STRONGLY MAGNETIZED CLASSICAL PLASMA MODELS

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reprint from the Proceedings of the 1972 Les Houches Summer School of Theoretical Physics, J. Peyraud and C. DeWitt, editors; published by Gordon & Breach, New York, 1974. (Univ. of Iowa Report 72-17)

CONTENTS

INTRODUCTION

Every beginning course in classical mechanics has as an exercise the derivation of the Rutherford scattering cross section for two classical Coulomb point charges. Many people are surprised to learn that this is still apparently an unsolved problem if a uniform dc magnetic field is assumed to be present. Since the interaction of classical point charges is fundamental for all discrete-particle processes that go on in most plasmas, this simple example should make clear that there are many things we do not know about such processes. The situation becomes worse if we add in the complexities associated with the long range of the Coulomb force, and the statistical mechanical side of the problem.

There is a seriously underpopulated border region of plasma physics that lies between controlled thermonuclear research and the older, less volatile society of classical statistical mechanics. This territory was initially explored in classic papers of Landau [1,2], Vlasov, [3], and Bogolyubov [4]. Ten or twelve years ago a few permanent settlements appeared to have been established there. Life on this rugged frontier proved to be harsh, however, and many of these settlements have now been abandoned. One hope of these lectures will be to recruit some vigorous pioneers who will be willing to try again at the neglected task of developing this important area.

Our subject matter may be roughly defined as that class of plasma processes for which the so-called "Vlasov approximation" is inadequate [5]. Such phenomena include: equations of state and other equilibrium thermodynamic relations, thermal relaxation phenomena, transport properties such as diffusion and electrical conduction, and microscopic statistical fluctuations in such quantities as the electric field and the charge density. All these may be loosely called "discrete particle" processes.

Convincing calculations of these quantities have been given in some cases for the case of a plasma in the absence of an external magnetic field. Calculations in the presence of a strong magnetic field are usually absent or less than convincing, though progress has been made within the last year or two. Reliable laboratory measurements of these quantities, that would pass muster in other branches of physics, are usually lacking, both with and without the external magnetic field.

Results in this area come slowly, and it is rather easy to make embarrassing conceptual mistakes, even though it is only classical physics we are dealing with. There is also great economic pressure for plasma physics to show results in a hurry, more rapidly than it is realistic to expect these rather deep fundamental problems to sort themselves out. To yield to this economic pressure is to gamble upon achieving practical success in plasma situations *without needing* to understand these phenomena. Such a gamble may be justified for some people, but I think it would be unwise for *all* of us to take it.

The question of why practitioners of orthodox statistical mechanics have largely stayed away from these problems is more difficult to answer. Their training and extensive experience with similar problems in neutral particle systems would appear to be an excellent preparation for plasmas. Perhaps it has to do with their perceptions of the sometimes desparate professional style of the plasma physics community. But for whatever reason, the classical statistical mechanics experts have largely remained aloof from considerations of systems with long range forces, one suspects much to the detriment of both subjects.

The subject of discrete-particle processes is much better understood in the absence of an externally-imposed dc magnetic field than in the presence of such a field. Many, if not most, of the important laboratory plasmas involve external magnetic fields in a fundamental way. The emphasis in this article will be almost exclusively on the situation in which a strong external magnetic field is present. A recent volume by the author [6] surveys the theory of the unmagnetized, or field-free plasma, and many other good survey volumes exist.

The following material is divided into four sections. In Section I, a number of results from the equilibrium statistical mechanics of two-dimensional plasmas are derived. The corresponding results for three dimensions are already well known. These results are independent of the presence of an external dc magnetic field, and are of interest in their own right as well as being essential to what follows. Section II is concerned with the non-equilibrium statistical mechanics of the electrostatic guiding-center plasma, a two-dimensional plasma model of great power and simplicity, recently introduced by Taylor and McNamara [7]. Section III concerns the generalization of this model to three dimensions. Section IV returns to two dimensions and relaxes the guiding-center model to include finite Larmor radius effects.

REFERENCES FOR THE INTRODUCTION

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2. LANDAU, L.D., *J. Physics (U.S.S.R.) 10,* 25 (1946).

3. VLASOV, A.A., *J. Physics (U.S.S.R.) 9,* 25 (1945).

4. BOGOLYUBOV, N.N., *J. Physics (U.S.S.R.) 10,* 257 (1946).

5. The "Vlasov approximation" is a limit in which the charges e of mass m are supposed to be infinitely subdivided in such a way that $e \to 0$, $m \to 0$, but e/m and $n_o e$ stay finite. n_o is the average number density. See, e.g. N. ROSTOKER and M.N. ROSENBLUTH, *Phys. Fluids 3,* 1 (1960), or the following reference.

6. MONTGOMERY, D., *Theory of the Unmagnetized Plasma,* New York, Gordon and Breach, (1971).

7. TAYLOR, J.B. and McNAMARA, B., *Phys. Fluids, 14,* 1492 (1971). This paper can be said to have originated the subject we are going to be developing, though it has many antecedents.

I. EQUILIBRIUM ENSEMBLE AVERAGES; STATIONARY CORRELATION FUNCTIONS

(A) Phase Space and the Gibbs Canonical Ensemble

As is the case with any system with a very large number of degrees of freedom, our description is necessarily probabilistic. The dynamical state of the system can be specified by identifying it with a point of the *phase space* of the system. The coordinates of any such point (X, say) are the positions and velocities of every particle. The dynamical state of the system is then equivalent to a random variable defined over the phase space. Equilibrium statistical mechanics [1] demonstrates that the normalized probability distribution of this random variable appropriate to a system in thermal equilibrium with a temperature $\theta = KT$ (in energy units) is

$$D_{eq.} \equiv \frac{\exp(-\mathcal{E}/\theta)}{\int dX \exp(-\mathcal{E}/\theta)} \qquad (1)$$

This is, of course, the canonical distribution of Gibbs; $\mathcal{E}$ is the total energy, uniquely defined for each point X of the phase space, and dX is the volume element.

Thermodynamic quantities and other macroscopic observables are interpretable as "ensemble averages", or expectation values computed with respect to $D_{eq.}$. Thus, if A(X) is any dynamical variable, its expectation value will be written as

$$\langle A \rangle = \int dX \; A(X) \; D_{eq.} \tag{2}$$

where the integration runs over the entire phase space accessible to the system.

Thermodynamic functions are derivable from the *partition function*, which, apart from quantum mechanical factors which will not be of interest to us here, is

$$Z = \int dX \; \exp(-\mathcal{E}/\theta) \tag{3}$$

In particular, the equation of state for the pressure [2] is

$$p = \theta \frac{\partial}{\partial V} \ln Z = \frac{\theta}{Z} \frac{\partial Z}{\partial V} \tag{4}$$

where V is the spatial volume in which the system is confined. If one wants to calculate extensive quantities (such as the Helmholtz free energy or the entropy) accurately, one has to put the additional factors into Eq. (3) correctly, but since they are volume-independent, they cancel out of Eq. (4).

For definiteness, we will usually consider N positive charges +e and N negative ones -e. For $\mathcal{E}$ we have the expression

$$\mathcal{E} = T + U \tag{5}$$

$$T = \sum_i \frac{1}{2} \; m_i \; \vec{v}_i^{\,2} \tag{6}$$

$$U = \sum_{i<j} \varphi_{ij} \tag{7}$$

T is the total kinetic energy and U is the total potential (interaction) energy. The Σ_i in (6) runs over all particles of both signs, and m_i and $\vec{v}_i$ are the mass and velocity of the ith particle. In (7), ϕ_{ij} is the two-body Coulomb potential between charged particles i and j. The sum runs over all pairs. We make the convenient and important restriction to *electrostatic interactions*, ignoring all retardation effects and magnetic interactions as being down by a factor of $0(v^2/c^2)$ from the ones we shall include. (This approximation is often called, somewhat incorrectly, the "low beta" approximation by plasma physicists. "Beta" is a rather important plasma quantity which is essentially the ratio of thermal energy density to magnetic energy density.) It is also important to note that the things that have been said so far are unchanged by the addition of an external, constant, and spatially uniform magnetic field of arbitrary strength, since $\mathcal{E}$ is unchanged by the addition of such a field. Thus, the calculation of such ensemble averages as Eq. (2) are *independent* of $\vec{B}$, the external field, so long as A is a time-independent function of X.

(B) Reduced Probability Distributions [2-6]

It is convenient to project the probability distribution (1) onto a subspace which is just the phase of s particles, where s is some small integer (1,2,3,...). Let $X_j \equiv (\vec{x}_j, \vec{v}_j)$ be the position and velocity of the jth charge. Then

$$\frac{f_s^{eq.}(X_1 \ldots X_s)}{V^s} \equiv \int dX_{s+1}\, dX_{s+2} \cdots D_{eq.} \tag{8}$$

is the *reduced probability distribution* in the phase space of s particles. The volume factor V^s is for later convenience, and the integration is over all the coordinates of all the remaining charges. To discuss two charged species, it is convenient to partition s into s_i "ions" (positive charges) and s_e "electrons" ($s_e + s_i = s$) and define

$$f^{eq.}_{s_i,s_e} = f^{eq.}_{s_i,s_e}(X^i_1 \cdots X^i_{s_i}; X^e_1 \cdots X^e_{s_e})$$

$$= V^{s_i+s_e} \int dX^i_{s_i+1} \cdots dX^i_N \, dX^e_{s_e+1} \cdots dX^e_N \, D_{eq.} \tag{9}$$

where $X^i_j = (\vec{x}^i_j, \vec{v}^i_j)$ are the phase space coordinates of the jth ion; a similar expression applies for electrons. The superscripts will indicate the species of the charged particles. Note that all the f_{s_i,s_e} are symmetric under the interchange of like particle coordinates.

It is immediately obvious that the velocity-space parts of the f_{s_i,s_e} are trivial and may be separated from the configuration-space parts:

$$f^{eq.}_{s_i,s_e} = \left\{ \prod_{\ell=1}^{s_i} \frac{\exp(-m_i \vec{v}^{i\,2}_\ell / 2\theta)}{(2\pi\theta/m_i)^{d/2}} \right\} \tag{10}$$

$$\cdot \left\{ \prod_{j=1}^{s_e} \frac{\exp(-m_e \vec{v}^{e\,2}_j / 2\theta)}{(2\pi\theta/m_e)^{d/2}} \right\} n_{s_i,s_e}$$

where d is the dimensionality of the system, and

$$n_{s_i,s_e} \equiv \frac{V^{s_i+s_e} \int d\vec{x}^i_{s_i+1} \cdots d\vec{x}^i_N \, d\vec{x}^e_{s_e+1} \cdots d\vec{x}^e_N \exp(-U/\theta)}{\int d\vec{x}^i_1 \cdots d\vec{x}^i_N \, d\vec{x}^e_1 \cdots d\vec{x}^e_N \exp(-U/\theta)} \tag{11}$$

is the configuration-space probability distribution of the subsystem.

It will be generally understood that we are dealing throughout with very large systems: $N \gg 1$, and $N/V \equiv n_o$, the number density. The "thermodynamic limit" of $N \to \infty$ and $V \to \infty$ may or may not be well defined for such expressions as Eqs. (3) and (4). It is not an entirely academic question as to whether such limits exist. Some rigorous work has been done in one dimension [7,8] and in three dimensions [9], and more is badly needed (in two dimensions, especially). We shall generally assume that the limit does exist; but that V and N are just very large but finite quantities is perhaps a slighly more acceptable way for us to look at the problem. The large-system limit is desirable in that we may hope that removing the system's boundaries to infinity will leave us with a "bulk" limit in which the surface contributions to extensive quantities will be negligible compared to the volume contributions, and that nothing that happens near the boundaries will affect the intensive quantities. (This may appear to be obvious, but it is not.) Making this assumption, it is easy to show that $n_{1,0} = n_{0,1} = \text{const.} = 1$, and that the higher n_{s_i, s_e} depend only on the separations of the particle coordinates and not on their absolute spatial locations.

(C) The Equilibrium BBGKY Hierarchy

Most of the macroscopic functions of interest can be expressed as integrals of $n_{2,0}, n_{0,2}$, and $n_{1,1}$, as it turns out. Any approximate method of calculating these quantities is therefore of some interest. A convenient starting place is the well-knwon *equilibrium BBGKY* [2-5] *hierarchy*, which can be readily obtained by taking gradients of Eq. (11). It is left as an exercise to show that, assuming periodic boundary conditions over a very large volume V, for any j between 1 and s_i,

$$\frac{\partial n_{s_i,s_e}}{\partial \vec{x}^i_j} + \frac{n_{s_i,s_e}}{\theta} \frac{\partial \Phi(s_i,s_e)}{\partial \vec{x}^i_j}$$

$$= -\frac{N - s_i}{\theta V} \int d\vec{x}^i_{s_i+1} \frac{\partial \varphi^{ii}\left(\vec{x}^i_j - \vec{x}^i_{s_i+1}\right)}{\partial \vec{x}^i_j} n_{s_i+1,s_e} \quad (12)$$

$$-\frac{N - s_e}{\theta V} \int d\vec{x}^e_{s_e+1} \frac{\partial \varphi^{ie}\left(\vec{x}^i_j - \vec{x}^e_{s_e+1}\right)}{\partial \vec{x}^i_j} n_{s_i,s_e+1}$$

The notation used is the following: ϕ^{ii} is the Coulomb potential interaction of the two ions, and ϕ^{ee} is that of two electrons. $\phi^{ii}(\vec{x}) = \phi^{ee}(\vec{x})$. $\phi^{ie} = \phi^{ei} = -\phi^{ee}$ is the Coulomb potential between an electron and an ion. $\Phi(s_i,s_e)$ is the total Coulomb potential energy of the first s_i ions and s_e electrons If s_e and s_i are not large, we can replace $(N - s_i)/V$ and $(N - s_e)/V$ by the average number density n_o, on the right-hand side of (12). A similar relation comes from taking gradients of Eq. (11) with respect to the electron coordinates:

$$\frac{\partial n_{s_i,s_e}}{\partial \vec{x}^e_j} + \frac{n_{s_i,s_e}}{\theta} \frac{\partial \Phi(s_i,s_e)}{\partial \vec{x}^e_j}$$

$$= -\frac{n_o}{\theta} \int d\vec{x}^i_{s_i+1} \frac{\partial \varphi^{ei}\left(\vec{x}^e_j - \vec{x}^i_{s_i+1}\right)}{\partial \vec{x}^e_j} n_{s_i+1,s_e}$$

$$-\frac{n_o}{\theta}\int d\vec{x}^{e}_{s_e+1}\;\frac{\partial\varphi^{ee}\left(\vec{x}^{e}_{j}-\vec{x}^{e}_{s_e+1}\right)}{\partial\vec{x}^{e}_{j}}\;n_{s_i,s_e+1} \tag{13}$$

where we have assumed $N \gg s_i$ or s_e. The normalization conditions to be obeyed by n_{s_i,s_e} are, from (11),

$$\int d\vec{x}^{i}_{1}\ldots d\vec{x}^{i}_{s_i}\,d\vec{x}^{e}_{1}\ldots d\vec{x}^{e}_{s_e}\,n_{s_i,s_e}=V^{s_i+s_e} \tag{14}$$

and $n_{0,0}=n_{1,0}=n_{0,1}=1$.

(D) Expansion in the "Plasma Parameter" ε

For the entirely general case, there is no particular advantage in writing the thermal equilibrium relations in the form of Eqs. (12) and (13). Rather, their advantage lies in the ease with which perturbation theory can be done on them. It turns out that for a wide range of parameters, the potential energy of two charges in a plasma which lie at a distance equal to the nearest-neighbor separation is much less than the average kinetic energy per particle, θ. This means that the interactions are mostly *weak* in the sense that ϕ^{ee}/θ, ϕ^{ii}/θ and ϕ^{ie}/θ can all be considered to be $\ll 1$ over most of the phase space. We can thus do a weak-coupling expansion in $\langle\phi\rangle_{avg.}/\theta$, (where $\langle\phi\rangle_{avg.}$ is some average interaction potential). The expansion is complicated slightly by the long range of the potential, which raises some of the integral terms in Eqs. (12), (13) by one order in $\langle\phi\rangle_{avg.}/\theta$.

The details of this expansion for the three-dimensional case have been adequately described several times (see e.g., Guernsey [10] or Montgomery [11]), and there is no purpose in doing so again here. We shall outline the development, however, for the two-dimensional [12] case, since so much of the later theory to be developed involves two-dimensional models. Equations (12) and (13) apply to one, two, or three dimensions, with appropriate modifications of the Coulomb potential $\phi^{ii}=\phi^{ee}=-\phi^{ei}=-\phi^{ie}$. In two dimensions, the charges are imagined as very long parallel "rods" of length ℓ and charge e, and the potential between two rods of positive charge +e each is, for example,

$$\varphi^{ii}(\vec{x}^i_\ell - \vec{x}^i_j) = -\frac{2e^2}{\ell} \ln |\vec{x}^i_\ell - \vec{x}^i_j| \tag{15}$$

with similar expressions for the other interactions.

The zeroth order (i.e., non-interacting) values of the n_{s_i,s_e} are seen to be those solutions of Eqs. (12) and (13) with all the potential energy parts neglected and which also satisfy (14). These are, obviously,

$$n^{(0)}_{s_i,s_e} = 1, \qquad \text{all } s_i \text{ and } s_e \tag{16}$$

The superscript zero means of zeroth order in powers of the interaction potential, and we imagine (16) as the first term of the series.

$$n_{s_i,s_e} = n^{(0)}_{s_i,s_e} + n^{(1)}_{s_i,s_e} + \ldots \tag{17}$$

We expect the higher-order terms of this series, the "correlation functions", all to vanish as the particle separations become very large, and $n_{s_i,s_e} \to n^{(0)}_{s_i,s_e}$ then.

The lowest order there is, according to (16), no correlation between the positions of the particles: what one would expect from non-interacting particles. We proceed to (12) and (13) to obtain the next terms in the series. However, the long range of the Coulomb potential means that one has to keep the next order terms in the integrals in Eq. (12), above those a purely formal estimate would indicate.

A self-consistent expansion procedure is readily found in which the correlations between larger numbers of particles is progressively weaker in powers of $\langle\phi\rangle/\theta$. The next order can be found in terms of *pair correlations*:

$$n^{(1)}_{s_i,s_e} = \sum_{k<\ell=1}^{s_i} p^{ii}(\vec{x}^i_k - \vec{x}^i_\ell)$$

$$+ \sum_{k<\ell=1}^{s_e} p^{ee}(\vec{x}^e_k - \vec{x}^e_\ell)$$
$$+ \sum_{k=1}^{s_i} \sum_{\ell=1}^{s_e} p^{ei}(\vec{x}^e_\ell - \vec{x}^i_k) \qquad (18)$$

where $p^{ii} = p^{ee} = -p^{ei}$.

With the assumptions, it can be shown that the first-order part of *all* of Eqs. (12) and (13) is satisfied if one sets $p^{ee}(\vec{x}) = e^2\mathcal{O}(x)$, $\phi^{ee}(\vec{x}) = e^2\psi(x)$, and

$$\frac{\partial \mathcal{O}(x_{12})}{\partial \vec{x}_{12}} + \frac{1}{\theta} \frac{\partial \psi(x_{12})}{\partial \vec{x}_{12}}$$
$$= -\frac{n_o e^2}{\theta} \int d\vec{x}_3 \frac{\partial \psi(x_{13})}{\partial \vec{x}_{13}} \mathcal{O}(x_{23}) \qquad (19)$$
$$-\frac{n_o e^2}{\theta} \int d\vec{x}_3 \frac{\partial \psi(x_{13})}{\partial \vec{x}_{13}} \mathcal{O}(x_{23})$$

In Eq. (19), $x_{12} \equiv |\vec{x}_1 - \vec{x}_2|$, and so on. It is a differentio-integral equation of the convolution type, and can be solved by the Fourier transformation. Setting

$$\mathcal{P}(x) = \int d\vec{k}\ \mathcal{P}(\vec{k})\ e^{i\vec{k}\cdot\vec{x}}$$

$$\psi(x) = \int d\vec{k}\ \psi(\vec{k})\ e^{i\vec{k}\cdot\vec{x}} \tag{20}$$

we find

$$\mathcal{P}(\vec{k}) = \frac{-\psi(\vec{k})/\theta}{1 + \dfrac{2n_o e^2}{\theta}(2\pi)^2\,\psi(\vec{k})} \tag{21}$$

or

$$p^{ee}(\vec{k}) = -\frac{\varphi^{ee}(\vec{k})/\theta}{1 + (2n_o/\theta)\,\varphi^{ee}(\vec{k})(2\pi)^2} \tag{22}$$

Poisson's equation, Fourier-transformed in two dimensions, give $\phi^{ee}(\vec{k}) = e^2/\pi\ell k^2$, so that we may finally write (22) as the classical *Debye-Hückel expression*:

$$p^{ee}(\vec{k}) = -\left(\frac{1}{8\pi^2 n_o}\right)\left(\frac{1}{1 + k^2\lambda_D^2}\right) \tag{23}$$

Here, we have introduced the very important *Debye length* into the problem, namely

$$\lambda_D^2 \equiv \frac{\ell\theta}{8\pi n_o e^2} \equiv K_D^{-2} \tag{24}$$

If we Fourier transform Eq. (23) back to $\vec{x}$-space, we get

$$p^{ee}(x) = -\frac{2e^2}{\ell\theta}\,K_o(x/\lambda_D) \tag{25}$$

where K_0 is a Bessel function of imaginary argument, and has the asymptotic form

$$p^{ee}(x) \xrightarrow[x \gg \lambda_D]{} - \frac{2e^2}{\ell\theta} \sqrt{\frac{\pi \lambda_D}{2x}} \exp(-x/\lambda_D) \tag{26}$$

We see that λ_D has the important physical significance of being the distance over which pair correlations persist, just as in three dimensions, though the functional form is somewhat different. Since $p^{ii} = p^{ee} = -p^{ei}$, we now have all the first-order solution.

The calculation of the correlation functions to higher order becomes quite messy, but as long as $p^{ee}(x)$ is $\ll 1$, Eqs. (17), (18), and (25) are a satisfactory approximation. It is readily seen that the condition for validity is thus $2e^2/\ell\theta \ll 1$, or what is equivalent,

$$\epsilon \equiv (4\pi n_0 \lambda_D^2)^{-1} \ll 1 \tag{27}$$

This condition is essentially that the number of charged rods per Debye square is very large. The well-known analogue [13] of this condition in three dimensions is that the number of point charges per Debye cube is very large.

(E) Further Comments on the Validity of the Expansion

The failure of the weak-interaction approximation over small regions of the phase space (at small particle separations) is indicated by the fact that the short-range form of Eq. (25), $p_{ee}(x \ll \lambda_D) \approx -(2e^2/\ell\theta)\ln(x/\lambda_D)$. This ceases to be $\ll 1$ at a distance $x_{min.}$ given by

$$-\frac{2e^2}{\ell} \ln(x_{min.}/\lambda_D) \approx \theta$$

or

$$x_{min.} \cong \lambda_D e^{-1/\epsilon} \tag{28}$$

Compared with the nearest-neighbor separation $n_0^{-1/3}$, this gives

$$\frac{x_{min.}}{n_o^{-\frac{1}{2}}} = \frac{\sqrt{\epsilon}}{4\pi} \exp(-1/\epsilon) \tag{29}$$

which is smaller than any power of ϵ, but is not zero. Because the separation of two charges is so seldom less than $x_{min.}$, it is reasonable to hope that the important features of the theory will be unchanged by ignoring the short range of $\phi^{ee}(x)$. To render certain integrals convergent, it is often convenient to work not with $\phi^{ee}(x)$, but with another function which differs from it only for $x \leq x_{min}$, namely

$$\varphi_{eff.}(x) = \int_0^{k_{max}} k\,dk \int_0^{2\pi} d\theta \; \frac{e^2 e^{ikx\cos\theta}}{\pi \ell k^2} \tag{30}$$

where $k_{max} = 1/x_{min}$. The length x_{min} has the physical interpretation in the unmagnetized case of being the distance of closest approach of two colliding thermal particles with zero impact parameter. To remove this somewhat unsatisfactory state of affairs at short range requires quantum mechanical modifications in three dimensions, and has not been carried out in two dimensions. It has been concluded by everyone who has seriously looked at the three-dimensional calculations that the dominant effects are correctly obtained from the modification of ϕ^{ee} of Eq. (3) at short distances. We shall have to hope that the same thing is true in two dimensions. The effect of this modification of the interaction on the pair correlations is just to cut off the Fourier integrals of such expressions as (23) at $|\vec{k}| = 1/x_{min}$

(F) Ensemble Averages of Macroscopic Functions [14]

To summarize our results to this point, we have calculated the four pair correlations in this problem in terms of a single function, which in Fourier transform language reads

$$p^{ee}(\vec{k}) = p^{ii}(\vec{k}) = -p^{ei}(\vec{k}) = -p^{ie}(\vec{k}) \tag{31}$$

$$= -\frac{(8\pi^2 n_o)^{-1}}{1 + k^2\lambda_D^2}$$

This is not an exact result, and is valid to the extent that the inequality (27) holds. We have immediately from it expressions for all the various n_{s_i,s_e}, from (17) and (18), to the same accuracy. We can calculate, in terms of it, additional quantities of physical interest. For example, the electric field

$$\vec{E}(\vec{x}) = -\frac{\partial}{\partial \vec{x}} \left\{ \sum_{j=1}^{N} \left(-\frac{2e}{\ell} \ln | \vec{x} - \vec{x}^{i}_{j} | \right) \right. \tag{32}$$

$$\left. + \sum_{k=1}^{N} \left(\frac{2e}{\ell} \ln | \vec{x} - \vec{x}^{e}_{k} | \right) \right\}$$

can be readily shown to have zero ensemble average

$$\langle \vec{E}(\vec{x}) \rangle = \int D_{eq.} \vec{E} \, dX = 0 \tag{33}$$

as can the net electrical charge density

$$\rho(\vec{x}) = \frac{e}{\ell} \sum_{j=1}^{N} \left[\delta(\vec{x} - \vec{x}^{i}_{j}) - \delta(\vec{x} - \vec{x}^{e}_{j}) \right] \tag{34}$$

For, using our previous expressions for p^{ee}, p^{ii}, p^{ie}, we can also show

$$\langle \rho(\vec{x}) \rangle = 0 \tag{35}$$

However, the vanishing of the ensemble averages of $\vec{E}$ and ρ should not be construed to mean they are zero, because, for example, $\langle \vec{E}^2 \rangle \neq 0$. Let us calculate the *autocorrelation* of the charge density,

$$\langle \rho(\vec{x}) \, \rho(\vec{x} + \vec{r}) \rangle = \int D_{eq.} \, \rho(\vec{x}) \, \rho(\vec{x} + \vec{r}) \, dX \tag{36}$$

For this purpose it is convenient to insert (35), (9), and (10) into (36), getting:

$$\langle \rho(\vec{x})\, \rho(\vec{x}+\vec{r}) \rangle$$

$$= \int dX \; \frac{e^2}{\ell^2} \sum_{j=1}^{N} \sum_{q=1}^{N} \left\{ \delta(\vec{x} - \vec{x}^{i}_{j}) - \delta(\vec{x} - \vec{x}^{e}_{j}) \right\}$$

$$\left\{ \delta(\vec{x} + \vec{r} - \vec{x}^{i}_{q}) - \delta(\vec{x} + \vec{r} - \vec{x}^{e}_{q}) \right\} D_{eq.}$$

$$= \frac{e^2}{\ell^2} \frac{N(N-1)}{V^2} \left\{ n_{2,0}(\vec{x},\vec{x}+\vec{r}) + n_{0,2}(\vec{x},\vec{x}+\vec{r}) \right\} \quad (37)$$

$$- \frac{e^2}{\ell^2} \frac{N^2}{V^2} \left\{ n_{1,1}(\vec{x},\vec{x}+\vec{r}) + n_{1,1}(\vec{x},\vec{x}+\vec{r}) \right\}$$

$$+ \frac{e^2}{\ell^2} \frac{N}{V} \left\{ n_{1,0}(\vec{x})\, \delta(\vec{r}) + n_{0,1}(\vec{x})\, \delta(\vec{r}) \right\}$$

Using $n_{1,0} = n_{0,1} = 1$, and passing to the limit of N,V very large, we use (18) and our previously derived expressions for the pair correlations to write:

$$\langle \rho(\vec{x})\, \rho(\vec{x}+\vec{r}) \rangle = 4 \frac{n_o^2 e^2}{\ell^2} p^{ee}(\vec{r}) + \frac{2 n_o e^2}{\ell^2} \delta(\vec{r}) \quad (38)$$

which depends only upon the separation of the points $\vec{r}$, as we know it should.

It is often useful to decompose expressions like Eq. (38) into a Fourier sum or integral, and write them in terms of their *spectral densities*. Thus,

$$\langle \rho(\vec{x})\ \rho(\vec{x}+\vec{r})\rangle = \int d\vec{k}\ S_\rho(\vec{k})\ e^{i\vec{k}\cdot\vec{r}} \tag{39}$$

where Fourier transformation of Eq. (38) and use of (23) shows that the spectral density $S_\rho(\vec{k})$ is

$$S_\rho(\vec{k}) = \frac{n_o e^2}{2\pi^2 \ell^2}\left(\frac{k^2\lambda_D^2}{1+k^2\lambda_D^2}\right) \tag{40}$$

Of considerable interest also will be the spectral density of the auto-correlation tensor of the electric field, $\vec{\vec{S}}_E(\vec{k})$, defined by

$$\langle \vec{E}(\vec{x})\ \vec{E}(\vec{x}+\vec{r})\rangle = \int d\vec{k}\ \vec{\vec{S}}_E(\vec{k})\ e^{i\vec{k}\cdot\vec{r}} \tag{41}$$

Two applications of Poisson's equation shows that $\vec{\vec{S}}_E$ and S_ρ are related by

$$\vec{\vec{S}}_E(\vec{k}) = \frac{16\pi^2 \vec{k}\vec{k}}{k^4}\ S_\rho(\vec{k})\ , \tag{42}$$

so that

$$\vec{\vec{S}}_E(\vec{k}) = \frac{\theta}{\ell\pi}\ \frac{\vec{k}\vec{k}}{k^2}\ \frac{1}{1+k^2\lambda_D^2}\ . \tag{43}$$

It will be instructive for the reader to compute other auto-correlations as exercices, such as the current-current auto-correlation, $\langle \vec{j}(\vec{x})\ \vec{j}(\vec{x}+\vec{r})\rangle$

(G) Field Energies, Thermal Energies, and a Warning

It is interesting to calculate the rms electric field strength

$$\langle \vec{E}^2 \rangle = \text{Trace} \int d\vec{k}\, \vec{\vec{S}}_E(\vec{k}) = \frac{2\theta}{\ell} \int_0^\infty \frac{k\, dk}{1 + k^2 \lambda_D^2} \tag{44}$$

Equation (44) contains a logarithmic divergence at large k, which is to be associated with small particle separations; it can be rendered finite by the cutoff (30). However, this is misleading, for (44) contains in it a term proportional to the infinite electrostatic self-energies of all the point charges, which originates in the $\delta(\vec{r})$ term in Eq. (38). Subtracting this off will give a more revealing quantity. The self energy of one charge is proportional to $E^2_{avg} \equiv (8\pi e^2/\ell^2 V) \int dk/k$, and the self field-energy for 2N of them is an additive quantity, so

$$\langle \vec{E}^2 \rangle_{self} = \frac{16\pi n_o e^2}{\ell^2} \int \frac{dk}{k}$$

and

$$\langle \vec{E}^2 \rangle - \langle \vec{E}^2 \rangle_{self} = -\frac{2\theta}{\ell \lambda_D^2} \int_0^\infty \frac{dk}{k(1 + k^2 \lambda_D^2)} \tag{45}$$

Note that Eq. (45) has now a small $\vec{k}$ divergence, which must be associated with *large* spatial separations. This has no analogue in the three-dimensional theory, even though (23) is essentially the same as the three-dimensional expression. This pathology has its origin in the fact that the Coulomb potential (15) has a divergence at infinity in two dimensions, but the three-dimensional expression does not. The divergence strongly suggests that we may anticipate other unexpected small-$\vec{k}$ divergences, and that at times we may want to limit the discussion to a large but finite plasma volume $V = L^2$, to provide an effective cutoff in k space at $k \sim 2\pi/L$.

An interesting ratio is the ratio of the electric field energy density to the thermal energy density $2n_o\theta/\ell$. Dividing (44) by $16\pi n_o\theta/\ell$ and cutting the integral off at $x_{min.} = \lambda_D e^{-1/\varepsilon}$

$$\frac{\langle \vec{E}^2 \rangle}{16\pi n_o \theta/\ell} = \frac{1}{8\pi n_o \lambda_D^2} \ln(\lambda_D/x_{min.})$$

$$= \frac{1}{2} \epsilon \ln e^{1/\epsilon} = \frac{1}{2}$$

This is not small small compared to unity for any ϵ. However, if we compute the comparable ratio for the part of $\langle \vec{E}^2 \rangle$ that is to be associated with interaction energy,

$$\frac{\langle \vec{E}^2 \rangle - \langle \vec{E}^2 \rangle_{self}}{16\pi n_o \theta/\ell} = -\frac{\epsilon}{2} \ln (L/2\pi\lambda_D) \tag{46}$$

As long as the right-hand side of (46) is small compared to unity it makes sense to speak about the field fluctuation energy as being small relative to the kinetic energy, which is the essential approximation that has been made. But it is clear that such formulas as (46) lead to grave difficulties if one tries to pass to the limit $L \to \infty$. Whether these are difficulties with the perturbation expansion (17), or correspond to more basic questions connected with the existence of the thermodynamic limit in two dimensions is something that at the time of this writing, apparently nobody knows. Definitive answers to these questions will involve investigations of a considerably higher degree of rigor than the one presented here. In the meantime, one wants to be especially cautious of attempts to evaluate extensive quantities, such as the total energy, by integrating intensive ones such as $\langle \vec{E}^2 \rangle$. The circumstance may arise (it does in one dimension [15]) in which small terms of $0(1/V)$ may be missed in the perturbation theory, which give no contribution to $\langle \vec{E}^2 \rangle$ in the limit, but which may nonetheless give finite or even divergent contributions to integrals over the entire volume.

(H) Equation of State, Collapse at Low Temperatures

As a next comment on the equilibrium theory, let us remark on one rather remarkable property of this system, namely that its equation of state is exactly calculable. This has apparently been discovered at least twice [16,17]. It is not a complicated manipulation [2] to reduce Eq. (4) to

$$p = 2\frac{N}{V}\theta - \frac{1}{2V^3}\sum_{i<j}\int d\vec{x}_i\, d\vec{x}_j\, \varphi'_{ij}(x_{ij})\, x_{ij}\, n_2(\vec{x}_i,\vec{x}_j) \qquad (47)$$

where

$$n_2(\vec{x}_i,\vec{x}_j) \equiv \frac{V^2 \int d^{8N-4}X\, D_{eq.}}{\int dX\, D_{eq.}} \qquad (48)$$

In (48), $\int d^{8N-4}X$ means the integral over all phase space coordinates except for $\vec{x}_i$ and $\vec{x}_j$. The $\sum_{i<j}$ is over all pairs of both signs.

Equation (47) applies to any force law ϕ_{ij}. The unique feature of the two-dimensional Coulomb potential is that $x_{ij}\phi'(x_{ij}) = -2e_ie_j/\ell$, independently of $\vec{x}_i$ and $\vec{x}_j$. What remains is just the normalization integral for the two-body distributions, and the problem has been reduced to counting particle pairs. This gives, after a little algebra

$$\frac{p}{2n_o\theta} = 1 - \frac{\varepsilon}{4} \qquad (49)$$

The equation of state, while becoming that of an ideal gas for high temperatures ($\varepsilon \ll 1$), has the feature of predicting a *collapse* ($p < 0$) at low temperatures, corresponding to $\varepsilon = 4$. This is nothing remarkable, since it has its origin in the fact that the ion-electron interaction (and thus the total energy) is unbounded from below. In fact, one sees from (3) that when $\vec{x}_i$ is close to $\vec{x}_j$, Z contains a factor

$$\sim \int d\vec{x}_{ij}\ \exp\left(\frac{2e_ie_j}{\ell\theta}\ln x_{ij}\right) \sim \int x_{ij}\, dx_{ij}\, x_{ij}^{2e_ie_j/\ell\theta}$$

If e_i and e_j are of unlike sign, this will diverge when $-(2^2/\ell\theta) + 1 < -1$, or $\varepsilon > 2$. If more than two particles are considered, the divergence may even occur at higher θ (see Knorr [18]). Some lower bound on the interaction would be required to eliminate this collapse, as quantum mechanics is required in three dimensions.

(I) Summary

Starting from the Gibbs distribution for an equilibrium system of electrostatically interacting charged rods aligned parallel, *approximate* expressions for the phase space probability distributions have been derived by an expansion in the plasma parameter (number of rods per Debye square). The program has been carried out in parallel to the familiar [10,13] expansion in three dimensions. These approximate probability distributions permit the calculation of ensemble averages of stationary fluctuation quantities and their spectral densities; these are independent of external magnetic field strength. The thermodynamic equation of state has been noted to be exact. Anomalies which may possibly result from taking the "thermodynamic" limit ($N \to \infty$ $V \to \infty$) have been noted. Up to this point the theory is the same with or without an external magnetic field.

II. THE TWO-DIMENSIONAL GUIDING-CENTER PLASMA

(A) Some General Remarks on Transport Processes

A central concern in statistical physics has always been the matter of *transport properties* [19]. That is, gradients in the macroscopic hydrodynamic parameters will result in the net transport of mass, momentum, energy, electric current, etc., across an imaginary surface in a gas, liquid, or plasma. This transport comes about as a direct consequence of the microscopic interactions among the particles, and is inevitably a problem in non-equilibrium statistical mechanics.

This problem has been of acute concern for strongly magnetized plasmas, because many controlled fusion schemes propose to confine a very hot plasma away from material walls by means of externally-imposed dc magnetic fields. Often, actual machines appear to confine plasmas less well than they are supposed to, and this "anomalous" transport of plasma has been a matter of concern for many years. It was addressed early by Bohm [20], disappeared from sight for several years, re-appeared about 1960 [21,22] when controlled fusion research became public knowledge, and has been the motivation for many of the exotic instability and "weak turbulence" calculations which were a major pre-occupation of plasma theory in the 1960's. The subject is far from exhausted, though it has become difficult of access by virtue of being surrounded with hundreds of rather diffuse calculations.

Actually, the transport may not be so "anomalous" as it might appear at first sight. Most of the more sober calculations [23] have started from a *kinetic equation*. By kinetic equation, one means a differentio-integral equation such as Boltzmann's equation, which will advance the time-dependent analogues of $f_{1,0}$ and $f_{0,1}$ of Eqs. (9) in time. A crucial ingredient of any kinetic equation is a *collision term* in which the irreversibility originates, which is a functional of the one-body distribution ($f_{1,0}$ or $f_{0,1}$), and into the derivation of which goes some microscopic model of the particle-particle interaction process. Different forms of the collision term arise from different microscopic models [24]. Thus uncorrelated, strong, two-body collisions lead to the Boltzmann equation, and weak two-body collisions lead to the Fokker-Planck equation. In the plasma case, however, most of the microscopic models used to describe the interactions of charged particles have *not* included the effects of an external magnetic field on the particle-particle interaction process. Putting in an external magnetic field renders the strongly-interacting two-body problem insoluble, so there is at present no analogue of Boltzmann's equation for the case in which an intense, external magnetic field is present. In a weak-interaction model of a plasma, a kinetic equation results [25–27] but is so complicated that even the derivation of it is quite strenuous, and heroic efforts will be required to extract many predictions about transport properties from it.

From the foregoing remarks we may conclude that the subject of transport properties in a strongly-magnetized plasma is still very much an open one. One expects significant gains in understanding only to the extent that clean and bold simplifying approximations can be found. A recent breakthrough in this direction was due to Taylor and McNamara [28]. They introduced a two-dimensional model of a plasma consisting of charged rods aligned parallel which are free to move perpendicularly to a strong dc magnetic field. They further simplified the dynamics by introducing the *guiding-center approximation* for the particle motion. Discussion of the various properties of this model will be the subject-matter of this chapter. Various features of the model can be generalized to three dimensions, but discussion of these is better deferred until after the two-dimensional model has been presented.

(B) The Guiding-Center Plasma ("Zero Gyro-Radius" Plasma)

The physical system is the same as that described in Section I: a collection of N very long rods of charge +e, length ℓ, mass m_I, and N more of charge -e, length ℓ, mass m_e aligned parallel to the z-axis of coordinates, and interacting through the Coulomb potential, with an interaction potential given by $\phi_{\ell m} = -(2e_\ell e_m/\ell)\ln|\vec{x}_\ell - \vec{x}_m|$ for any two charges.

The equation of motion for the velocity of a given charged rod of charge-to-mass ratio e_j/m_j is

$$\frac{d\vec{v}_j}{dt} = \frac{e_j}{m_j}\left(\vec{E} + \frac{\vec{v}_j}{c} \times \vec{B}\right) \tag{50}$$

where $\vec{v}_j$, $\vec{E}$ are two-dimensional vectors lying in the xy plane, $\vec{B} = B\hat{e}_z$, with B = a constant, and $\vec{E}$ is evaluated at the location of the particle. $\vec{E} = -\nabla\phi$ obeys the two-dimensional Poisson equation,

$$\nabla^2\phi = -4\pi\sum_j \frac{e_j}{\ell}\,\delta(\vec{x} - \vec{x}_j) \tag{51}$$

where the sum runs over all charges of both signs. Equations (50) and (51), together with $d\vec{x}_j/dt = \vec{v}_j$ and the appropriate boundary conditions, constitute a complete dynamical description of the system. This dynamical description is still quite complicated, and we shall have more to say about it later. At present, we wish to discuss an even more simplified version of the dynamics, namely, the *guiding-center approximation.*

The reader is referred to Northrop [29] for a discussion of the guiding center approximation; in practice, it amounts to neglecting the left-hand side of Eq. (50), so that $\vec{E} + (\vec{v}_j/c) \times \vec{B} \cong 0$, or in this two-dimensional geometry,

$$\vec{v}_j = c\,\vec{E}(\vec{x}_j,t) \times \vec{B}/B^2 \tag{52}$$

The $\vec{v}_j$ in (52) is to be distinguished from the $\vec{v}_j$ in (50); (52) describes the motion of the *guiding center* of the particle. This guiding center motion has superimposed on it the "fast" gyration of the particle with characteristic gyro-frequency $\Omega_j = e_jB/m_jc$ and characteristic radius of the order of $r_j = \sqrt{\theta/m_j}/\Omega_j$, for a particle of thermal energy θ. The "fast" motion is averaged out in writing down Eq. (52). The conditions

for the validity of Eq. (52) as a satisfactory approximation are that, most importantly, the characteristic time τ and length λ over which $\vec{E}(\vec{x},t)$ varies significantly must satisfy the inequalities

$$\lambda \gg r_j$$
$$\Omega_j \tau \gg 1 \tag{53}$$

The actual microscopic electric field in the plasma will be de-composable into components which do and do not satisfy the inequalities (53). Only the former will be treated accurately by (52). At the present time, no satisfactory quantitative estimate exists as to the errors which result from this approximation.

(C) Connection with Ideal Hydrodynamics

The replacement of the differential equation (50) by the algebraic equation (52) is a considerable simplification. It also has the unexpected benefit of leaving us with a Hamiltonian system, the Hamiltonian of which is essentially the same as the potential energy of the original system! To see this, we digress briefly on the hydrodynamics of incompressible, inviscid fluids in two dimensions. In the theory of parallel "line" vortex motion [30], it is shown that the fluid velocity field which results from a collection of line vortices of vortex strength $\vec{K}_j = K_j \hat{e}_z$ located at positions $\vec{x}_j$ is

$$\vec{v}(\vec{x}) = \sum_j \frac{\vec{K}_j}{2\pi} \times \frac{\vec{x} - \vec{x}_j}{|\vec{x} - \vec{x}_j|^2} \tag{54}$$

The vortices move along streamlines, so in particular the velocity of the ith vortex is

$$\vec{v}_i = \sum_{j \neq i} \frac{\vec{K}_j}{2\pi} \times \frac{\vec{x}_{ij}}{x_{ij}^2} \tag{55}$$

Now consider (50), (51), and (52). The electric field seen by charge i is

$$\vec{E}(\vec{x}_i) = \sum_{j \neq i} \frac{2e_j}{\ell} \frac{\vec{x}_{ij}}{x_{ij}^2} \tag{56}$$

so its velocity is

$$\dot{\vec{v}}_i = \sum_{j \neq i} \frac{2ce_j}{\ell B} \frac{\vec{x}_{ij}}{x_{ij}^2} \times \frac{\vec{B}}{B} \tag{57}$$

If we identify $(2\pi)^{-1}$ times the vortex strength, $K_j/2\pi$, with $-2ce_j/\ell B$, then the two mathematical descriptions become *identical*. We may use language appropriate to the guiding center plasma interchangably with language appropriate to line vortex motion, providing that

$$K_j = -4\pi c e_j/\ell B \tag{58}$$

That the system is Hamiltonian with Hamiltonian

$$H = -\frac{1}{2\pi} \sum_{i<j} K_i K_j \ell n |\vec{x}_i - \vec{x}_j| \tag{59}$$

can readily be seen by picking canonical coordinates $[\vec{x}_j \equiv (x_j, y_j)]$ q_j, p_j:

$$(q_j, p_j) = |K_j|^{\frac{1}{2}} (x_j, y_j \text{ sgn } K_j) \tag{60}$$

for each vortex (or charge). It is readily verified that (55) is the same as

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} \tag{61}$$

Thus the canonical coordinates and momenta are essentially just the Cartesian spatial coordinates x_i, y_i of the charges (or vortices), with a change in sign of y_i for the negative vortices.

Since H is not explicitly time dependent, it is a constant of the motion. The canonical distribution is again given by

Eq. (1), since H is up to a constant factor, just the interaction energy $\sum_{i<j} \phi_{ij}$ of all the pairs of charges. All the stationary thermal equilibrium correlation functions and spectral densities calculated in Section I are identical to the corresponding quantities for the guiding center plasma. Of course time dependent quantities will evolve much differently, and we pass now to a consideration of some of these.

(D) Formulation of the Problem of Spatial Diffusion in the Green-Kubo Picture

A measure of the rate of spatial transport can be obtained in the following simple situations. Suppose a very tenuous distribution of labeled particles, so rarified that their interaction with each other is negligible, is released near $\vec{x} = 0$. As time goes on, due to the fluctuating microscopic electric field, the charges (they will hereafter be called "test" particles) will be moved away from the origin. The position after time t for one of them will be

$$\vec{x} = \int_0^t \vec{v}(\tau)\, d\tau \tag{62}$$

where $\vec{v}(t)$ is the velocity. By (52), this is

$$\vec{x} = c \int_0^t \frac{\vec{E}(\tau) \times \vec{B}}{B^2}\, d\tau \tag{63}$$

where $\vec{E}(\tau) = \vec{E}[\vec{x}(\tau),\tau]$ is the electric field evaluated at the instantaneous test particle position.

The path (63) is erratic and complicated, and to evaluate it would involve knowing, among other things, the exact microscopic electric field $\vec{E}(\vec{x},t)$. An ensemble average prediction is the best we can hope for. So we imagine a set of similar plasmas, distributed according to the Gibbs distribution (1), and release a collection of test particles near $\vec{x} = 0$ at $t = 0$ in each one. Hereafter we address ourselves to the question of calculating ensemble averages over this ensemble [31].

Since $\langle \vec{E} \rangle = 0$, by Eq. (33), we have at once that $\langle \vec{x} \rangle = 0$, which simply says that particle transport is isotropic. A measure of the amount of diffusion is $\langle \vec{x}^2 \rangle \neq 0$; thus

$$\langle \vec{x}^2 \rangle = \frac{c^2}{B^2} \int_0^t d\tau_1 \int_0^t d\tau_2\, \langle \vec{E}(\tau_1) \cdot \vec{E}(\tau_2) \rangle \tag{64}$$

Auto-correlations like $\langle \vec{E}(\tau_1) \cdot \vec{E}(\tau_2) \rangle$ play a central role in what follows. If the ensemble is stationary, they will depend upon $\tau_2 - \tau_1$ only, so that $\langle \vec{E}(\tau_1) \cdot \vec{E}(\tau_2) \rangle = \langle \vec{E}(0) \cdot \vec{E}(\tau_2 - \tau_1) \rangle$, and it is expected that for all physical systems, $\langle \vec{E}(0) \vec{E}(\tau) \rangle$ will $\to 0$ as $\tau \to \pm \infty$. It is also to be expected that $\langle \vec{E}(0) \vec{E}(\tau) \rangle$ is an *even* function of τ. We shall assume so.

Defining $\tau_2 - \tau_1 \equiv \tau$, $(\tau_2 + \tau_1)/2 \equiv T$, the integral (64) can be rewritten as

$$\langle \vec{x}^2 \rangle = \frac{c^2}{B^2} \int dT \int d\tau \ \langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle \qquad (65)$$

The region of integration in (65) is as shown below (Fig. II-1).

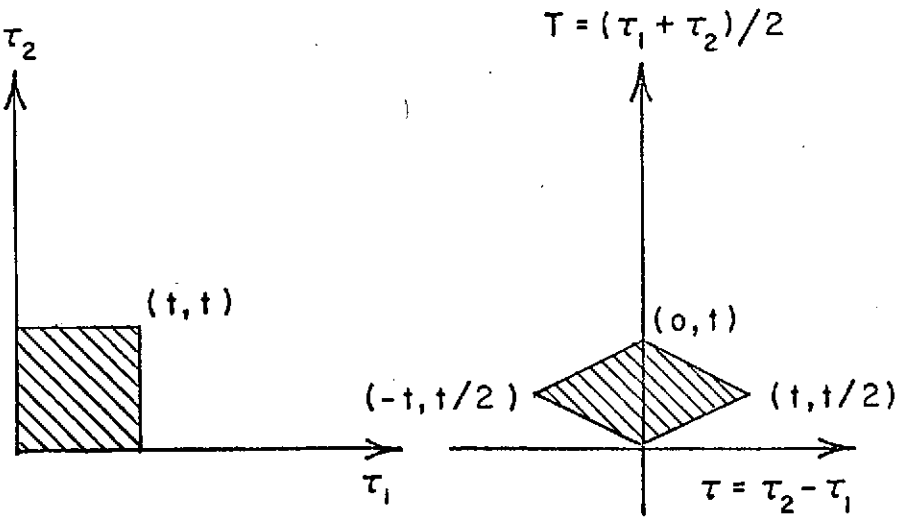

Figure II-1. Regions of integration for Eqs. (64) and (65). For large t, the non-vanishing part of the integrand will lie in the neighborhood of the τ=0 axis.

Now *if* $\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle$ goes to zero sufficiently rapidly (more rapidly than $1/\tau$) for large τ, we may consider the case when t is >> the time in which $\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle$ goes to zero, and write, from (65),

$$\langle \vec{x}^2 \rangle \cong \frac{c^2}{B^2} \int_0^t d\tau \int_{-\infty}^{\infty} d\tau \; \langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle \tag{66}$$

$$= t \left\{ \frac{2c^2}{B^2} \int_0^{\infty} \langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle \, d\tau \right\}$$

The ratio $\langle \vec{x}^2 \rangle / 2t$ thus becomes time-independent for long times and this ratio D,

$$D = \frac{c^2}{B^2} \int_0^{\infty} \langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle \, d\tau \tag{67}$$

is called the *diffusion coefficient*. It is generally thought of as a good measure of the diffusiveness of a medium. (It is also frequently defined by $\langle \vec{x}^2 \rangle / 2td$, where d is the dimensionality of the system).

Recently a lot of interest has arisen in the possibility that $\langle \vec{x}^2 \rangle$ may grow *faster* than t as $t \to \infty$ in two dimensions for a variety of systems [32]. For example, if $\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle$ were to fall off as $O(1/\tau)$ for large τ, detailed examination of (65) reveals that $\langle \vec{x}^2 \rangle \sim t \ln t$ for large t. This has been hard for many people to accept, but it has also shown up now in the plasma case. The reason the $\langle \vec{x}^2 \rangle \sim t$ behavior is so well accepted is that it can be derived *regardless of dimensionality* [31], from a random-walk model in which a random walker takes a large number n of uncorrelated steps per unit time, each step of very small length σ, with $n\langle \sigma^2 \rangle = \text{const.}$. (D is in fact, just proportional to $n\langle \sigma^2 \rangle$.) This is true for virtually any distribution law for the steps. If $\langle \vec{x}^2 \rangle$ turns out to grow *faster* than t for systems of dimensionality less than three, then it means that the venerable random walk model is probably less accurate than we had supposed for these systems.

In any case, we will be concerned in the following pages with attempts to calculate the function

$$D(t) \equiv \frac{c^2}{B^2} \int_0^t \langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle \, d\tau \tag{68}$$

for the guiding-center plasma whose dynamics are given by (51) and (52). *If* the integral converges, it will be called the "diffusion coefficient",

$$D = D(\infty) \qquad \text{(if it exists!)} \; . \tag{69}$$

But if the integral D(t) fails to approach a limit, it still is an interesting measure of the way the test particles diffuse.

(E) Evaluation of $\int_0^t \langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle \, d\tau$ *and D*

It is convenient to evaluate (68) in a large but finite volume $V = L^2$. $\vec{E}(\tau) = \vec{E}[\vec{x}(\tau),\tau]$, and the exact electric field due to the 2N charges is

$$\vec{E}(\vec{x},t) = - \sum_{\vec{k}} \frac{4\pi i\vec{k}}{k^2} \rho_{\vec{k}}(t) \, e^{i\vec{k}\cdot\vec{x}} \tag{70}$$

where the Fourier transform of the charge density is

$$\rho_{\vec{k}}(t) = \frac{1}{V} \int d\vec{x} \; e^{-i\vec{k}\cdot\vec{x}} \, \rho(\vec{x},t) \tag{71}$$

Using (34), and inserting the result into (70) gives

$$\vec{E}(\vec{x},t) = - \sum_{\vec{k}} \sum_{j} \frac{e_j}{\ell} \frac{4\pi i\vec{k}}{k^2 V} \, e^{i\vec{k}\cdot[\vec{x}-\vec{x}_j(t)]} \tag{72}$$

where the sum is over all charges of both signs, and the values of $\vec{k}$ have components which are integer multiples of $2\pi/L$ (we impose periodic boundary conditions).

$\vec{E}(\tau) = \vec{E}[\vec{x}(\tau), \tau]$ is then

$$\vec{E}(\tau) = \sum_{\vec{k},j} \frac{4\pi e_j \vec{k}}{ik^2 \ell V} \, e^{i\vec{k}\cdot[\vec{x}(\tau)-\vec{x}_j(\tau)]} \tag{73}$$

where $\vec{x}(\tau)$ is the trajectory of a particle which starts at the origin: $\vec{x}(0) = 0$. For shorthand, (73) will also be written as

$$\vec{E}(\tau) = \sum_{\vec{k}} \vec{E}_{\vec{k}}(\tau)\, e^{i\vec{k}\cdot\vec{x}(\tau)} \tag{74}$$

in an obvious notation.

The problem is then one of evaluating

$$\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle = \sum_{\vec{k}_1,\vec{k}_2} \left\langle \vec{E}_{\vec{k}_1}(0) \cdot \vec{E}_{\vec{k}_2}(\tau) \exp\{i\vec{k}_2 \cdot \vec{x}(\tau)\} \right\rangle \tag{75}$$

This result is exact, and we cannot go further without an approximation or an hypothesis. One which can be made [28] but which some day will have to be given careful scrutiny is to neglect the correlations between the positions of the test particles and those of the background plasma, the $\vec{x}_j(\tau)$. That assumption reduces (75) to

$$\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle \cong \sum_{\vec{k}_1,\vec{k}_2} \left\langle \vec{E}_{\vec{k}_1}(0) \cdot \vec{E}_{\vec{k}_2}(\tau) \right\rangle \left\langle e^{i\vec{k}_2 \cdot \vec{x}(\tau)} \right\rangle \tag{76}$$

Further, noting that $\langle \vec{E}_{\vec{k}_1}(t_1)\, \vec{E}_{\vec{k}_2}(t_2) \rangle = 0$ unless $\vec{k}_1 + \vec{k}_2 = 0$ for a spatially-uniform ensemble, and that $\vec{E}_{-\vec{k}}(t) = \vec{E}^*_{\vec{k}}(t)$ since $\vec{E}$ is real, we have, from (76),

$$\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle = \sum_{\vec{k}} \left\langle \vec{E}^*_{\vec{k}}(0) \cdot \vec{E}_{\vec{k}}(\tau) \right\rangle \left\langle e^{i\vec{k}\cdot\vec{x}(\tau)} \right\rangle \tag{77}$$

The next step necessarily involves expressing the two terms on the right-hand side of (77) in an approximate way in terms of $\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle$, to provide an equation to be solved for this quantity. The last term can be written as

$$\langle \exp[-i\vec{k} \cdot \vec{x}(\tau)] \rangle$$

$$= \left\langle \exp\left[- \frac{ic\vec{k}}{B^2} \cdot \int_0^\tau d\xi\ \vec{E}(\xi) \times \vec{B} \right] \right\rangle$$

$$= \lim_{M\to\infty} \left\langle \exp\left[\frac{ic\vec{k} \times \vec{B}}{B^2} \Delta t \cdot \sum_{m=1}^{M} \vec{E}(m) \right] \right\rangle \tag{78}$$

where $\vec{E}(\vec{m})$ is the electric field between times $(m - 1)\tau/M$ and $m\tau/M$. Δt is τ/M.

The computation of the expectation value in (78) involves not just the probability distribution of the electric field seen by the particle, but the *joint* probability distribution of the electric field at a whole sequence of times, $\vec{E}(1)$, $\vec{E}(2)$, $\vec{E}(3)$. . . If we call this joint probability distribution $P_M[\vec{E}(1),\vec{E}(2),\ldots \vec{E}(M)]$, then formally

$$\langle \exp[-i\vec{k} \cdot \vec{x}(\tau)] \rangle$$

$$= \lim_{M\to\infty} \int d\vec{E}(1) \ldots d\vec{E}(M)\ P[\vec{E}(1) \ldots \vec{E}(M)] \tag{79}$$

$$\exp\left[\frac{ic\vec{k} \times \vec{B}}{B^2} \Delta t \cdot \sum_{m=1}^{M} \vec{E}(m) \right]$$

It has not been possible to establish an expression for $P[\vec{E}(1) \ldots \vec{E}(M)]$ from first principles[33]. A reasonable conjecture is that it is a *jointly normal probability distribution* [34,35]; that is,

$$P[\vec{E}(1) \ldots \vec{E}(M)] = \eta \exp\left[- \sum_{i,j=1}^{M} \vec{\vec{a}}_{ij} : \vec{E}(i)\ \vec{E}(j) \right] \tag{80}$$

where η is a normalizing constant and the $\vec{\vec{a}}_{ij}$ are constant 2×2 positive-definite dyadics. A generalization of the central limit theorem, for example, makes (80) look extremely plausible [35]. In the present context, the theorem says the following. Consider $\vec{E}(1),\vec{E}(2), \ldots$ to be the sum $\Sigma_j[\vec{E}_j(1),\vec{E}_j(2),\ldots]$.

Here $\vec{E}_j(n)$ is the electric field at the test particle produced by particle j in time interval n, and the Σ_j runs over all the plasma charges. Then, under very weak conditions on the moments of the probability distributions of the individual $\vec{E}_j(n)$, and the additional assumption that the $\vec{E}_j(n)$ are uncorrelated with each other for different j, the probability distribution of the Σ_j (where j goes from 1 to 2N) converges to (80) for $N \to \infty$. This neglect of correlation between the trajectories of the plasma particles is the key approximation, and is made quite plausible by the weakness of the pair correlation [Eqs. (21) to (25)] for small plasma parameter.

It is shown in Appendix I that the result of substituting (80) into (79) is

$$\langle \exp[-i\vec{k} \cdot \vec{x}(\tau)] \rangle$$

$$\lim_{M\to\infty} \exp\left[-\frac{c^2}{2} \sum_{m_1=1}^{M} \sum_{m_2=1}^{M} \langle \vec{E}(m_1)\, \vec{E}(m_2) \rangle : \frac{(\vec{k} \times \vec{B})(\vec{k} \times \vec{B})}{B^4} \Delta t\, \Delta t \right]$$

$$= \exp\left[-\frac{c^2 k^2}{2B^2} \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \langle \vec{E}(\tau_1)\, \vec{E}(\tau_2) \rangle : (\hat{k} \times \hat{B})(\hat{k} \times \hat{B}) \right]$$

$$= \exp\left[-\frac{c^2 k^2}{4B^2} \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \langle \vec{E}(\tau_1) \cdot \vec{E}(\tau_2) \rangle \right] \qquad (81)$$

Inserting (81) into (77) yields the result that

$$\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle$$

$$\sum_{\vec{k}} \left\langle \vec{E}^*_{\vec{k}}(0) \cdot \vec{E}_{\vec{k}}(\tau) \right\rangle \exp\left\{ -\frac{c^2 k^2}{4B^2} \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \langle \vec{E}(\tau_1) \cdot \vec{E}(\tau_2) \rangle \right\} \qquad (82)$$

Equation (82) is a satisfactory starting point for calculation of $\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle$ as soon as we have an acceptable expression for $\langle \vec{E}^*_{\vec{k}}(0) \cdot \vec{E}_{\vec{k}}(\tau) \rangle$. First, note that

$$\langle \vec{E}^*_{\vec{k}}(0) \cdot \vec{E}_{\vec{k}}(\tau) \rangle$$

$$= \sum_{j,\ell} \frac{16\pi^2 e_j e_\ell}{\ell^2 k^2 V^2} \left\langle e^{i\vec{k}\cdot[\vec{x}_j(0)-\vec{x}_\ell(\tau)]} \right\rangle \tag{83}$$

$$= \sum_{j,\ell} \frac{16\pi^2 e_j e_\ell}{\ell^2 k^2 V^2} \left\langle \exp\left\{ i\vec{k} \cdot \left[\vec{x}_j(0) - \vec{x}_\ell(0) - c\int_0^\tau d\xi \, \frac{\vec{E}_\ell(\xi) \times \vec{B}}{B^2} \right] \right\} \right\rangle$$

By $\vec{E}_\ell(\tau)$, we mean the electric field seen by the ℓth particle, of charge e_ℓ. In (83), we can treat the ℓth particle in the same approximations we treated the test particles in Eqs. (76) through (82), getting

$$\langle \vec{E}^*_{\vec{k}}(0) \cdot \vec{E}_{\vec{k}}(\tau) \rangle$$

$$\cong \sum_{j,\ell} \frac{16\pi^2 e_j e_\ell}{\ell^2 k^2 V} \left\langle \exp i\vec{k} \cdot [\vec{x}_j(0) - \vec{x}_\ell(0)] \right\rangle \cdot$$

$$\left\langle \exp \frac{ic\vec{k} \times \vec{B}}{B^2} \cdot \int_0^\tau \vec{E}_\ell(\xi)\, d\xi \right\rangle$$

$$= \left\langle \vec{E}^*_{\vec{k}}(0) \cdot \vec{E}_{\vec{k}}(0) \right\rangle \exp\left\{ -\frac{c^2 k^2}{4B^2} \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \, \langle \vec{E}(\tau_1) \cdot \vec{E}(\tau_2) \rangle \right\} \tag{84}$$

Since the $\vec{E} \times \vec{B}$ drift is independent of both the mass and sign of the charge, the statistical properties of $\vec{E}_\ell(\xi)$ will be the same for all ℓ.

Inserting (84) into (82) gives

$$\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle \tag{85}$$

$$= \sum_{\vec{k}} \langle |\vec{E}_{\vec{k}}|^2 \rangle \exp \left\{ - \frac{c^2 k^2}{2 B^2} \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \langle \vec{E}(\tau_1) \cdot \vec{E}(\tau_2) \rangle \right\}$$

Since $\langle |\vec{E}_{\vec{k}}|^2 \rangle$ is a stationary quantity obtainable from the methods of Section I, Eq. (85) is now a closed equation that can be used to determine $\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle$. Defining $R(\tau) \equiv (c^2/2B^2)$.

$\int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \, Q(\tau_2 - \tau_1)$, where $Q(\tau) \equiv \frac{1}{2} \langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle$

$dR(\tau)/d\tau = (c^2/B^2) \int_0^\tau Q(\xi) \, d\xi$, and $d^2R/d\tau^2 = c^2 \, Q(\tau)/B^2$, which means that (85) can be expressed as

$$2 \frac{B^2}{c^2} \frac{d^2 R}{d\tau^2} = \sum_{\vec{k}} \langle |\vec{E}_{\vec{k}}|^2 \rangle \exp [-2 k^2 R(\tau)] \tag{86}$$

From (65), R(t) has the physical interpretation of being $\langle \vec{x}^2 \rangle / 4$, where $\langle \vec{x}^2 \rangle$ is the mean square spread after time t of the distribution of test particles.

Equation (86) has a first integral

$$\frac{1}{2} \left(\frac{dR}{d\tau} \right)^2 - \frac{c^2}{2 B^2} \sum_{\vec{k}} \langle |\vec{E}_{\vec{k}}|^2 \rangle \left\{ \frac{1 - \exp(-2 k^2 R)}{2 k^2} \right\} = 0 \tag{87}$$

where the fact that $dR(0)/d\tau = 0$ has been used to evaluate the constant of integration. To proceed further with Eq. (87) requires numerical integration, if $R(\tau)$ is desired. This integration is discussed in a later section. We can extract the long time behavior analytically, however, by noting that $R(\tau) \xrightarrow[\tau\to\infty]{} +\infty$, so that

$$\left(\frac{dR}{d\tau}\right)^2 \xrightarrow[\tau\to\infty]{} \frac{c^2}{2B^2} \sum_{\vec{k}} \frac{\langle |\vec{E}_{\vec{k}}|^2 \rangle}{k^2} \tag{88}$$

or

$$D(t) = \frac{2c^2}{B^2} \int_0^t Q(\tau)\, d\tau = 2\,\frac{dR(t)}{dt}$$

approaches the limit

$$D(\infty) = 2\left[\frac{c^2}{2B^2} \sum_{\vec{k}} \frac{\langle |\vec{E}_{\vec{k}}|^2 \rangle}{k^2}\right]^{\frac{1}{2}} \tag{89}$$

Up to a factor of $\sqrt{2}$, this is the result of Taylor and McNamara [28].

Several remarks are in order concerning (89) before we pass to the details of its evaluation. First, it varies as $1/B$, confirming the elusive conjecture of Bohm [20]. Secondly, it will contain a factor $\sqrt{\varepsilon}$, since it contains $\sum_{\vec{k}} \langle |\vec{E}_{\vec{k}}|^2 \rangle / k^2$, and will vanish in the Vlasov limit ($\varepsilon \to 0$). Finally, the expression will *diverge* in the infinite volume limit, since the minimum value of $|\vec{k}|$ is $2\pi/L$, and a factor $1/k^2$ occurs in the $\Sigma_{\vec{k}}$. This is a reflection of the fact that the long-wavelength, small-k contributions to (89) are the dominant terms, in contrast to "classical" diffusion models based on random two-body encounters.

We may estimate (89) for a large volume $V = L^2$ by replacing the sum over discrete values of $\vec{k}$ by an integral over continuous $\vec{k}$ according to the prescription

$$\sum_{\vec{k}} \langle |\vec{E}_{\vec{k}}|^2 \rangle \longrightarrow \int d\vec{k} \ \text{Trace} \ \vec{\vec{S}}_E(\vec{k})$$

(90)

$$= 2\pi \int k \, dk \ \frac{\theta}{\ell\pi} \ \frac{1}{1 + k^2 \lambda_D^2}$$

where use has been made of (43). The integral in (90) is to be cut off at $k_{min.} = 2\pi/L$ from below. The result of using (90) in (89) for the diffusion coefficient is

$$D(\infty) = \frac{2c}{B} \left[\frac{\theta}{\ell} \int_{k_{min}}^{\infty} \frac{dk}{k(1 + k^2 \lambda_D^2)} \right]^{\frac{1}{2}}$$

(91)

$$\cong \frac{2c}{B} \left(\frac{\theta}{\ell} \right)^{\frac{1}{2}} \ell n \, (L/2\pi \lambda_D)^{\frac{1}{2}}$$

$$= \sqrt{2} \, \frac{c\theta}{eB} \, e^{\frac{1}{2}} \, [\ell n \, (L/2\pi \lambda_D)]^{\frac{1}{2}}$$

The Bohm result, never justified in detail, was $D(\infty) = \frac{1}{16} \frac{c\theta}{eB}$.

Notice, however, the following limitation in the derivation of Eqs. (89) and (91). It was arrived at by assuming that the terms involving $\exp(-2k^2R)$ in (87) had become $<< 1$. Since $k_{min.} = 2\pi/L$ and $R = \langle \vec{x}^2 \rangle / 4$, this is equivalent to

$$2 \, \frac{\langle \vec{x}^2 \rangle}{4} \ \frac{4\pi^2}{L^2} \gg 1$$

or $\langle \vec{x}^2 \rangle \gg L^2/2\pi^2 \cong L^2/20$. However, when $\langle \vec{x}^2 \rangle$ becomes $\gtrsim L^2$, the test particles will have encountered the boundary once,

and in view of the periodic boundary conditions, the diffusion model with its monotonically increasing $\langle \vec{x}^2 \rangle$ will have broken down. Thus there is a relatively narrow temporal interval,

$$\frac{L^2}{20} \ll 2t\, D(\infty) \ll L^2 \tag{92}$$

in which (89) and (91) may be expected to apply. Both before and after this interval, the development of $\langle \vec{x}^2 \rangle$ will be more complicated.

Finally, we remark that if one attempts to apply (86) to *unbounded* two-dimensional plasmas, a variation [36] of $\langle \vec{x}^2 \rangle \sim t\sqrt{\ln t}$ occurs for large values of t.

(F) Liouville Equation, BBGKY Hierarchy, and the Time-Dependent "Vlasov" Limit [12]

The diffusion calculation could proceed in a relatively unsystematic way, with the assumption (80) as the key ingredient For other purposes, a more systematic formalism for the kinetic theory of the two-dimensional guiding center plasma is desirable. Proceeding in parallel to the well-known program of the non-equilibrium BBGKY [2-6,11] hierarchy, it is useful to construct a description based on the *Liouville equation*.

The phase-space probability distribution of any Hamiltonian system obeys a Liouville equation, which simply expresses the constancy of that probability distribution as seen by an observer who moves along any phase-space trajectory traced out by the system as it moves through its phase space. Since we have seen the system to be Hamiltonian in Eqs. (59) to (61), we can write down the Liouville equation immediately as

$$\frac{dD}{dt} = \left(\frac{\partial}{\partial t} + \sum_i \frac{d\vec{x}_i(t)}{dt} \cdot \frac{\partial}{\partial \vec{x}_i} \right) D = 0 \tag{93}$$

where $D = D(\vec{x}_1, \vec{x}_2, \ldots, t)$ is the probability distribution of all the particles in their phase space. Noting that $d\vec{x}_i(t)/dt = c\,\vec{E}(\vec{x}_i,t) \times \vec{B}/B^2$, (93) becomes

$$\left(\frac{\partial}{\partial t} + \sum_i \frac{c\vec{E}(\vec{x}_i,t) \times \vec{B}}{B^2} \cdot \frac{\partial}{\partial \vec{x}_i} \right) D = 0, \tag{94}$$

where $\vec{E}(\vec{x}_i,t) = + \frac{\partial}{\partial \vec{x}_i} \sum_{j\neq i} \frac{2e_j}{\ell} \ln |\vec{x}_i - \vec{x}_j|$, and the summations run over all charges of both signs. The single most important solution of (94) is, of course, $D_{eq.} = \exp\left(-\sum_{i<j} \varphi_{ij}/\theta\right)/\int dX \exp\left(-\sum_{i<j} \varphi_{ij}/\theta\right)$, though for a non-equilibrium situation, D will *not* be $D_{eq.}$. We shall generally assume, however, that the various correlation functions have the same *orders of magnitude* as the thermal equilibrium correlations of Section I. We also assume D to be symmetric under interchange of like particle coordinates. The reduced probability distributions are defined in analogy to Eqs. (9) as

$$f_{s_i,s_e} = V^{s_i+s_e} \int d\vec{x}^i_{s_i+1} \cdots d\vec{x}^i_N \, d\vec{x}^e_{s_e+1} \cdots d\vec{x}^e_N \, D \qquad (95)$$

where it is to be noted that

$$f_{s_i,s_e} = f_{s_i,s_e}(\vec{x}^{\,i}_1 \cdots \vec{x}^{\,i}_{s_i}, \vec{x}^{\,e}_1 \cdots \vec{x}^{\,e}_{s_e}, t)$$

is a function of *position space coordinates* only (the dimensionality of the phase space is now 4N).

Integrating (94) over a big box of volume $V = L^2$ gives:

$$\left\{ \frac{\partial}{\partial t} + \sum_{j=1}^{s_i} \frac{c\,\vec{E}_j^{(s_i,s_e)} \times \vec{B}}{B^2} \cdot \frac{\partial}{\partial \vec{x}^i_j} + \sum_{k=1}^{s_e} \frac{c\,\vec{E}_k^{(s_i,s_e)} \times \vec{B}}{B^2} \cdot \frac{\partial}{\partial \vec{x}^e_k} \right\} f_{s_i,s_e}$$

$$= n_o \sum_{j=1}^{s_i} \int d\vec{x}^i_{s_i+1} \left[\frac{\partial \varphi^{ii}\left(\vec{x}^i_j - \vec{x}^i_{s_i+1}\right)}{\partial \vec{x}^i_j} \times \frac{\vec{B}}{B^2} \right] \cdot \frac{\partial f_{s_i+1,\cdots}}{\partial \vec{x}^i_j}$$

$$+ n_o \sum_{k=1}^{s_e} \int d\vec{x}^{i}_{s_i+1} \left[\frac{\partial \varphi^{ie}\left(\vec{x}^{e}_{k} - \vec{x}^{i}_{s_i+1}\right)}{\partial \vec{x}^{e}_{k}} \times \frac{\vec{B}}{B^2} \right] \cdot \frac{\partial f_{s_i+1, s_e}}{\partial \vec{x}^{e}_{k}}$$

$$+ n_o \sum_{j=1}^{s_e} \int d\vec{x}^{e}_{s_e+1} \left[\frac{\partial \varphi^{ei}\left(\vec{x}^{i}_{j} - \vec{x}^{e}_{s_e+1}\right)}{\partial \vec{x}^{i}_{j}} \times \frac{\vec{B}}{B^2} \right] \cdot \frac{\partial f_{s_i, s_e+1}}{\partial \vec{x}^{i}_{j}}$$

$$+ n_o \sum_{k=1}^{s_e} \int d\vec{x}^{e}_{s_e+1} \left[\frac{\partial \varphi^{ee}\left(\vec{x}^{e}_{k} - \vec{x}^{e}_{s_e+1}\right)}{\partial \vec{x}^{e}_{k}} \times \frac{\vec{B}}{B^2} \right] \cdot \frac{\partial f_{s_i, s_e+1}}{\partial \vec{x}^{e}_{k}} \qquad (96)$$

This is the *non-equilibrium* BBGKY hierarchy for the system, and is the analogue of (13). By $\vec{E}_j(s_i, s_e)$, we mean $-\frac{\partial}{\partial \vec{x}^{i}_{j}} \cdot \Phi(s_i, s_e)$, where $\Phi(s_i, s_j)$ is the $\sum_{i<j} \phi_{ij}$ for the first s_i ions and the first s_e electrons. Similarly $\vec{E}_k(s_i, s_e) = -\partial \Phi(s_i, s_e)/\partial \vec{x}^{e}_{k}$.

Equations (94) and (96) are not as complicated as they look, and we can profitably introduce abbreviations for the linear differential and integral operators which appear in them. Thus, (94) will sometimes be written as $\left(\frac{\partial}{\partial t} + H_o\right) D = 0$, and (96) as

$$\left[\frac{\partial}{\partial t} + H(s_i, s_e)\right] f_{s_i, s_e} = \left(L^{ii}_{s_i s_e} + L^{ie}_{s_i s_e}\right) f_{s_i+1, s_e} + \left(L^{ei}_{s_i s_e} + L^{ee}_{s_i s_e}\right) f_{s_i, s_e+1}$$

in an obvious notation. H_o is the

Liouville operator for all 2N particles, and $H(s_i,s_e)$ is the Liouville operator for s_i ions and s_e electrons.

The BBGKY hierarchy (96) is analogous to the well-known hierarchy for ordinary plasmas but it has important differences which are worth mentioning. It is interesting to consider the "Vlasov limit" in which the correlations vanish; the corresponding thermal equilibrium limit (see Section I) is that in which $\varepsilon \to 0$. Thus we write

$$f_{s_i,s_e} = \left[\prod_{j=1}^{s_i} f_{1,0}(\vec{x}^i_j,t)\right]\left[\prod_{k=1}^{s_e} f_{0,1}(\vec{x}^e_j,t)\right] \text{ and all of Eqs. (96)}$$

become satisfied if

$$\frac{\partial f_{1,0}}{\partial t}(\vec{x}^i_1,t) + \frac{c\,\vec{E}(\vec{x}^i_1,t) \times \vec{B}}{B^2} \cdot \frac{\partial f_{1,0}(\vec{x}^i_1,t)}{\partial \vec{x}^i_1} = 0$$

$$\frac{\partial f_{0,1}}{\partial t}(\vec{x}^e_1,t) + \frac{c\,\vec{E}(\vec{x}^e_1,t) \times \vec{B}}{B^2} \cdot \frac{\partial f_{0,1}(\vec{x}^e_1,t)}{\partial \vec{x}^e_1} = 0 \qquad (97)$$

where

$$\vec{E}(\vec{x},t) = +\frac{\partial}{\partial \vec{x}}\, n_o \int \left[f_{1,0}(\vec{x}',t) - f_{0,1}(\vec{x}',t)\right] \frac{2e}{\ell} \ln|\vec{x} - \vec{x}'|\, d\vec{x} \qquad (98)$$

Equations (97) and (98) are the analogues of the classical Vlasov-Poisson system for the two-dimensional guiding-center plasma. The spatially-uniform state, $f_{1,0} = f_{0,1} = 1$ and $\vec{E} = 0$, is obviously a solution. However, the analogue of the Landau problem, in which one linearizes about the spatially uniform state, is apparently insoluble; for the first non-vanishing terms in (97) are intrinsically of second order in powers of the departure from spatial uniformity.

Equations (97), (98) are completely equivalent to the equations of motion for incompressible, inviscid flow. They can be made to look simpler by subtracting Eqs. (97) from each other to get

$$\frac{\partial \rho}{\partial t} + \frac{c\,\vec{E} \times \vec{B}}{B^2} \cdot \frac{\partial \rho}{\partial \vec{x}} = 0 \tag{99}$$

where

$$\frac{\partial}{\partial \vec{x}} \cdot \vec{E} = 4\pi\rho \tag{100}$$

Very little information exists on the solutions of the system (99), (100), although a number of numerical simulations exist which are relevant [37-39]. Taylor and Thompson [40] have argued for the existence of a large class of *oscillatory* solutions of this system with a frequency which vanishes as the amplitude of $|\vec{E}|^2$ vanishes. This derivation, however, is not wholly transparent. Any oscillations which do exist *will* be intrinsically nonlinear.

Another peculiar feature of the hierarchy (96) is the apparent absence of any tendency toward thermal equilibrium. For the conventional hierarchy for a normal plasma, assuming an initially spatially uniform state leads to rapidly-developing pair correlations. These in turn relax to functionals of $f_{1,0}$ and $f_{0,1}$ which, when substituted into the equations for $\partial f_{1,0}/\partial t$ and $\partial f_{0,1}/\partial t$, lead to the Balescu-Lenard equation [7,41-43]. From this, an approach to thermal equilibrium can be proved. The situation is quite different for the guiding-center plasma. It is clear by inspection that the completely uncorrelated state, in which $f_{s_i,s_e} = \prod_{j=1}^{s_i} \prod_{k=1}^{s_e} f_{1,0}(\vec{x}^i_j)\, f_{0,1}(\vec{x}^e_k)$ with $f_{1,0} = f_{0,1} = 1$, is a *time-independent* solution of (96)! It is not known whether there is *any* sense in which the solutions of (96) relax to the thermal equilibrium values (11). This is one of the really mysterious parts of the theory. A number of time independent solutions with non-thermal equilibrium correlations for (96) can be constructed, but there is no particular reason for preferring one over the other.

(G) Calculation of the Electrical Conductivity by the Kubo Method [44,45]

A calculation of a second transport coefficient, the electrical conductivity, proceeds most naturally from the Liouville equation, Eq. (94) will be abbreviated as

$$\left(\frac{\partial}{\partial t} + H_o\right) D = 0 \tag{94a}$$

The addition of a spatially uniform electric field $\vec{E}_o e^{-i\omega t}$ modifies (94a) to read

$$\left(\frac{\partial}{\partial t} + H_o\right) D = - \frac{c \vec{E}_o \exp(-i\omega t) \times \vec{B}}{B^2} \cdot \left\{ \sum_{j=1}^{N} \left(\frac{\partial D}{\partial \vec{x}_j^{i}} + \frac{\partial D}{\partial \vec{x}_j^{e}} \right) \right\} \equiv - \exp(-i\omega t) H_1 D \tag{101}$$

where the right-hand side of Eq. (101) defines the linear operator H_1. We now regard $\vec{E}_o$ as *weak*, and seek a perturbation solution to (101) in the form $D = D^{(0)} + D^{(1)} + \ldots$, where the superscript indicates the order in H_1. Thus $D^{(0)}$ obeys

$$\left(\frac{\partial}{\partial t} + H_o\right) D^{(0)} = 0 \tag{102}$$

and $D^{(1)}$ obeys

$$\left(\frac{\partial}{\partial t} + H_o\right) D^{(1)} = - \exp(-i\omega t) H_1 D^{(0)} \tag{103}$$

The appropriate solution to (102) is the Gibbs distribution

$$D^{(0)} = \eta \exp\left(- \sum_{i<j} \varphi_{ij}/\theta\right) \tag{104}$$

and the $\sum_{i<j} \phi_{ij}$ runs over all pairs of particles of both signs. As usual, η is chosen so that $\int D^{(0)} \, dX = 1$. Since (102) is simply a statement of the constancy of $D^{(0)}$ along a trajectory in the total phase space, it is just a statement that the formal solution

$$D^{(0)}(t) = e^{-tH_o} D^{(0)}(0) \tag{105}$$

is just a function of the constants of the motion [6]. $\sum_{i<j} \phi_{ij}$ has already been noted to be such a constant, and (104) is the solution appropriate to thermal equilibrium.

We imagine the external uniform field $\vec{E}_o$ to be turned on abruptly at t = 0. The solution to (103) can be found by integrating the equivalent equation

$$\frac{\partial}{\partial t}\left(e^{tH_o} D^{(1)}\right) = - e^{tH_o} e^{-i\omega t} H_1 e^{-tH_o} D^{(0)} \qquad (106)$$

between t = 0 and t = t. Setting $D^{(1)} = 0$ at t = 0, noting that $e^{-tH_o} D^{(0)} = D^{(0)}$, and rearranging the integral gives

$$D^{(1)} = - e^{i\omega t} \int_o^t d\tau\, e^{i\omega\tau} e^{-\tau H_o} H_1 D^{(0)} \qquad (107)$$

Now for any dynamical variable whose value is determined by the phase space coordinates (A, say), the expectation value $\langle A \rangle = \int A\, D\, dX$. For variables which have expectation values which vanish in thermal equilibrium, we have the result that to first order in E_o, $\langle A \rangle = \int A\, D^{(1)}\, dX$. Introducing the explicit form of H_1, we have

$$H_1 D^{(0)} = + \frac{c \vec{E}_o \times \vec{B}}{B^2} \cdot \sum_j \left[\frac{e \vec{E}(\vec{x}_j^{\,i})}{\theta} - \frac{e \vec{E}(\vec{x}_j^{\,e})}{\theta}\right] D^{(0)} \qquad (108)$$

$$= - \frac{1}{\theta} \vec{E}_o \cdot \sum_j e(\vec{v}_j^{\,i} - \vec{v}_j^{\,e})\, D^{(0)}$$

We now note that the effect of the operator $e^{-\tau H_o}$ applied to any function of the coordinates obeying Liouville's equation is to trace those coordinates back τ units along the phase space trajectory and assign them their values there. Thus, for instance $e^{-\tau H_o} \vec{v}_j^{\,i} = \vec{v}_j^{\,i}(-\tau)$, where $\vec{v}_j^{\,i}(-\tau)$ is the velocity that the jth ion would have had τ seconds ago if it now has

velocity $\vec{v}_j^{\,i} = \vec{v}_j^{\,i}(0)$. Therefore

$$\exp(-\tau H_o)\, H_1\, D^{(0)} = -\frac{1}{\theta}\vec{E}_o \cdot \sum_j e\left[\vec{v}_j^{\,i}(-\tau) - \vec{v}_j^{\,e}(-\tau)\right] D^{(0)} \tag{109}$$

since $D^{(0)}$ is constant along the phase space trajectory. Inserting (109) into (107) and the result into the expression for <A> gives

$$\langle A\rangle = \frac{\exp(-i\omega t)}{\theta}\int_o^\infty d\tau\, \exp(i\omega\tau) \tag{110a}$$

$$\langle A \sum_j e\,\vec{E}_o \cdot [\vec{v}_j^{\,i}(-\tau) - \vec{v}_j^{\,e}(-\tau)]\rangle$$

We have not yet committed ourselves to an explicit expression for the dynamical variable A. To calculate the electric current density produced in response to the electric field $\vec{E}_o$, we pick A a function which makes <A> equal to the volume current density $\langle\vec{j}\rangle$. We must remember to apply this procedure only far away from the plasma boundary. Thus

$$A \to \vec{A} = \sum_j e(\vec{v}_j^{\,i} - \vec{v}_j^{\,e})/\ell L^2$$

Noting that $\langle\vec{v}(0)\cdot\vec{v}(\tau)\rangle$ will be an even function of τ for all particles of both signs,

$$\langle\vec{A}\rangle = \langle\vec{j}\rangle_\omega\, e^{-i\omega t}$$

$$= \frac{e^2\exp(-i\omega t)}{\ell L^2\theta}\int_o^\infty d\tau\, e^{i\omega\tau}$$

$$\sum_{j,\ell} \langle [\vec{v}^{i}_{j}(0) - \vec{v}^{e}_{j}(0)]\ [\vec{v}^{i}_{\ell}(\tau) - \vec{v}^{e}_{\ell}(\tau)]\rangle \cdot \vec{E}_{o} \qquad (110b)$$

The coefficient of $\vec{E}_o \exp(-i\omega t)$ in (110b) can be identified as the *conductivity tensor* $\vec{\vec{\sigma}}(\omega)$:

$$\vec{\vec{\sigma}}(\omega) = \frac{e^2}{\ell\,\theta\,L^2} \int_0^\infty d\tau\, e^{i\omega\tau} \sum_{j,\ell} \langle [\vec{v}^{i}_{j}(0) - \vec{v}^{e}_{j}(0)]\ [\vec{v}^{i}_{\ell}(\tau) - \vec{v}^{e}_{\ell}(\tau)]\rangle \qquad (111)$$

The terms in (111) with $j \neq \ell$ and not of the same species can be shown to give a contribution which is down by a factor of $0(1/N)$ from those *with* $j = \ell$. Therefore $\vec{\vec{\sigma}}(\omega)$ becomes, since all the $j = \ell$ terms of the same species are equal.

$$\vec{\vec{\sigma}}(\omega) = \frac{2Ne^2}{\ell L^2 \theta} \int_0^\infty d\tau\, e^{i\omega\tau} \langle \vec{v}(0)\ \vec{v}(\tau)\rangle$$

$$= \frac{2n_o e^2 c^2}{\ell\,\theta\,B^4}\ \vec{B} \times \int_0^\infty d\tau\, e^{i\omega\tau} \langle \vec{E}(0)\ \vec{E}(\tau)\rangle \times \vec{B} \qquad (112)$$

The $\langle \vec{E}(0)\ \vec{E}(\tau)\rangle$ is the same quantity whose trace appears in Eq. (64), and the same methods can be used to calculate it. Under the apparently straightforward assumption that $\langle \vec{E}(0)\ \vec{E}(\tau)\rangle$ is diagonal (for it to be otherwise would imply a preferred direction in space), we have

$$\vec{\vec{\sigma}}(\omega) = \vec{\vec{I}}\,\sigma(\omega)$$

where

$$\sigma(\omega) = \frac{2n_o e^2 c^2}{\ell\ \theta\ B^2} \int_0^\infty Q(\tau)\, e^{i\omega\tau}\, d\tau \qquad (113)$$

with $Q(\tau)$ given by $\frac{1}{2}\langle \vec{E}(0) \cdot \vec{E}(\tau) \rangle$, as before.

In the dc limit, $\omega = 0$,

$$\sigma(0) = \frac{2 n_o e^2 c^2}{\ell\, \theta\, B^2} \int_0^\infty Q(\tau)\, d\tau \tag{114}$$

$$= \frac{n_o e^2}{\ell\, \theta}\, D(\infty)$$

showing a direct proportionality with the diffusion coefficient $D(\infty)$ of Eq. (89). Such a proportionality between the conductivity and diffusion coefficient was called by Kubo [44] a "generalized Einstein relation", for systems whose dynamics are governed by Newton's laws. But it was far from obvious that such a proportionality would exist for the guiding-center plasma.

The ac conductivity requires a Fourier transform of $Q(\tau)$, which is not obtainable analytically in closed form. For that, one needs to solve Eq. (87) numerically. This has been done, and the details are reported by Montgomery and Tappert [45]. Typical results are shown, in dimensionless units whose interpretation we can ignore here, in Figs. II-2 and II-3. The same remarks about divergences for $L \to \infty$ apply as were applicable in the discussion of the diffusion coefficient.

(Note added in proof: Recent numerical simulations by G. Joyce and the author have indicated that the conductive behavior of the guiding center plasma may be somewhat more complicated than the present development would indicate. A transverse electric field induces large scale vortex motion of a non-local character in a wide variety of simulations. We are also indebted to Dr. Leaf Turner for pointing out to us certain shortcomings of the present derivation of a local conductivity.)

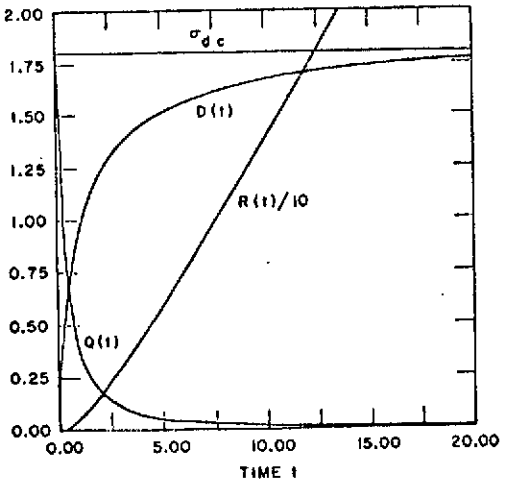

Figure II-2. Numerical solution of Eq. (87), from Montgomery and Tappert [45]. Dimensionless units are: k/K_D, $R/2K_D^2$, $t[8\pi n_0 ce\epsilon^{\frac{1}{2}}]/\ell B$. Notice that R increases linearly with t almost from the beginning.

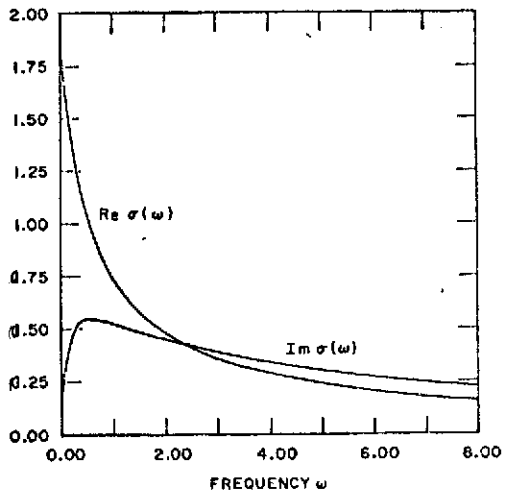

Figure II-3. Real and imaginary parts of the conductivity from Eq. (113). The units of σ are $n_0 ce\epsilon^{\frac{1}{2}}/\ell B$.

(H) The "Negative Temperature" Instability for the Guiding-Center Plasma

A rather unusual feature of the guiding-center plasma is that for interaction energies $\sum_{i<j} \phi_{ij}$ greater than a certain amount, statistical mechanics predicts no stable, thermal equilibrium, spatially uniform state. In the discussion immediately after Eq. (49), the absence of such a state for sufficiently *low* energies was noted. It was noted some time ago by Onsager [46] that, for different reasons, no equilibrium states exist for *high* enough energies.

Onsager's prediction was made in the context of the hydrodynamic formulation of the problem, Eqs. (59) through (61). We can understand it in the following way. Equation (3) for the partition function can be rewritten as (again omitting factors which are not of interest here)

$$Z = \int dX \exp(-\mathcal{E}/\theta) = \int d\mathcal{E}\, \Omega(\mathcal{E}) \exp(-\mathcal{E}/\theta) \tag{115}$$

where ε is now just the numerical value of the Hamiltonian (59), $d\mathcal{E} = \prod_{i=1}^{2N} (dq_i\, dp_i)$ is the phase space volume element for all the vortices (charges) of both signs, and $\Omega(\mathcal{E})$ is the *structure function*, or phase space volume per unit $\mathcal{E}$. $\Omega(\mathcal{E}) = d\Phi(\mathcal{E})/d\mathcal{E}$, where $\Phi(\mathcal{E})$ is the total phase space volume with values of H less than $\mathcal{E}$: $\Phi(\mathcal{E}) = \int_{-\infty}^{\mathcal{E}} d\mathcal{E}'\, \Omega(\mathcal{E}')$.

For most thermodynamic systems, $\Omega(\mathcal{E})$ increases rapidly with increasing $\mathcal{E}$ for very large N, and $\exp(-\mathcal{E}/\theta)$ decreases rapidly for very large N. The competition between the two means that the integrand in (115) will be rather sharply peaked about some value $\mathcal{E} = \mathcal{E}_0$. Indeed, this is nothing but the statement that the fluctuations in energy in the ensemble are small, and that most of the systems are sharply concentrated in energy near the mean value, which is the only reason statistical mechanics works at all. Thus we can write

$$Z = \int d\mathcal{E} \exp[\psi(\mathcal{E}) - \mathcal{E}/\theta] \tag{116}$$

where $\psi(\mathcal{E}) = \ln\Omega(\mathcal{E})$. The integrand in (116) can be well approximated near $\mathcal{E}_0$ by $\exp[\psi(\mathcal{E}_0) - \mathcal{E}_0/\theta + \psi''(\mathcal{E}_0)(\mathcal{E} - \mathcal{E}_0)^2/2]$, and is negligibly small away from $\mathcal{E}_0$. The integral can be performed to give

$$Z = \exp\left[\psi(\mathcal{E}_0) - \mathcal{E}_0/\theta\right] \left|\psi''(\mathcal{E}_0)\right|^{-\frac{1}{2}} \sqrt{2\pi} \tag{117}$$

but the theory of the canonical ensemble also shows

$$Z = \exp\left[\frac{S(\mathcal{E}_0)}{K} - \mathcal{E}_0/\theta + \text{const.}\right] \tag{118}$$

where $\mathcal{E}_0$ is the thermodynamic internal energy of the system being represented and $S(\mathcal{E}_0)$ is the corresponding entropy. The maximum occurs at $1/\theta = \psi'(\mathcal{E}_0) = \Omega'(\mathcal{E}_0)/\Omega(\mathcal{E}_0)$. Noting that ψ and S are both O(N) quantities, comparing (118) and (117) gives for the entropy

$$S(\mathcal{E}_0) = K \ln \Omega(\mathcal{E}_0) + \text{const.} \tag{119}$$

plus terms negligible compared to O(N), and for the temperature,

$$\frac{1}{\theta} = \frac{\Omega'(\mathcal{E}_0)}{\Omega(\mathcal{E}_0)} \tag{120}$$

Equations (119) and (120) are standard formulas but they have some peculiar implications for the system described by Eqs. (59)-(61). For notice that each one of the 2N particles (vortices) has available to it a maximum phase space volume $L^2|K|$. All 2N of them have a phase-space volume $L^{4N}|K|^{2N}$. This means that $\Phi(\mathcal{E})$, which is a non-decreasing function of $\mathcal{E}$ by definition, increases monotonically from 0 to $L^{4N}|K|^{2N}$ as $\mathcal{E}$ goes from $-\infty$ to $+\infty$. This further implies a maximum (at $\mathcal{E} = \mathcal{E}_m$, say) in $\Omega(\mathcal{E})$, where $\Omega'(\mathcal{E})$ goes from positive to negative. Equation (120) then implies that if $\mathcal{E}_0 > \mathcal{E}_m$, the temperature θ is *negative*.

Landau and Lifshitz [47] show quite generally that a stationary thermal equilibrium state is not to be expected for negative temperatures, provided the component parts of the system are free to move with respect to each other. That condition is fulfilled for the present system, so that no stationary state is expected when the value of H exceeds the quantity $\mathcal{E}_m$. One can get a qualitative picture of what the states of the system are for $H = \mathcal{E} > \mathcal{E}_m$ by considering the limit of very large values of H. These clearly correspond to states in which the charges (vortices) of unlike sign as they can get. The

macroscopic flow pattern is that of a pair of large counter-rotating vortices, spatially separated in different parts of the volume.

Recently, such a flow pattern emerged [37] rather unexpectedly in a numerical solution to the two-dimensional equations of incompressible fluid flow at high Reynolds number (essentially Eqs. (99) and (100) with a small viscous damping). An "ergodic boundary" separating those initial conditions where the large pair of vortices is formed from those where they are not was proposed by Deem and Zabusky. A possible explanation in terms of the "negative temperature" instability was offered by Montgomery [48], who also observed that the prediction was not totally applicable to the Deem-Zabusky simulation in view of the lack of a distinction between "self" and "interaction" energies in the continuum representation (Onsager's prediction applies only to a "line" vortex model, and $H = \mathcal{E}$ is the *interaction* energy). Cook and Taylor [49] had previously offered an explanation of the Deem-Zabusky results in terms of the impossibility of relaxation to the thermal equilibrium energy spectrum [essentially Eq. (44)], as compatible with conservation of *enstrophy*, for certain sets of initial conditions. [Enstrophy is the volume integral of the square of the vorticity, or for the plasma interpretation, the square of the charge density. It is one among many constants of the motion for the ideal incompressible fluid, where the integral of any power of the vorticity is also such a constant. Due to the non-square-integrability of the delta function, it is ill-defined for the discrete-charge or discrete-vortex system].

Joyce and Montgomery [50] performed a simulation of the discrete vortex situation in which the equations of motion of individual vortices were advanced numerically in time. At high initial interaction energies (energies >> the self energies, in the particle-in-cell representation), the Deem-Zabusky phenomenon reappeared: formation of a pair of large vortices composed of many of the smaller discrete ones.

A calculation due to Taylor [51] attempts to determine the threshold for the negative temperature instability by evaluation of the stationary-phase approximation to $\frac{d}{d\varepsilon_o} \ln\Omega(\varepsilon_o$ The rather remarkable result emerges that, when the infinite self-energy is taken into account, $\mathcal{E}_m = 0$. [there is also a somewhat confusing attempt to connect the result with the previous Cook-Taylor [49] calculation for *continuous* fluids. We are unable to appreciate this argument, and shall remark no more on it here]. An improved calculation of $\mathcal{E}_m$ is due to C.E. Seyler, Jr., in Physical Review Letters, 32 515 (1974).

The result $\mathcal{E}_m = 0$ would be a remarkable conclusion for the following reason. Since the plasma parameter does not enter the calculation, we may assume that the result holds down to $\mathcal{E} = 0$. But $\mathcal{E} = 0$ is the "Vlasov" limit, or for the fluid case, the limit of ideal, incompressible, inviscid hydrodynamics. The spatially-uniform, field-free, quiescent state would be by this criterion *unstable*. The interaction energy per particle in the Vlasov limit is zero. Therefore, *any* perturbation on the spatially-uniform state would have the effect of adding a positive interaction energy per particle. (The energy $\int dx \langle \vec{E}^2 \rangle / 8\pi$ is a positive-definite functional of any charge density perturbation). Thus *any* perturbation from uniformity would put the system on the negative-temperature side of the boundary.

Finally, it may be speculated that the energy at which p becomes < 0 according to (49) might lie *above* $\mathcal{E}_m$. This eventuality, which seems unlikely, would deny the existence in principle of a stable spatially-uniform thermodynamic equilibrium state of the guiding-center plasma in two dimensions for *any* energy.

From these various calculations, it is to be concluded that the guiding center plasma in two dimensions is an extraordinarily rich statistical-mechanical system, one about which much remains to be discovered. It is also one whose pathologies and singular features should make the investigator wary until we have developed a better intuitive understanding of the system than now exists.

III. GUIDING-CENTER PLASMAS IN THREE DIMENSIONS

(A) *Dynamical Description, Canonical Ensemble, and Time-Independent Fluctuations*

The two-dimensional guiding-center plasma model developed in Sections I and II is of intrinsic interest for statistical physics, but any practical significance it may have depends in large part upon finding a way to generalize it to three dimensions. This program is still in its infancy, and though some of the results which have been achieved will now be summarized, they will very probably have been considerably enlarged upon by the time this material appears in print.

The natural generalization to three dimensions involves allowing point charges to move with the $c\vec{E} \times \vec{B}/B^2$ drift perpendicularly to a constant uniform magnetic field $\vec{B}$, but to

move according to Newton's laws in the parallel direction. If the velocity of the jth charge is $\vec{v}_j$, we decompose it (and other vectors) into parallel and perpendicular parts,

$$\vec{v}_j = v_{j\parallel}\,\hat{b} + \vec{v}_{j\perp} \tag{121}$$

where $\hat{b} \equiv \vec{B}/B$ is in the z-direction, say, and $\hat{b} \cdot \vec{v}_{j\perp} = 0$. For $v_{j\perp}$, we have

$$\vec{v}_{j\perp} = c\vec{E}(\vec{x}_j,t) \times \vec{B}/B^2 \tag{122}$$

where $\vec{E}(\vec{x}_j,t)$ is the electric field evaluated at the instantaneous position $\vec{x}_j$ of the jth charged particle ($d\vec{x}_j/dt = \vec{v}_j$). For the parallel motion, we have

$$m_j \frac{dv_{j\parallel}}{dt} = e_j\,\hat{b} \cdot \vec{E}(\vec{x}_j,t) \tag{123}$$

Once again, we use the electrostatic approximation (more for convenience than anything else) and determine $\vec{E}$ by Poisson's equation, or equation, or equivalently, by its solution

$$\vec{E}(\vec{x},t) = \sum_j e_j \frac{(\vec{x} - \vec{x}_j)}{|\vec{x} - \vec{x}_j|^3} \tag{124}$$

where the sum runs over all charges of both signs. (Again we take N positive ions and N electrons inside a large volume V. Now, $V = L^3$.)

Equations (121)-(124) form a closed system. The dynamics they express can be given a Hamiltonian form; this will guarantee, among other things, the existence of a Liouville equation and a BBGKY hierarchy, with their attendant possibilities for doing a systematic perturbation theory. Letting the jth particle have coordinates $\vec{x}_j = (x_j,y_j,z_j)$, the Hamiltonian function can be chosen to be

$$H = \sum_{i<j} e_i e_j \left[\left(\sqrt{\frac{c}{B|e_i|}}\; q_i - \sqrt{\frac{c}{B|e_j|}}\; q_j\right)^2\right.$$

$$+\left(\sqrt{\frac{c}{B\,|e_i|}}\;p_i - \sqrt{\frac{c}{B\,|e_j|}}\;p_j\right)^2 + (z_i - z_j)^2\Bigg]^{-\frac{1}{2}} \tag{125}$$

$$+\sum_j p_{zj}^2/2m_j$$

where the summations run over all charges of both signs, and the canonically conjugate Hamiltonian variables for the perpendicular motion are

$$q_j = \sqrt{\frac{B\,|e_j|}{c}}\;x_j$$

$$p_j = \sqrt{\frac{B\,|e_j|}{c}}\;y_j\;\operatorname{sgn} e_j \tag{126}$$

For the parallel motion, the canonical variables are just z_j and p_{zj}. It is straightforward to show that Hamiltonian's equations $\dot{q}_i = \partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial q_i$ reproduce (122), while $\dot{z}_j = \partial H/\partial p_{zj}$, $\dot{p}_{zj} = -\partial H/\partial z_j$ reproduce (123) and the definition of $v_{jz} \equiv v_{jz}$.

Expressed in terms of the positions and velocities, H becomes

$$H = \sum_{i<j} \varphi_{ij} + \sum_j m_j\, v_{jz}^2/2 \tag{127}$$

where ϕ_{ij} is the two-body Coulomb potential $e_i\, e_j/|\vec{x}_i - \vec{x}_j|$. The canonical distribution of Gibbs will be

$$D_{eq.} = \frac{e^{-H/\theta}}{\int dX\; e^{-H/\theta}} \tag{128}$$

where, if H is expressed in the form (127), $\int dX$ means an integral over the $2N \cdot 4 = 8N$ coordinates (x_j, y_j, z_j, v_{jz}), where j runs over all charges of both signs.

That the equilibrium phase space probability density has the form that it does [Eq. (128)] means that, in three dimensions as in two, the thermal equilibrium configuration-space probability density (using superscripts to label particle species, now) has the form:

$$n_{N,N}(\vec{x}_1^{\,i} \cdots \vec{x}_N^{\,i};\ \vec{x}_1^{\,e} \cdots \vec{x}_N^{\,e})/V^{2N}$$

$$= \frac{\exp[-\Phi(N,N)/\theta]}{\int d\vec{x}_1^{\,i} \cdots d\vec{x}_N^{\,i}\ d\vec{x}_1^{\,e} \cdots d\vec{x}_N^{\,e}\ \exp[-\Phi(N,N)/\theta]}$$

$$\equiv \exp[-\Phi(N,N)/\theta]/Q_{2N} \qquad (129)$$

Equation (129) is the same as for the ordinary plasma in three dimensions, with or without an external magnetic field. Here, $\Phi(N,N) = \sum_{i<j} e_i\, e_j/x_{ij}$ is the total potential energy of N ions and N electrons. All the reduced probability distributions

$$n_{s_i,s_e} = V^{s_i+s_e} \int d\vec{x}^{\,i}_{s_i+1} \cdots d\vec{x}^{\,i}_N \quad d\vec{x}^{\,e}_{s_e+1} \cdots d\vec{x}^{\,e}_N \quad \frac{\exp[-\Phi(N,N)/\theta]}{Q_{2N}}$$

will be the same, too, as will all single-time thermal equilibrium ensemble averages. This simplifies matters considerably, since these quantities are familiar ones and have been for several years (see, for a review, reference 13). They are, as before, independent of B. Without proof we shall list a number of readily-proved results obtained from $n_{2,0}$ $n_{0,2}$, and $n_{1,1}$.

Interest frequently focuses on the case $n_0\lambda_D^3 >> 1$, where now $n_0 = N/V$ is the average density (particles/cm^3) for both species. The Debye length λ_D in three dimensions is defined

by $\lambda_D^2 = \theta/8\pi n_0 e^2$. For $\varepsilon = (4\pi n_0 \lambda_D^3)^{-1} << 1$, we have $n_{1,0} = n_{0,1} = 1$ exactly, and to $O(\varepsilon)$,

$$n_{2,0} = 1 + p^{ii}(\vec{x}_1^i - \vec{x}_2^i)$$

$$n_{0,2} = 1 + p^{ee}(\vec{x}_1^e - \vec{x}_2^e) \tag{130}$$

$$n_{1,1} = 1 + p^{ie}(\vec{x}_1^i - \vec{x}_2^e)$$

Here, the pair correlation $p^{ii} = p^{ee} = -p^{ei} = -p^{ie}$ is given by

$$p^{ee}(\vec{x}) \cong -\frac{e^2}{\theta}\,\frac{e^{-|\vec{x}|/\lambda_D}}{|\vec{x}|}$$

$$= \int d\vec{k}\, e^{i\vec{k}\cdot\vec{x}}\, p^{ee}(\vec{k}) \tag{131a}$$

where

$$p^{ee}(\vec{k}) = -\frac{1}{16\pi^3 n_o}\,\frac{1}{1 + k^2\lambda_D^2} \tag{131b}$$

From such expressions, other quantities such as $\langle \vec{E}(\vec{x})\ \vec{E}(\vec{x}') \rangle$ may be similarly calculated. Thus, the autocorrelation of the electrical charge density $\rho(\vec{x})$ is

$$\langle \rho(\vec{x})\ \rho(\vec{x}') \rangle = 4e^2 n_o^2\, p^{ee}(\vec{x} - \vec{x}')$$

$$+ 2n_o e^2\, \delta(\vec{x} - \vec{x}')$$

$$= \int S_\rho(\vec{k})\ e^{i\vec{k}\cdot\vec{x}}\ d\vec{k} \tag{132}$$

where

$$S_\rho(\vec{k}) = \frac{n_o e^2}{4\pi^3}\ \frac{k^2\lambda_D^2}{1 + k^2\lambda_D^2} \tag{133}$$

Also

$$\langle \vec{E}(\vec{x})\ \vec{E}(\vec{x}') \rangle = \int \vec{\vec{S}}_E(\vec{k})\ e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}\ d\vec{k}$$

where

$$\vec{\vec{S}}_E(\vec{k}) = \frac{4 n_o e^2}{\pi}\ \frac{\vec{k}\vec{k}}{k^2}\ \frac{\lambda_D^2}{1 + k^2\lambda_D^2} \tag{134}$$

from which

$$\frac{\langle \vec{E}^2 \rangle}{8\pi} = \frac{\theta}{2} \int \frac{d\vec{k}}{(2\pi)^3}\ \frac{1}{1 + k^2\lambda_D^2} \tag{135}$$

Equation (135) contains a divergence at large $\vec{k}$ which, as in two dimensions, may be associated with the infinite self-energies of the 2N particles

$$\frac{\langle \vec{E}^2 \rangle_{self}}{8\pi} = \frac{n_o e^2}{2\pi^2} \int \frac{d\vec{k}}{k^2} = \frac{\theta}{2} \int \frac{d\vec{k}}{(2\pi)^3}\ \frac{1}{k^2\lambda_D^2} \tag{136}$$

The *interaction energy* can then be measured by

$$\frac{\langle \vec{E}^2 \rangle - \langle \vec{E}^2 \rangle_{self}}{8\pi} = -\frac{\theta}{2} \int \frac{d\vec{k}}{(2\pi)^3}\ \frac{1}{k^2\lambda_D^2(1 + k^2\lambda_D^2)} \tag{137}$$

so that the ratio of interaction energy to thermal energy is approximately

$$\frac{[\langle \vec{E}^2 \rangle - \langle \vec{E}^2 \rangle_{self}]/8\pi}{3 n_o \theta} = -\frac{1}{24\pi n_o \lambda_D^3} = -\frac{\epsilon}{6} \qquad (138)$$

The smallness of this quantity is the basis of the weak-interaction approximation in which the kinetic theory of a plasma is usually treated. The limit $\epsilon \to 0$ for the non-thermal equilibrium problem (where n_o and θ/m are defined in terms of an average number density and average thermal velocity) is usually called the "Vlasov limit". The non-equilibrium properties of this limit have occupied the attentions of more plasma theoriests than any other single topic.

To render $\langle \vec{E}^2 \rangle$ finite, we may cut offthe radial k-integration in (135) in the usual way at the inverse distance of closest approach of a thermal particle, $k_{max.} = \theta/e^2$. This gives $\langle \vec{E}^2/8\pi \rangle / 3 n_o \theta = 2\pi/3$, which, it should be noted, is *not* small [unlike (138)] for any value of ϵ. The failure to distinguish between interaction energy and the spurious self-energy of point Coulomb charges can lead to serious conceptual errors.

(B) Transverse Diffusion in Three Dimensions [52]

The coefficient of transverse spatial diffusion, for the three-dimensional guiding-center plasma, can be written in the same way as (67), in terms of the part of $\vec{E}$ that is perpendicular to $\vec{B}$:

$$D_\perp = \frac{c^2}{B^2} \int_0^\infty \langle \vec{E}_\perp(0) \cdot \vec{E}_\perp(\tau) \rangle \, d\tau \qquad (139)$$

Again, the problem is one of evaluating the electric field auto-correlation $\langle \vec{E}(0) \vec{E}(\tau) \rangle$, where $\vec{E}(\tau)$ is the electric field seen at time τ by a "test" ion which is located at the z-axis at $\tau = 0$. $\vec{E}(\tau)$ is related to the Eulerian electric field $\vec{E}(\vec{x},t)$ by

$$\vec{E}(\tau) = \vec{E}[\vec{x}(\tau), \tau] \qquad (140)$$

and

$$\vec{E}(\vec{x},t) = \sum_{\vec{k}} \sum_{j} \frac{4\pi \vec{k}}{i k^2} \frac{e_j}{V} \exp i\vec{k} \cdot [\vec{x} - \vec{x}_j(t)]$$

where $\vec{x}(\tau)$ is the orbit of the test particle. $\vec{x}_j(t)$ is the location of the jth charge in the plasma at time t. We have toFourier-analyze $\vec{E}(x,t)$ in a large volume V, rather than an infinite volume, for reasons which will become clear as we go on. Equation (141) will also be written as

$$\vec{E}(\vec{x},t) = \sum_{\vec{k}} \vec{E}_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}} \quad , \tag{142}$$

so that (139) becomes

$$D_{\perp} = \frac{c^2}{B^2} \int_0^{\infty} \sum_{\vec{k}} \langle \vec{E}^{*}_{\perp\vec{k}}(0) \cdot \vec{E}_{\perp\vec{k}}(\tau)\, e^{i\vec{k}\cdot\vec{x}(\tau)} \rangle \, d\tau \tag{143}$$

since $\langle \vec{E}_{\vec{k}_1} \vec{E}_{\vec{k}_2} \rangle = 0$ for $\vec{k}_1 \neq -\vec{k}_2$ for a spatially uniform ensemble.

If we once again ignore the correlation between the spatial locations of the test ions and the plasma particles [it will be small if it is of the same order as (131)], we have a relation analogous to (77), and

$$D_{\perp} = \frac{c^2}{B^2} \int_0^{\infty} \sum_{\vec{k}} \langle \vec{E}^{*}_{\perp\vec{k}}(0) \cdot \vec{E}_{\perp\vec{k}}(\tau) \rangle \langle e^{i\vec{k}\cdot\vec{x}(\tau)} \rangle \, d\tau \tag{144}$$

Once again, we can evaluate $D_{\perp}$ if we can evaluate $\langle \exp i\vec{k} \cdot x(\tau) \rangle$. The same methods will be applicable in evaluating

$$\langle \vec{E}^{*}_{\perp\vec{k}}(0) \cdot \vec{E}_{\perp\vec{k}}(\tau) \rangle$$

$$= \sum_{ij} \frac{16\pi^2 e_i e_j}{k^2 V^2} \frac{k_\perp^2}{k^2} \langle \exp i\vec{k} \cdot [\vec{x}_i(\tau) - \vec{x}_j(0)] \rangle \qquad (145)$$

The program to be followed closely parallels that for two dimensions. For $\vec{x}(\tau)$, we have

$$\vec{x}(\tau) = \vec{x}(0) + c \int_0^\tau \frac{\vec{E}(\tau') \times \vec{B}\, d\tau'}{B^2}$$

$$+ \hat{b}\, v_\parallel \tau + \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \frac{e_i \hat{b} \cdot \vec{E}(\tau'')\, \hat{b}}{m_i} \qquad (146)$$

In (146), the "test" particle has been assumed to be an ion with charge-to-mass ratio e_i/m_i. Equation (146) follows from integrating (122) and (123). $\vec{x}(0)$ is the initial position and $v_\parallel$ the initial velocity of the test ion. $\vec{x}(0)$ can be set equal to zero with no loss of generality. $v_\parallel$ is a statistically distributed quantity which may be assumed to obey the Maxwell-Boltzmann distribution.

Evaluating $\langle \exp i\vec{k} \cdot \vec{x}(\tau)$ is no simple matter, and Eq. (146) is considerably more complicated than (78). About the best that has been done so far in evaluating it is the following. Because $\vec{x}(\tau)$ is the position of a random variable which is initially localized near $\vec{x}(0) = 0$, and because its probability distribution $P[\vec{x}(\tau),\tau]$ is expected to spread out with time,

$$\langle \exp i\vec{k} \cdot \vec{x}(\tau) \rangle = \int d\vec{x}(\tau)\, e^{i\vec{k}\cdot\vec{x}(\tau)}\, P[\vec{x}(\tau),\tau]$$

will damp with increasing τ. $\langle \exp i\vec{k} \cdot \vec{x}(\tau)$ is the "characteristic function" of the random variable $\vec{x}(\tau)$, in the language of probability. Roughly speaking, the damping will occur for two reasons: motion parallel to the field lines, and motion perpendicular to them. Only the latter mechanism was operative in the two-dimensional case. The latter is inhibited

by increasing B, but the former is not. The damping due to the parallel motion will be more extreme than that due to the perpendicular motion. Therefore, for very strong magnetic fields, it is useful to forget the perpendicular drift *except* when $\vec{k}$ is normal to $\vec{B}$. This is tantamount to assuming

$$\langle \exp i\vec{k} \cdot \vec{x}(\tau) \rangle$$

$$= \begin{cases} \left\langle \exp \left(\dfrac{-i c\vec{k} \times \vec{B}}{B^2} \cdot \int_0^\tau \vec{E}(\tau')\, d\tau' \right) \right\rangle & \text{if } \vec{k} \cdot \vec{B} = 0 \\ \left\langle \exp \left\{ i \vec{k} \cdot \left[\hat{b} v_\parallel \tau + \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \dfrac{e_i \hat{b}\hat{b} \cdot \vec{E}(\tau'')}{m_i} \right] \right\} \right\rangle & \text{if } \vec{k} \cdot \vec{B} \neq 0 \end{cases} \tag{147}$$

The first of these two expressions can be approximated by the assumption which led to Eq. (81), but the second is more complicated.

That the statistical properties of the parallel electric field seen by a particle can be rather different from the properties of the perpendicular field can be seen as follows. Even if the parallel electric field obeys some jointly normal distribution such as (80), there is the important difference that it can *accelerate* a particle, whereas the perpendicular components cannot. While the perpendicular motion can be visualized as a limit of small increments in position space, the parallel motions are a sum of small increments in *velocity* space. This means that eventually some of the particles will be moving very fast. Such fast-moving particles will, as is well known, eventually begin losing energy due to the radiation of plasma oscillations, and due to the relatively infrequent close collisions with other particles. Therefore, the electric field seen in the parallel direction can*not* in general be well represented by an expression such as (80) in which

the probability of the electric field seen by a particle is given by a velocity-independent expression.

The parallel motion is closer in some ways to the conventional [31] Langevin approximation of the theory of Brownian motion in one dimension. There, a test particle in one dimension obeys the equation

$$\frac{dv}{dt} = -F(v) + A(t) \tag{148}$$

with a frictional "drag" coefficient $F(v)$ and a stationary random force field A [analogous to our $\delta\vec{b} \cdot \vec{E}(t)$]. The statistics of A can be given by some such expression as (80). For $F(v) = \beta v$, where β^{-1} is a constant long compared to the time necessary for $\langle A(0)\, A(t)\rangle$ to go to zero, the stochastic solutions to (148) are tractable [31]. However, we know that for the plasma case, $F(v)$ is a considerably more complicated nonlinear function of v. Methods apparently do not currently exist for the calculation of such quantities as $\langle \exp[ik \int_0^t v(t')dt']\rangle$ when (148) governs $v(t)$, with a more elaborate nonlinear form for $F(v)$. Therefore we must admit an inability to treat the second expressions in (147) properly over the whole range of τ, and seek an approximation which will render it tractable, even if the approximation is not entirely satisfactory.

One possible procedure is to neglect the contribution of the parallel electric field in the second of Eqs. (147). We are treating only the low-frequency, long-wavelength parts of the electric field spectrum correctly anyway. For this reason, one may expect that the mechanism of *free streaming* of particles down the field lines will be dominant in destroying the correlations between the initial electric field seen by the particle and that field at a later time. The parallel motion of a particle will endure for something of the order of a mean free path, and we may imagine that the auto-correlation length for the electric field will be $\lesssim$ this mean free path. If we ignore $E_\cdot$, we get for $\vec{k} \cdot \vec{B} \neq 0$,

$$\langle \exp i\vec{k} \cdot \vec{x}(\tau)\rangle_{k_{\parallel}\neq 0} \cong \langle \exp(i\vec{k} \cdot \hat{b}\, v_{\parallel}\, \tau)\rangle$$

$$= \int_{-\infty}^{\infty} dv_{\parallel} \left(\frac{m_i}{2\pi\theta}\right)^{\frac{1}{2}} \exp\left(\frac{-m_i v_{\parallel}^2}{2\theta}\right) \exp(i k v_{\parallel} \tau) \tag{149}$$

$$= \exp(-k_{\parallel}^2 \theta \tau^2/2m_i) = \exp(-k_{\parallel}^2 V_{ion}^2 \tau^2/2),$$

where we have assumed the test ions to be distributed with a thermal equilibrium velocity distribution. $V_{ion}^2 \equiv \theta/m_i$

We shall make the approximation (149) and explore its consequences, both for the test particles and the plasma particles. Later we shall remark upon the possibilities of relaxing it. For the case of purely perpendicular $\vec{k}$'s, a development parallel to that of

$$\langle \exp i\vec{k} \cdot \vec{x}(\tau)\rangle_{k_{\parallel}=0}$$

$$= \exp\left(\frac{-c^2 k_{\perp}^2}{4B^2} \int_0^{\tau} d\tau_1 \int_0^{\tau} d\tau_2 \langle \vec{E}_{\perp}(\tau_1) \cdot \vec{E}_{\perp}(\tau_2)\rangle\right) \tag{150}$$

Before substituting (150) and (149) into (144), we need expressions for the $\langle \vec{E}^*_{\perp k}(0) \cdot \vec{E}_{\perp k}(\tau)\rangle$. Making the same approximations that led to (149) and (150) on the plasma charges in (145) gives

$$\langle \vec{E}^{*}_{\perp \vec{k}}(0) \cdot \vec{E}_{\perp \vec{k}}(\tau) \rangle \cong$$

$$\begin{cases} \displaystyle\sum_{ij} \frac{16\pi^2 e_i e_j}{k^2 V^2} \frac{k_\perp^2}{k^2} \langle \exp i\vec{k} \cdot [\vec{x}_i(0) - \vec{x}_j(0)] \rangle \cdot \\ \qquad \exp\left(\frac{-k_\parallel^2 V_{oj}^2 \tau^2}{2}\right) \quad \text{for } k_\parallel \neq 0 \\ \displaystyle\sum_{ij} \frac{16\pi^2 e_i e_j}{k_\perp^2 V^2} \exp\left(\frac{-c^2 k_\perp^2}{4 B^2} \int_0^\tau d\tau_1 \int_0^\tau d\tau_2 \langle \vec{E}_\perp(\tau_1) \cdot \vec{E}_\perp(\tau_2) \rangle\right) \\ \qquad \cdot \langle \exp i\vec{k} \cdot [\vec{x}_i(0) - \vec{x}_j(0)] \rangle \quad \text{for } k_\parallel = 0 \end{cases} \tag{151}$$

In (151), $V_{oj}^2 \equiv \theta/m_j$ is the square of the thermal speed for the species to which charge j belongs. We have also implicitly assumed that the jointly normal probability distribution of the perpendicular electric field is the same for all the particles.

Finally, we need $\langle \exp i \vec{k} \cdot [\vec{x}_i(0) - \vec{x}_j(0)] \rangle$, which is $\int d\vec{x}_1\, d\vec{x}_2\, n_2(\vec{x}_1, \vec{x}_2)\, e^{-i\vec{k}\cdot\vec{x}_{12}/V^2}$, where n_2 is the probability distribution (130) of whatever two particles are involved. If i and j are the same particle, the $\langle\ \rangle$ is clearly unity. If they are different, $i \neq j$, then

$$\langle \exp i\vec{k} \cdot [\vec{x}_i(0) - \vec{x}_j(0)] \rangle$$

$$= \delta_{Kr}(\vec{k}) - \frac{1}{2 n_o V} \left(\frac{K_D^2}{k^2 + K_D^2} \right) \quad \text{sgn } e_i \text{ sgn } e_j \tag{152}$$

(152) is just (129)-(131), expressed in a finite-volume representation.

Inserting (152) and (151) into the relation

$$\langle \vec{E}_\perp(0) \cdot \vec{E}_\perp(\tau) \rangle = \sum_{\vec{k}} \langle \vec{E}^*_{\perp\vec{k}}(0) \cdot \vec{E}_{\perp\vec{k}}(\tau) \rangle \langle e^{i\vec{k}\cdot\vec{x}(\tau)} \rangle \tag{153}$$

we have

$$\langle \vec{E}_\perp(0) \cdot \vec{E}_\perp(\tau) \rangle$$

$$= \sum_{i=j} \sum_{k_\parallel \neq 0}^{\vec{k}} \frac{16\pi^2 e_i e_j}{k^2 V^2} \frac{k_\perp^2}{k^2} \exp\left(\frac{-k_\parallel^2 V_{oj}^2 \tau^2}{2} \right) \exp\left(\frac{-k_\parallel^2 V_{ion}^2 \tau^2}{2} \right)$$

$$+ \sum_{i=j} \sum_{k_\parallel = 0}^{\vec{k}} \frac{16\pi^2 e_i e_j}{k^2 V^2} \exp\left(\frac{-c^2 k_\perp^2}{2 B^2} \int_o^\tau d\tau_1 \int_o^\tau d\tau_2 \langle \vec{E}_\perp(\tau_1) \cdot \vec{E}_\perp(\tau_2) \rangle \right).$$

$$+\sum_{i\neq j}\sum_{k_\parallel\neq 0}^{\vec{k}}\frac{16\pi^2 e_i e_j}{k^2 V^2}\frac{k_\perp^2}{k^2}\exp\left(\frac{-k_\parallel^2 V_{oj}^2 \tau^2}{2}\right)$$

$$\left\{\delta_{Kr}(\vec{k}) - \frac{1}{2n_o V}\frac{K_D^2}{k^2+K_D^2}\,\mathrm{sgn}\, e_i e_j\right\}\exp\frac{-k_\parallel^2 V_{ion}^2 \tau^2}{2}$$

$$+\sum_{i\neq j}\sum_{k_\parallel=0}^{\vec{k}}\frac{16\pi^2 e_i e_j}{k^2 V^2}\exp\left(\frac{-c^2 k_\perp^2}{2B^2}\int_0^\tau d\tau_1\int_0^\tau d\tau_2\,\langle\vec{E}_\perp(\tau_1)\cdot\vec{E}_\perp(\tau_2)\rangle\right)$$

$$\cdot\left\{\delta_{Kr}(\vec{k}) - \frac{1}{2n_o V}\frac{K_D^2}{k^2+K_D^2}\,\mathrm{sgn}\, e_i e_j\right\} \qquad (154)$$

This is an integral equation for $\langle\vec{E}_\perp(0)\cdot\vec{E}_\perp(\tau)\rangle$. If we had defined the Fourier transform of the Coulomb potential $\phi^{ee}(\vec{k}) = \lim_{\lambda\to 0} e^2/2\,\pi^2(k^2+\lambda^2)$ more carefully, the $\delta_{Kr}(\vec{k})$ terms would not have been present in $\langle\vec{E}^*_{\vec{k}}(0)\cdot\vec{E}_{\vec{k}}(\tau)\rangle$, and they will not contribute to the summation. Hereafter they are dropped. (This is only one of the places where one has to be alert for the pathologies that follow from the fact that the Coulomb potential is neither square integrable nor absolutely integrable, and so has no Fourier transform in a rigorous sense, except as such a limit as the one just described.) The Σ_{ij} can be performed in (154) and the result is:

$$\langle \vec{E}_\perp(0) \cdot \vec{E}_\perp(\tau) \rangle$$

$$= \sum_{\substack{\vec{k} \\ k_\parallel \neq 0}} \frac{16\pi^2 e^2 n_o k_\perp^2}{V k^2 (k^2 + K_D^2)} \exp\left(\frac{-k_\parallel^2 V_{ion}^2 \tau^2}{2}\right) \tag{155}$$

$$\cdot \left[\exp\left(\frac{-k_\parallel^2 V_{ion}^2 \tau^2}{2}\right) + \exp\left(\frac{-k_\parallel^2 V_{oe}^2 \tau^2}{2}\right)\right]$$

$$+ \sum_{k_\parallel = 0} \frac{32\pi^2 e^2 n_o}{V(k^2 + K_D^2)} \exp\left(\frac{-c^2 k_\perp^2}{2 B^2} \int_o^\tau d\tau_1 \int_o^\tau d\tau_2 \langle \vec{E}_\perp(\tau_1) \cdot \vec{E}_\perp(\tau_2)\right.$$

Equation (155) can be simplified by defining, as in the two-dimensional case,

$$\frac{1}{2} \langle \vec{E}_\perp(0) \cdot \vec{E}_\perp(\tau) \rangle \equiv Q_\perp(\tau)$$

$$R_\perp(\tau) \equiv \frac{c^2}{2 B^2} \int_o^\tau d\tau_1 \int_o^\tau d\tau_2 \, Q_\perp(\tau_2 - \tau_1) \tag{156}$$

$$\frac{d^2 R_\perp(\tau)}{d \tau^2} = \frac{c^2 Q_\perp(\tau)}{B^2}$$

These definitions can be used in (155) to give

$$2Q_{\perp}(\tau) = \frac{16\pi^2 e^2 n_o}{V} \sum_{\substack{\vec{k} \\ k_{\parallel}\neq 0}} \frac{k_{\perp}^2/k^2}{(k^2 + K_D^2)}$$

$$\cdot \left[\exp\left(\frac{-k_{\parallel}^2 V_{ion}^2 \tau^2}{2}\right) + \exp\left(\frac{-k_{\parallel}^2 V_{oe}^2 \tau^2}{2}\right)\right] \exp\left(\frac{-k_{\parallel}^2 V_{ion}^2 \tau^2}{2}\right) \tag{157}$$

$$+ \frac{32\pi^2 e^2 n_o}{V} \sum_{\substack{\vec{k} \\ k_{\parallel}=0}} \frac{1}{k_{\perp}^2 + K_D^2} \exp\left[-2k_{\perp}^2 R_{\perp}(\tau)\right]$$

Defining $\epsilon_b \equiv 16\pi^2 e^2 c^2/B^2 V K_D^2$, this becomes, replacing τ by t,

$$\frac{d^2 R_{\perp}(t)}{dt^2} = \epsilon_b \sum_{\substack{\vec{k} \\ k_{\parallel}=0}} \frac{\exp[-2k_{\perp}^2 R_{\perp}(t)]}{1 + k_{\perp}^2 \lambda_D^2} + \epsilon_b \sum_{\substack{\vec{k} \\ k_{\parallel}\neq 0}} f(\vec{k},t) \tag{158}$$

where $f(\vec{k},t)$ is a known function of t:

$$f(\vec{k},t) \equiv \frac{1}{2} \frac{k_{\perp}^2}{k^2} \frac{\exp -k_{\parallel}^2 V_{ion}^2 t^2/2}{1 + k^2 \lambda_D^2}$$

$$\cdot \left[\exp\left(\frac{-k_{\parallel}^2 V_{ion}^2 t^2}{2}\right) + \exp\left(\frac{-k_{\parallel}^2 V_{oe}^2 t^2}{2}\right)\right] \tag{159}$$

Equation (158) is the analogue of Eq. (86) for three dimensions. It is considerably more complicated because of the last term, however, and so far has not been solved in the general case. It can be integrated once, noting that $dR_\perp(0)/dt = 0$, to yield:

$$\frac{dR_\perp(t)}{dt} = \epsilon_b \sum_{k_\parallel=0}^{\vec{k}} \int_0^t d\tau \frac{\exp[-2k_\perp^2 R_\perp(\tau)]}{1 + k_\perp^2 \lambda_D^2} \tag{160}$$

$$+ \epsilon_b \sum_{k_\parallel\neq 0}^{\vec{k}} \int_0^t f(\vec{k},t)\, d\tau$$

and we may show how to extract the leading term in powers of 1/B from Eq. (160).

The only place the magnetic field enters (158) or (160) is in the denominator of the small quantity ϵ_b. We seek the leading term, in powers of ϵ_b, of the solution to (158) and (160). We shall see presently that for large t, $R_\perp(t)$ varies proportionally to $\epsilon_b^{\frac{1}{2}} t$, which makes the first term on the right of (160) an $O(\epsilon_b^{\frac{1}{2}})$ term. Since the second term on the right of (160) is $O(\epsilon_b)$, the result follows that it can be neglected to lowest significant order, either in (160) or in (158).

If we neglect $f(\vec{k},t)$ in (158), we get

$$\frac{d^2 R_\perp(t)}{dt^2} \cong \epsilon_b \sum_{k_\parallel=0}^{\vec{k}} \frac{\exp[-2k_\perp^2 R_\perp(t)]}{1 + k_\perp^2 \lambda_D^2}$$

or equivalently,

$$\frac{1}{2}\left(\frac{dR_\perp(t)}{dt}\right)^2 = \frac{1}{2}\epsilon_b \sum_{\vec{k}_\perp} \frac{[1 - \exp(-2k_\perp^2 R_\perp)]}{k_\perp^2(1 + k_\perp^2 \lambda_D^2)} \tag{161}$$

Since $R_\perp$ and its first two derivatives are always positive for $t > 0$, last term in (161) becomes negligible as $t \to \infty$, and

$$\frac{1}{2}\left(\frac{d R_\perp(\infty)}{dt}\right)^2 = \frac{\epsilon_b}{2}\sum_{\vec{k}_\perp} \frac{1}{k_\perp^2(1 + k_\perp^2 \lambda_D^2)} \tag{162}$$

But $D_\perp/2$ is just $d R_\perp(\infty)/dt$, so

$$D_\perp = 2\epsilon_b^{\frac{1}{2}}\left[\sum_{\vec{k}_\perp} \frac{1}{k_\perp^2(1 + k_\perp^2 \lambda_D^2)}\right]^{\frac{1}{2}} + \text{terms of } O(\epsilon_b) \tag{163}$$

Equation (163) is the three-dimensional analogue of (89). To approximate it by an integral for a large volume, we make the replacement [as in (90)]:

$$\sum_{\vec{k}_\perp} \longrightarrow \frac{V^{\frac{2}{3}}}{2\pi}\int_{k_{min}}^{\infty} k_\perp \, d k_\perp$$

where $k_{min} = 2\pi/L = 2\pi/V^{\frac{1}{3}}$ is the lower limit of $k_\perp$ integration which results from the finite box size.

Carrying out the integration and introducing the definition of ϵ_b, (163) becomes

$$D_\perp = \frac{c\theta}{eB}\left(\frac{1}{2\pi n_o \lambda_D^3}\right)^{\frac{1}{2}}\left(\frac{\lambda_D}{L}\ln\frac{L}{2\pi\lambda_D}\right)^{\frac{1}{2}} \tag{164}$$

Equation (164) is strikingly similar to (91). It contains the factors $c\theta/eB$, the square root of the plasma parameter, and a volume-dependent factor. Only the volume-dependent factor is different: it diverges slowly as $L \to \infty$ in two dimensions, but it approaches zero slowly in three dimensions. This slow

approach to zero puts a constraint on the volume in order that the second term of (160) really be negligible compared to the first. Replacing the second $\Sigma_{\vec{k}}$ in (160) by an integral (which must be cut off at $k_{max.} = \theta/e^2$ to avoid an unphysical short-range divergence) the condition that the term involving $f(\vec{k},t)$ be negligible is that

$$\frac{B^2}{n_o m_i c^2} \gg \frac{1}{32 n_o \lambda_D^3} \frac{L/2\pi\lambda_D}{\ln(L/2\pi\lambda_D)} \left[\ln\left(\frac{L\, k_{max}}{2\pi}\right)\right]^4 \quad (165)$$

This can be regarded as a constraint on either B or L, that (164) shall adequately represent the coefficient of transverse diffusion. The limits of large B and large V are not interchangeable. At fixed volume, however large, (164) becomes accurate as $B \to \infty$. But at fixed B, however large, (164) ceases to be accurate as $V \to \infty$, because (165) ceases to hold.

Arguments which led to the inequality (92) on the time over which true diffusion-like behavior can be observed likewise complicate the picture for the solutions of (161).

(C) Problems Associated with a Convergent Infinite-Volume Theory

The form of Eq. (164) leaves little doubt of its inadequacy if we are interested in passing to the rigorous limit of an infinite volume, $V \to \infty$. One might at first guess that the way to obtain the infinite-volume limit would be to drop the *first* term in (160) and simply write

$$D_\perp(\infty) = 2 \lim_{t\to\infty} \frac{d\, R_\perp(t)}{d\, t}$$

$$= 2\, \epsilon_b\, V \int \frac{d\,\vec{k}}{(2\pi)^3} \int_0^\infty f(\vec{k},\tau)\, d\tau \quad (166)$$

However, upon substituting (159) into (166), one finds that the integral diverges, and so what superficially appears to be a nice expression with the correct "classical" dependence $\sim B^{-2}$ is actually useless.

The reasons for this are not difficult to see, but they are not easy to correct. The reasons lie in the separations, represented by Eqs. (147), (149), and (150), of the damping of $\langle \exp i\vec{k} \cdot \vec{x}(\tau) \rangle$ into purely perpendicular and purely parallel contributions. The point is that as V increases and the allowed values of $\vec{k}$ become more and more dense, this separation will become less and less realistic for very small but non-zero values of $k_{\|}$. The damping due to the $\vec{E} \times \vec{B}$ drift will compete with the free-streaming damping below a value of $k_{\|}$ that can be estimated by setting $k_{\|}^2 V_{ion}^2 t^2 \cong k^2 D_{\perp} t$ at a time when both are of order unity; i.e., below $k_{\|} \approx k_{\perp}^2 D_{\perp}/V_{ion}$. Similarly the parallel electric field which has been unceremoniously dropped up to now from the second of Eqs. (147) can become important in the small $k_{\|}$ region. It has been pointed out recently by Vahala [53] that mild assumptions on the statistics of the parallel electric field can lead to a term in (147) which has the essential temporal behavior $\sim \exp\{-k_{\|}^2 t^3$ times a quantity which $\to 0$ as $\varepsilon \to 0\}$. This mechanism, essentially a parallel collision damping, is a third approximately equal competitor in the role of damping $f(\vec{k},\tau)$ to zero for large τ at small $k_{\|}$.

It should be apparent that a rather complex limiting process, involving the three small parameters ε, ε_b, and $1/V$ is involved; a simple formula which covers all cases is too much to hope for in the near future. But formula (164), supplemented by the inequality (165), may be supposed to be a good measure of the regime in which the guiding-center plasma exhibits Bohm-like diffusion in thermal equilibrium. What are not yet clear are the infinite volume or finite 1/B limits in which "classical" diffusion may be expected. The "classical" diffusion formulas, though very familiar, lack convincing derivations in a strongly magnetized plasma. One may expect this part of the picture to clear up before long, but jumping to premature conclusions is to be avoided.

(D) A Systematic Kinetic Theory of the Three-Dimensional Guiding-Center Plasma

Since the system is Hamiltonian [Eqs. (125)-(127)], it will obey a Liouville equation. From this a BBGKY hierarchy can be derived, and provides a possibility of doing a systematic kinetic theory via an expansion in the plasma parameter ε. This in some ways is similar to the theory for a one-dimensional

plasma, but the correspondence is not perfect. We shall adopt a compact notation, dispensing with the superscripts to identify charged species, as in the previous two sections.

For the phase space coordinates of the jth particle, we write $\underset{\sim}{X}_j \equiv (x_j, y_j, z_j, p_{zj}) \equiv (\vec{x}_j, p_{zj})$, where $p_{zj} = m_j v_{zj}$. The Liouville equation then reads

$$\left(\frac{\partial}{\partial t} + \sum_j \frac{d \underset{\sim}{X}_j}{dt} \cdot \frac{\partial}{\partial \underset{\sim}{X}_j} \right) D = 0 \tag{167}$$

where the probability distribution D is a function of all the $\underset{\sim}{X}_j$'s for all particles of both species, and is assumed to be symmetric under the interchange of like particle coordinates. Written out in detail, (167) is

$$\left\{ \frac{\partial}{\partial t} + \sum_j \left(\frac{c \vec{E}_j \times \vec{B}}{B^2} \cdot \frac{\partial}{\partial \vec{x}_j} + v_{zj} \frac{\partial}{\partial z_j} + \frac{e_j}{m_j} E_{j\parallel} \frac{\partial}{\partial v_{zj}} \right) \right\} D = 0 \tag{168}$$

Here, $\vec{E}_j = \Sigma_{\ell \neq j} e_\ell (\vec{x}_j - \vec{x}_\ell) / |\vec{x}_j - \vec{x}_\ell|^3$, and $E_{j\parallel} = \hat{b} \cdot \vec{E}_j$ is the z-component of the electric field at the location of particle j. Summations run over all particles of both signs.

The reduced probability distributions are

$$f_s \equiv V_s \int D \, d \underset{\sim}{X}_{s+1} \, d \underset{\sim}{X}_{s+2} \cdots d \underset{\sim}{X}_{2N} \tag{169}$$

and integrating (168) yields

$$\frac{\partial f_s}{\partial t} + \sum_{j=1}^{s} \left(\frac{c \vec{E}_j^s \times \vec{B}}{B^2} \cdot \frac{\partial}{\partial \vec{x}_j} + v_{zj} \frac{\partial}{\partial z_j} + \frac{e_j}{m_j} E_{j\parallel}^s \frac{\partial}{\partial v_{zj}} \right) f_s$$

$$= - c n_o \sum_{\text{species}} \sum_{i=1}^{s} \int d \underset{\sim}{X}_{s+1}$$

(170)

$$\left[\left(\frac{\vec{B}}{B^2} \times \frac{\partial \tilde{\varphi}_{i,s+1}}{\partial \vec{x}_i} \right) \cdot \frac{\partial f_{s+1}}{\partial \vec{x}_i} + \frac{e_i}{m_i c} \frac{\partial \tilde{\varphi}_{i,s+1}}{\partial z_i} \cdot \frac{\partial f_{s+1}}{\partial v_{zi}} \right]$$

Here,

$$\vec{E}_j^s = \sum_{\substack{\ell=1 \\ \ell \neq j}}^{s} \frac{e_\ell (\vec{x}_j - \vec{x}_\ell)}{|\vec{x}_j - \vec{x}_\ell|^3}$$

$$\tilde{\varphi}_{i,s+1} = e_{s+1} / |\vec{x}_i - \vec{x}_{s+1}|$$

$$n_o = N/V$$

and the $\sum_{\text{species}}$ means to sum over the (s+1) st particle, first treating it as an ion and then as an electron. Equation (170) is a generalization of (96), but because of the velocity dependence, greatly exceeds (96) in dynamical content.

The "Vlasov approximation" amounts to ignoring all the correlations and writing $f_s = \prod_{i=1}^{s} f_1(i)$, as usual. This gives, for the jth species,

$$\frac{\partial f_1}{\partial t} + v_z \frac{\partial f_1}{\partial z} + \frac{e_j E_z}{m_j} \frac{\partial f_1}{\partial v_z} + \frac{c \vec{E} \times \vec{B}}{B^2} \cdot \frac{\partial f_1}{\partial \vec{x}} = 0$$

(171)

where

$$\vec{E}(\vec{x},t) = -\frac{\partial}{\partial \vec{x}} \sum_{\text{species}} n_o e \int \frac{f_1(\vec{x}',v_z',t)}{|\vec{x}-\vec{x}'|}\, d\vec{x}'\, dv_z' \tag{172}$$

This is the limit of the *Vlasov-Poisson* system in a very strong magnetic field. An equivalent limit was treated some time ago by Harris [54] and Rosenbluth [55], although only after the solution to the full linearized Vlasov-Poisson system had been carried out.

Standard methods* [13] can be applied to solve the linearized version of (171)-(172). That is, we keep only linear terms in the departure from the spatially-uniform, field-free state. Linearized, Eq. (171) takes the form

$$\frac{\partial f_{i,e}^{(1)}}{\partial t} + v_z \frac{\partial f_{i,e}^{(1)}}{\partial z} \pm \frac{e E_z}{m_{i,e}} \frac{\partial f_{i,e}^{(0)}}{\partial v_z} = 0 \tag{173}$$

where i,e stands for ions and electrons. Poisson's equation (172) takes the form

$$\vec{E}(\vec{x},t) = -\frac{\partial}{\partial \vec{x}} \sum_{\text{i and e}} (\pm e n_o) \int \frac{f_{i,e}^{(1)}(\vec{x}',v_z',t)\, d\vec{x}'\, dv_z'}{|\vec{x}-\vec{x}'|} \tag{174}$$

$f_{i,e}^{(0)} = f_{i,e}^{(0)}(v_z)$ only are the spatially uniform equilibria we are perturbing about, and $f_{i,e}^{(1)}(\vec{x}, v_z, t)$ are the perturbations. Assuming $\vec{E}, f_{i,e}^{(1)} \sim \exp(i\vec{k}\cdot\vec{x})$ and that $f_{i,e}^{(1)}(\vec{x}, v_z, 0) = g_{i,e}(v_z)\, e^{i\vec{k}\cdot\vec{x}}$, the solution to (173) is (dropping $e^{i\vec{k}\cdot\vec{x}}$):

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* L. D. Landau, *J. Phys. (U.S.S.R)* *10*, 25 (1946).

$$f_{i,e}^{(1)}(v_z,t) = g_{i,e}(v_z)\, e^{-ik_z v_z t}$$

$$\mp \frac{e}{m_{i,e}} \frac{\partial f_{i,e}^{(0)}(v_z)}{\partial v_z} \int_0^t d\tau\, E_z(\tau)\, e^{ik_z v_z(\tau - t)} \tag{175}$$

Substituting (175) into (174) gives

$$i\vec{k} \cdot \vec{E}(t) = ikE(t) = 4\pi e n_o \int_{-\infty}^{\infty} dv_z$$

$$\left\{ [g_i(v_z) - g_e(v_z)]\, e^{-ik_z v_z t} - \frac{e}{m} \left(\frac{\partial f_i^{(0)}}{\partial v_z} + \frac{\partial f_e^{(0)}}{\partial v_z} \right) \right. \tag{176}$$

$$\left. \cdot \int_0^t d\tau\, E_z(\tau)\, e^{-ik_z v_z(t-\tau)} \right\}$$

Since $E_z(t) = (k_z/k)\, E(t)$, (176) is an integral equation of the convolution type for $E(t)$ which can be solved by standard techniques [13]. The result is conveniently obtained as a Laplace transform

$$E(\tau) = \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{ds}{2\pi i}\, E(s)\, e^{st} \tag{177}$$

where s is the complex Laplace transform variable, and

$$E(s) = \frac{-\int \frac{dv_z}{v_z - is/k_z} \frac{4\pi e n_o}{k k_z} [g_e(v_z) - g_i(v_z)]}{D(\vec{k},s)} \qquad (178)$$

$D(\vec{k},s)$ is the dielectric function, defined by (the usual [13] remarks about the contours of s and v_z integration apply):

$$D(\vec{k},s) = 1 - i \sum_j \frac{\omega_{pj}^2}{k^2} k_z \int \frac{dv_z \, \partial f_j^{(0)}(v_z)/\partial v_z}{s + i k_z v_z} \qquad (179)$$

The Σ_j in (179) is a sum over species (two terms) and $\omega_{pj}^2 \equiv 4\pi n_o e^2/m_j$, $j = i$ or e. The vanishing of $D(\vec{k},s)$ at $s = -i\omega(\vec{k}) + \gamma(\vec{k})$ determines the angular frequencies $\omega(\vec{k})$ at which the medium can support electrostatic oscillations; the corresponding Landau damping decrement is $-\gamma(\vec{k})$. Antecedents of Eq. (179) were first derived by Harris [54] and Rosenbluth [55].

Equation (179) becomes identical with that for an unmagnetized plasma upon replacing k_z by k, v_z by the component of $\vec{v}$ along $\vec{k}$, and letting $f_j^{(0)}(v_z)$ be replaced by the velocity distribution function of that component of $\vec{v}$. It follows that from all the previous detailed work on the solutions of $D(\vec{k},s) = 0$ we can immediately infer the character of the possible modes of oscillation of the 3-dimensional guiding center plasma.

In summary, there are two important branches of the ω versus $\vec{k}$ curve for Maxwellian distributions, one associated with electron oscillations and one associated with ion acoustic waves. For $k \ll K_D$, the electron branch has $\omega \approx \omega_{pe} k_z/k$, and goes over into ordinary electron plasma oscillations for parallel propagation ($k_z = k$). The ion acoustic branch is heavily damped unless $T_e \gg T_i$, in which case its dispersion relation

becomes $\omega/k \approx (\omega_{pi}(k_z/k))/(k^2 + K_{De}^2)^{1/2}$, where $K_{De}^2 \equiv 4\pi n_o e^2/\theta_e$, and this also goes over into it unmagnetized value at $k_z = k$. The damping decrements depend linearly, for the case $\gamma^2 \ll \omega^2$, on the values of $\partial f_{i,e}^{(0)}/\partial v_z(v_z = \omega/k_z)$, and are

large when these quantities are large. The most important difference between the solution of $D(\vec{k},s) = 0$ here and in the unmagnetized case is perhaps the possibility that undamped electrostatic oscillations at large angles ($k_z/k << 1$) can exist at considerably lower values than the electron plasma frequency: $\omega \approx \omega_{pe}\, k_z/k$. Since $\omega/k_z \approx \omega_{pe}/k$, these are still essentially undamped as long as ω_{pe}/k is greater than an electron thermal speed. For sufficiently large wavelength (small k) and large angles, these can lie below the ion gyrofrequency in a real plasma and can render the drift approximation more satisfactory than it would otherwise be.

Other kinetic theory problems can be approached through (170). For example, it may be expected that *kinetic equations* for the evolution of non-Maxwellian one-body distributions, such as the Balescu-Lenard equation, might be extracted from an expansion in ϵ for the spatially uniform state. The strong similarity to the one-dimensional case suggests, however, that a messy, $O(\epsilon^2)$, calculation might be required in order to predict relaxation to thermal equilibrium. A calculation of the spectral density of the fluctuations in the next sub-section points to a possible pathological feature of any such expansion, and re-emphasizes the very special role played in the guiding-center plasma by wave numbers with $k_z/k << 1$.

(E) Fluctuations for the Three-Dimensional Guiding-Center Plasma

It is possible, for the spatially-uniform three-dimensional guiding-center plasma, to calculate the auto-correlation function of the electric field in the Eurlerian representation:

$$\langle \vec{E}(\vec{x},t)\, \vec{E}(\vec{x}+\vec{r},t+\tau)\rangle = \int d\vec{k}\, d\omega\, \vec{S}_E(\vec{k},\omega)\ e^{i(\vec{k}\cdot\vec{r}+\omega\tau)} \quad (180)$$

The calculation can be carried out one of two ways: either by the Rostoker method of super-posed "dressed" test particles [14] or through the two-time BBGKY hierarchy [13], which can be shown to lead to the same results. Both methods are by now well known, and we shall use the former since it is simpler.

The calculation is instructive because it illustrates clearly the singular behavior associated with those components of $\vec{k}$ which lie nearly in the perpendicular direction to $\vec{B}$. The unusual physical behavior associated with $k << K_D$ and $\vec{k}\cdot\vec{B} = 0$ has already been shown to give the dominant contribution to the "anomalous" diffusion coefficient (164). A calculation of $\vec{S}_E(\vec{k},\omega)$ from the conventional expansion in the strength of the interaction sheds additional light on the unusual features of

this small, crucial region of $\vec{k}$-space.

"Test particles", in the sense they were introduced into kinetic theory by Rostoker [14] and Thompson and Hubbard [56], are uncorrelated, freely-streaming Coulomb charges carrying a modified potential. The modification of the potential is achieved by Fourier transforming it in space and time, then dividing the transformed potential $\phi(\vec{k},\omega)$ by $D(\vec{k},i\omega)$. Such fluctuation quantities as $S_E(\vec{k},\omega)$ are then obtained by superposing *uncorrelated* particles with this potential. The foundations of the method are discussed, e.g., in detail in Ref. 13, where it is shown to be connected with the more rigorous derivations proceeding from the two-time BBGKY hierarchy.

For particles of species j, the exact number density for non-interacting charges is:

$$n_j(\vec{x},t) = \sum_{i=1}^{N} \delta(\vec{x} - \vec{x}^j_{io} - \vec{v}^j_{io}\, t) \quad .$$

For the 3-dimensional guiding-center plasma, $\vec{v}^j_{io} = \hat{b}\, v^j_{io}$, but $\vec{x}^j_{io}$ can be anything. The Fourier-transformed charge density

$$n_j(\vec{k},t) = \sum_{i=1}^{N} (2\pi)^{-3} \exp -i\vec{k} \cdot (\vec{x}^j_{io} + \vec{v}^j_{io}\, t) \tag{181}$$

will give

$$\langle n_j(\vec{k},t)\; n_j(\vec{k}',t')\rangle$$

$$\left\langle \sum_{i,\ell=1}^{N} (2\pi)^{-6} \exp[-i\vec{k} \cdot (\vec{x}^j_{io} + \vec{v}^j_{io}\, t) - i\vec{k}' \cdot (\vec{x}^j_{\ell o} + \vec{v}_{\ell o}\, t')] \right\rangle$$

(182)

If the various charges are uncorrelated, the only terms which will contribute to the electric field spectrum for large t, t' are those with $i = \ell$, so

$$\langle n_j(\vec{k},t)\, n_j(\vec{k}',t+\tau)\rangle$$

$$= \frac{N_j}{V}\,\frac{\delta(\vec{k}+\vec{k}')}{(2\pi)^3}\int d\vec{v}^{\,j}_{io}\; e^{-i\vec{k}\cdot\vec{v}^{\,j}_{io}\tau}\; f_j(\vec{v}^{\,j}_{io}) \tag{183}$$

plus terms which do not contribute to $\overrightarrow{\overrightarrow{S}}_E$. The velocity integral for the 3D guiding-center plasma is a single integration, so we can write, for the guiding center plasma

$$\langle n_j(\vec{x},t)\, n_j(\vec{x}+\vec{r},\, t+\tau)\rangle$$

$$= \int d\vec{k}\, d\omega\, S_{jj}(\vec{k},\omega)\, \exp[i\vec{k}\cdot\vec{r} + i\omega\tau] \tag{184}$$

where

$$S_{jj}(\vec{k},\omega) = \frac{n_{oj}}{(2\pi)^3}\,\frac{f_j(-\omega/k_z)}{|k_z|} \tag{185}$$

plus terms which do not contribute to $\overrightarrow{\overrightarrow{S}}_E$. $n_{oj} \equiv N_j/V = n_o$ for both ions and electrons. Summing (185) over species and applying Poisson's equation, we have the spectral density of the electric field fluctuations for "bare" (i.e., uncorrelated and non-interacting) particles:

$$\vec{\vec{S}}_E(\vec{k},\omega)$$

(186)

$$= \sum_j \frac{2 n_{oj} e_j^2}{\pi} \frac{\vec{k}\vec{k}}{k^4} \frac{f_j(-\omega/k_z)}{|k_z|}$$

("bare" particles).

The canonical recipe now requires division of (186) by $|D(\vec{k},i\omega)|^2$ to give:

$$\vec{\vec{S}}_E(\vec{k},\omega)$$

(187)

$$= \sum_j \frac{2 n_{oj} e_j^2}{\pi} \frac{\vec{k}.\vec{k}}{k^4} \frac{1}{|k_z|} \frac{f_j(-\omega/k_z)}{|D(\vec{k},i\omega)|^2}$$

The corresponding expression for the three-dimensional *un*magnetized plasma is [13,14]

$$\vec{\vec{S}}_E(\vec{k},\omega)\Big|_{B=0}$$

(188)

$$= \sum_j \frac{2 n_{oj} e_j^2}{\pi} \frac{\vec{k}\vec{k}}{k^4} \frac{1}{k} \frac{F_j(-\omega/k)}{|D(\vec{k},i\omega)|^2}$$

where F_j is the velocity distribution function for the component of velocity along $\vec{k}$.

Equations (187) and (188) are similar, but there is a

striking difference. Namely, for ω/k_z = const., k finite, and $k_z \to 0$, (187) diverges; whereas (188) does not. In other words, (187) predicts that $<|E(\vec{k},\omega)|^2>$ will be in no sense small for $\vec{k} \perp \vec{B}$, even though the plasma parameter may be small.

Superficially more complicated, but similar, divergences plague the full three-dimensional plasma in a very strong magnetic field [57]. There exists at the present time no satisfactory calculation of $\vec{\vec{S}}_E(\vec{k},\omega)$ for those components of $\vec{k}$ nearly perpendicular to $\vec{B}$. The weak-coupling procedures simply will not work. In view of the already displayed importance of these components for plasma transport properties, the problem of calculating them accurately appears to be of the utmost importance. Because of the discussion following Eq. (166), it is not reasonable to expect this calculation to be easy.

(F) Summary

We have outlined a basis for the systematic calculation of statistical mechanical quantities for a plasma in which the plasma particles $\vec{E} \times \vec{B}$ drift transverse to a strong dc magnetic field, but respond according to Newton's laws in the parallel direction. The leading term in the coefficient of transverse spatial diffusion for a large but finite plasma has been calculated, and has been shown to fall off as $0(1/B)$, even in thermal equilibrium. Spectral density calculations for the electric field fluctuations based on the more conventional weak-coupling expansion are shown to lead to an unphysical divergence for those $\vec{k}$ vectors nearly perpendicular to $\vec{B}$. It is exactly these components which are involved in "anomalous" transport properties. The conventional expansion in the plasma parameter is therefore inadequate in such calculations.

IV. FINITE GYRORADIUS EFFECTS AND THERMAL RELAXATION IN TWO DIMENSIONS

(A) Preliminary Considerations; Liouville Equation; Hierarchy

Though the guiding-center approximation in two dimensions has greatly facilitated calculation of thermal equilibrium transport coefficients, certain other processes associated with finite values of the plasma parameter apparently lie outside its scope. Mot importantly, there is apparently no tendency toward thermal equilibrium in the conventional sense. Any uncorrelated spatially uniform state is an exact solution of (96):

setting $f_{s_i,s_e} = \prod_{i=1}^{s_i} \prod_{e=1}^{s_e} f_{1,0}(i)\, f_{0,1}(e)$ gives $\partial/\partial t \equiv 0$ in (96)

if $f_{1,0} = f_{0,1} = \text{const.} = 1$. There is no tendency, as there is for other Hamiltonian systems that obey a hierarchy, for the pair correlations to develop and go to their thermal equilibrium values [such as (25)]. Moreover, any state with vanishing triplet correlations and with two-particle distributions which depend only upon the magnitudes of the separations of the particles is *also* a time independent solution of the equations from (96) for which $(s_i,s_e) = (1,1)$, $(2,0)$, and $(0,2)$. Therefore from the *dynamical* point of view, there is no particular reason to prefer (43) over any other spectral density. For example, the purely random distribution, with all pair correlations $\equiv 0$ and $\langle \vec{E}^2 \rangle \equiv \langle \vec{E}^2 \rangle_{self}$ has been studied in some detail numerically by Taylor and McNamara [28].

Clearly what is required in order to study thermalization processes in a two-dimensional magnetized plasma is a relaxation of the guiding-center approximation. By averaging over the "fast" gyromotion, one arrives at a system of dynamical equations which, among other things, makes the potential energy a constant of the motion. Since total energy is also conserved this makes it impossible for a ratio of potential to total energy which is different from that demanded by thermal equilibrium to adjust itself. This is an additional proof of the impossibility of complete thermalization within the framework of guiding-center theory.

In this section we shall examine the possibility of predicting thermal relaxation for a plasma made up of two-dimensional charges whose equation of motion is

$$\frac{d\vec{v}_j}{dt} = \frac{e_j}{m_j}\left[\vec{E}(\vec{x}_j,t) + \frac{\vec{v}_j}{c} \times \vec{B}\right]$$

instead of (52). To keep the description simple, we shall also make the popular approximation of a uniform immobile ion background, so that only the negative charges are involved in the dynamics: all $e_j = -e$. The Liouville equation is simple to write down, and the BBGKY hierarchy which results from integrating over the phase space of the last N-s charges is

$$\left\{ \frac{\partial}{\partial t} + \sum_{j=1}^{s} \left(\vec{v}_j \cdot \frac{\partial}{\partial \vec{x}_j} + \frac{e_j}{m_j} \frac{\vec{v}_j}{c} \times \vec{B} \cdot \frac{\partial}{\partial \vec{v}_j} \right) \right.$$

$$\left. - \frac{1}{m} \sum_{i<j=1}^{s} \frac{\partial \varphi_{ij}}{\partial \vec{x}_{ij}} \cdot \left(\frac{\partial}{\partial \vec{v}_i} - \frac{\partial}{\partial \vec{v}_j} \right) \right\} f_s \qquad (189)$$

$$= \frac{n_o}{m} \sum_{i=1}^{s} \int d\vec{x}_{s+1} \, d\vec{v}_{s+1} \, \frac{\partial \varphi_{i,s+1}}{\partial \vec{x}_i} \cdot \frac{\partial f_{s+1}}{\partial \vec{v}_i}$$

where ϕ_{ij} is given by (15). $f_s = f_s(\vec{x}_1 \, \vec{v}_1 \ldots \vec{x}_s \, \vec{v}_s; t)$ is a function of the 4s phase space variables necessary to describe the first s particles, and is normalized so that $\int f_s \, d\vec{x}_1 \, d\vec{v}_1 \ldots d\vec{x}_s \, d\vec{v}_s = V^s = L^{2s}$.

Equation (189) looks much more like the familiar hierarchies that have been investigated than does (96), but the magnetic field term introduces some non-trivial complications.

It is realistic to confine our attention to the spatially uniform and gyrotropic limit, so that $f_1 = f_1(v_1^2/2, t)$ only, and $f_2 = f_2(\vec{x}_1 - \vec{x}_2, \vec{v}_1, \vec{v}_2, t)$ only. The s = 1 and s = 2 equations become, from (189),

$$\frac{\partial f_1}{\partial t} = \frac{n_o}{m} \int d\vec{x}_2 \, d\vec{v}_2 \, \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial f_2}{\partial \vec{v}_1} \qquad (190)$$

$$\left\{ \frac{\partial}{\partial t} + \vec{v}_1 \cdot \frac{\partial}{\partial \vec{x}_1} + \vec{v}_2 \cdot \frac{\partial}{\partial \vec{x}_2} - \frac{e}{m} \frac{\vec{v}_1 \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{v}_1} \right.$$

$$\left. - \frac{e}{m} \frac{\vec{v}_2 \times \vec{B}}{c} \cdot \frac{\partial}{\partial \vec{v}_2} - \frac{1}{m} \frac{\partial \varphi_{12}}{\partial \vec{x}_{12}} \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2} \right) \right\} f_2$$

$$= \frac{n_o}{m} \int d\vec{x}_3 \, d\vec{v}_3 \left(\frac{\partial \varphi_{13}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} + \frac{\partial \varphi_{23}}{\partial \vec{x}_2} \cdot \frac{\partial}{\partial \vec{v}_2} \right) f_3 \tag{191}$$

Another useful way of writing (191) compactly is

$$\left(\frac{\partial}{\partial t} + H_2 \right) f_2 = L_2 f_3 \tag{192}$$

where the linear operator H_2 is the *Liouville operator for two particles*, and L_2 is defined by the right-hand side of (191).

(B) Boltzmann's Equation for the Two-Dimensional Strongly-Magnetized Plasma

Equations (190) and (191) are general for any two-body potential ϕ_{12}. Their analogues in three dimensions (usually minus the magnetic field) are the starting point for the most satisfactory existing derivations of *kinetic equations*: approximate equations of the form $\partial f_1/\partial t$ = {a functional of f_1 only} which predict relaxation to a Maxwellian f_1. Expansion in the density, for example, leads to the Boltzmann equation; expansion in the potential leads to the Fokker-Planck equation; and expansion in the plasma parameter leads to the Balescu-Lenard equation. It is natural to attempt these expansions on (190) and (191).

The perturbation theory is not entirely straightforward in the three-dimensional case, and is less so for (190) and (191). Normally the complications result from the presence of disparate time scales (first articulated by Bogolyubov [2])

in the underlying physical processes. This must be incorporated into the mathematics either formally [2,58], or by a fortunate insight [59]. The complication of more than one time scale is present in (190)-(191) also, but another difficulty is more fundamental. The conventional kinetic equations have in common that the two-body distribution f_2 evolves in terms of f_1 and what is essentially a two-particle dynamical process that goes to completion in a time short compared with $f_1/\partial f_1/\partial t$. For instance, in the derivation of Boltzmann's equation [60,61], the two-body process is a collision in which two particles approach each other, collide, and separate in well-defined stages during the duration of which f_1 is sensibly constant. For two colliding charges in two dimensions, the situation is much different in the presence of a strong magnetic field, as we shall now show.

Two equal charges in two dimensions obey the equations

$$\frac{d\vec{v}_1}{dt} = -\frac{e}{m}\left(\vec{E}_{12} + \frac{\vec{v}_1}{c} \times \vec{B}\right)$$

$$\frac{d\vec{v}_2}{dt} = -\frac{e}{m}\left(\vec{E}_{21} + \frac{\vec{v}_2}{c} \times \vec{B}\right) \tag{193}$$

where $\vec{E}_{12} = -(2e/\ell)\,\vec{x}_{12}/x_{12}^2$, and $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$. Adding Eqs. (193) gives, since $\vec{E}_{12} = -\vec{E}_{21}$

$$\frac{d\vec{V}}{dt} = -\frac{e}{mc}\,\vec{V} \times \vec{B} \tag{194}$$

where $\vec{V} \equiv (\vec{v}_1 + \vec{v}_2)/2$ is the center-of-mass velocity of the two charges. Equation (194) is the equation of motion of a free particle in a uniform magnetic field and so the solution for the center-of-mass motion is immediate.

Subtracting Eqs. (193) gives

$$\frac{d\vec{v}_{12}}{dt} = -\frac{2e}{m}\vec{E}_{12} - \frac{e\vec{v}_{12}}{mc} \times \vec{B}$$

$$= \frac{4e^2}{\ell m}\frac{\vec{x}_{12}}{x_{12}^2} - \frac{e\vec{v}_{12}}{mc} \times \vec{B} \qquad (195)$$

where $\vec{v}_{12} \equiv \vec{v}_1 - \vec{v}_2$ is the relative velocity of the two charges. Two constants of the motion of (195) are (as can be readily verified)

$$\frac{m\vec{v}_{12}^2}{2} - \frac{4e^2}{\ell} \ln |\vec{x}_{12}| = \sum = \text{const.}$$

$$m\hat{b} \cdot (\vec{x}_{12} \times \vec{v}_{12}) - \frac{eB}{2c} x_{12}^2 = P_\theta = \text{const.} \qquad (196)$$

Expressing $\vec{x}_{12} = (r \cos\theta, r \sin\theta)$ in terms of polar coordinates and eliminating $\dot{\theta}$, we have

$$\frac{m}{2}\dot{r}^2 + \text{"V(r)"} = \mathcal{E} = \text{const.} \qquad (197)$$

where the "effective potential" is

$$\text{"V(r)"} \equiv \frac{P_\theta^2}{2mr^2} + \frac{e^2B^2r^2}{8mc^2} - \frac{4e^2}{\ell}\ln r \,. \qquad (198)$$

The solution to (197) is

$$\pm \int \frac{dr}{\sqrt{\mathcal{E} - \text{"V(r)"}}} = \sqrt{\frac{2}{m}} \int dt$$

and since "V(r)" has a single minimum and $\to \infty$ as $r \to 0$ or $r \to \infty$,

the r motion is always bounded and periodic, with limits defined by the intersection of "V(r)" with $\mathcal{E}$.

Since $\dot{\theta} = P_\theta/mr^2 + eB/2mc$, $\dot{\theta}$ is periodic with the same period as r, and θ can be written as a constant times t plus a periodic function of t with the same period as the r motion and with average value zero. The motion may or may not encircle the origin, but there is always a maximum radius r which is reached periodically.

This shows that a significant difference exists between the basic two-body interaction in a magnetized two-dimensional plasma and a three-dimensional one (or a two-dimensional one with $\vec{B} = 0$). There is no time before which one can confidently assume that the spatial correlations of two colliding charges vanish. This renders suspect many of the assumptions of the BBGKY kinetic theory. Nevertheless, alternative equally-plausible assumptions have yet to be suggested, so it is still of interest to take the kinetic theory as far as possible in parallel to its usual form.

The easiest expansion to perform on Eqs. (190)-(192) is in powers of the density. To lowest order in the density, (192) is just the two-particle Liouville equation

$$\left(\frac{\partial}{\partial t} + H_2\right) f_2 \approx 0 \tag{199}$$

whose solution is any function of the constants of the motion of the two-body problem defined by (193). If we go ahead and ignore the pair correlation at $t = 0$, $f_2 \approx f_1 f_1$ then, the solution to (199)

$$f_2(\vec{x}_1,\vec{v}_1,\vec{x}_2,\vec{v}_2,\tau) = f_1[\vec{v}_1^2(-\tau)/2,0]\; f_1[\vec{v}_2^2(-\tau)/2,0] \tag{200}$$

where $\vec{v}_1(-\tau)$, $\vec{v}_2(-\tau)$ are the solutions to (193) that lead to two particles, being at $\vec{x}_1,\vec{v}_1,\vec{x}_2,\vec{v}_2$, at time τ. Substitution of (200) into (190) would lead to secularity for reasons which are well-known [11,61,62]. This is avoided by the method of Dupree [59] or its more systematic formulation in the method of multiple time scales [11,58,61].

One notes that an equally accurate solution to (199) is obtained by allowing f_1 to have a slow dependence on the time. Equation (190) predicts $\frac{\partial f_1}{\partial t} = 0(n_0)$, and so a slowly-varying f_1 results for low densities. If we write a formal dimensionless

expansion parameter $\tilde{\varepsilon}$ in front of n_o to remind us that it is treated as very small, (200) can to equal accuracy be replaced by

$$f_2(\vec{x}_1,\vec{v}_1,\vec{x}_2,\vec{v}_2,\tau) \cong f_1[\vec{v}_1^2(-\tau)/2,\tilde{\varepsilon}\tau]\ f_1[\vec{v}_2^2(-\tau)/2,\tilde{\varepsilon}\tau] \quad (201)$$

since when substituted into (199), the only additional terms generated are of $O(\tilde{\varepsilon})$. At the end of the calculation, we can then let $\tilde{\varepsilon} \to 1$.

The canonical recipe would be to substitute (201) into (190) after passing to the limit $\tau \to \infty$, $\tilde{\varepsilon}\tau$ finite. That is, one would write

$$\frac{\partial f_1(\vec{v}_1^2/2,\tilde{\varepsilon}t)}{\partial(\tilde{\varepsilon}t)} = \lim_{\tau\to\infty} \frac{n_o}{m} \int d\vec{x}_2\, d\vec{v}_2\, \frac{\partial \varphi_{12}}{\partial \vec{x}_{12}} \cdot \frac{\partial}{\partial \vec{v}_1} f_1\left[\frac{\vec{v}_1^2(-\tau)}{2},\tilde{\varepsilon}t\right] f_1\left[\frac{\vec{v}_2^2(-\tau)}{2},\tilde{\varepsilon}t\right] \quad (202)$$

In the $\vec{B} = 0$ case, (202) is nothing but Boltzmann's equation [2], and only a few additional manipulations are required [6] to bring it into the form given by Boltzmann. For the case under discussion, however, it is not so simple. The limit of the integrand does not exist; it is an oscillatory function of τ. The period of oscillation does vary from point to point in the two-particle phase space so that the limit of the integral may exist even though the limit of the integrand may not. But this has not been proved. An additional *ad hoc* step which makes the limit of (202) well defined is that of *time averaging*, reasonable in view of the success of the Bogolyubov "method of averaging" [63,64] in other situations. It consists of replacing (202) by its time average

$$\frac{\partial f_1(v_1^2/2, \tilde{\epsilon}t)}{\partial(\tilde{\epsilon}t)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T d\tau \, \frac{n_0}{m}$$

$$\int d\vec{x}_2 \, d\vec{v}_2 \, \frac{\partial \varphi_{12}}{\partial \vec{x}_{12}} \cdot \frac{\partial}{\partial \vec{v}_1} f_1[\vec{v}_1^2(-\tau)/2, \tilde{\epsilon}t] \, f_1[\vec{v}_2^2(-\tau)/2, \tilde{\epsilon}t] \tag{203}$$

If the limit in (202) exists, it is identical to (203).

We propose (203) [or (202)] as the analogue of Boltzmann's equation for the two-dimensional magnetized plasma [12]. We suspect that it obeys an H-theorem and conservation laws, but have not been able to prove any of these.

(C) The Weak-Coupling Approximation; The Fokker-Planck Equation

A second expansion of (191) which lies in the mainstream of kinetic theory is an expansion in powers of the interaction. That is, we shall treat the terms containing a two-body potential as of one order higher in a formal expansion parameter $\tilde{\epsilon}$ than the terms which do not. Corresponding to this (and gleaning an insight from the equilibrium theory), we expect the correlation functions to get successively smaller in powers of $\tilde{\epsilon}$ as well. Since $\frac{\partial f_1}{\partial t} = O(\tilde{\epsilon}^2)$,

$$f_1(1) = f_1(\vec{v}_1^2/2, \tilde{\epsilon}^2 t)$$

$$f_2(1,2) = f_1(\vec{v}_1^2/2, \tilde{\epsilon}^2 t) \, f_1(\vec{v}_2^2/2, \tilde{\epsilon}^2 t)$$

$$+ \tilde{\varepsilon}\, P(\vec{x}_1,\vec{v}_1,\vec{x}_2,\vec{v}_2;t,\tilde{\varepsilon}^2 t)$$

$$f_3(1,2,3) = f_1(\vec{v}_1^2/2,\tilde{\varepsilon}^2 t)\, f_1(\vec{v}_2^2/2,\tilde{\varepsilon}^2 t)\, f_1(\vec{v}_3^2/2,\tilde{\varepsilon}^2 t) \tag{204}$$

$$+ \tilde{\varepsilon}\{f_1(\vec{v}_1^2/2,\tilde{\varepsilon}^2 t)\, P(\vec{x}_2,\vec{v}_2,\vec{x}_3,\vec{v}_3;t,\tilde{\varepsilon}^2 t) + \text{cyclic permutations}\}$$

$$+ O(\tilde{\varepsilon}^2)$$

and so on.

Dropping terms of $O(\tilde{\varepsilon}^2)$ from (191) after substituting in (204) gives

$$\left(\frac{\partial}{\partial t} + \vec{v}_1\cdot\frac{\partial}{\partial \vec{x}_1} + \vec{v}_2\cdot\frac{\partial}{\partial \vec{x}_2} - \frac{e}{mc}\vec{v}_1 \times B\cdot\frac{\partial}{\partial \vec{v}_1} - \frac{e}{mc}\vec{v}_2 \times \vec{B}\cdot\frac{\partial}{\partial \vec{v}_2}\right) P$$

$$= \frac{1}{m}\frac{\partial \varphi_{12}}{\partial \vec{x}_{12}}\cdot\left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2}\right) f_1(\vec{v}_1^2/2,\tilde{\varepsilon}^2 t)\, f_1(\vec{v}_2^2/2,\tilde{\varepsilon}^2 t) \tag{205}$$

for the pair correlation P.

The left-hand side of (205) can be recognized as the total time derivative dP/dt *computed along the noninteracting trajectories in the two-particle phase space.* Integrating (205) from 0 to t and assuming that no pair correlations exist initially, we get

$$P = -\int_0^t d\tau \; \frac{2e^2}{\ell m} \frac{\vec{x}_{12}(\tau) \cdot \vec{v}_1(\tau)}{x_{12}^2(\tau)} \; f_1'(v_1^2/2, \tilde{\epsilon}^2 t) \, f_1(v_2^2/2, \tilde{\epsilon}^2 t)$$

$$+ (1 \leftrightarrow 2) \tag{206}$$

In (206), the prime means differentiation with respect to $v_1^2/2$, and $(1 \leftrightarrow 2)$ means the same term with coordinates of particles 1 and 2 interchanged.

Since $\vec{x}_{12}(\tau)$, $\vec{v}_1(\tau)$, $\vec{v}_2(\tau)$ are simple periodic functions of τ, readily expressible in terms of x_1, v_1, x_2, and v_2, the integral

$$I(1) = \frac{2e^2}{\ell m} \int_0^t d\tau \; \frac{\vec{x}_{12}(\tau) \cdot \vec{v}_1(\tau)}{x_{12}^2(\tau)} \tag{207}$$

can be done. I(1) has the physical interpretation of being m^{-1} times the kinetic energy transferred, to lowest order in perturbation theory, by particle 2 to particle 1. As long as I(1) remains $\ll \vec{v}_1^2/2$, the perturbation theory remains valid.

The integral (207) is messy, but can be carried out in terms of elementary functions. It is performed in detail in Vahala's thesis [12]. Taken over an integral number of gyroperiods $T_o \equiv 2\pi mc/eB$, it is

$$I(1) = \begin{cases} 0 \;, & x_{12}^2 + \dfrac{2\,\vec{x}_{12} \cdot (\hat{b} \times \vec{v}_{12})}{\Omega} > 0 \\[2ex] \dfrac{t}{T_o} \dfrac{4\pi e^2}{\ell m} \dfrac{\vec{v}_1 \cdot [\vec{v}_{12} \times \hat{b}]}{v_{12}^2} \;, & x_{12}^2 + \dfrac{2\,\vec{x}_{12} \cdot (\hat{b} \times \vec{v}_{12})}{\Omega} < 0 \end{cases} \tag{208}$$

Here, $\Omega \equiv eB/mc$. The content of (208) is roughly that only particles initially close enough together contribute to P, and this contribution grows secularly with time.

We must confess lack of a systematic understanding of what the proper use of (208) in (190) is. The formal substitution of (208) into (190) leaves a secularly-growing term which is unphysical. It was previously dealt with [12] by the somewhat *ad hoc* procedure of cutting off the integration at an upper limit corresponding to $mI(1) \sim \theta$. Here we shall use a time averaging procedure motivated by "stroboscopic" perturbation theory [65] and obtain the same answer, up to a numerical factor.

The motivating physical inequality is that in one gyroperiod, f_1 should have changed by very little. Therefore, (190) will be replaced by

$$\frac{\partial f_1}{\partial t} = \tilde{\epsilon} \frac{n_o}{m} \int d\vec{x}_2 \, d\vec{v}_2 \, \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} \bar{f}_2$$

$$= \tilde{\epsilon}^2 \frac{n_o}{m} \int d\vec{x}_2 \, d\vec{v}_2 \, \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1} \bar{P} \qquad (209)$$

where the bar over f_2 and P indicates an average over a gyroperiod. Since in the region where it is non-zero, $\overline{I(1)} = (2\pi e^2/\ell m) \cdot \hat{v}_1 \cdot (\hat{v}_{12} \times \hat{b})/v_{12}^2$, and since

$$P = -I(1) \, f_1'(v_1^2/2, \tilde{\epsilon}^2 t) \, f_1(v_2^2/2, \tilde{\epsilon}^2 t) + (1 \leftrightarrow 2) \qquad (210)$$

we may substitue (208) and (210) into (209) to get a kinetic equation

$$\frac{\partial f_1}{\partial(\tilde{\epsilon}^2 t)}\left(\vec{v}_1^{\,2}/2, \tilde{\epsilon}^2 t\right)$$

$$= \frac{n_o}{m}\int d\vec{x}_2\, d\vec{v}_2\, \frac{\partial \varphi_{12}}{\partial \vec{x}_1} \cdot \frac{\partial}{\partial \vec{v}_1}\left\{ -\overline{I(1)}\, f_1'(v_1^2/2, \tilde{\epsilon}^2 t)\, f_1(v_2^2/2, \tilde{\epsilon}^2 t) + (1 \leftrightarrow 2)\right\} \tag{211}$$

with the region of integration to run over all the region defined by the second inequality in (208).

The integrals in (211) can be performed [12] and the result is

$$\frac{\partial f_1(v_1^2/2, \tilde{\epsilon}^2 t)}{\partial(\tilde{\epsilon}^2 t)} = -\frac{\partial}{\partial \vec{v}_1} \cdot \int d\vec{v}_2\, \vec{\vec{Q}}(\vec{v}_1, \vec{v}_2) \cdot \left(\frac{\partial}{\partial \vec{v}_1} - \frac{\partial}{\partial \vec{v}_2}\right) \tag{212}$$

$$f_1(v_1^2/2, \tilde{\epsilon}^2 t)\, f_1(v_2^2/2, \tilde{\epsilon}^2 t)$$

where the dyadic $\vec{\vec{Q}}$ is given by

$$\vec{\vec{Q}} = -\frac{4\pi^2 n_o e^4}{m^2 \ell^2 \Omega}\, \frac{(\vec{v}_{12} \times \hat{b})(\vec{v}_{12} \times \hat{b})}{v_{12}^2} \tag{213}$$

$$= -\frac{4\pi^2 n_o e^4}{m^2 \ell^2 \Omega}\, \frac{v_{12}^2 \vec{\vec{I}} - \vec{v}_{12}\vec{v}_{12}}{v_{12}^2}$$

Equation (212) is a standard form in which the Fokker-Planck equation has been written before. It is easy to prove from it conservation of particles, momentum (zero in this case) and kinetic energy. It also obeys an H-theorem, and thus predicts an approach to a Maxwellian f_1 as $t \to \infty$:

$$f_1(v_1^2/2, \tilde{\epsilon}^2 t) \xrightarrow[\tilde{\epsilon}^2 t \to \infty]{} \frac{m}{2\pi\theta} \exp(-m\vec{v}_1^2/2\theta)$$

where θn_0 is the initial kinetic energy density. The relaxation that occurs is predicted to occur on a time scale $t_{rel.}$, where

$$\frac{1}{t_{rel.}} \approx \frac{1}{4 n_0 \lambda_D^2} \frac{\omega_{pe}^2}{\Omega}$$

That $t_{rel.}$ shall be long compared to other time scales involved is a necessary condition for the applicability of the guiding-center model to situations other than thermal equilibrium, as done, for example, by Taylor and McNamara [28] for the case of a random initial distribution (P = 0).

Up to an arbitrary numerical factor associated with the cut-off of I(1), Eq. (212) is the same as that given by Vahala and Montgomery [12]. [See, however, reference [67].]

(D) The "Weak, Long-Range", or Balescu-Lenard, Limit

Note that no long-range cutoff of the potential was required to render (213) finite, unlike the corresponding situation in the unmagnetized case. Once can thus question the necessity for going to the "weak long-range" limit usually thought necessary to account properly for the long-range part of the Coulomb force. Equation (212) may thus be surmised to represent the "Balescu-Lenard limit" as well as the "Fokker-Planck limit". Nevertheless, the Balescu-Lenard limit was investigated in some detail in Vahala's thesis [12]. P was re-calculated by adding in terms

$$- \frac{n_o}{m} \frac{\partial f_1}{\partial \vec{v}_1} \cdot \int d\vec{x}_3 \, d\vec{v}_3 \, \frac{\partial \varphi_{13}}{\partial \vec{x}_1} \, P(\vec{x}_2, \vec{v}_2, \vec{x}_3, \vec{v}_3, t)$$

$$+ (1 \leftrightarrow 2)$$

on the left-hand side of (205). The calculation is still manageable, though very lengthy. The rather surprising result is that following the standard procedures, $\partial f_1/\partial t = 0$, through terms of first order in the plasma parameter. This not entirely expected vanishing of the Balescu-Lenard collision term has as yet no satisfactory physical interpretation, and requires additional investigation.

(E) The "Test Particle" Problem; Absence of Dynamic Screening on Vlasov Time Scales

One of the most important physical characteristics of the unmagnetized Vlasov plasma in both two and three dimensions is the separation of "Vlasov" and "collisional" time scales. These can be roughly associated with processes which occur at finite rates even when the plasma parameter ϵ goes to zero and those which do not. In the former category, we can put plasma oscillations, Landau damping, and relaxation to the "kinetic stage" of Bogolyubov, all of which occur on time scales measured by the inverse plasma frequency ω_p^{-1}. In the latter class is relaxation to thermal equilibrium, which occurs on time scales $\sim \omega_p^{-1} \epsilon^{-1}$ in two dimensions, and $\sim \omega_p^{-1} \epsilon^{-1}/\ln\epsilon^{-1}$ in three.

Many of the novel features of two-dimensional strongly magnetized plasmas (and the nearly perpendicular $\vec{k}$-vectors in three) are associated with the fact that these two time scales no longer enjoy a clear separation. There are no relaxation processes which occur for components of the fluctuation spectra with $\vec{k}$ perpendicular to $\vec{B}$ that occur on a time scale independent of ϵ.

This can be illustrated by adapting a very standard kind of calculation for three-dimensional plasma [56] to the two-dimensional magnetized plasma: the linear response of a Vlasov plasma to a "test" charge introduced at $t = 0$.

The Vlasov equation obtained from (189) by ignoring all correlations is

$$\left[\frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} - \frac{e}{m}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \cdot \frac{\partial}{\partial \vec{v}}\right] f_1 = 0 \tag{214}$$

where

$$\vec{E}(\vec{x},t) = \frac{2e}{\ell} n_o \int f(\vec{x}',\vec{v}',t) \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^2} d\vec{x}' d\vec{v}' \tag{215}$$

It is soluble, as a linearized initial value problem, by standard methods. If the initial conditions are such that a test charge e_t is introduced at $\vec{x}_o$ with velocity $\vec{v}_o = (v_o \cos\theta, v \sin\theta)$ at $t = 0$, the electric field which results for $t > 0$ is

$$\vec{E}(\vec{x},t) = \int d\vec{k}\ e^{i\vec{k}\cdot\vec{x}} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{ds}{2\pi i}\ e^{st}\ \vec{E}_{\vec{k}}(s) \tag{216}$$

where

$$\vec{E}(\vec{k},s)$$

$$= \frac{\left(\vec{k} i e_t/\pi k^2 \ell\right)}{D(\vec{k},s)}\ e^{-i\vec{k}\cdot\vec{x}_o + (ik v_o/\Omega_e)\sin(\Theta-\alpha)} \sum_{n=-\infty}^{\infty} \frac{J_n\left(\frac{k v_o}{\Omega_e}\right) e^{in(\Theta-\alpha}}{s - in\Omega_e} \tag{217}$$

The new symbols appearing in (217) are $\Omega_e \equiv |eB/mc|$, $\vec{k} = (k \cos\alpha, k \sin\alpha)$, and the dielectric function $D(\vec{k},s)$ is

$$D(\vec{k},s) = 1 - \frac{2\pi i\, \omega_{pe}^2\, \Omega_e}{k^2} \sum_{n=-\infty}^{\infty} \frac{n}{s + in\Omega} \int_0^{\infty} v dv\, f_o'(v^2/2)\, J_n^2\left(\frac{kv}{\Omega_e}\right) \tag{218}$$

J_n is the Bessel function of its argument and $f_o = (2\pi\theta)^{-1}$. $\exp(-mv^2/2\theta)$ is the two-dimensional Maxwellian; $\omega_{pe} \equiv 4\pi n_o e^2/\ell m$.

If the charge is stationary, $v_o = 0$, and $kV_e \ll \Omega_e$, $|s| \ll \Omega_e$, (217) simplifies considerably to

$$\vec{E}_{\vec{k}}(s) \cong \frac{i e_t \vec{k}}{\pi k^2 \ell} \frac{e^{-i\vec{k}\cdot\vec{x}_o}}{s\left(1 + \dfrac{\omega_{pe}^2}{\Omega_e^2}\right)} \tag{219}$$

When we Fourier-Laplace invert (219) we get, for the electric field produced by the test charge

$$\vec{E}(\vec{x},t) \cong \frac{2 e_t (\vec{x} - \vec{x}_o)}{\ell |\vec{x} - \vec{x}_o|^2} \frac{1}{1 + \dfrac{\omega_{pe}^2}{\Omega_e^2}}, \qquad \text{all } t, \tag{220}$$

which is the vacuum value divided by the low-frequency limit of the dielectric function $1 + \omega_{pe}^2/\Omega_e^2$.

If, however, we had considered the limit $\Omega_e \to 0$ in (217) we would have gotten instead of (220),

$$\vec{E}(\vec{x},t) \cong \frac{2 e_t (\vec{x} - \vec{x}_o)}{\ell |\vec{x} - \vec{x}_o|} \sqrt{\frac{\pi K_D}{2 |\vec{x} - \vec{x}_o|}}\, e^{-K_D |\vec{x} - \vec{x}_o|} \tag{221}$$

for $\omega_{pe} t \gg 1$ and $K_D |\vec{x} - \vec{x}_o| \gg 1$.

The differences between (220) and (221) are considerable. The latter is Debye shielded and has an effective range $\sim K_D^{-1} = \lambda_D$, while the former is not, and has an effective range of the order of the size of the system.

We know, of course, that the field of the test charge will eventually be shielded; the thermal equilibrium theory tells us that. The point is that it occurs on time scales which are not identifiable with Vlasov's equation, but with processes which slow down to stationary as $\varepsilon \to 0$.

(F) Comments on the Problem of Spatial Diffusion for Finite Larmor Radii

Dawson, Okuda, and Carlile have published a preliminary [66] account of a calculation of a two-dimensional coefficient of spatial diffusion which endeavors to include finite Larmor radius effects. The essentially new result, for which some evidence from numerical simulation is presented, is that the Taylor-McNamara result [Eqs. (89) and (91) of this article] appears divided by the square root of the dielectric function at zero frequency, $\left[1 + \sum_j \omega_{pj}^2/\Omega_j^2\right]^{\frac{1}{2}}$. The development, however, does not aspire to the deductive character of the Taylor-McNamara development and its generalizations, and a number of the steps are little more than order-of-magnitude estimates. These fortunately combine to produce exactly the right numerical coefficient to agree with Taylor-McNamara in the limit $B \to \infty$. A supplementary derivation based on a two-particle encounter model of diffusion is used to interpret the data for lower values of B and a "classical" [i.e., $O(1/B^2)$ diffusion coefficient is claimed for these lower values. This derivation omits contributions from the very-long-wavelength components of the fluctuation spectrum, however, and so contains no volume dependence. Since it is precisely these components that led to the volume divergence in Eq. (91) and in the calculations of Ref. 32, one suspects they may play a similar role in the magnetized finite-Larmor-radius plasma. This "classical" expression therefore probably needs, at best, an inequality on the plasma volume in order that it be in fact the dominant term.

One can hope for a clarification of these, and other, important unanswered questions connected with the finite gyroradius plasma in the next few years.

ACKNOWLEDGMENTS

This work was supported by NASA Grant NGL-16-001-043 at the University of Iowa.

Valuable discussions with the following individuals are gratefully acknowledged: Dr. George Vahala (in collaboration with whom many of these results were obtained), Dr. Frederick Tappert, Dr. Glenn Joyce, Dr. C. -S. Liu, and Dr. Bruce Langdor An air-mail preprint of the important manuscript of Taylor and McNamara, shortly before the English postal strike of 1970-71, enabled us to begin work on these problems many months earlier than we would otherwise have been able to do. Mrs. Rozanne Huff skillfully performed the very demanding typing.

APPENDIX I: DERIVATION OF EQ. (81)

We are interested in evaluating expressions of the type

$$I \equiv \int d\vec{E}_1\, d\vec{E}_2 \cdots d\vec{E}_M\, P(\vec{E}_1 \vec{E}_2 \cdots \vec{E}_M) \exp\left(i \sum_{m=1}^{M} \vec{\lambda}_m \cdot \vec{E}_m\right) \tag{A1}$$

where

$$P(\vec{E}_1 \vec{E}_2 \cdots \vec{E}_M) = \eta \exp\left(- \sum_{i,j=1}^{M} \vec{\vec{a}}_{ij} : \vec{E}_i \vec{E}_j\right) \tag{A2}$$

and η is chosen so that

$$\int d\vec{E}_1\, d\vec{E}_2 \cdots P(\vec{E}_1 \vec{E}_2 \cdots) = 1 \tag{A3}$$

The $\vec{\vec{a}}_{ij}$ are positive definite dyadics which satisfy $\vec{x} \cdot \vec{\vec{a}}_{ij} \cdot \vec{x} > 0$ for any non-zero $\vec{x}$.

The $(\vec{E}_1, \vec{E}_2, \ldots \vec{E}_M)$ can be thought of as a column vector whose ith element is $\mathcal{E}_i$, say. The $\vec{\vec{a}}_{ij}$ can be thought of as a real, symmetric, positive definite matrix whose ijth element is A_{ij}, say. Finally the $(\vec{\lambda}_1, \vec{\lambda}_2, \ldots \vec{\lambda}_M)$ can be thought of as a column vector whose jth element is ℓ_j, say. We can then write the desired integral I as

$$I = \eta \int \left(\prod_j d\mathcal{E}_j\right) \exp\left\{- \sum_{ij} A_{ij}\, \mathcal{E}_i\, \mathcal{E}_j + i \sum_j \ell_j\, \mathcal{E}_j\right\} \tag{A4}$$

The exponentials in (A4) are scalars, and a unitary transformation of the $\mathcal{E}_i$'s always exists which will make A_{ij} diagonal in the new representation. Call the unitary transformation U_{ij}. Thus

$$\varepsilon_i' = \sum_j U_{ij}\, \varepsilon_j$$

$$A_{ij}' = \sum_{kn} U_{ik}\, A_{kn} (U^{-1})_{nj} \tag{A5}$$

where $A_{ij}' = A_i'\, \delta_{ij}$, with $A_i' > 0$, all i.

The Jacobian of a unitary transformation is unity, so we can write (A4) as

$$I = \eta \int \left(\prod_j d\varepsilon_j' \right) \exp\left\{ - \sum_j A_j'\, \varepsilon_j'^2 + i \sum_j \ell_j'\, \varepsilon_j' \right\} \tag{A6}$$

where $\ell_j' = \sum_k U_{jk}\, \ell_k$. The integrals in (A6) are now elementar and

$$I = \eta \prod_j \left\{ \int_{-\infty}^{\infty} d\varepsilon_j'\, e^{-\left(\sqrt{A_j'}\, \varepsilon_j' - i\ell_j'\varepsilon_j'/2\sqrt{A_j'} \right)^2} e^{-\ell_j'^2/4A_j'} \right\}$$

$$= \eta \prod_j \left(\frac{\pi^{\frac{1}{2}}}{\sqrt{A_j'}}\; e^{-\ell_j'^2/4A_j'} \right) \tag{A7}$$

Now $\eta \prod_j (\pi^{\frac{1}{2}}/\sqrt{A_j'}) = 1$, and

$$\prod_j \exp(-\ell_j'^2/4A_j') = \exp\left(-\sum_{ij} \ell_i (A^{-1})_{ij}\, \ell_j/4 \right) \quad , \text{ so}$$

$$I = \exp\left(-\sum_{ij} \ell_i (A^{-1})_{ij}\, \ell_j/4 \right) \tag{A8}$$

We also note from (A4) that

$$- \left. \frac{\partial^2 I}{\partial \ell_i \, \partial \ell_j} \right|_{\text{all } \ell_k = 0} = \langle \mathcal{E}_i \, \mathcal{E}_j \rangle$$

so

$$\langle \mathcal{E}_i \, \mathcal{E}_j \rangle = (A^{-1})_{ij}/2$$

from (A8), and

$$I = \exp\left(- \frac{1}{2} \sum_{ij} \ell_i \langle \mathcal{E}_i \, \mathcal{E}_j \rangle \, \ell_j \right) \tag{A9}$$

If we now revert to the vector notation,

$$I = \exp\left(- \frac{1}{2} \sum_{i,j=1}^{M} \vec{\lambda}_i \, \vec{\lambda}_j : \langle \vec{E}_i \, \vec{E}_j \rangle \right) \tag{A10}$$

To derive Eq. (81) of the text from (A10), replace $\vec{\lambda}_i$ by $(c\vec{k} \times \vec{B}/B^2) \, \Delta t$, all for i.

Equation (A2) has the property that integrating it over all the $\vec{E}_i$ but one leaves a Gaussian distribution in that $\vec{E}_i$. However, the converse is not true: that the electric field have a Gaussian distribution at any instant is not a sufficient condition that the joint distribution have the form (A2).

Considerable use has been made of this theorem in statistical physics in recent years. See, e.g., the review article by A.J.F. Siegert, in *Statistical Mechanics at the Turn of the Decade*, E.G.D. Cohen, ed.; New York, Marcel Dekker, 1971.

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