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Solution of the Equation of Heat Conduction With Time-Dependent Sources: Programmed Application to Planetary Thermal History

James E. Conel

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16. Abstract

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Preface

The work described in this report was performed by the Space Sciences Division of the Jet Propulsion Laboratory.

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Foreword

A need to investigate solutions to the inhomogeneous equation of heat conduction with time-dependent sources and the so-called linearized radiation (or insulation) boundary condition arose directly from a study of the constraints on lunar thermal history posed by systematic analysis of returned lunar samples and geophysical data reported by Conel et al. in 1972 (see Ref. 1). A great deal of numerical modelling was carried out to support and guide us to the principal conclusions presented in that report; these specific model results will be published elsewhere. The present document describes only the mathematical problem involved and its numerical solution.

Studies of planetary thermal history have, with time, evolved mathematical models of ever-increasing complexity. Added complications have involved inclusion of radiative transfer with the (radiative) thermal conductivity as a prescribed function of temperature, inclusion of latent heat associated with phase changes, simulated convection in molten and solid zones, redistribution of heat sources upon solidification according to specified laws, etc. Most of these refinements render the mathematical problem nonlinear, and hence force numerical integration of the heat equation from the outset. Sophisticated models also require specification of an increasing number of material parameters that are often poorly determined. In my opinion, it is hard to justify an increase in model complexity when the fundamental data to be used are not well known. An increase in the degrees of freedom accompanying construction of elaborate model schemes also allows an investigator to achieve his desired result by more and more diverse means. To avoid some of these possible sources of difficulty, I have purposely chosen to deal with comparatively simple analytic solutions to the equation of heat conduction. My attitude is that until it is compellingly shown that elementary procedures and simple assumptions fail to explain the observations, it is not worthwhile abandoning such procedures and assumptions.

Ordinarily, there is no place for publication of the details of complex computer codes in the scientific literature. I consider this an unfortunate necessity, since it may mean that there is no way of checking the procedure or accuracy of a lengthy numerical exercise, and thus no basis for judging its validity. I have prepared the present report to compensate in part for any such deficiency in my own work. I hope in addition that the documentation and code may prove useful to others with their special problems.

Acknowledgment

I want to thank John B. Morton of the Jet Propulsion Laboratory for coding this program.

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Abstract

A computer program (Program SPHERE) solving the inhomogeneous equation of heat conduction with radiation boundary condition on a thermally homogeneous sphere is described. The source terms are taken to be exponential functions of the time. Thermal properties are independent of temperature. The solutions are appropriate to studying certain classes of planetary thermal history. Special application to the Moon is discussed.

Solution of the Equation of Heat Conduction with Time-Dependent Sources: Programmed Application to Planetary Thermal History

I. Introduction

This report is a concise documentation of a programmed solution to the inhomogeneous equation of heat conduction in spherical coordinates with time-dependent sources (Program SPHERE). The application here is in the study of planetary thermal history, and in particular the Moon (see Ref. 1). The sphere is homogeneous in density and thermal properties. The particular example given is specialized from a general case for radial distribution of heat sources and initial temperature given by Lowan (Ref. 2). The boundary condition at the outer surface is the so-called linearized radiation boundary condition or "Fourier's problem of the third kind"; in this particular instance the body is considered to radiate to a medium at constant temperature. This condition also applies to problems where a *thin* skin of insulating material exists on the exterior of the body (Ref. 3) such that the heat capacity of the skin can be neglected. This is tantamount to saying that if a change in temperature occurs on the inner insulating boundary, then the exterior medium responds "instantaneously" to establish a linear temperature gradient in the insulation itself. In this case the outer surface temperature of the insulation is held fixed, which corresponds mathematically to the case previously discussed where radiation is to a medium at constant temperature.

Two points should be emphasized in applying these results: (1) The *surface* temperature solved for is the temperature of the medium *just beneath* the insulating layer, and (2) Instead of dealing with the more complicated problem of varying *surface* temperature (i.e., the temperature at the outer physical surface), we have chosen the boundary temperature to be fixed, in most cases taking it to be the average value of whatever the expected sinusoidal variation might be on an atmosphereless body rotating with given angular velocity about its own axis, and around the Sun. The boundary temperature, of course, can be assigned arbitrarily. Thus, the details of the fourth power nonlinearity in the usual boundary conditions are not dealt with; at the same time, these details are completely unimportant in understanding thermal problems of deep planetary interiors. (They may, of course, become important in applications where near-surface regolith conditions are of interest; in this case numerical techniques would be advisable from the outset, although viable alternative procedures, still requiring computers for hard answers, are useful (see Ref. 4 for details)).

The reason for utilizing this general approach in analyzing problems involving insulation is that the full, more complicated problem of dealing with the thermal properties of the outer layer is avoided by a simple mathematical trick. At the same time, we have been forced to neglect any radioactively generated heat contribution in the insulating blanket itself. This could normally be dealt with in practice by an approximate separate calculation of equating sources to flow, since the insulating layer is usually thin relative to the planetary radius.

An additional feature has been added to the original Lowan calculations. In considering the physical problem of lunar thermal history, it became evident that the boundary condition might necessarily be a function of time. The specific problem treated is that of having the insulation vary in thickness as a step function of time at some specified time $t' > 0$. In this instance, $t = 0$ corresponds to the time of formation of the Moon, 4.6×10^9 years ago. In the lunar example, $t' \simeq 1 \times 10^9$ years (3.1536×10^{16} s), although in general the value of t' may be anything, in many practical circumstances zero. When a *nonzero* t' is used in the program, the new origin of time coordinates is such that $t = 0$ corresponds to t', and the calculations are made in specified, time incremental steps Δt with the first being $t' + \Delta t$. Thus, for example, if $t' = 1 \times 10^9$ years, the time corresponding to the present would be 3.6×10^9 years, and if 10 time-increments were needed, $\Delta t = 3.6 \times 10^8$ years. The first calculated value would correspond to an actual time from the standpoint of radioactive source strength of 1.0×10^9 years plus 3.6×10^8 years or 1.36×10^9 years. Once t' and Δt have been assigned, a specific portion of the time-history may be examined by redefining the origin of time, and specifying N, the number of time steps, in a suitable fashion. Suppose, for example, that the thermal regime at a single time 4.5×10^9 years is wanted. The time origin (program statement 999) is simply redefined as TIME = 1.41912D17 − Δt, and N is set equal to one.

II. Statement of Problem

The formal boundary value problem for temperature T as a function of r and t which we solve, is as follows:

$$\frac{\partial T}{\partial t} - \frac{K}{\rho c}\frac{2}{r}\frac{\partial T}{\partial r} - \frac{K}{\rho c}\frac{\partial^2 T}{\partial r^2} = \phi(r,t) \tag{1}$$

$$\lim_{t \to 0} T(r,t) = f(r) \tag{2}$$

$$\frac{\partial T}{\partial r} = 0, \qquad r = 0 \tag{3}$$

$$\frac{\partial T}{\partial r} + hT = hT_s, \qquad r = R \qquad (0 < t < t') \tag{4}$$

In Eqs. (1) through (4),

ρ = density in g/cm^{-3}

c = specific heat in cal g^{-1} °K^{-1}

K = thermal conductivity in cal cm^{-1} sec^{-1} °K^{-1}

The factor $h = K'/Kd$, where

K' = thermal conductivity of insulation

d = thickness of insulation in cm

At time t' the thickness of insulation is assumed to change to d', and Eq. (4) becomes

$$\frac{\partial T}{\partial r} + h'T = h'T_s, \qquad t \geq t' \tag{5}$$

The source function $\phi(r,t)$ pertains to heat production from exponential decay of radioactive sources and is given by

$$\phi(r,t) = \frac{1}{\rho c} \sum_{j=1}^{4} \rho A_j(\overline{T},r)\, H_j \exp\left[\lambda_j(\overline{T} - t)\right] \tag{6}$$

where $A_j(\overline{T},r)$ are abundances (in g/g) of radionuclide species j at time $\overline{T} = 4.6 \times 10^9$ years, i.e., the present. The H_j are rates of heat generation in cal g^{-1} sec^{-1} and λ_j are the decay constants. The radionuclides considered are ^{40}K, ^{232}Th, ^{235}U, and ^{238}U (j = 1,2,3,4). The following relations between these have been assumed or are to be specified:

$$[^{40}\mathrm{K}] = 1.19 \times 10^{-4}\,[\mathrm{K}]; \frac{[^{232}\mathrm{Th}]}{[\mathrm{U}]} = 4; \frac{[^{235}\mathrm{U}]}{[^{238}\mathrm{U}]} = \frac{1}{137.7}; \frac{[\mathrm{K}]}{[\mathrm{U}]} = \beta \tag{7}$$

The parameter β may be specified arbitrarily but has not in the present problem been taken as a function of r. The final variable of the source function is [U], and may be specified arbitrarily. In all of the above, [•] signifies concentration in g/g.

III. Solution of the Differential Equation

As a result of the boundary condition (Eq. 5), the solution for $T(r,t)$ subsequent to t' requires the solution of a new boundary-value problem with a new "initial" temperature distribution $f'(r)$ appropriate to whatever problem is embodied in the solutions to Eqs. (1) through (4) and Eq. (5). The formal solution to the system of Eqs. (1) through (4) is given by Lowan (Ref. 2; see his Eq. 21′) as

$$T(r,t) - T_s = \frac{2}{r}\sum_{n=1}^{\infty} \frac{(\bar{h}^2 + \lambda_n^2)\sin\lambda_n r}{R\lambda_n^2 + \bar{h}(R\bar{h}+1)} \exp(-\kappa\lambda_n^2 t)\left\{\int_{0^-}^{R} \xi f(\xi)\sin\lambda_n\xi d\xi + \int_0^R \xi\sin\lambda_n\xi d\xi \int_0^t \phi(\xi,\tau)\exp(-\kappa\lambda_n^2 t)\, d\tau\right\} \tag{8}$$

where

$$\bar{h} = \frac{K'}{Kd} - \frac{1}{R} \tag{9}$$

and the λ_n are roots of

$$\alpha\cot\alpha R + \bar{h} = 0 \tag{10}$$

The conductivity of the insulating layer is K'; R is the planetary radius. It is ordinarily not known how many roots of this equation may be required to achieve a given stability to the convergence of the series in Eq. (8), so it is possible to specify the maximum number of solutions to Eq. (10) independently. These are ordinarily computed first and stored. In a typical problem using double-precision arithmetic, 75 to 100 terms may be summed for any given radial and time increment, so that at least this many λ_n are required. As a safety factor, the value MAXIT, defined in the Appendix, is usually given a value like 500. The summation over n is continued until ten consecutive values in the series give values differing by less than 10^{-3}. This criterion naturally depends upon uniform convergence of the series. The question of nonuniformity of convergence has not been formally investigated. Thusfar, however, no numerical instability has been noted in any calculation carried out to date.

IV. Specification of Initial Temperature and Distribution of Radioactivity

In problems dealing with the Moon, we have considered spherical shell geometries like that shown in Fig. 1.

For initial temperature, the Moon is divided into two radial, spherically symmetric zones:

$$f(\xi) = T(r,0) = T_0 + [T_m(r_1) - T_0]\frac{r^2}{r_1^2}, \qquad 0 \le r \le r_1 \tag{11}$$

$$= T_m(R) + T_m'(R - r), \qquad r_1 \le r \le R \tag{12}$$

Here, T_0 is equal to the initial temperature of accreting material (on the independent spherical accretion model) adjusted appropriately for any increase due

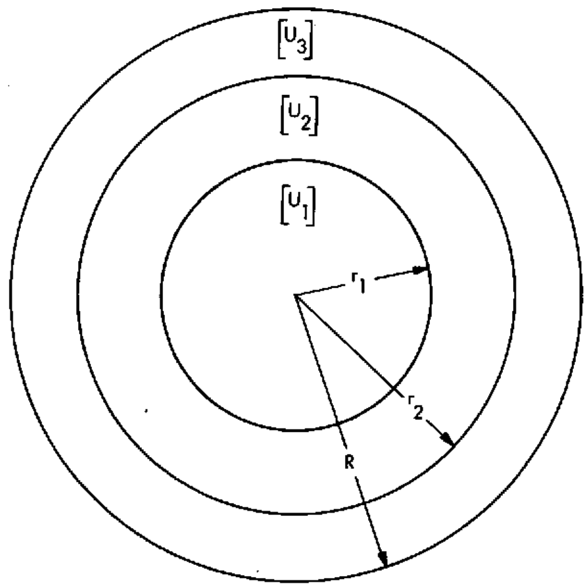

Fig. 1. Three-shell radioactive sphere

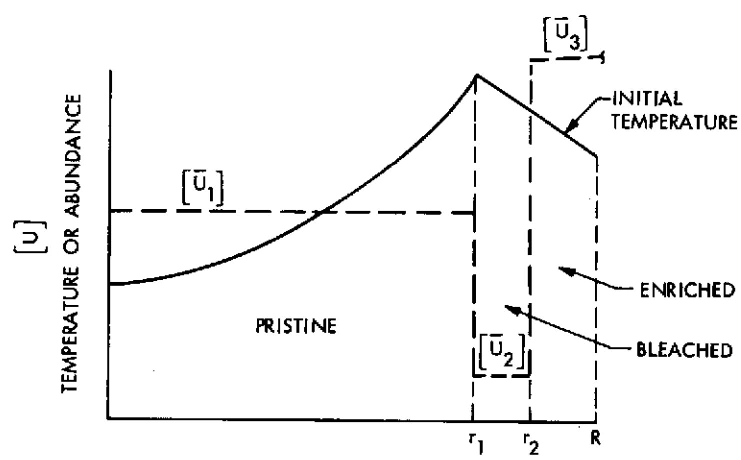

Fig. 2. Initial temperature and radionuclide distributions

to adiabatic compression. At 1 AU at the present time, for example, uncompressed material would have a temperature of about 400°K while, for lunar mass, adiabatic compression would account for about a 50°K increase. The central temperature T_0 for such problems is thus near 450°K. In Eqs. (11) and (12), $T_m(r_1)$ is the melting point of postulated lunar material at radius r_1, $T_m(R)$ is the melting point at the surface, and T'_m is the melting point gradient in °K/cm.

The graphical form of Eqs. (11) and (12) describing independent accretion and surficial melting and differentiation is shown in Fig. 2. For numerical purposes, these quantities, where required, have been taken from experimental values obtained from returned lunar samples. Thus, Ringwood and Essene (Ref. 5) would give the following values for Apollo 11 basalt:

$$T_m(R) = 1360°K$$

$$T_m(r_1) = 1480°K$$

$$T'_m = 4.8 \times 10^{-6} \ °K/cm$$

Equations (11) and (12) may be modified to describe other models of planetary origin. If a lunar-sized object is taken to be at the melting point throughout initially, the starting temperature in the absence of phase changes will be quite nearly a parabolic function of radius. To describe such a temperature profile, in Eq. (11) we set r_1 equal to R, and $T_m(r_1)$ equal to $T_m(R)$, the melting point at the surface; in Eq. (12), $T_m(R) = T'_m = 0$. The quantity T_0 is then interpreted as the melting point at $r = 0$.

The abundances of radioactive species on the accretion model are taken to be *constant* values within each zone, not exponentially decreasing functions of depth as assumed, for example, by Hanks and Anderson (Ref. 6). While there is field evidence for decrease of radioactive sources with depth in the Earth according to a law such as $[U] = [U_0]\, e^{-\mu z}$ (z positive downward), this has been substantiated only for crustal regions and most firmly in relatively localized batholithic portions of the crust. At any rate, if the portions of the Moon involved in any primordial melting are small, i.e., $R - r_2 << R$, then the detailed distribution of sources near the surface is of no consequence as far as the deeper temperature is concerned.

The value of $[U_1]$ is equal to the *present* "primordial" U abundance, $[U_2]$ is equal to the present U abundance of the depleted zone, and $[U_3]$ is equal to the present U abundance in the "crust," and determined as follows: $[U_1]$ is the assumed primordial value of "untouched" or undifferentiated lunar material; $[U_2]$ and $[U_3]$ then follow by mass conservation arguments once it has been hypothesized what fraction $\bar{f}$ of $[U_1]$ has been removed from the bleached zone by a differentiation process. If complete bleaching has occurred, then $\bar{f} = 1$ for zone two and the entire mass of U in the zone between r_2 and r_1 has been concentrated into the zone between r_2 and R.

With the concentration $[U_1]$ specified, together with the factor $\bar{f}$, we get for a homogeneous body

$$[U_2] = (1 - \bar{f})\,[U_1] \tag{13}$$

and

$$[U_3] = \frac{R^3 - r_2^3 + \bar{f}(r_2^3 - r_1^3)}{R^3 - r_2^3}\,[U_1] \tag{14}$$

Note that Eqs. (13) and (14) allow treatment of a variety of two-zone models as well. If the uranium abundance of postulated primordial material is $[U_1]$ and the planet is conceived to differentiate into two portions with core of radius r_2 and (present) abundance $[U'_1]$, we set $\bar{f}$ equal to $1 - [U'_1]/[U_1]$, and r_1 equal to zero. The factor $\bar{f}$ must be calculated by hand. It does not matter mathematically that $[U_1]$ is specified for a region of zero volume. To obtain the temperature increase from radioactive sources alone, T_s must be set equal to zero, and the initial temperature distribution taken zero throughout as well.

Specification of the U abundance has been emphasized here and we have relied upon connections between U, Th, and K cited earlier to specify abundances of the other species involved, and their corresponding heat generations. It may well be that such systematic connections do not exist in all instances (except the $^{235}U/^{238}U$ ratio). Thus β, the [K]/[U] ratio from the orbital gamma ray data, can be shown to vary *laterally* over the Moon's surface. There is also *no* guarantee that it is constant throughout the lunar interior. A similar situation exists on the Earth where xenoliths from the deep interior (Ref. 7) have lunar-like rather than terrestrial-like [K]/[U] ratios. So the real distributions may be ones of great complexity.

We have noted that there is room for considerable computational flexibility in the distributions of initial temperature or radioactivity in spite of specific analytic forms taken here. The user is cautioned, however, that calculations for decay of initial temperature or temperature changes for radioactive heating must be done sequentially *unless* boundaries of zones in the initial profiles coincide, as they are shown to do in Figs. 1 and 2. The assumption of spherical symmetry is always made.

Integration of Eq. (8) is routine for the conditions given in Eqs. (11) and (12) and the constant radioactivity specified, i.e.,

$$\begin{aligned}[U] &= [U_1], \qquad 0 < r < r_1 \\ &= [U_2], \qquad r_1 < r < r_2 \\ &= [U_3], \qquad r_2 < r < R\end{aligned}$$

The result is

$$T(r,t) - T_s = \frac{2}{r}\sum_{n=1}^{\infty} \frac{(\bar{h}^2 + \alpha_n^2)\sin\alpha_n r}{R\alpha_n^2 + \bar{h}(R\bar{h} + 1)}[I_1(n) + I_2(n)]\exp(-\kappa\alpha_n^2 t) \tag{15}$$

where

$$I_1(n) = I_{11}(n) + I_{12}(n) + I_{13}(n) \tag{16}$$

$$I_{11}(n) = \frac{T_0}{\alpha_n^2}(\sin\alpha_n r_1 - \alpha_n r_1 \cos\alpha_n r_1) + \left[\frac{T_m(r_1) - T_0}{r_1^2\alpha_n^4}\right] \tag{17}$$

$$\begin{aligned}I_{12}(n) = {} & \frac{[T_m(R) + T'_m R]}{\alpha_n^2}(\sin\alpha_n R - \sin\alpha_n r_1 - \alpha_n R\cos\alpha_n R + \alpha_n r_1 \cos\alpha_n r_1) \\ & - \frac{T'_m}{\alpha_n^3}[2\alpha_n(R\sin\alpha_n R - r_1\sin\alpha_n r_1) - (\alpha_n^2 R^2 - 2)\cos\alpha_n R \\ & + (\alpha_n^2 r_1^2 - 2)\cos\alpha_n r_1)]\end{aligned} \tag{18}$$

$$I_{13}(n) = -\frac{T_s}{\alpha_n^2}(\sin \alpha_n R - \alpha_n R \cos \alpha_n R) \quad (19)$$

$$I_2(n) = \frac{1}{\rho c}\sum_{j=1}^{4} C_j(n,t)\,[S_{21}(j,n) + S_{22}(j,n) + S_{33}(j,n)] \quad (20)$$

with

$$C_j(n,t) = \frac{1 - \exp[-(\lambda_j - \kappa\alpha_n^2)t]}{(\lambda_j - \kappa\alpha_n^2)\alpha_n^2} \quad (21)$$

$$S_{21}(j,n) = s_{1j}(\sin \alpha_n r_1 - \alpha_n r_1 \cos \alpha_n r_1) \quad (22)$$

$$S_{22}(j,n) = s_{2j}(\sin \alpha_n r_2 - \sin \alpha_n r_1 - \alpha_n r_2 \cos \alpha_n r_2 + \alpha_n r_1 \cos \alpha_n r_1) \quad (23)$$

$$S_{23}(j,n) = s_{3j}(\sin \alpha_n R - \sin \alpha_n r_2 - \alpha_n R \cos \alpha_n r + \alpha_n r_2 \cos \alpha_n r_2) \quad (24)$$

The α_n are consecutive positive roots of Eq. (10). The s_{ij} are given by

$$s_{ij} = \rho A_{ij} H_j \exp(\lambda_j \overline{T}), \qquad (j = 1,2,3,4) \quad (25)$$

where the A_{ij} are abundances of species j in the ith shell and $\overline{T} = 4.6 \times 10^9$ years or an appropriate age for the object in question.

The heat generation and decay constants used are taken from Clark (Ref. 8) and are listed in Table 1 in calories and in calories per gram units.

Table 1. Radionuclide heat generation and decay constants

j	Nuclide	H_j (cal sec^{-1} g^{-1})	λ_j (sec^{-1})
1	^{40}K	6.658×10^{-9}	1.682×10^{-17}
2	^{232}Th	6.314×10^{-9}	0.1582×10^{-17}
3	^{235}U	1.363×10^{-7}	3.082×10^{-17}
4	^{238}U	2.25×10^{-8}	0.488×10^{-17}

V. Solution for Change in Boundary Insulation at Time t'

If the insulation changes at time $t' > 0$ to a value such that the constant $\overline{h}$ changes from $\overline{h}$ to $\overline{h}_1$ (corresponding to insulation thickness changes d to d_1, all other parameters remaining invariant), the solution for the subsequent temperature is $\widetilde{T}(r,t)(t > t')$:

$$\widetilde{T}(r,t) - T_s = \frac{2}{r}\sum_{n=1}^{\infty} \psi(n) \sin \alpha_n r \left\{ \sum_{m=1}^{\infty} \psi(m) \right.$$

$$\times \left[I_1(m) \exp[-\kappa(\alpha_m^2 t' + \alpha_n^2 t)] \right.$$

$$+ \frac{4}{K} \sum_{j=1}^{4} \bar{C}(j,m,n;t,t')\, Q(j,m) \Big] U(m,n)$$

$$+ \frac{\kappa}{K} \sum_{j=1}^{4} \bar{C}'(j,n;t)\, Q'(j,n) \Big\} \tag{26}$$

where

$$\psi(m) = \frac{\bar{h}^2 + \alpha_m^2}{R\alpha_m^2 + \bar{h}(R\bar{h} + 1)}, \qquad (\alpha_m \text{ roots of } \alpha \cot \alpha R + \bar{h} = 0) \tag{27}$$

$$\psi(n) = \frac{\bar{h}_1^2 + \alpha_n^2}{R\alpha_n^2 + \bar{h}_1(R\bar{h}_1 + 1)}, \qquad (\alpha_n \text{ roots of } \alpha \cot \alpha R + \bar{h}_1 = 0) \tag{28}$$

$$I_1(m) = I_{11}(m) + I_{12}(m) + I_{13}(m) \tag{29}$$

where I_{11}, I_{12}, I_{13}, are given by Eqs. (17), (18) and (19), changing m for n. Note however that in Eq. (17), $T_m(r_1)$ refers to the melting temperature at r_1; there is no summation over m implied. Also in Eq. (26)

$$\bar{C}(j,m,n;t,t') = \frac{\exp\left[-\kappa(\alpha_m^2 t' + \alpha_n^2 t)\right] - \exp\left[-(\lambda_j t' + \kappa\alpha_n^2 t)\right]}{(\lambda_j - \kappa\alpha_m^2)\, \alpha_m^2} \tag{30}$$

$$\bar{C}'(j,n;t) = \frac{\exp(-\kappa\alpha_n^2 t) - \exp(-\lambda_j t)}{(\lambda_j - \kappa\alpha_n^2)\, \alpha_n^2} \tag{31}$$

$$U(m,n) = \frac{\sin(\alpha_m - \alpha_n)R}{(\alpha_m - \alpha_n)} - \frac{\sin(\alpha_m + \alpha_n)R}{(\alpha_m + \alpha_n)} \tag{32}$$

$$Q(j,m) = S_{21}(j,m) + S_{22}(j,m) + S_{23}(j,m) \tag{33}$$

$$S_{21}(j,m) = s_{1j}(\sin \alpha_m r_1 - \alpha_m r_1 \cos \alpha_m r_1) \tag{34}$$

$$S_{22}(j,m) = s_{2j}(\sin \alpha_m r_2 - \sin \alpha_m r_1 - \alpha_m r_2 \cos \alpha_m r_2 + \alpha_m r_1 \cos \alpha_m r_1) \tag{35}$$

$$S_{23}(j,m) = s_{3j}(\sin \alpha_m R - \sin \alpha_m r_2 - \alpha_m R \cos \alpha_m R + \alpha_m r_2 \cos \alpha_m r_2) \tag{36}$$

with the s_{ij} given by Eq. (25).

The $[A_{ij}]$ ($i = 1,2,3$ for core, mantle, and crust; $j = 1,2,3,4$ for nuclide species) are

$$[A_{ij}] = \begin{bmatrix} 1.19 \times 10^{-4}\beta[\mathrm{U}_1] & 1.19 \times 10^{-4}\beta[\mathrm{U}_2] & 1.19 \times 10^{-4}\beta[\mathrm{U}_3] \\ 3.7\,[\mathrm{U}_1] & 3.7\,[\mathrm{U}_2] & 3.7\,[\mathrm{U}_3] \\ \dfrac{[\mathrm{U}_1]}{138.7} & \dfrac{[\mathrm{U}_2]}{138.7} & \dfrac{[\mathrm{U}_3]}{138.7} \\ \dfrac{137.7}{138.7}[\mathrm{U}_1] & \dfrac{137.7}{138.7}[\mathrm{U}_2] & \dfrac{137.7}{138.7}[\mathrm{U}_3] \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \\ A_{14} & A_{24} & A_{34} \end{bmatrix} \tag{37}$$

and the relations between $[\mathrm{U}_1]$, $[\mathrm{U}_2]$, $[\mathrm{U}_3]$ as given by Eqs. (13) and (14) are

$$Q'(j,n) = S'_{21}(j,n;t') + S'_{22}(j,n;t') + S'_{23}(j,n;t') \tag{38}$$

$$\left.\begin{aligned} S'_{21}(j,n;t') &= s'_{1j}(\sin\alpha_n r_1 - \alpha_n r_1 \cos\alpha_n r_1) \\ S'_{22}(j,n;t') &= s'_{2j}(\sin\alpha_n r_2 - \sin\alpha_n r_1 - \alpha_n r_2 \cos\alpha_n r_2 + \alpha_n r_1 \cos\alpha_n r_1) \\ S'_{23}(j,n;t') &= s'_{3j}(\sin\alpha_n R - \sin\alpha_n r_2 - \alpha_n R \cos\alpha_n R + \alpha_n r_2 \cos\alpha_n r_2) \end{aligned}\right\} \tag{39}$$

$$s'_{ij} = \rho A_{ij} H_j \exp[\lambda_j(\overline{T} - t')] \tag{40}$$

A "complete" discussion of a planetary thermal history thus involves a two-stage calculation with the present program: 1) the interval $0 \leq t \leq t'$, and 2) the interval $t' < t < \overline{T}$ (present).

Planetary density and [K]/[U] ratio are to be specified internally in the main program.

VI. Input Data

The following input data are required to make a calculation, in the order and format specified.

(1) First card: Maximum number of time steps, maximum number of radial steps (16I5).

(2) Second card: Time increment Δt (seconds); T_s, T_0, $T_m(R)$, T'_m (5D15.5).

(3) Third card: r_1, r_2, R, κ, K (5D15.5).

(4) Fourth card: Maximum value of ordinate for temperature in degrees Kelvin in plots (example, 3500°K) (5D15.5).

(5) Fifth card: d, K', t', d' (5D15.5).

(6) Sixth card: $[U_1], \bar{f}$ (5D15.5).

The output consists of a printout of the input data (Fig. 3) and the temperature at specific times given as a function of radius (Fig. 4), and plots (Fig. 5), which each give initial temperature, as well as temperature at time t.

The maximum number of radial steps in a calculation as well as the maximum number of iterations MAXIT in the main program (solutions to transcendental equations), are specified only in the main program. All input data are in calories and cgs units. Temperatures are printed in degrees Kelvin and times in seconds and years. Radii are given in centimeters.

While we have dealt with homogeneous spheres, the solution given by Lowan (Ref. 2) is general enough to allow K, the thermal conductivity, to vary with radius. Whether the problem dealt with can be solved analytically depends upon whether expressions given in the original paper can be integrated. There is also no necessary restriction on forms for distribution of radioactivity; uniform distributions have been used here lacking any reason to suppose otherwise in the Moon, but exponential distributions could be handled as well. Problems involving thermal properties varying with temperature must be treated numerically.

A listing of the main program and subroutines is given in the Appendix.

```
    N     M
    3    20

TIME INCREMENT   SURFACE TEMPERATURE   'INITIAL' TEMPERATURE OF ACCRETING MATERIAL   MELTING POINT AT ZERO PRESSURE
  .1500000+017            .2500000+003                              .4500000+003                      .1360000+004

MELTING POINT GRADIENT          TMR1    'CORE' RADIUS   'MANTLE' RADIUS   PLANETARY RADIUS    THERMAL DIFFUSIVITY
         .4800000-005   .1480000+004    .1488000+009     .1714000+009        .1738000+009          .1640000-001

THERMAL CONDUCTIVITY   MAXIMUM ABSCISSA VALUE IN PLOTS
       .1100000-001                     .3500000+004

THICKNESS OF INSULATING LAYER   THERMAL CONDUCTIVITY OF INSULATING LAYER
               .1000000+000                                   .1000000-004

     'T-PRIME'    NEW THICKNESS OF INSULATING LAYER
  .3000000+017                       .1000000+004

           U1              F
  .3000000-007   .1000000+001

SOURCE TERMS
     .17743-014     .29500-014     .82040-014     .45085-014
     .00000         .00000         .00000         .00000
     .16174-013     .26890-013     .74783-013     .41097-013

      DECAY CONSTANTS
K40:       .1682000-016
TH232:     .1582000-017
U235:      .3082000-016
U238:      .4880000-017

ROOTS OF EQUATION: TAN(ALPH*R3) + (ALPH/H) = 0.
          1      .107672323506-007
          2      .278012319980-007
          3      .456100055506-007
          4      .635674580711-007
          5      .815768264820-007
```

Fig. 3. Listing of "input data" and first five roots of transcendental equation tan $(\alpha \cdot R) + \alpha/h = 0$

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TIME (SEC)	TIME (YEARS)	NO. OF TERMS IN SUMMATION
.7500000+017	.2378234+010	
RADIUS	T1+VS	
.8690000+007	.2339513+04	62
.1738000+008	.2341481+04	47
.2607000+008	.2343264+04	41
.3476000+008	.2344303+04	38
.4345000+008	.2341946+04	36
.5214000+008	.2333929+04	28
.6083000+008	.2320373+04	26
.6952000+008	.2292757+04	26
.7821000+008	.2251677+04	25
.8690000+008	.2193342+04	25
.9559000+008	.2111307+04	24
.1042800+009	.2001994+04	24
.1129700+009	.1866203+04	23
.1216600+009	.1700109+04	25
.1303500+009	.1503824+04	24
.1390400+009	.1279133+04	27
.1477300+009	.1026541+04	27
.1564200+009	.7675109+03	35
.1651100+009	.5251613+03	41
.1738000+009	.2893931+03	75

Fig. 4. Listing of radius and temperature for accompanying plots for $t = 7.5 \times 10^{16}$ sec and [U] = 30 ppb

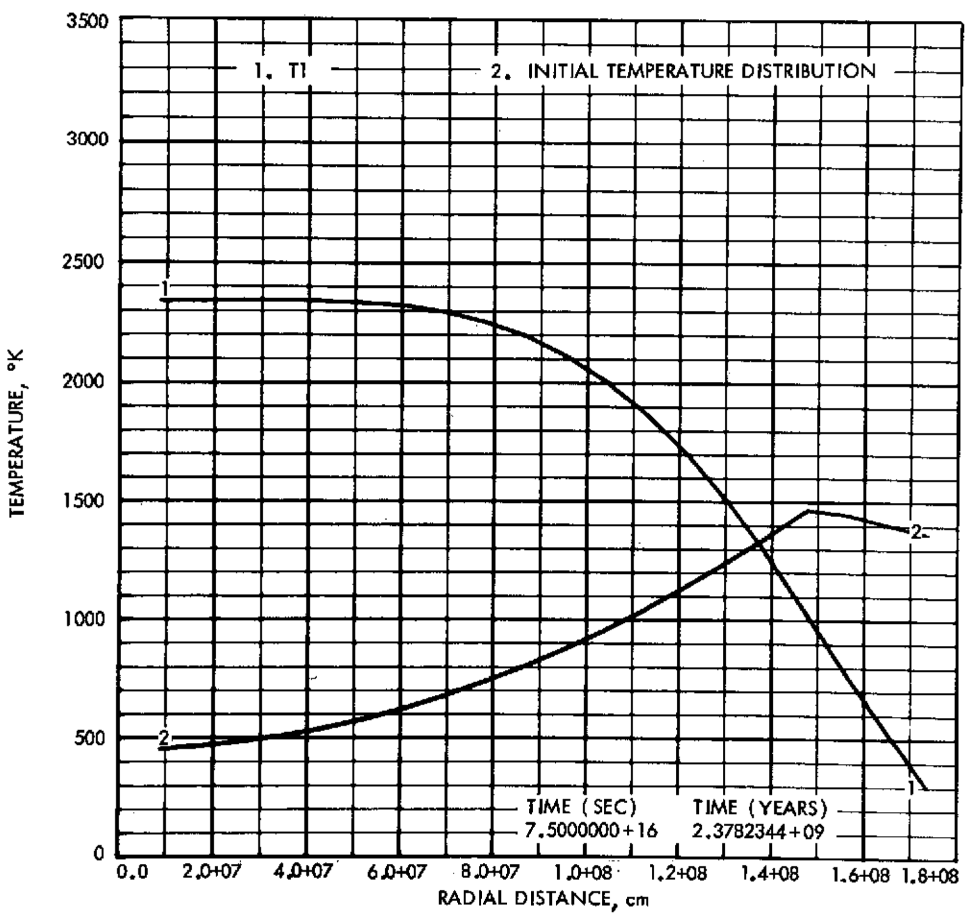

Fig. 5. Temperature distribution in three-zone model at time indicated for 30 ppb U

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Appendix

Program SPHERE

```
'RUN,/R   JEC3,J5N25Z,SPHERE,06,400/0,,183/516  ,  CONEL
'SC4020 BLDG/183,BOX/516,CAMERA/9IN,FRAMES/90
'FOR,IS MAIN,MAIN
C
C PARAMETER' MAXR -- MAXIMUM NUMBER OF RADIAL STEPS.
C            MAXIT -- MAXIMUM NUMBER OF ITERATIONS
C
      PARAMETER MAXR=40
      PARAMETER MAXIT=500
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION A(3,4),H(4)
      COMMON /BLK1/ ALPHA(MAXIT),ALPHAM(MAXIT),S(3,4),XLAMDA(4),
     1              PSI(MAXIT),PSIBAR(MAXIT),XI1(MAXIT),
     2              S21(MAXIT),S22(MAXIT),S23(MAXIT),
     3              SBAR1(MAXIT),SBAR2(MAXIT),SBAR3(MAXIT),
     4              TPR,TIME,XKKAPPA,XKOXK,R3,SPR(3,4)
      INTEGER JX(3),JY(3),NP(3),INTERP(3)
      REAL XY(MAXR,4),ROW2(2)
      REAL YMAX
      INTEGER SYMBOL(3),TITLE2(14),XNAME(14),YNAME(10),ROW1(2,2)
      DIMENSION ICONRG(4),SUM0(4),SUM1(4),KOUNT(4),IFLAG(4)
      DATA XLAMDA/1.682D-17,.1582D-17,3.082D-17,.488D-17/
      DATA PI/3.14159265358979300/
      DATA JX/1,1,1/JY/2,3,4/INTERP/1,1,1/
      DATA SYMBOL/6H1     ,6H2     ,6H3     /
      DATA TITLE2/6H     ,6H     ,6H     ,6H1. T1 ,6H     ,6H     ,
     1            6H2. INI,6HTIAL T,6HEMPERA,6HTURE D,6HISTRIB,6HUTION ,
     2            6H     ,6H     /
      DATA XNAME/6H     ,6H     ,6H     ,6H     ,6H     ,6H     ,
     1           6HRADIAL,6H DISTA,6HNCE   ,6H     ,6H     ,6H     ,
     2           6H     ,6H     /
      DATA YNAME/6H     ,6H     ,6H     ,6H     ,6HTEMPER,6HATURE ,
     1          6H     ,6H     ,6H     ,6H     /
      DATA IDIM/MAXR/
      TAU=1.435D17
      RHO=3.34D0
      BETA=2.D3
      H(1)=6.658D-9
      H(2)=6.341D-9
      H(3)=1.363D-7
      H(4)=2.250D-8
      ROW1(1,1)=6HTIME (
      ROW1(2,1)=6HSEC)
      ROW1(1,2)=6HTIME (
      ROW1(2,2)=6HYEARS)
      EPS=1.D-3
      NCOUNT=9
C
C INPUT' N -- NUMBER OF TIME STEPS
C        M -- NUMBER OF RADIAL STEPS
C
   10 READ(5,500,END=400) N,M
      WRITE(6,600) N,M
      IF(M.LE.MAXR) GO TO 15
      MTEMP=MAXR
      WRITE(6,607) MTEMP
      STOP
C
C INPUT' DELT -- TIME INCREMENT
C        VS -- SURFACE TEMPERATURE
C        TO -- 'INITIAL' TEMPERATURE OF ACCRETING MATERIAL
C        TMO -- MELTING POINT AT ZERO PRESSURE
C        TMPR -- MELTING POINT GRADIENT
C        R1 -- 'CORE' RADIUS
C        R2 -- 'MANTLE' RADIUS
C        R3 -- PLANETARY RADIUS
C        XKAPPA -- THERMAL DIFFUSIVITY
C        XK -- THERMAL CONDUCTIVITY
C        ABMAX -- MAXIMUM ORDINATE VALUE IN PLOTS (MINIMUM VALUE IS ASSUMED
C                 TO BE 0)
C
```

```
C
   15  READ(5,501) DELT,VS,TO,TMO,TMPR,R1,R2,R3,XKAPPA,XK
       READ(5,501) ABMAX
       TMR1=TMO+TMPR*(R3-R1)
       WRITE(6,601) DELT,VS,TO,TMO,TMPR,TMR1,R1,R2,R3,XKAPPA,XK,ABMAX
       YMAX=ABMAX
C
C INPUT' D -- THICKNESS OF INSULATING LAYER
C        XKPR -- THERMAL CONDUCTIVITY OF INSULATING LAYER
C         TPR -- 'T-PRIME'
C         D1  -- NEW THICKNESS OF INSULATING LAYER
C
       READ(5,501) D,XKPR,TPR,D1
       WRITE(6,608) D,XKPR
       WRITE(6,610) TPR,D1
C
C INPUT' U1
C        F
C
       READ(5,501) U1,F
       WRITE(6,613) U1,F
       U2=(1.D0-F)*U1
       U3=U1*(R3**3-R2**3+F*(R2**3-R1**3))/(R3**3-R2**3)
       FAC1=1.19D-4*BETA
       FAC2=137.7D0/138.7D0
       FAC3=1.D0/138.7D0
       A(1,1)=FAC1*U1
       A(2,1)=FAC1*U2
       A(3,1)=FAC1*U3
       A(1,2)=3.7*U1
       A(2,2)=3.7*U2
       A(3,2)=3.7*U3
       A(1,3)=FAC3*U1
       A(2,3)=FAC3*U2
       A(3,3)=FAC3*U3
       A(1,4)=FAC2*U1
       A(2,4)=FAC2*U2
       A(3,4)=FAC2*U3
       DIF=TAU-TPR
       DO 16 J=1,4
       EX1=DEXP(XLAMDA(J)*TAU)
       EX2=DEXP(XLAMDA(J)*DIF)
       DO 16 I=1,3
       S(I,J)=RHO*A(I,J)*H(J)*EX1
       SPR(I,J)=RHO*A(I,J)*H(J)*EX2
   16  CONTINUE
       WRITE(6,602) ((S(I,J),J=1,4),I=1,3)
       WRITE(6,603) XLAMDA
       DO 17 I=1,3
       NP(I)=M
   17  CONTINUE
       XKOXK=XKAPPA/XK
       R1R1=R1*R1
       R3R3=R3*R3
       SIGMA=XKPR/(XK*D)
       HO=SIGMA-1./R3
       HH=HO*HO
       TIME=0.D0
C      STATEMENT 999 DEFINES  NEW TIME ORIGIN ,T-DELT
  999  TIME=1.387584D17
       DELTR=R3/M
C
C OBTAIN ROOTS OF' TAN(X*R3)+(X/HO)=0.
C
       DO 19 I=1,MAXIT
       ALPHA(I)=ALPH(I,HO,R3,JFLAG)
       IF(JFLAG.EQ.0) GO TO 18
       WRITE(6,609)
       GO TO 10
   18  AK=ALPHA(I)
       AKAK=AK*AK
       AKR1=AK*R1
       AKR2=AK*R2
       AKR3=AK*R3
       AAR1R1=AKAK*R1R1
       SAKR1=DSIN(AKR1)
```


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```
      CAKR1=DCOS(AKR1)
      SAKR2=DSIN(AKR2)
      CAKR2=DCOS(AKR2)
      SAKR3=DSIN(AKR3)
      CAKR3=DCOS(AKR3)
      PSI(I)=(HH+AKAK)/(R3*AKAK+H0*(R3*H0+1.))
      XI11=(T0/AKAK)*(SAKR1-AKR1*CAKR1)
     1    +((TMR1-T0)/(AAR1R1*AKAK))*(3.*(AAR1R1-2.)*SAKR1-
     2                               AKR1*(AAR1R1-6.)*CAKR1)
      XI12=((TM0+TMPR*R3)/AKAK)*(SAKR3-SAKR1-AKR3*CAKR3+AKR1*CAKR1)
     1    -(TMPR/(AK*AKAK))*(2.*AK*(R3*SAKR3-R1*SAKR1)
     2                       -(AKAK*R3R3-2.)*CAKR3+(AAR1R1-2.)*CAKR1)
      XI13=(-VS/AKAK)*(SAKR3-AKR3*CAKR3)
      XI1(I)=XI11+XI12+XI13
      S21(I)=SAKR1-AKR1*CAKR1
      S22(I)=SAKR2-SAKR1-AKR2*CAKR2+AKR1*CAKR1
      S23(I)=SAKR3-SAKR2-AKR3*CAKR3+AKR2*CAKR2
   19 CONTINUE
  190 SIGMA1=XKPR/(XK*D1)
      H1=SIGMA1-1./R3
      H1H1=H1*H1
C
C OBTAIN ROOTS OF' TAN(X*R3)+(X/H1)=0.
C
      DO 230 I=1,MAXIT
      ALPHAM(I)=ALPH(I,H1,R3,JFLAG)
      IF(JFLAG.EQ.0) GO TO 210
      WRITE(6,609)
      GO TO 10
  210 AK=ALPHAM(I)
      AKAK=AK*AK
      AKR1=AK*R1
      AKR2=AK*R2
      AKR3=AK*R3
      SAKR1=DSIN(AKR1)
      CAKR1=DCOS(AKR1)
      SAKR2=DSIN(AKR2)
      CAKR2=DCOS(AKR2)
      SAKR3=DSIN(AKR3)
      CAKR3=DCOS(AKR3)
      PSIBAR(I)=(H1H1+AKAK)/(R3*AKAK+H1*(R3*H1+1.))
      SBAR1(I)=SAKR1-AKR1*CAKR1
      SBAR2(I)=SAKR2-SAKR1-AKR2*CAKR2+AKR1*CAKR1
      SBAR3(I)=SAKR3-SAKR2-AKR3*CAKR3+AKR2*CAKR2
  230 CONTINUE
      DO 280 I=1,N            'LOOP OVER TIME
      TIME=TIME+DELT
      TIMEYR=TIME/3.1536D07
      ROW2(1)=TIME
      ROW2(2)=TIMEYR
      R=0.D0
      WRITE(6,611) TIME,TIMEYR
      DO 270 J=1,M            'LOOP OVER RADIAL DISTANCE
      R=R+DELTR
C
C INITIALIZE CONVERGENCE FLAGS AND SUMS
C
      ICONRG(1)=0
      KOUNT(1)=0
      IFLAG(1)=0
      SUM0(1)=0.D0
      SUM1(1)=0.D0
      DO 250 K=1,MAXIT
      AK=ALPHAM(K)
      AKAK=AK*AK
      AKR=AK*R
      SAKR=DSIN(AKR)
      X1=X1EX(AKAK,AK,K)
      SUM1(1)=SUM1(1)+PSIBAR(K)*SAKR*X1
      IF(DABS((SUM1(1)-SUM0(1))/SUM1(1)).GT.EPS) GO TO 240
      KM1=K-1
      IF(IFLAG(1).NE.KM1) KOUNT(1)=1
      IF(IFLAG(1).EQ.KM1) KOUNT(1)=KOUNT(1)+1
      IFLAG(1)=K
  240 SUM0(1)=SUM1(1)
      ICONRG(1)=K
      IF(KOUNT(1).GT.NCOUNT) GO TO 260
```

```
250  CONTINUE
260  T1=2.*SUM1(1)/R
     XY(J,1)=R
     XY(J,2)=T1+VS
     IF(R.GT.R1) GO TO 262
     XY(J,3)=T0+(TMR1-T0)*R*R/R1R1
     GO TO 264
262  XY(J,3)=TM0+TMPR*(R3-R)
264  WRITE(6,612) R,XY(J,2),ICONRG(1)
270  CONTINUE
     CALL KCPL(XY,IDIM,0,JX,JY,NP,INTERP,SYMBOL,TITLE2,XNAME,YNAME,
    1          ROW1,ROW2,2)
     CALL GRIDY(-.1,YMAX,0,0,0,6H
     CALL KCPL1(2,2,1)
280  CONTINUE
     GO TO 10
400  STOP
500  FORMAT(16I5)
501  FORMAT(5D15.5)
502  FORMAT(4D15.5)
600  FORMAT('0     N      M'/1H ,2I5)
601  FORMAT('0TIME INCREMENT   SURFACE TEMPERATURE    ''INITIAL'' TEMPER
    1ATURE OF ACCRETING MATERIAL   MELTING POINT AT ZERO PRESSURE'/1H ,
    2D14.7,8X,D14.7,32X,D14.7,19X,D14.7/'0MELTING POINT GRADIENT
    3      TMR1      ''CORE'' RADIUS    ''MANTLE'' RADIUS    PLANETARY RADIU
    4S    THERMAL DIFFUSITIVITY'/1H ,8X,D14.7,2X,D14.7,3X,D14.7,4X,D14.7
    5,5X,D14.7,10X,D14.7/'0THERMAL CONDUCTIVITY    MAXIMUM ABSISSA VALUE
    6 IN PLOTS'/1H ,6X,D14.7,19X,D14.7)
602  FORMAT('0SOURCE TERMS'/1H ,4D15.5/1H ,4D15.5/1H ,4D15.5)
603  FORMAT('0        DECAY CONSTANTS'/' K40'',4X,D14.7/' TH232'',2X,
    1          D14.7/' U235'',3X,D14.7/' U238'',3X,D14.7)
604  FORMAT('0             TIME'/1H ,D14.7/'0          RADIUS         T1+VS
    1          TR+VS     V=TI+TR+VS')
605  FORMAT(1H ,4D14.7,4I10)
606  FORMAT('0    PARTIAL(TI)    PARTIAL(TR)         FLUX 1        FLUX 2
    1    TOTAL FLUX'/1H ,5D14.7)
607  FORMAT('0NUMBER OF RADIAL STEPS EXCEEDS MAXIMUM DIMENSION'/
    1         ' MAXIMUM DIMENSION'',I4)
608  FORMAT('0THICKNESS OF INSULATING LAYER   THERMAL CONDUCTIVITY OF I
    1NSULATING LAYER'/1H ,15X,D14.7,29X,D14.7)
609  FORMAT('0FAULTY ROOT OBTAINED FROM SOLUTION TO' TAN(X*R3)+(X/R3)=0
    1. COMPUTATIONS TERMINATED.')
610  FORMAT('0      ''T-PRIME''      NEW THICKNESS OF INSULATING LAYER'/
    1          1H ,D14.7,23X,D14.7)
611  FORMAT('0     TIME (SEC)        TIME (YEARS)'/1H ,D14.7,5X,D14.7/
    1          1H0,8X,'RADIUS',9X,'T1+VS')
612  FORMAT(1H ,D14.7,E14.7,28X,I10)
613  FORMAT('0                U1                F'/1H ,2D14.7)
     END
'FOR,IS X1EX,X1EX
      DOUBLE PRECISION FUNCTION X1EX(AKAK,AK,K)
      PARAMETER MAXIT=500
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON /BLK1/ ALPHA(MAXIT),ALPHAM(MAXIT),S(3,4),XLAMDA(4),
     1              PSI(MAXIT),PSIBAR(MAXIT),XI1(MAXIT),
     2              S21(MAXIT),S22(MAXIT),S23(MAXIT),
     3              SBAR1(MAXIT),SBAR2(MAXIT),SBAR3(MAXIT),
     4              TPR,TIME,XKAPPA,XKOXK,R3,SPR(3,4)
      DATA EPS/1.D-5/NCOUNT/9/
      KOUNT=0
      IFLAG=0
      SUM0=0.D0
      SUM1=0.D0
      DO 20 I=1,MAXIT
      AI=ALPHA(I)
      AIAI=AI*AI
      DIF=AI-AK
      SUM=AI+AK
      SDR3=DSIN(DIF*R3)
      SSR3=DSIN(SUM*R3)
      POWER1=-XKAPPA*AIAI*TPR-XKAPPA*AKAK*TIME
      POWER2=-XKAPPA*(AIAI*TPR+AKAK*TIME)
      EX=DEXP(POWER1)
      EX2=DEXP(POWER2)
      XI1EX=XI1(I)*EX
      XI2EX=0.D0
```

```
      DO 8 J=1,4
      POWER3=-(XLAMDA(J)*TPR+XKAPPA*AKAK*TIME)
      CBAR=(EX2-DEXP(POWER3))/((XLAMDA(J)-XKAPPA*AIAI)*AIAI)
      Q=S(1,J)*S21(I)+S(2,J)*S22(I)+S(3,J)*S23(I)
      XI2EX=XI2EX+CBAR*Q
    8 CONTINUE
      XI2EX=XI2EX*XKOXK
      SUM1=SUM1+PSI(I)*(XI1EX+XI2EX)*((SDR3/DIF)-(SSR3/SUM))
      IF(DABS((SUM1-SUM0)/SUM1).GT.EPS) GO TO 10
      IM1=I-1
      IF(IFLAG.NE.IM1) KOUNT=1
      IF(IFLAG.EQ.IM1) KOUNT=KOUNT+1
      IFLAG=I
      IF(KOUNT.GT.NCOUNT) GO TO 40
   10 SUM0=SUM1
   20 CONTINUE
      WRITE(6,30)
   30 FORMAT('0FAILURE TO CONVERGE IN XIBAR1')
   40 POWER2=-XKAPPA*AKAK*TIME
      EX2=DEXP(POWER2)
      DO 50 J=1,4
      POWER3=-XLAMDA(J)*TIME
      CBARPR=(EX2-DEXP(POWER3))/((XLAMDA(J)-XKAPPA*AKAK)*AKAK)
      QPR=SPR(1,J)*SBAR1(K)+SPR(2,J)*SBAR2(K)+SPR(3,J)*SBAR3(K)
      SUM1=SUM1+XKOXK*CBARPR*QPR
   50 CONTINUE
      X1EX=SUM1
      RETURN
      END
'FOR,IS ALPH,ALPH
      DOUBLE PRECISION FUNCTION ALPH(K,H,R3,JFLAG)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C
C FUNCTION ROUTINE WHICH CALCULATES SUCCESSIVE POSITIVE ROOTS OF THE
C FUNCTION' TAN(ALPHA*R3) + (ALPHA/H) = 0.
C
      DATA PI/3.14159265358979324D0/
      DATA NCOUNT/4/
      KOUNT=1
      ICHECK=0
      IF(K.NE.1) GO TO 10
      WRITE(6,5)
    5 FORMAT('0ROOTS OF EQUATION' TAN(ALPH*R3) + (ALPH/H) = 0.')
      JFLAG=0
      STEP=PI/R3
      STEPO2=STEP/2.
      EPS=H/1000.
      X=STEPO2+EPS
      ONEOH=1./H
      GO TO 20
   10 X=K*STEP-STEPO2+EPS
   20 DO 30 I=1,200
      IM1=I-1
      THETA=X*R3
      TEMP=1./DCOS(THETA)
      F=DTAN(THETA)+X/H
      FPR=R3*TEMP*TEMP+ONEOH
      XNEW=X-F/FPR
      IF(DABS((XNEW-X)/XNEW).GT.1.D-12) GO TO 25
      IF(ICHECK.NE.IM1) KOUNT=1
      IF(ICHECK.EQ.IM1) KOUNT=KOUNT+1
      ICHECK=I
      IF(KOUNT.GT.NCOUNT) GO TO 50
   25 X=XNEW
   30 CONTINUE
      WRITE(6,40) X,XNEW,F
   40 FORMAT('0NEWTON METHOD FAILED TO SATISFY TERMINATING CRITERIA
     1          X               XNEW                    F'/1H ,52X,3D16.8)
      STOP
   50 ALPH=XNEW
      WRITE(6,55) K,XNEW
   55 FORMAT(1H ,I10,D22.12)
      XMID=K*PI/R3
      XLBND=XMID-STEPO2
      UBOUND=XMID+STEPO2
      IF(XLBND.LT.X.AND.X.LT.UBOUND) RETURN
```


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```
      WRITE(6,60) K,XLBND,X,UBOUND
   60 FORMAT('0SOLUTION NOT PROPERLY BOUNDED      K      LOWER BOUND
     1          X      UPPER BOUND'/1H ,30X,I5,3D16.8)
      JFLAG=1
      RETURN
      END
'MAP,L
 LIB LIB*PLOTS
'XQT
    3    20
        1.5D16 250.             450.             1360.                    4.8D-6
     1.488D08       1.714D08        1.738D08          1.64D-2         1.10D-2
 3500.
 100000.         .00001                     .3D+17 1000.
      .3000D-07 1.
        1.5D16 130.             250.             1373                     5.0D-6
    3    20
        1.5D16 250.             450.             1360.                    4.8D-6
     1.488D08       1.714D08        1.738D08          1.64D-2         1.10D-2
 3500.
 100000.         .00001                     .3D+17 1000.
      .4000D-07 1.
```


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