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DYNAMIC ANALYSIS OF THE GEOS SATELLITE

Final Technical Report

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(Spin Stabilization of Flexible Satellites about Nontrivial Equilibrium Positions)

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Abstract

The stability problem of the GEOS satellite has been solved. A computer simulation indicates lack of stability, a fact that can be attributed to the lack of bending stiffness of the cables. Whereas small cable bending stiffness can render the system stable, the first natural frequency of oscillation of the spacecraft is likely to be very low, so that the cables can represent a potential problem area.

Introduction

The GEOS satellite (the "simple model") consists of a rigid core, one pair of radial booms, one pair of cables with tip masses, and two pairs of axial booms, as shown in Fig. 1. The latter two pairs of booms are not strictly axial, as they are inclined with respect to the equatorial plane at angles other than 90°. The satellite spins freely in space with constant angular velocity Ω. The interest lies in the stability of motion when the spacecraft is perturbed slightly from the uniform spin equilibrium state.

The stability of force-free satellites with flexible appendages, such as that considered here, has been investigated on several previous occasions (Refs. 1, 2, 3). Such systems are described by both ordinary and partial differential equations and are referred to as hybrid. The formulation presented in Refs. 1, 2, 3 is perfectly valid for the GEOS satellite. Hence, we shall dispense with the details and only outline the method of approach.

It was shown in Refs. 1, 2, 3 that, under certain circumstances, the Liapunov direct method with the Hamiltonian as a Liapunov functional can be used to test the stability of hybrid dynamical systems. The main problem is how to treat continuous elastic members. The Liapunov direct method has been used widely in conjunction with discrete systems. To test the stability of an equilibrium point, the testing function must satisfy one of several stability or instability theorems. If it does, then the testing function is said to be a Liapunov function and appropriate stability, or instability, conclusions can be drawn. If it does not, the analysis is inconclusive. The stability analysis consists of testing the sign properties of the testing function. The problem in applying the Liapunov direct method to hybrid systems lies in the difficulty of testing the sign properties of the testing functional (as opposed to the testing function). The author of this report has developed and used three different approaches to treat the problem of hybrid systems, namely, (1) the method of testing density functions, (2) the method of integral coordinates, and (3) the assumed modes method. The method of testing density functions works directly with the hybrid dynamical system but is quite often unduly restrictive. On the other hand, the remaining two methods are based on discretization schemes, which implies that the testing functional is replaced by a testing function. In particular, the method of integral coordinates involves the definition of new generalized coordinates representing certain integrals appearing in the testing functional, as well as the use of Schwarz's inequality for functions, to eliminate the spatial dependence from the testing functional. The difficulty in using this method is that the definition of integral coordinates is not always possible. Moreover, the method generally yields conservative results. The assumed modes method discretizes the system by representing the continuous displacements by finite series of space-dependent admissible functions multiplied by time-dependent generalized coordinates. Integration over the elastic domains eliminates the spatial dependence, so that the testing functional reduces also in this case to a testing function. The main criticisms of the method are the truncation effect, which generally leads to more conservative stability criteria, and the amount of labor involved in deriving the criteria.

Another aspect of the stability analysis is the definition of equilibrium. In certain cases, the equilibrium is one in which not all the coordinates are zero (see Ref. 4). In such cases, the equilibrium is referred to as nontrivial, and it is necessary first to solve for the nontrivial equilibrium and then to expand the testing function about this equilibrium. Note that a typical example of nontrivial equilibrium for flexible spacecraft is that in which the flexible parts are deformed under centrifugal force.

This report presents a stability investigation of the GEOS satellite by the assumed modes method. In considering the stability of small motions about nontrivial equilibrium, it is shown later that if the analysis performed by ignoring the motion of the mass center indicates stability, then the system remains stable if the motion of the mass center is included.

Derivation of the Testing Functional

Let us define a set of body axes xyz as the principal axes of the body in nominal undeformed state. We shall refer to these axes as a global system. In addition, let us define sets of axes $x_i y_i z_i$ $(i = 1,2,\ldots,8)$ such that x_i is directed along the length of the elastic members in undeformed state and y_i and z_i are perpendicular to x_i. The set of axes $x_i y_i z_i$ will be referred to as a local system. The motion of the spacecraft can be described by the rotational coordinates $\theta_j(t)$ $(j = 1,2,3)$ of the global system xyz and by the elastic displacements $v_i(x_i,t)$ and $w_i(x_i,t)$ $(i = 1,2,\ldots,8)$ relative to the local system $x_i y_i z_i$. In general, the displacements of the elastic members cause the mass center of the spacecraft to move relative to its nominal position, where the latter is identified as the origin of xyz. It is shown in Ref. 3, however, that this shift in the position of the mass center can be ignored without affecting adversely the stability criteria. Moreover, assuming that the mass center of the spacecraft moves in a known orbit in space, the kinetic energy of rotation about the mass center can be written in the matrix form

$$T = \frac{1}{2}\{\omega\}^T[J]\{\omega\} + \{\omega\}^T\{K\} + \frac{1}{2}\sum_{i=1}^{8}\int_{m_i}\{\dot{u}_i'\}^T\{\dot{u}_i'\}dm_i \tag{1}$$

where $\{\omega\}$ is the column matrix of the angular velocity components and $[J]$ is the inertia matrix of the deformed body. Moreover, $\{K\}$ is the angular momentum matrix due to elastic velocities alone and $\{\dot{u}_i'\}$ is the matrix of the elastic velocities relative to $x_iy_iz_i$.

The potential energy is entirely due to elastic deformations and can be written in the form

$$V_{EL} = \frac{1}{2}\sum_{i=1}^{8}\int_0^{\ell_i} EI_i \left[\left(\frac{\partial^2 v_i}{\partial x_i^2}\right)^2 + \left(\frac{\partial^2 w_i}{\partial x_i^2}\right)^2\right] dx_i$$

$$+ \frac{1}{2}\sum_{i=1}^{8}\int_0^{\ell_i} P_{xi}\left[\left(\frac{\partial v_i}{\partial x_i}\right)^2 + \left(\frac{\partial w_i}{\partial x_i}\right)^2\right]dx_i \tag{2}$$

where EI_i and P_{xi} $(i = 1,2,\ldots,8)$ are bending stiffnesses and axial forces, respectively. The functions $v_i(x_i,t)$ and $w_i(x_i,t)$ are subject to given boundary conditions. Note that in the case of the members 3 and 4 the bending stiffness is zero (or nearly zero) and one of the boundary conditions at $x_i = \ell_i$ depends on the tip mass m_i.

Because this is a natural system, the Hamiltonian is simply

$$H = T + V_{EL} \tag{3}$$

But the spacecraft is torque-free, so that the angular momentum about the mass center must be conserved. It is shown in Ref. 1 that the conservation of the angular momentum can be expressed in the matrix form

$$[J]\{\omega\} + \{K\} = \{\beta\} \tag{4}$$

where $\{\beta\}$ is the matrix of the conserved angular momentum. Introducing

Eq. (4) into Eq. (3), in conjunction with Eq. (1), the Hamiltonian reduces to

$$H = T_2 + T_o + V_{EL} \tag{5}$$

where

$$T_2 = \frac{1}{2} \sum_{i=1}^{8} \int_{m_i} \{\dot{u}_i'\}^T \{\dot{u}_i'\} dm_i \tag{6}$$

is a quadratic function of the elastic velocities and

$$T_o = \frac{1}{2} \{\beta\}^T [J]^{-1} \{\beta\} \tag{7}$$

depends on the generalized coordinates alone. Moreover, only two of the angular coordinates θ_j are present in T_o. This can be easily explained by means of the following argument. Assuming that initially the direction of the angular momentum vector coincides with the inertial axis Z and that its magnitude is β, then after some perturbation the angular momentum vector can be written in the matrix form $\{\beta\} = \beta\{\ell_Z\}$, where $\{\ell_Z\}$ is the column matrix of the direction cosines between axis Z and axes xyz. These direction cosines can be expressed in terms of only two angular coordinates.

For the system to be stable in the neighborhood of the equilibrium, it is necessary that the Hamiltonian be positive definite (see Ref. 1). But T_2 is positive definite by definition, so that H is positive definite if the functional

$$\kappa = T_o + V_{EL} \tag{8}$$

is positive definite. The testing of κ for positive definiteness is hindered by the fact that κ is a functional and not a function, as it involves the continuous variables v_i and w_i in integral form. We shall circumvent this difficulty by using the assumed modes method.

The term T_0 in Eq. (8) involves the motion x_c, y_c, z_c of the mass center. Generally, these terms complicate the stability analysis enormously without affecting materially the stability statement. Indeed, it is not difficult to show that if the system is judged as being stable in the sense of Liapunov on the basis of an analysis that ignores x_c, y_c, and z_c, then the same conclusion is valid for the actual motion. To this end, let us write

$$[J] = [J]_u - [J]_c \tag{9}$$

where $[J]_u$ is the inertia matrix obtained by ignoring x_c, y_c, and z_c and

$$[J]_c = \begin{bmatrix} y_c^2 + z_c^2 & -x_c y_c & -x_c z_c \\ -x_c y_c & x_c^2 + z_c^2 & -y_c z_c \\ -x_c z_c & -y_c z_c & x_c^2 + y_c^2 \end{bmatrix} \tag{10}$$

Whereas the matrices $[J]$ and $[J]_u$ are positive definite, the matrix $[J]_c$ is only positive. From Eq. (9), it follows that for any arbitrary vector $\{\alpha\}$ the quadratic forms associated with $[J]$ and $[J]_u$ satisfy the inequality

$$\{\alpha\}^T[J]\{\alpha\} \leq \{\alpha\}^T[J]_u\{\alpha\} \tag{11}$$

From Appendix B, however, we conclude that

$$\{\beta\}^T[J]^{-1}\{\beta\} \geq \{\beta\}^T[J]_u^{-1}\{\beta\} \tag{12}$$

Next, let us introduce the functional

$$\kappa_1 = \frac{1}{2}\{\beta\}^T[J]_u^{-1}\{\beta\} + V_{EL} \tag{13}$$

By virtue of inequality (12), we conclude that

$$\kappa \geq \kappa_1 \tag{14}$$

so that, if κ_1 is positive definite the system is asymptotically stable.

The preceding statement is true irrespective of the magnitude of x_c, y_c and z_c, although when they are large the stability criteria derived by using κ_1 insteady of κ can be very restrictive. In most practical cases, however, x_c, y_c, and z_c are one order of magnitude smaller than the elastic displacements themselves, in which case appreciable simplification of the stability analysis is achieved by ignoring them, without sacrificing accuracy.

Calculation of Nontrivial (Deformed) Equilibrium

a. Problem formulation

The equilibrium state to be considered is that in which the spacecraft spins about the symmetry axis with constant angular velocity Ω, as shown in Fig. 1. In that state the radial booms remain undeformed, but the axial booms undergo bending deformations in two perpendicular directions as a result of the centrifugal forces. The distributed centrifugal forces are equal to the negative of the distributed mass multiplied by the centripetal accelerations. Hence, we wish to calculate first the centripetal accelerations. Considering boom i (i = 5,6,7,8), we can write the position vector of any point on the boom in the form $\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i$, where $\underset{\sim}{h}_i$ is the vector from the satellite center to the point of attachment of the boom, $\underset{\sim}{r}_i$ is the vector from the point of attachment to any arbitrary point on the boom, and $\underset{\sim}{u}_i$ is the corresponding displacement vector. Denoting by $\underset{\sim}{i}_i$, $\underset{\sim}{j}_i$, $\underset{\sim}{k}_i$ the unit vector along the local axes x_i, y_i, z_i, the position vectors are as follows

$$\underset{\sim}{h}_i+\underset{\sim}{r}_i+\underset{\sim}{u}_i = (h_{xi}+x_i-z_c\sin\alpha_i)\underset{\sim}{i}_i + (h_{yi}+v_i)\underset{\sim}{j}_i + (h_{zi}+w_i-z_c\cos\alpha_i)\underset{\sim}{k}_i \;, \quad i=5,6,7,8 \qquad (15)$$

Recognizing that $\underset{\sim}{\Omega} = \Omega\underset{\sim}{k}$, where $\underset{\sim}{k} = \underset{\sim}{i}_i\sin\alpha_i + \underset{\sim}{k}_i\cos\alpha_i$, the centripetal accelerations are

$$\underset{\sim}{a}_i = \underset{\sim}{\Omega}\times[\underset{\sim}{\Omega}\times(\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i)] = \Omega^2\{[(h_{zi} + w_i - z_c\cos\alpha_i)\sin\alpha_i$$
$$-(h_{xi} + x_i - z_c\sin\alpha_i)\cos\alpha_i]\cos\alpha_i\underset{\sim}{i}_i - (h_{yi} + v_i)\underset{\sim}{j}_i$$
$$-[(h_{zi} + w_i - z_c\cos\alpha_i)\sin\alpha_i - (h_{xi} + x_i - z_c\sin\alpha_i)\cos\alpha_i]\sin\alpha_i\underset{\sim}{k}_i\} \qquad (16)$$

Hence, neglecting the relatively small quantities w_i, the centrifugal axial forces become

$$P_{xi} = -\int_{x_i}^{\ell_i} \rho_i\Omega^2[h_{zi}\sin\alpha_i - (h_{xi} + x_i)\cos\alpha_i]\cos\alpha_i dx_i$$

$$= \rho_i\Omega^2\{\tfrac{1}{2}[(h_{xi}+\ell_i)^2-(h_{xi}+x_i)^2]\cos\alpha_i-h_{zi}(\ell_i-x_i)\sin\alpha_i\}\cos\alpha_i$$

$$i = 5,6,7,8 \qquad (17)$$

where ρ_i is the constant mass density. On the other hand, the transverse distributed forces are

$$p_{yi} = \rho_i\Omega^2(h_{yi} + v_i)$$

$$p_{zi} = \rho_i\Omega^2[(h_{zi}+w_i)\sin\alpha_i - (h_{xi}+x_i)\cos\alpha_i]\sin\alpha_i \quad i = 5,6,7,8 \qquad (18)$$

The differential equations and the boundary conditions for the equilibrium deformations $v_{io}(x_i)$ and $w_{io}(x_i)$ are

$$EI\,\frac{d^4v_{io}}{dx_i^4} - \frac{d}{dx_i}\left(P_{xi}\,\frac{dv_{io}}{dx_i}\right) = p_{yi} \;, \quad i = 5,6,7,8 \qquad (19a)$$

$$v_{io}(0) = v'_{io}(0) = 0 \;, \quad v''_{io}(\ell_i) = v'''_{io}(\ell_i) = 0, \quad i = 5,6,7,8 \qquad (19b)$$

$$EI\,\frac{d^4w_{io}}{dx_i^4} - \frac{d}{dx_i}\left(P_{xi}\,\frac{dw_{io}}{dx_i}\right) = p_{zi} \;, \quad i = 5,6,7,8 \qquad (20a)$$

$$w_{io}(0) = w'_{io}(0) = 0, \quad w''_{io}(\ell_i) = w'''_{io}(\ell_i) = 0, \quad i=5,6,7,8 \qquad (20b)$$

Let the solution of Eqs. (19) have the form

$$v_{io}(x_i) = \sum_{j=1}^{p} a_{ij}\phi_{ij}(x_i) \qquad i = 5,6,7,8 \tag{21}$$

where $\phi_{ij}(x_i)$ are the modes of the fixed-base cantilever beam, in which the first index denotes the beam number and the second the mode number. The explicit expression of $\phi_{ij}(x_i)$ is (see Ref. 6, Sec. 5-10)

$$\begin{aligned}\phi_{ij}(x_i) = A_{ij}[&(\cos \beta_{ij}\ell_i + \cosh \beta_{ij}\ell_i)(\sin \beta_{ij}x_i - \sinh \beta_{ij}x_i) \\ &- (\sin \beta_{ij}\ell_i + \sinh \beta_{ij}\ell_i)(\cos \beta_{ij}x_i - \cosh \beta_{ij}x_i)]\end{aligned} \tag{22}$$

where the amplitudes A_{ij} are such that the functions $\phi_{ij}(x_i)$ are orthonormal, i.e., they satisfy relations

$$\int_0^{\ell_i} \rho_i \, \phi_{ij}(x_i) \, \phi_{ik}(x_i)dx_i = \delta_{jk} \tag{23}$$

where δ_{jk} is the Kronecker delta. Inserting Eqs. (21) and (22) into Eqs. (19), we conclude that the coefficients a_{ij} must satisfy the algebraic equations

$$\sum_{j=1}^{p} a_{ij}(\omega_{ij}^2 \, \delta_{jk} + \int_0^{\ell_i} P_{xi} \frac{d\phi_{ij}}{dx_i}\frac{d\phi_{ik}}{dx_i} \, dx_i) = \int_0^{\ell_i} p_{yi}\phi_{ik}dx_i$$

$$i = 5,6,7,8; \; k = 1,2,\ldots,p \tag{24}$$

Similarly, letting the solution of Eqs. (20) be

$$w_{io}(x_i) = \sum_{j=1}^{p} b_{ij} \, \phi_{ij}(x_i), \qquad i = 5,6,7,8 \tag{25}$$

we arrive at the algebraic equations

$$\sum_{j=1}^{p} b_{ij}(\omega_{ij}^2 \, \delta_{jk} + \int_0^{\ell_i} P_{xi} \frac{d\phi_{ij}}{dx_i}\frac{d\phi_{ik}}{dx_i} \, dx_i) = \int_0^{\ell_i} p_{zi}\phi_{ik}dx_i$$

$$i = 5,6,7,8; \; k = 1,2,\ldots,p \tag{26}$$

to be satisfied by the coefficients b_{ij}.

Problem Discretization by the Assumed-Modes Method

Next let us transform the functional κ_1 into a function and, to this end, let us derive an explicit expression for T_o. First, we recognize that the inertia matrix $[J]_u$ can be written in the general form

$$[J]_u = [J]_r + \sum_{i=1}^{n} [\ell_i]^T[J_i][\ell_i] \tag{27}$$

where $[J]_r$ is the inertia matrix of the rigid hub and $[J_i]$ is the inertia matrix of the member i in terms of local coordinates. Its elements are

$$\begin{aligned}
J_{i11} &= \int \rho_i[(h_{yi} + v_{io} + v_{i1})^2 + (h_{zi} + w_{io} + w_{i1})^2]dx_i \\
J_{i22} &= \int \rho_i[(h_{xi} + x_i)^2 + (h_{zi} + w_{io} + w_{i1})^2]dx_i \\
J_{i33} &= \int \rho_i[(h_{xi} + x_i)^2 + (h_{yi} + v_{io} + v_{i1})^2]dx_i \\
J_{i12} &= J_{i21} = -\int \rho_i(h_{xi} + x_i)(h_{yi} + v_{io} + v_{i1})dx_i \\
J_{i13} &= J_{i31} = -\int \rho_i(h_{xi} + x_i)(h_{zi} + w_{io} + w_{i1})dx_i \\
J_{i23} &= J_{i32} = -\int \rho_i(h_{yi} + v_{io} + v_{i1})(h_{zi} + w_{io} + w_{i1})dx_i
\end{aligned} \tag{28}$$

where v_{io} and w_{io} are the equilibrium elastic displacements and v_{i1} and w_{i1} are small perturbations. Moreover $[\ell_i]$ is the matrix of direction cosines between the local coordinates $x_iy_iz_i$ and the global coordinates xyz. It will prove convenient to separate the various orders of magnitude in $[J]_u$. To this end, let us write

$$[J]_u = [J]_o + [J]_1 + [J]_2 \tag{29}$$

where

$$[J]_o = \sum_{i=0}^{n} [\ell_i]^T [J_i]_o [\ell_i] \tag{30a}$$

in which

$$[J]_r = [J_o]_o = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{30b}$$

is the inertia matrix of the rigid hub in which A, B, and C are the principal moments of inertia about x, y, and z, respectively. Note that the above statement implies that the global axes xyz are principal axes for the spacecraft. Moreover,

$$\begin{aligned}
(J_{i11})_o &= \int \rho_i \, [(h_{yi} + v_{io})^2 + (h_{zi} + w_{io})^2] dx_i \\
(J_{i22})_o &= \int \rho_i \, [(h_{xi} + x_i)^2 + (h_{zi} + w_{io})^2] dx_i \\
(J_{i33})_o &= \int \rho_i \, [(h_{xi} + x_i)^2 + (h_{yi} + v_{io})^2] dx_i \\
(J_{i12})_o &= (J_{i21})_o = - \int \rho_i (h_{xi} + x_i)(h_{yi} + v_{io}) dx_i \\
(J_{i13})_o &= (J_{i31})_o = - \int \rho_i (h_{xi} + x_i)(h_{zi} + w_{io}) dx_i \\
(J_{i23})_o &= (J_{i32})_o = - \int \rho_i (h_{yi} + v_{io})(h_{zi} + w_{io}) dx_i
\end{aligned} \tag{30c}$$

where v_{io} and w_{io} are given by Eqs. (21) and (25). Recalling Eqs. (23), we can write

$$\begin{aligned}
\int \rho_i v_{io}(x_i) dx_i &= \sum_{j=1}^{p} a_{ij} \int \rho_i \phi_{ij}(x_i) dx_i \\
\int \rho_i w_{io}(x_i) dx_i &= \sum_{j=1}^{p} b_{ij} \int \rho_i \phi_{ij}(x_i) dx_i \\
\int \rho_i v_{io}^2(x_i) dx_i &= \sum_{j=1}^{p} \sum_{k=1}^{p} \int \rho_i a_{ij} a_{ik} \phi_{ij}(x_i) \phi_{ik}(x_i) dx_i = \sum_{j=1}^{p} a_{ij}^2
\end{aligned} \tag{31}$$

$$\int \rho_i w_{io}^2(x_i)dx_i = \sum_{j=1}^{p} b_{ij}^2$$

$$\int \rho_i x_i v_{io}(x_i)dx_i = \sum_{j=1}^{p} a_{ij}\int \rho_i x_i \phi_{ij}(x_i)dx_i \qquad \text{(31 cont'd.)}$$

$$\int \rho_i x_i w_{io}(x_i)dx_i = \sum_{j=1}^{p} b_{ij}\int \rho_i x_i \phi_{ij}(x_i)dx_i$$

$$\int \rho_i v_{io}(x_i)w_{io}(x_i)dx_i = \sum_{j=1}^{p}\sum_{k=1}^{p}\int \rho_i a_{ij}b_{ik}\phi_{ij}\phi_{ik}dx_i = \sum_{j=1}^{p}\sum_{k=1}^{p} a_{ij}b_{ik}\delta_{ik} = \sum_{j} a_{ij}b_{ij}$$

Next, let us write

$$[J]_1 = \sum_{i=1}^{n} [\ell_i]^T[J_i]_1[\ell_i] \tag{32}$$

and introduce the generalized coordinates

$$\theta_j = q_j(t) \quad , \quad j = 1,2$$

$$\begin{aligned}
v_{11} &= \sum_{j=3}^{p+2} \phi_{1j}(x_1)q_j(t) & w_{11} &= \sum_{j=p+3}^{2p+2} \psi_{1j}(x_1)q_j(t) \\
v_{i1} &= \sum_{j=2(i-1)p+3}^{(2i-1)p+2} \phi_{ij}(x_i)q_j(t) & w_{i1} &= \sum_{j=(2i-1)p+3}^{2ip+2} \psi_{ij}(x_i)q_j(t) \\
v_{n1} &= \sum_{j=2(n-1)p+3}^{(2n-1)p+2} \phi_{nj}(x_n)q_j(t) & w_{n1} &= \sum_{j=(2n-1)p+3}^{2np+2} \psi_{nj}(x_n)q_j(t)
\end{aligned} \tag{33}$$

Then, we have

$$[J_i]_1 = \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)\begin{bmatrix} 2\int \rho_i(h_{yi}+v_{io})\phi_{ij}(x_i)dx_i & -\int \rho_i(h_{xi}+x_i)\phi_{ij}(x_i)dx_i & 0 \\ -\int \rho_i(h_{xi}+x_i)\phi_{ij}(x_i)dx_i & 0 & 2\int\rho_i(h_{yi}+w_{io})\phi_{ij}(x_i)dx_i \\ 0 & -\int\rho_i(h_{yi}+w_{io})\phi_{ij}(x_i)dx_i & 2\int\rho_i(h_{yi}+v_{io})\phi_{ij}(x_i)dx_i \end{bmatrix}$$

$$+ \sum_{j=(2i-1)p+3}^{2ip+2} q_j(t)\begin{bmatrix} 2\int\rho_i(h_{zi}+w_{io})\psi_{ij}dx_i & 0 & -\int\rho_i(h_{xi}+x_i)\psi_{ij}dx_i \\ 0 & 2\int\rho_i(h_{zi}+w_{io})\psi_{ij}(x_i)dx_i & -\int\rho_i(h_{yi}+v_{io})\psi_{ij}(x_i)dx_i \\ -\int\rho_i(h_{xi}+x_i)\psi_{ij}(x_i)dx_i & -\int\rho_i(h_{yi}+v_{io})\psi_{ij}(x_i)dx_i & 0 \end{bmatrix} \tag{34}$$

so that

$$[J]_1 = \sum_{i=1}^{n}[\ell_i]^T[J_i]_1[\ell_i] = \sum_{i=1}^{n}\left\{\sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)[\ell_i]^T[J_i]_{1j}^{v}[\ell_i] + \sum_{j=(2i-1)p+3}^{2ip+2} q_j(t)[\ell_i]^T[J_i]_{1j}^{w}[\ell_i]\right\} \tag{35}$$

where $[J_i]_{1j}^{v}$ and $[J_i]_{1j}^{w}$ are the corresponding matrices in Eq. (34). Similarly, we can write

$$[J]_2 = \sum_{i=1}^{n} [\ell_i]^T[J_i]_2[\ell_i] \tag{36}$$

where

$$[J_i]_2 = \sum_{j=2(i-1)p+3}^{(2i-1)p+2}\ \sum_{k=2(i-1)p+3}^{(2i-1)p+2} q_j(t)q_k(t)\begin{bmatrix} \int\rho_i\phi_{ij}(x_i)\phi_{ik}(x_i)dx_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \int\rho_i\phi_{ij}(x_i)\phi_{ik}(x_i)dx_i \end{bmatrix}$$
$$+ \sum_{j=(2i-1)p+3}^{2ip+2}\ \sum_{k=(2i-1)p+3}^{2ip+2} q_j(t)q_k(t)\begin{bmatrix} \int\rho_i\psi_{ij}(x_i)\psi_{ik}(x_i)dx_i & 0 & 0 \\ 0 & \int\rho_i\psi_{ij}(x_i)\psi_{ik}(x_i)dx_i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+ \sum_{j=2(i-1)p+3}^{(2i-1)p+2}\ \sum_{k=(2i-1)p+3}^{2ip+2} q_j(t)q_k(t)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \int\rho_i\phi_{ij}(x_i)\psi_{ik}(x_i)dx_i \\ 0 & \int\rho_i\phi_{ij}(x_i)\psi_{ik}(x_i)dx_i & 0 \end{bmatrix} \tag{37}$$

If we choose the functions $\phi_{ij}(x_i)$, $\phi_{ik}(x_i)$ and $\psi_{ij}(x_i)$, $\psi_{ik}(x_i)$ such that

$$\begin{aligned} \int \rho_i\phi_{ij}(x_i)\phi_{ik}(x_i)dx_i &= \delta_{jk} \\ \int \rho_i\psi_{ij}(x_i)\phi_{ik}(x_i)dx_i &= \delta_{jk} \\ \int \rho_i\psi_{ij}(x_i)\psi_{ik}(x_i)dx_i &= \delta_{jk} \end{aligned} \tag{38}$$

then

$$[J_i]_2 = \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j^2(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sum_{j=(2i-1)p+3}^{2ip+2} q_j^2(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t) q_{j+p}(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{39}$$

so that

$$[J]_2 = \sum_{i=1}^{n} [\ell_i]^T [J_i]_2 [\ell_i] = \sum_{i=1}^{n} \left\{ \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j^2(t)\ [\ell_i]^T [J_i]_2^V [\ell_i] \right.$$

$$\left. + \sum_{j=(2i-1)p+3}^{2ip+2} q_j^2(t) [\ell_i]^T [J_i]_2^W [\ell_i] + \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t) q_{j+p}(t) [J_i]_2^{VW} \right\} \tag{40}$$

Note that $[J]_0$ can contain static elastic displacements caused by centrifugal forces resulting from steady spin, whereas $[J]_1$ and $[J]_2$ contain oscillations about the deformed equilibrium. To evaluate $[J]_1$ and $[J]_2$, we need the matrices $[\ell_i]$. From Fig. 1, we conclude that the matrices of the direction cosines are as follows:

$$[\ell_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [\ell_2] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\ell_3] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [\ell_4] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{41}$$

$$[\ell_5] = \begin{bmatrix} 0 & \cos\alpha_5 & \sin\alpha_5 \\ -1 & 0 & 0 \\ 0 & -\sin\alpha_5 & \cos\alpha_5 \end{bmatrix}, \qquad [\ell_6] = \begin{bmatrix} 0 & -\cos\alpha_6 & \sin\alpha_6 \\ 1 & 0 & 0 \\ 0 & \sin\alpha_6 & \cos\alpha_6 \end{bmatrix}$$

$$[\ell_7] = \begin{bmatrix} \cos\alpha_7 & 0 & \sin\alpha_7 \\ 0 & 1 & 0 \\ -\sin\alpha_7 & 0 & \cos\alpha_7 \end{bmatrix}, \qquad [\ell_8] = \begin{bmatrix} -\cos\alpha_8 & 0 & \sin\alpha_8 \\ 0 & -1 & 0 \\ \sin\alpha_8 & 0 & \cos\alpha_8 \end{bmatrix}$$

and note that $\alpha_5 = \alpha_6$ and $\alpha_7 = \alpha_8$.

Assuming that $[J]_1$ and $[J]_2$ are small compared to $[J]_0$, we can write the following approximation for the inverse of $[J]^{-1}$:

$$[J]^{-1} = [K] \cong [K]_0 + [K]_1 + [K]_2 \tag{42}$$

where the subscripts 0, 1 and 2 once again identify the order of magnitude of the quantities involved, in which

$$\begin{aligned} [K]_0 &= [J]_0^{-1} \\ [K]_1 &= -[J]_0^{-1}[J]_1[J]_0^{-1} \\ [K]_2 &= -[J]_0^{-1}[J]_2[J]_0^{-1} + [J]_0^{-1}[J]_1[J]_0^{-1}[J]_1[J]_0^{-1} \end{aligned} \tag{43}$$

From Eqs. (35) and (43), we conclude that

$$\begin{aligned} [K]_1 = - [K]_0[J]_1[K]_0 = - \sum_{i=1}^{n} \Bigg\{ & \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)\ [K]_0[\ell_i]^T[J_i]_{1j}^{v}[\ell_i][K]_0 \\ & + \sum_{j=(2i-1)p+3}^{2ip+2} q_j(t)[K]_0[\ell_i]^T[J_i]_{1j}^{w}[\ell_i][K]_0 \Bigg\} \end{aligned} \tag{44}$$

Next, let

$$\begin{aligned} [A_i]_{1j} &= [K]_0[\ell_i]^T[J_i]_{1j}^{v}[\ell_i] \\ [B_i]_{1j} &= [K]_0[\ell_i]^T[J_i]_{1j}^{w}[\ell_i] \end{aligned} \tag{45}$$

Then

$$[K]_1 = - \sum_{i=1}^{n} \Bigg\{ \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)\ [A_i]_{1j}[K]_0 + \sum_{j=(2i-1)p+3}^{2ip+2} q_j(t)[B_i]_{1j}[K]_0 \Bigg\} \tag{46}$$

Moreover, using Eqs. (40) and (43), we can write

$$\begin{aligned} [K]_2 &= - [K]_0[J]_2[K]_0 + [K]_0[J]_1[K]_0[J]_1[K]_0 \\ &= - \sum_{i=1}^{n} \Bigg\{ \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j^2(t)[K]_0[\ell_i]^T[J_i]_2^{v}[\ell_i][K]_0 \end{aligned}$$

$$+\sum_{j=(2i-1)p+3}^{2ip+2} q_j^2(t)[K]_o[\ell_i]^T[J_i]_2^W[\ell_i][K]_o$$

$$\left.+\sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)q_{j+p}(t)[K]_o[\ell_i]^T[J_i]_2^{VW}[\ell_i][K]_o\right\}$$

$$+\left\{\sum_{i=1}^{n}\left(\sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)[A_i]_{1j}+\sum_{j=(2i-1)p+3}^{2ip+2} q_j(t)[B_i]_{1j}\right)\right\}^2 [K]_o \qquad (47)$$

so that, letting

$$[A_i]_2 = [K]_o[\ell_i]^T[J_i]_2^V[\ell_i]$$

$$[B_i]_2 = [K]_o[\ell_i]^T[J_i]_2^W[\ell_i] \qquad (48)$$

$$[C_i]_2 = [K]_o[\ell_i]^T[J_i]_2^{VW}[\ell_i]$$

we obtain

$$[K]_2 = -\sum_{i=1}^{n}\left\{\sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j^2(t)\ [A_i]_2[K]_o+\sum_{j=(2i-1)p+3}^{2ip+2} q_j^2(t)\ [B_i]_2[K]_o\right.$$

$$\left.+\sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)q_{j+p}(t)\ [C_i]_2[K]_o\right\}$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\sum_{\ell=2(i-1)p+3}^{(2i-1)p+2}\ \sum_{m=2(j-1)p+3}^{(2j-1)p+2} q_\ell(t)q_m(t)[A_i]_{1\ell}[A_j]_{1m}[K]_o\right.$$

$$+\sum_{\ell=(2i-1)p+3}^{2ip+2}\ \sum_{m=(2j-1)p+3}^{2jp+2} q_\ell(t)q_m(t)\ [B_i]_{1\ell}[B_j]_{1m}[K]_o$$

$$+\sum_{\ell=2(i-1)p+3}^{(2i-1)p+2}\ \sum_{m=(2j-1)p+3}^{2jp+2} q_\ell(t)q_m(t)\ [A_i]_{1\ell}[B_j]_{1m}[K]_o$$

$$\left.+\sum_{\ell=(2i-1)p+3}^{2ip+2}\ \sum_{m=2(j-1)p+3}^{(2j-1)p+2} q_\ell(t)q_m(t)\ [B_i]_{1\ell}[A_j]_{1m}[K]_o\right\} \qquad (49)$$

Introducing the notation

$$[R_i]_{1j} = [A_i]_{1j}[K]_o \quad , \qquad [S_i]_{1j} = [B_i]_{1j}[K]_o$$

$$[R_i]_2 = [A_i]_2[K]_o \quad , \quad [S_i]_2 = [B_i]_2[K]_o \quad , \quad [T_i]_2 = [C_i]_2[K]_o \qquad (50)$$

Eqs. (46) and (49) can be rewritten in the form

$$[K]_1 = - \sum_{i=1}^{n} \left\{ \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)\,[R_i]_{1j} + \sum_{j=(2i-1)p+3}^{2ip+2} q_j(t)\,[S_i]_{1j} \right\} \quad (51)$$

and

$$[K]_2 = - \sum_{i=1}^{n} \left\{ \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j^2(t)\,[R_i]_2 + \sum_{j=(2i-1)p+3}^{2ip+2} q_j^2(t)\,[S_i]_2 \right.$$

$$\left. + \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t) q_{j+p}(t)\,[T_i]_2 \right\} + \text{small terms} \quad (52)$$

Next, we wish to write the expression for κ_1. Assuming that the orientation of the global system xyz is obtained from the inertial space XYZ by the rotations θ_3 about z, θ_1 about x, and θ_2 about y, then the direction cosines between axes xyz and Z are as follows: $\ell_{xZ} = -\cos\theta_1 \sin\theta_2$, $\ell_{yZ} = \sin\theta_1$, $\ell_{zZ} = \cos\theta_1 \cos\theta_2$. For small angles θ_1 and θ_2, the column matrix $\{\ell_Z\}$ can be approximated by

$$\{\ell_Z\} \cong \{\ell_Z\}_o + \{\ell_Z\}_1 + \{\ell_Z\}_2 \quad (53a)$$

where, recalling that $\theta_1 = q_1$ and $\theta_2 = q_2$, we have

$$\{\ell_Z\}_o = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \quad \{\ell_Z\}_1 = \begin{Bmatrix} -q_2 \\ q_1 \\ 0 \end{Bmatrix}, \quad \{\ell_Z\}_2 = \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{2}(q_1^2+q_2^2) \end{Bmatrix} \quad (53b)$$

Recognizing that $\beta = C\Omega$, the functional κ_1 can be approximated by

$$\kappa_{1E} \cong \frac{1}{2}C^2\Omega^2(\{\ell_Z\}_1^T[K]_o\{\ell_Z\}_1 + 2\{\ell_Z\}_2^T[K]_o\{\ell_Z\}_o$$

$$+ 2\{\ell_Z\}_1^T[K]_1\{\ell_Z\}_o + \{\ell_Z\}_o^T[K]_2\{\ell_Z\}_o) + V_{EL} \quad (54)$$

in which

$$\{\ell_z\}_1[K]_0\{\ell_z\}_1 = (K_{11})_0 q_2^2 + (K_{22})_0 q_1^2 + 2(K_{12})_0 q_1 q_2$$

$$2\{\ell_z\}_2^T[K]_0\{\ell_z\}_0 = -(K_{33})_0(q_1^2 + q_2^2)$$

$$2\{\ell_z\}_1^T[K]_1\{\ell_z\}_0 = -2q_2(K_{13})_1 + 2q_1(K_{23})_1$$

$$= -2 \sum_{i=1}^{n}\left\{ \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)\,[-(R_{i13})_{1j}q_2 + (R_{i23})_{1j}q_1] \right. \quad (55)$$
$$\left. + \sum_{j=(2i-1)p+3}^{2ip+2} q_j(t)\,[-(S_{i13})_{1j}q_2 + (S_{i23})_{1j}q_1] \right\}$$

$$\{\ell_z\}_0^T[K]_2\{\ell_z\}_0 = -\sum_{i=1}^{n}\left\{ \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j^2(t)(R_{i33})_2 \right.$$
$$\left. + \sum_{j=(2i-1)p+3}^{2ip+2} q_j^2(t)(S_{i33})_2 + \sum_{j=2(i-1)p+3}^{(2i-1)p+2} q_j(t)q_{j+p}(t)(T_{i33})_2 \right\}$$

But, by virtue of the fact that the functions ϕ_{ij} and ψ_{ij} satisfy corresponding eigenvalue problems, the elastic potential energy satisfies the inequality

$$V_{EL} \geq \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j=2(i-1)p+3}^{(2i-1)p+2} \Lambda_{ij}^2 q_j^2 + \sum_{k=(2i-1)p+3}^{2ip+2} \Lambda_{ik}^2 q_k^2 \right) \quad (56)$$

where Λ_{ij} and Λ_{ik} are the natural frequencies associated with the modes ϕ_{ij} and ψ_{ik}. Replacing V_{EL} in Eq. (54) by the expression on the right side of inequality (56), the system can be regarded as asymptotically stable if

$$\kappa_{2E} = \frac{1}{2}C^2\Omega^2\{q\}^T[H]\{q\} \quad (57)$$

is positive definite where [H] is the Hessian matrix given by

$$[H] = \begin{bmatrix} (K_{22})_0-(K_{33})_0 & (K_{12})_0 & \cdots -(R_{i23})_{1j} & \cdots (S_{i23})_{1k} & \cdots -(R_{n23})_{1\ell} & \cdots -(S_{n23})_{1m} \\ & (K_{11})_0-(K_{33})_0 & \cdots (R_{i13})_{1j} & \cdots (S_{i13})_{1k} & \cdots (R_{n13})_{1\ell} & \cdots (S_{n13})_{1m} \\ & & (\Lambda_{ij}^*)^2 & \cdots 0 & \cdots 0 & \cdots 0 \\ & \text{symmetric} & & (\Lambda_{ik}^*)^2 & \cdots 0 & \cdots 0 \\ & & & & (\Lambda_{n\ell}^*)^2 & \cdots 0 \\ & & & & & (\Lambda^*_{nm})^2 \end{bmatrix} \quad (58)$$

where

$$(\Lambda^*_{ij})^2 = \frac{\Lambda^2_{ij}}{C^2\Omega^2} - (R_{i33})_2$$
$$(\Lambda^*_{ik})^2 = \frac{\Lambda^2_{ik}}{C^2\Omega^2} - (S_{i33})_2 \qquad \begin{cases} j = 2(i-1)p+3, \ \ldots, \ (2i-1)p+2 \\ k = (2i-1)p+3, \ \ldots, \ 2ip+2 \end{cases} \tag{59a}$$

$$(\Lambda^*_{n\ell})^2 = \frac{\Lambda^2_{n\ell}}{C^2\Omega^2} - (R_{n33})_2$$
$$(\Lambda^*_{nm})^2 = \frac{\Lambda^2_{nm}}{C^2\Omega^2} - (S_{n33})_2 \qquad \begin{cases} \ell = 2(n-1)p+3, \ \ldots, \ (2n-1)p+2 \\ m = (2n-1)p+3, \ \ldots, \ 2in+2 \end{cases} \tag{59b}$$

The function κ_{2E} is positive definite if the matrix [H], which in turn requires that all the eigenvalues of [H] be positive. A computer program has been written for the calculation of [H] and for the evaluation of its eigenvalues. The program is described in the next section.

Description of the Computer Program

The computer program follows in detail the equations derived for Hessian matrix. Some explanations of all its subroutines are given below:

1. The standard Gauss-Jordan is used for the inversion of matrices. The corresponding subroutine is called MINV.
2. Subroutine HSBG reduces an n by n real matrix A to an upper almost triangular form by a similarity transformation. Each row is reduced in turn, starting from the last one, by applying right elimination matrix, and similarity is achieved by also applying the left inverse transformation. Thus the eigenvalues of A are preserved. Similarity transformations are using elementary elimination matrices with partial pivoting.
3. Subroutine ATEIG computes the eigenvalues of a real upper almost triangular matrix (Hessenberg form) using the double QR iteration of Francis

(Ref. 7). If all the eigenvalues of the matrix are positive, then the matrix is positive definite.

4. Subroutine GMPRD is used to multiply two general matrices to form a resultant general matrix.

5. Subroutine GTPRD is used to premultiply a general matrix by the transpose of another general matrix. The transpose of A is not actually calculated. Instead, elements of matrix A are taken columnwise rather than rowwise for postmultiplication by matrix B.

6. DRTMI determines a root of the general nonlinear equation $f(x) = 0$ in the range of x from $x_{\ell i}$ up to x_{ri} ($x_{\ell i}$, x_{ri} given by input) by means of Mueller's iteration scheme of successive bisection and inverse parabolic interpolation. The procedure assumes $f(x_{\ell i}) \cdot f(x_{ri}) \leq 0$. Convergence is quadratic if the derivative of $f(x)$ at root is not equal to zero.

All the subroutines described above could be found in the System/360 Scientific Subroutine Package.

The function SIMPS is used to evaluate numerically all the integrals by n repeated applications of Simpson's rule, where n is given by the NASR variable in the program. Because all the chosen admissible functions involve only well-behaved curves, use of Simpson's rule for all the integrations is justified.

Function FCT contains the equation $\cos \beta x \cosh \beta x + 1$ which is the characteristic equation of the nonrotating cantilever beam.

Function THI is used to calculate the value of an admissible function for either beam or cable with given amplitude, frequency and arguement.

Function DTHI is the derivative of function THI and function TTHI is the integral of function THI.

Total moment of inertia [J] and the moment of inertia $[J_i]_1$ and $[J_i]_2$

for each beam and each cable are calculated numerically by subroutine NFREQ. One identification variable (ID) indicates that the input data belongs to either a cable or a beam. A corresponding procedure is used to determine the coefficients of the admissible functions associated with the nontrivial equilibrium.

Function DIJ and its entry functions provide some products of the independent variable x, the admissible functions and their derivatives, which are all involved in the centrifugal terms appearing in the differential equation of nontrivial equilibrium.

Finally, all the elements of Hessian matrix are obtained numerically in the main program and the property of the matrix is tested by solving all the eigenvalues of the matrix using the QR iteration method.

Numerical Results

The preceding computer program has been used to test the stability of the GEOS satellite (the "simple model"). The numerical data (per letter of Dr. Peter Kulla dated 9 May 1974) is as follows:

$\ell_1 = \ell_2 = 2.66\text{m}$, $h_{x1} = h_{x2} = 0.73\text{m}$, $h_{y1} = h_{y2} = 0$, $h_{z1} = h_{z2} = -0.5\text{m}$

$\alpha_1 = \alpha_2 = 0$, $\rho_1 = \rho_2 = 1.127\ \text{kgm}^{-1}$; $\omega_1 = \omega_2 = 2\ \text{Hz}$;

$\ell_3 = \ell_4 = 20\text{m}$, $h_{x3} = h_{x4} = 0.73\text{m}$, $h_{y3} = h_{y4} = 0$, $h_{z3} = h_{z4} = 0.15\text{m}$

$\alpha_3 = \alpha_4 = 0$, $\rho_3 = \rho_4 = 0.03\ \text{kgm}^{-1}$, $m_3 = m_4 = 0.1\text{kg}$;

$\ell_5 = \ell_6 = 3\text{m}$, $h_{x5} = h_{x6} = 0.8 \sin 27° + 0.5 \cos 27°\ \text{m}$, $h_{y5} = h_{y6} = -0.42\text{m}$

$h_{z5} = h_{z6} = 0.8 \cos 27° - 0.5 \sin 27°$, $\alpha_5 = \alpha_6 = 27°$, $\rho_5 = \rho_6 = 0.733\text{kgm}^{-1}$

$\omega_5 = \omega_6 = 3\text{Hz};$

$\ell_7 = \ell_8 = 1.5\text{m}$, $h_{x7} = h_{x8} = 0.8 \sin 45° + 0.5 \cos 45°\text{m}$, $h_{y7} = h_{y8} = -0.42\text{m}$

$h_{z7} = h_{z8} = 0.8 \cos 45° - 0.5 \sin 45°\text{m}$, $\alpha_7 = \alpha_8 = 45°$, $\rho_7 = \rho_8 = 1.132\text{kgm}^{-1}$

$\omega_7 = \omega_8 = 5\text{Hz};$

$$A = I_{xx} + 2\int_0^{\ell_3} \rho_3[h_{z3}^2 + (h_{x3} + x_3)^2]dx_3 + 2m_3[h_{z3}^2 + (h_{x3} + \ell_3)^2] = 354.13806\text{kgm}^2$$

$$B = I_{yy} = 125\text{kgm}^2$$

$$C = I_{zz} + 2\int_0^{\ell_3} \rho_3(h_{x3} + x_3)^2 dx_3 + 2m_3(h_{x3} + \ell_3)^2 = 397.10606\text{kgm}^2$$

$$\Omega = 1 \text{ rad s}^{-1}$$

First, the nontrivial equilibrium configuration was evaluating by using two terms in series (21) and (25). The results are as follows:

$a_{i1} = -0.13710 \times 10^{-2}\ \text{mkg}^{1/2}$
$a_{i2} = -0.19418 \times 10^{-4}\ \text{mkg}^{1/2}$ $\qquad v_{io}(\ell_i) = -0.18229 \times 10^{-2}$ m

$b_{i1} = -0.36107 \times 10^{-2}\ \text{mkg}^{1/2}$
$b_{i2} = -0.22324 \times 10^{-4}\ \text{mkg}^{1/2}$ $\qquad w_{io}(\ell_i) = -0.48396 \times 10^{-2}$m

$i = 5.6$

$a_{i1} = -0.43397 \times 10^{-3}\ \text{mkg}^{1/2}$
$a_{i2} = -0.61305 \times 10^{-5}\ \text{mkg}^{1/2}$ $\qquad v_{io}(\ell_i) = -0.65666 \times 10^{-3}$ m

$b_{i1} = -0.92787 \times 10^{-3}\ \text{mkg}^{1/2}$
$b_{i2} = -0.74565 \times 10^{-5}\ \text{mkg}^{1/2}$ $\qquad w_{io}(\ell_i) = -0.14127 \times 10^{-2}$ m

$i = 7.8$

Using the above results, and using two terms in the series (33), a 34 × 34 Hessian matrix was obtained. The matrix failed the test of positive definiteness, a fact that can be traced to the cables.

Conclusions

The stability problem associated with a spin-stabilized satellite similar in configuration to the GEOS satellite has been formulated and programmed for digital computation. The formulation is capable of accommodating satellites with a somewhat different configuration than the GEOS, in the sense that the number of elastic members and their orientation relative to the spacecraft is arbitrary.

For the given configuration, one eigenvalue was found to be negative, so that on the basis of the Liapunov direct method the spacecraft cannot be judged as being stable. By inference, the system can be regarded as being unstable. This lack of stability can be traced to the fact that the lowest natural frequency of in-plane vibration of the cables is close to zero (see Appendix A). This is based on the assumption that the cables do not possess bending stiffness. In view of the negative stability statement obtained, a study of the effect of small cable bending stiffness on the spacecraft stability appears warranted. However, even if an analysis including small cable bending stiffness indicates stability, the first natural frequency of oscillation of the spacecraft is likely to be very low, so that the cables can represent a potential problem area.

References

1. Meirovitch, L., "A Method for the Liapunov Stability Analysis of Force-Free Dynamical Systems", AIAA Journal, Vol. 9, No. 9, 1971, pp. 1695-1701.
2. Meirovitch, L., "Liapunov Stability Analysis of Hybrid Dynamical Systems with Multi-Elastic Domains", International Journal of Non-Linear Mechanics, Vol. 7, 1972, pp. 425-433.
3. Meirovitch, L. and Calico, R. A., "Stability of Motion of Force-Free Spinning Satellites with Flexible Appendages", Journal of Spacecraft and Rockets, Vol. 9, No. 4, 1972, pp. 237-245.
4. Meirovitch, L., "Liapunov Stability Analysis of Hybrid Dynamical Systems in the Neighborhood of Nontrivial Equilibrium", AIAA Journal, Vol. 12, No. 7, 1974, pp. 889-898.
5. Meirovitch, L., "Dynamic Analysis of the GEOS Satellite", Preliminary Report, NASA Research Grant NGR 47-004-126, July 1974.
6. Meirovitch, L., Analytical Methods in Vibrations, The Macmillan Co., New York, 1967, Secs. 5-10 and 10-4.
7. Francis, J. G. F., "The QR Transformation, Parts I and II", Computer Journal, Vol. 4, 1961, 1962, pp. 265-271, 332-345.

a. Rotating Beam

Let us consider a rotating beam in transverse vibration, as shown in Fig. 2. The eigenvalue problem is defined by the differential equation (see Ref. 6)

$$EI \frac{d^4\phi}{dx^4} - \frac{1}{2}\rho\Omega^2 \frac{d}{dx}\{[(h+\ell)^2 - (h+x)^2]\frac{d\phi}{dx}\} = \Lambda^2\rho\phi \ , \ 0 < x < \ell \qquad (A.1)$$

where (x) is subject to the boundary conditions

$$\phi = 0 \text{ and } \frac{d\phi}{dx} = 0 \text{ at } x = 0 \qquad (A.2)$$

$$\frac{d^2\phi}{dx^2} = 0 \text{ and } \frac{d^3\phi}{dx^3} = 0 \text{ at } x = \ell \qquad (A.3)$$

There is no closed-form solution of the eigenvalue problem (A.1) - (A.3). Hence, we wish to obtain an approximate solution. To this end, we use the Rayleigh-Ritz method and assume a solution in the form of the series

$$\phi(x) = \sum_{r=1}^{n} a_r u_r(x) \qquad (A.4)$$

where $u_r(x)$ are comparison functions, namely, functions satisfying all the boundary conditions of the problem but not the differential equation (otherwise they would be eigenfunctions). We choose as comparison functions for the rotating bar the eigenfunctions of the nonrotating cantilever beam, obtained by setting $\Omega = 0$ in Eq. (A.1). These functions are (see Ref. 6)

$$u_r(x) = A_r[(\sin\beta_r L - \sinh\beta_r L)(\sin\beta_r x - \sinh\beta_r x)$$

$$+ (\cos\beta_r L + \cosh\beta_r L)(\cos\beta_r x - \cosh\beta_r x)] \ , \ r = 1,2,\ldots,n \qquad (A.5)$$

where the coefficients A_r are arbitrary β_r are the eigenvalues of the problem; they satisfy the characteristic equation

$$\cos\beta_r L \cosh\beta_r L + 1 = 0 \tag{A.6}$$

The functions $u_r(x)$ are orthogonal. Moreover, it will prove convenient to remove the arbitrariness from $u_r(x)$ $(r = 1,2,\ldots,n)$ and determine the coefficients A_r uniquely by normalizing the functions $u_r(x)$ so as to satisfy

$$\int_0^\ell \rho u_r(x) u_s(x) dx = \delta_{rs} \quad , \qquad r,s = 1,2,\ldots,n \tag{A.7}$$

where δ_{rs} is the Kronecker delta.

It can be shown that the Rayleigh-Ritz method, in conjunction with the normalized comparison functions $u_r(x)$, lead to the special eigenvalue problem

$$[k]\{a\} = \Lambda^2\{a\} \tag{A.8}$$

where the matrix $[k]$ is real and symmetric; its elements have the values

$$k_{rs} = EI\int_0^\ell \frac{d^2u_r}{dx^2}\frac{d^2u_s}{dx^2} dx + \frac{1}{2}\rho\Omega^2\int_0^\ell [(h+\ell)^2 - (h+x)^2] \frac{du_r}{dx}\frac{du_s}{dx} dx$$

$$= \omega_r^2\delta_{rs} + \frac{1}{2}\rho\Omega^2\int_0^\ell [(h+\ell) - (h+x)^2] \frac{du_r}{dx}\frac{du_s}{dx} dx \quad , \quad r,s = 1,2,\ldots,n \tag{A.9}$$

in which ω_r are the natural frequencies of the nonrotating beam. The solution of the eigenvalue problem (A.8) and (A.9) consists of the eigenvalues Λ_i^2, which are the squares of the estimated natural frequencies of the rotating beam, and the eigenvectors $\{a^{(i)}\}$ $(i = 1,2,\ldots,n)$. It follows that the estimated eigenfunctions are

$$\phi_i(x) = \sum_{r=1}^{n} a_r^{(i)} u_r(x) \tag{A.10}$$

Eigenvalue problems of the type (A.8) and (A.9) must be solved for radial members such as 1 and 2. For members 5, 6, 7, and 8 the eigenvalue problem must be modified to account for the inclination of the bar and the resulting transverse loads.

b. Rotating cable with tip mass

The eigenvalue problem for a rotating cable is similar to that of the rotating beam shown in Fig. 2, except that the bending stiffness is equal to zero. In addition, we are interested in the case in which the cable has a tip mass m. The corresponding eigenvalue problem is defined by the differential equation

$$-\Omega^2 \frac{d}{dx}\left\langle\left\{\frac{1}{2}\rho[(h+\ell)^2 - (h+x)^2] + m(h+\ell)\right\}\frac{d\phi}{dx}\right\rangle = \Lambda^2\rho\phi\ , \ 0 < x < \rho \quad (A.11)$$

where $\phi(x)$ is subject to the boundary conditions

$$\phi = 0 \text{ at } x = 0 \quad (A.12)$$

$$-m(h+\ell)\Omega^2 \frac{d\phi}{dx} + m\Lambda^2\phi = 0 \quad \text{at } x = \ell \quad (A.13)$$

The eigenvalue problem (A.11) - (A.13) has no closed-form solution either. The eigenvalue problem of the rotating string with no tip mass and with $h = 0$, however, is satisfied by the Legendre functions. The Legendre functions of odd degree can be used as admissible functions for the eigenvalue problem (A.11) - (A.13) as they solve a similar problem and satisfy the boundary condition at $x = 0$. Note that admissible functions need satisfy only the geometric boundary conditions of the problem. Hence, let us assume an approximate solution in the form

$$\phi(x) = \sum_{r=1}^{n} a_r P_{2r-1}(x) \quad (A.14)$$

where

$$\begin{aligned} P_1(x) &= \frac{x}{\ell} \\ P_3(x) &= \frac{1}{2}[5(\frac{x}{\ell})^3 - 3\frac{x}{\ell}] \\ P_5(x) &= \frac{1}{8}[63(\frac{x}{\ell})^5 - 70(\frac{x}{\ell})^3 + 15\frac{x}{\ell}] \end{aligned} \quad (A.15)$$

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are known as Legendre polynomials. They possess the orthogonality property

$$\int_0^{\ell} P_j(x)P_k(x)dx = 0 \quad , \qquad j,k = 1,2,... \tag{A.16}$$

and they satisfy the relation

$$\int_0^{\ell} P_j^2(x)dx = \frac{\ell}{2j+1} \quad , \qquad j = 1,2,... \tag{A.17}$$

The Rayleigh-Ritz method leads to the eigenvalue problem

$$[k]\{a\} = \Lambda^2[m]\{a\} \tag{A.18}$$

where the matrices [k] and [m] are real and symmetric. Their elements are

$$k_{rs} = \Omega^2\int_0^{\ell} \{\tfrac{1}{2}\rho[(h+\ell)^2 - (h+x)^2] + m(h+\ell)\}P'_{2r-1}(x)P'_{2s-1}(x)dx \qquad r,s = 1,2,...,n \tag{A.19}$$

and

$$m_{rs} = \rho\int_0^{\ell} P_{2r-1}(x)P_{2s-1}(x)dx + mP_{2r-1}(\ell)P_{2s-1}(\ell)$$

$$= \frac{\rho\ell}{2(2r-1)+1}\,\delta_{rs} + mP_{2r-1}(\ell)P_{2s-1}(\ell) \tag{A.20}$$

The solution of the eigenvalue problem (A.18) - (A.20) consists of the eigenvalues Λ_i^2, which are the squares of the estimated natural frequencies of the rotating string with a tip mass, and the eigenvectors $\{a^{(i)}\}$ $(i = 1,2,...,n)$. It follows that the estimated eigenfunctions are

$$\phi_i(x) = \sum_{r=1}^{n} a_r^{(i)}P_{2r-1}(x) \tag{A.21}$$

The eigenvalue problem (A.18) - (A.20) must be solved for member 3, which yields automatically the solution also for member 4.

Appendix B - Theorems on Inequalities for Quadratic Forms

Theorem. Given two matrices A and B which are symmetric and positive definite over real number field R. If $\underset{\sim}{x}^T A \underset{\sim}{x} \geq \underset{\sim}{x}^T B \underset{\sim}{x}$ for any vector $\underset{\sim}{x}$ over R, then $\underset{\sim}{x}^T A^{-1} \underset{\sim}{x} \leq \underset{\sim}{x}^T B^{-1} \underset{\sim}{x}$.

Proof: Because A is symmetric, there is an orthonormal matrix U such that

$$A^{-1/2} = U\lambda^{-1/2}U^T \tag{B.1}$$

where λ is a diagonal matrix with its elements equal to the eigenvalue of the matrix A. The effect of the operation

$$C = A^{-1/2}BA^{-1/2} \tag{B.2}$$

is to transform the symmetric and positive definite matrix B into a matrix C which is also symmetric and positive definite, namely,

$$C^T = (A^{-1/2}BA^{-1/2})^T = A^{-1/2}B^TA^{-1/2} = A^{-1/2}BA^{-1/2} = C \tag{B.3}$$

Similarly, there exists an orthonormal matrix V such that

$$V^TCV = V^TA^{-1/2}BA^{-1/2}V = \mu \tag{B.4}$$

where μ is a diagonal matrix.

Introducing the linear transformation

$$\underset{\sim}{p} = V^TA^{1/2}\underset{\sim}{x} \tag{B.5}$$

into the inequality $\underset{\sim}{x}^T A \underset{\sim}{x} \geq \underset{\sim}{x}^T B \underset{\sim}{x}$, we obtain

$$\underset{\sim}{p}^T V^T A^{-1/2} A A^{-1/2} V \underset{\sim}{p} \geq \underset{\sim}{p}^T V^T A^{-1/2} B A^{-1/2} V \underset{\sim}{p} \tag{B.6}$$

which reduces to

$$\underset{\sim}{p}^T I \underset{\sim}{p} \geq \underset{\sim}{p}^T \underset{\sim}{\mu} \underset{\sim}{p} \tag{B.7}$$

where I is the identity matrix. Because A and B are positive definite, all the elements of the diagonal matrix μ are positive. It follows from inequality (8.7) that

$$\underset{\sim}{p}^T I^{-1} \underset{\sim}{p} \leq \underset{\sim}{p}^T \mu^{-1} \underset{\sim}{p} \tag{B.8}$$

Moreover, recalling inequalities (B.6) and (B.7), it follows that

$$\underset{\sim}{p}^T V^T A^{1/2} A^{-1} A^{1/2} V \underset{\sim}{p} \leq \underset{\sim}{p}^T V^T A^{1/2} B^{-1} A^{1/2} V \underset{\sim}{p} \tag{B.9}$$

Next, let

$$y = A^{1/2} V \underset{\sim}{p} = A^{1/2} V V^T A^{1/2} \underset{\sim}{x} = A \underset{\sim}{x} \tag{B.10}$$

so that inequality (B.9) reduces to

$$\underset{\sim}{y}^T A^{-1} \underset{\sim}{y} \leq \underset{\sim}{y}^T B^{-1} \underset{\sim}{y} \tag{B.11}$$

Because A is symmetric and positive definite, we can show that A can be regarded as a linear transformation mapping the linear space into itself. This concludes the proof that $\underset{\sim}{x}^T A^{-1} \underset{\sim}{x} \leq \underset{\sim}{x}^T B^{-1} \underset{\sim}{x}$.

Corollary. Given two matrices A and B which are symmetric and positive definite over real number field R. Then $\underset{\sim}{x}^T A \underset{\sim}{x} \geq \underset{\sim}{x}^T B \underset{\sim}{x}$ for any vector $\underset{\sim}{x}$ over R if and only if every eigenvalue μ_i $(i = 1,\ldots,n)$ of $A^{-1}B$ is such that $1 \geq \mu_i > 0$.

Consider the series

$$I + A^{-1}B + (A^{-1}B)^2 + \ldots + (A^{-1}B)^m + \ldots \tag{B.12}$$

For convergence it is clearly necessary that $\lim_{m \to \infty} (A^{-1}B)^m \to 0$. This condition is also sufficient, for if $\lim_{m \to \infty} (A^{-1}B)^m \to 0$ if follows that $|\mu_r| < 1$, and therefore $I - A^{-1}B$ does not vanish and $(I-A^{-1}B)^{-1}$ exists. But

$$[I + A^{-1}B + (A^{-1}B)^2 + \ldots + (A^{-1}B)^m](I - A^{-1}B) = I - (A^{-1}B)^{m+1} \tag{B.13}$$

so that postmultiplication of Eq. (B.13) by $(I - A^{-1}B)^{-1}$ yields

$$I + A^{-1}B + (A^{-1}B)^2 + \ldots + (A^{-1}B)^m = (I - A^{-1}B)^{-1} - (A^{-1}B)^{m+1}(I - A^{-1}B)^{-1} \quad (B.14)$$

As $m \to \infty$, Eq. (B.14) reduces to

$$I + A^{-1}B + (A^{-1}B)^2 + \ldots + (A^{-1}B)^m + \ldots = (I - A^{-1}B)^{-1} \quad (B.15)$$

But postmultiplication of Eq. (B.15) by A^{-1} gives

$$A^{-1} + A^{-1}BA^{-1} + (A^{-1}B)^2A^{-1} + \ldots + (A^{-1}B)^mA^{-1} + \ldots = (I - A^{-1}B)^{-1}A^{-1} = [A(I - A^{-1}B)]^{-1} \quad (B.16)$$

Hence,

$$A^{-1} + A^{-1}BA^{-1} + (A^{-1}B)^2A^{-1} + \ldots + (A^{-1}B)^mA^{-1} + \ldots = (A - B)^{-1} \quad (B.17)$$

so that if two symmetric and positive definite matrices A and B satisfy the inequality $\tilde{x}^T A \tilde{x} > \tilde{x}^T B \tilde{x}$ for any vector $\tilde{x}$ over R, then the series expansion (B.17) is valid and convergent.

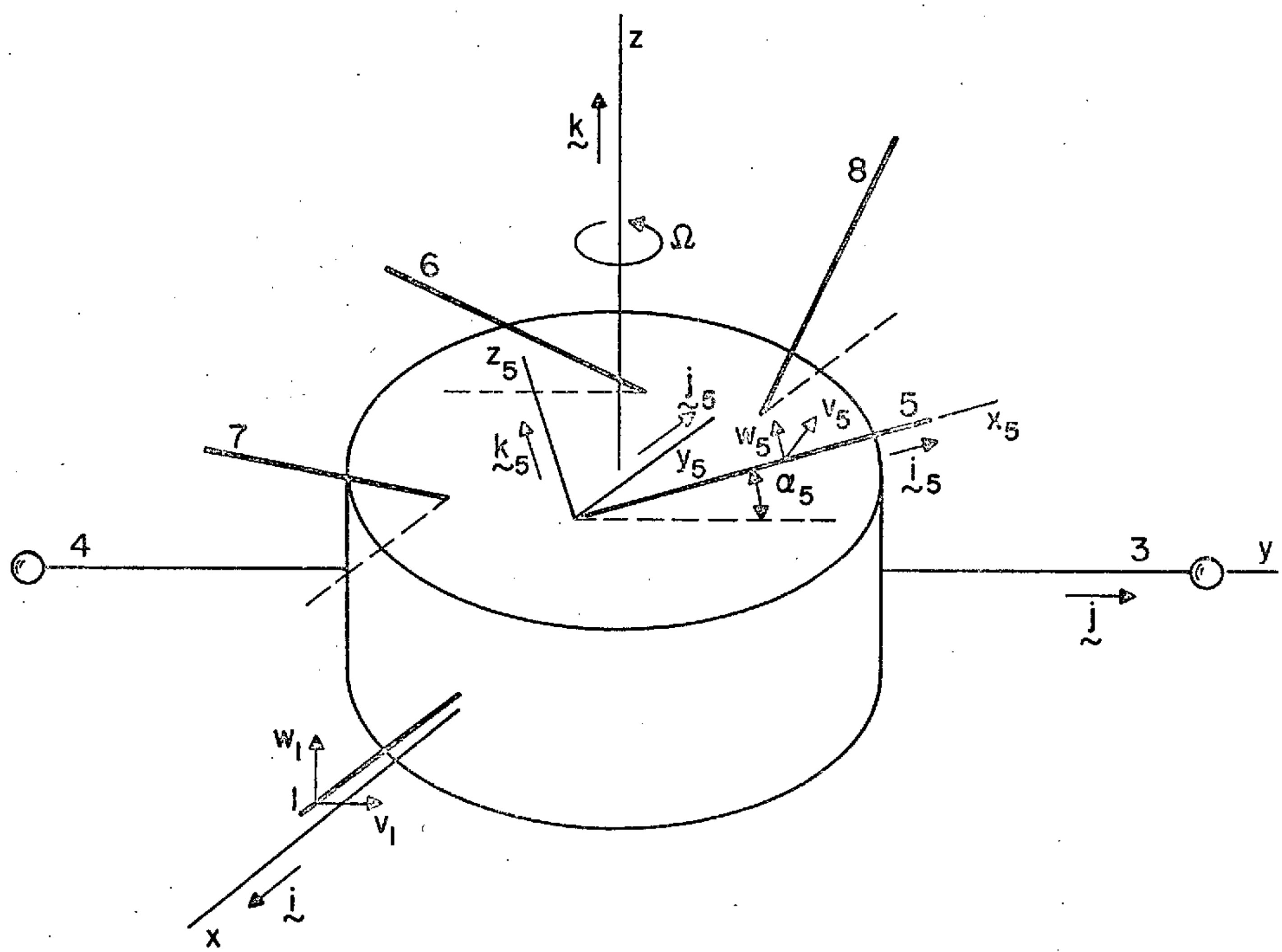

FIGURE I. THE GEOS SATELLITE

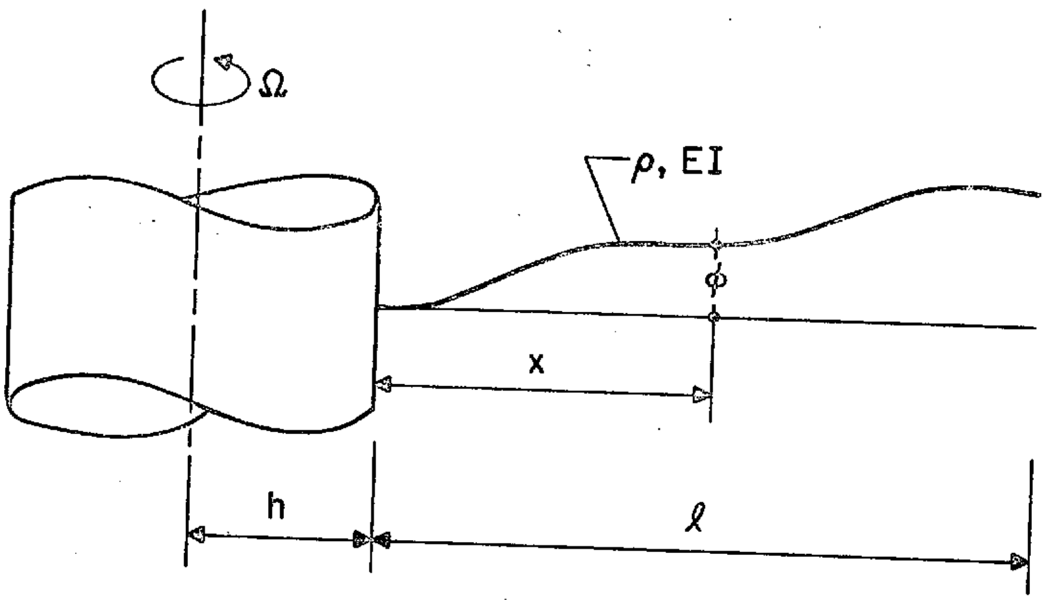

FIGURE 2. ROTATING BEAM