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# ULTRASONIC DISPERSION ($\Delta v/v$) DETERMINED FROM MECHANICAL RESONANCE FREQUENCY SHIFTS †

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ABSTRACT. With standing wave ultrasonic techniques, small changes in phase velocity which result from changes in some external parameter (e.g., temperature or magnetic field) have traditionally been determined by observing shifts in the mechanical resonance frequency of a composite resonator. Some previous investigators have assumed that the fractional change in velocity $\Delta v/v$ is equal to the fractional change in frequency $\Delta\nu/\nu$. We discuss quantitatively the errors involved in such an approach. We show that using the relation $\Delta v/v = \Delta\nu/\nu$ results in substantial inaccuracies when the loading effect of the transducer(s) cannot be neglected. Substantially improved formulas for determining the dispersion are presented and one of these is shown to be much more accurate than all previous approximations. The results of simulated and actual experiments over wide ranges of dispersion $\Delta v/v$, transducer loading parameter $\delta \equiv \rho_t \ell_t/\rho_s \ell_s$, and frequency are analyzed in order to compare the errors inherent in the various approximations.

## I. Introduction

In a variety of ultrasonic experiments one monitors changes in acoustic phase velocity in a specimen that result from variations in some external parameter, e.g., magnetic field, pressure, or temperature. Standing wave ultrasonic techniques are well suited to measurements of this sort.[1-3] Fractional changes $\Delta v/v$ in phase velocity can be determined from measurements of the shifts in frequency of a standing wave mechanical resonance. To analyze the data, the relation $\Delta v/v \approx \Delta\nu/\nu$ is commonly used, where $\Delta\nu/\nu$ is the fractional change in mechanical resonance frequency. Because measurements are usually made on composite resonators consisting of a specimen plus one or two transducers, this "uncorrected formula" is only approximate. The uncorrected formula results in substantial errors in the estimation of the ultrasonic dispersion $\Delta v/v$ for a variety of cases of experimental interest.

In this paper, we present substantially more accurate formulas for determining the dispersion $\Delta v/v$ from standing wave ultrasonic measurements. Using numerical simulations we compare quantitatively the errors resulting from use of the new approximations with those resulting from use of the uncorrected formula. In order to demonstrate the significance of the new dispersion formulas, experimental data are analyzed using the conventional and the improved formulas, and the resulting values for the dispersion are compared.

## II. Theory

### A. Reflection Case (One Transducer)

We consider an ultrasonic resonator consisting of a specimen (properties labeled with subscript s) and one transducer (subscript t). The velocity of sound $v_s$ is given by

$$v_s = 2\ell_s \nu_s^n/n \quad , \tag{1}$$

where $\ell_s$ is the length of the sample and $\nu_s^n$ is the frequency of the $n^{th}$ (sample only) mechanical resonance. In the limit that the transducer has no effect on the mechanical resonance frequencies of the sample, $\nu_s^n$ is equal to $\nu_c^n$, the measured mechanical resonance frequency. Thus,

$$\Delta v_s/v_s = \Delta\nu_c^n/\nu_c^n + \Delta\ell_s/\ell_s \quad . \tag{2}$$

Limiting the discussion to cases where the last term in Eq. (2) can be neglected, one obtains the "uncorrected formula" for the dispersion

$$\Delta v_s/v_s \approx \Delta\nu_c^n/\nu_c^n \quad . \tag{3}$$

Equations (2) and (3) ignore the loading effects of the transducer. For the case where the transducer loading parameter $\delta = \rho_t\ell_t/\rho_s\ell_s$ is not too large, a more accurate expression for the mechanical resonance frequency $\nu_c^n$ of the composite resonator is[4]

$$\nu_c^n = \nu_s^n + \delta(\nu_t - \nu_s^n) \quad , \tag{4}$$

where $\nu_t$ is the (unloaded) transducer resonance frequency. Using Eqs. (1)-(4) and restricting to the case where $\nu_c^n$ is not too far from $\nu_t$, one obtains after some manipulation the "(1+δ) formula" for the dispersion,

$$\Delta v_s/v_s = (\Delta\nu_c^n/\nu_c^n)(1+\delta) \quad . \tag{5}$$

Anticipating the results of Section III, Eqs. (3) and (5) typically exhibit errors substantially larger than those resulting from experimental inaccuracies.

In order to develop a more accurate formula for the dispersion $\Delta v_s/v_s$, we begin with the one transducer resonance condition[5]

$$\tan\left[\frac{2\pi\ell_s\nu_c}{v_s}\right] = -\frac{\rho_t v_t}{\rho_s v_s} T \quad . \tag{6}$$

Here $T = \tan(\pi\nu_c/\nu_t)$, and $\rho_t$ and $\rho_s$ are the densities of the transducer and sample, respectively. For convenience we have suppressed the superscript n which specifies the particular mechanical resonance which is being monitored. After a change in the external parameter (e.g., magnetic field), the same resonance equation relates the new velocity $v_s^*$ to the new composite resonance frequency $\nu_c^*$,

$$\tan\left[\frac{2\pi\ell_s\nu_c^*}{v_s^*}\right] = -\frac{\rho_t v_t}{\rho_s v_s^*} T^* \quad , \tag{7}$$

where $T^* = \tan(\pi\nu_c^*/\nu_t)$. (We are assuming negligible changes in $\ell_s$.)

We seek the value of $v_s^*$, given the values of $\nu_c^*$, $\nu_c$, $v_s$, and the remaining parameters. Subtracting Eq. (6) from Eq. (7) and simplifying, one obtains

$$\tan\left[2\pi\ell_s\left(\frac{\nu_c^*}{v_s^*} - \frac{\nu_c}{v_s}\right)\right] = \frac{X\cos^2(2\pi\ell_s\,\nu_c/v_s)}{1 + \frac{X}{2}\sin(4\pi\ell_s\,\nu_c/v_s)} \tag{8}$$

where $X = -\frac{\rho_t v_t}{\rho_s}\left(\frac{T^*}{v_s^*} - \frac{T}{v_s}\right)$. The argument of the tangent function on the left side of Eq. (8) is small over a wide range of parameters, so we keep only the first



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term in the series expansion. The resulting equation is quadratic in $v_s^*$,

$$A(v_s^*)^2 + B\, v_s^* + C = 0 \,, \tag{9a}$$

where $$A = \nu_c + T\delta\nu_t M, \tag{9b}$$

$$B = -T^* v_s \delta\nu_t M - NT - v_s \nu_c^*, \tag{9c}$$

$$C = NT^* \nu_s \tag{9d}$$

and where

$$M = \frac{\ell_s \nu_c}{v_s} \sin(4\pi\ell_s \nu_c/v_s) + \frac{1}{\pi}\cos^2(2\pi\ell_s\, \nu_c/v_s), \tag{10a}$$

$$N = \ell_s \delta\nu_t\, \nu_c^* \sin(4\pi\ell_s \nu_c/v_s) \quad . \tag{10b}$$

The resulting one transducer formulas for $v_s^*$ and the dispersion $\Delta v_s/v_s$ are given by

$$v_s^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{11a}$$

$$\Delta v_s/v_s = (v_s^* - v_s)/v_s \quad . \tag{11b}$$

(The plus sign in Eq. (11a) is required.)

In Section III, we investigate the behavior of the uncorrected formula [Eq. (3)], the $(1+\delta)$ formula [Eq. (5)], and the present result [Eq. (11)] using both numerical simulations and experimental data.

B. Transmission Case (Two Transducers)

For a composite resonator consisting of a sample and two transducers, the appropriate resonance equation is[6]

$$\tan\theta_s - 2\left(\frac{r-1}{r+1}\right)\tan\theta_t - \left(\frac{r-1}{r+1}\right)^2 \tan^2\theta_t \tan\theta_s = 0, \tag{12}$$

where

$$\theta_s = 2\pi\ell_s \nu_c^n/v_s \quad ,$$

$$\theta_t = \pi\nu_c^n/\nu_t \quad ,$$

$$r = (\rho_s v_s - \rho_t v_t)/(\rho_s v_s + \rho_t v_t).$$

This equation can be factored into two simpler resonance equations which are individually quite similar to the resonance equation for the one transducer case [Eq. (7)]:

$$\tan\left[\frac{\pi\ell_s \nu_c}{v_s}\right] = -\frac{\rho_t v_t}{\rho_s v_s} T \tag{13}$$

$$\cot\left[\frac{\pi\ell_s \nu_c}{v_s}\right] = \frac{\rho_t v_t}{\rho_s v_s} T \tag{14}$$

The solutions to Eq. (13) (the "tangent set" of resonances) describe only alternate members of the full set of resonances implicitly defined by Eq. (12). The solutions of Eq. (14) (the "cotangent set" of resonances) represent the other half of the full set of resonances.

Using Eq. (13) and analytical techniques similar to those outlined for the one transducer case, an improved dispersion formula for the tangent set of resonances is obtained. The result is identical in form with Eqs. (9) and (11), but requires the following redefinitions of M and N, replacing Eq. (10):

$$M = \frac{\ell_s \nu_c}{v_s} \sin(2\pi\ell_s \nu_c/v_s) + \frac{2}{\pi}\cos^2(\pi\ell_s \nu_c/v_s) \tag{15a}$$

$$N = \ell_s \delta\nu_t \nu_c^* \sin(2\pi\ell_s \nu_c/v_s) \tag{15b}$$

The dispersion formula defined by Eqs (9), (11), and (15) does not apply to the cotangent set of resonances. The corresponding formula for determining the dispersion from transmission measurements on a cotangent resonance [Eq. (14)] is given by Eqs. (9) and (11) with a second redefinition of M and N:

$$M = \frac{-\ell_s \nu_c}{v_s} \sin(2\pi\ell_s \nu_c/v_s) + \frac{2}{\pi}\sin^2(\pi\ell_s \nu_c/v_s) \tag{16a}$$

$$N = -\ell_s \delta\nu_t \nu_c^* \sin(2\pi\ell_s \nu_c/v_s) \quad . \tag{16b}$$

In order to analyze data in the two transducer case, one must determine whether the mechanical resonance being monitored is a member of the tangent or the cotangent set. One first calculates the number C, given by

$$C = \nu_t/\Delta\nu_s \;, \tag{17}$$

where $\Delta\nu_s \equiv v_s/2\ell_s$. Let C' = [C], the greatest integer less than or equal to C'. If C' is odd, the first resonance $\nu_c$ above $\nu_t$ is a tangent resonance. If C' is even, the first resonance $\nu_c$ above $\nu_t$ is a cotangent resonance. This allows an unequivocal determination of whether dispersion monitored with any particular mechanical resonance should be calculated using Eqs. (9), (11) and (15), or using Eqs. (9), (11), and (16). In Section III we investigate the behavior of the various approximations to $\Delta v_s/v_s$ over a wide range of conditions.

## III. Discussion

In this section, we examine the behavior of the uncorrected formula [Eq. (3)] , the $(1+\delta)$ formula [Eq. (5)] , and the present result [Eqs. (9)-(11)]. A numerical simulation technique is used to investigate the errors that result from the use of each of the three approximations to $\Delta v_s/v_s$. Data from an actual dispersion experiment are also analyzed. Since the dispersion formulas for the transmission case [Eqs. (9), (11), and (15) or (16)] exhibit a behavior very similar to that for the reflection case, only the results of the analysis of the reflection case [Eqs. (9)-(11)] are presented.

In order to study analytically the errors resulting from the various approximations, we used computer iteration to find to 5 parts in $10^{13}$ the solutions $\nu_c$ of Eq. (6) with assumed values for $v_s$, $\rho_s$, $\ell_s$, $v_t$, $\rho_t$, and $\nu_t$. An assumed value of dispersion $\Delta v_s/v_s \equiv (v_s^* - v_s)/v_s$ defines the shifted phase velocity $v_s^*$. Repeating the iteration process with this new value for the phase velocity yields values for the shifted mechanical resonance frequencies $\nu_c^*$. The approximate formulas [Eqs. (3), (5), and (9)-(11)] were used to obtain estimates of $\Delta v_s/v_s$ for each $\nu_c$, $\nu_c^*$ pair. The percent error for each approximation is defined with respect to the assumed value of $\Delta v_s/v_s$. The parameters were chosen so that the errors of the various approximations could be studied as functions of the magnitude of the dispersion $\Delta v_S/v_S$, the size of $\delta$, and the distance in frequency of the mechanical resonance $\nu_c$ from $\nu_t$.

In Figures 1 and 2 we present the results of our analysis of Eqs. (3), (5), and (9)-(11), with the magnitude of the dispersion $\Delta v_S/v_S$ as the independent variable. Figure 1 treats the case where $\nu_c$ is close to $\nu_t$. (In this case $\nu_c$ was chosen to be the first mechanical resonance above $\nu_t$.) Results for two values of $\delta$ are presented: $\delta$ small (0.005), a value typical of experiments in solids, and $\delta$ large (0.2), a value typical of liquid experiments. In any specific experiment, $\delta$ is constant, so the three curves for each value of $\delta$ are to be viewed as a group.

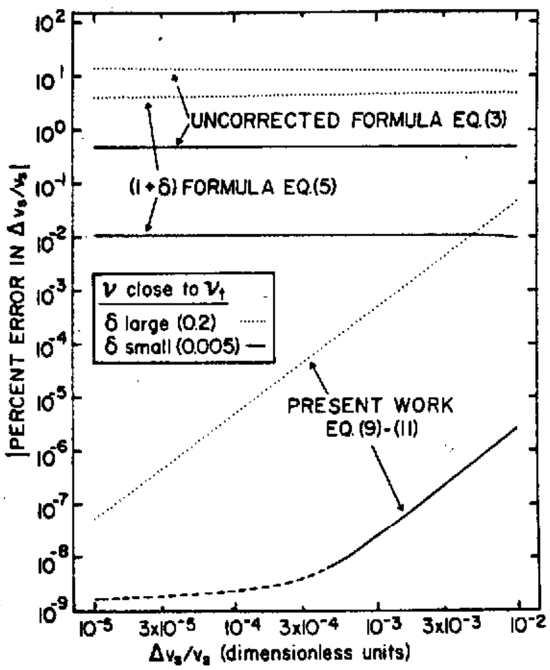


Figure 1. Absolute value of the percent error in $\Delta v_S/v_S$ versus $\Delta v_S/v_S$ for each of the three approximate formulas and for two values of $\delta \equiv \rho_t \ell_t/\rho_s \ell_s$, using the first mechanical resonance $\nu_c$ above $\nu_t$.

For $\delta$ large or small, the uncorrected formula [Eq. (3)] and the $(1+\delta)$ formula [Eq. (5)] exhibit errors that are essentially constant over a wide range of $\Delta v_S/v_S$. This result is discussed below. For $\delta = 0.005$, use of the uncorrected formula results in errors of approximately 0.5%, while use of the $(1+\delta)$ formula results in errors of about 0.01%. By contrast, the present result [Eqs. (9)-(11)] is far superior, yielding errors from 3 to at least 6 orders of magnitude smaller than either of the other approximations. Since data for $\nu_c$ and $\nu_t^s$ accurate to parts in $10^6$ or $10^7$ are available from experiments,[3] the increased accuracy provided by Eqs. (9-11) is required. (Since the final accuracy of our double-precision computer calculations is limited to about $10^{-8}$%, we plot a dashed line where the error becomes less than this value. In this region, we can only establish an upper bound for the error.) For the larger value of $\delta = 0.2$, the errors for all three approximations increase, but the qualitative features of the curves are maintained.

In Figure 2 we present the percent error as a function of $\Delta v_S/v_S$ using a resonance $\nu_c$ that is far from $\nu_t$. (The 6th mechanical resonance above $\nu_t$ was chosen.) The shapes of the various error curves are similar to those presented in Figure 1, with the results of the present work offering a dramatic improvement over previous approximations.

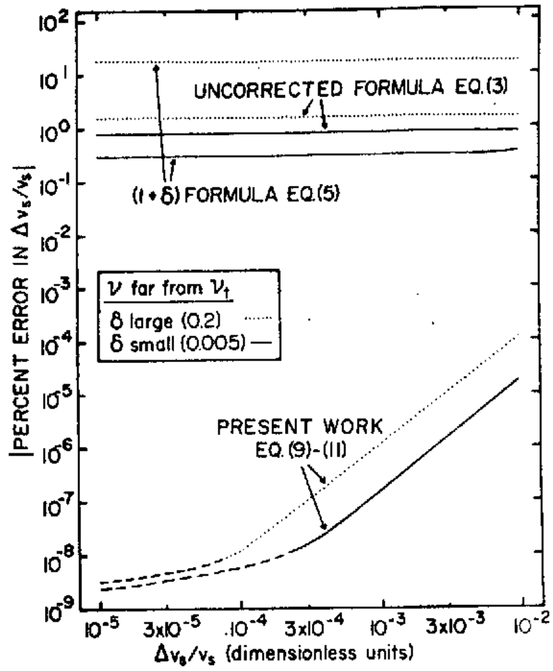


Figure 2. Absolute value of the percent error in $\Delta v_S/v_S$ versus $\Delta v_S/v_S$, with $\nu_c$ chosen to be the sixth mechanical resonance above $\nu_t$.

The calculations required for the use of the improved formula for the dispersion are more complex than those required for the use of previous approximations. An examination of the behavior of the uncorrected formula [Eq. (3) and Figs. 1 and 2], however, indicates that under a variety of conditions Eqs. (9)-(11) need only be applied once for any given experiment. One selects data corresponding to a small value of dispersion and uses Eqs. (9)-(11) to compute an approximate value of $\Delta v_S/v_S$. The nearly exact value for $\Delta v_S/v_S$ provided by this single application of Eqs. (9)-(11) yields a simple multiplicative factor which can be applied to the mechanical resonance frequency shifts $(\Delta\nu/\nu)$ to obtain the dispersion. (This is analogous to the use of the $(1+\delta)$ formula, with an "effective $\delta$" obtained using Eqs. (9)-(11) for a particular set of experimental parameters.)

In Figures 3 and 4 are presented the percent errors of the various approximations as functions of the transducer loading parameter $\delta$ for two discrete choices of $\Delta v_S/v_S$: $\Delta v_S/v_S$ large ($10^{-2}$), and $\Delta v_S/v_S$ small ($5 \times 10^{-5}$). As can be anticipated from the horizontal curves in Figures 1 and 2, the behavior of the uncorrected formula and the $(1+\delta)$ formula is independent of the size of $\Delta v_S/v_S$. The mechanical resonances were chosen to be the 1st and 6th above $\nu_t$ in Figures 3 and 4, respectively. (The cusp-like behavior near $\delta = 0.04$ in Figure 4 is due to a change in sign of the error.)

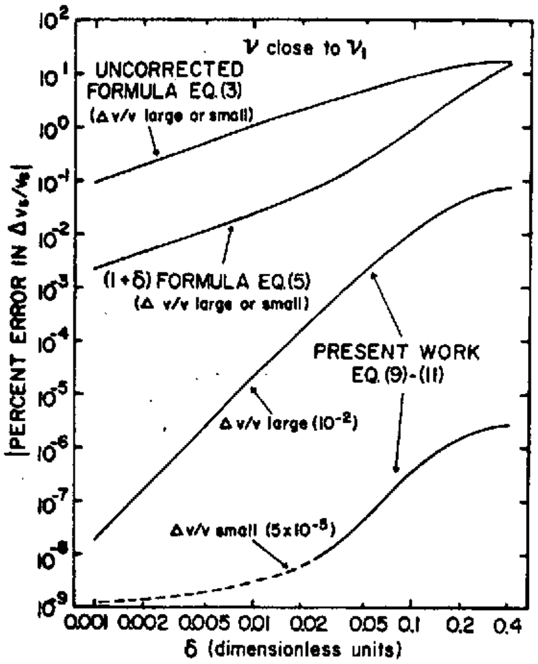


Figure 3. Absolute value of the percent error in $\Delta v_s/v_s$ versus $\delta \equiv \rho_t\ell_t/\rho_s\ell_s$ for $\Delta v_s/v_s$ large ($10^{-2}$) and small ($5 \times 10^{-5}$). $\nu_c$ close to $\nu_t$.

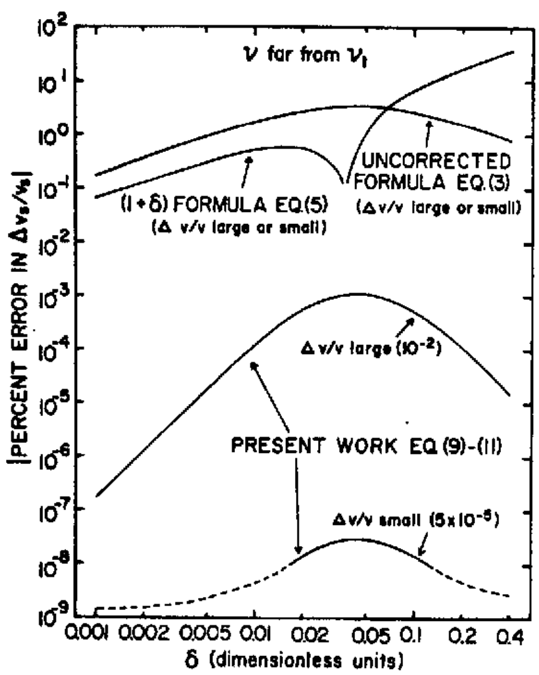


Figure 4. Absolute value of the percent error in $\Delta v_s/v_s$ versus $\delta$ for $\Delta v_s/v_s$ large and small. $\nu_c$ far from $\nu_t$.

The absolute values of the percent errors for the several approximations are plotted in Figures 5, 6, and 7 as functions of the frequency $\nu_c$ relative to the unloaded transducer resonance frequency $\nu_t$ (taken as 5 MHz here). (Although smooth curves are shown, in a particular experiment only discrete values of $\nu_c$ occur, corresponding to peaks of mechanical resonances. The resonances are spaced at 200 kHz intervals for the parameters used in Figures 5, 6, and 7.) Results are presented for small dispersion with small and large values of $\delta$ in Figure 5, and for large dispersion with $\delta = 0.005$ in Figure 6 and with $\delta = 0.2$ in Figure 7. The results of the present work are seen to represent substantial improvements in accuracy for all choices of parameters.

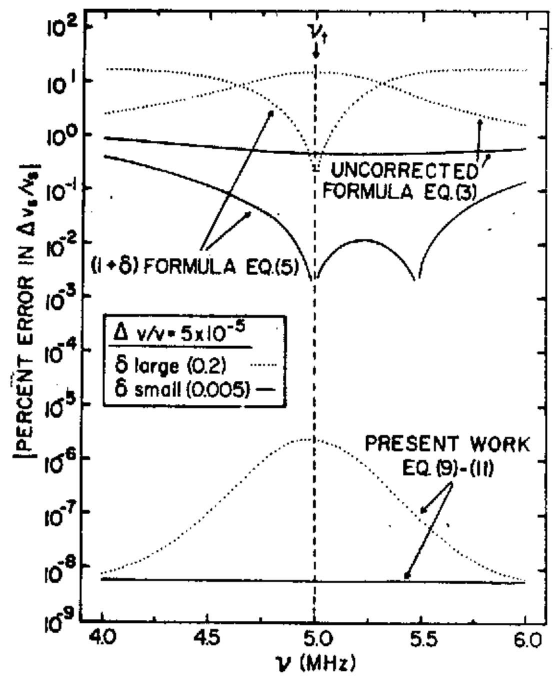


Figure 5. Absolute value of the percent error in $\Delta v_s/v_s$ versus mechanical resonance frequency $\nu_c$ for $\Delta v_s/v_s = 5 \times 10^{-5}$.

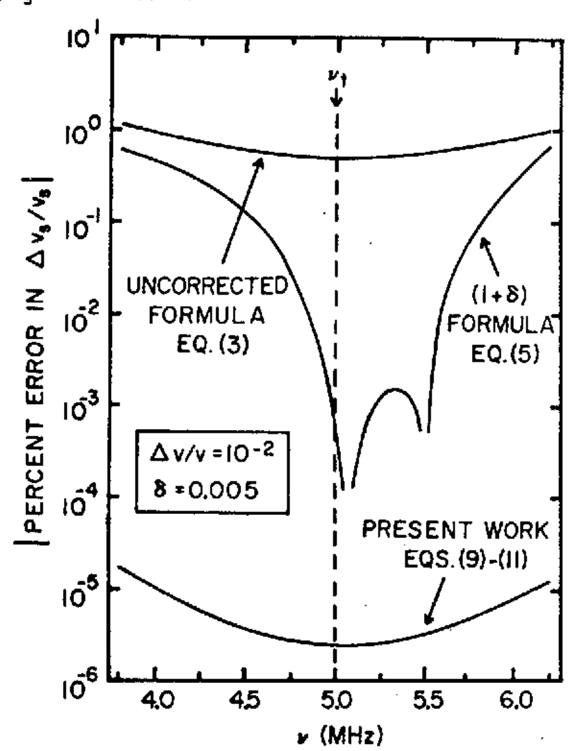


Figure 6. Absolute value of the percent error in $\Delta v_s/v_s$ versus mechanical resonance frequency $\nu_c$ with $\Delta v_s/v_s$ large ($10^{-2}$) and $\delta$ small (0.005).

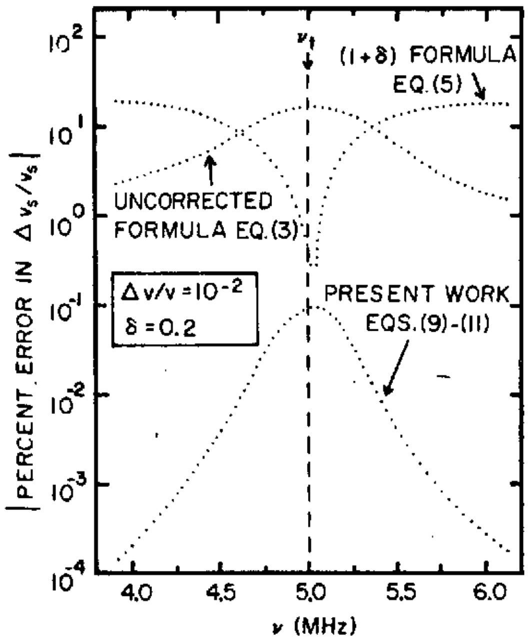


Figure 7. Absolute value of the percent error in $\Delta v_S/v_S$ versus frequency $\nu_C$ with $\Delta v_S/v_S$ large ($10^{-2}$) and $\delta$ large (0.2).

As an example of the use of the present work to analyze the results of an actual experiment, we consider magnetoelastic dispersion measurements in single crystal Ni.[7] The specimen under study is a disk of length 0.1887 cm with a diameter to thickness ratio of about 10:1. The resulting $\delta$ for a 10.00 MHz quartz transducer is 0.0311. Crystalline axes [001], <111>, and [1$\bar{1}$0], and an external magnetic field $H_0$ lie in the plane of the disk. We define $\theta_0$ as the angle between $H_0$ and the [001] direction. Transverse ultrasonic waves of frequency $\sim$8.9 MHz were propagated along the [110] axis perpendicular to the plane of the disk. The frequency $\nu_C$ of a particular mechanical resonance was measured as a function of $H_0$ and $\theta_0$. The ultrasonic dispersion $\Delta v_S/v_S$ was calculated using $\theta_0 = 90^\circ$, $H_0$ = 10 kOe as the uncoupled magnetic field orientation. In Figure 8, the data have been reduced using the uncorrected formula and the present work. Even on this linear plot, the improvement provided by the present work is evident.

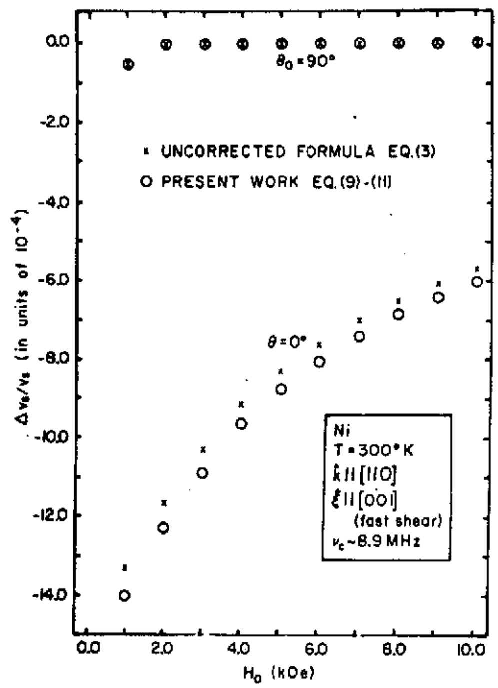


Figure 8. Ultrasonic dispersion ($\Delta v_S/v_S$) in bulk single crystal Ni versus magnetic field $H_0$ for $\theta_0 = 0^\circ$ and $\theta_0 = 90^\circ$. Dispersion determined using the uncorrected formula and the present work.

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## REFERENCES


[1] R. L. Melcher and D. I. Bolef, Phys. Rev. 186, 491 (1969).

[2] P. A. Fedders, I. Wu, J. G. Miller, and D. I. Bolef, Phys. Rev. Letters 32, 1443 (1974).

[3] D. I. Bolef and J. G. Miller, in Physical Acoustics, Vol. 8, edited by W. P. Mason and R. N. Thurston (Academic Press, New York, 1971).

[4] J. G. Miller and D. I. Bolef, J. Appl. Phys. 39, 4589 (1968).

[5] H. I. Ringermacher, W. E. Moerner, and J. G. Miller, J. Appl. Phys. 45, 549 (1974).

[6] H. I. Ringermacher, W. E. Moerner, and J. G. Miller, Proc. IEEE 1974 Ultrasonics Symposium.

[7] V. E. Stubblefield, W. E. Moerner, P. A. Fedders, J. G. Miller, and D. I. Bolef, Proc. IEEE 1974 Ultrasonics Symposium.