       DD1=GG*(WCDT*S+GDT*C)
40       DD2=GG*(WCDT*C-GDT*S)
41       WDT=WWP*WPDT1
42       EX0(1)=PULSE(0)
43       EX1(1)=PULSE(1)
44       EX1(2)=EX0(1)/(1.0+0.5*WPDT(2))
45       REFEX(1)=EX1(2)-EX0(1)
46       REFEY(1)=EY1(2)
```

```
47 C       FINITE DIFFERENCE SOLUTION STARTS HERE
48         DO 11 N=2, NSTEP
49         EX2(1)=PULSE(N)
50         EY2(1)=EY1(2)
51         IF(N .LT. IEND1)MOVING=N+1
52         IF(N .GT. IEND1)MOVING=MOVING-1
53         DO 10 I=2, MOVING
54         Q1=WPDT(I)
55         Q2=1.0+0.5*Q1
56         W2=2.0*Q1
57         SOLD=SX(I)
58         SX(I)=(GGC*SX(I)+GGS*CX(I))+0.5*(GDT*EX1(I)+DD1*EX0(I))
59         CX(I)=(GGC*CX(I)-GGS*SOLD)+0.5*(WCDT*EX1(I)+DD2*EX0(I))
60         SOLD=SY(I)
61         SY(I)=(GGC*SY(I)+GGS*CY(I))+0.5*(GDT*EY1(I)+DD1*EY0(I))
62         CY(I)=(GGC*CY(I)-GGS*SOLD)+0.5*(WCDT*EY1(I)+DD2*EY0(I))
63         SORCEX=EX1(I-1)
64         IF(I.EQ.2) SORCEX=EX1(1)+(EX0(2)-EINC)
65         SSX=SX(I)-CY(I)
66         EX2(I)=-EX0(I)+(EX1(I+1)+SORCEX  -Q1*EX1(I)+W2*SSX)/Q2
67         SSY=CX(I)+SY(I)
68         EY2(I)=-EY0(I)+(EY1(I+1)+EY1(I-1)-Q1*EY1(I)+W2*SSY)/Q2
69    10   CONTINUE
70 C       PICK UP THE REFLECTED WAVEFORM & UPDATE THE FIELDS
71         REFEX(N)=EX2(2)-EX1(1)
72         REFEY(N)=EY2(2)
73         EINC=EX0(1)
74         DO 11 I=1, MOVING
75         EX0(I)=EX1(I)
76         EX1(I)=EX2(I)
77         EY0(I)=EY1(I)
78    11   EY1(I)=EY2(I)
79         WRITE(6,12) (REFEX(N),N=1, NSTEP)
80    12   FORMAT(100(10F7.4/))
81         WRITE(6,12) (REFEY(N),N=1, NSTEP)
82         END
83         FUNCTION PULSE(N)
84         COMMON IP, WDT, WPDT1
85         DATA K, N0,NN2, TP/4, 11, 22, 6.2832/
86 C       GAUSSIAN PULSE PARAMETERS: N0=K*2.6;  NN2=2*N0
87         PULSE=0.0
88         IF(IP .EQ. 1) PULSE=1.0
89         IF(IP .EQ. 2) PULSE=SIN(N*WDT)
90         IF(IP.EQ.3.AND.N.LE.NN2)
91        1PULSE=EXP(-((N-N0)/(K*1.4142))**2)/SQRT(TP*(K*WPDT1)**2)
92         RETURN
93         END
```

```
 1 C       ********** COMPUTER PROGRAM 6 **********
 2 C       THIS PROGRAM COMPUTES THE REFLECTED WAVEFORM OF A
 3 C       UNIT-STEP SIGNAL NORMALLY INCIDENT TO A THREE-LAYERED
 4 C       DISPERSIVE MEDIUM, AIR/ICE/WATER, IN WHICH WATER
 5 C       ASSUMES A DEBYE MODEL.
 6         DIMENSION F0(510),F1(510),F2(510),SUM(510),WAFORM(2000)
 7         READ(8,-) NSTEP1, W0DT, EPS1, EPS20, EPS2N, COND0
 8         IEND=2+NSTEP1/2
 9         IEND1=IEND-1
10         WRITE(6, 102) NSTEP1, W0DT, EPS1, EPS20, EPS2N, COND0
11   102 FORMAT(' # OF TIME STEPS RUN = ',I5/,' W0DT = ',F8.5/,
12       1' DIELECTRIC CONSTANT(2ND LAYER) = ',F5.2/,
13       A' LOW & HIGH FREQ DIELECTRIC CONSTS. OF WATER=',2F9.3/,
14       2' COND0(COND*TAU0/EPSAIR) OF WATER = ',F10.2//)
15         EXP1=EXP(-W0DT)
16         EPS0=1.0
17         CONDR=COND0/EPS2N+(EPS20-EPS2N)/EPS2N
18         Q1=1.0+0.5*CONDR*W0DT
19         Q2=1.0-0.5*CONDR*W0DT
20         Q3=W0DT*W0DT*(EPS20-EPS2N)/EPS2N
21         Q4=Q3*W0DT
22         EPS21=SQRT(EPS1/EPS2N)
23         Q5=(EPS21+Q1)/2.0
24         Q6=(EPS21+Q2)/2.0
25         DO 11 I=2, IEND
26         SUM(I)=0.0
27         F0(I)=0.0
28   11    F1(I)=0.0
29 C       REFLECTION & TRANSMISSION COEFF. AT THE AIR/ICE BOUNDARY
30         DUM=SQRT(EPS0)+SQRT(EPS1)
31         T01=2.0*SQRT(EPS0)/DUM
32         T10=2.0*SQRT(EPS1)/DUM
33         R01=(SQRT(EPS0)-SQRT(EPS1))/DUM
34 C       INCIDENT FIELD TO THE WATER SURFACE
35         F0(1)=1.0*T01
36         F1(1)=1.0*T01
37         F1(2)=EPS21*F0(1)/Q5
38         WAFORM(1)=R01+(F1(2)-T01)*T10
39 C       FINITE DIFFERENCE SOLUTIONS START HERE
40         DO 200 N=2, NSTEP1
41         F2(1)=F1(2)
42 C       I=2 IS THE ICE/WATER INTERFACE
43         I=2
44         SUM(I)=SUM(I)*EXP1+(F0(I)*EXP1+F1(I))/2.0
45         F2(I)=(-Q6*F0(I)+F1(I+1)+EPS21*F1(I-1)+0.5*(Q3*F1(I)-
46       1      Q4*SUM(I)))/Q5
47 C       EFFICIENT MOVING TIME WINDOW CALCULATIONS ADOPTED
48         IF(N .LT. IEND1) MOVING=N+1
49         IF(N .GT. IEND1) MOVING=MOVING-1
```

```
50          DO 20 I=3, MOVING
51          SUM(I)=SUM(I)*EXP1+(F0(I)*EXP1+F1(I))/2.0
52          F2(I)=(-Q2*F0(I)+F1(I-1)+F1(I+1)+Q3*F1(I)-Q4*SUM(I))/Q1
53     20   CONTINUE
54  C       UPDATE THE FIELDS AT A NEW TIME STEPS
55          DO 21 I=1, MOVING
56          F0(I)=F1(I)
57     21   F1(I)=F2(I)
58  C       PICK UP THE REFLECTED WAFORM AT THE AIR/ICE INTERFACE
59          WAFORM(N)=R01+(F1(2)-T01)*T10
60    200   CONTINUE
61          WRITE(6, 23) R01
62     23   FORMAT(' REFLECTION OF THE AIR/ICE BOUNDARY = ',F10.4//)
63          WRITE(6, 105)
64    105   FORMAT(' REFLECTED WAFORM OF A UNIT-STEP SIGNAL FROM',
65         1' THE ICE/WATER INTERFACE SEEN IN FREE SPACE'//)
66          WRITE(6,22) (WAFORM(N),N=1,NSTEP1)
67     22   FORMAT(400(10F8.4/))
68          END
```