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PLASTIC FLOW AROUND RIGID SPHERICAL INCLUSIONS

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Abstract

The extent of plastic flow in a spherical solid (assumed to be homogeneous and elastically and plastically isotropic) surrounding a concentric rigid sphere as a function of applied external pressure is calculated. The applied pressure necessary to cause plastic deformation throughout the solid is obtained.

INTRODUCTION

When a solid matrix surrounding a rigid sphere is pressurized, the rigid particle acts as a stress concentrator. The radial pressure at the interface at the rigid sphere exceeds the external hydrostatic pressure. When the applied pressure is increased sufficiently, yielding of the matrix begins at the interface. This is a well known phenomenon and can be observed around inclusions in solids where dislocations are punched out.[1] The pressure which causes the onset of yielding is available from earlier calculation from linear elasticity theory and Tresca or von Mises yield criteria. In the present paper we calculate the extent of plastic flow versus applied pressure and finally obtain the applied pressure necessary to cause plastic deformation throughout the solid. The present analysis assumes that there is no strain hardening.

In the present paper external pressurization of a medium surrounding a rigid spherical inclusion is studied. Two methods are used. The first, developed here, leads more easily to the solution than the second, which is the technique used by Hill[2] for studying the internal pressurization of a hollow sphere.

THEORETICAL

Consider the matrix of external radius b surrounding a rigid sphere of radius a. The boundary conditions are that the radial component of stress is given by

$$\sigma_{rr} = -P_{ext} \text{ at } r=b \tag{1}$$

and the radial component of displacement is given by

$$u = 0 \text{ at } r = a \tag{2}$$

Elastic Behavior

At sufficiently small values of P_{ext}, the material behaves everywhere in a linear elastic manner. The stresses then have the form

$$\sigma_{rr} = A + \frac{B}{r^3} \tag{3}$$

$$\sigma_{\theta\theta} = A - \frac{B}{2r^3} \tag{4}$$

and the radial displacement is given by

$$u = r\varepsilon_{\theta\theta} = \frac{r}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{\theta\theta} + \sigma_{rr})\right]. \tag{5}$$

Here $\varepsilon_{\theta\theta}$ is the circumferential strain, E is Young's modulus, ν is Poisson's ratio and we have used the fact that $\sigma_{\phi\phi} = \sigma_{\theta\theta}$.

From these five equations, the state of stress can be readily ascertained. The pressure which causes the onset of yielding can be obtained using Tresca's or von Mises' yield criteria (which are identical for the sphere):

$$\sigma_{\theta\theta} - \sigma_{rr} = \sigma_o \tag{6}$$

where σ_o is the yield stress in simple tension. This leads to the result that yielding (designated by an asterisk) begins when

$$P_{ext}^{*} = \left\{\frac{(1+\nu)}{3(1-2\nu)} + \frac{2K^{-3}}{3}\right\} \sigma_o \tag{7}$$

where $K = b/a$.

When yielding begins the contact pressure at $r=a$ for $\nu=1/3$

$$P_{int}^{*} = 2\sigma_o \tag{8}$$

(We note that $P_{int}^{*} = \frac{3}{2}P_{ext}^{*}$ for large K.)

Elasto-Plastic

Here the material is elastic for $r \geq c$ and plastic for $r \leq c$. In the plastic region the stresses are determined by Equation (6) and the boundary conditions. The elastic strains in the plastic region are obtained from these stresses using Hooke's laws. It is assumed that strain hardening is absent, i.e., σ_o does not vary with the extent of plastic deformation. The stresses are continuous at the elastic-plastic boundary, $r=c$.

The boundary conditions for the elastic region are

$$\sigma_{\theta\theta} - \sigma_{rr} = \sigma_o \text{ at } r=c \tag{9}$$

and

$$\sigma_{rr} = -P_{ext} \text{ at } r=b. \tag{10}$$

The solution for the elastic region is

$$\sigma_{rr} = -P_{ext} - \frac{2}{3}\sigma_o \frac{c^3}{b^3}\left(\frac{b^3}{r^3} - 1\right). \tag{11}$$

and

$$\sigma_{\theta\theta} = -P_{ext} + \frac{2}{3}\sigma_o \frac{c^3}{b^3}\left(\frac{b^3}{2r^3} + 1\right). \tag{12}$$

The displacement in the elastic region is

$$u = \frac{r}{E}\left[(1-2\nu)\left(-P_{ext} + \frac{2}{3}\sigma_o c^3/b^3\right) + \frac{(1+\nu)}{3}\sigma_o c^3/r^3\right]. \tag{13}$$

The general solution in the plastic region can be obtained from the equation of radial equilibrium, namely

$$\frac{r d\sigma_{rr}}{dr} + 2(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{14}$$

and Equation (6). The result is

$$\sigma_{rr} = 2\sigma_o \ell n\, r + D$$

where D is an integration constant which can be evaluated by the requirement of continuity of σ_{rr} at r=c. Then we have

$$\sigma_{rr} = -2\sigma_o \ell n\frac{c}{r} - P_{ext} - \frac{2}{3}\sigma_o\left(1 - \frac{c^3}{b^3}\right). \tag{15}$$

The local volume dilation in the plastic region is given by

$$\frac{\Delta V}{V_o} = \frac{(1-2\nu)}{E}\,(\sigma_{\theta\theta}+\sigma_{\phi\phi}+\sigma_{rr}) = \frac{(1-2\nu)}{E}(2\sigma_o+3\sigma_{rr}) \quad . \tag{16}$$

The total change in volume of the plastic region (whose inner radius is constant a and outer radius is c) is therefore

$$\int_a^c \frac{(1-2\nu)}{E}(2\sigma_o+3\sigma_{rr})\,4\pi r^2 dr = 4\pi c^2 u(c) \quad . \tag{17}$$

The value of σ_{rr} is given by Equation (15) while u(c) follows from (13) with r set equal to c. It follows that

$$P_{ext} = \frac{(1-\nu)}{(1-2\nu)}(c/a)^3\sigma_o - 2\sigma_o \ln\left(c/a\right) + \frac{2}{3}\,(c/b)^3\sigma_o - \frac{2}{3}\sigma_o \quad . \tag{18}$$

For the fully plastic condition (labelled with double asterisk) c→b and (for ν=1/3)

$$P_{ext}^{**} = 2\sigma_o K^3 - 2\sigma_o \ln K, \tag{19}$$

and

$$P_{int}^{**} - P_{ext}^{**} = 2\sigma_o \ln K. \tag{20}$$

For the case where K is only slightly greater than one (the thin shell) we have from (19) $\lim_{K\to 1} P_{ext} = 2\sigma_o$
as required since for a thin shell σ_{rr} does not vary across the shell and the entire shell yields simultaneously as given by (8).

The ratio P^{**}_{ext}/P^{*}_{ext} vs K for ν=1/3 is shown in Table 1.

Table 1. Ratio of pressure needed to cause plastic flow throughout matrix over pressure needed for onset of plastic flow versus radius ratio. Based on Poisson ratio of one-third.

K	$P^{**}_{ext}/2\sigma_o$	$P^{*}_{ext}/2\sigma_o$	P^{**}_{ext}/P^{*}_{ext}
1	1	1	1
2	7.31	0.71	10.3
3	25.90	0.68	38.1

As K becomes modestly large the pressure needed to cause plastic flow throughout the matrix is enormous relative to the pressure required for the commencement of plastic flow.

The extent of plastic flow for a given applied pressure and given K can readily be written in terms of f_m, the fraction of the matrix which has undergone plastic deformation. Here

$$1-f_m = (b^3-c^3)/(b^3-a^3) \quad . \tag{21}$$

Then (for ν=1/3) from (18)

$$\frac{P_{ext}}{2\sigma_o} = (K^3-1)f_m+1 - \frac{1}{3}\ell n[(K^3-1)f_m+1]-\frac{1}{3}(1-f_m)(1-K^{-3}) . \tag{22}$$

Curves of f_m vs $P_{ext}/2\sigma_o$ for K=1, 2 and 3 are shown in Figure 1.

For the cases considered here, the strains everywhere are five percent or less (assuming $3(1-2\nu)\sigma_o/E \lesssim 1/100$) so the neglect of strain hardening is justified unless the matrix has an unusually rapid rate of strain hardening.

<u>Alternate Method</u>

This problem can also be solved following the technique described by Hill.[2] The elastic stresses and displacements are identical as those derived above. In the plastic region, the equation of radial equilibrium again leads to the radial stress as in Equation (15). Now following Hill's definition of the velocity υ of a particle, we find

$$\upsilon = \frac{\partial u/\partial c}{1-(\partial u/\partial r)} \qquad (23)$$

in the elastic region; and

$$\frac{\partial \upsilon}{\partial r}+\frac{2\upsilon}{r} = \frac{(1-2\nu)}{E}\left(\frac{\partial}{\partial c}+\upsilon\frac{\partial}{\partial r}\right)(\sigma_{rr}+2\sigma_{\theta\theta}) \qquad (24)$$

in the plastic region. These velocities must be continuous at r=c, the plastic-elastic interface.

Using the equations for the stresses in the plastic region, (9) and (15), we can solve Equation (24) if we make the same approximations that Hill makes; i.e. neglecting second and higher orders of σ_o/E, and neglecting $\partial u/\partial r$ with respect to 1. Matching the velocities in the two regions gives the velocity in the plastic region,

$$\upsilon = \frac{3(1-\nu)\sigma_o}{E}\,\frac{c^2}{r^2} - \frac{2(1-2\nu)\sigma_o}{E}\left(1-\frac{c^3}{b^3}+\frac{c}{2\sigma_o}\,\frac{\partial P_{ext}}{\partial c}\right) r/c \qquad (25)$$

when P_{ext} is here written as a function of c.

Since υ is defined as dr/dc we can rewrite Equation (25) in terms of da/dc, the velocity of a particle at the inner boundary. With an incompressible material in the center da/dc is just zero, so we are left with an ordinary differential equation.

Solving this we find

$$P_{ext}=\frac{2}{3}\,\sigma_o(c/b)^3-2\sigma_o\ell n(c/a)+\frac{(1-\nu)}{(1-2\nu)}\,\sigma_o(c/a)^3+F \tag{26}$$

where F is the constant of integration. We can calculate F from the boundary condition that $P_{ext}=P^*_{ext}$ for c=a. This gives

$$P_{ext}=\frac{(1-\nu)}{(1-2\nu)}(c/a)^3\sigma_o-2\sigma_o\ell n(c/a)+\frac{2}{3}(c/b)^3\sigma_o-\frac{2}{3}\sigma_o \quad . \tag{27}$$

For the fully plastic conditions

$$P^{**}_{ext}=\frac{(1-\nu)}{(1-2\nu)}\sigma_o K^3-2\sigma_o\ell nK \quad . \tag{28}$$

This result agrees with Equation (19) in spite of the approximations used in obtaining (28).

CONCLUSIONS

The pressure required to cause plastic deformation throughout a homogeneous elastically and plastically isotropic sphere surrounding a concentric rigid sphere is shown to be very large relative to the pressure required for the onset of plastic flow. In fact, the ratio of these pressures approaches $1.5K^3$ for no strain hardening, where K is the radius ratio, for large K. Note that the ratio already approximates the large K limit for K=2 and K=3. For such values of K, strain hardening effects usually can be neglected, while for much larger K strain hardening would have to be taken into account as would also nonlinear elastic effects.

For values of K of two or larger, the volume fraction of the matrix, which has undergone plastic deformation, increases nearly linearly with pressure above the critical pressure for the onset of yielding.

Acknowledgment

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References

1. D. A. Jones, and J. W. Mitchell, Phil. Mag. 3, 1 (1958).
2. R. Hill, The Mathematical Theory of Plasticity, Oxford at the Clarendon Press, London, (1950) Chapter V, p. 103.

Figure Legends

Figure 1. Volume fraction of matrix which is plastically deformed versus pressure for vaious K values. Here $\nu=1/3$.

Figure 2. Ratio of internal pressure to external pressure vs applied pressure for K=2 and 3 for $\nu=1/3$. The single asterisk indicates the point of initiation of plastic flow. The double asterisk indicates the point at which the sphere is fully plastic. The dashed line is the region in which Equation (20) holds.

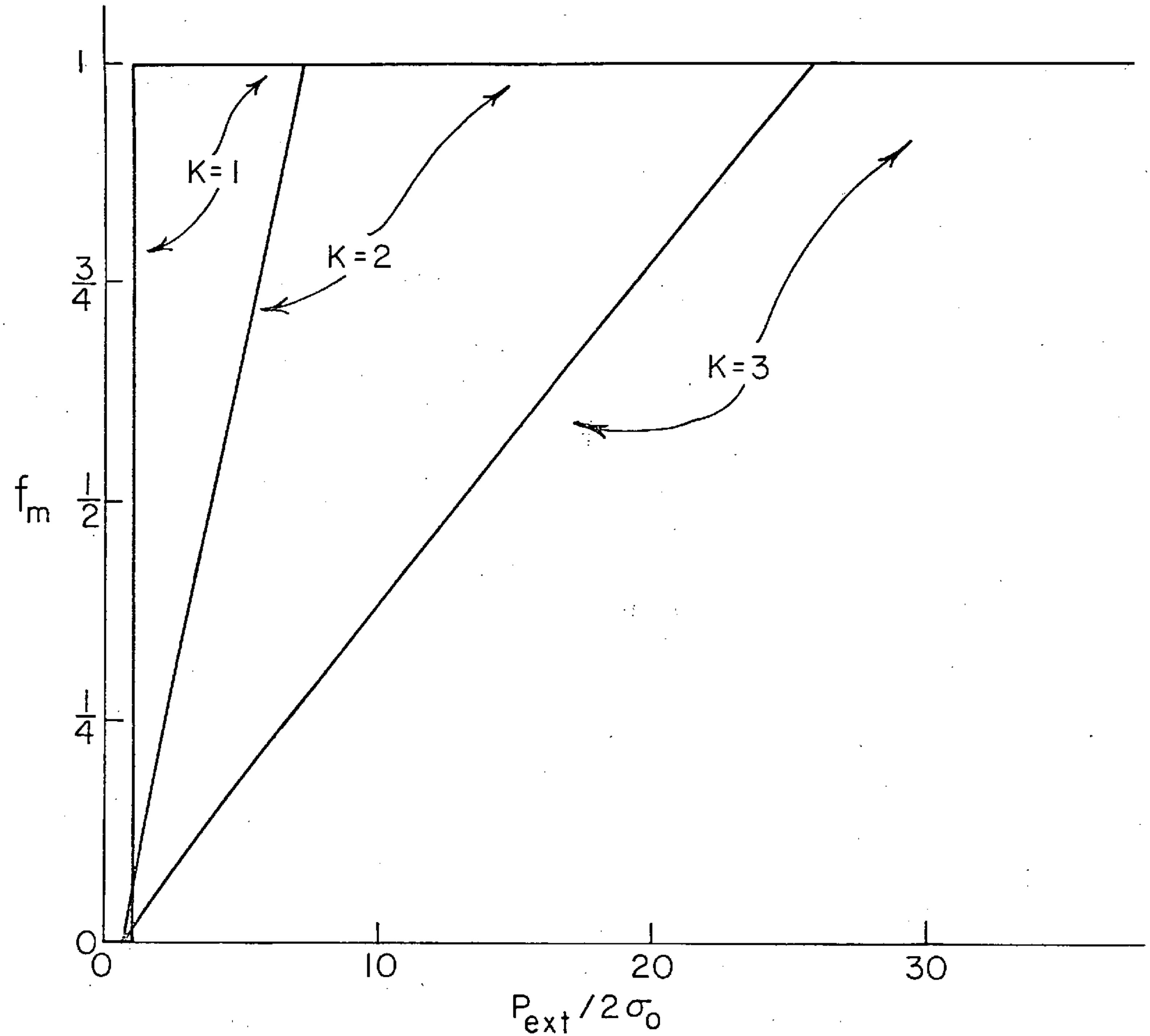

Figure 1.

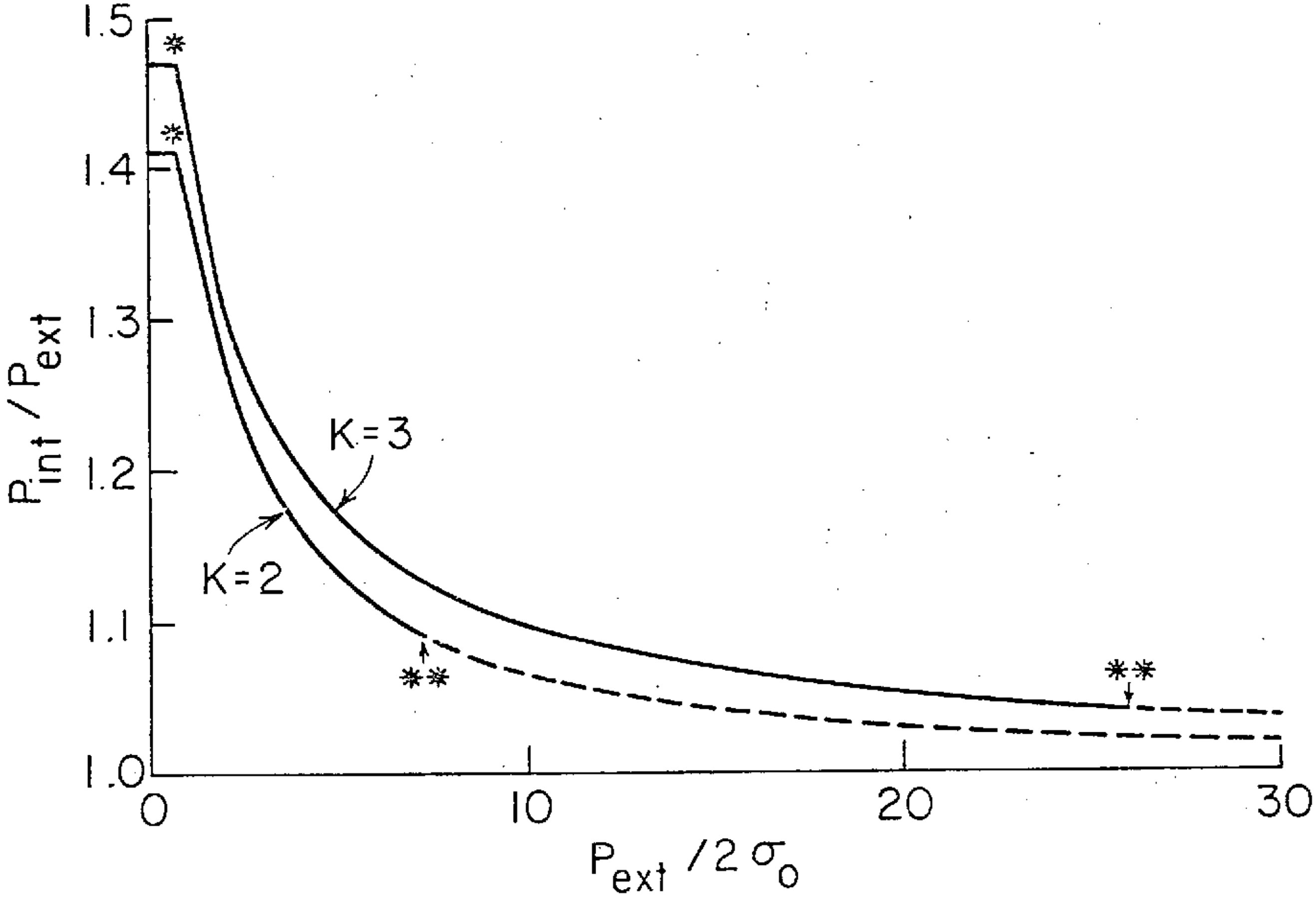

Figure 2.