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SUPPRESSION OF NONLINEAR OSCILLATIONS IN COMBUSTORS WITH PARTIAL LENGTH ACOUSTIC LINERS

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16. Abstract

An analytical model is formulated for a three-dimensional nonlinear stability problem in a rocket motor combustion chamber. The chamber is modeled as a right circular cylinder with a short (multi-orifice) nozzle, and an acoustic liner covering an arbitrary portion of the cylindrical periphery. The combustion is concentrated at the injector and the gas flow field is characterized by a mean Mach number. The unsteady combustion processes are formulated using the Crocco time lag model. The resulting equations are solved using a Green's function method combined with numerical evaluation techniques. The influence of acoustic liners on the nonlinear waveforms is predicted. Nonlinear stability limits and regions where triggering is possible are also predicted for both lined and unlined combustors in terms of the combustion parameters.

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Foreword

The research described herein, which was conducted at Colorado State University during the period December 1, 1970 to November 30, 1974 was supported by NASA Grant NGR 06-002-095. The work was done under the management of the NASA Project Manager, Dr. Richard J. Priem, Chemical Rockets Division, NASA Lewis Research Center.

ABSTRACT

An analytical model is formulated for a three-dimensional nonlinear stability problem in a rocket motor combustion chamber. The chamber is modeled as a right circular cylinder with a short (multiorifice) nozzle, and an acoustic liner covering an arbitrary portion of the cylindrical periphery. The combustion is concentrated at the injector and the gas flow field is characterized by a mean Mach number. The unsteady combustion processes are formulated using the Crocco time lag model. The resulting equations are solved using a Green's function method combined with numerical evaluation techniques. The influence of acoustic liners on the nonlinear waveforms is predicted. Nonlinear stability limits and regions where triggering is possible are also predicted for both lined and unlined combustors in terms of the combustion parameters.

NOMENCLATURE

Symbol	Definition
A	constant defined in Equation 14
a	speed of sound
$B_{1,2}$	constant defined after Equations 18-2, 24
C	constant defined after Equation 18-2
G	Green's function
i	unit complex ($\sqrt{-1}$)
j	amplitude ordering parameter
k	Specific acoustic impedance, harmonic ordering parameter
ℓ	radial wave number
M	mean flow Mach number
m	tangential wave number
$\dot{m}$	mass flow rate
n	interaction index, longitudinal wave number, unit outward normal
P	time dependent pressure
p	time independent pressure
Q	constant of integration defined in Equation (7-0)
R	radius of the chamber
r	radial coordinate
T	chamber temperature
t	time
u	longitudinal perturbation velocity
V	chamber perturbation velocity

Symbol	Definition
W	transverse perturbation velocity
y	stretched time
z	axial coordinate

Greek

α	second order coefficient matrix
β	specific acoustic admittance
γ	ratio of specific heats, integral defined before Equation 25
δ	Dirac delta function, integral defined before Equation 25
ε	amplitude parameter
η	acoustic frequency
θ	tangential coordin te
Λ	normalization constant, defined after Equation 17
λ	root of equation $J_m'(\gamma_{\ell m}) = 0$
μ	linear coefficient matrix
ρ	density
τ	sensitive time lag
Φ	time dependent potential function
ϕ	time independent potential function
ψ	normalization constant, defined after Equation 18
ω	chamber frequency

Superscripts

*	dimensional quantity
$\bar{}$	mean quantity
$\tilde{}$	linear solution to an unlined chamber

Symbol	Definition
$\hat{}$	dominant acoustic mode
$^{1,2,3,\ldots}$	order of perturbation

Subscripts

$_i$	injector
$_c$	cylindrical
$_N$	nozzle
$_{1,2,3,\ldots}$	order of the harmonic oscillation

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INTRODUCTION

Combustion instability is generally classified into three types. Low frequency instability (ten to three hundred hertz) is usually characterized as a coupling between the propellant feed system and combustion chamber. High frequency instability (seven hundred to ten thousand hertz) is the most destructive and is usually characterized by an in-phase coupling between the combustion processes and the pressure oscillations in the combustion chamber. The frequencies of these oscillations are close to the acoustic frequencies for the combustion chamber. Intermediate frequency instability, thought to be generated by a coupling of entropy waves and pressure oscillations, fills the frequency gap left by the two previous types of instability and is not frequently observed. This investigation is directed toward a better understanding of high frequency combustion instability.

A rocket motor can be broken down into four areas of interest, the combustion processes, the nozzle, a damping device, and the gas dynamic field in the combustion chamber. The combustion processes, even in steady operation, are extremely complex and are not well understood. The processes involve injection, atomization, mixing, vaporization, and chemical kinetics. A hueristic approach to modeling the combustion processes in a gross way useful to the engineer was suggested by Theodore von Kármán in 1941. His basic hypothesis was that the combustion processes could be collected to form a time lag which represented the residence time of the propellants from injection to combustion.(1) This early work led to the Crocco sensitive time lag theory which formulates a relationship between the thermodynamic variables of the chamber and the time lag for an element of propellant. The time lag, τ, represents the residence time before combustion for which the propellant element is exposed to gasdynamic oscillations. The effect of the gasdynamic oscillations is correlated by the interaction index, n. This parameter is an indicator of the propellant elements' sensitivity to the chamber oscillations.[2,3,4] This theory has been experimentally verified and the results show good agreement with the theory for the cases studied.[5]

The combustion parameters (n and τ) are used to determine stability limits for a combustion chamber. Neutral stability limits separate the stability map into two regions. The first of these is one in which oscillations grow in time and are termed unstable, in the second one the oscillations decay in time and are stable. A combustor with combustion parameters that lie on the neutral stability curve are termed neutrally stable.

The behavior of oscillatory flow through the convergent portion of a nozzle has received considerable attention. Tsien[6] treated the linear small amplitude, problem for a one-dimensional, isothermal nozzle flow. Crocco[4] extended the analysis by dropping the isothermal assumption. Crocco and Sirignano[7] later extended the analyses to consider three-dimensional flow in nozzles of varying geometry. The problems of finite amplitude oscillations (the nonlinear problem) has also been analyzed for three-dimensional oscillations of moderate amplitude by Zinn and Crocco.[9,10]

The use of a damping device is of considerable importance in rocket engines. These devices may consist of baffles, acoustic resonator damping devices, or a combination of the two types. Baffles are blades that are placed perpendicular to the injector face. The mechanisms by which baffles work is not understood, and they must be treated as part of the gas dynamic problem within the combustor. The baffle problem will not be included in this report. Acoustic resonator damping devices are a category of devices which may be attached or machined into the cylindrical periphery of the combustion chamber. The resonant chamber for these devices may be modeled as Helmholtz resonator, quarter-wave, or half-wave tube. The theoretical analysis for small amplitude oscillations in Helmholtz resonators was done by Lord Rayleigh.[11] The theory has been extended to include nonlinear influences and empirically corrected through the years.[12,13,14] Recent analyses have theoretically evaluated different resonator devices to include such effects as finite amplitude oscillations, mean and oscillatory chamber flows, liner-mean-through flow, flow in liner backing distances (for Helmholtz geometries), and differences in the mean gas properties of the combustion chamber and the resonanting device.[16,17,18,19] Experimental results show good agreement with the above theories.[20]

The gas dynamics of the flow in the combustion chamber is dependent on the combustion processes, the nozzle, and the acoustic absorber (and the baffles). By choosing particular models to describe the combustion, nozzle, and acoustic resonator behaviors, the stability of the flow in the combustion chamber can, in many cases, be determined.

The stability of small amplitude oscillations in combustion chambers has been extensively investigated using the time lag model to represent the combustion process.[4] Early studies concentrated on the influence of the combustion process and nozzle interaction on the gas dynamics. Oberg and Kuluva[21] investigated the influence of a full length liner on a combustion chamber with respect to frequency and decay rate. However, their work did not include the influences of mean flow in the chamber, of combustion, or of the nozzle. A later full length liner investigation was made by Priem and Rice[22] which included the influences of mean flow, combustion concentrated at the injector, and the nozzle. A similar invetigation to the one above with the combustion distributed axially was performed by Sirignano.[23] These investigations showed that acoustic liners change both the stability of wave oscillations and their frequency of oscillation. These studies were fundamentally sound, but they were not realistic. In practice an acoustic liner does not cover the entire surface of a combustor but only a fraction of it. The linear analysis of the partially lined combustion chamber for both concentrated and distributed combustion, with mean flow, liner and nozzle influences has recently been completed at Colorado State University.[24,25] This latter analysis determined the influence of partial length acoustic liners at arbitrary locations and of arbitrary length of neutral stability limits for the combustor.

The major failure of the linear analyses just described is their limitation to very small amplitude oscillations and their inability to predict waveforms similar to experimental results. Allied with this is their inability to predict the possibility of triggered instability. Triggering is defined as the initiation of finite amplitude oscillations by the introduction of a disturbance of sufficient magnitude. This may occur accidentally, through the combustion process, or intentionally by the introduction of a high pressure gas pulse or an explosive charge. While it has been shown that linear analysis produces stability limit results that are in agreement with experiment, the results do not accurately predict the waveforms. The experimental waveforms are found to have sharp peaks and shallow valleys, or even discontinuities which the linear analyses do not model at all.

Due to the complex nature of the problem the investigation of the nonlinear (finite amplitude) instability is not as fully developed as the investigation of linear instability. The behavior of finite amplitude transverse waves in a circular cylinder was investigated by Maslen and Moore.(26) Their investigation included only fluid mechanical effects and did not include the influences of combustion, mean flow, nozzle, or damping devices. They concluded from their work that transverse waves do not steepen into shock waves as longitudinal waves must. Their results are of interest to the field of combustion instability because the predicted nonlinear waveforms are similar to those obtained experimentally in rocket engines.

The success of the sensitive time lag concept in linear analyses has led to its extension into the nonlinear regime. The nonlinear analyses may be broken down into two groups. The first group studied the behavior of one-dimensional longitudinal oscillations. Sirignano(27) assumed concentrated combustion and that the chamber was terminated by a "short" (quasi-steady) nozzle. He demonstrated the existance of periodic, finite amplitude, longitudinal waves near the linear stability limit. These solutions were unstable with respect to the linear stability limits indicating the possibility of triggering. This analysis was extended by Mitchell[28] to include the possibility of discontinuous wave forms. He included both concentrated and distributed combustion and, again, he used the "short" nozzle approximation. Finite amplitude, periodic, discontinuous oscillations were shown to exist for both types of combustion. It was also demonstrated that intrinsically stable waves may be triggered to produce unstable waveforms. A later analysis by Zinn and Lores[29] with distributed combustion and "short" nozzle was developed to include the transient behavior of a longitudinal oscillation. The authors were able to show the transient and limit cycle behavior and waveforms, and found that in some regions the waveforms exhibit shock wave characteristics. They were also able to show that the final form of the oscillations was independent of the initial disturbance.

The logical extension of the one-dimensional longitudinal analysis to three dimensions was made by Zinn.[10] His model assumed concentrated combustion and was terminated by his previously discussed three-dimensional nozzle model. Zinn showed the existence of three-dimensional, finite amplitude, periodic oscillations. He was able to prove the possibility of triggering unstable oscillations, however due to the complicated nature of the final form of his equations, he was unable to determine triggering amplitudes. Powell[30] using a different method of analysis studied essentially the same problem. For both concentrated and distributed combustion with the chamber terminated by a short nozzle he was able to obtain solutions to his equations and to determine nonlinear stability limits. One main drawback in the two previous analyses is that they are forced by their respective analytic techniques to assume a solution in one of the spatial directions. In other words, they assume a priori the final waveform for one of the dimensions. A more complete survey of the work in linear and nonlinear combustion instability may be found in any of the excellent survey papers.[23,31,32,33]

From the list of previous accomplishments in the area of nonlinear instability, it is evident that it is desirable to find an analytical solution to the problem in which no approximations need be made about the form of the spacial waveforms and to include the influence of partial length acoustic liners on the final waveforms. This is, indeed, the goal of this analysis.

In the present work the combustion process will be represented by the Crocco time lag model. In addition the combustion will be taken to be concentrated at the injector surface. This latter assumption is equivalent to saying that the region in which combustion takes place is small when compared to the length of the chamber, which has been shown experimentally to be relatively accurate in some cases.[34] The model will include mean flow effects on the oscillations and the chamber will be terminated by a "short" nozzle. Since it is not the purpose of this analysis to design tuned acoustic liners for a given chamber configuration the acoustic liner boundary conditions will be represented simply, through an admittance coefficient. The use of the admittance coefficient is consistent with the previous work in the field since results of theory and experiments on the effect of acoustic liners are typically given in terms of this parameter. It is an outgrowth of the early work in acoustics and its extension to the area of combustion instability. The axial location and the length of liner will remain arbitrary in the analysis.

The Green's function technique used herein has been successfully applied to acoustic problems many times in the literature (for example Ref. 35). One of the first demonstrations of its applicability to the combustion instability was made by Culick[36] in his study of solid propellant rockets. The solution technique used herein closely parallels that used by Oberg,[21] Mitchell, et al.[24] and Baer, et al.[25] This technique is extended to determine an analytical solution for the nonlinear equations with discontinuous boundary conditions.

In summary, this report presents a formulation of the three-dimensional nonlinear stability problem (subject to the aforementioned model assumptions), solves the resulting equations using applied analysis and numerical techniques, predicts the influence of acoustic liners on the nonlinear waveforms, determines nonlinear stability limits in terms of the combustion parameters n and τ, and predicts regions of operation where triggering is possible for both lined and unlined combustors.

THEORY

In order to facilitate the modeling of the flow in a combustor certain simplifications must be made in order to obtain governing equations which can be solved. The combustor model (Fig. 1) chosen for this analysis has the combustion zone concentrated at the injector, a uniform cylindrical cross section, a finite Mach number mean flow in the combustor, and is terminated by a "short" nozzle. An acoustic liner of arbitrary location and impedance may cover a portion of the cylindrical periphery, while the remaining uncovered chamber surface is assumed to be an acoustically hard wall, that is to have an infinite impedance.

The concentrated combustion assumption implies that the combustor is long compared to the region where the combustion takes place, or that the combustion zone is immediately adjacent to the injector surface. This assumption serves to separate the gasdynamics from the combustion processes and thus leads to considerable analytical simplification in the gasdynamic governing equations (discussed later on) because of the fact that the combustion processes appear only as a boundary condition and not in the governing equations themselves. The response of the combustion zone mass generation rate according to the Crocco sensitive time lag ($n-\tau$) model,[4] is assumed to be pressure dependent only.

The short nozzle or multi-orifice nozzle[7] approximation (a collection of small nozzles in the combustor exit plane) assumes that the individual nozzle dimensions are small compared to the chamber dimensions and the through-flow time is small compared to the period of the chamber oscillations. Therefore at each instant the individual nozzle behaves as if the flow were steady or that a constant Mach number condition exists at the entrance to the nozzle.

The damping influence of acoustic liners for non-vanishing oscillation amplitudes is a result of the formation of jets at the exit of the passage connecting the acoustic liner's resonant cavity and the combustion chamber and the eventual dissipation of the jets' kinetic energy in the combustor and the cavity, and to the frictional effects in the connecting passage. The effectiveness of an acoustic liner is a function of frequency, and maximum damping is obtained around the resonant frequency of the liner where the pressure amplitude in the resonant cavity is at a maximum. The resonant frequency of the acoustic liner is a function of the state variables and the physical dimensions of the acoustic liner. In this analysis the acoustic liners will be assumed to be tuned for all frequencies. That is, it will be assumed that the physical dimensions of the damping device will be adjusted so that the resonant frequency of the liner corresponds to the frequency of the combustor or that the impedance is independent of the chamber frequency.

The flow of gas in the combustor is three-dimensional and it will be assumed the oscillations in the flow field are periodic and that one frequency is dominant in the chamber. The concentrated combustion model permits the gasdynamic field to be treated as a single constituent, source free gas. It is further assumed that the gasdynamic field is homentropic, irrotational, and calorically perfect. Transport phenomena, such as diffusion, viscosity, and heat conduction, are neglected in the governing equations. The absence of diffusion is consistent with the single constituent gas assumption. The absence of heat conduction and viscosity lead to conservative results since they result in damping of pressure oscillations. Two final forms of natural damping are neglected. Droplet drag is neglected because of the concentrated combustion assumption, and wall friction is neglected, which is consistent with the inviscid assumption.

Using the model described above, a mathematical representation is obtained by writing the conservation equations and the boundary equations for the combustor.

The conservation equations are written as follows

Continuity:

$$\frac{\partial \rho^*}{\partial t^*} + \vec{\nabla} \cdot (\rho^* \vec{V}^*) = 0$$

Momentum:

$$\rho^* \frac{D\vec{V}^*}{Dt^*} + \vec{\nabla} P^* = 0$$

Homentropic Condition:

$$P^* = C\rho^{*\gamma}$$

State (perfect gas relation):

$$p^* = \rho^* R^* T^*$$

The following scheme is now used to put the governing equations in a nondimensional form. The state variables are nondimensionalized by their respective steady state or mean values. The characteristic velocity and time will be the mean of sonic velocity ($\bar{a}^*$) and the wave travel time ($\bar{a}^*/R^*$), respectively, while chamber dimensions are nondimensionalized with respect to the chamber radius. The dimensionless quantities are

$$r = \frac{r^*}{R^*}\ , \quad z = \frac{z^*}{R^*}\ , \quad t = \frac{t^* R^*}{\bar{a}^*}\ , \quad \rho = \frac{\rho^*}{\bar{\rho}^*}$$

$$P = \frac{P^*}{\bar{p}^*}\ , \quad T = \frac{T^*}{\bar{T}^*}\ , \quad V = \frac{V^*}{\bar{a}^*}\ .$$

The nondimensional governing equations are

Continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0 \tag{1}$$

Momentum:

$$\rho \frac{D\vec{V}}{Dt} + \frac{1}{\gamma} \vec{\nabla P} = 0 \tag{2}$$

Homentropic Condition:

$$P = \rho^{\gamma} \tag{3}$$

State:

$$P = \rho T \tag{4}$$

The irrotationality assumption permits the representation of the velocity vector in terms of a potential function.

$$\vec{V} = \vec{\nabla \Phi} \tag{5}$$

By substitution of Equations (3) and (5) into Equations (1) and (2), it is possible to combine the conservation equations into a single partial differential equation in terms of the velocity potential.

$$Q\nabla^2\Phi - \frac{\partial^2 \Phi}{\partial t^2} = \frac{\partial}{\partial t} (\vec{\nabla\Phi} \cdot \vec{\nabla\Phi}) + \frac{\gamma-1}{2} (\vec{\nabla\Phi} \cdot \vec{\nabla\Phi})\nabla^2\Phi$$

$$+ \frac{1}{2} \nabla\Phi \cdot \vec{\nabla}(\vec{\nabla\Phi} \cdot \vec{\nabla\Phi}) + (\gamma-1)\frac{\partial \Phi}{\partial t} \nabla^2\Phi \tag{6}$$

where Q is a constant of integration. The pressure is expressed as a function of the velocity potential as follows

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\vec{\nabla\Phi} \cdot \vec{\nabla\Phi}) + \frac{1}{(\gamma-1)} P^{\frac{\gamma-1}{\gamma}} = Q \tag{7}$$

Lacking a direct solution of the above equations it is assumed that P and Φ can be expanded in an infinite series in powers of some amplitude parameter, ε, and that each order of the function Φ may be represented by a Fourier series in time, t. By making use of the periodicity and mean flow Mach number assumptions the velocity potential is then represented by

$$\Phi = Mz + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \varepsilon^j \phi_{(k)}^{(j)} e^{i(k\omega)t} \tag{8}$$

where M is the mean flow Mach number, z is the nondimensional axial coordinate, i is the unit complex ($\sqrt{-1}$), j is the ordering parameter, ω is the complex chamber frequency, k is the frequency ordering parameter, t is the nondimensional time, and ϕ is the time dependent potential function.

The pressure is defined similarly

$$P = 1 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \varepsilon^j \, p_{(k)}^{(j)} e^{i(k\omega)t} \tag{9}$$

The chamber frequency is also represented by an infinite series in the amplitude parameter

$$\omega = \omega^{(0)} + \sum_{j=1}^{\infty} \varepsilon^j \omega^{(j)} \tag{10}$$

Before going further it is convenient to introduce a new variable y, where y is the product of the frequency and time

$$y = \omega t \tag{11}$$

The boundary condition on Equation (6) is $\vec{\nabla}\Phi \cdot \vec{n} = \beta P$ where $\vec{n}$ is the outward normal for the appropriate surface, and β is an average specific acoustic admittance for that surface. The surfaces are defined as the injector plane, the nozzle exit plane, and the cylindrical periphery. The cylindrical periphery can consist of two or more surfaces, depending on the location of the acoustic liner. The acoustic admittance changes discontinuously with the change in surfaces.

The injector boundary condition is derived using the time-lag model to represent the combustion processes. This model relates the rate of mass generation due to combustion to the steady-state mass flow as follows

$$\dot{m} = \overline{\dot{m}}(1 - \frac{d\tau}{dt})$$

where $\frac{d\tau}{dt}$ represents the rate of change of the time lag, and $\dot{m}$ is the mass flow rate. This relationship assumes that the injection rate is constant at all times. The evaluation of $\frac{d\tau}{dt}$ will not be repeated here, because it is available and clearly presented in Crocco's work.[4] The final form of Crocco's results will be accepted as a postulate in this report. His relationship is

$$\frac{d\tau}{dt} = - n(P'(t) - P'(t - \tau))$$

where $P' = P - 1$. By arithmetic manipulation we obtain the following relationship between the admittance and the combustion parameters n and τ.

$$\beta_i = M(\frac{1}{\gamma} - n \frac{(P'(t) - P'(t-\tau))}{p'(t)}) \tag{12}$$

Terms of p'^2 and higher are ignored in this derivation. Nonlinear extensions to the above model have been formulated by Sirignano[27] and Zinn.[10] The linear formulation is used here because of its simplicity and because corrections to the primary combustion parameters n and τ can be calculated through β_i in a simple way.

The chamber exit plane or nozzle entrance plane boundary condition is derived using the short or quasi-steady nozzle approximation. This approximation assumes that the dimensions of the nozzle are small compared to chamber dimensions. Such a quasi-steady condition can be approximated by a uniform distribution of many small individual nozzles over the chamber exit plane. The quasi-steady nozzle condition[8] is

$$\frac{\vec{\nabla}\Phi \cdot \vec{n}}{p^{(\frac{\gamma-1}{2\gamma})}} = A\left\{1 + \frac{(\gamma-1)}{2} \frac{\vec{W}^2}{p^{(\frac{\gamma-1}{2\gamma})}}\right\}^{\frac{\gamma+1}{2(\gamma-1)}} \tag{13}$$

where A is a constant.

On the cylindrical periphery the acoustic admittance need not be uniform along the axial length, and when a partial length liner is present the admittance must change discontinuously. When a liner is not present or on the segment of the cylindrical surface which is not covered by a liner the acoustic admittance is assumed to be identically zero. This implies that the surface is an acoustically hard wall, that does not deflect due to the pressure oscillations. The acoustic impedance is a coefficient which describes the interaction of sound waves with a locally reacting surface. The acoustic impedance for the absorber is related to the acoustic admittance by the following equation

$$\beta_c = \frac{1}{\gamma k} \tag{14}$$

where k is the specific acoustic impedance for the absorber. The acoustic impedance is the term generally used in acoustic literature for theory and experiment. In experiment (both acoustic and rocket test firings) it is easier to determine the impedance. The reacting surface in this case is the acoustic absorber. The impedance is a function of absorber dimensions and the frequency of oscillation within the absorber. There are many theories, linear and nonlinear, which can be used to calculate the impedance. However, the nonlinear and some of the linear theories use experimental data to define some of the parameters used in the theory, because of this a typical acoustic impedance is arbitrarily chosen and used. This impedance will be constant, which implies that as the frequency changes the absorber dimensions change such that the main chamber frequency and the absorber frequency are the same. The absorber dimensions are not of interest in this report, the reader is referred to the work of Mitchell, et al.[36] for the procedure used in the designing an acoustic absorber for a combustion chamber.

The expansions for Φ, p and ω in terms of ε and harmonic content are now substituted into the governing equations, Equations (6) and (7) and the result is separated according to powers of ε. In this work terms through $O(\varepsilon^3)$ were considered. Similarly, the boundary conditions on the bounding surfaces are also expanded. Equations and appropriate boundary conditions for $O(\varepsilon)$, $O(\varepsilon^2)$ and $O(\varepsilon^3)$ are thus obtained. These equations and boundary conditions are lengthy. They are written down completely in Appendix A.

For the j^{th} order in ε there are j linear equations and boundary conditions for j harmonic components of $\phi^{(j)}$. That is, to $O(\varepsilon)$ there is a single linear equation for $\phi^{(1)}$. To $O(\varepsilon^2)$ there are two linear equations, one for $\phi^{(2)}_{(1)}$ and one for $\phi^{(2)}_{(2)}$. To $O(\varepsilon^3)$ there are three linear equations for $\phi^{(3)}_{(1)}$, $\phi^{(3)}_{(2)}$ and $\phi^{(3)}_{(3)}$. The general form of these equations is

$$\nabla^2 \Phi^{(j)}_{(k)} - \omega^{(0)^2} \frac{\partial^2 \Phi^{(j)}_{(k)}}{\partial y^2} = F_L(\Phi^{(j)}_{(k)}) + F_{N.L}(\Phi^{(j-1)}, \Phi^{(j-2)} \ldots) \quad ,$$

where $\Phi^{(j)}_{(k)} = \phi^{(j)}_{(k)} e^{iky}$, $F_L(\Phi^{(j)}_{(k)})$ is a linear function of $\Phi^{(j)}_{(k)}$ and $F_{N.L}(\Phi^{(j-1)}, \Phi^{(j-2)})$ is a nonlinear function of lower order solutions.

In order to convert the governing equations and their boundary conditions into integral equations, Green's functions corresponding to the appropriate frequency values are introduced. The Green's functions satisfy the equation

$$\nabla^2 G^{(k)} + (k\omega^{(0)})^2 G^{(k)} = \delta(\vec{r}/\vec{r}_o) \tag{15}$$

where $\nabla\vec{G}^{(k)} \cdot \vec{n} = 0$ on all boundaries and δ is the Dirac delta function. The Green's function is represented as an orthonormal eigenfunction expansion for the chamber with no mean flow, no combustion, and solid walls. The form of this expansion is given in Appendix A.

The partial differential equations for all orders are then converted to integral equations using the appropriate Green's function and following standard techniques.[34] Details of this transformation as well as the complete integral equations are given in Appendix A.

To lowest order the linear integral equation is exactly the same as the one solved previously by Mitchell et al.[24] Solution is obtained by successive approximation and results finally in a combustion zone eigenvalue, $\beta_i^{(0)}$ (or $n^{(0)}$ and $\tau^{(0)}$) and a solution for $\phi^{(1)}$ of the following form

$$\phi^{(1)} = \tilde{\phi} + \cos \hat{m}\theta \sum_{\ell} \sum_{n} \mu_{\ell n} J_m(\lambda_{\ell m} r) \cos \frac{n\pi z}{L}$$

where $\tilde{\phi}$ is a solution obtained by separation of variables for the unlined chamber, and $\mu_{\ell n}$ is a two dimensional matrix determined by the successive approximation technique.

To second order (in ε) two integral equations result for the two harmonic components $\phi_{(1)}^{(2)}$ and $\phi_{(2)}^{(2)}$. Solution of the integral equation for $\phi_{(1)}^{(2)}$ determines the form of $\phi_{(1)}^{(2)}$ and the first order corrections to β_i and ω, $\beta_i^{(1)}$ and $\omega^{(1)}$. This solution is found to be (see Appendix A) $\phi_{(1)}^{(2)} = 0$, $\beta_i^{(1)} = 0$, $\omega^{(1)} = 0$. That is, to second order no change in frequency or combustion response eigenvalues is predicted, and the first harmonic component of the second order solution is identically zero.

The solution of the integral equation for $\phi_{(2)}^{(2)}$ results in the following expression for $\phi_{(2)}^{(2)}$

$$\phi_{(2)}^{(2)} = \sum_{\ell} \sum_{m} \sum_{n} \alpha_{\ell mn} \Omega_{\ell mn}$$

The three dimensional matrix $\alpha_{\ell mn}$ is evaluated directly by integration of nonlinear functions of $\phi^{(1)}$ over the appropriate boundary surfaces and bounded volume of the chamber. No iteration or successive approximation is necessary since $\phi^{(1)}$ is now a known function. Most of the integrals can be evaluated analytically, however the ones involving triple products of Bessel functions were evaluated numerically following a technique discussed in Appendix B.

To third order three integral equations for $\phi^{(3)}_{(1)}$, $\phi^{(3)}_{(2)}$ and $\phi^{(3)}_{(3)}$ may be written down. In this work only the first harmonic component for $\phi^{(3)}_{(1)}$ was considered. The reason for this is that it is this equation which serves to determine the second order corrections to the eigenvalues β_i and ω, $\beta_i^{(2)}$ and $\omega^{(2)}$. It is the determination of this eigenvalue relationship (which specifies the nonlinear stability behavior of the combustor) which is the main goal of this research. The third order corrections to the nonlinear waveforms are not determined, mainly from a profound awe of the computational complexities involved. Thus, the third order equations are invoked only to determine the nonlinear eigenvalue corrections; the waveforms are determined only to second order accuracy. The determining third order eigenvalue relationship is found to be (see Appendix A)

$$\gamma\beta_i^{(2)} \iint_{s_i} \Omega_N \left[i\omega^{(0)}\phi^{(1)} + M \frac{\partial\phi}{\partial z}^{(1)} \right] ds_i$$

$$= \beta^{(0)} \iint_{s_o} \Omega_N H(\phi^{(1)}, \phi^{(2)}_{(2)}, \omega^{(2)}) ds_o - \iiint_{V_o} \Omega_N F(\phi^{(1)}, \phi^{(2)}_{(2)}, \omega^{(2)}) dV_o \tag{16}$$

where both H and F are known nonlinear function of lower order solutions.

The method used to determine $\beta^{(2)}$ and $\omega^{(2)}$ is to specify a particular combustor configuration, to solve the linear equations for the waveform and the acoustic admittance at the injector, to solve the second order equation for the nonlinear waveform, and finally to solve the first harmonic third order equation for the second order correction to the acoustic admittance at the injector. A demonstration of the success of the technique in predicting nonlinear waveforms and nonlinear stability limits in combustors with and without partial length acoustic liners is presented in the next section.

RESULTS AND CONCLUSIONS

The analytical method developed in this investigation can be used to predict pressure waveforms similar to experimentally observed waveforms and to predict regions where triggering is possible for both unlined and lined combustors. A particular combustor configuration will be used to demonstrate the effectiveness of the method. The combustor configuration chosen to demonstrate the technique is one in which several of the parameters chosen have adverse effects on the calculations. The combustor chosen has a long chamber ($L/R = 2.7$), a high mean flow Mach number (0.33), and an acoustic liner with strong damping influence ($k = 5$) covering the chamber walls from the injector to one-third of the chamber length. Many combustors are shorter, have a lower mean flow Mach number, and although the admittance coefficient for the liner may be larger, the liner typically covers a much smaller portion of the periphery. All these changes cause positive influences on the calculational rapidity and convergence of the technique. The configuration chosen thus serves as an upper limit, as far as severity of calculational problems are concerned, for a wide range of combustors of interest.

After the model parameters are fixed the linear equation is solved.

A solution to the unlined chamber is found by Bernoulli's method (separation of variables) and then the effect of the acoustic liner on the injector acoustic admittance and the waveform is determined by a modified Green's function technique using Picard's method (successive functional approximations).

Once the linear injector admittance and waveform are determined the second order correction to the waveform can be determined. Application of the Green's function method leads directly to a solution involving only the eigenfunctions (Ω_N) and a coefficient matrix (B_N). The coefficient matrix (B_N) involves integrals of the products of the linear solution $(\phi^{(1)})$ and of the eigenfunctions. To solve for the second order coefficients the integrals involving the spacial variables θ and z are evaluated analytically. There is no closed form solution for the majority of the integrals involving the Bessel function and these are evaluated numerically (see Appendix B).

Finally, with the second order coefficient matrix calculated, the second order correction to the injector admittance $(B_i^{(2)})$ can be determined through the use of the eigenvalue relationship, Equation (16). Use of the expression $\beta_i = -M[n(1 - e^{i\omega\tau}) - \frac{1}{\gamma}]$ then results in two equations relating the three quantities $n^{(2)}$, $\tau^{(2)}$ and $\omega^{(2)}$.

In the calculations performed $n^{(2)}$ and $\tau^{(2)}$ were required to lie on the normal to the first order stability curve, for each point on that curve (in other words for each value of $\omega^{(0)}$). For a given value of ε, then, the corresponding values of $\omega^{(2)}$, $n^{(2)}$, $\tau^{(2)}$ and D, the normal displacement from the linear neutral stability curve could be calculated.

Results of such calculations for the combustor chosen with no liner are shown in Figure 2. The linear stability curve separates the region of stable oscillations (outside the curve) from the region of unstable oscillations (inside the curve). The stability curve is parameterized in frequency with frequency increasing from right to left. For convenience the ratio of the frequency to the first transverse acoustic frequency is used to define the frequency points along the linear curve. The linear stability curve has a low minimum value for $n^{(0)}$ which implies that the combustor at this point is very sensitive to disturbances. The linear curve also exhibits mixed mode behavior in the range of interest for the combustion parameters. That is, there is a transition from first transverse at low frequencies to the combined first transverse-first longitudinal at higher frequencies. In the figure this transition appears as the loop with the region to the right of the loop being predominately first transverse acoustic and the region to the left of the loop being the combined first transverse-first longitudinal acoustic mode.

The various types of nonlinear stability behavior predicted will now be discussed, again using the frequency ratio as parameter. For frequency ratios less than about .93 all normal displacements from the linear stability curve are found to be negative (inside the neutral stability curve). A region of stable finite amplitude oscillations is consequently indicated inside the neutral stability curve for this part of the stability map. Small amplitude oscillations grow in amplitude until they reach the amplitude predicted for the n and τ of interest.

For frequency ratios between about .93 and .965 along the neutral stability limit normal displacements are found to be positive (outside the neutral stability curve) and to depend on ε approximately quadradically. This type of behavior is shown in Figure 3 for frequency ratios of .94 and .95. The region outside the neutral stability curve for this range of frequency ratios is a region where triggering is predicted. Though this region is a linearly stable region, finite amplitude oscillations are predicted from the nonlinear analysis. The conclusion to be drawn is that, though the combustor is stable to small amplitude disturbances, perturbations of sufficient amplitude may excite finite amplitude periodic oscillations. For example, at a point displaced .075 units from the point on the neutral stability curve corresponding to a frequency ratio of .95, a disturbance of amplitude .175 is required to trigger the finite amplitude oscillations (see Figure 3).

Also shown in this region are dashed lines corresponding to the locus of points requiring the same amplitude disturbance to excite oscillations. The further away from the neutral stability curve one of these constant amplitude lines is located, the more sensitive is the location on the n,τ plane to triggering. In particular at the frequency ratios of approximately .93 and .965 all lines of constant ε pass through the neutral stability curve and no triggering is possible.

Proceeding outward from the neutral stability curve at approximately a 45° angle from the point corresponding to a frequency ratio of .965 is a solid-dashed line. This line represents the approximate location of a triggering limit. Along this line disturbances of very large amplitude (beyond the amplitude for which this analysis may be considered accurate) are necessary to trigger finite amplitude oscillations. Below this curve no triggering is predicted.

For values of the frequency ratio of from .965 to 1.01 both positive and negative normal displacements are predicted. This behavior is seen in Figure 4. For a given positive normal displacement two, one, or zero finite amplitude solutions may be found. When two solutions are predicted the lower amplitude solution is assumed to be unstable, the upper stable. Then, disturbances with amplitudes less than that of the lower solution decay while those with amplitudes greater than that of the lower solution either grow (if they are below the upper solution) or decay (if they are above the upper solution) until this amplitude is the same as that of the stable (upper) solution. The lower solution serves as a limiting minimum amplitude for triggering in this case. The limiting displacement for this type of behavior occurs when the lower and upper solution coincide. The locus of these limiting displacements serves as a triggering limit in that, for displacements beyond this locus, no triggering is possible. The locus is shown as a solid-dashed line in Figure 2. Note that this line is very close to the neutral stability limit for this region.

Negative displacements are also predicted in this region and are presumed to indicate the existence of stable finite amplitude oscillations of relatively large amplitude (see Figure 4). The dashed continuations of the various curves in Figures 3 and 4 indicate regions of sufficient amplitude that higher order corrections, not considered here may be quite important.

The frequency regime from 1.01 on corresponds to the mixed mode region mentioned earlier. Here again a triggering region along with lines of constant amplitude (dashed lines) and a triggering limit solid-dashed line are shown.

The results for the combustor with an acoustic liner are presented in Figures 5, 6 and 7 and are similar to those of Figures 2, 3 and 4 except that the stability curves are shifted to larger values of n. This indicates that a substantial overall stabilizing effect is produced by the liner. It can be seen in Figure 5 that the low frequency triggering regime (frequency ratio less than about .95) is relatively smaller when a liner is present. In addition the lines of constant ε are more closely spaced indicating that larger amplitude distrubances are necessary for triggering when a liner is present. This is also shown in Figure 6. (Compare with Figure 3.)

The moderate frequency triggering region which was present without the liner is completely eliminated when the liner is in place. No two solution or two normal displacement region occurs.

In the mixed mode high frequency region the size of the triggering region is reduced and the required triggering amplitudes are increased when a liner is present as was the case for the lower frequency triggering region.

The conclusion here is that the addition of an acoustic liner not only increases the linear stability of the chamber (the stability curves are shifted upward) but also decreases substantially the sensitivity of the combustor to triggering. The liner thus improves both the linear and nonlinear stability of the combustor.

The nonlinear waveforms were determined by adding the nonlinear waveform perturbation to the linear waveform. In Figure 8 a typical experimental pressure oscillation (from Ref. 23) is shown. The oscillation displays a sharp peak with a distorted valley. Figures 9 and 10 are the pressure waveforms as a function of time at the injector for two typical frequencies for an unlined chamber. In Figure 9 the combustor frequency is below the resonant frequency and in Figure 10 the combustor frequency is above the resonant frequency. Figure 9 shows some nonlinear behavior in the slight steepening of the wave with a slightly sharper peak and shallower valley. Figure 10 has a nonlinear waveform with steepening of the waveform at the peak and a shallow valley. The results in Figure 10 are similar to these presented in Maslen and Moore[26] in their acoustic work, and the results presented in Powell and Zinn[30] in their nonlinear stability work.

For a lined combustor the pressure-time plots (Figs. 11 and 12) are similar to those for the unlined chamber. In Figure 11 the nonlinear waveform coincides with the linear waveform. The liner has a sufficiently strong influence at this value of frequency so that there is no distortion. At the higher frequency (Figure 12) the influence of the liner is not as strong and in fact the wave steepening is more pronounced than in the unlined chamber. In Figures 13-16 pressure is plotted versus the axial dimension. Figure 13 shows that at a frequency below the acoustic frequency for an unlined chamber the nonlinear wave shows steepening and a sharper peak than the linear wave with an increase in the maximum and a decrease in the minimum pressure amplitude. For the same chamber at a frequency above the acoustic frequency (Figure 14) the linear wave is a very flat first longitudinal wave. The nonlinear wave shifts to a more pronounced waveform similar to a second longitudinal wave with a decrease in the maximum pressure amplitude. Figure 15 shows the nonlinear wave exhibiting typical nonlinear behavior with steepening, shallower valleys, and sharper peaks for a lined chamber with increases and decreases in the maximum and minimum pressure amplitudes. Figure 16 is the lined chamber at a frequency above the acoustic frequency. The nonlinear wave exhibits typical nonlinear behavior with a decrease in the maximum perturbation pressure.

The analytical technique presented in this report is quite general and from the results the following conclusions are drawn:

1. A Green's function technique can be successfully applied in the solution of nonlinear combustion stability problems.
2. The technique can predict experimentally observed nonlinear phenomena.
    a. regions where finite amplitude disturbances may be trigger instability in a linearly stable engine.
    b. nonlinear pressure waveforms similar to those experimentally observed.
3. The technique is applicable to a wide range of combustor configurations as determined by the ability to handle the "limiting" case used.
4. The addition of an acoustic liner to a combustor improves both the linear stability (higher combustion zone response

is required) and the nonlinear stability. (Regions where triggering is possible are reduced and larger amplitudes are required.)

5. The technique requires a reasonable expenditure of computer time to determine the nonlinear behavior of a given combustor behavior.

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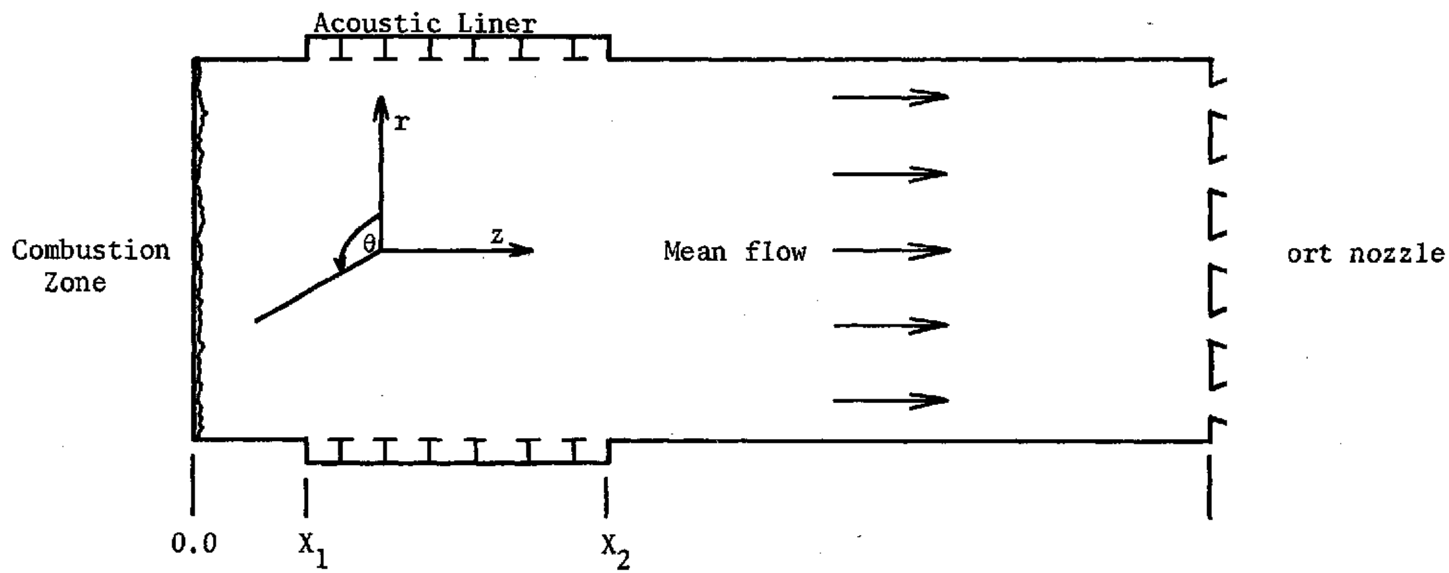

Figure 1. Rocket motor combustion chamber model with concentrated combı   on and "short" (multi-orifice) nozzle.

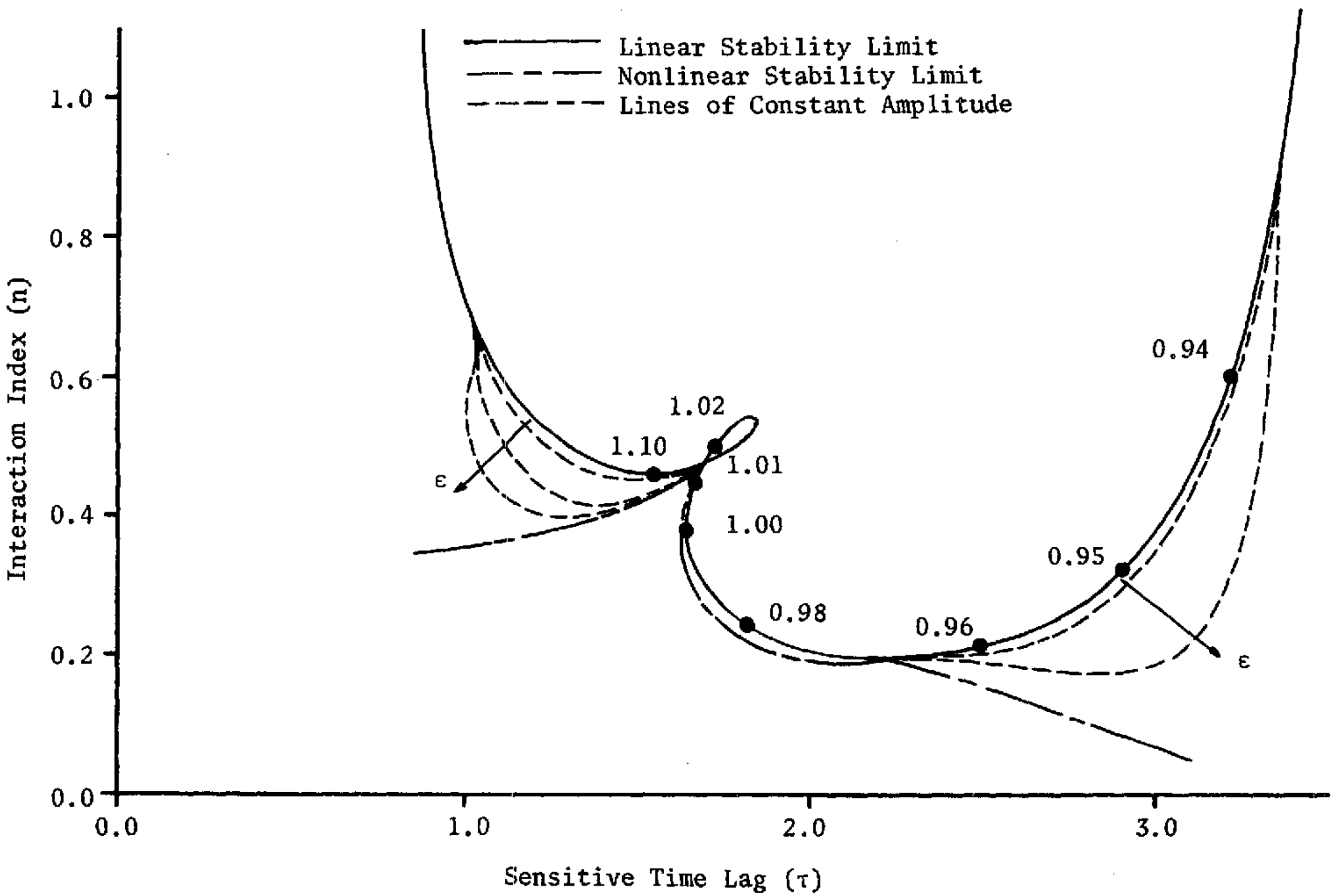

Figure 2. Stability map for an unlined combustor. Chamber length = 2.7, mean flow Mach number = 0.33, ratio of specific heats = 1.2.

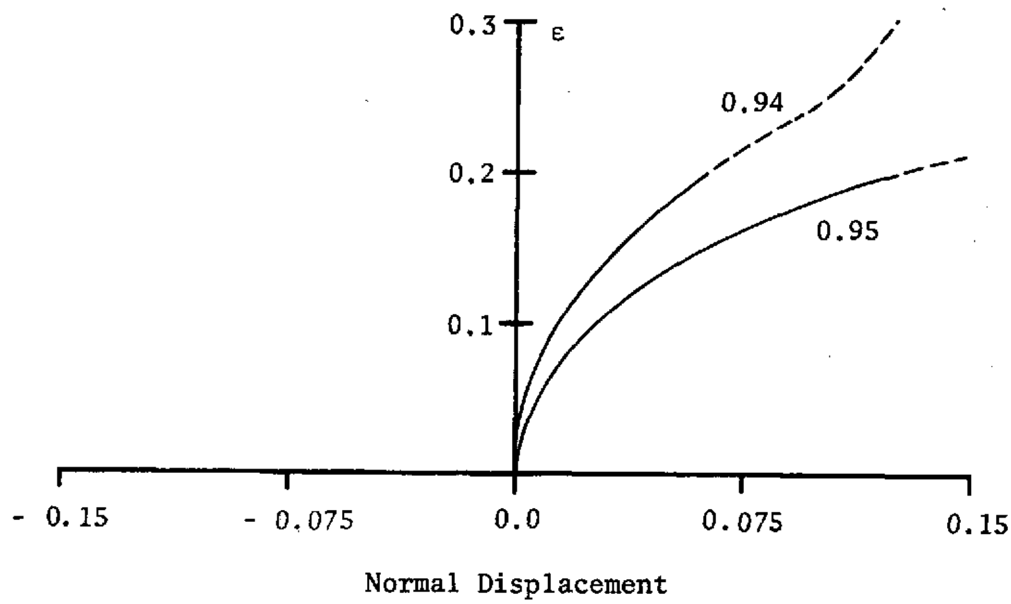

Figure 3. Normal displacement for an unlined combustor for various frequencies along the linear neutral stability curve.

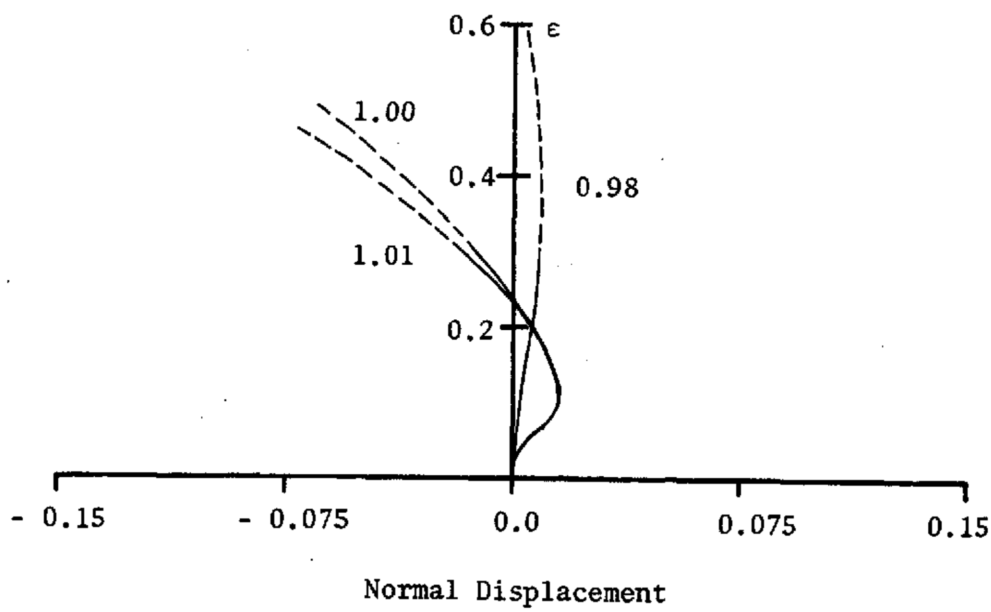

Figure 4. Normal displacement for an unlined combustor for various frequencies along the linear neutral stability curve.

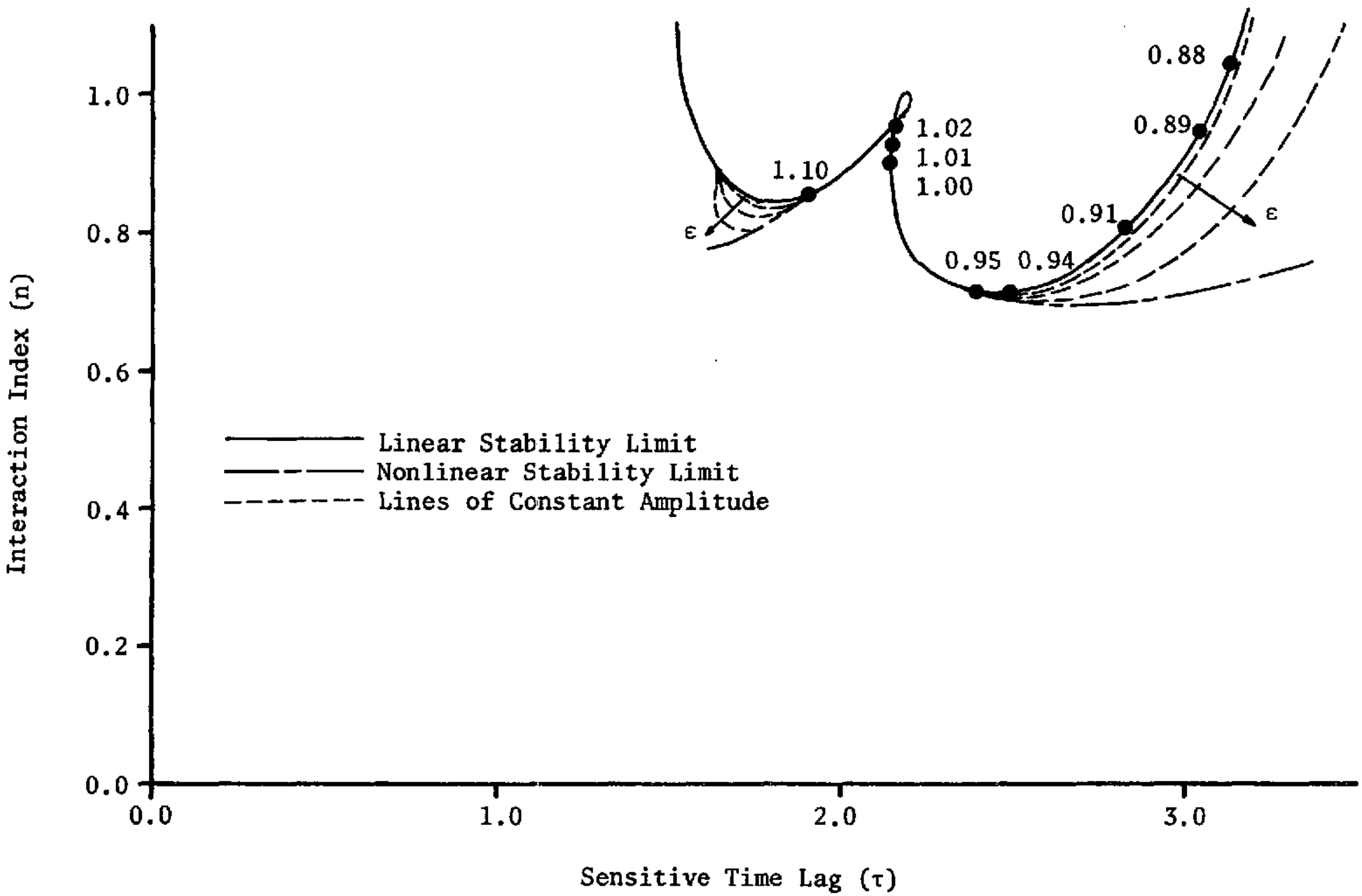

Figure 5. Stability map for a lined combustor. Chamber length = 2.7, mean flow Mach number = 0.33, ratio of specific heats = 1.2, X_1 = 0.0, X_2 = 0.90, k = 5.0.

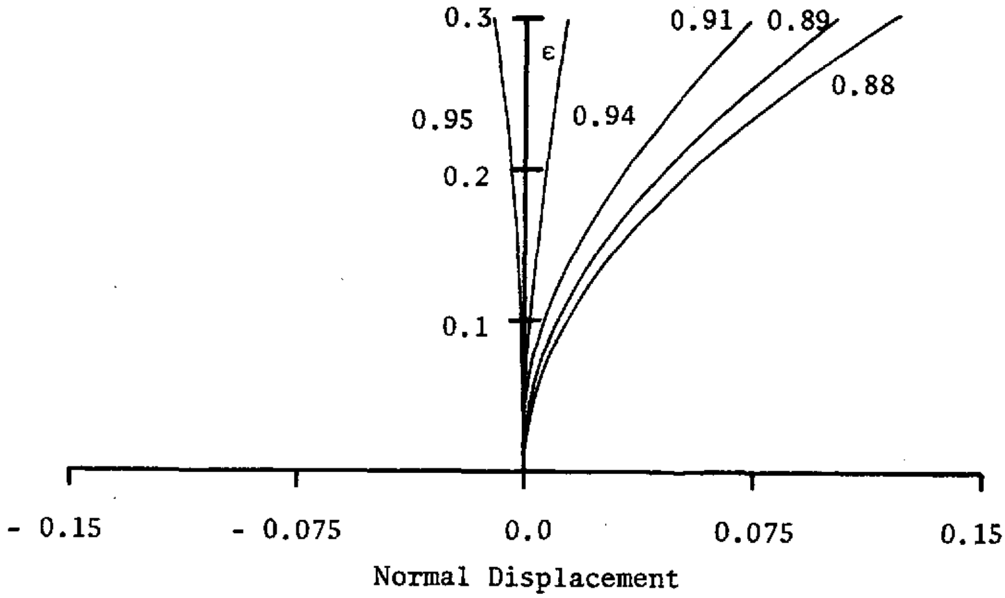

Figure 6. Normal displacement for a lined combustor for various frequencies along the linear neutral stability curve.

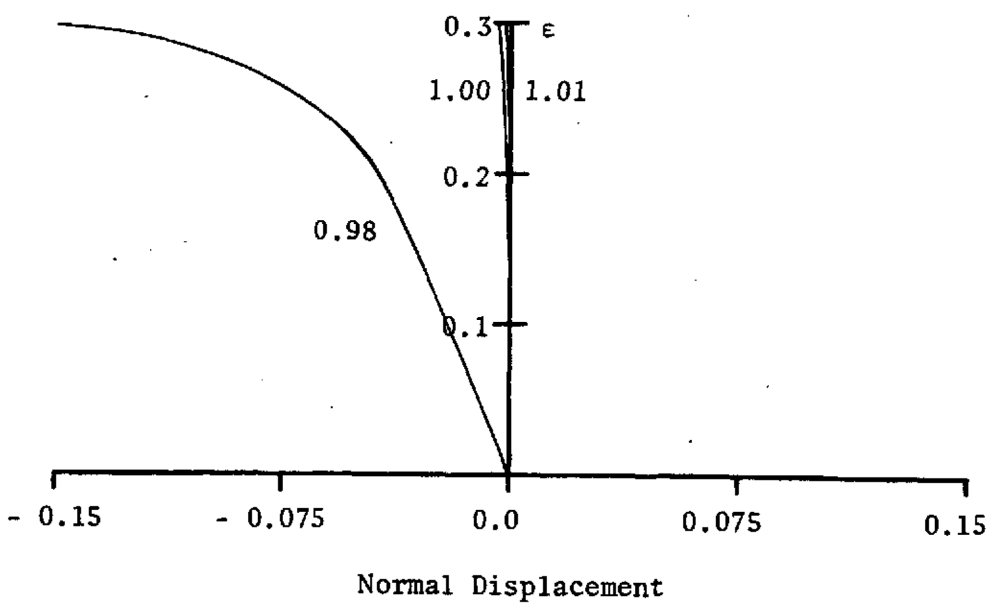

Figure 7. Normal displacement for a lined combustor for various frequencies along the linear neutral stability curve.

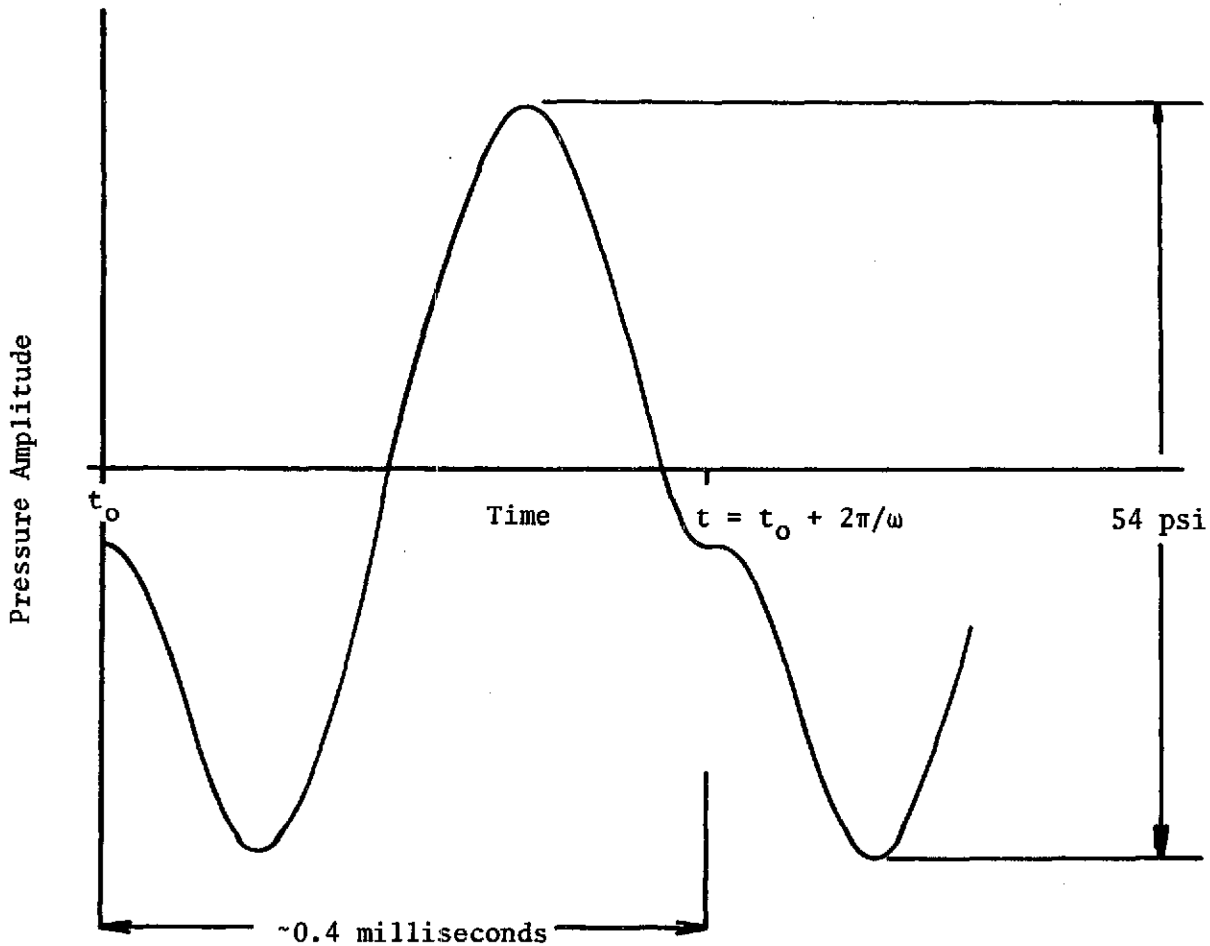

Figure 8. Experimental pressure waveforms versus time (at the injector) (see Ref. 23, p. 188).

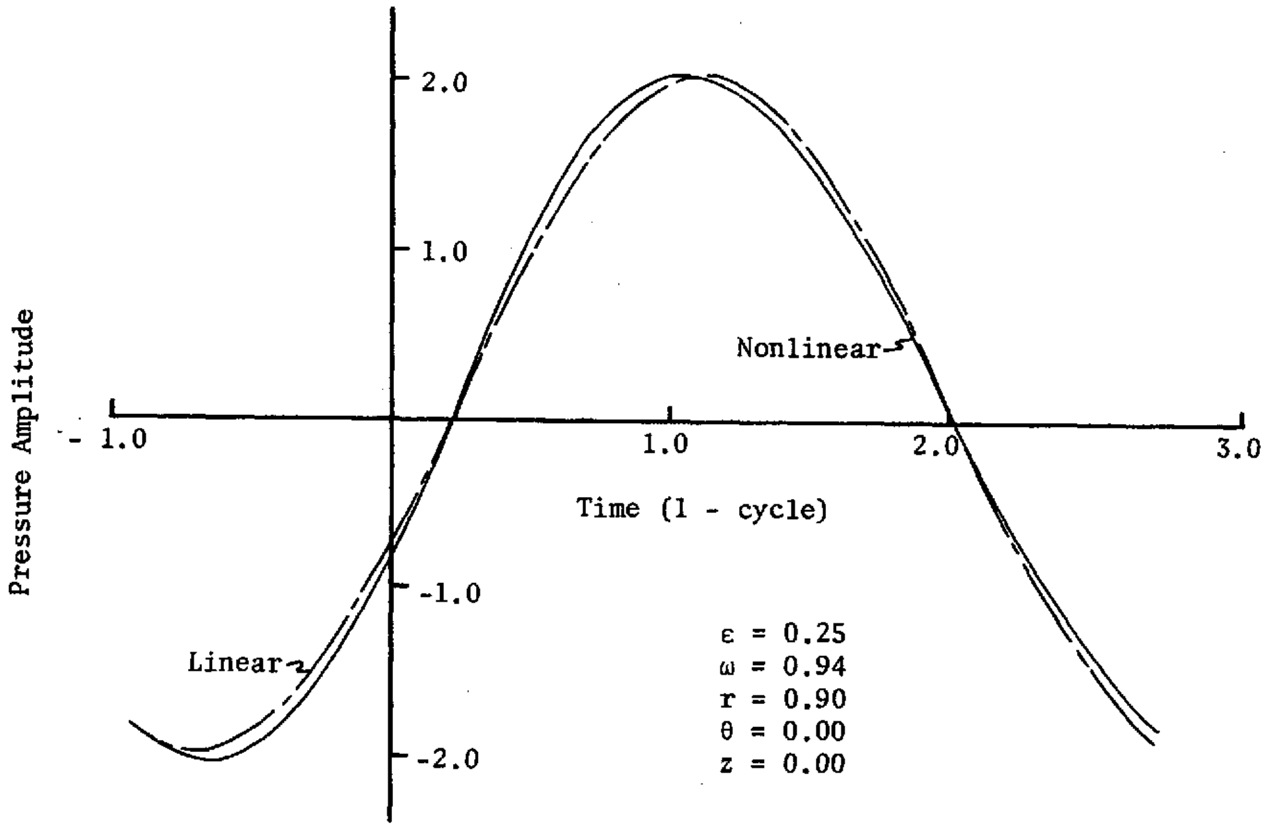

Figure 9. Pressure versus time for an unlined combustor below resonant frequency.

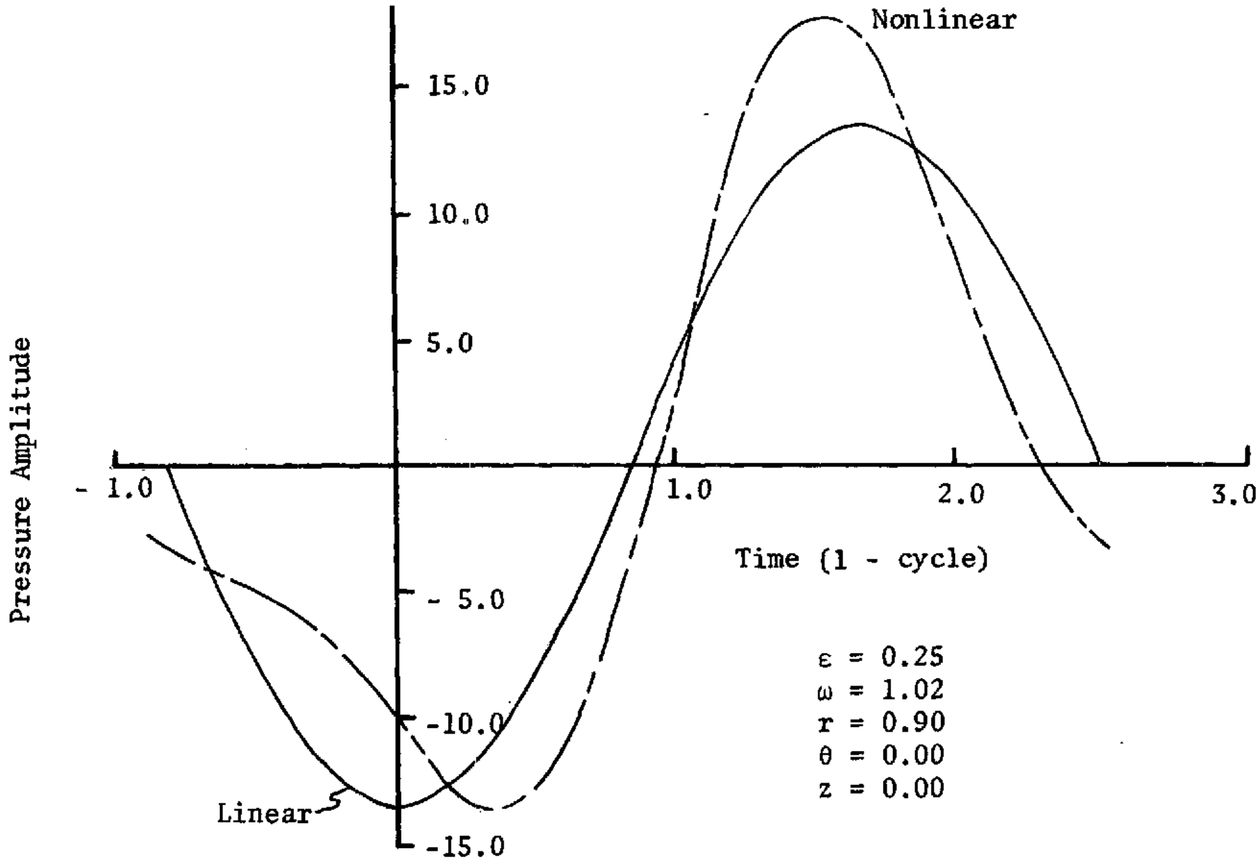

Figure 10. Pressure versus time for an unlined combustor above the resonant frequency.

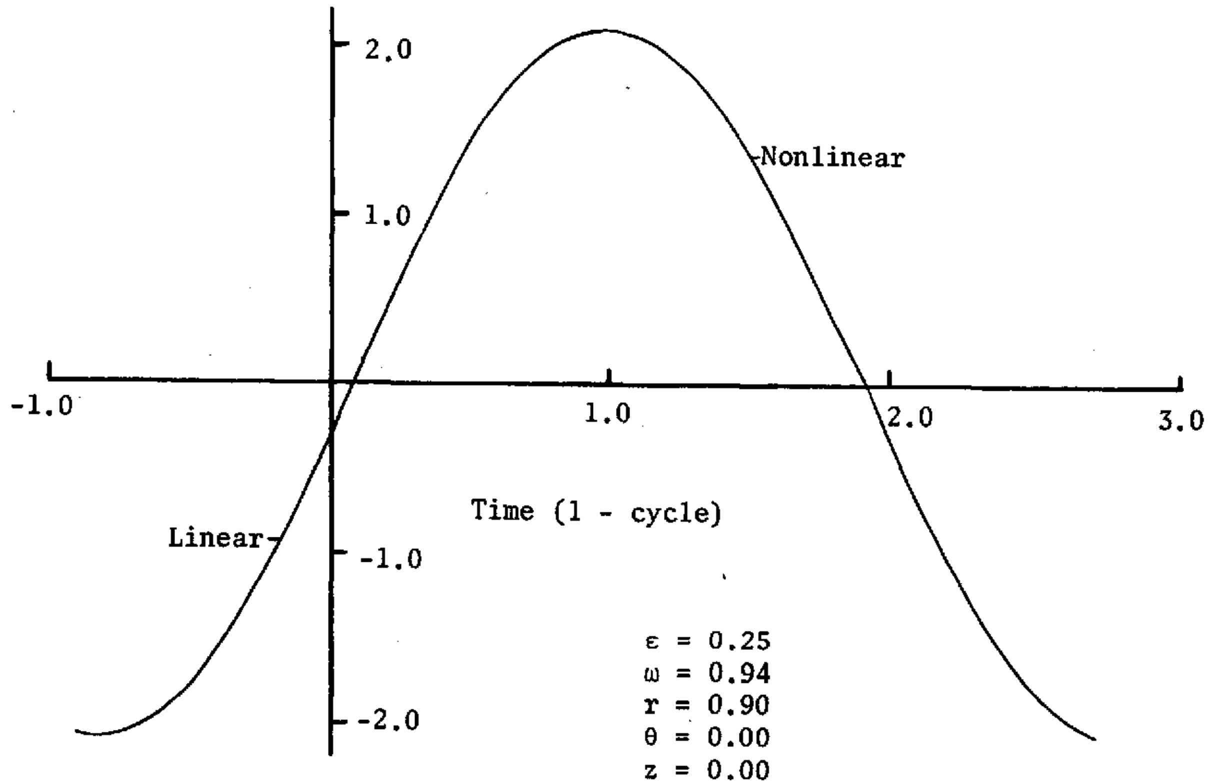

Figure 11. Pressure versus time for a lined chamber below the resonant frequency.

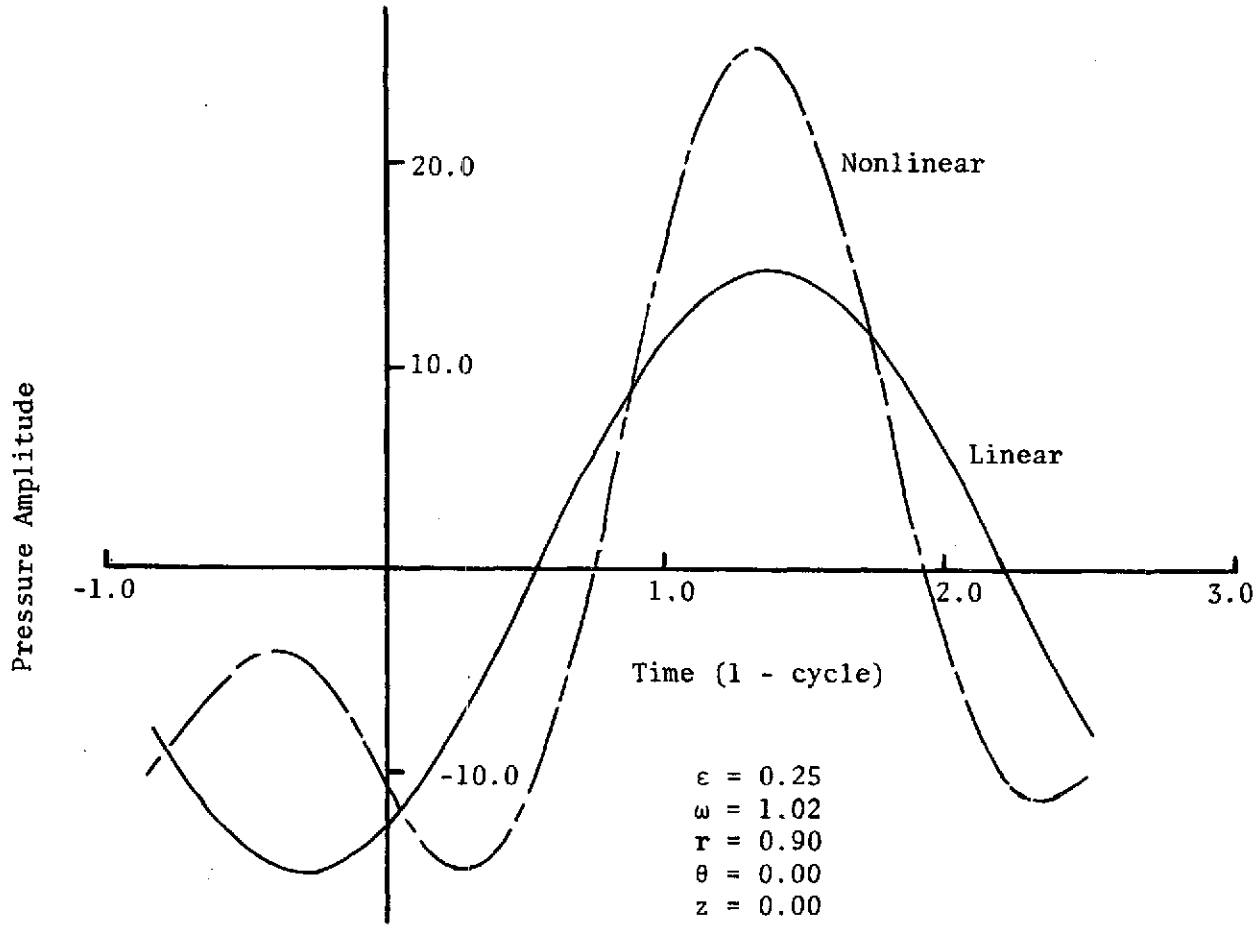

Figure 12. Pressure versus time for a lined combustor above the resonant frequency.

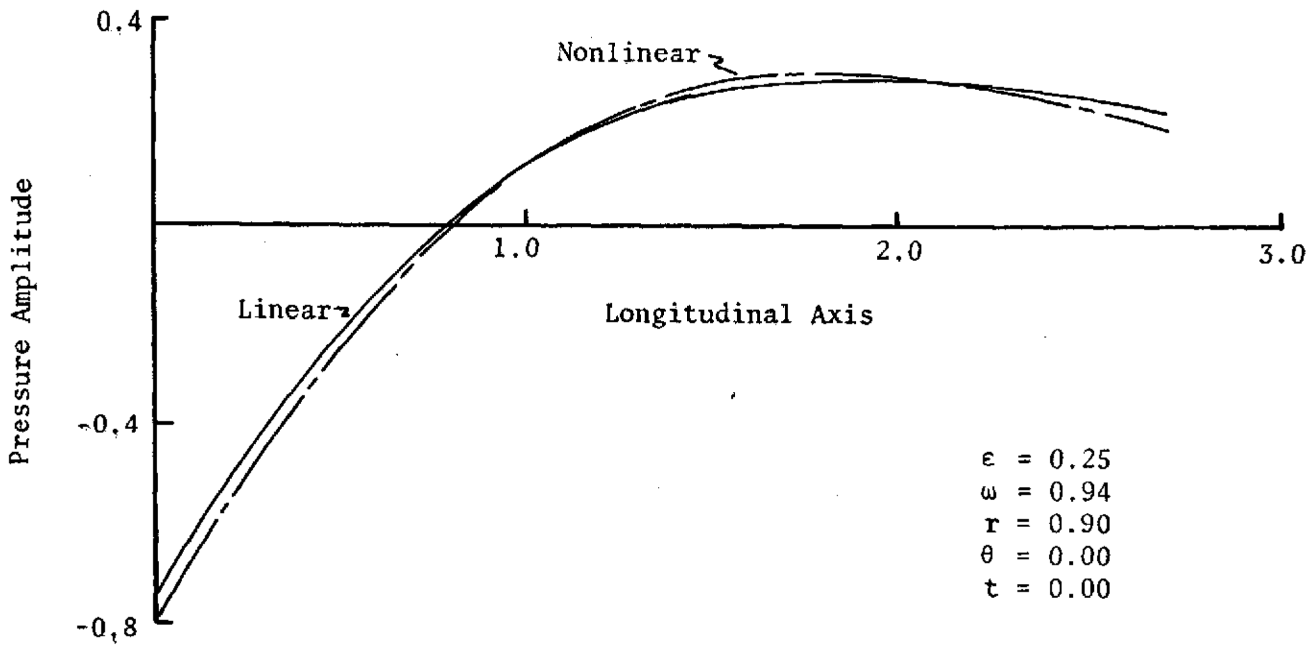

Figure 13. Pressure versus axial coordinate for an unlined combustor below the resonant frequency.

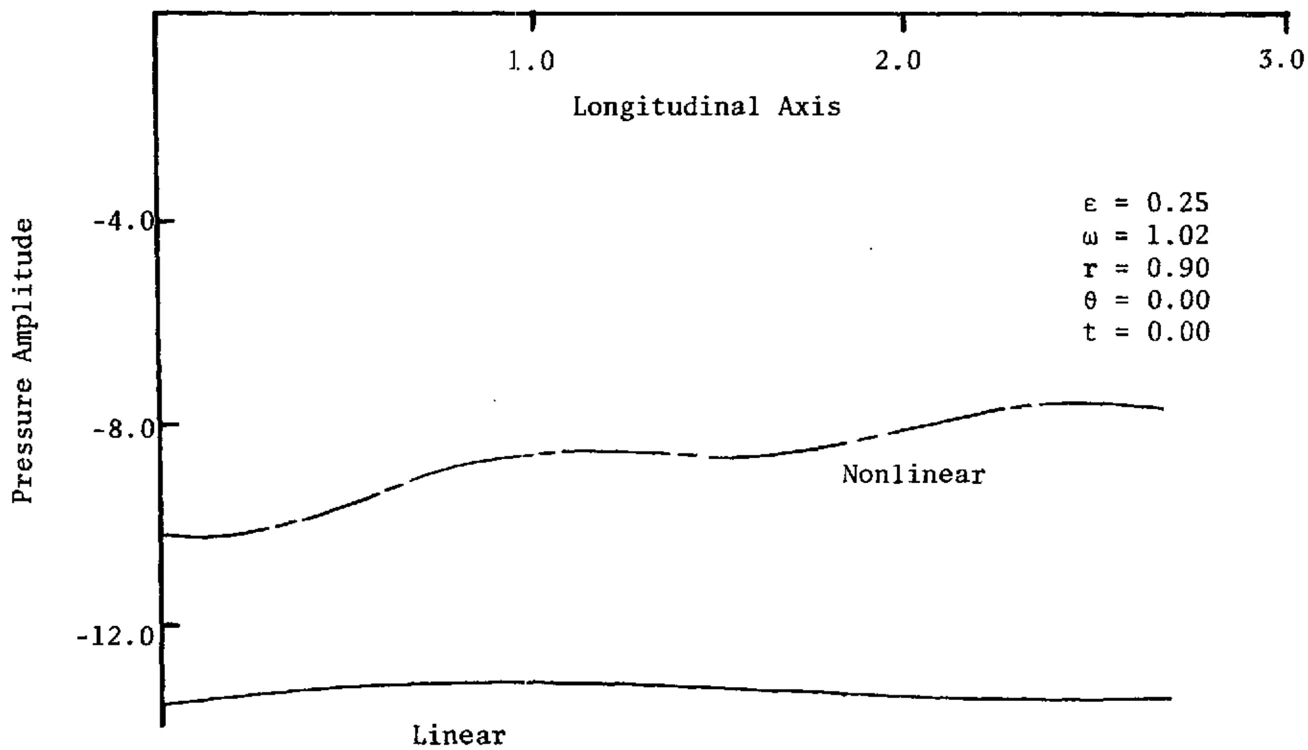

Figure 14. Pressure versus axial coordinate for an unlined combustor above the resonant frequency.

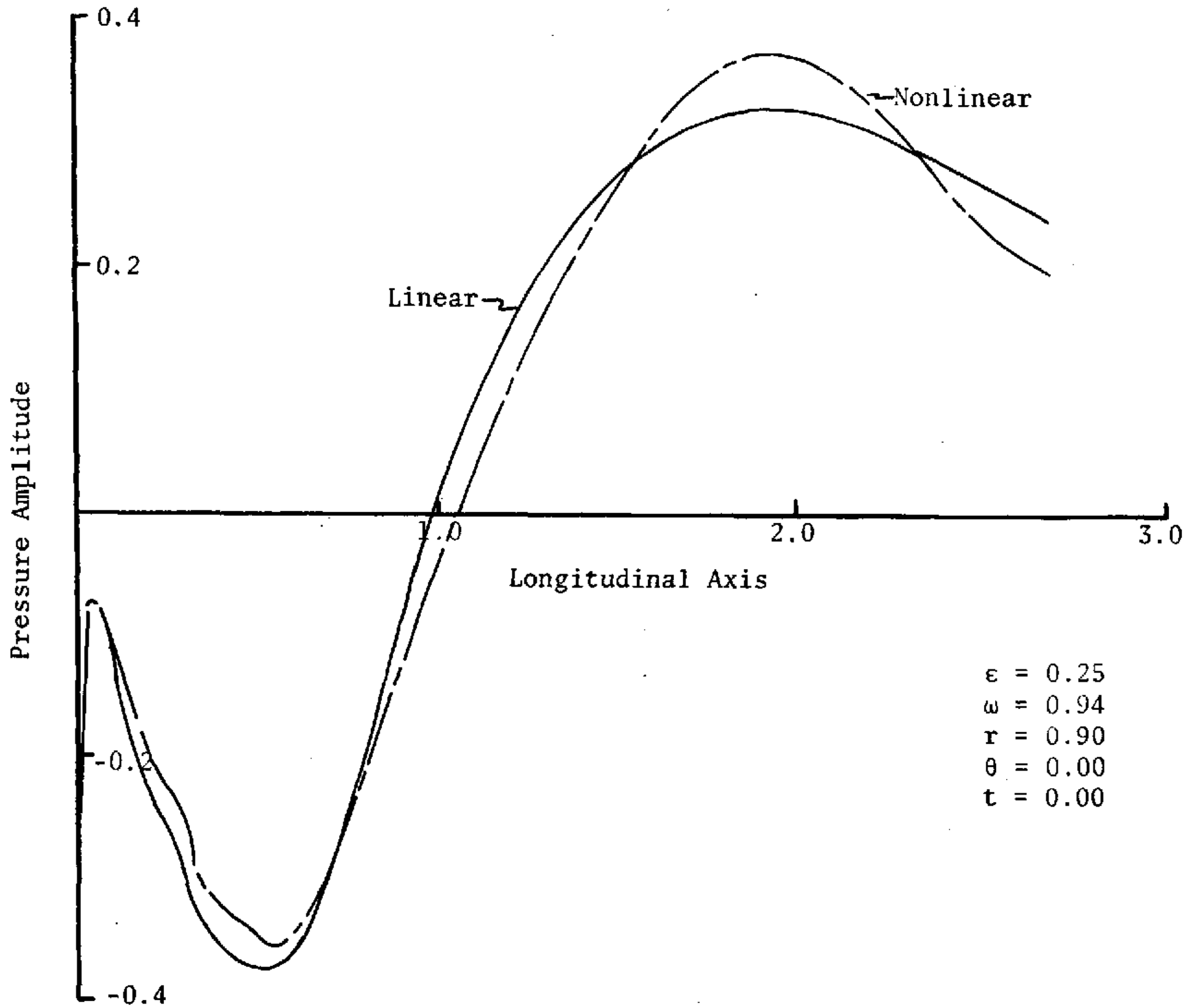

Figure 15. Pressure versus axial coordinate for a lined combustor below the resonant frequency.

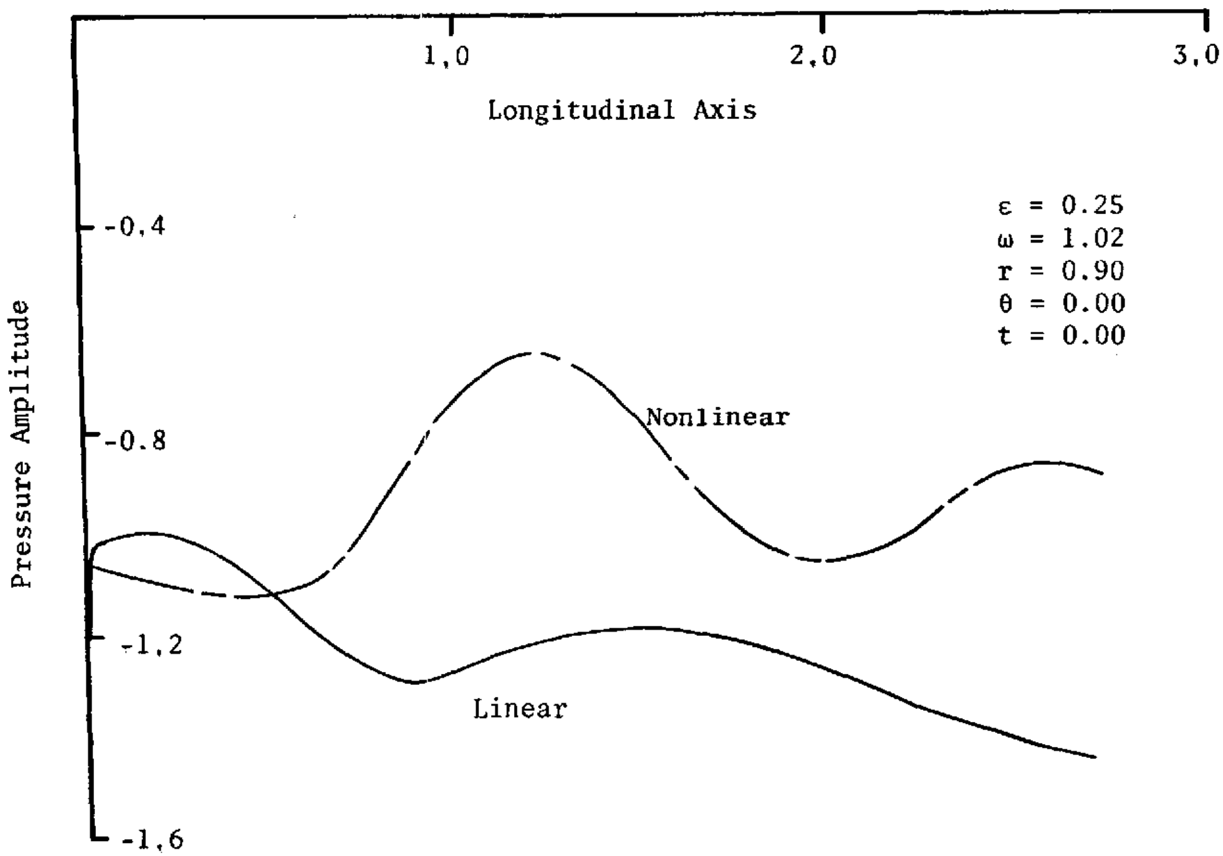

Figure 16. Pressure versus axial coordinate for a lined combustor above the resonant frequency.

Appendix A

This Appendix presents the differential and integral equations, boundary conditions, and eigenvalue relationships developed for the application of this technique.

Substitution of the expansions in ε discussed in the Theory section into the governing equations Equations (6) and (7) leads to the following equations to the appropriate orders in ε

$\underline{O(1)}$:

$$\nabla^2\phi^{(0)} = 0$$

$$Q = 1 + \frac{\gamma-1}{2} M^2$$

$\underline{O(\varepsilon)}$:

$$\nabla^2\phi^{(1)} - \omega^{(0)^2}\frac{\partial^2\phi^{(1)}}{\partial y^2} = 2\omega^{(0)}M\,\frac{\partial^2\phi^{(1)}}{\partial z\partial y} + M^2\,\frac{\partial^2\phi^{(1)}}{\partial z^2}$$

$$p^{(1)} = -\gamma[\omega^{(0)}\,\frac{\partial\phi^{(1)}}{\partial y} + M\,\frac{\partial\phi^{(1)}}{\partial z}]$$

$\underline{O(\varepsilon^2)}$:

$$\nabla^2\phi^{(2)}_{(k)} - \omega^{(0)^2}\,\frac{\partial^2\phi(k)}{\partial y^2} = 2\omega^{(0)}\,M\,\frac{\partial^2\phi^{(2)}_{(k)}}{\partial z\partial y} + M^2\,\frac{\partial^2\phi^{(2)}_{(k)}}{\partial z^2}$$

$$+ (M\,\frac{\partial}{\partial z}\,\omega^{(0)}\,\frac{\partial}{\partial y})(\vec{\nabla}\phi^{(1)}\cdot\vec{\nabla}\phi^{(1)})$$

$$+ (\gamma-1)[\omega^{(0)}\,\frac{\partial\phi^{(1)}}{\partial y}\,\nabla^2\phi^{(1)} + M\,\frac{\partial\phi^{(1)}}{\partial z}\,\nabla^2\phi^{(1)}]$$

$$+ 2\omega^{(1)}[\omega^{(0)}\,\frac{\partial^2\phi^{(1)}}{\partial y^2} + M\,\frac{\partial^2\phi^{(1)}}{\partial z\partial y}]$$

$$p^{(2)}_{(k)} = \frac{p^{(1)^2}}{2\gamma} - \gamma[M\,\frac{\partial\phi^{(2)}_{(k)}}{\partial z} + \omega^{(0)}\,\frac{\partial\phi^{(2)}_{(k)}}{\partial y} + \frac{1}{2}\,\vec{\nabla}\phi^{(1)}\cdot\vec{\nabla}\phi^{(1)}$$

$$+ \omega^{(1)}\,\frac{\partial\phi^{(1)}}{\partial y}]$$

$\underline{0(\varepsilon^3)}$: $\quad \nabla^2\Phi^{(3)}_{(k)} - \omega^{(0)^2} \frac{\partial^2\Phi^{(3)}_{(k)}}{\partial y^2} = 2M\omega^{(0)} \frac{\partial^2\Phi^{(3)}_{(k)}}{\partial z\partial y} + M^2 \frac{\partial^2\Phi^{(3)}_{(k)}}{\partial z^2}$

$$+ 2(\omega^{(0)} \frac{\partial}{\partial y} + M \frac{\partial}{\partial z})(\vec{\nabla}\Phi^{(1)} \cdot \vec{\nabla}\Phi^{(2)}) + (\gamma-1) \Big\{\nabla^2\Phi^{(1)}$$

$$(M \frac{\partial\Phi^{(2)}_{(k)}}{\partial z} + \omega^{(0)} \frac{\partial\Phi^{(2)}_{(k)}}{\partial y}) + \nabla^2\Phi^{(2)}_{(k)} (M \frac{\partial\Phi^{(1)}}{\partial z} + \omega^{(0)} \frac{\partial\Phi^{(1)}}{\partial y})$$

$$+ \frac{1}{2} \nabla^2\Phi^{(1)} (\vec{\nabla}\Phi^{(1)} \cdot \vec{\nabla}\Phi^{(1)})\Big\} + \frac{1}{2} \vec{\nabla}\Phi^{(1)} \cdot \vec{\nabla}(\vec{\nabla}\Phi^{(1)} \cdot \vec{\nabla}\Phi^{(1)})$$

$$+ 2\omega^{(0)}\omega^{(1)} \frac{\partial^2\Phi^{(2)}_{(k)}}{\partial y^2} + (2\omega^{(0)}\omega^{(2)} + \omega^{(1)^2}) \frac{\partial^2\Phi^{(1)}}{\partial y^2}$$

$$+ 2\omega^{(1)} M[\frac{\partial^2\Phi^{(2)}_{(k)}}{\partial z\partial y} + \frac{1}{2} \frac{\partial}{\partial y} (\vec{\nabla}\Phi^{(1)} \cdot \vec{\nabla}\Phi^{(1)})] + \omega^{(2)} M \frac{\partial^2\Phi^{(1)}}{\partial z\partial y}$$

$$+ (\gamma-1)[\omega^{(1)}\nabla^2\Phi^{(1)} \frac{\partial\Phi^{(1)}}{\partial y}]$$

$$p^{(3)}_{(k)} = \frac{p^{(2)}_{(k)}p^{(1)}}{\gamma} - \frac{\gamma+1}{6\gamma^2} p^{(1)^3} - \gamma\Big\{M\frac{\partial\Phi^{(3)}_{(k)}}{\partial z} + \omega^{(0)} \frac{\partial\Phi^{(3)}_{(k)}}{\partial y}$$

$$+ \omega^{(1)} \frac{\partial\Phi^{(2)}_{(k)}}{\partial y} + \omega^{(2)} \frac{\partial\Phi^{(1)}}{\partial y} + \vec{\nabla}\Phi^{(1)} \cdot \vec{\nabla}\Phi^{(2)}_{(k)}\Big\}$$

The associated boundary conditions are written

$\underline{0(\varepsilon)}$: $\quad \vec{\nabla}\Phi^{(1)} \cdot \vec{n} = \beta^{(0)}p^{(1)}$

$\underline{0(\varepsilon^2)}$: $\quad \vec{\nabla}\Phi^{(2)}_{(k)} \cdot \vec{n} = \beta^{(0)}p^{(2)}_{(k)} + \beta^{(1)}p^{(1)}$

$\underline{0(\varepsilon^3)}$: $\quad \vec{\nabla}\Phi^{(3)}_{(k)} \cdot \vec{n} = \beta^{(0)}p^{(3)}_{(k)} + \beta^{(1)}p^{(2)}_{(k)} + \beta^{(2)}p^{(1)}$

At the liner $\beta^{(0)} = \frac{1}{\gamma k}$, $\beta^{(1)} = \beta^{(2)} = 0$, where k is the acoustic impedance and is taken to be a real constant. At the nozzle $\beta^{(0)} = \frac{\gamma-1}{2} M$,

$\beta^{(1)} = \beta^{(2)} = 0$, where M is the Mach number. At the injector (combustion zone) $\beta^{(0)}$, $\beta^{(1)}$, and $\beta^{(2)}$ are complex eigenvalues to be determined by the periodicity requirement.

The partial differential equations along with their associated boundary conditions are converted to integral equations using a Green's function represented by an eigenfunction expansion as follows

$$G^{(k)} = \Sigma_\ell \Sigma_m \Sigma_n \frac{\Omega_{\ell mn}(\vec{r})\Omega_{\ell mn}(\vec{r}_o)}{(k\omega^{(0)})^2 - \eta^2_{\ell mn}} .$$

$\Omega_{\ell mn}$ is an orthornormal eigenfunction given by

$$\Omega_{\ell mn} = \frac{\cos m\theta}{\Lambda^{1/2}_{\ell mn}} J_m(\lambda_{\ell m} r) \cos \frac{n\pi z}{L}$$

and $\eta^2_{\ell mn}$ is the frequency of the closed chamber

$$\eta^2_{\ell mn} = \lambda^2_{\ell m} + \frac{n^2\pi^2}{L^2} .$$

The quantity $\lambda_{\ell m}$ is a root of the equation

$$J'_m(\lambda_{\ell m} r)\Big|_{r=1} = 0 \quad \text{and} \quad \Lambda_{\ell mn}$$

is the normalization constant which is determined from

$$\iiint_V \Omega^2_{\ell mn} \, dV = 1.$$

Utilization of this Green's function in the governing equation and following standard techniques[34] converts the partial differential equations to integral equations for each power of ε and each harmonic component. To lowest order $(O(\varepsilon))$

$$\phi^{(1)} = \iint_{S_o} G_N^{(1)}(\vec{r}/\vec{r}_o)\beta[\gamma i\omega^{(0)}\phi^{(1)} + \gamma M \frac{\partial\phi^{(1)}}{\partial z}]\, d\, S_o$$

$$+ \iiint_{\Psi_o} G_N^{(1)}(\vec{r}/\vec{r}_o)[M^2 \frac{\partial^2\phi^{(1)}}{\partial z^2} + 2i\omega^{(0)}M \frac{\partial\phi^{(1)}}{\partial z}]d\, \Psi_o \tag{18-1}$$

$$\omega^{(0)^2} - \eta_N^2 = \iiint_{\Psi_o} \Omega_N(M^2 \frac{\partial^2\phi^{(1)}}{\partial z^2} + 2i\omega^{(0)}\, M \frac{\partial\phi^{(1)}}{\partial z})\, d\, \Psi_o$$

$$+ \iint_{S_o} \Omega_N[\gamma i\omega^{(0)}\phi^{(1)} + \gamma M \frac{\partial\phi^{(1)}}{\partial z}]\, d\, S_o \tag{18-2}$$

where Ω_N is one of the acoustic eigenfunctions for the chamber with no mean flow, no combustion zone, liner or nozzle, η_N is the eigenvalue for the frequency of oscillation associated with the eigenfunction Ω_N, and $G_N^{(1)}$ is a Green's function for the chamber with one term (Ω_N) removed from the series for $G^{(1)}$. The solution to the linear equations is given formally by

$$\phi^{(1)} = \tilde{\phi} + \cos\hat{m}\,\theta\, \Sigma_\ell\, \Sigma_n\, \mu_{\ell n} J_m(\lambda_{\ell m} r)\cos\frac{n\pi z}{2}$$

where

$$\tilde{\phi} = \frac{\cos\hat{m}\,\theta}{\Lambda^{\frac{1}{2}}_{\hat{\ell}\hat{m}\hat{n}}\Psi} J_{\tilde{m}}(\lambda_{\hat{\ell}\hat{m}} r)(e^{izB} + Ce^{izB_2})$$

$$B_{1,2} = \frac{\omega^{(0)}M \pm [\omega^{(0)^2}M^2 + (M^2-1)(\lambda_{\hat{\ell}\hat{m}} - \omega^{(0)^2})]^{\frac{1}{2}}}{(1-M^2)}$$

$$C = -e^{iL(B_1-B_2)}\frac{[B_1 + \beta_n\gamma(\omega^{(0)} + MB_1)]}{[B_2 + \beta_n\gamma(\omega^{(0)} + MB_2)]}$$

and Ψ is a normalization constant determined by

$$\iiint_{V} \tilde{\phi}\ \Omega_{\ell mn}\ dV = 1.$$

The coefficients $\mu_{\ell n}$ and the specific acoustic admittance at the injector are determined by a successive approximation technique (essentially Picard's method). The subscripts ℓ,m,n represent the radial, tangential, and longitudinal wave numbers, respectively. The reader is referred to the papers of Mitchell, et al.(24) and Espander(37) for a more detailed description of the above solution technique.

To second order in ε two harmonic components exist, $\phi^{(2)}_{(1)}$ and $\phi^{(2)}_{(2)}$. The equation for $\phi^{(2)}_{(1)}$ is

$$\phi^{(2)}_{(1)} = \iiint_{V_o} G_N^{(1)}(\vec{r}/\vec{r}_o)[2i\omega^{(0)}M \frac{\partial\phi^{(2)}_{(1)}}{\partial z} M \frac{\partial^2\phi^{(2)}_{(1)}}{\partial z^2}$$

$$+ \omega^{(1)}(iM \frac{\partial\phi^{(1)}}{\partial z} - \omega^{(0)}\phi^{(1)})]dV_o$$

$$+ \iint_{S_o} G_N^{(1)}(\vec{r}/\vec{r}_o)\{\gamma\beta^{(0)}[M \frac{\partial\phi^{(2)}_{(1)}}{\partial z} + i\omega^{(0)}\phi^{(2)}_{(1)}$$

$$+ \frac{i\omega^{(1)}}{2}\phi^{(1)}] + \gamma\beta^{(1)}[M \frac{\partial\phi^{(1)}}{\partial z} + i\omega^{(0)}\phi^{(1)}]\}dS_o$$

where the surface integral represents the integral over all three surfaces.

This equation in turn is divided into an eigenvalue relationship for $\beta^{(1)}$ and $\omega^{(1)}$ and a linear homogeneous integral relationship for $\phi^{(1)}_{(2)}$. These are, respectively

$$\beta_i^{(1)} = \omega^{(1)}\{\iiint_{V_o} \Omega_N(\vec{r}_o)[\omega^{(0)}\phi^{(1)} - iM \frac{\partial\phi^{(1)}}{\partial z}]\ dV_o$$

$$- \iint_{S_o} \frac{\gamma i\beta^{(0)}}{2} \Omega_N(\vec{r}_o)\phi^{(1)} dS_o\} / \{\iint_{S_i} \gamma\Omega_N(\vec{r}_o)(i\omega^{(0)}\phi^{(1)}$$

$$+ M \frac{\partial\phi^{(1)}}{\partial z}) dS_i\}$$

and

$$\phi^{(2)}_{(1)} = \iiint_{V_o} G_N^{(1)}(\vec{r}/\vec{r}_o)[2i\omega^{(0)} M \frac{\partial\phi^{(2)}_{(1)}}{\partial z} + M^2 \frac{\partial^2\phi^{(2)}_{(1)}}{\partial z^2}] dV_o$$

$$+ \iint_{S_o} G_N^{(1)}(\vec{r}/\vec{r}_o)\gamma\beta^{(0)} [M \frac{\partial\phi^{(2)}_{(1)}}{\partial z} + i\omega^{(0)}\phi^{(2)}_{(1)}] dS_o$$

The only consistent solution to the eigenvalue relationship is $\beta_i^{(1)} = \omega^{(1)} = 0$. The solution to the integral equation for $\phi^{(2)}_{(1)}$ on the other hand is $\phi^{(2)}_{(1)} = \alpha\phi^{(1)}$ where α is arbitrary. In particular α may be identically zero. Thus the first harmonic second order equations imply $\beta_i^{(1)} = \omega^{(1)} = \phi^{(2)}_{(1)} = 0$.

The integral relationship for $\phi^{(2)}_{(2)}$ is

$$\phi^{(2)}_{(2)} = \iiint_{V_o} G_N^{(2)}[f_1(\phi^{(2)}_{(2)}) + g_1(\phi^{(1)})]dV_o + \iint_{S_o} G_N^{(2)}[f_2(\phi^{(2)}_{(2)}$$

$$+ g_2(\phi^{(1)})]dS_o$$

where

$$f_1(\phi^{(2)}_{(2)}) = 2i(2\omega^{(0)})M \frac{\partial\phi^{(2)}_{(2)}}{\partial z} + M \frac{\partial^2\phi^{(2)}_{(2)}}{\partial z^2}$$

$$f_2(\phi^{(2)}_{(2)}) = \gamma\beta^{(0)}[M \frac{\partial\phi^{(2)}_{(2)}}{\partial z} + i(2\omega^{(0)})\phi^{(2)}_{(2)}]$$

$$g_1(\phi^{(1)}) = \frac{(\gamma-1)}{4} [(M \frac{\partial}{\partial z} + i\omega^{(0)})\phi^{(1)}]\nabla^2\phi^{(1)}$$

$$+ \frac{1}{2} (M \frac{\partial}{\partial z} + i2\omega^{(0)}) (\vec{\nabla}\phi^{(1)} \cdot \vec{\nabla}\phi^{(1)})$$

$$g_2(\phi^{(1)}) = \frac{\gamma\beta^{(0)}}{8} [\vec{\nabla}\phi^{(1)} \cdot \vec{\nabla}\phi^{(1)} - (i\omega^{(0)}\phi^{(1)} + M \frac{\partial\phi^{(1)}}{\partial z})^2]$$

The equation may be rewritten so that the terms involving $\phi^{(2)}_{(2)}$ appear on the left hand side of the equation and the terms remaining are a function only of $\phi^{(1)}$.

$$\phi^{(2)}_{(2)} - \iiint_{V_o} G_N^{(2)} f_1(\phi^{(2)}_{(2)}) dV_o - \iint_{S_o} G_N^{(2)} f_2(\phi^{(2)}_{(2)}) dS_o$$

$$= \iiint_{V_o} G_N^{(2)} g_1(\phi^{(1)}) dV_o + \iint_{S_o} G_N^{(2)} g_2(\phi^{(1)}) dS_o$$

The right hand side of this equation can be rewritten as R.H.S. = $\Sigma_\ell \Sigma_m \Sigma_n B_{\ell mn} \Omega_{\ell mn}$, where

$$B_{\ell mn} = \Sigma_N [\iiint_{V_o} \frac{\Omega_N(\vec{r}_o)}{((2\omega^{(0)})^2 - \eta_N^2)} g_1(\phi^{(1)}) dV_o$$

$$+ \iint_{S_o} \frac{\Omega_N(\vec{r}_o)}{((2\omega^{(0)})^2 - \eta_N^2)} g_2(\phi^{(1)}) dS_o]$$

Assuming that a particular solution can be represented as a triple infinite series, $\phi^{(2)}_{(2)} = \Sigma_N \alpha_N \Omega_N(\vec{r})$, substitution into Equation (20) yields

$$\Sigma_i \Sigma_j \Sigma_k \alpha_{ijk} (1 + \gamma_{ijk} + \delta_{ijk}) \Omega_{ijk}(\vec{r}) = \Sigma_\ell \Sigma_m \Sigma_n B_{\ell mn} \Omega_{\ell mn}(\vec{r})$$

where

$$\gamma_{ijk} = -\iiint_{V_o} G^{(2)}_{ijk}\, f_1(\Omega_{ijk}(\vec{r}_o))\, dV_o$$

and

$$\delta_{ijk} = -\iint_S G^{(2)}_{ijk}\, f_2(\Omega ijk(\vec{r}_o))\, dS_o$$

Due to the orthonormality of the eigenfunctions, the coefficients $\alpha_{\ell mn}$ are defined as

$$\alpha_{\ell mn} = \frac{B_{\ell mn}}{(1 + \delta_{\ell mn} + \gamma_{\ell mn})} \tag{25}$$

The integrals appearing in the coefficient $B_{\ell mn}$, $\gamma_{\ell mn}$, and $\delta_{\ell mn}$ can be evaluated analytically, except for the Bessel integrals which must be evaluated numerically. The Bessel integrals involve triple products of the Bessel functions of different orders. The evaluation of these integrals is discussed in Appendix B.

To third order in ε three harmonic component equations occur. As discussed in the theory section, only the one for $\phi^{(3)}_{(1)}$ is of interest since it yields the relationship between the eigenvalues $\beta^{(2)}_i$ and $\omega^{(2)}$. The equation for $\phi^{(3)}_{(1)}$ is

$$\phi^{(3)}_{(1)} = \iiint_{V_o} G^{(1)}_N(\vec{r}/\vec{r}_o)\Big[(2i\omega^{(0)}\, M\, \frac{\partial \phi^{(3)}_{(1)}}{\partial z} + M^2\, \frac{\partial^2 \phi^{(3)}_{(1)}}{\partial z^2})$$

$$+ F(\vec{r}_o)\Big]dV_o + \iint_{S_o} G^{(1)}_N(\vec{r}/\vec{r}_o)\gamma\beta^{(0)}\Big[(i\omega^{(0)}\phi^{(3)}_{(1)} + M\,\frac{\partial \phi^{(3)}_{(1)}}{\partial z})$$

$$- \beta^{(0)} H(\vec{r}_o)\Big]dS_o + \gamma\beta^{(2)}_i \int_{S_i} G^{(1)}_N(\vec{r}/\vec{r}_o)\Big[i\omega^{(0)}\phi^{(1)} + M\,\frac{\partial\phi^{(1)}}{\partial z}\Big]dS_i$$

where

$$F(\vec{r}) = \frac{1}{2}\,(i\omega^{(0)} + M\frac{\partial}{\partial z})\vec{\nabla}\phi^{(2)} \cdot \vec{\nabla}\bar{\phi}^{(1)} + (\frac{\gamma-1}{4})\,\nabla^2\bar{\phi}^{(1)}(M\frac{\partial\phi^{(2)}}{\partial z}$$

$$+ 2i\omega^{(0)}\phi^{(2)}) + \nabla^2\phi^{(2)}(M\frac{\partial\bar{\phi}^{(1)}}{\partial z} + i\omega^{(0)}\bar{\phi}^{(1)})$$

$$+ \frac{1}{2}\,(\vec{\nabla}\phi^{(1)} \cdot \vec{\nabla}\bar{\phi}^{(1)})\nabla^2\phi^{(1)} + \frac{1}{4}\,(\vec{\nabla}\phi^{(1)} \cdot \vec{\nabla}\phi^{(1)})\nabla^2\phi^{(1)}$$

$$+ \frac{1}{8}\,\vec{\nabla}\phi^{(1)} \cdot \vec{\nabla}(\vec{\nabla}\phi^{(1)} \cdot \vec{\nabla}\bar{\phi}^{(1)}) + \frac{1}{2}\,\vec{\nabla}\bar{\phi}^{(1)} \cdot \vec{\nabla}(\vec{\nabla}\phi^{(1)} \cdot \vec{\nabla}\phi^{(1)})$$

$$+ \omega^{(2)}[\frac{iM}{2}\frac{\partial\phi^{(1)}}{\partial z} - \omega^{(0)}\phi^{(1)}]$$

and

$$H(\vec{r}) = \frac{p^{(2)}p^{(1)}}{4\gamma} - \frac{(\gamma+1)}{16\gamma}\,{p^{(1)}}^2\bar{p}^{(1)} - \frac{\gamma}{2}[i\omega^{(2)}\phi^{(1)} + \vec{\nabla}\phi^{(2)} \cdot \vec{\nabla}\bar{\phi}^{(1)}]$$

Following essentially the same separation procedure as was used in the development of the second order eigenvalue relationship the following equation relating $\beta_i^{(2)}$ and $\omega^{(2)}$ results

$$\gamma\beta_i^{(2)} \iint_{S_i} \Omega_N[i\omega^{(0)}\phi^{(1)} + M\frac{\partial\phi^{(1)}}{\partial z}]dS_i$$

$$= \beta^{(0)} \iint_{S_o} \Omega_N\, H(\vec{r}_o)dS_o - \iiint_{\Psi_o} \Omega_N\, F(\vec{r}_o)d\Psi_o$$

where $F(\vec{r}_o)$ and $H(\vec{r}_o)$ are the functions defined previously.

Appendix B

The integrals appearing in Equations 25 and 30 are Lommel-type integrals involving products of Bessel functions to the cubic and quartic powers. The integration scheme chosen to evaluate these integrals was an improved Gaussian Quadrature technique by Kronrod.[37] In Gaussian Quadrature one gives up the freedom of arbitrary end-points (a,b) for the interval and fixes them at (-1,1) by a change of variables from x to y by $x = \frac{1}{2}(b-a)\,y + \frac{1}{2}(b+a)$. Assuming the original integrand $\phi(x)$ is a continuous function, a new integrand is introduced as a polynomial in y.

$$\int_a^b \phi(x)dx = \frac{b-a}{2}\int_{-1}^{1} \phi(y)dy = \alpha_o\phi(y_o) + \alpha_1\phi(y_1) + \ldots$$

where $\phi(y) = y^n L_n(y)$ and $L_n(y)$ is the Legendre polynomial. The α_i are evaluated (and corresponding values of y_i) by evaluating $\phi(y)$ for increasing values of n and evaluating the weighting functions (α_n) and the nodes (y_n). The technique is accurate up to a polynomial of order exactly $2n - 1$. Kronrod's method is similar except that a second orthogonal function is introduced and the weights and nodes are evaluated.

$$\int_a^b \phi(x)dx = \frac{b-a}{2}\int_{-1}^{+1} \phi_n(y)\ \psi_{n+1}(y)dy = \beta_o\phi(y_o)$$

where $\psi_{n+1}(y) = \gamma_o + \gamma_1 x^1 + \gamma_2 x^2 + \ldots + x^{n+1}$. This quadrature method has an accuracy of $3n+1$ for even n and $3n+2$ for odd n.

The forms of the integrals to be evaluated are

$$\int_0^1 r^n J_m(\lambda_{\ell m} r)\ J_{\hat{m}}(\lambda_{\hat{\ell}\hat{m}} r)\ J_{\bar{m}}(\lambda_{\bar{\ell}\bar{m}} r)dr \qquad (1)$$

$$\int_0^1 r^n J_m(\lambda_{\ell m} r)\ J_{\hat{m}}(\lambda_{\hat{\ell}\hat{m}} r)\ J_{\bar{m}}(\lambda_{\bar{\ell}\bar{m}} r)\ J_{m'}(\lambda_{\ell' m'} r)dr \qquad (2)$$

where $-1 < n$ (an integer) $\leq +1$

$0 \leq m, \hat{m}, \bar{m}, m' \leq 2$

$J'_m(\lambda_{\ell m} r)\Big|_{r=1} = 0$

The values for $\lambda_{\ell m}$ were obtained from tabulated values in References 38, 39, and 40.

Three studies were made to determine the accuracy of the Improved Quadrature and the number of nodes needed. Fettis[41] evaluated the integral

$\int_0^1 t\ J_0(\lambda_{\Omega m} r)\ J_1(\lambda_{\ell m} r)\ J_1(\lambda_{\ell m} r)dr$ for particular values of the subscripts using Gaussian Quadrature. His results are published to five place accuracy with an upper bound on the error of $0(10^{-4})$. For most of the cases run nine nodes were the maximum needed to reproduce the same solutions to the published five decimal places. The second study involved a comparison using Simpson's rule and Improved Quadrature. The Simpson's rule algorithm (Conte[42]) had an internal error control which was set at one tenth of one percent. The results for both integrals (1) and (2) were identical within the error limit. However the Simpson's rule algorithm was slower than Improved Quadrature by a factor of ten. The third check on convergence was an internal check made by increasing the number of nodes and calculating the error with respect to the largest node case run (21 nodes). (This corresponds to a 65th order polynomial.) Depending on the integral evaluated the calculation time varied from 3 to 21 milliseconds for 21 nodes on a third generation CDC computer. Using this internal comparison it was found that 9 nodes gave an error which was less than 0.001 percent when compared with the 21 node solution. The 9 node Improved Quadrature technique was chosen for evaluating integrals (1) and (2).

Appendix C

This section describes the use of the computer program, COMBADM, to determine regions of triggering and nonlinear stability limits for a given set of input parameters which characterize the combustion chamber. The main program calls three main subroutines LINEAR, NONLINE and EIGEN. LINEAR calculates the linear coefficient matrix, μ, and the injector admittance, β_i. When the calculation for the linear solution are completed all variables that are not necessary later are destroyed.

NONLINE calls the routines which calculate the second order coefficient matrix. On the call to NONLINE a hollerith variable is used to determine whether or not the linear cross product terms are calculated. If the hollerith variable equals CROSSbPROD the cross product terms are calculated. For any other hollerith variable they are not calculated. After the calculations are complete the program returns to the main program.

EIGEN is now called by the main program. This is the subroutine which calculates the second order correction to the injector admittance and the second order correction to the frequency, $\omega^{(2)}$. It should be noted that a numerical root solving technique is used to find $\omega^{(2)}$ and since the equation is transcendental care must be taken in assuring that the proper root is found. Several numerical root solving techniques were tried and it was found that the Secant Method[43] was the most dependable. The subroutine SOLSION is the root finding technique.

The input parameters are:

1. LENGNO - the problem identifier
2. F - the initial frequency ratio of the combustor

3. ALENGTH - the length to radius ratio for the combustor
4. AMACH - mean flow Mach Number
5. GAMMA - ratio of specific heats
6. G - complex number specifying the acoustic impedance of the nozzle
7. AK - complex number specifying the acoustic impedance of the liner
8. X1 - the beginning of the liner opening with respect to the injector
9. X2 - the end of the liner opening with respect to the injector
10. ERROR1 - the maximum acceptable error in the real part of the combustion admittance
11. ERROR2 - the maximum acceptable error in the imaginary part of the combustion admittance
12. LHAT - radial acoustic mode number
13. MHAT - transverse acoustic mode number
14. NHAT - longitudinal acoustic mode number
15. LDEX - specifies the number of radial terms in the coefficient matrix
16. NDEX - specifies the number of longitudinal terms in the coefficient matrix
17. MATRIX - identifier used to determine whether or not the coefficient matrix is printed out
18. FINC - the increment size of the frequency ratio

19. FMAX - the frequency for which a combustion admittance is calculated

20. EPSILON - initial wave amplitude

21. LOWLIM - minimum acceptable value of ω^2

22. OMEGING - increment of ω^2 to start root search

23. UPLIM - maximum acceptable value of ω^2

Sample Calculation

An example of the output from COMBADM is shown on the page immediately following the listing of the program. It is suggested that the output be used for comparison to insure correct operation of the program. In the output the linear neutral stability point for the unlined combustor is denoted as BETAWI and the linear neutral stability point for the lined combustor to the specified accuracy is the last value of BETAI(J) listed. The n and τ values listed next are given for the lined combustor first and the unlined combustor next.

The nonlinear (second order) coefficient matrix is printed out next for the unlined combustor. The n and τ values above the line following the coefficient matrix are for $\omega^{(2)} = 0$. The numerical values inside the lines are the second order corrections. These are followed by the corrected values for the frequency, the admittance, n, and τ. The normal displacement is the distance between the linear stability point and the stability point for the given amplitude parameter. The quantities G and G1 are the error checks for the root ω^2. Following the results for the unlined combustor are the results for the lined combustor which are presented in the same form as above.

The form of the input for COMBADM is shown immpediately following the list of the program. The input is on five data cards as follows:

Card 1

Columns	Variable	Type
1-10	LENGNO	Alphanumeric word
11-20	Re(F)	Real number
21-30	Im(F)	Real number
31-40	ALENGTH	Real number
41-50	AMACH	Real number
51-60	GAMMA	Real number
61-70	Re(G)	Real number
71-80	Im(G)	Real number

Card 2

Columns	Variable	Type
1-10	Re(AK)	Real number
11-20	Im(AK)	Real number
21-30	X1	Real number
31-40	X2	Real number
41-50	ERROR1	Real number
51-60	ERROR2	Real number
61-63	LHAT	Integer
64-66	MHAT	Integer
67-69	NHAT	Integer
70-72	LDEX	Integer
73-75	NDEX	Integer

76-80	MATRIX*	Alphanumeric word

Card 3

Columns	Variable	Type
1-10	Re(FINC)	Real number
11-20	Im(FINC)	Real number
21-30	Re(FMAX)	Real number
31-40	Im(FMAX)	Real number

Card 4

Columns	Variable	Type
26-35	EPSILON	Real number

Card 5

Columns	Variable	Type
1-10	LOWLIM	Real number
11-20	OMEGINC	Real number
21-30	UPLIM	Real number

* If the coefficient matrix is to be printed out, then MATRIX is printed as YESBB, where B signifies a blank.

Nomenclature

		COMBADM
$\omega^{(0)}$		OMEGA
$\omega^{(2)}$		OMEGA2
ω		OMEGAC
M		AMACH
γ		GAMMA
$\gamma_{\ell mn}$	Function	GAMM
$\delta_{\ell mn}$	Function	DELTA
η	Function	ETA
$J'_m(\lambda_{\ell m})$	Function	BESPRIM
$J_m(\lambda_{\ell m})$	Function	BESSEL
$\Lambda^{\frac{1}{2}}_{\ell mn}$	Function	FNORM
ψ	Function	PSI
B_1		BETA1
B_2		BETA2
C		C
i		I
$\tilde{\beta}_i$		BETAW1
$\beta_i^{(0)}$		BETAI or BETA1S
$\beta_i^{(2)}$		BETAI2
β_i		BETA1C
β_c		BETAC
β_N		BETAN
$B_{\ell mn}$		MUSND and MUCROSS

$\mu\alpha_{\ell mn}$	MUSND
ℓmn	AMU
n	N or B
τ	TAU
ε	EPSILON

```
      PROGRAM COMBADM( TAPE1, TAPE2, TAPE5, TAPE6, OUTPUT=101 )
      REAL LENGTH , IJ1 , IJ2 , IJ3 , IJ4
      REAL N , NVEC , NSAVE(3) , TAUSAVE(3)
      COMPLEX PSI , FNORM , AN
      COMPLEX F, G, AK, FINC, FMAX, BETASTR
      COMPLEX BETA1, BETA2, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1 AK2, PSI2, AMU, BETAIS, C12, I
      COMPLEX D1 , D2 , D3 , D4 , D5 , D6
      COMMON / BLKA / BETA1, BETA2, AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
     1, AK2, BETAN, ALMDA, BETAI, BETAC, BETAWI,     LHAT, MHAT, NHAT,
     1PSI2,X1,X2
      COMMON / BLKB / AMU( 3,30,2),BETAIS( 2),I,MATRIX
      COMMON / BLKC / LENGTH , D1(30) , D2(30) , D3(30) , D4(30) , D5(30
     $) , D6(30) , A1(2)
      COMMON / BLOCK / C12(30,30),ISET,LDEX,NDEX,ERROR1,ERROR2
      COMMON / NORM / N(3) , TAU(3) , NVEC , TAUVEC , PRINT
C ********************************************************************
C ***
C *** THE FIRST THREE DATA CARDS ARE THE GENERAL ENGINE CHARACTERISTICS
C *** THE FOURTH DATA CARD IS FOR THE POSITION AND CASE IN PRESS AND VEL
C *** THE NEXT DATA CARD SPECIFIES THE INITIAL TIME , THE TIME INCREM ,
C *** AND THE EPSILON
C ***
C ***
C *** LENGNO IS THE PROBLEM IDENTIFIER
C *** F IS THE COMPLEX FREQUENCY RATIO
C *** ALENGTH IS THE CHAMBER LENGTH TO RADIUS RATIO
C *** AMACH IS THE CHAMBER MEAN FLOW MACH NUMBER
C *** GAMMA IS THE RATIO OF THE SPECIFIC HEATS
C *** G IS THE COMPLEX NOZZLE ADMITTANCE
C *** AK IS THE COMPLEX LINER ADMITTANCE
C *** X1 IS THE INITIAL LINER LOCATION
C *** X2 IS THE LOCATION OF THE END OF THE LINER
C *** ERROR1 IS THE MAXIMUM ALLOWABLE ERROR IN THE REAL PART OF THE SOLN
C *** ERROR2 IS THE MAXIMUM ALLOWABLE ERROR IN THE IMAG PART OF THE SOLN
C *** LHAT IS THE PRIMARY RADIAL MODE
```

```
C *** MHAT IS THE PRIMARY TANGENTIAL MODE
C *** NHAT IS THE PRIMARY LONGITUDINAL MODE
C *** LDEX AND NDEX SPECIFY THE SIZE OF THE COEFFICIENT MATRIX IN THE L
C *** AND N DIRCTIONS RESPECTIVELY
C *** IF MATRIX IS YES THE COEFFICIENT MATRICIES ARE PRINTED OUT
C *** FINC IS THE FREQUENCY RATIO INCREMENT
C *** FMAX IS THE MAXIMUM VALUE WHICH THE FREQUENCY RATIO MAY TAKE
C *** EPSILON IS THE AMPLITUDE PARAMETER
C *** LOWLIM IS THE INITIALIZTION FOR THE OMEGA(2) SEARCH
C *** OMEGINC IS THE FREQUENCY INCREMENT FOR OMEGA(2)
C *** UPLIM IS THE MAXIMUM VALUE THAT OMEGA(2) MAY TAKE
C ***
C ************************************************************************
      WRITE(6,1000)
 1000 FORMAT(*PM NEW RIBBON PLZ *)
    2 READ(5,200)LENGNO,F,ALENGTH,AMACH,GAMMA,G,AK,X1,X2,ERROR1,ERROR2,
     1LHAT,MHAT,NHAT,LDEX,NDEX,MATRIX,FINC,FMAX
      IF( EOF , 5 ) 50 , 3
    3 BETAC = 1./GAMMA/AK
      G = CMPLX((GAMMA+1.)/2./GAMMA,0.0 )
      BETAN = AMACH * ( G - 1. / GAMMA )
      JX  = 30
      JY = JX - 1
      PRINT = 10H
      IF ( BETAC .EQ. CMPLX(0.0,0.0) .OR. X2 .EQ. 0.0 ) JY = 0
      CALL LINEAR ( LENGNO , F , G , AK , FINC , FMAX , JX )
      CALL NTAU ( 1./GAMMA-BETAIS(1)/AMACH , OMEGA )
      NSAVE(1) = N              $ TAUSAVE(1) = TAU
      CALL NTAU ( 1./GAMMA-BETAWI/AMACH , OMEGA )
      N(3) = N      $ TAU(3) = TAU
      F = F + FINC
      JX = 30
      CALL LINEAR ( LENGNO , F , G , AK , FINC , FMAX , JX )
      CALL NTAU ( 1./GAMMA-BETAIS(1)/AMACH , OMEGA )
      NSAVE(2) = N              $ TAUSAVE(2) = TAU
      CALL NTAU ( 1./GAMMA-BETAWI/AMACH , OMEGA )
```

```
      N(2) = N        $ TAU(2) = TAU
      CALL WRYTE
      F = F + FINC
    4 CALL LINEAR ( LENGNO , F , G , AK , FINC , FMAX , JX )
      CALL NTAU ( 1./GAMMA-BETAIS(1)/AMACH , OMEGA )
      NSAVE(3) = N               $ TAUSAVE(3) = TAU
      CALL NTAU ( 1./GAMMA-BETAWI/AMACH , OMEGA )
      CALL WRYTE
      CALL NORMAL
      CALL REED
      MATRIX = 3HYES
      LENGTH = ALENGTH
      BETAI = BETAWI
      BETASTR = BETAC
      BETAC = CMPLX ( 0.0 , 0.0 )
      CALL INIT ( C12 , AMU(1,1,2) , D1 , 3 , 30 )
      WRITE(6,600)
      A1 ( 1 ) = 3.14159265358979
      A1 ( 2 ) = A1 ( 1 ) / 2.
      WRITE(6,601)
      CALL CONST
      CALL NONLINE ( 10HLINEAR     )
      CALL EIGEN
      GO TO 6
    6 CONTINUE
      IF ( JY .EQ. 0 .OR. JX .EQ. 30 ) GO TO 5
      BETAI = BETAIS( 2 )
      BETAC = BETASTR
      WRITE(6,602)
      CALL CONST
      CALL NONLINE (10HCROSS      )
      N(3) = NSAVE(1)           $ TAU(3) = TAUSAVE(1)
      NSAVE(1) = N(2)           $ TAUSAVE(1) = TAU(2)
      N(2) = NSAVE(2)           $ TAU(2) = TAUSAVE(2)
      NSAVE(2) = N(1)           $ TAUSAVE(2) = TAU(1)
      N(1) = NSAVE(3)           $ TAU(1) = TAUSAVE(3)
```

```
      CALL NORMAL
      CALL EIGEN
      N(3) = NSAVE(1)            $ TAU(3) = TAUSAVE(1)
      NSAVE(1) = N(2)            $ TAUSAVE(1) = TAU(2)
      N(2) = NSAVE(2)            $ TAU(2) = TAUSAVE(2)
      NSAVE(2) = N(1)            $ TAUSAVE(2) = TAU(1)
    1 F = F + FINC
      JX = 30
      IF ( F .GT. FMAX ) 2 , 4
    5 N(3) = N(2)                $ TAU(3) = TAU(2)
      IF ( JX .EQ. 30 ) GO TO 50
      N(2) = N(1)                $ TAU(2) = TAU(1)    $ GO TO 1
   50 CALL EXIT
  200 FORMAT(A10,7F10.0/6F10.0,5I3,A5/4F10.0)
  201 FORMAT(2A10,3F10.0,2I10)
  202 FORMAT(//)
  600 FORMAT(*1*135(1H=)/* *46(1H=),43X,46(1H=)/* *46(1H=)*  THE FOLLOWI
     $NG ARE THE NONLINEAR RESULTS  *46(1H=)/* *46(1H=),43X,46(1H=)/* *1
     $35(1H=))
  601 FORMAT(*0*50(1H^)/*  SECOND ORDER, NO LINER */* *50(1HV))
  602 FORMAT(*0*50(1H^)/*  SECOND ORDER, W/ LINER */* *50(1HV))
      END
```

```
      FUNCTION BESPRIM (M, L)
      DIMENSION A(10,5)
C**** THESE ARE THE ROOTS OF THE DERIVATIVE OF THE BESSEL FUNCTION OF OR
C**** SET EQUAL TO ZERO
      A(1 ,1) = 0.00000000
      A(2 ,1) = 3.83170597
      A(3 ,1) = 7.01558667
      A(4 ,1) = 10.17346814
      A(5 ,1) = 13.32369194
      A(6 ,1) = 16.47063005
      A(7 ,1) = 19.61585851
      A(8 ,1) = 22.76008438
      A(9 ,1) = 25.90367209
      A(10,1) = 29.04682853
C**** THESE ARE THE ROOTS OF THE DERIVATIVE OF THE BESSEL FUNCTION OF OR
C**** SET EQUAL TO ZERO
      A(1 ,2) = 1.84118378
      A(2 ,2) = 5.33144277
      A(3 ,2) = 8.53631637
      A(4 ,2) = 11.70600490
      A(5 ,2) = 14.86358863
      A(6 ,2) = 18.01552786
      A(7 ,2) = 21.16436986
      A(8 ,2) = 24.31132686
      A(9 ,2) = 27.45705057
      A(10,2) = 30.60192297
C**** THESE ARE THE ROOTS OF THE DERIVATIVE OF THE BESSEL FUNCTION OF OR
C**** SET EQUAL TO ZERO
      A(1 ,3) = 3.05423693
      A(2 ,3) = 6.70613319
      A(3 ,3) = 9.96946782
      A(4 ,3) = 13.17037086
      A(5 ,3) = 16.34752232
      A(6 ,3) = 19.51291278
      A(7 ,3) = 22.67158177
      A(8 ,3) = 25.82603714
```

```
      A(9 ,3) = 28.97767277
      A(10,3) = 32.12732702
C**** THESE ARE THE ROOTS OF THE DERIVATIVE OF THE BESSEL FUNCTION OF O
C**** SET EQUAL TO ZERO
      A(1 ,4) = 4.20118894
      A(2 ,4) = 8.01523660
      A(3 ,4) = 11.34592431
      A(4 ,4) = 14.58584829
      A(5 ,4) = 17.78874787
      A(6 ,4) = 20.97247694
      A(7 ,4) = 24.14489743
      A(8 ,4) = 27.31005793
      A(9 ,4) = 30.47026881
      A(10,4) = 33.62694918
C**** THESE ARE THE ROOTS OF THE DERIVATIVE OF THE BESSEL FUNCTION OF O
C**** SET EQUAL TO ZERO
      A(1 ,5) = 5.31755313
      A(2 ,5) = 9.28239629
      A(3 ,5) = 12.68190844
      A(4 ,5) = 15.96410704
      A(5 ,5) = 19.19602880
      A(6 ,5) = 22.40103227
      A(7 ,5) = 25.58975968
      A(8 ,5) = 28.76783622
      A(9 ,5) = 31.93853934
      A(10,5) = 35.10391668
    1 BESPRIM = A(L, M)
      RETURN
      ENTRY BESSEL
C**** THESE ARE THE BESSEL NUMBERS OF ORDER ZERO  FOR THE ZEROS OF THE F
C**** FUNCTION
      A(1 ,1) = 1.00000000
      A(2 ,1) = -0.4027588095
      A(3 ,1) = 0.301128303
      A(4 ,1) = -0.249704877
      A(5 ,1) = 0.218359407
```

```
      A(6 ,1) = -0.19645371
      A(7 ,1) = 0.180063375
      A(8 ,1) = -0.167184690
      A(9 ,1) = 0.156724985
      A(10,1) = -0.148011108
C**** THESE ARE THE BESSEL NUMBERS OF ORDER  ONE  FOR THE ZEROS OF THE B
C**** FUNCTION
      A(1 ,2) = 0.5818649368
      A(2 ,2) = -0.3461258542
      A(3 ,2) = 0.2732981131
      A(4 ,2) = -0.233304416
      A(5 ,2) = 0.207012651
      A(6 ,2) = -0.188017488
      A(7 ,2) = 0.173459050
      A(8 ,2) = -0.161838211
      A(9 ,2) = 0.152282069
      A(10,2) = -0.144242905
C**** THESE ARE THE BESSEL NUMBERS OF ORDER  TWO  FOR THE ZEROS OF THE B
C**** FUNCTION
      A(1 ,3) = 0.4864961885
      A(2 ,3) = -0.3135283099
      A(3 ,3) = 0.2547441235
      A(4 ,3) = -0.220881581
      A(5 ,3) = 0.197937434
      A(6 ,3) = -0.181010000
      A(7 ,3) = 0.167835534
      A(8 ,3) = -0.157195167
      A(9 ,3) = 0.148363778
      A(10,3) = -0.140878333
C**** THESE ARE THE BESSEL NUMBERS OF ORDER THREE FOR THE ZEROS OF THE B
C**** FUNCTION
      A(1 ,4) = 0.4343942763
      A(2 ,4) = -0.2911584413
      A(3 ,4) = 0.240738175
      A(4 ,4) = -0.210965204
      A(5 ,4) = 0.190419022
```

```
      A(6 ,4) = -0.175048405
      A(7 ,4) = 0.162954965
      A(8 ,4) = -0.153102409
      A(9 ,4) = 0.144866574
      A(10,4) = -0.137844513
C**** THESE ARE THE BESSEL NUMBERS OF ORDER FOUR  FOR THE ZEROS OF THE
C**** FUNCTION
      A(1 ,5) = 0.3996514545
      A(2 ,5) = -0.2743809949
      A(3 ,5) = 0.229590468
      A(4 ,5) = -0.202763849
      A(5 ,5) = 0.184029896
      A(6 ,5) = -0.169878516
      A(7 ,5) = 0.158655372
      A(8 ,5) = -0.149451156
      A(9 ,5) = 0.141714307
      A(10,5) = -0.135086328
      GO TO 1
      END
```

```
      SUBROUTINE BETAIJ ( J )
      COMPLEX AI50,AI51,AI52,AI53,AI55,AI55J,DUM1,PSI,AMU,BETAIS,COMEGA,
     1DUMMY,I, C12,SUM1,SUM2,SUM3,SUM4 , AN
      COMPLEX BETA1, BETA2, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1AK2, PSI2, FNORM, EPSIRZN
      COMMON / BLKA / BETA1, BETA2, AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
     1, AK2, BETAN, RLMDA, BETAI, BETAC, BETAWI,    LHAT, MHAT, NHAT,
     1PSI2,X1,X2
      COMMON / BLKB / AMU( 3,30,2),BETAIS( 2),I,MATRIX
      COMMON / BLOCK / C12(30,30),ISET,LDEX,NDEX,ERROR1,ERROR2
      NHAI = NHAT + 1
      COMEGA=CMPLX(-AIMAG(OMEGA),REAL(OMEGA))
      SUM1=SUM2=SUM3=SUM4=CMPLX(0.0,0.0)
      DO 10 LP = 1 , LDEX
      DUM =BESSEL(MHAT+1,LP)
      DO 10 NP = 1 , NDEX
      SUM4=SUM4 + AMU(LP,NP,  1)*DUM *C12(NHAT+1,NP)
   10 CONTINUE
      DO 20 NP = 1 , NDEX
      DUM1=AMU(LHAT,NP,  1)
      SUM1=SUM1 + DUM1
      SUM2=SUM2 + DUM1*(-1.)**(NP +1)
      IF(NP.EQ.NHAT +1) GOTO 20
      SUM3=SUM3 + DUM1*2.*COMEGA*(NP-1)**2/((NP-1)**2 -NHAT**2)
     1*(1. -(-1.)**(NP +NHAT -1))
   20 CONTINUE
      AI50=BETAI*AK1/PSI(DUMMY)/EPSIZN(LHAT,MHAT,NHAT,ALENGTH,RLMDA)
      AI51=BETAN*GAMMA*COMEGA*((-1.)**NHAT)*SUM2*FNORM(LHAT,MHAT,NHAT,
     1RLMDA,ALENGTH)/EPSIZN(LHAT,MHAT,NHAT,ALENGTH,RLMDA)
      AI52=(-1.)*AMACH*SUM3*FNORM(LHAT,MHAT,NHAT,RLMDA,ALENGTH)/EPSIZN(
     1LHAT,MHAT,NHAT,ALENGTH,RLMDA)
      AI53=BETAC*FNORM(LHAT,MHAT,NHAT,RLMDA,ALENGTH)*BESSEL(MHAT+1,LHAT)
     1*SUM4/EPSIRZN(LHAT,MHAT,NHAT,ALENGTH,RLMDA)
      AI55=(AK1/PSI(DUMMY) + GAMMA*COMEGA*FNORM(LHAT,MHAT,NHAT,RLMDA,
     1ALENGTH)*SUM1)/EPSIZN(LHAT,MHAT,NHAT,ALENGTH,RLMDA)
      BETAIS(2)=(AI50 -AI51 - AI52 - AI53)/AI55
```

```
      WRITE(6,100)J, BETAIS(2)
      AN = 1. / GAMMA - BETAIS(2) / AMACH
      WRITE(6,101) AN
      AI50 = BETAIS(2) - BETAIS(1)
      IF( ABS(REAL(AI50)/REAL(BETAIS( 1 ))).LE.ERROR1 .AND.ABS(AIMAG(AI5
     10)/AIMAG(BETAIS( 1 ))).LE.ERROR2) GO TO 2
      BETAIS(1) = BETAIS(2)
      RETURN
    2 ISET = 1
      F = REAL ( OMEGA ) / 1.84118378
      RETURN
  100 FORMAT(*0 BETAI(*,I2,*) = *,2G21.14)
  101 FORMAT(*+*,70X,* N = *,2G21.14,*I*)
      END
```

```
      SUBROUTINE CALMU( JP )
      COMPLEX AMU,G1,G2,PSI,DUMMY,I,DUM3,C12
      COMPLEX DUM1,DUM2,BETAIS,SUMS,S,COMEGA,SUM1,SUM2,SUM3,SUM4
      COMPLEX BETA1, BETA2, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1AK2, PSI2, FNORM
      COMMON / BLKA / BETA1, BETA2, AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
     1, AK2, BETAN, RLMDA, BETAI, BETAC, BETAWI,    LHAT, MHAT, NHAT,
     1PSI2,X1,X2
      COMMON / BLKB / AMU( 3,30,2),BETAIS( 2),I,MATRIX
      COMMON / BLOCK / C12(30,30),ISET,LDEX,NDEX,ERROR1,ERROR2
      ISET = 0
      DO 50 L = 1 , LDEX
      RLMDA1= BESPRIM(MHAT+1,L)
      AC= BESSEL(MHAT+1,LHAT)/BESSEL(MHAT+1,L)
      DO 50 NX = 1 , NDEX
      N=NX-1
      IF(L.EQ.LHAT.AND.N.EQ.NHAT) GOTO 40
      DUM1= CMPLX(0.,0.)
      IF(L.EQ.LHAT) DUM1=(BETAI-BETAWI)*AK1/(EPSIZN(LHAT,MHAT,N,ALENGTH,
     1RLMDA))
      DUM1=   (BETAC*(G1(X2,N)-G1(X1,N))/EPSIKAT(LHAT,MHAT,N,ALENGTH,RLMD
     1A1)*AC+DUM1)
      DUM2=DUM1/((OMEGA**2-ETA(LHAT,MHAT,NX,ALENGTH,RLMDA1))*PSI(DUMMY))
      GOTO 50
   40 DUM2=  CMPLX ( 0.0 ,0.0 )
   50 AMU(L,NX,1)=DUM2/FNORM(LHAT,MHAT,NHAT,RLMDA,ALENGTH)
      COMEGA=CMPLX(-AIMAG(OMEGA),REAL(OMEGA))
      NX = 1
      DUM1=CMPLX(0.0,0.0)
      DUM2=CMPLX(0.0,0.0)
      N = NHAT
      NX = NHAT + 1
      DO 2 NY = 1 , NDEX
      NP=NY-1
      IF(N.EQ.NP) 8,7
    7 C12(NX,NY)=GAMMA/2./(N**2-NP**2)*(ALENGTH*COMEGA/3.1415926*((N+NP)
```

```
 1*(SIN((N-NP)*3.1415926*X2/ALENGTH)-SIN((N-NP)*3.1415926*X1/ALENGTH
 1))+(N-NP)*(SIN((N+NP)*3.1415926*X2/ALENGTH)-SIN((N+NP)*3.1415926*X
 11*ALENGTH)))-AMACH*NP*((NP+N)*(COS((NP-N)*3.1415926*X2/ALENGTH)-CO
 1S((NP-N)*3.1415926*X1/ALENGTH))+(NP-N)*(COS((NP+N)*3.1415926*X2/AL
 1ENGTH)-COS((NP+N)*3.1415926*X1/ALENGTH))))
  GO TO 2
8 IF(NP.EQ.0)9,10
9 C12(NX,NY)=(X2-X1)*COMEGA*GAMMA
  GO TO 2
10 C12(NX,NY)=GAMMA*(((SIN(2.*N*3.1415926*X2/ALENGTH)-SIN(2.*N*3.1415
 1926*X1/ALENGTH))/4./N/3.1415926*ALENGTH+(X2-X1)/2.)*COMEGA-AMACH/2
 1.*(SIN(N*3.1415926*X2/ALENGTH)**2-SIN(N*3.1415926*X1/ALENGTH)**2))
2 CONTINUE
  CALL BETAIJ( 2 )
  IF(JP.EQ.1) RETURN
  DO 1 NX = 1 , NDEX
  N = NX - 1
  DO 1 NY = 1 , NDEX
  NP=NY-1
  IF(N.EQ.NP) 3,4
4 C12(NX,NY)=GAMMA/2./(N**2-NP**2)*(ALENGTH*COMEGA/3.1415926*((N+NP)
 1*(SIN((N-NP)*3.1415926*X2/ALENGTH)-SIN((N-NP)*3.1415926*X1/ALENGTH
 1))+(N-NP)*(SIN((N+NP)*3.1415926*X2/ALENGTH)-SIN((N+NP)*3.1415926*X
 11*ALENGTH)))-AMACH*NP*((NP+N)*(COS((NP-N)*3.1415926*X2/ALENGTH)-CO
 1S((NP-N)*3.1415926*X1/ALENGTH))+(NP-N)*(COS((NP+N)*3.1415926*X2/AL
 1ENGTH)-COS((NP+N)*3.1415926*X1/ALENGTH))))
  GO TO 1
3 IF(NP.EQ.0)5,6
5 C12(NX,NY)=(X2-X1)*COMEGA*GAMMA
  GO TO 1
6 C12(NX,NY)=GAMMA*(((SIN(2.*N*3.1415926*X2/ALENGTH)-SIN(2.*N*3.1415
 1926*X1/ALENGTH))/4./N/3.1415926*ALENGTH+(X2-X1)/2.)*COMEGA-AMACH/2
 1.*(SIN(N*3.1415926*X2/ALENGTH)**2-SIN(N*3.1415926*X1/ALENGTH)**2))
1 CONTINUE
  DO 60 J=2,JP
  DO 80 LS = 1 , LDEX
```

```
      DO 80 NX = 1 , NDEX
      NS=NX-1
      RLMDA2=BESPRIM( MHAT + 1, LS)
      DUM3 = BETAC*EPSIZN(LHAT,MHAT, NS ,ALENGTH,RLMDA)*BESSEL(MHAT+1,
     1LHAT)/EPSIKAT(LS,MHAT,NS,ALENGTH,RLMDA2)/BESSEL(MHAT+1,LS)
      DUM1 = CMPLX( 0.0, 0.0 )
      SUMS=CMPLX(0.,0.)
      IF( LS .EQ. LHAT .AND. NS .EQ. NHAT ) 15, 16
   15 AMU(LHAT, NX,2) = CMPLX( 0.0 , 0.0 )
      GO TO 80
   16 SUM1=SUM2=SUM3=SUM4=CMPLX(0.0,0.0)
      DO 90 LP = 1 , LDEX
      RLMDA3=BESSEL(MHAT +1,LP)
      RLMDA4=BESSEL(MHAT +1,LHAT)
      DO 90 NY = 1 , NDEX
      SUM4=SUM4 + AMU(LP,NY,  1)*C12(NX,NY)
   90 CONTINUE
      SUM4=SUM4*DUM3*RLMDA3/RLMDA4
      DO 21 NP = 1 , NDEX
      DUM1=AMU(LS,NP,  1)
      SUM1=SUM1 +DUM1
      SUM2=SUM2 + DUM1*(-1.)**(NP +1)
      IF(NP.EQ.NX) GOTO 21
      SUM3=SUM3 + DUM1*2.*COMEGA*(NP-1)**2 *(1.-(-1.)**(NP+NX))/(
     1(NP-1)**2 -NS**2 )
   21 CONTINUE
      SUM1=BETAIS(2)*GAMMA*COMEGA*SUM1
      SUM2=BETAN*GAMMA*COMEGA*(-1.)**NS*SUM2
      SUM3=SUM3 + AMACH*(3.1415926*NS)**2./2./ALENGTH*AMU(LS,NX,  1)
      SUM3=(-1.)*AMACH*SUM3
      DUM1=CMPLX(0.0,0.0)
      IF(LS.EQ.LHAT) DUM1=(BETAIS(2) - BETAWI)*AK1
      DUM1=DUM1 +DUM3*(G1(X2,NS) -G1(X1,NS))
      DUM1=DUM1/PSI(DUMMY)/FNORM(LHAT,MHAT,NHAT,RLMDA,ALENGTH)
      AMU(LS,NX,2)=(DUM1 +SUM1 +SUM2 +SUM3 + SUM4)/(OMEGA**2  - ETA(LHAT
     1,MHAT,NX,ALENGTH,BESPRIM(MHAT+1,LS)))/EPSIZN(LHAT,MHAT,NS,ALENGTH,
```

```
  2RLMDA)
80 CONTINUE
   IF ( J .EQ. 2 ) GO TO 14
18 DO 11 L = 1, LDEX
   DO 11 N = 1 , NDEX
11 AMU(L,N,1) = AMU(L,N,2)
14 IF ( MATRIX .NE. 3HYES ) GO TO 13
   KS = 1
   KF = LDEX
   WRITE(6,100) J, KS, KF
   DO 12 N = 1 , NDEX
12 WRITE(6,101) N, ( AMU(L,N,1), L = KS, KF )
13 CONTINUE
   IF ( AMU(LDEX,NDEX,1) .NE. AMU(LDEX,NDEX,2) ) GO TO 18
   CALL BETAIJ ( J+1 )
   IF ( ISET .NE. 0 ) GO TO 17
60 CONTINUE
17 CONTINUE
   JP = J
100 FORMAT(*0 N            J EQUAL *,I2,*      L  EQUAL *,I1,* TO *,I2)
101 FORMAT(* *,I3,*  *,10G13.6)
   RETURN
   END
```

```
      FUNCTION EPSIKAT(L,M,N,ALENGTH,RLMDA)
      DUM1 = ALENGTH
      IF( N     .NE. 0) DUM1 = DUM1/2.
      IF( M     .EQ. 0) GOTO 1
      EPSIKAT = (0.5-M    *M    /(2.*RLMDA*RLMDA))*DUM1
      RETURN
    1 EPSIKAT = DUM1/2.
      RETURN
      ENTRY EPSIZN
      IF ( N .EQ. 0 ) GO TO 2
      EPSIKAT = ALENGTH / 2.
      RETURN
    2 EPSIKAT= ALENGTH
      RETURN
      ENTRY ETA
      EPSIKAT = (N-1)*(N-1) * 3.1415926 *3.1415926 / (ALENGTH * ALENGTH)
     1+RLMDA * RLMDA
      RETURN
      END
```

```
      COMPLEX FUNCTION G1 (X,N)
      REAL LENGTH
      COMPLEX BETA1, BETA2, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1 AK2, PSI2
      COMPLEX DUM1, GDUM, BETA
      COMPLEX MU , BETAIS , I
      COMPLEX D1 , D2 , D3 , D4 , D5 , D6
      COMMON / BLKA / BETA1, BETA2, AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
     1, AK2, BETAN, BLMDA, BETAI, BETAC, BETAWI,    LHAT, MHAT, NHAT,
     1 PSI2,X1, X2
      COMMON / BLKB / MU(3,30,2) , BETAIS( 2) , I , MATRIX
      COMMON / BLKC / LENGTH , D1(30) , D2(30) , D3(30) , D4(30) , D5(30
     $) , D6(30) , A1(2)
      GDUM(A,I,BETA,X)=(CEXP(I*BETA*CMPLX(X,0.0))*(I*BETA*CMPLX(COS(A*X)
     $,0.0)+CMPLX(A*SIN(A*X),0.0)))/(CMPLX(A*A,0.0)-BETA*BETA)
      A = N * 3.14159265358979 / ALENGTH
      G1 = (OMEGA+AMACH*BETA1)*GDUM(A,I,BETA1,X)
      IF( N .EQ. 0 .AND. CABS( BETA2) .LE. 1.E-100 ) GO TO 1
      G1 = I*GAMMA*(G1+(OMEGA+AMACH*BETA2)*C*GDUM(A,I,BETA2,X))
      RETURN
    1 BETA = C * OMEGA * X
      G1 = GAMMA * ( G1 + BETA ) * I
      RETURN
      ENTRY G1FRST
      A = N * 3.14159265358979 / LENGTH
      IF( N .EQ. 0 .AND. CABS( BETA2) .LE. 1.E-100 ) GO TO 5
      G1 = ((AMACH*AMACH-1.)*BETA1*BETA1+2.*OMEGA*AMACH*BETA1+OMEGA*OMEG
     $A   -MHAT*MHAT)*GDUM(A,I,BETA1+BETA1,X)      +((AMACH*AMACH   -1.)*
     $2.*C*BETA1*BETA2+2.*OMEGA*AMACH*C*(BETA1+BETA2)+(OMEGA*OMEGA-MHAT*
     $MHAT)*2.*C)*GDUM(A,I,BETA1+BETA2,X)     +((AMACH*AMACH   -1.
     $)*C*C*BETA2*BETA2+2.*OMEGA*AMACH*BETA2*C*C+(OMEGA*OMEGA-MHAT*MHAT)
     $*C*C)*GDUM(A,I,BETA2+BETA2,X)
      RETURN
    5 G1 = ((AMACH*AMACH-1.)*BETA1*BETA1+2.*OMEGA*AMACH*BETA1+OMEGA*OMEG
     $A   -MHAT*MHAT)*GDUM(A,I,BETA1+BETA1,X)      +((AMACH*AMACH   -1.)*
     $2.*C*BETA1*BETA2+2.*OMEGA*AMACH*C*(BETA1+BETA2)+(OMEGA*OMEGA   -MH
     $MHAT)*2.*C)*GDUM(A,I,BETA1+BETA2,X)        +(OMEGA*OMEGA-MHAT*MHA
     $T)*C*C*X
      RETURN
      END
```

```
      COMPLEX FUNCTION EPSIRZN(L,M,N,ALENGTH,RLMDA)
      COMPLEX DUM2
      DUM1 = 1.
      IF(M.EQ.0) DUM1 = .5
      DUM2=FNORM(L,M,N,RLMDA,ALENGTH)**2
      EPSIRZN=DUM1 * DUM2 / 3.14159265358979
      RETURN
      END
```

```
      COMPLEX FUNCTION FNORM(L.M,N,RLMDA,ALENGTH)
      DUM1 = 3.1415926
      IF( M .EQ. 0 ) DUM1 = 6.2831852
      DUM1=DUM1*EPSIKAT(L,M,N,ALENGTH,RLMDA)*BESSEL(M+1,L)**2
      FNORM = CSQRT ( CMPLX( DUM1 , 0.0 ) )
      RETURN
      END
```

```
      FUNCTION F0(X)
      F0=.79788456-.00000077*(3./X)-.00552740*(3./X)**2
     1-.00009512*(3./X)**3+.00137237*(3./X)**4
     2-.00072805*(3./X)**5+.00014476*(3./X)**6
      RETURN
      ENTRY F1
      F0=.79788456+.00000156*(3./X)+.01659667*(3./X)**2
     1+.00017105*(3./X)**3-.00249511*(3./X)**4+.00113653*(3./X)**5
     2-.00020033*(3./X)**6
      RETURN
      END
```

```
      COMPLEX FUNCTION G2(N)
      REAL LENGTH
      COMPLEX BETA1, BETA2, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1 AK2, PSI2
      COMPLEX MU , AMP , I
      COMPLEX D1 , D2 , D3 , D4 , D5 , D6
      COMPLEX GBUM , BETA
      COMMON / BLKA / BETA1, BETA2, AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
     1, AK2, BETAN, RLMDA, BETAI, BETAC, BETAWI,    LHAT, MHAT, NHAT,
     1 PSI2,X1, X2
      COMMON / BLKB / MU(3,30,2) , AMP( 2) ,I , MATRIX
      COMMON / BLKC / LENGTH , D1(30) , D2(30) , D3(30) , D4(30) , D5(30
     $) , D6(30) , A1(2)
      GBUM(A,BETA,ALENGTH,N)=(BETA*(CEXP(BETA*CMPLX(ALENGTH,0.0))*CMPLX(
     $-1.0,0.0)**(N+2)+CMPLX(-1.0,0.0)))/(CMPLX(A,0.0)+BETA*BETA)
      A = N*N*3.14159265358979*3.14159265358979/ALENGTH/ALENGTH
      G2 = -BETA1*(AMACH*BETA1+2.*OMEGA)*GBUM(A,BETA1*I,ALENGTH,N)
      IF( N .EQ. 0 .AND. CABS( BETA2) .LE. 1.E-10 ) RETURN
      G2 = G2-C*BETA2*(AMACH*BETA2+2.*OMEGA)*GBUM(A,BETA2*I,ALENGTH,N)
      RETURN
      ENTRY PSI
      A = NHAT*NHAT*3.14159265358979*3.14159265358979/ALENGTH/ALENGTH
      G2 = GBUM ( A , I * BETA1 , ALENGTH , NHAT )
      IF( NHAT .EQ. 0 .AND. CABS( BETA2) .LE. 1.E-100) GO TO 3
      G2 = (G2+C*GBUM(A,I*BETA2,ALENGTH,NHAT))/EPSIZN(LHAT,MHAT,NHAT,ALE
     $NGTH,RLMDA)
      RETURN
    3 G2  = ( G2  + C * ALENGTH ) / EPSIZN(LHAT,MHAT,NHAT,ALENGTH,RLMDA)
      RETURN
      END
```

```
      REAL FUNCTION INTEGRL (F,A,B,LHAT,L,LBAR,MHAT,MBAR)
      DIMENSION W(11) , X(11)
      DATA(W(   I),I=1,11)/.02129101837554,.05761665831124,.0934003982782
     15,.12052016961432,.13642490095628,.14149370892875,.13642490095628,
     2,.12052016961432,.09340039827825,.05761665831124,.02129101837554/
      DATA(X(   I),I=1,11)/.00795731995258,.04691007703067,.1229166367145
     18,.23076534494716,.36018479341911,.50000000000000,.63981520658089,
     2,.76923465505284,.87708336328542,.95308992296933,.99204268004742/
      CONST = B - A
      ARG1 = BESPRIM(MHAT,LHAT)
      ARG2 = BESPRIM(MHAT,L)
      ARG3 = BESPRIM(MBAR,LBAR)
      INTEGRL = W(   1)*F(A+CONST*X(   1),ARG1,ARG2,ARG3)
      DO 1 I = 2,11
    1 INTEGRL = INTEGRL + W(   I)*F(A+CONST*X(   I),ARG1,ARG2,ARG3)
      RETURN
      END
```

```
      REAL FUNCTION I1 ( X , RLMDA , RLMDAL , RLMBAR )
C ***
C *** THIS FUNCTION SUBPROGRAM ASSUMES THAT MBAR EQUALS TWO MHAT
C *** AND THAT MHAT EQUALS ONE
C ***
      REAL J0 , J1 , J2
      J2( ARG ) = 2.*J1(ARG)/ARG-J0(ARG)
C ***
C *** THESE ARE THE INTEGRALS FOR M EQUAL ZERO
C ***
                  I1=J1(RLMDA*X)*J1(RLMDAL*X)*J0(RLMBAR*X)/X $ RETURN
      ENTRY I2 $ I1=J1(RLMDA*X)*J1(RLMDAL*X)*J0(RLMBAR*X)*X $ RETURN
      ENTRY I3 $ I1=J0(RLMDAL*X)*J1(RLMDA*X)*J0(RLMBAR*X) $ RETURN
      ENTRY I4 $ I1=J0(RLMDA*X)*J1(RLMDAL*X)*J0(RLMBAR*X) $ RETURN
      ENTRY I5 $ I1=X*J0(RLMDA*X)*J0(RLMDAL*X)*J0(RLMBAR*X) $ RETURN
C ***
C *** THESE ARE THE INTEGRALS FOR M EQUAL TWO
C ***
      ENTRY I6 $ I1=J1(RLMDA*X)*J1(RLMDAL*X)*J2(RLMBAR*X)/X $ RETURN
      ENTRY I7 $ I1=J1(RLMDA*X)*J1(RLMDAL*X)*J2(RLMBAR*X)*X $ RETURN
      ENTRY I8 $ I1=J0(RLMDAL*X)*J1(RLMDA*X)*J2(RLMBAR*X) $ RETURN
      ENTRY I9 $ I1=J0(RLMDA*X)*J1(RLMDAL*X)*J2(RLMBAR*X) $ RETURN
      ENTRY I10 $ I1=X*J0(RLMDA*X)*J0(RLMDAL*X)*J2(RLMBAR*X) $ RETURN
      ENTRY I11 $ I1 = X*J0(RLMDA*X)*J0(RLMBAR*X)*J1(RLMDA*X)     $ RETURN
      ENTRY I12 $ I1 = J0(RLMBAR*X)*J1(RLMDA*X)**2                $ RETURN
      ENTRY I13 $ I1 = X*J0(RLMDA*X)*J1(RLMDA*X)*J2(RLMBAR*X)     $ RETURN
      ENTRY I14 $ I1 = J1(RLMDA*X)**2*J2(RLMBAR*X)                $ RETURN
      ENTRY I15 $ I1 = J1(RLMDA*X)**2*J1(RLMBAR*X)                $ RETURN
      ENTRY R1 $ I1 = X*(J0(RLMDA*X)*J1(RLMDA*X))**2        $ RETURN
      ENTRY R2 $ I1 = J0(RLMDA*X)*J1(RLMDA*X)**3            $ RETURN
      ENTRY R3 $ I1 = J1(RLMDA*X)**4/X                      $ RETURN
      ENTRY R4 $ I1 = J1(RLMDA*X)**4*X                      $ RETURN
      ENTRY R5 $ I1 = J0(RLMDA*X)**3*J1(RLMDA*X)            $ RETURN
      ENTRY R6 $ I1 = (J0(RLMDA*X)*J1(RLMDA*X))**2/X        $ RETURN
      ENTRY R7 $ I1 = J0(RLMDA*X)*J1(RLMDA*X)**3/X/X        $ RETURN
      ENTRY R8 $ I1 = J1(RLMDA*X)**4/X/X/X                  $ RETURN
      ENTRY R9 $ I1 = (1.-RLMBAR*RLMBAR/RLMDA/RLMDA)*BESSEL(IFIX(RLMBAR+
     $                1.),IFIX(RLMDAL))**2/2.               $ RETURN
      END
```

```
      REAL FUNCTION J0(X)
      IF(X-3.)50,50,51
   50 J0=1.-2.2499997*(X/3.)**2+1.2656208*(X/3.)**4-.3163866*(X/3.)**6
     1+.0444479*(X/3.)**8-.0039444*(X/3.)**10+.0002100*(X/3.)**12
      RETURN
   51 J0=(F0(X)*COS(TH0(X)))/SQRT(X)
      RETURN
      ENTRY J1
      IF(X-3.)52,52,53
   52 J0=(.5-.56249985*(X/3.)**2+.21093573*(X/3.)**4-.03954289*(X/3.)
     1**6+.00443319*(X/3.)**8-.00031761*(X/3.)**10+.00001109*(X/3.)**12
     2)*X
      RETURN
   53 J0=(F1(X)*COS(TH1(X)))/SQRT(X)
      RETURN
      END
```

```
 SUBROUTINE LINEAR ( LENGNO , F , G , AK , FINC , FMAX , JX )
 COMPLEX BETA1, BETA2, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
1 AK2, PSI2, AMU, BETAIS, C12
 COMPLEX G, AN, AK, I, A4, A5, BETA, DUM1, DUM2, PSI, DUMMY, G1,
1 G2,F,AKA,FINC,FMAX
 COMMON / BLKA / BETA1, BETA2, AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
1, AK2, BETAN, RLMDA, BETAI, BETAC, BETAWI,    LHAT, MHAT, NHAT,
1PSI2,X1,X2
 COMMON / BLKB / AMU( 3,30,2),BETAIS( 2),I,MATRIX
 COMMON / BLOCK / C12(30,30),ISET,LDEX,NDEX,ERROR1,ERROR2
 I = CMPLX ( 0.0 , 1.0 )
 RLMDA = BESPRIM ( MHAT + 1 , LHAT )
 NHAI = NHAT + 1
 OMEGA=F*1.84118378
 BETA=CSQRT(OMEGA*AMACH*OMEGA*AMACH+(AMACH*AMACH-1.)*(RLMDA**2 -
1OMEGA*OMEGA))/(1.-AMACH*AMACH)
 BETA1=(OMEGA*AMACH)/(1.-AMACH*AMACH)+BETA
 BETA2=(BETA1)-2.*BETA
 C = - CEXP ( I * ALENGTH * ( BETA1 - BETA2 ) )
 C = C * (BETA1 + BETAN * GAMMA * (OMEGA + AMACH * BETA1 )) / (
1 BETA2 + BETAN * GAMMA * (OMEGA + AMACH * BETA2 ))
 PSI2 = PSI( DUMMY )**2
 AK1=(OMEGA+       AMACH *BETA1+C*(OMEGA+       AMACH*BETA2))*GAMMA
 AK1= CMPLX(-AIMAG(AK1),REAL(AK1))
 DUM1 = CMPLX(-AIMAG(BETA1),REAL(BETA1))
 DUM1 = DUM1*ALENGTH
 DUM2 = CMPLX(-AIMAG(BETA2),REAL(BETA2))
 DUM2 = DUM2*ALENGTH
 AK2 = ((OMEGA+AMACH*BETA1)*CEXP(DUM1)+(OMEGA+AMACH*BETA2)*C*CEXP(
1DUM2))*GAMMA
 AK2 = CMPLX(-AIMAG(AK2),REAL(AK2))
 A4 = AK1/EPSIZN(LHAT,MHAT,NHAT,ALENGTH,RLMDA)/PSI(DUMMY)
 A5=BETAC*(G1(X2,NHAT)-G1(X1,NHAT))/EPSIKAT(LHAT,MHAT,NHAT,ALENGTH,
1RLMDA)/PSI(DUMMY)
 BETAWI = (OMEGA**2 - ETA(LHAT,MHAT,NHAI,ALENGTH,RLMDA)-(AMACH*G2(
1NHAT)/EPSIZN(LHAT,MHAT,NHAT,ALENGTH,RLMDA)+BETAN*AK2/EPSIZN(LHAT,
```

```
     1MHAT,NHAT,ALENGTH,RLMDA)*(-1.)**(NHAT+2))/PSI(DUMMY))/A4
      BETAI = BETAWI - A5 / A4
      BETAIS( 2 ) = BETAIS ( 1 ) = BETAI
      WRITE(6,100) LENGNO
      WRITE(6,110) LDEX , NDEX
      WRITE(6,116) AMACH
      WRITE(6,117) GAMMA
      WRITE(6,118) OMEGA
      WRITE(6,119) ALENGTH
      WRITE(6,101) LHAT, MHAT, NHAT
      WRITE(6,115) X1,X2
      WRITE(6,113) F
      WRITE(6,102) BETAN
      WRITE(6,120) G
      WRITE(6,121) BETAC
      WRITE(6,108) AK
      WRITE(6,105) BETAWI
      AN = 1. / GAMMA - BETAWI / AMACH
      WRITE(6,103) AN
      BETAWI = ( BETA1 + C * BETA2 ) / ( GAMMA * (OMEGA * ( 1. + C ) +
     1 AMACH * ( BETA1 + C * BETA2 ) ) )
      WRITE(6,112) BETAWI
      AN = 1. / GAMMA - BETAWI / AMACH
      WRITE(6,103) AN
      WRITE(6,109)    BETAIS(1)
      AN = 1. / GAMMA - BETAIS(1) / AMACH
      WRITE(6,103) AN
      IF ( X2.LE.1.E-10.OR.CABS(BETAC).LE.1.E-10) GO TO 1
      JY = JX - 1
      CALL CALMU( JY )
    1 JX = JY + 1
      IF( JX .EQ. 30 ) 2 , 3
    2 WRITE(6,122)
    3 RETURN
  100 FORMAT(*1 THIS IS ENGINE NUMBER *A10)
  101 FORMAT(*0 THE PRIMARY MODE ASSUMED IS LHAT = *,I2,* MHAT = *,I2,*
```

C-2

```
   INHAT = *,I2)
102 FORMAT(*0 BETAN = *,2G21.14)
103 FORMAT(*+*,70X,* N = *,2G21.14,*I*)
105 FORMAT(*0 BETAWI = *,2G21.14)
107 FORMAT(*0*,G21.14,* WAS THE TIME FOR THE ITERATION *,G21.14)
108 FORMAT(*+*,70X,* K = *,2G21.14)
109 FORMAT(*0 BETAI( 1) = *,2G21.14)
110 FORMAT(*+*,70X,* THE COEFFICIENT MATRIX IS *,I2,* X*,I3)
112 FORMAT(*0 BETAWID = *,2G21.14)
113 FORMAT(*0 W/W0 = *,2G21.14)
115 FORMAT(*+*,70X,* THE LINER BEGINS AT *,F6.4,* AND ENDS AT *,F6.4)
116 FORMAT(*0 THE MACH NUMBER IS *,G21.14)
117 FORMAT(*+*,70X,* THE RATIO OF SPECIFIC HEATS IS *,G21.14)
118 FORMAT(*0 THE FREQUENCY IS *, 2G21.14)
119 FORMAT(*+*,70X,* THE LENGTH OF THE COMBUSTOR IS *,G21.14)
120 FORMAT(*+*,70X,* G = *, 2G21.14)
121 FORMAT(*0 BETAC = *, 2G21.14)
122 FORMAT(*1*135(1H*)/* *47(1H*)* THE COMBUSTION RESPONSE DID NOT CON
   $VERGE *46(1H*)/* *135(1H*))
    END


   FUNCTION THO(X)
   THO=X-.78539816-.04166397*(3./X)-.00003954*(3./X)**2+.00262573*
  1(3./X)**3-.00054125*(3./X)**4-.00029333*(3./X)**5+.00013558*
  2(3./X)**6
   RETURN
   ENTRY TH1
   THO=X-2.35619449+.12499612*(3./X)+.00005650*(3./X)**2
  1-.00637879*(3./X)**3+.00074348*(3./X)**4+.00079824*(3./X)**5
  2-.00029166*(3./X)**6
   RETURN
   END
```

```
      SUBROUTINE CONST
      REAL I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,INTEGRL,LENGTH
      COMPLEX D1 , D2 , D3 , D4 , D5 , D6
      COMPLEX A , B , D ,I5I6(30,3,2) , G1FRST
      COMPLEX B1,    B2, I, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1 AK2, PSI2, MU , AMP , C12,FNORM
      COMMON / BLKA / B1    , B2    , AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
     1, AK2, BETAN, RLMDA, BETAI, BETAC, BETAWI,    LHAT, MHAT, NHAT,
     1PSI2,X1,X2
      COMMON / BLKB / MU(3,30,2) , AMP( 2) ,I , MATRIX
      COMMON / BLKC / LENGTH , D1(30) , D2(30) , D3(30) , D4(30) , D5(30
     $) , D6(30) , A1(2)
      COMMON / BLOCK / C12(30,30),ISET,LDEX,NDEX,ERROR1,ERROR2
      EQUIVALENCE ( D1 ( 1 ) , I5I6 ( 1 ,1 , 1 ) )
      EXTERNAL I1,I2,I3,I4,I5,I6,I7,I8,I9,I10
      D(A,B,X,N) = (A*(CEXP(A*CMPLX(X,0.0))*CMPLX(-1.0,0.0)**(N+2)+CMPLX
     $(-1.0,0.0)))/(A*A+B*B)
C ***
C ***    NOTE THAT IN THIS SUBROUTINE BETA1 , BETA2  BECOME B1 , B2
C ***    RESPECTIVELY.
C ***
      B3 = B1 + B2
      DO 1 NX = 1 , NDEX
      N = NX - 1
      B = CMPLX ( N * 3.14159265358979 / LENGTH , 0.0 )
      D1(NX)=B1*D(2.*I*B1,B,LENGTH,N)+B3*C    *D(I*B3,B,LENGTH,N)+C    *C
     $  *B2*D(I*2.*B2,B,LENGTH,N)
      D2(NX)=B1*B1*B1*D(2.*I*B1,B,LENGTH,N)+B1*B2*B3*C    *D(I*B3,B,LENGT
     $H,N)+B2*B2*B2*C    *C    *D(2.*I*B2,B,LENGTH,N)
      D3(NX)=D(2.*I*B1,B,LENGTH,N)+C*2.*D(I*B3,B,LENGTH,N)+C    *C    *D(I
     $*2.*B2,B,LENGTH,N)
      D4(NX)=B1*B1*D(2.*I*B1,B,LENGTH,N)+C*(B1*B1+B2*B2)*D(I*B3,B,LENGTH
     $,N)+C*C *B2*B2*D(2.*I*B2,B,LENGTH,N)
    1 D5(NX)=B1*B1*D(2.*I*B1,B,LENGTH,N)+2.*C    *B1*B2*D(I*B3,B,LENGTH,N
     $)+C    *C    *B2*B2*D(2.*I*B2,B,LENGTH,N)
      IF ( CABS ( B2 ) .LE. 1.E-100 ) 4 , 5
```

```
    4 D1( I)=B1*D(2.*I*B1,B,LENGTH,0)+B3*C    *D(I*B3,B,LENGTH,0)
      D2( I)=B1*B1*B1*D(2.*I*B1,B,LENGTH,0)
      D3( I)=D(2.*I*B1,B,LENGTH,0)+C*2.*D(I*B3,B,LENGTH,0)+C*C*LENGTH
      D4( I)=B1*B1*D(2.*I*B1,B,LENGTH,0)+C*(B1*B1+B2*B2)*D(I*B3,B,LENGTH
     $,0)
      D5( I)=B1*B1*D(2.*I*B1,B,LENGTH,0)
    5 DO 2 L = 1 , LDEX
      MU(L,11,2) = CMPLX(INTEGPL(I1,0.0,1.0,LHAT,LHAT,L,2,1) , 0.0)
      MU(L,12,2) = CMPLX(INTEGPL(I2,0.0,1.0,LHAT,LHAT,L,2,1) , 0.0)
      MU(L,13,2) = CMPLX(INTEGPL(I3,0.0,1.0,LHAT,LHAT,L,2,1) , 0.0)
      MU(L,15,2) = CMPLX(INTEGPL(I5,0.0,1.0,LHAT,LHAT,L,2,1) , 0.0)
      MU(L,16,2) = CMPLX(INTEGPL(I6,0.0,1.0,LHAT,LHAT,L,2,3) , 0.0)
      MU(L,17,2) = CMPLX(INTEGPL(I7,0.0,1.0,LHAT,LHAT,L,2,3) , 0.0)
      MU(L,18,2) = CMPLX(INTEGPL(I8,0.0,1.0,LHAT,LHAT,L,2,3) , 0.0)
    2 MU(L,20,2) = CMPLX(INTEGPL(I10,0.0,1.0,LHAT,LHAT,L,2,3) , 0.0)
C
C *** CALCULATION OF INTEGRALS I5 AND I6 , AND THEN SUM
C
      NBAI = NBAR + 1
      MU(2,1,2) = CEXP(B1*I*LENGTH)
      MU(3,3,2) = CEXP(B2*I*LENGTH)
      MU(1,1,2)=CMPLX(1.0,0.)/PSI2/FNORM(LHAT,MHAT,NHAT,RLMDA,LENGTH)**2
      MU(2,2,2) = BETAI*(1.+C)**2
      MU(1,4,2) = BETAN*(MU(2,1,2)+C*MU(3,3,2))**2
      MU(2,3,2) = BETAI*((B1+C*B2)**2*(AMACH*AMACH-1.)+2.*OMEGA*AMACH*(1
     $.+C)*(B1+C*B2)+OMEGA*OMEGA*(1.+C)*(1.+C))
      MU(2,4,2)= BETAN*((B1*MU(2,1,2)+B2*C*MU(3,3,2))**2*(AMACH*AMACH-1.
     $)+2.*OMEGA*AMACH*(MU(2,1,2)+C*MU(3,3,2))*(B1*MU(2,1,2)+B2*C*MU(3,3
     $,2))+OMEGA*OMEGA*(MU(2,1,2)+C*MU(3,3,2))**2)
      MU(3,1,2) = BESSEL(MHAT+1,LHAT)**2 * BETAC
      DO 3 NX = 1 , NDEX
      N = NX - 1
      MU(1,2,2) = (1.-GAMMA)/4. * (AMACH*AMACH*AMACH*D2(NX)+2.*OMEGA*AMA
     $CH*AMACH*(D5(NX)+D4(NX)/2.)+3.*OMEGA*OMEGA*AMACH*D1(NX)+OMEGA**3
     $ *D3(NX)) - (AMACH*D2(NX)+OMEGA*D5(NX))/2.
      MU(1,3,2) =  (AMACH*D1(NX) + OMEGA * D3(NX))/2.
```

```
      MU(3,2,2) = G1FRST( X2 , N ) - G1FRST( X1 , N )
      MU(3,4,2) = CMPLX ( ( -1. ) ** N , 0.0 )
      MU(1,5,2) = MU(2,2,2)+MU(1,4,2)*MU(3,4,2)
      MU(2,5,2) = MU(2,3,2)+MU(2,4,2)*MU(3,4,2)
      DO 3 L = 1 , LDEX
      I5I6(NX,L,1)=A1(1)*MU(1,1,2)/FNORM(L,0,N,BESPRIM(1,L),LENGTH)*(
     $MU(L,12,2)*I*MU(1,2,2)         +I*MU(1,3,2)*(RLMDA*(RLMDA*MU(L,15,2)
     $-2.*MHAT*MU(L,13,2))+CMPLX(MHAT*MHAT*2.,0.0)*MU(L,11,2))
     $+GAMMA/8.*(MU(1,5,2)*(RLMDA*(RLMDA*MU(L,15,2)-2.*MHAT*MU(L,13,2))
     $+CMPLX(MHAT*MHAT*2.,0.0)*MU(L,11,2))+MU(L,12,2)*MU(2,5,2)
     $+MU(3,1,2)*BESSEL(1,L)*MU(3,2,2)))
    3 I5I6(NX,L,2)=A1(2) * MU(1,1,2) / FNORM(L,2,N,BESPRIM(3,L),LENGTH)
     $*(MU(L,17,2)*I*MU(1,2,2)     +I*MU(1,3,2)*RLMDA*(RLMDA*MU(L,20,2)
     $-2.*MHAT*MU(L,18,2))           +GAMMA/8.*(MU(1,5,2)*RLMDA*(RLMDA*
     $MU(L,20,2)-2.*MHAT*MU(L,18,2))             +MU(L,17,2)*MU(2,5,2)
     $+MU(3,1,2)*BESSEL(3,L)*MU(3,2,2)))
      RETURN
      END
```

```
      COMPLEX FUNCTION  GAMM ( L , M , N )
      REAL LENGTH
      COMPLEX F OF X , I
      COMPLEX BETA1, BETA2, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1 AK2, PSI2, MU , AMP , C12
      COMMON / BLKA / BETA1, BETA2, AMACH, GAMMA, OMEGA,  LENGTH, C, AK1
     1, AK2, BETAN, RLMDA, BETAI, BETAC, BETAWI,    LHAT, MHAT, NHAT,
     1PSI2,X1,X2
      COMMON / BLKB / MU ( 3 , 30 , 2 ) , AMP ( 2 ) , I , MATRIX
      COMMON / BLOCK / C12(30,30),ISET,LDEX,NDEX,ERROR1,ERROR2
      GAMM = AMACH*AMACH*N*N*3.14159265358979*3.14159265358979*3.1415926
     $5358979/LENGTH/FNORM(L,M,N,BESPRIM(M+1,L),LENGTH)**2*BESSEL(M+1,L)
     $**2
      IF ( M ) 2 , 1 , 2
    1 RETURN
    2 GAMM = GAMM*(1.-M*M/BESPRIM(M+1,L)**2)/2. $ RETURN
      ENTRY DELTA
      GAMM = -GAMMA*I*2.*OMEGA*3.14159265358979*BESSEL(M+1,L)**2/FNORM(L
     $,M,N,BESPRIM(M+1,L),LENGTH)**2
      IF ( N ) 4 , 3 , 4
    3 F OF X = CMPLX ( X2 - X1 , 0.0 ) $ GO TO 5
    4 F OF X = CMPLX(0.5,0.0)*(X2-X1+LENGTH/2./N/3.14159265358979*(SIN(N
     $*2.*3.14159265358979*X2/LENGTH)-SIN(2.*N*3.14159265358979*X1/LENGT
     $H))-AMACH/I/2./OMEGA*(SIN(N*3.14159265358979*X2/LENGTH)**2-SIN(N*3
     $.14159265358979*X1/LENGTH)**2))
    5 IF ( M ) 7 , 6 , 7
    6 GAMM = GAMM*(BETAI+BETAN*(-1.)**N+BETAC*2.*F OF X) $ RETURN
    7 GAMM = GAMM*((BETAI+BETAN*(-1.)**N)*(1.-M*M/BESPRIM(M+1,L)**2)/2.+
     $BETAC*F OF X) $ RETURN
      END
```

```
      SUBROUTINE INIT ( A , B , C , L , N )
      DIMENSION A(1) , B(1) , C(1)
      INDEX = L*N*2
      DO 1 I = 1 , INDEX
    1 B(I) = 1777 0000 0000 0000 0000B
      INDEX = INDEX * 2
      DO 2 I = 1 , INDEX
    2 C(I) = 1777 0000 0000 0000 0000B
      INDEX = N*N*2
      DO 3 I = 1 , INDEX
    3 A(I) = 0.0
      RETURN
      END
```

```
      SUBROUTINE NONLINE ( LOCATE )
      REAL LENGTH
      COMPLEX GAMM , DELTA, MUSND, MUCROSS(3,30,2), FNORM, COR
      COMPLEX BETA1, BETA2, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1 AK2, PSI2, AMU, BETAIS, C12, I, ETA
      COMMON / BLKA / BETA1, BETA2, AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
     1, AK2, BETAN, PLMDA, BETAI, BETAC, BETAWI,    LHAT, MHAT, NHAT,
     1PSI2,X1,X2
      COMMON / BLKB / AMU( 3,30,2),BETAIS( 2),I,MATRIX
      COMMON / BLKC / LENGTH , MUSND(30,3,2) , A1(2)
      COMMON / BLOCK / C12(30,30),ISET,LDEX,NDEX,ERROR1,ERROR2
      EQUIVALENCE (C12(1,1),MUCROSS(1,1,1))
C ***
C *** CALCULATE NECESSARY CONSTANTS
C ***
      IF ( LOCATE .EQ. 10HCROSS PROD ) CALL X PROD
      DO 1 NX = 1 , NDEX
      N = NX - 1
      DO 1 L = 1 , LDEX
      COR = 4.*OMEGA*OMEGA-ETA(L,0,NX,LENGTH,BESPRIM(1,L))
      MUSND(NX,L,1) =(MUSND(NX,L,1) + MUCROSS(L,NX,1))/ ( COR + GAMM(L,0
     $,N) + DELTA(L,0,N) ) / FNORM(L,0,N,BESPRIM(1,L),LENGTH)
      COR = 4.*OMEGA*OMEGA-ETA(L,2,NX,LENGTH,BESPRIM(3,L))
    1 MUSND(NX,L,2) =(MUSND(NX,L,2) + MUCROSS(L,NX,2))/ ( COR + GAMM(L,2
     $,N) + DELTA(L,2,N) ) / FNORM(L,2,N,BESPRIM(3,L),LENGTH)
      IF(MATRIX.NE.3HYES) GO TO 4
      WRITE(6,600) LDEX
      DO 2 NX = 1 , NDEX
      N = NX - 1
    2 WRITE(6,601) N , (MUSND(NX,L,1),L=1,LDEX)
      WRITE(6,602) LDEX
      DO 3 NX = 1 , NDEX
      N = NX - 1
    3 WRITE(6,601) N , (MUSND(NX,L,2),L=1,LDEX)
    4 RETURN
  600 FORMAT(*0 N*12X                   *L EQUAL 1 TO *I2,6X* M EQUAL 0 *)
  601 FORMAT(* *I3*  *10G13.6)
  602 FORMAT(*0 N*12X                   *L EQUAL 1 TO *I2,6X* M EQUAL 2 *)
      END
```

```
      SUBROUTINE X PROD
      REAL I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,INTEGRL,LENGTH
      COMPLEX MUCROSS(3,30,2)
      COMPLEX MUSND , ZCROSS1 , ZCROSS2 , PSI
      COMPLEX B1,    B2, I, BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1 AK2, PSI2, MU , AMP , C12,FNORM
      COMMON / BLKA / B1    , B2    , AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
     1, AK2, BETAN, RLMDA, BETAI, BETAC, BETAWI,     LHAT, MHAT, NHAT,
     1PSI2,X1,X2
      COMMON / BLKB / MU(3,30,2) , AMP( 2) ,I , MATRIX
      COMMON / BLKC / LENGTH , MUSND(30,3,2) , A1(2)
      COMMON / BLOCK / C12(30,30),ISET,LDEX,NDEX,ERROR1,ERROR2
      EQUIVALENCE (C12(1,1) , MUCROSS(1,1,1) )
      EXTERNAL I1,I2,I3,I4,I5,I6,I7,I8,I9,I10
C $$$
C $$$ THESE ARE THE CONSTANTS TO BE USED IN THIS PART OF THE PROGRAM
C $$$
      WRITE(6,600)
      MU(1,1,2)=CMPLX(1.0,0.0)/PSI(0)/FNORM(LHAT,MHAT,NHAT,RLMDA,LENGTH)
      MU(2,1,2) = CMPLX((1.0-GAMMA)/4.0 , 0.0 )
      MU(3,1,2) = I * OMEGA
      MU(1,2,2) = AMACH*AMACH*B1*B1+2.*OMEGA*AMACH*B1+OMEGA*OMEGA
      MU(2,2,2) = (AMACH*AMACH*B2*B2+2.*OMEGA*AMACH*B2+OMEGA*OMEGA)*C
      MU(3,4,2) = CMPLX( 3.14159265358979 / LENGTH , 0.0 )
      MU(3,2,2) = AMACH*MU(3,4,2)
      MU(1,15,2) = MU(3,2,2) * MU(3,2,2)
      MU(1,3,2) = (AMACH*B1+OMEGA)*I
      MU(2,3,2) = (AMACH*B2+OMEGA)*I*C
      MU(3,3,2) = OMEGA*OMEGA
      MU(1,4,2) = (2.*OMEGA+AMACH*B1)*I
      MU(2,4,2) = (2.*OMEGA+AMACH*B2)*I*C
      MU(1,5,2) = MU(1,4,2)*B1/I
      MU(2,5,2) = MU(2,4,2)*B2/I
      MU(3,5,2) = B1*MU(3,2,2)
      MU(1,6,2) = C*B2*MU(3,2,2)
      MU(2,15,2) = CMPLX(2.0,0.0)*MU(3,1,2)
```

```
      MU(2,6,2)  =  CMPLX(GAMMA/4.0,0.0)
      MU(3,6,2)  =  BETAI*(CMPLX(1.0,0.)+C)
      MU(2,8,2)  =  CEXP(I*B1*LENGTH)
      MU(3,8,2)  =  CEXP(I*B2*LENGTH)*C
      MU(1,7,2)  =  BETAN*(MU(2,8,2)+MU(3,8,2))
      MU(2,7,2)  =  OMEGA*BETAI*(AMACH*(B1+C*B2)+OMEGA*(CMPLX(1.0,0.0)+C))
      MU(3,7,2)  =  OMEGA*BETAN*(AMACH*(B1*MU(2,8,2)+B2*MU(3,8,2))+OMEGA*(
     $MU(2,8,2)+MU(3,8,2)))
      MU(1,8,2)  =  BETAC*BESSEL(MHAT+I,LHAT)
      MU(2,8,2)  =  I*MU(3,4,2)*(CMPLX(1.-AMACH*AMACH,0.0)*B1-OMEGA*AMACH)
      MU(3,8,2)=I*MU(3,4,2)*(CMPLX(1.-AMACH*AMACH,0.0)*B2-OMEGA*AMACH)*C
      MU(1,9,2)  =  -OMEGA*(AMACH*B1+OMEGA)
      MU(2,9,2)  =  -OMEGA*(AMACH*B2+OMEGA)*C
      DO 3 NX = 1 , NDEX
      NBAR = NX - 1
      DO 3 LBAR = 1 , LDEX
      DO 1 L = 1 , LDEX
      MU(L,21,2)=CMPLX(INTEGRL(I1,0.0,1.0,LHAT,L,LBAR,2,1),0.0)
      MU(L,22,2)=CMPLX(INTEGRL(I2,0.0,1.0,LHAT,L,LBAR,2,1),0.0)
      MU(L,23,2)=CMPLX(INTEGRL(I3,0.0,1.0,LHAT,L,LBAR,2,1),0.0)
      MU(L,24,2)=CMPLX(INTEGRL(I4,0.0,1.0,LHAT,L,LBAR,2,1),0.0)
      MU(L,25,2)=CMPLX(INTEGRL(I5,0.0,1.0,LHAT,L,LBAR,2,1),0.0)
      MU(L,26,2)=CMPLX(INTEGRL(I6,0.0,1.0,LHAT,L,LBAR,2,3),0.0)
      MU(L,27,2)=CMPLX(INTEGRL(I7,0.0,1.0,LHAT,L,LBAR,2,3),0.0)
      MU(L,28,2)=CMPLX(INTEGRL(I8,0.0,1.0,LHAT,L,LBAR,2,3),0.0)
      MU(L,29,2)=CMPLX(INTEGRL(I9,0.0,1.0,LHAT,L,LBAR,2,3),0.0)
    1 MU(L,30,2)=CMPLX(INTEGRL(I10,0.0,1.0,LHAT,L,LBAR,2,3),0.0)
      MU(2,14,2)  =  CMPLX  ( 0.0  ,  0.0  )
      MU(3,14,2)  =  CMPLX  ( 0.0  ,  0.0  )
      DO 2 NY = 1 , NDEX
      N = NY - 1
      MU(3,9,2)  =  MU(3,2,2)*CMPLX(FLOAT(N),0.0)
      MU(1,10,2)  =  MU(3,9,2)*MU(3,9,2)
      MU(2,10,2)  =  MU(3,4,2)*CMPLX(FLOAT(N),0.0)
      MU(3,10,2)  =  CMPLX((-1.)**N,0.0)
      MU(2,16,2)  =  MU(2,15,2)*MU(3,9,2)
```

```
 MU(3,15,2)  =  B1*MU(3,9,2)*I
 MU(1,16,2)  =  C*B2*MU(3,9,2)*I
 MU(3,11,2)  =  ZCROSS1(I*B1,N,NBAR,MU(3,4,2),LENGTH)-
$               ZCROSS1(I*B1,N,NBAR,MU(3,4,2),0.0)
 MU(1,12,2)  =  ZCROSS1(I*B2,N,NBAR,MU(3,4,2),LENGTH)-
$               ZCROSS1(I*B2,N,NBAR,MU(3,4,2),0.0)
 MU(2,12,2)  =  ZCROSS2(I*B1,N,NBAR,MU(3,4,2),LENGTH)-
$               ZCROSS2(I*B1,N,NBAR,MU(3,4,2),0.0)
 MU(3,12,2)  =  ZCROSS2(I*B2,N,NBAR,MU(3,4,2),LENGTH)-
$               ZCROSS2(I*B2,N,NBAR,MU(3,4,2),0.0)
 MU(1,13,2)  =  ZCROSS1(I*B1,N,NBAR,MU(3,4,2),X2)-
$               ZCROSS1(I*B1,N,NBAR,MU(3,4,2),X1)
 MU(2,13,2)  =  ZCROSS1(I*B2,N,NBAR,MU(3,4,2),X2)-
$               ZCROSS1(I*B2,N,NBAR,MU(3,4,2),X1)
 MU(3,13,2)  =  ZCROSS2(I*B1,N,NBAR,MU(3,4,2),X2)-
$               ZCROSS2(I*B1,N,NBAR,MU(3,4,2),X1)
 MU(1,14,2)  =  ZCROSS2(I*B2,N,NBAR,MU(3,4,2),X2)-
$               ZCROSS2(I*B2,N,NBAR,MU(3,4,2),X1)
 DO 2 L = 1 , LOEX
 MU(1,11,2)  =  CMPLX(BESPRIM(MHAT+1,L),0.0)
 MU(2,11,2)  =  MU(1,11,2)*CMPLX(RLMDA,0.0)
 MU(2,14,2)  =  MU(2,14,2)  +MU(L,NY,1)*(MU(2,1,2)*MU(L,22,2)*
$(MU(1,2,2)*      (MU(3,1,2)*MU(3,11,2)-MU(3,9,2)*MU(2,12,2))
$+MU(2,2,2)*(MU(3,1,2)*MU(1,12,2)-MU(3,9,2)*MU(3,12,2))
$+MU(1,3,2)*((MU(3,3,2)+MU(1,10,2))*MU(3,11,2)+MU(2,16,2)
$*MU(2,12,2))+MU(2,3,2)*((MU(3,3,2)+MU(1,10,2))*MU(1,12,2)+
 $MU(2,16,2)*MU(3,12,2)))+((RLMDA*MU(1,11,2)*MU(L,25,2)-MHAT*
$(MU(1,11,2)*MU(L,23,2)+RLMDA* MU(L,24,2))+CMPLX(2.*MHAT*MHAT,0.0)
$*MU(L,21,2))
$*(MU(1,4,2)*MU(3,11,2)+MU(2,4,2)*MU(1,12,2)-MU(3,9,2)*(MU(2,12,2)
$+C*MU(3,12,2)))+            MU(2,10,2)*MU(L,22,2)*
$(MU(1,5,2)*MU(2,12,2)+MU(2,5,2)*MU(3,12,2)-MU(3,15,2)*MU(3,11,2)-
$MU(1,16,2)*MU(1,12,2)))/2.
$+MU(2,6,2)*((RLMDA*MU(1,11,2)*MU(L,25,2)-MHAT*(MU(1,11,2)*
$ MU(L,23,2)+RLMDA*MU(L,24,2))+CMPLX(2.*MHAT*MHAT,0.0)*MU(L,21,2))
$*(MU(3,6,2)+MU(3,10,2)*MU(1,7,2))+MU(L,22,2)*
```

```
      $(MU(2,7,2)+MU(3,10,2)*MU(3,7,2))-      MU(1,8,2)*BESSEL(MHAT+1,L)*
      $BESSEL(1,LBAR)*(MU(2,8,2)*MU(3,13,2)+MU(3,8,2)*MU(1,14,2)
      $+MU(1,9,2)*MU(1,13,2)+MU(2,9,2)*MU(2,13,2))))
    2 MU(3,14,2) = MU(3,14,2) +MU(L,NY,1)*(MU(2,1,2)*MU(L,27,2)*
      $(MU(1,2,2)*      (MU(3,1,2)*MU(3,11,2)-MU(3,9,2)*MU(2,12,2))
      $+MU(2,2,2)*(MU(3,1,2)*MU(1,12,2)-MU(3,9,2)*MU(3,12,2))
      $+MU(1,3,2)*((MU(3,3,2)+MU(1,10,2))*MU(3,11,2)+MU(2,16,2)
      $*MU(2,12,2))+MU(2,3,2)*((MU(3,3,2)+MU(1,10,2))*MU(1,12,2)+
      $MU(2,16,2)*MU(3,12,2)))+((RLMDA*MU(1,11,2)*MU(L,30,2)-MHAT*
      $(MU(1,11,2)*MU(L,28,2)+RLMDA* MU(L,29,2)))
      $*(MU(1,4,2)*MU(3,11,2)+MU(2,4,2)*MU(1,12,2)-MU(3,9,2)*(MU(2,12,2)
      $+C*MU(3,12,2)))+             MU(2,10,2)*MU(L,27,2)*
      $(MU(1,5,2)*MU(2,12,2)+MU(2,5,2)*MU(3,12,2)-MU(3,15,2)*MU(3,11,2)-
      $MU(1,16,2)*MU(1,12,2)))/2.
      $+MU(2,6,2)*((RLMDA*MU(1,11,2)*MU(L,30,2)-MHAT*(MU(1,11,2)*
      $ MU(L,28,2)+RLMDA*MU(L,29,2)))
      $*(MU(3,6,2)+MU(3,10,2)*MU(1,7,2))+MU(L,27,2)*
      $(MU(2,7,2)+MU(3,10,2)*MU(3,7,2))-      MU(1,8,2)*BESSEL(MHAT+1,L)*
      $BESSEL(3,LBAR)*(MU(2,8,2)*MU(3,13,2)+MU(3,8,2)*MU(1,14,2)
      $+MU(1,9,2)*MU(1,13,2)+MU(2,9,2)*MU(2,13,2))))
      MU(3,21,2) = A1(1)/FNORM(LBAR,0,NBAR,BESPRIM(1,LBAR),LENGTH)
      MU(3,22,2)=A1(2)/FNORM(LBAR,2,NBAR,BESPRIM(3,LBAR),LENGTH)
      MUCROSS(LBAR,NX,1) = MU(1,1,2)*MU(3,21,2)*MU(2,14,2)
    3 MUCROSS(LBAR,NX,2) = MU(1,1,2)*MU(3,22,2)*MU(3,14,2)
      RETURN
  600 FORMAT(* *50(1H*)/*  WITH THE CROSS PRODUCT TERMS*/* *50(1H*))
      END
```

```
      COMPLEX FUNCTION ZCROSS1 (A,M,N,PI,X)
      DIMENSION PI(2)
      COMPLEX A, ZB, ZD, ZF
C *******************************************************************
C *** NOTE : IN THIS FUNCTION SUBPROGRAM PI IS DEFINED AS PI( 3.14 ETC )
C *** DIVIDED BY LENGTH
C *******************************************************************
      ZA (P,X) = SIN(P*X)/P/2.
      ZB (A,P,X) = (A*SIN(P*X)-P*COS(P*X))/(A*A+P*P)
      ZC (P,X) = (X+SIN(P*X)/P)/2.
      ZD (A,X) = (A*X-1.)/A/A
      ZE (P,X) = -COS(P*X)/P/2.
      ZF (A,P,X) = (A*COS(P*X)+P*SIN(P*X))/(A*A+P*P)
      ZG (P,X) = SIN(P*X)**2/P/2.
      IF ( M .EQ. N ) GO TO 1
      PA = (M-N) * PI                         $ PB = (M+N) * PI
      ZCROSS1 = CEXP(A*X)*(ZA(PA,X)+ZA(PB,X)-A/2.*(ZB(A,PA,X)/PA+ZB(A,PB
     $,X)/PB))                                $ RETURN
    1 IF ( CABS(A) .LT. 1.E-100 ) GO TO 3
      IF ( N .EQ. 0 ) GO TO 2
      PA = 2.*N*PI
      ZCROSS1 = CEXP(A*X)/2.*(CMPLX(1.0,0.0)/A+ZF(A,PA,X)) $ RETURN
    2 ZCROSS1 = CEXP(A*X)/A                   $ RETURN
    3 IF ( N .NE. 0 ) GO TO 4
      ZCROSS1 = CMPLX( X , 0.0 )              $ RETURN
    4 PA = 2.*N*PI
      ZCROSS1 = CMPLX(ZC(PA,X),0.0)           $ RETURN
      ENTRY ZCROSS2
      IF ( M .EQ. 0 ) GO TO 7
      IF ( M .EQ. N ) GO TO 5
      PA = (M-N) * PI                         $ PB = (M+N) * PI
      ZCROSS1 = CEXP(A*X)*(ZE(PA,X)+ZE(PB,X)+A/2.*(ZF(A,PA,X)/PA+ZA(PB,X
     $)/PB))                                  $ RETURN
    5 IF ( CABS(A) .LT. 1.E-100 ) GO TO 6
      PA = 2.*N*PI
      ZCROSS1 = CEXP(A*X)*ZB(A,PA,X)/CMPLX(2.0,0.0)          $ RETURN
    6 ZCROSS1 = ZG(N*PI,X)                    $ RETURN
    7 ZCROSS1 = CMPLX ( 0.0 , 0.0 )           $ RETURN
      END
```

```
      COMPLEX FUNCTION ZNTEGRL(NHAT,N,A,PIORL,X,TYPE)
      INTEGER TYPE
      COMPLEX A
      P = NHAT * PIORL          $        Q = N * PIORL
      GO TO (1,5), TYPE
    1 IF(NHAT.EQ.N.AND.N.EQ.0) GO TO 2
      IF(NHAT.EQ.N.AND.N.EQ.0.OR.NHAT.EQ.0) GO TO 3
      IF(NHAT.EQ.N.AND.N.NE.0) GO TO 4
      ZNTEGRL = CEXP(A*X)*(CMPLX(SIN((P-Q)*X)-(P-Q)+SIN((P+Q)*A)/(P+Q),0
     $.0)-A*((A*SIN((P-Q)*X)-(P-Q)*COS(P-Q)*X))/(A*A+(P-Q*(P-Q))+(A*SIN(
     $(P+Q)*X)-(P+Q)*COS((P+Q)*X))/(A*A+(P+Q)*(P+Q))))/CMPLX(2.0,0.0)
      RETURN
    2 ZNTEGRL = CEXP(A*X)/A                                     $ RETURN
    3 ZNTEGRL = CEXP(A*X)/(A*A+P*P)*(A*COS(P*X)+P*SIN(P*X))$ RETURN
    4 ZNTEGRL = CEXP(A*X)/(A*A+4.*P*P)*((A*COS(P*X)+2.*P*SIN(P*X))*COS(P
     $*X)+2.*P*P/A)                                             $ RETURN
    5 IF ( N .EQ. 0 ) GO TO 6
      IF(NHAT.EQ.N.AND.NHAT.EQ.0) GO TO 7
      IF(NHAT.EQ.N.AND.N.NE.0) GO TO 8
      ZNTEGRL = -CEXP(A*X)/CMPLX(2.0,0.0)*(COS((Q-P)*X)/(Q-P)+COS((Q+P)*
     $X)/(Q+P)-A*((A*COS((Q-P)*X)+(Q-P)*SIN((Q-P)*X)/(A*A+(Q-P)*(Q-P))+A
     $*COS((Q+P)*X)+(Q+P)*SIN((Q+P)*X))/(A*A+(Q+P)*(Q+P))))$ RETURN
    6 ZNTEGRL = CMPLX(0.0,0.0)                                  $ RETURN
    7 ZNTEGRL = CEXP(A*X)*(A*SIN(Q*X)-Q*COS(Q*X))/(A*A+Q*Q)$ RETURN
    8 ZNTEGRL = CEXP(A*X)/CMPLX(2.0,0.0)*(A*SIN(2.*P*X)-CMPLX(2.*P,0.0)*
     $COS(2.*P*X))/(A*A+CMPLX(4.*P*P,0.0))                     $ RETURN
      END
```

```
      SUBROUTINE EIGEN
      INTEGER REDEFIN
      REAL LOWLIM, LENGTH, THETA(3), INTEGRL
      REAL I1, I2, I3, I4, I5, I6, I7, I8, I9, I10, I11, I12, I13, I14,
     $I15
      COMPLEX A, E1, E2, E3, E4, B1CON, B2CON, B3, ZINTGRL, Z(24), CCON,
     $FNORM, PSI, WMB1, WMB2, CMOD, B1MOD, B2MOD, WB1M, WB2M, WMB1CON,
     $WMB2CON, PSISTR, PSICON, BETAI2, OMEGA2, OMEGAC, BETAIC, ANC, R(18
     $),ZINT(4), REVAL(5), ZNTEGRL, SLOPE, BETASTR
      COMPLEX B1    , B2    , BETAN, BETAC, BETAI, BETAWI, OMEGA, C, AK1,
     1 AK2, PSI2, MU, BETAIS, C12, I, MUSND
      COMMON / BLKA / B1    , B2    , AMACH, GAMMA, OMEGA, ALENGTH, C, AK1
     1, AK2, BETAN, RLMDA, BETAI, BETAC, BETAWI,     LHAT, MHAT, NHAT,
     1PSI2,X1,X2
      COMMON / BLKB / MU(3,30,2) , BETAIS(2) ,I , MATRIX
      COMMON / BLKC / LENGTH , MUSND(30,3,2) , A1(2)
      COMMON / BLOCK / C12(30,30),ISET,LDEX,NDEX,ERROR1,ERROR2
      COMMON / ROOT / BETAI2 , BETASTR , SLOPE
      EQUIVALENCE (R(1),MU(1,1,2)) , (Z(1),MU(1,7,2)) , (ZINT(1),MU(1,15
     $,2)) , (REVAL(1),MU(2,16,2))
      EXTERNAL I1, I2, I3, I4, I5, I6, I7, I8, I9, I10, I11, I12, I13,
     $I14, I15
      EXTERNAL R1, R2, R3, R4, R5, R6, R7, R8, R9
      DATA PI / 3.14159265358979 /
      ZINTGRL (A,P,UPLIM,LOWLIM) = (CEXP(A*UPLIM)*(A*COS(P*UPLIM)+P*SIN(
     $P*UPLIM))-CEXP(A*LOWLIM)*(A*COS(P*LOWLIM)+P*SIN(P*LOWLIM)))/(A*A+P
     $*P)
C ********************************************************************
C *****************                                       ************
C *****************  NOTE, IN THIS ROUTINE THE INTERACTION ************
C *****************  INDEX (N) HAS BEEN REPLACED BY THE    ************
C *****************  ALPHA CHARACTER B, SO AS NOT TO BE    ************
C *****************  CONFUSED WITH THE MATRIX INDEX (N).   ************
C *****************                                       ************
C ********************************************************************
      B1CON = CONJG(B1)
```

```
      B1MOD = CABS(B1)**2
      B2CON = CONJG(B2)
      B2MOD = CABS(B2)**2
      B3 = B1 + B2
      CCON = CONJG ( C )
      CMOD = CABS ( C ) **2
      CONST = NHAT * PI / LENGTH
      E1 = I*(2.*B1-B1CON)
      E2 = I*(2.*B2-B2CON)
      E3 = I*B1
      E4 = I*B2
      OMEGA2 = CMPLX(0.0,0.0)
      PIORL = PI / LENGTH
      PSISTR = PSI2 * FNORM(LHAT,MHAT,NHAT,RLMDA,LENGTH)**2
      PSICON = CONJG(CSQRT(PSISTR))
      WB1M = OMEGA - AMACH * B1CON
      WB2M = OMEGA - AMACH * B2CON
      WMB1 = OMEGA + AMACH * B1
      WMB2 = OMEGA + AMACH * B2
      WMB1CON = OMEGA + AMACH * B1CON
      WMB2CON = OMEGA + AMACH * B2CON
      WRITE(6,602)
C *************************************************************************
C *****************                                          ***************
C *****************  THESE ARE THE PHI(1) ** 3 PRODUCTS      ***************
C *****************                                          ***************
C *************************************************************************
C ***
C *** THESE ARE THE THETA-INTEGRALS
C ***
      THETA(1) = 0.75 * PI
      THETA(2) = PI / 4.
      THETA(3) = PI
C ***
C *** THESE ARE THE R-INTEGRALS
C ***
```

```
      Z(1) = CMPLX(INTEGRL(R1,0.0,1.0,LHAT,1,1,2,2),0.0)
      Z(2) = CMPLX(INTEGRL(R2,0.0,1.0,LHAT,1,1,2,2),0.0)
      Z(3) = CMPLX(INTEGRL(R3,0.0,1.0,LHAT,1,1,2,2),0.0)
      Z(4) = CMPLX(INTEGRL(R4,0.0,1.0,LHAT,1,1,2,2),0.0)
      Z(5) = CMPLX(INTEGRL(R5,0.0,1.0,LHAT,1,1,2,2),0.0)
      Z(6) = CMPLX(INTEGRL(R6,0.0,1.0,LHAT,1,1,2,2),0.0)
      Z(7) = CMPLX(INTEGRL(R7,0.0,1.0,LHAT,1,1,2,2),0.0)
      Z(8) = CMPLX(INTEGRL(R8,0.0,1.0,LHAT,1,1,2,2),0.0)
      R(1) = RLMDA*RLMDA*Z(1)-2.*MHAT*RLMDA*Z(2)+MHAT*MHAT*Z(3)
      R(2) = Z(3)
      R(3) = Z(4)
      R(4) = RLMDA*RLMDA*(-RLMDA*RLMDA*Z(1)+2.*MHAT*RLMDA*Z(2)-MHAT*MHAT
     $*Z(3)-RLMDA*Z(5)+MHAT*(3.+MHAT)*Z(6))-MHAT*MHAT*(3.+2.*MHAT)*Z(7)*
     $RLMDA
      R(5) = RLMDA*RLMDA*Z(6)-2.*RLMDA*MHAT*Z(7)+MHAT*MHAT*Z(8)
      R(6) = RLMDA*Z(7)-MHAT*Z(8)
      R(7) = Z(8)
      R(8) = CMPLX(R9(1.,RLMDA,FLOAT(LHAT),FLOAT(MHAT)),0.0)
      REVAL(1) = CMPLX(RLMDA*RLMDA*R1(1.0,RLMDA,0.0,0.0)-2.*MHAT*RLMDA*
     $R2(1.0,RLMDA,0.0,0.0)+MHAT*MHAT*R3(1.0,RLMDA,0.0,0.0),0.0)
      REVAL(2) = CMPLX(R3(1.0,RLMDA,0.0,0.0),0.0)
      REVAL(3) = CMPLX(R4(1.0,RLMDA,0.0,0.0),0.0)
      REVAL(4) = CMPLX(R9(1.0,RLMDA,0.0,0.0),0.0)
C ***
C *** THESE ARE THE Z-INTEGRALS
C ***
      ZINT(1) = ZINTGRL( E1,CONST,LENGTH,0.0)
      ZINT(2) = ZINTGRL( E2,CONST,LENGTH,0.0)
      ZINT(3) = ZINTGRL( E3,CONST,LENGTH,0.0)
      ZINT(4) = ZINTGRL( E4,CONST,LENGTH,0.0)
      Z(1) = -(WMB1*WMB1*(ZINT(1)+ZINT(3)*(C+CCON)+CMOD*ZINT(4))+
     $          WMB2*WMB2*(ZINT(3)+ZINT(4)*(C+CCON)+CMOD*ZINT(2)))
      Z(2) = -(WMB1*WMB1*(B1MOD*ZINT(1)+ZINT(3)*(C*B2*B1CON+B1*CCON*B2CO
     $N)+CMOD*B2MOD*ZINT(4))+
     $          WMB2*WMB2*(B1MOD*ZINT(3)+ZINT(4)*(C*B2*B1CON+B1*CCON*B2CO
     $N)+CMOD*B2MOD*ZINT(2)))
```

```
      Z(3) = -(WMB1*WMB1*(ZINT(1)+2.*C*ZINT(3)+C*C*ZINT(4))+
     $       CCON*WMB2*WMB2*(ZINT(3)+2.*C*ZINT(4)+C*C*ZINT(2)))
      Z(4) = WMB1*WMB1*(B1*B1*ZINT(1)+2.*B1*B2*C*ZINT(3)+C*B2*C*B2*ZINT(
     $4))+
     $      CCON*WMB2*WMB2*(B1*B1*ZINT(3)+2.*B1*B2*C*ZINT(4)+C*B2*C*B2*
     $ZINT(2))
      Z(5) = ZINT(1)+(2.*C+CCON)*ZINT(3)+C*(2.*CCON+C)*ZINT(4)+C*CMOD*ZI
     $NT(2)
      Z(6) = B1CON*(B1*ZINT(1)+C*B3*ZINT(3)+C*C*B2*ZINT(4))+
     $      CCON*B2CON*(B1*ZINT(3)+C*B3*ZINT(4)+C*C*B2*ZINT(2))
      Z(7) = -(B1*B1*ZINT(1)+B1*(2.*C*B2+B1*CCON)*ZINT(3)+B2*(C*C*B2+2.*
     $B1*CMOD)*ZINT(4)+C*CMOD*B2*B2*ZINT(2))
      Z(8) = -(B1*B1*B1MOD*ZINT(1)+(C*B2*B3*B1MOD+CCON*B2CON*B1*B1*B1)*
     $ZINT(3)+(C*C*B2*B2*B2*B1CON+CMOD*B1*B3*B2MOD)*ZINT(4)+C*CMOD*B2*B2
     $*B2MOD*ZINT(2))
      Z(9) = B1MOD*B1MOD*ZINT(1)+(2.*C*B1MOD*B1CON*B2+CCON*B1*B1*B2CON*B
     $2CON)*ZINT(3)+(C*C*B2*B2*B1CON*B1CON+2.*CMOD*B2MOD*B2CON*B1)*ZINT(
     $4)+CMOD*CMOD*B2MOD*B2MOD*ZINT(2)
      Z(10) = -((OMEGA+AMACH*B1/2.)*ZINT(3)+C*(OMEGA+AMACH*B2/2.)*ZINT(4
     $))
C ***
C *** REDEFINE THE INTEGRATION LIMITS FROM X1 TO X2
C ***
      IF( CABS(BETAC).LE.1.E-10.OR.X2-X1.LE.1.E-10) GO TO 2
      ASSIGN 2 TO REDEFIN
      ZINT(1) = ZINTGRL(E1,CONST,X2,X1)
      ZINT(2) = ZINTGRL(E2,CONST,X2,X1)
      ZINT(3) = ZINTGRL(E3,CONST,X2,X1)
      ZINT(4) = ZINTGRL(E4,CONST,X2,X1)
    1 Z(11) = I*(WMB1*WMB1*WMB1CON*ZINT(1)+WMB1*(2.*C*WMB1CON*WMB2+CCON*
     $WMB1*WMB2CON)*ZINT(3)+WMB2*C*(2.*WMB1*WMB2CON+C*WMB2*WMB1CON)*ZINT
     $(4)+C*C*WMB2*WMB2*WMB2CON*ZINT(2))
      Z(12) = -I*(WMB1CON*ZINT(1)+(2.*C*WMB1CON+CCON*WMB2CON)*ZINT(3)+C*
     $(C*WMB1CON+2.*CCON*WMB2CON)*ZINT(4)+C*CMOD*WMB2CON*ZINT(2))
      Z(13) = I*(B1*B1*WMB1CON*ZINT(1)+(2.*B2*C*WMB1CON+B1*CCON*WMB2CON)
     $*ZINT(3)*B1+B2*(WMB1CON*C*C*B2+WMB2CON*CCON*2.*B1*C)*ZINT(4)+C*WMB
```

```
     $2CON*CMOD*B2*B2*ZINT(2))
      Z(14) = ZINT(3)+C*ZINT(4)
      Z(22) = B1*ZINT(1)+(C*B3+CCON*B1)*ZINT(3)+(CMOD*B3+C*C*B2)*ZINT(4)
     $+C*CMOD*B2*ZINT(2)
      Z(23) = -I*(B1CON*ZINT(1)        +(CMPLX(2.0,0.0)*C*B1CON    +CCON*B
     $2CON       )*ZINT(3)+(CMPLX(2.0,0.0)    *CMOD*B2CON+C*C       *B1CON)
     $*ZINT(4)+C    *CMOD*B2CON*ZINT(2))
      Z(24) = B1*B1MOD*ZINT(1)+(CMPLX(2.0,0.0)*C*B2*B1MOD+CCON*B2CON*B1*
     $B1)*ZINT(3)+(C*C*B2*B2*B1CON+CMPLX(2.0,0.0)*CMOD*B2MOD*B1)*ZINT(4)
     $+C*B2*CMOD*B2MOD*ZINT(2)
      GO TO REDEFIN
C ***
C *** EVALUATE THE VOLUME INTEGRALS
C ***
C *** TERMS 4 AND 5
C ***
    2 BETAI2 = ((GAMMA-1.)*(2.*(Z(1)*(THETA(1)*R(1)+THETA(2)*MHAT*MHAT*
     $R(2))+THETA(1)*R(3)*Z(2))+Z(3)*(THETA(1)*R(1)+MHAT*MHAT*THETA(2)*
     $R(2))+THETA(1)*R(3)*Z(4))
C ***
C *** TERM 6
C ***
     $+    (2.*Z(5)*(THETA(1)*R(4)+MHAT*MHAT*THETA(2)*(2.*R(5)-R(6)-MHAT*
     $MHAT*R(7)))+Z(6)*(THETA(1)*(3.          )*R(1)+3.*MHAT*MHAT*THETA(
     $2)*R(2))+Z(7)*(THETA(1)*R(1)+MHAT*MHAT*THETA(2)*R(2))-THETA(1)*R(3
     $)*(Z(8)+Z(9)))
C ***
C *** TERM 7
C ***
     $+    (Z(5)*(THETA(1)*R(4)+MHAT*MHAT*THETA(2)*(2.*R(5)-R(6)-MHAT*MHA
     $T*R(7)))+Z(4)*MHAT*MHAT*THETA(2)*R(2)+Z(6)*(THETA(1)*R(1)+MHAT*MHA
     $T*THETA(2)*R(2))+THETA(1)*R(1)*Z(7)+Z(8)*THETA(1)*R(3)))/CMPLX(16.
     $,0.0)/PSISTR/PSICON
      SLOPE = THETA(3)*R(8)*Z(10)/CSQRT(PSISTR)/I
      BETAI2 = - BETAI2
C ***
```

```
C *** EVALUATE THE SURFACE INTEGRALS
C ***
C *** FIRST DO THE LINER INTEGRALS
C ***
      IF( CABS(BETAC).LE.1.E-10.OR.X2-X1.LE.1.E-10) GO TO 5
      BETAI2 = BETAI2 + GAMMA*BETAC/CMPLX(2.0,0.0)*((-THETA(1)*REVAL*Z(1
     $1)+(THETA(1)*REVAL(1)+MHAT*MHAT*THETA(2)*REVAL(2))*Z(12)+THETA(1)*
     $REVAL(3)*Z(13)+
     $(GAMMA+1.)/CMPLX(8.0,0.0)*(THETA(1)*REVAL(3)*(I*OMEGA*OMEGA*OMEGA*
     $Z(5)+OMEGA*OMEGA*AMACH*(CMPLX(2.0,0.0)*Z(22)-Z(23))+I*OMEGA*AMACH*
     $AMACH*(CMPLX(2.0,0.0)*Z(6)-Z(7))+AMACH*AMACH*AMACH*Z(24))))/PSISTR
     $/PSICON)
      SLOPE = SLOPE + GAMMA*BETAC/CMPLX(2.0,0.0)*(THETA(3)*REVAL(4)*Z(14
     $)/CONJG(PSICON))
C ***
C ***EVALUATE THE INJECTOR INTEGRALS
C ***
    5 ASSIGN 3 TO REDEFIN
      ZINT(1) = ZINT(2) = ZINT(3) = ZINT(4) = CMPLX ( 1.0 , 0.0 )
      GO TO 1
    3 BETAI2 = BETAI2 + GAMMA*BETAI/CMPLX(2.0,0.0)*((-THETA(1)*R(3)*Z(11
     $)+(THETA(1)*R(1)+MHAT*MHAT*THETA(2)*R(2))*Z(12)+THETA(1)*R(3)*Z(13
     $)+
     $(GAMMA+1.)/CMPLX(8.0,0.0)*(THETA(1)*R(3)*(I*OMEGA*OMEGA*OMEGA*Z(5)
     $+OMEGA*OMEGA*AMACH*(CMPLX(2.0,0.0)*Z(22)-Z(23))+I*OMEGA*AMACH*AMAC
     $H*(CMPLX(2.0,0.0)*Z( 6)-Z(7))+AMACH*AMACH*AMACH*Z(24))))/PSISTR/PS
     $ICON)
      SLOPE = SLOPE + GAMMA*BETAI*CMPLX(2.0,0.0)*(THETA(3)*R(8)*Z(14)/CO
     $NJG(PSICON))
C ***
C *** EVALUATE THE NOZZLE INTEGRALS
C ***
      ASSIGN 4 TO REDEFIN
      ZINT(1) = CEXP(E1*LENGTH)*CMPLX((-1.)**NHAT,0.0)
      ZINT(2) = CEXP(E2*LENGTH)*CMPLX((-1.)**NHAT,0.0)
      ZINT(3) = CEXP(E3*LENGTH)*CMPLX((-1.)**NHAT,0.0)
```

```
      ZINT(4) = CEXP(E4*LENGTH)*CMPLX((-1.)**NHAT,0.0)
      GO TO 1
    4 BETAI2 = BETAI2 + GAMMA*BETAN/CMPLX(2.0,0.0)*((-THETA(1)*R(3)*Z(11
     $)+(THETA(1)*R(1)+MHAT*MHAT*THETA(2)*R(2))*Z(12)+THETA(1)*R(3)*Z(13
     $)+
     $(GAMMA+1.)/CMPLX(8.0,0.0)*(THETA(1)*R(3)*(I*OMEGA*OMEGA*OMEGA*Z(5)
     $+OMEGA*OMEGA*AMACH*(CMPLX(2.0,0.0)*Z(22)-Z(23))+I*OMEGA*AMACH*AMAC
     $H*(CMPLX(2.0,0.0)*Z( 6)-Z(7))+AMACH*AMACH*AMACH*Z(24)))))/PSISTR/PS
     $ICON)
      SLOPE = SLOPE + GAMMA*BETAN*CMPLX(2.0,0.0)*(THETA(3)*R(8)*Z(14)/CO
     $NJG(PSICON))
C ************************************************************************
C *****************                                           ***********
C *****************  THESE ARE THE PHI(2) , PHI(1) PRODUCTS   ***********
C *****************                                           ***********
C ************************************************************************
      DO 15 L = 1 , LOEX
C ***
C *** THE THETA  INTEGRALS REMAIN THE SAME AS THOSE DEFINED ABOVE
C ***
C *** THESE ARE THE R-INTEGRALS
C ***
      R(9) = CMPLX(RLMDA* INTEGRL(I11,0.0,1.0,LHAT,LHAT,L,MHAT+1,1)-MHAT
     $*INTEGRL(I12,0.0,1.0,LHAT,LHAT,L,MHAT+1,1) ,0.0)
      R(10) = CMPLX(RLMDA* INTEGRL(I13,0.0,1.0,LHAT,LHAT,L,MHAT+1,3)-MHA
     $T*INTEGRL(I14,0.0,1.0,LHAT,LHAT,L,MHAT+1,3) ,0.0)
      R(11) = CMPLX(INTEGRL(I1,0.0,1.0,LHAT,LHAT,L,MHAT+1,1),0.0)
      R(12) = CMPLX(INTEGRL(I6,0.0,1.0,LHAT,LHAT,L,MHAT+1,3),0.0)
      R(13) = CMPLX(INTEGRL(I2,0.0,1.0,LHAT,LHAT,L,MHAT+1,1),0.0)
      R(14) = CMPLX(INTEGRL(I7,0.0,1.0,LHAT,LHAT,L,MHAT+1,3),0.0)
      R(15) = CMPLX(BESPRIM(1,L)*(INTEGRL(I15,0.0,1.0,LHAT,LHAT,L,MHAT+1
     $,1)-BESPRIM(1,L)*INTEGRL(I2,0.0,1.0,LHAT,LHAT,L,MHAT+1,1)),0.0)
      R(16) = CMPLX(-BESPRIM(3,L)*(INTEGRL(I15,0.0,1.0,LHAT,LHAT,L,MHAT+
     $1,3)*BESPRIM(3,L)*INTEGRL(I2,0.0,1.0,LHAT,LHAT,L,MHAT+1,3))+6.*INT
     $EGRL(I6,0.0,1.0,LHAT,LHAT,L,MHAT+1,3),0.0)
      R(17) = CMPLX(-BESPRIM(1,L)*INTEGRL(I15,0.0,1.0,LHAT,LHAT,L,MHAT+1
```

```
     $,1),0.0)
      R(18) = CMPLX( BESPRIM(3,L)*INTEGRL(I15,0.0,1.0,LHAT,LHAT,L,MHAT+
     $,3)-2.0*INTEGRL(I6,0.0,1.0,LHAT,LHAT,L,MHAT+1,3),0.0)
      REVAL(1) = CMPLX(RLMDA* I11(1.0,RLMDA,RLMDA,BESPRIM(1,L))-MHAT*I1
     $(1.0,RLMDA,RLMDA,BESPRIM(1,L)) ,0.0)
      REVAL(2) = CMPLX(RLMDA* I13(1.0,RLMDA,RLMDA,BESPRIM(3,L))-MHAT*I1
     $(1.0,RLMDA,RLMDA,BESPRIM(3,L)) ,0.0)
      REVAL(3) = CMPLX(I6(1.0,RLMDA,RLMDA,BESPRIM(3,L)),0.0)
      REVAL(4) = CMPLX(I2(1.0,RLMDA,RLMDA,BESPRIM(1,L)),0.0)
      REVAL(5) = CMPLX(I7(1.0,RLMDA,RLMDA,BESPRIM(3,L)),0.0)
      DO 15 NX = 1 , NDEX
      N = NX - 1
      ZINT(1) = ZNTEGRL(NHAT,N,-I*B1CON,PIORL,LENGTH,1)-
     $          ZNTEGRL(NHAT,N,-I*B1CON,PIORL,0.,1)
      ZINT(2) = ZNTEGRL(NHAT,N,-I*B2CON,PIORL,LENGTH,1)-
     $          ZNTEGRL(NHAT,N,-I*B2CON,PIORL,0.,1)
      ZINT(3) = ZNTEGRL(NHAT,N,-I*B1CON,PIORL,LENGTH,2)-
     $          ZNTEGRL(NHAT,N,-I*B1CON,PIORL,0.,2)
      ZINT(4) = ZNTEGRL(NHAT,N,-I*B2CON,PIORL,LENGTH,2)-
     $          ZNTEGRL(NHAT,N,-I*B2CON,PIORL,0.,2)
C ***
C *** THESE ARE THE Z VOLUME INTEGRALS
C ***
      Z(15) = I*(WB1M*ZINT(1)+CCON*WB2M*ZINT(2))-AMACH*PIORL*N*(ZINT(3)+
     $CCON*ZINT(4))
      Z(16) =    (WB1M*B1CON*ZINT(3)+CCON*B2CON*WB2M*ZINT(4))-AMACH*PIORL
     $*N*(B1CON*ZINT(1)+CCON*B2CON*ZINT(2))*I
      Z(17) = -I*OMEGA*      2.0      *(WMB1CON*WMB1CON*ZINT(1)+CCON*WMB2
     $CON*WMB2CON*ZINT(2))+AMACH*PIORL*N*(WMB1CON*WMB1CON*ZINT(3)+CCON*W
     $MB2CON*WMB2CON*ZINT(4))
      Z(18) = -I*((OMEGA-AMACH*B1)*ZINT(1)+(OMEGA-AMACH*B2)*ZINT(2)*CCON
     $)
C ***
C *** THESE ARE THE Z SURFACE INTEGRALS
C ***
      IF ( CABS(BETAC).LE.1.E-10.OR.X2-X1.LE.1.E-10) GO TO 12
```

```
      ASSIGN 12 TO REDEFIN
      ZINT(1) = ZNTEGRL(NHAT,N,-I*B1CON,PIORL,X2,1)-
     $          ZNTEGRL(NHAT,N,-I*B1CON,PIORL,X1,1)
      ZINT(2) = ZNTEGRL(NHAT,N,-I*B2CON,PIORL,X2,1)-
     $          ZNTEGRL(NHAT,N,-I*B2CON,PIORL,X1,1)
      ZINT(3) = ZNTEGRL(NHAT,N,-I*B1CON,PIORL,X2,2)-
     $          ZNTEGRL(NHAT,N,-I*B1CON,PIORL,X1,2)
      ZINT(4) = ZNTEGRL(NHAT,N,-I*B2CON,PIORL,X2,2)-
     $          ZNTEGRL(NHAT,N,-I*B2CON,PIORL,X1,2)
   11 Z(19) = OMEGA*(WMB1CON*ZINT(1)+CCON*WMB2CON*ZINT(2))+
     $        I*N*AMACH*PIORL/2.*(WMB1CON*ZINT(3)+CCON*WMB2CON*ZINT(4))
      Z(20) = ZINT(1)+CCON*ZINT(2)
      Z(21) = -I*(B1CON*ZINT(1)+CCON*B2CON*ZINT(2))
      GO TO REDEFIN
C ***
C *** EVALUATE THE VOLUME INTEGRALS
C ***
   12 BETAI2 = BETAI2-(MUSND(NX,L,1)*THETA(3)*((R(9)*Z(15)-N*PIORL*R(13)
     $*Z(16))/     2.0      +        (GAMMA-1.)/4.0       *(R(13)*Z(17)+Z(18)
     $*(R(15)+R(17)-N*N*PIORL*PIORL*R(13))))+MUSND(NX,L,2)*(THETA(2)*((R
     $(10)*Z(15)-N*PIORL*R(14)*Z(16))/       2.0       +       (GAMMA-1.)/4.
     $      *(R(14)*Z(17)+Z(18)*(R(16)+R(18)-4.*R(12)-N*N*PIORL*PIORL*R(1
     $4)))))+       THETA(3)/2.*MHAT      *R(12)*Z(15)))/PSICON
C ***
C *** EVALUATE THE SURFACE INTEGRALS
C ***
C *** FIRST DO THE LINER INTEGRALS
C ***
      IF ( CABS(BETAC).LE.1.E-10.OR.X2-X1.LE.1.E-10) GO TO 13
      BETAI2 = BETAI2 + BETAC*      GAMMA/2.0      /PSICON*(MUSND(NX,L,1)
     $*THETA(3)*(REVAL(4)*Z(19)-(REVAL(1)*Z(20)-N*PIORL*REVAL(4)*Z(21)))/
     $      2.0      )+MUSND(NX,L,2)*(THETA(2)*(REVAL(5)*Z(19)-(REVAL(2)*
     $Z(20)-N*PIORL*REVAL(5)*Z(21))/       2.0       )-       MHAT*THETA(3)/
     $2.0       *REVAL(3)*Z(20)))
C ***
C *** EVALUATE THE INJECTOR INTEGRALS
```

```
C ***
   13 ASSIGN 14 TO REDEFIN
      ZINT(1) = ZINT(2) = ZINT(3) = ZINT(4) = CMPLX(1.0,0.0)
      GO TO 11
   14 BETAI2 = BETAI2 + BETAI*      GAMMA/2.0      /PSICON*(MUSND(NX,L,1)
     $*THETA(3)*(R(13)*Z(19)-(R(9)*Z(20)-N*PIORL*R(13)*Z(21))/       2.0
     $     )+MUSND(NX,L,2)*(THETA(2)*(R(14)*Z(19)-(R(10)*Z(20)-N*PIORL*R(
     $14)*Z(21)))-       MHAT*THETA(3)/2.0      *R(12)*Z(20)))
C ***
C *** EVALUATE THE NOZZLE INTEGRALS
C ***
      ASSIGN 15 TO REDEFIN
      ZINT(1) = CEXP(-I*B1CON)*      (-1.)**(NHAT+N)
      ZINT(2) = CEXP(-I*B2CON)*      (-1.)**(NHAT+N)
      ZINT(3) = CEXP(-I*B1CON)*     ((-1.)**NHAT*(1.-(-1.)**N))
      ZINT(4) = CEXP(-I*B2CON)*     ((-1.)**NHAT*(1.-(-1.)**N))
      GO TO 11
   15 BETAI2 = BETAI2 + BETAN*      GAMMA/2.0      /PSICON*(MUSND(NX,L,1)
     $*THETA(3)*(R(13)*Z(19)-(R(9)*Z(20)-N*PIORL*R(13)*Z(21))/       2.0
     $     )+MUSND(NX,L,2)*(THETA(2)*(R(14)*Z(19)-(R(10)*Z(20)-N*PIORL*R(
     $14)*Z(21)))-       MHAT*THETA(3)/2.0      *R(12)*Z(20)))
      BETASTR = BETAI2*EPSIZN(LHAT,MHAT,NHAT,LENGTH,RLMDA)*PSI(0)/FNORM(
     $LHAT,MHAT,NHAT,RLMDA,LENGTH)/AK1
      SLOPE = (SLOPE*BETASTR/BETAI2)*I
       WRITE(6,700) SLOPE
      CALL SOLSION ( OMEGA , BETAI , GAMMA , AMACH , PI )
  602 FORMAT(*0*///)
  700 FORMAT(*0 THE SLOPE IS *2G21.14)
      RETURN
      END
```

```
      SUBROUTINE NTAU ( AN , OMEGA )
      REAL N , NVEC
      COMMON / NORM / N(3) , TAU(3) , NVEC , TAUVEC , PRINT
      DIMENSION AN ( 2 )
      PIE2 = 2. * 3.14159265358979323846
      ANR = AN ( 1 )
      ANI = AN ( 2 )
      N = ( ANR**2 + ANI**2 ) / 2. / ANR
      IF ( ANI .LT. 0.0 ) 5, 6
    5 TAU = (PIE2-ACOS(1.-ANR/N))/OMEGA
      GO TO 7
    6 TAU = ACOS( 1.-ANR / N ) / OMEGA
    7 IF ( PRINT .EQ. 10HNO PRINT   ) RETURN
      WRITE(6,100) N(1) , TAU(1)
  100 FORMAT(*0 THE INTERACTION INDEX IS *G21.14,23X* THE SENSITIVE TIME
     $ LAG IS * G21.14)
      RETURN
      ENTRY NORMAL
      DET OF A = TAU(1)*TAU(1)*(TAU(2)-TAU(3)) + TAU(2)*TAU(2)*(TAU(3)-T
     $AU(1)) + TAU(3)*TAU(3)*(TAU(1)-TAU(2))
      ANR = (N(1)*(TAU(2)-TAU(3)) + N(2)*(TAU(3)-TAU(1)) + N(3)*(TAU(1)-
     $TAU(2))) / DET OF A
      ANI = (TAU(1)*TAU(1)*(N(2)-N(3)) + TAU(2)*TAU(2)*(N(3)-N(1)) + TAU
     $(3)*TAU(3)*(N(1)-N(2))) / DET OF A
      ANR = 2.*ANR*TAU(2)+ANI          $ PIE2 = SQRT(ANR*ANR+1.)
      TAUVEC = ANR / PIE2     $ NVEC = -1. / PIE2
      IF ( TAU .LT. TAU(2) ) GO TO 1
      NVEC = -NVEC                $ TAUVEC = - TAUVEC
    1 WRITE(6,600) NVEC , TAUVEC
  600 FORMAT(*0 NVEC = *G21.14*  TAUVEC = *G21.14)
      RETURN
      END
```

```
      SUBROUTINE SOLSION ( OMEGA , BETAI , GAMMA , AMACH , PI )
      REAL NVEC , N2 , LOWLIM
      COMPLEX OMEGA , BETAI , BETAI2, OMEGA2 , OMEGAC , BETAIC , ANC
      COMPLEX OMEGSTR, OMEGCOR, BETASTR, WORWO , SLOPE
      COMMON / NORM / B(3) , TAU(3) , NVEC , TAUVEC , PRINT
      COMMON / ROOT / BETAI2 , BETASTR , SLOPE
      F(N2,NVEC,TAU2,TAUVEC,SIGN) = N2*N2*(NVEC*NVEC-1.) + TAU2*TAU2*(TA
     5UVEC*TAUVEC-1.) + N2*TAU2*NVEC*TAUVEC*2.
      REWIND 5
      READ(5,501)
      READ(5,500) EPSILON
      B(3) = B(1)               $ TAU(3) = TAU(1)
   30 OMEGA2 = CMPLX ( 0.0 ,0.0 )
      SOLN = 0.0
      BETAI2 = BETASTR
      BETAI2 = BETASTR - OMEGA2*SLOPE
      OMEGAC = OMEGA+EPSILON*EPSILON*OMEGA2
      BETAIC = BETAI+EPSILON*EPSILON*BETAI2
      ANC = 1./GAMMA-BETAIC/AMACH
      CALL NTAU ( ANC , OMEGAC )
   20 CONTINUE
      OMEGINC = -.10
      OMEGA2 = CMPLX ( -10.0 , 0.0 )      $ OMEGSTR = OMEGA2
      SOLN = 1.0
      PRINT = 10HNO PRINT
      KOUNT = 0
      IVEC = 1
   21 BETAI2 = BETASTR - OMEGA2*SLOPE
      OMEGAC = OMEGA+EPSILON*EPSILON*OMEGA2
      BETAIC = BETAI+EPSILON*EPSILON*BETAI2
      ANC = 1./GAMMA-BETAIC/AMACH
      CALL NTAU ( ANC , OMEGAC )
      GO TO ( 22 , 23 ) , IVEC
   22 G = F(B-B(2),NVEC,TAU-TAU(2),TAUVEC,SOLN)
      IVEC = 2
      OMEGA2 = OMEGA2 + CMPLX ( OMEGINC , 0.0 )
```

```
      GO TO 21
   23 G1 = F(B-B(2),NVEC,TAU-TAU(2),TAUVEC,SOLN)
      IF ( G1* G1.LT. 1.E-18 .OR. (G1-G)**2 .LT. 1.0E-100 ) GO TO 24
      BETAI2 = (OMEGSTR*G1-OMEGA2*G)/(G1-G)
      OMEGSTR = OMEGA2          $ OMEGA2 = BETAI2
      KOUNT = KOUNT + 1
      IF ( KOUNT .GT. 500 ) GO TO 25
      G = G1          $ GO TO 21
   24 CONTINUE
      WORW0 = OMEGAC / CMPLX(1.84118378,0.0)
      WRITE(6,606)
      WRITE(6,600) BETAI2, OMEGA2, EPSILON, OMEGAC, WORW0, BETAIC, ANC
      PRINT = 10H                  $ WRITE(6,605) B(1) , TAU(1)
      DISP = SQRT((B-B(2))*(B-B(2))+(TAU-TAU(2))*(TAU-TAU(2)))
      WRITE(6,608) DISP
      WRITE(6,701) KOUNT
      N2 = B - B(2) $ TAU2 = TAU - TAU(2) $ WRITE(6,702)N2,TAU2,G,G1
      IF ( SOLN ) 28 , 20 , 29
   28 EPSILON = EPSILON + 0.10
      IF ( EPSILON .LT. 0.51 ) GO TO 30
      B(1) = B(3)                  $ TAU(1) = TAU(3)
      RETURN
   25 WRITE(6,607) EPSILON
      GO TO 24
   29 OMEGA2 = CMPLX ( 10.0 , 0.0 )
      IVEC = 1      $ OMEGINC = -.10
      SOLN = -1.0                  $ KOUNT = 0
      PRINT = 10HNO PRINT          $ GO TO 21
  500 FORMAT(25X,F10.0)
  501 FORMAT(//)
  600 FORMAT(*0 BETAI2 EQUALS *2G21.14,12X* OMEGA2 EQUALS *2G21.14//
     $*  EPSILON EQUALS *F10.5//
     $*  OMEGAC EQUALS *2G21.14,12X* FREQUENCY RATIO EQUALS *2G21.14//
     $*  BETAIC EQUALS *2G21.14,12X* N CORRECTED EQUALS *2G21.14)
  605 FORMAT(*0 THE INTERACTION INDEX IS *G21.14,23X* THE SENSITIVE TIME
     $ LAG IS *G21.14)
  606 FORMAT(*0*135(1H$))
  607 FORMAT(*0*50(1H*)* FOR EPSILON EQUAL *F4.2* IT DID NOT CONVERGE*)
  608 FORMAT(*0 NORMAL DISPLACEMENT = *G21.14)
  701 FORMAT(*0 KOUNT EQUAL *I10)
  702 FORMAT(* N2 = *G21.14* TAU2 = *G21.14* G = *G21.14* G1 = *G21.14)
      END
```

```
      SUBROUTINE WRYTE
      INTEGER TAPE NO
      COMMON / BLKA / A(31)
      COMMON / BLKB / B(367)
      COMMON / BLKC / C(363)
      COMMON / BLOCK / D(1805)
      DATA TAPE NO / 1 /
      GO TO ( 1 , 2 ) , TAPENO
    1 TAPE NO = 2
      REWIND 1
      WRITE(1) (A(I),I=1,31) , (B(I),I=1,367) , (C(I),I=1,363) , (D(I),I
     $=1,1805)
      REWIND 1
      RETURN
    2 TAPE NO = 1
      REWIND 2
      WRITE(2) (A(I),I=1,31) , (B(I),I=1,367) , (C(I),I=1,363) , (D(I),I
     $=1,1805)
      REWIND 2
      RETURN
      ENTRY REED
      GO TO ( 4 , 3 ) , TAPENO
    3 READ(2)  (A(I),I=1,31) , (B(I),I=1,367) , (C(I),I=1,363) , (D(I),I
     $=1,1805)
      RETURN
    4 READ(1)  (A(I),I=1,31) , (B(I),I=1,367) , (C(I),I=1,363) , (D(I),I
     $=1,1805)
      RETURN
      END
```

THE FORM OF THE INPUT DATA IS

```
REFERENCE 0.930     0.00      2.700000000.330000001.20      0.9166    0.00
5.0       0.0       0.0       0.90      0.010     0.010       1  1  0  3 30NO
0.010000000.000000000.900000000.00000000
 THE VALUE OF EPSILON IS 0.10
-10.0     +0.10     +10.0
```

THIS IS ENGINE NUMBER REFERENCE

THE MACH NUMBER IS   .33000000000000

THE FREQUENCY IS   1.7307127532000   0.

THE PRIMARY MODE ASSUMED IS LHAT = 1 MHAT = 1 NHAT = 0

W/WO =   .94000000000000   0.

BETAN = 2.7499999999995E-02 0.

BETAC =   .16666666666667   0.

BETAWI =   .22841574691221   .12798019395453

BETAWID =   .22841574691221   .12798019395453

BETAI(1) = -.12561752604581   5.4219240529716BE-02

BETAI(2) = -8.8637142022792OE-02 .23427026213829

BETAI(3) = -4.74063589737710E-02 .21136353458143

BETAI(4) = -5.61807157129248E-02 .20974951898918

BETAI(5) = -5.30512371758145E-02 .21508343588394

BETAI(6) = -5.10830488334757E-02 .21411940711915

BETAI(7) = -5.10069049770763E-02 .21421061176020

THE INTERACTION INDEX IS   .70709480293063

THE INTERACTION INDEX IS   .60330643846113

THE COEFFICIENT MATRIX IS 3 X 30

THE RATIO OF SPECIFIC HEATS IS   1.2000000000000

THE LENGTH OF THE COMBUSTOR IS   2.7000000000000

THE LINER BEGINS AT 0.0000 AND ENDS AT .9000

G =   .91666666666666   0.

K =   5.0000000000000   0.

N =   .14116440329633   -.38781876955917   I

N =   .14116440329633   -.38781876955917   I

N =   1.2139925031691   -.16430072887793   I

N =   1.1019307334024   -.70990988526754   I

N =   .97698896658718   -.64049555933767   I

N =   1.0035779264028   -.63560460299751   I

N =   .99409465810853   -.65176798752709   I

N =   .98813045101053   -.64884668623986   I

N =   .98789971205174   -.64912306593999   I

THE SENSITIVE TIME LAG IS   2.48663532235916

THE SENSITIVE TIME LAG IS   3.2269990766139

THE FOLLOWING ARE THE NONLINEAR RESULTS

SECOND ORDER, NO LINER

L EQUAL 1 TO 3 M EQUAL 0

N						
0	5.476586E-03	2.623434E-02	-3.170864E-02	-4.822118E-03	6.959646E-04	4.319266E-04
1	2.501978E-02	1.856020E-02	-2.770545E-02	1.118276E-02	7.942642E-04	-6.749603E-05
2	2.510291E-02	1.055734E-03	-9.595699E-03	1.008094E-02	3.159720E-04	-2.569781E-04
3	4.306227E-02	-3.837226E-03	-3.400710E-03	4.206667E-03	1.123933E-04	-1.561282E-04
4	-1.794550E-02	9.224667E-03	-1.838304E-03	1.643988E-03	7.239246E-05	-9.773413E-05
5	-5.076112E-03	2.141271E-03	-9.515721E-04	6.812070E-04	4.859863E-05	-6.538249E-05
6	-2.680891E-03	9.841346E-04	-5.859316E-04	3.181605E-04	4.221376E-05	-4.733224E-05
7	-1.589505E-03	6.195492E-04	-3.358497E-04	1.714042E-04	3.273437E-05	-3.501692E-05
8	-1.138500E-03	3.988629E-04	-2.315352E-04	9.754164E-05	2.973775E-05	-2.661022E-05
9	-8.055139E-04	3.055401E-04	-1.477137E-04	6.540520E-05	2.397362E-05	-2.076098E-05
10	-6.448912E-04	2.214886E-04	-1.120084E-04	4.265162E-05	2.195692E-05	-1.624314E-05
11	-4.946722E-04	1.850300E-04	-7.741544E-05	3.295727E-05	1.804652E-05	-1.322593E-05
12	-4.197798E-04	1.426167E-04	-6.246527E-05	2.341124E-05	1.663394E-05	-1.060611E-05
13	-3.373175E-04	1.251356E-04	-4.607033E-05	1.975064E-05	1.388604E-05	-8.960944E-06
14	-2.967367E-04	1.001497E-04	-3.944000E-05	1.481827E-05	1.290012E-05	-7.339381E-06
15	-2.657974E-04	9.070368E-05	-3.006500E-05	1.321476E-05	1.091857E-05	-6.396755E-06
16	-2.216116E-04	7.446936E-05	-2.673689E-05	1.028430E-05	1.022844E-05	-5.331694E-06
17	-1.975476E-04	6.996082E-05	-2.100584E-05	9.513605E-06	8.762556E-06	-4.767385E-06
18	-1.721546E-04	5.767562E-05	-1.922842E-05	7.597132E-06	8.274853E-06	-4.030592E-06
19	-1.480432E-04	5.429699E-05	-1.545439E-05	7.208339E-06	7.163405E-06	-3.680104E-06
20	-1.377680E-04	4.605495E-05	-1.446722E-05	5.866417E-06	6.814560E-06	-3.147507E-06
21	-1.199545E-04	4.391304E-05	-1.183345E-05	5.669369E-06	5.952483E-06	-2.923342E-06
22	-1.128450E-04	3.766229E-05	-1.127535E-05	4.681238E-06	5.699936E-06	-2.523968E-06
23	-9.923661E-05	3.627746E-05	-9.350904E-06	4.586997E-06	5.017487E-06	-2.377413E-06
24	-9.417993E-05	3.139387E-05	-9.037503E-06	3.830955E-06	4.832694E-06	-2.068663E-06
25	-8.350662E-05	3.049167E-05	-7.578514E-06	3.794454E-06	4.282568E-06	-1.971429E-06
26	-7.982582E-05	2.658315E-05	-7.410047E-06	3.198363E-06	4.146182E-06	-1.726590E-06
27	-7.125797E-05	2.599868E-05	-6.270230E-06	3.195288E-06	3.695569E-06	-1.661618E-06
28	-6.854219E-05	2.280814E-05	-6.190064E-06	2.713856E-06	3.594283E-06	-1.463271E-06
29	-6.154016E-05	2.243760E-05	-5.277301E-06	2.730430E-06	3.219931E-06	-1.419921E-06

L EQUAL 1 TO 3 M EQUAL 2

N						
0	5.490727E-02	-8.866755E-03	9.745175E-04	5.868550E-04	-1.751481E-04	-1.257070E-04
1	4.508200E-02	-2.602548E-02	1.099791E-03	-1.052863E-04	-2.119571E-04	6.509926E-06
2	1.799997E-02	-1.564632E-02	4.339918E-04	-3.571572E-04	-8.824058E-05	6.728852E-05
3	6.315333E-03	-3.157533E-03	1.539769E-04	-2.139433E-04	-3.187117E-05	4.406729E-05
4	1.575643E-03	5.897529E-04	9.894216E-05	-1.319623E-04	-2.070757E-05	2.978788E-05
5	-1.367031E-05	8.482113E-04	6.618869E-05	-8.689787E-05	-1.399154E-05	2.166582E-05
6	-4.514522E-04	6.412530E-04	5.693546E-05	-6.184957E-05	-1.266609E-05	1.725140E-05
7	-5.100318E-04	4.635920E-04	4.374177E-05	-4.510782E-05	-1.021470E-05	1.391803E-05
8	-4.943682E-04	3.360711E-04	3.926162E-05	-3.379138E-05	-9.933980E-06	1.162529E-05
9	-4.267222E-04	2.577637E-04	3.135940E-05	-2.611841E-05	-8.445911E-06	9.732204E-06
10	-3.879992E-04	1.968220E-04	2.843851E-05	-2.023179E-05	-8.291173E-06	8.237132E-06
11	-3.233293E-04	1.609349E-04	2.320983E-05	-1.639605E-05	-7.160313E-06	7.029550E-06
12	-2.898522E-04	1.280326E-04	2.124095E-05	-1.306261E-05	-6.997237E-06	5.982839E-06
13	-2.463893E-04	1.096818E-04	1.764474E-05	-1.101794E-05	-6.077920E-06	5.189148E-06
14	-2.231965E-04	8.983507E-05	1.631031E-05	-8.985150E-06	-5.910110E-06	4.438717E-06
15	-1.918979E-04	7.957461E-05	1.375771E-05	-7.831544E-06	-5.154456E-06	3.910873E-06
16	-1.758588E-04	6.654349E-05	1.284289E-05	-6.507994E-06	-4.999709E-06	3.366016E-06
17	-1.528918E-04	6.042914E-05	1.097570E-05	-5.824658E-06	-4.379949E-06	3.012696E-06
18	-1.416030E-04	5.135316E-05	1.033872E-05	-4.913579E-06	-4.246677E-06	2.609480E-06
19	-1.243322E-04	4.749881E-05	8.934316E-06	-4.492388E-06	-3.737991E-06	2.370645E-06
20	-1.162182E-04	4.086240E-05	8.483561E-06	-3.835618E-06	-3.628442E-06	2.066169E-06

```
21   -1.029195E-04 3.834944E-05 7.400540E-06-3.567827E-06-3.209436E-06 1.902933E-06
22   -9.696979E-05 3.331521E-05 7.076857E-06-3.076073E-06-3.121698E-06 1.668162E-06
23   -8.650825E-05 3.163234E-05 6.223230E-06-2.901892E-06-2.774449E-06 1.555401E-06
24   -8.206689E-05 2.769987E-05 5.987781E-06-2.522016E-06-2.705280E-06 1.370669E-06
25   -7.368036E-05 2.655155E-05 5.301903E-06-2.407032E-06-2.415310E-06 1.292072E-06
26   -7.031305E-05 2.340563E-05 5.128947E-06-2.105900E-06-2.361345E-06 1.143943E-06
27   -6.347760E-05 2.261294E-05 4.568542E-06-2.029491E-06-2.117206E-06 1.088811E-06
28   -6.089029E-05 2.004603E-05 4.440577E-06-1.785575E-06-2.075439E-06 9.679765E-07
29   -5.523731E-05 1.949631E-05 3.975921E-06-1.734942E-06-1.868166E-06 9.291971E-07


 THE SLOPE IS  -.26334228821269        1.3680352949050

 THE INTERACTION INDEX IS   .64198611058035                          THE SENSITIVE TIME LAG IS   3.2588120513033
$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
 BETAI2 EQUALS    .55576107101676      -.76630625472799              OMEGA2 EQUALS   .49202914749944      0.

 EPSILON EQUALS      .10000

 OMEGAC EQUALS   1.7356330446750      0.                             FREQUENCY RATIO EQUALS   .94267235217278      0.

 BETAIC EQUALS   .23397335762238       .12031713140725               N CORRECTED EQUALS   .12432315872006      -.36459736790074

 THE INTERACTION INDEX IS   .59678136399513                          THE SENSITIVE TIME LAG IS   3.2414331063256

 NORMAL DISPLACEMENT =  1.58403854279776E-02

 KOUNT EQUAL          22
N2 = -6.52507446599415E-03 TAU2 =  1.44340297117651E-02 G = -1.52831505220372E-09 G1 = -5.83765308228429E-10
$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
 BETAI2 EQUALS    .55588459735059      -.76694796094876              OMEGA2 EQUALS   .49249821889194      0.

 EPSILON EQUALS      .10000

 OMEGAC EQUALS   1.7356377353889      0.                             FREQUENCY RATIO EQUALS   .94267489983477      0.

 BETAIC EQUALS   .23397459288572       .12031071434504               N CORRECTED EQUALS   .12431941549783      -.36457792225769

 THE INTERACTION INDEX IS   .59673856200702                          THE SENSITIVE TIME LAG IS   3.2414161702489

 NORMAL DISPLACEMENT =  1.58426509776387E-02

 KOUNT EQUAL          26
N2 = -6.56787645410262E-03 TAU2 =  1.44170936350889E-02 G = -1.24575802668350E-09 G1 = -4.75836561559390E-10

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 SECOND ORDER, W/ LINER
vvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvvv

 N             L EQUAL 1 TO  3        M EQUAL 0
  0    5.109542E-03 2.792276E-02-3.241071E-02-6.509633E-03 5.745803E-04 1.095831E-03
  1    2.389123E-02 2.182683E-02-2.917372E-02 8.445653E-03 6.274194E-04 1.074706E-03
  2    2.232905E-02 5.487209E-03-1.059782E-02 8.447743E-03 3.168123E-04 4.373469E-04
  3    2.712535E-02 7.651223E-03-4.094402E-03 3.566390E-03 2.804066E-04 1.030347E-04
  4   -1.431050E-02-5.032299E-03-2.442632E-03 1.486464E-03 3.924817E-04-7.117815E-05
  5   -3.123203E-03-2.149497E-03-1.293528E-03 5.663515E-04 2.095619E-04-6.891315E-05
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8	-7.039016E-04	-8.449697E-04	-1.715214E-04	-6.023984E-06	-9.935337E-06	4.036249E-05
9	-4.486261E-04	-6.927471E-04	-1.445856E-04	-6.048763E-06	2.760096E-05	1.956820E-05
10	-3.483257E-04	-4.057507E-04	-1.421220E-04	-8.257095E-06	5.491946E-05	5.673362E-06
11	-2.538259E-04	-3.220271E-04	-1.002563E-04	-6.798585E-06	4.431451E-05	4.294080E-06
12	-2.304602E-04	-2.769272E-04	-6.023890E-05	-1.050042E-05	1.677986E-05	9.282497E-06
13	-1.847132E-04	-2.273979E-04	-2.692709E-05	-1.018551E-05	-4.151601E-06	1.228129E-05
14	-1.665212E-04	-2.005113E-04	-2.277739E-05	-1.137470E-05	-3.321151E-06	1.163936E-05
15	-1.283664E-04	-1.686454E-04	-2.594381E-05	-9.191915E-06	8.241889E-06	8.441177E-06
16	-1.155054E-04	-1.520646E-04	-3.159401E-05	-8.767690E-06	1.757747E-05	6.110300E-06
17	-9.393182E-05	-1.306913E-04	-2.503670E-05	-6.928080E-06	1.496740E-05	5.090282E-06
18	-9.168932E-05	-1.197554E-04	-1.654994E-05	-6.973356E-06	6.292518E-06	5.451183E-06
19	-7.817907E-05	-1.043037E-04	-7.793395E-06	-5.980736E-06	-1.042284E-06	5.544169E-06
20	-7.493956E-05	-9.656838E-05	-7.549912E-06	-6.160315E-06	-6.830574E-07	5.349255E-06
21	-6.144959E-05	-8.497433E-05	-9.372549E-06	-5.236974E-06	3.797885E-06	4.486923E-06
22	-5.832579E-05	-7.950753E-05	-1.232062E-05	-5.149901E-06	7.791919E-06	3.878239E-06
23	-4.942652E-05	-7.[illegible]72341E-05	-1.024118E-05	-4.280672E-06	6.839449E-06	3.368082E-06
24	-4.960520E-05	-6.675075E-05	-7.249125E-06	-4.247662E-06	3.211949E-06	3.329133E-06
25	-4.339177E-05	-5.980756E-05	-3.647792E-06	-3.662267E-06	-1.177493E-07	3.174026E-06
26	-4.280435E-05	-5.676739E-05	-3.780088E-06	-3.744798E-06	5.996416E-08	3.096934E-06
27	-3.620635E-05	-5.114861E-05	-4.697872E-06	-3.267800E-06	2.178681E-06	2.751492E-06
28	-3.534269E-05	-4.884776E-05	-6.324922E-06	-3.275298E-06	4.217328E-06	2.542447E-06
29	-3.061065E-05	-4.430263E-05	-5.380067E-06	-2.803739E-06	3.766481E-06	2.264466E-06

N L EQUAL 1 TO 3 M EQUAL 2

N						
0	5.272515E-02	-5.956541E-03	1.144470E-03	-1.298522E-04	-2.863508E-04	2.273800E-04
1	4.193029E-02	-2.206012E-02	1.357230E-03	-1.318516E-03	-3.803107E-04	6.122587E-04
2	1.517787E-02	-1.585454E-02	4.977185E-04	-1.044438E-03	-1.307637E-04	4.292651E-04
3	4.792353E-03	-6.538881E-03	1.426215E-05	-3.937355E-04	4.724295E-05	1.576817E-04
4	1.935442E-03	-2.882926E-03	-1.237782E-04	-4.925419E-05	1.067581E-04	6.370956E-07
5	6.449676E-04	-1.548242E-03	-8.926825E-05	2.175480E-05	7.716002E-05	-2.853547E-05
6	-1.043793E-04	-9.433949E-04	3.277207E-05	-1.860183E-05	4.292870E-06	6.683205E-06
7	-3.379501E-04	-6.480821E-04	9.939204E-05	-4.550792E-05	-4.254553E-05	3.531779E-05
8	-3.224479E-04	-4.986830E-04	8.608664E-05	-3.279953E-05	-3.967102E-05	3.117414E-05
9	-1.914437E-04	-4.004164E-04	2.553422E-05	-5.908087E-06	-4.791933E-06	8.607683E-06
10	-1.296580E-04	-3.386060E-04	-1.219840E-05	1.119999E-05	2.018807E-05	-7.912454E-06
11	-1.369378E-04	-2.808008E-04	-1.153740E-05	1.974193E-05	1.761115E-05	-8.995415E-06
12	-1.363861E-04	-2.405091E-04	1.425758E-05	7.819060E-06	-3.960075E-06	-1.512899E-06
13	-1.369919E-04	-2.012484E-04	3.000055E-05	9.149818E-06	-1.932486E-05	4.183278E-06
14	-1.268804E-04	-1.763056E-04	2.665842E-05	3.552449E-06	-1.764943E-05	3.619350E-06
15	-9.060700E-05	-1.513193E-04	9.611257E-06	5.245775E-06	-3.838977E-06	-3.909414E-07
16	-7.329397E-05	-1.364198E-04	-1.765013E-06	6.937439E-06	6.350708E-06	-3.550603E-06
17	-6.145008E-05	-1.194269E-04	-2.095170E-06	6.643039E-06	5.863993E-06	-3.631573E-06
18	-6.875369E-05	-1.089392E-04	6.376049E-06	5.738332E-06	-2.610488E-06	-1.989217E-06
19	-6.613983E-05	-9.594413E-05	1.194867E-05	4.499974E-06	-8.883873E-06	-5.166254E-07
20	-6.338706E-05	-8.818085E-05	1.108113E-05	4.198249E-06	-8.242394E-06	-4.972391E-07
21	-4.905569E-05	-7.839423E-05	4.621986E-08	4.069174E-06	-2.187889E-06	-1.279467E-06
22	-4.290264E-05	-7.303358E-05	9.328852E-08	4.285704E-06	2.425321E-06	-2.038642E-06
23	-3.691962E-05	-6.571644E-05	-1.874741E-07	4.000258E-06	2.335956E-06	-2.000580E-06
24	-4.022210E-05	-6.173597E-05	3.507887E-06	3.748012E-06	-1.614869E-06	-1.544592E-06
25	-3.831508E-05	-5.578828E-05	6.020357E-06	3.235554E-06	-4.631731E-06	-1.039974E-06
26	-3.753343E-05	-5.259042E-05	5.749182E-06	3.064248E-06	-4.381412E-06	-9.777572E-07
27	-3.836194E-05	-4.773186E-05	2.675551E-06	2.833238E-06	-1.333939E-06	-1.123779E-06
28	-2.772258E-05	-4.534255E-05	4.813302E-07	2.846212E-06	1.045595E-06	-1.345668E-06
29	-2.428152E-05	-4.148637E-05	2.813373E-07	2.648508E-06	1.048751E-06	-1.295646E-06

NVEC = -.99842661286642 TAUVEC = 5.60740467604330E-02

THE SLOPE IS -.50450142588528 1.4820197171086

THE INTERACTION INDEX IS .70583873874668

THE SENSITIVE TIME LAG IS 2.4844325867333

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BETAI2 EQUALS -5.693979277721 68E-02-6.71601683625633E-03          OMEGA2 EQUALS -5.12061062438897E-02 0.
EPSILON EQUALS     .10000
OMEGAC EQUALS   1.7302006921376      0.                            FREQUENCY RATIO EQUALS   .939721B8487210     0.
BETAIC EQUALS -5.15763029048484E-02  .21414345159184               N CORRECTED EQUALS   .98962516031772    -.64891955027829
THE INTERACTION INDEX IS   .70756817673149                          THE SENSITIVE TIME LAG IS   2.4866293693257
NORMAL DISPLACEMENT =  4.73411246830655E-04
KOUNT EQUAL        20
N2 =  4.73373800861765E-04 TAU2 = -5.95426598692939E-06 G = -1.11089916470696E-09 G1 = -4.24322386190641E-10
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BETAI2 EQUALS -5.64410242473306E-02-8.18119563979991E-03           OMEGA2 EQUALS -5.02174697320870E-02 0.
EPSILON EQUALS     .10000
OMEGAC EQUALS   1.7302105785027      0.                            FREQUENCY RATIO EQUALS   .93972725444207     0.
BETAIC EQUALS -5.15713152195494E-02  .21412879980380               N CORRECTED EQUALS   .98961004611984    -.64887515092060
THE INTERACTION INDEX IS   .70753475591433                          THE SENSITIVE TIME LAG IS   2.4865869885060
NORMAL DISPLACEMENT =  4.42600167614358E-04
KOUNT EQUAL        21
N2 =  4.39952983700920E-04 TAU2 = -4.83350856512743E-05 G = -1.45677941342082E-09 G1 = -5.56445245766485E-10
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