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ON THE EFFICIENT COMPUTATION OF RECURRENCE RELATIONS

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A new parallel algorithm for the solution of a general linear recurrence is described. Its relation to the work of Kogge and Stone is discussed.

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ON THE EFFICIENT COMPUTATION OF RECURRENCE RELATIONS

Recently much progress has been made in the formulation of parallel algorithms which compute the terms of a sequence (y_i) defined by

$$(1)\qquad \begin{aligned} &y_0 \text{ given,} \\ &y_i = f_i(y_0, y_1,\ldots,y_{i-1}),\ i = 1,\ldots,N. \end{aligned}$$

The germinal point of this work is the now well-known "log-sum" algorithm which computes $\Sigma_{i=1}^{N} a_i$ in $\lceil \log_2 N \rceil$ parallel addition steps, given $\lceil N/2 \rceil$ processors. Here the underlying recurrence is

$$\begin{aligned} &y_0 = o \\ &y_i = y_{i-1} + a_i,\ i = 1,\ldots,N; \end{aligned}$$

y_N is the desired result.

Two apparently distinct generalizations of the log-sum algorithm have appeared. Kogge and Stone [1] have considered the case

$$(2)\qquad \begin{aligned} &y_0 = b_0 \\ &y_i = f(b_i,\ g(a_i,\ y_{i-1})),\ i = 1,\ldots,N, \end{aligned}$$

where f is associative, g distributes over f, and there is a function h such that $g(x,g(y,z)) = g(h(x,y),z)$. Seemingly restricted to first order recurrences, by a suitable mapping m^{th} order recurrences are also treated. Heller [2] has studied the case

$$(3)\qquad \begin{aligned} &y_0 = h_0 \\ &y_i = \Sigma_{j=o}^{i-1}\ a_{ij}y_j + h_i,\ i = 1,\ldots,N. \end{aligned}$$

This problem is equivalent to the solution of a lower triangular linear system of equations. In this note we give an improved parallel algorithm for (3) and show a relationship between the two generalizations.

Rewrite (3) as $(I-L)y=h$, where L is a strictly lower triangular matrix, and I is the identity. y and h are $(N + 1)$-vectors. Since $L^{N+1} = 0$,

$$(I-L)^{-1} = (I+L+L^2+\cdots+L^N)$$
$$= (I+L^{2^m})(I+L^{2^{m-1}})\cdots(I+L)$$

where $2^m \leq N < 2^{m+1}$. Thus we have the algorithm:

$$\{x_0 \leftarrow h;\ LI \leftarrow I\};$$
$$\underline{\text{for}}\ i \leftarrow 0\ \underline{\text{step}}\ 1\ \underline{\text{until}}\ m-1\ \underline{\text{do}}$$
$$\{x_{i+1} \leftarrow (I+L^{2^i})\ x_i;$$
$$L^{2^{i+1}} \leftarrow L^{2^i}\ L^{2^i};$$
$$LI \leftarrow (I+L^{2^i})LI\};$$
$$\{y \leftarrow (I+L^{2^m})\ x_m\ ; LI \leftarrow (I+L^{2^m})LI\}.$$

The algorithm is sequential in i, and within the braces operations are performed concurrently. When completed, we have the desired y, and $(I-L)^{-1}$ is stored in LI. LI may now be used to compute y' given h'. It is easily shown that, with $O(N^3)$ processors, the calculation may be done in $m^2 + 3m + 1$ parallel steps of addition and multiplication. (We use the fact that matrix products may be computed in logarithmic time with sufficiently many processors.) The previous result required $O(N^4)$ processors and $m^2 + 4m + 2$ operation steps.

We now turn to the Kogge - Stone results. Rewrite (2) as

$$(2') \quad \begin{aligned} y_0 &= b_0 \\ y_i &= a_i \otimes y_{i-1} \oplus b_i,\ i=1,\ldots,N. \end{aligned}$$

Here g is replaced by the binary operation $\otimes$, and f by $\oplus$. Assume that $\oplus$ is associative, $\otimes$ distributes over $\oplus$, and there is a $\otimes'$ such that $a \otimes (b \otimes c) = (a \otimes' b) \otimes c$. Let α be a symbol distinct from all others, and define $\alpha \oplus x = x \oplus \alpha = x$, $\alpha \otimes x = x \otimes \alpha$ α for all x. Define an operator L on (N+1)-vectors by

$$(Lz)_0 = \alpha$$

$$(Lz)_i = a_i \otimes z_{i-1}, \; i = 1,\ldots,N.$$

Then $y = Ly \oplus b$ by (2'). It is observed that L is an additive operator since $\otimes$ distributes over $\oplus$ and by the definition of α. Moreover, $L^{N+1} = \alpha$, since for any z, and $i = 1,\ldots,N+1$, $(L^i z)_0 = \alpha$ and for $1 \leq j < i$, $(L^i z)_j = (L(L^{i-1} z))_j = a_j \otimes (L^{i-1} z)_{j-1} = a_j \otimes \alpha = \alpha$. Therefore,

$$
\begin{aligned}
y &= Ly \oplus b = L(Ly \oplus b) \oplus b = L^2 y \oplus (L \oplus I)b \\
&= \cdots = L^{N+1} y \oplus (L^N \oplus L^{N-1} \oplus \cdots \oplus I)b \\
&= (L^N \oplus L^{N-1} \oplus \cdots \oplus I)b \\
&= (L^{2^m} \oplus I)(L^{2^{m-1}} \oplus I)\cdots(L \oplus I)b.
\end{aligned}
$$

Since $L^3 = (L^2)L = L(L^2)$, $\otimes'$ behaves as an associative operation, and so

$$
\begin{aligned}
(L^{2^i} y)_j &= \alpha, \quad 0 \leq j < 2^i \\
&= a_j \otimes (a_{j-1} \otimes (\ldots \otimes (a_{j-2^i+1} \otimes y_{j-2^i})\cdots) \\
&= (a_j \otimes' a_{j-1} \otimes' \cdots \otimes' a_{j-2^i+1}) \otimes y_{j-2^i}, \quad 2^i \leq j \leq N.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(L^{2^{i+1}} y)_j &= \alpha, \quad 0 \leq j < 2^{i+1} \\
&= [(a_j \otimes' \cdots \otimes' a_{j-2^i+1}) \\
&\quad \otimes' (a_{j-2^i} \otimes' \cdots \otimes' a_{j-2^{i+1}+1})] \otimes y_{j-2^{i+1}}, \quad 2^{i+1} \leq j \leq N,
\end{aligned}
$$

and the "coefficients" of $L^{2^{i+1}}$ may be computed from the "coefficients" of L^{2^i} in one $\otimes'$ operation step. Thus an algorithm similar to the previous one may be used to compute y. If the operator $(L^N \cdots \oplus I)$ is not formed, the computation time is $O(\log_2 N)$ with $O(N)$ processors. In fact, if $y' = Ly' \oplus b'$, it is less efficient to directly apply $(L^N \oplus \cdots \oplus I)$ than to use the above method.

The general recurrence (1) may be written as $y = L_1 y$, where L_1 is a strictly lower triangular operator in the sense that, for any z, $(L_1 z)_i$ is independent of $z_i, z_{i+1}, \ldots, z_N$. By an induction argument L_1^{N+1} is a constant operator, and so the solution may be found by $y = L_1^{n+1} z$ for any z. The special cases (2) and (3) allow the simple computation of the powers of L_1 when $L_1 z = Lz \oplus b$, and L is linear. Kung [3] has shown that for non-linear recurrences, it is not possible, in general, to decrease the computation time by more than a constant factor by the use of parallelism.

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