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STABILITY ANALYSIS OF NONLINEAR SYSTEMS BY MULTIPLE TIME SCALING

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ABSTRACT

The asymptotic solution for the transient analysis of a general nonlinear system in the neighborhood of the stability boundary is obtained by using the multiple-time-scaling asymptotic- expansion method. The nonlinearities are assumed to be of algebraic nature. Terms of order ε^3 (where ε is the order of amplitude of the unknown) are included in the solution. The solution indicates that there always exists a limit cycle. The limit cycle is stable (unstable) and exists above (below) the stability boundary if the nonlinear terms are stabilizing (destabilizing). Extension of the solution to include fifth order nonlinear terms is also presented. Comparisons with harmonic balance and with multiple-time-scaling solution of panel flutter equations are also included.

FOREWARD

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SECTION I

INTRODUCTION

1.1 Regular Perturbation Methods

This work deals with the stability analysis of nonlinear systems by a singular perturbation method, the multiple time scaling method. In order to motivate the use of this method, it is convenient to consider first the regular perturbation method. For the sake of simplicity, this method is used for solving a simple equation, the Duffin equation.

$$\ddot{X} + X + \varepsilon X^3 = 0 \qquad 0 < \varepsilon \ll 1 \tag{1.1}$$

which does not yield any instability, since it is conservative. The analysis of this equation is useful to understand the meaning of certain terms (secular terms) and hence to motivate the use of the singular perturbation methods. As well known, Eq. (1.1) represents a mass-spring system (unit mass with non-linear spring) and has a periodic solution (elliptic functions). It may be noted that Eq. (1.1) has an energy integral

$$\frac{\dot{X}^2}{2} + \frac{X^2}{2} + \varepsilon \frac{X^4}{4} = \text{constant} \tag{1.2}$$

which implies that the energy (and hence the amplitude of vibration) remains bounded.

According to a well known theorem (Poincare theorem[1]), the solution of Eq. (1.1) depends analytically upon the parameter and hence may be expressed in the form

$$x = x(t,\varepsilon) = \sum_0^\infty \varepsilon^n x_n(t)$$
$$\ddot{x} = \ddot{x}(t,\varepsilon) = \sum_0^\infty \varepsilon^n \ddot{x}_n(t) \tag{1.3}$$

where

$$x_n(t) = \frac{1}{n!} \left.\frac{\partial^n x}{\partial \varepsilon^n}\right|_{\varepsilon=0} \tag{1.4}$$

The regular perturbation method consists in the use of Eq. (1.3) for solving Eq. (1.1). Combining Eqs. (1.1) and (1.3) yields

$$\sum_0^\infty \varepsilon^n \ddot{x}_n + \sum_0^\infty \varepsilon^n x_n + \varepsilon \left(\sum_0^\infty \varepsilon^n x_n\right)^3 \equiv$$
$$(\ddot{x}_0 + x_0) + \varepsilon(\ddot{x}_1 + x_1 + x_0^3) + \varepsilon^2(\ddot{x}_2 + x_2 + 3x_0^2 x_1) + \ldots = 0 \tag{1.5}$$

which implies

$$\ddot{x}_0 + x_0 = 0 \tag{1.6a}$$

$$\ddot{x}_1 + x_1 + x_0^3 = 0 \tag{1.6b}$$

$$\ddot{x}_2 + x_2 + 3x_0^2 x_1 = 0 \tag{1.6c}$$

The solution of Eq. (1.6a) is:

$$x_0 = a e^{it} + \bar{a} e^{-it} \tag{1.7}$$

where a is a complex member and $\bar{a}$ is its conjugate. Combining Eqs. (1.6b) and (1.7), one obtains

$$\ddot{x}_1 + x_1 + (a^3 e^{i3t} + 3a^2\bar{a} e^{it} + 3a\bar{a}^2 e^{-it} + \bar{a}^3 e^{-i3t}) = 0 \tag{1.8}$$

The solution of Eq. (1.8) is

$$x_1 = \left[b e^{it} + \frac{1}{8} a^3 e^{i3t} + i\frac{3}{2} a^2\bar{a} t e^{it}\right] + \text{C.T.} \tag{1.9}$$

where C.T. represents the Conjugate Term of the term in brackets.

Hence, the solution is given by

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2) = \Big[(a+\varepsilon b)\, e^{it} + \varepsilon \tfrac{1}{8} a^3 e^{i3t} + \varepsilon i \tfrac{3}{2} a^2 \bar{a} t\, e^{it}\Big] + C.T. + O(\varepsilon^2) \qquad (1.10)$$

The third term on the right hand side grows in time beyond any limit. Terms of this type are called <u>secular terms.</u>[1] Clearly, if the series is truncated to terms of second order, the solution does not have the expected properties since it is not periodic and does not remain bounded in time.

1.2 Singular Perturbation Methods

In order to circumvent the above mentioned problem, several methods were introduced. These methods are called singular perturbation methods and are based upon the concept of asymptotic expansions.[2] Well-known are the Krylov and Boliubov[3] method (averaging method) and the Cole and Kevorkian[4] method (two-variable expansion). More recent are the multiple scaling method (Refs. 5-9) and the Lie transform method (Refs. 10-14). These last two methods are used in Ref. 15 to analyze the stability of nonlinear systems. The multiple time scaling is considered here.* This method consists of assuming that the solution be of the form

$$\begin{aligned} x &= x_0(t_0, t_1, t_2, \ldots) + \varepsilon x_1(t_0, t_1, t_2, \ldots) + \ldots \\ &= \sum_0^\infty \varepsilon^n x_n(t_0, t_1, t_2, \ldots) \end{aligned} \qquad (1.11)$$

*A comparison of the multiple time scaling with the harmonic balance method is given in Appendix A.

where

$$t_m = \varepsilon^m t \tag{1.12}$$

are the multiple scales and are treated as independent variables. The functional dependence of the x_n upon the t_m is obtained by imposing that the solution contains no secular terms. It may be noted that Eq. (1.12) implies that

$$\frac{d}{dt} = \sum_0^\infty \frac{\partial}{\partial t_n} \frac{dt_n}{dt} = \sum_0^\infty \varepsilon^n \frac{\partial}{\partial t_n} \tag{1.13}$$

Combining Eqs. (1.1), (1.11), and (1.13), one obtains

$$\left[\frac{\partial^2}{\partial t_0^2} + \varepsilon\, 2\frac{\partial^2}{\partial t_0 \partial t_1} + \varepsilon^2\left(\frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_0 \partial t_2}\right) \dots\right]\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2\right)$$

$$+ \left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots\right) + \varepsilon x_0^3 + \varepsilon^2 3 x_0^2 x_1 + \dots = 0 \tag{1.14}$$

which implies

$$\frac{\partial^2 x_0}{\partial t_0^2} + x_0 = 0 \tag{1.15a}$$

$$\frac{\partial^2 x_1}{\partial t_0^2} + 2\frac{\partial^2 x_0}{\partial t_0 \partial t_1} + x_1 + x_0^3 = 0 \tag{1.15b}$$

$$\frac{\partial^2 x_2}{\partial t_0^2} + 2\frac{\partial^2 x_1}{\partial t_0 \partial t_1} + \frac{\partial^2 x_0}{\partial t_1^2} + 2\frac{\partial^2 x_0}{\partial t_0 \partial t_2} + x_2 + 3x_0^2 x_1 = 0 \tag{1.15c}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The solution of (1.15a) is

$$x_0 = a e^{it_0} + \bar{a} e^{-it_0} \tag{1.16}$$

where a and its conjugate $\bar{a}$ are functions of t_1, t_2, ...

Combining Eqs. (1.15b) and (1.16), one obtains

$$\frac{\partial^2 x_1}{\partial t_0} + x_1 = -\left[2i\frac{\partial a}{\partial t_1} e^{it_0} + a^3 e^{i3t_0} + 3a^2\bar{a} e^{it}\right] + \text{C.T.} \tag{1.17}$$

where C.T. indicates the conjugate term. In order to avoid secular terms, the condition

$$2i\frac{\partial a}{\partial t_1} + 3a^2\bar{a} = 0 \tag{1.18}$$

must be imposed. The solution of Eq. (1.18) is obtained by setting

$$a = |a|e^{i\varphi} \tag{1.19}$$

and separating real and imaginary parts. This yields

$$\frac{\partial\varphi}{\partial t_1} = \frac{3}{2}|a|^2 \tag{1.20a}$$

$$\frac{\partial|a|}{\partial t_1} = 0 \tag{1.20b}$$

or

$$a = |a|e^{i(\frac{3}{2}|a|^2 t_1 + \varphi_0)} = a_0 e^{i\frac{3}{2}a_0\bar{a}_0 t_1} \tag{1.21}$$

where

$$a_0 = |a|e^{i\varphi_0} \tag{1.22}$$

is a function of t_2, t_3, ... If Eq. (1.18) is satisfied, the solution of Eq. (1.17) is given by

$$X_1 = be^{it_0} + \frac{1}{8}a^3 e^{i3t_0} + \text{C.T.} \tag{1.23}$$

Combining Eq. (1.11), (1.16), (1.21) and (1.23), one obtains

$$X = a_0 e^{i(t_0 + \frac{3}{2}a_0\bar{a}_0 t_1)} + \varepsilon\left(be^{it_0} + \frac{1}{8}a_0^3 e^{i3t_0}\right) + O(\varepsilon^2) + \text{C.T.} \tag{1.24}$$

which is equivalent to

$$X = (a_0 + \varepsilon b + \ldots)e^{i\omega t} + \varepsilon\left(\frac{a_0^3}{8} + \ldots\right)e^{i3\omega t} + \text{C.T.} + O(\varepsilon^2) \tag{1.25}$$

with

$$\omega = 1 + \varepsilon \tfrac{3}{2} a_0 \bar{a}_0 + \ldots \tag{1.26}$$

The analysis may be continued with the solution of Eq. (1.15). This yields another term in the expression for ω and a harmonic of the type $e^{i5\omega t}$.

1.3 Comments

It may be noted that the Taylor series of Eq. (1.26) yields

$$x = a_0 e^{it} + \varepsilon\left[b e^{it} + \tfrac{1}{8} a^3 e^{i3t} + i\tfrac{3}{2} a_0^2 \bar{a}_0 t e^{it}\right] + O(\varepsilon^2) \tag{1.27}$$

in agreement with Eq. (1.10). Hence, "the appearance of the secular terms does not mean that the series solution does actually diverge if one considers the whole series, just in the same manner in which the whole series $t - (t^3/3!) + (t^5/5!)\ldots$ converges to sin (t). As it is impossible to build enough terms by successive approximations to be able to ascertain this fact, the appearance of secular terms renders the method impracticable," (Ref. 1, p. 219). The term te^{it} is due to the fact that the difference between two sinusoidal functions having amplitude and frequency slightly different, grows (initially) linear in t. Obviously, solutions which contain secular terms are meaningful only if the complete series is known. Hence, they have limited interest from the practical point of view.

It may be noted that the Duffin Equation has been considered only as an example in order to introduce the concepts of secular terms and singular perturbation methods. The results obtained above may be obtained with more elementary methods, such as Lindstedt's small perturbation method (Ref. 1, p. 224).

SECTION II

ONE-DEGREE-OF-FREEDOM SYSTEMS

2.1 Van der Pol Equation

As it will be shown in Section III, a characteristic behavior of nonlinear systems in the neighborhood of the stability boundary is given by the existence of limit cycles. The simplest equation to yield a limit cycle solution is the Van der Pol equation.

Consider the two forms of the Van der Pol equation[2]

$$\frac{d^2x}{dt^2} - \varepsilon \frac{dx}{dt} + \varepsilon \frac{1}{3}\left(\frac{dx}{dt}\right)^3 + x = 0 \tag{2.1}$$

and

$$\frac{d^2y}{dt^2} - \varepsilon \frac{dy}{dt}\left(1 - y^2\right) + y = 0 \tag{2.2}$$

Equation (2.2) can be obtained by differentiating Eq. (2.1) with respect to t and setting

$$y = \frac{dx}{dt} \tag{2.3}$$

For simplicity, the multiple scaling method is applied to Eq. (2.1). Similar results are obtained for Eq. (2.2) (Ref. 5). Combining Eqs. (1.11), (1.13), and (2.1) yields

$$\left(\frac{\partial^2}{\partial t_0^2} + \varepsilon\, 2\frac{\partial^2}{\partial t_0 \partial t_1} + \ldots\right)\left(x_0 + \varepsilon x_1 + \ldots\right) + \left(x_0 + \varepsilon x_1 + \ldots\right)$$
$$- \varepsilon\left(\frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \ldots\right)\left(x_0 + \varepsilon x_1 + \ldots\right) + \varepsilon\left[\left(\frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \ldots\right)\left(x_0 + \varepsilon x_1 + \ldots\right)\right]^3 = 0 \tag{2.4}$$

which implies

$$\frac{\partial^2 x_0}{\partial t_0^2} + x_0 = 0 \tag{2.5a}$$

$$\frac{\partial^2 x_1}{\partial t_0^2} + x_1 = -2\frac{\partial^2 x_0}{\partial t_0 \partial t_1} + \frac{\partial x_0}{\partial t_0} - \frac{1}{3}\left(\frac{\partial x_0}{\partial t_0}\right)^3 \tag{2.5b}$$

. . . .

The solution of Eq. (2.5a) is given by

$$x_0 = a e^{it_0} + \bar{a} e^{-it_0} \tag{2.6}$$

where a and $\bar{a}$ are functions of t_1, t_2, ... Combining Eqs. (2.5b) and (2.6), one obtains

$$\frac{\partial^2 x_1}{\partial t_0^2} + x_1 = -2i\frac{\partial a}{\partial t_1} e^{it_0} + i a e^{it_0} + \frac{1}{3} i a^3 e^{i3t_0} - i a^2 \bar{a} e^{it_0} + C.T. \tag{2.7}$$

The condition for avoiding secular terms is

$$-2\frac{\partial a}{\partial t_1} + a - a^2\bar{a} = 0 \tag{2.8}$$

By setting

$$a = |a| e^{i\varphi} \tag{2.9}$$

and separating real and imaginary parts, one obtains

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$$2\frac{\partial |a|}{\partial t_1} - |a| + |a|^3 = 0 \tag{2.10a}$$

$$\frac{\partial \varphi}{\partial t_1} = 0 \tag{2.10b}$$

or $\varphi = \varphi_0$ and*

$$a = \sqrt{\frac{1}{1+Ke^{-t_1}}}\, e^{i\varphi_0} \tag{2.12}$$

where k and φ_0 are functions of t_2, ... Hence, the solution of Eq. (2.1) is given by

$$x = \sqrt{\frac{1}{1+Ke^{-t_1}}}\left(e^{i(t+\varphi_0)} + e^{-i(t+\varphi_0)}\right) + O(\varepsilon) \tag{2.13}$$

The function

$$f_1(t) = \sqrt{\frac{1}{1+Ke^{-t_1}}} = \sqrt{\frac{1}{1+S_k e^{-\tau}}} \tag{2.14}$$

---

*Setting $u = \frac{1}{|a|^2}$, one obtains $\frac{\partial u}{\partial t_1} = -2\frac{1}{|a|^3}\frac{\partial |a|}{\partial t_1} = -\frac{1}{|a|^3}\left(|a| - |a|^3\right) = -u+1$ or $u = 1 + Ke^{-t_1}$ which is equivalent to Eq. (2.12).

with

$$S_K = \mathrm{sgn}(K) \tag{2.15a}$$

$$T = t_1 - \ln|K| \tag{2.15b}$$

is plotted in Fig. 1 which shows clearly that the solution of Eq. (2.13) tends to a limit cycle

$$X_0(t) = 2\cos(t+\varphi_0) \tag{2.16}$$

independent of the initial conditions (except for the phase angle φ_0).

2.2 Generalized Van der Pol Equations

A one-degree-of-freedom system which has many characteristics of the N-degrees-of-freedom systems (see Section III) is the generalized Van der Pol equation.

$$\ddot{X} + X = -2\varepsilon\left[\left(\beta_R \dot{X} - \beta_I X\right) + \frac{1}{3}\left(\gamma_R \dot{X}^3 - \gamma_I X^3\right)\right] \tag{2.17}$$

Combining Eqs. (1.11), (1.13), and (2.17) and separating different powers of ε yields

$$\frac{\partial^2 X_0}{\partial t_0^2} + X_0 = 0 \tag{2.18a}$$

$$\frac{\partial^2 X_1}{\partial t_0^2} + X_1 = -2\left[\frac{\partial^2 X_0}{\partial t_0 \partial t_1} + \beta_R \frac{\partial X_0}{\partial t_0} - \beta_I X_0 + \frac{1}{3}\gamma_R\left(\frac{\partial X_0}{\partial t_0}\right)^3 - \frac{1}{3}\gamma_I X_0^3\right] \tag{2.18b}$$

$$\cdots$$

The solution of Eq. (2.18) is given by Eq. (2.6), which when combined with Eq. (2.18b) yields

$$\frac{\partial^2 X_1}{\partial t_0^2} + X_1 = -2\Big[i\frac{\partial a}{\partial t_1}e^{it_0} + \beta_R i a e^{it_0} - \beta_I a e^{it_0} + \frac{1}{3}\gamma_R\left(-i a^3 e^{i3t_0} + 3 i a^2 \bar{a} e^{it_0}\right) - \frac{1}{3}\gamma_I\left(a^3 e^{i3t_0} + 3a^2\bar{a}\, e^{it_0}\right)\Big] + \text{C.T.} \tag{2.19}$$

The condition for the elimination of the secular terms is

$$\frac{\partial a}{\partial t_1} + (\beta_R + i\beta_I)a + (\gamma_R + i\gamma_I)a^2\bar{a} = 0 \tag{2.20}$$

By using Eq. (2.9) and separating real and imaginary parts, one obtains

$$\frac{\partial |a|}{\partial t_1} + \beta_R |a| + \gamma_R |a|^3 = 0 \tag{2.21a}$$

$$\frac{\partial \varphi}{\partial t_1} + \beta_I + \gamma_I |a|^2 = 0 \tag{2.21b}$$

The solution of Eq. (2.21a) is*

$$|a| = \sqrt{\frac{-\beta_R/\gamma_R}{1+Ke^{2\beta_R t_1}}} \tag{2.22}$$

with k function of t_2. Substituting into Eq. (2.21b) and integrating yields

$$\varphi = -\left(\frac{\beta_I}{\beta_R} - \frac{\gamma_I}{\gamma_R}\right)\beta_R t_1 + \frac{\gamma_I}{\gamma_R} \ln |A| + \varphi_0 \tag{2.23}$$

with φ_0 function of t_2. Finally, combining Eqs. (1.11) and (1.13) with Eqs. (2.6), (2.22), and (2.23), one obtains

$$x = |a| 2\cos\left[(1+\varepsilon\omega_1)t + \frac{\gamma_I}{\gamma_R} \ln|a| + \varphi_0\right] \tag{2.24}$$

with $\omega_1 = -\beta_I + \gamma_I \beta_R/\gamma_R$.

The behavior of the solution depends upon the signs of β_R and γ_R

$$S_\beta = \text{sgn}(\beta_R) \tag{2.25a}$$

$$S_\gamma = \text{sgn}(\gamma_R) \tag{2.25b}$$

*See footnote to Eq. (2.12).

Note that Eq. (2.22) is equivalent to

$$\frac{|a|}{|\beta_R/\gamma_R|^{\frac{1}{2}}} = \sqrt{\frac{-S_\beta S_\gamma}{1+S_\kappa e^{S_\beta \tau}}} \tag{2.26}$$

with

$$\tau = 2|\beta_R|\,t_1 + S_\beta \ln|K| \tag{2.27}$$

The discussion of the solution behavior is divided into four cases, depending upon the signs of β_R and γ_R.

Consider first the case analogous to the Van der Pol equation, namely $\beta_R \leq 0$ and $\gamma_R \geq 0$ (destabilizing linear terms and stabilizing nonlinear terms). In this case, Eq. (2.26) yields

$$\frac{|a|}{|\beta_R/\gamma_R|^{\frac{1}{2}}} = f_1(\tau) = \sqrt{\frac{1}{1+S_\kappa e^{-\tau}}} \tag{2.28}$$

with $f_1(\tau)$ plotted in Fig. 1. In this case, the solution tends to a limit cycle given by

$$x = 2|a|_\infty \cos(\omega t + \varphi_\infty) \tag{2.29}$$

with

$$|a|_\infty = \sqrt{-\beta_R/\gamma_R} \tag{2.30a}$$

$$\omega = 1 + \varepsilon\omega_1 \tag{2.30b}$$

$$\varphi_\infty = \frac{\gamma_I}{\gamma_R} \ln|a|_\infty + \varphi_0 \tag{2.30c}$$

Next, consider the case with both (linear and nonlinear) terms stabilizing ($\beta_R > 0$ and $\gamma_R > 0$). In this case, Eq. (2.26) yields

$$\frac{|a|}{|\beta_R/\gamma_R|^{\frac{1}{2}}} = f_2(\tau) = \sqrt{\frac{-1}{1+S_\kappa e^{\tau}}} \tag{2.31}$$

The function $f_2(\tau)$ (real only for $S_\kappa < 0$, $\tau > 0$), is plotted in

Fig. 2, from which it is evident that the solution is unconditionally stable.

Third, consider the case with both (linear and nonlinear) terms destabilizing ($\beta_R < 0$ and $\gamma_R < 0$). In this case, Eq. (2.26) yields

$$\frac{|a|}{|\beta_R/\gamma_R|^{\frac{1}{2}}} = f_3(\tau) = \sqrt{\frac{-1}{1+S_k e^{-\tau}}} \qquad (2.32)$$

The function $f_3(\tau)$ (real only for $K < 0$, $\tau < 0$) is shown in Fig. 3, from which it is apparent that the solution is unconditionally unstable, and goes to infinity in finite time.

Finally consider the case with stabilizing linear terms ($\beta_R > 0$) and destabilizing nonlinear terms ($\gamma_R < 0$). In this case, Eq. (2.26) yields

$$\frac{a}{\sqrt{-\beta_R/\gamma_R}} = f_4(\tau) = \sqrt{\frac{1}{1+S_k e^{\tau}}} \qquad (2.33)$$

The function $f_4(\tau)$ is plotted in Fig. 4, from which it is apparent that there exists an unstable limit cycle given by

$$x = 2|a|_{-\infty} \cos(\omega t + \varphi_{-\infty}) \qquad (2.34)$$

with

$$|a|_{-\infty} = \sqrt{-\beta_R/\gamma_R} \qquad (2.35a)$$

$$\omega = 1 + \varepsilon\omega_1 \qquad (2.35b)$$

$$\varphi_{-\infty} = \frac{\gamma_I}{\gamma_R} \ln|a|_{-\infty} + \varphi_0 \qquad (2.35c)$$

Equations (2.35) are analogous to Eqs. (2.30). The subscript $-\infty$ is used to indicate that the unstable limit cycle is reached as $t \to -\infty$.

SECTION III

N-DEGREES-OF-FREEDOM SYSTEMS

3.1 Introduction

Consider an autonomous N-degrees-of-freedom system of the type

$$\dot{\mathbf{x}} + \mathbf{A}(\lambda)\mathbf{x} = \mathbf{f}(\mathbf{x}) \tag{3.1}$$

where $\mathbf{A}$ is a matrix, function of a parameter λ, and $\mathbf{f}(\mathbf{x})$ is the vector of the nonlinear terms. Assume that the linear system

$$\dot{\mathbf{x}} + \mathbf{A}(\lambda)\mathbf{x} = 0 \tag{3.2}$$

is such that all the eigenvalues p_i of the matrix, A given by

$$\mathrm{Det}\left(p\,\mathbf{I} + \mathbf{A}(\lambda)\right) = 0 \tag{3.3}$$

are stable (negative real part) for $\lambda < \lambda_o$, while for $\lambda = \lambda_o$ there exists a pair of eigenvalues $p_i = \pm i\omega$ and for ~~$\lambda \geq \lambda_o$~~ this pair of eigenvalues becomes unstable (positive real part). The value λ_o defines the stability limit. For $\lambda = \lambda_o$, the solution of Eq. (3.2) is given by

$$\mathbf{x} = \mathbf{u}\,a\,e^{i\omega t} + \bar{\mathbf{u}}\,\bar{a}\,e^{-i\omega t} + \mathrm{D.T.} \tag{3.4}$$

where D.T. indicates the Damped Terms (eigenvalues with negative real part), $\mathbf{u}$ is the eigenvector relative to the eigenvalue $\lambda = i\omega$, that is the nontrivial solution of the system,

$$\left(i\omega\,\mathbf{I} + \mathbf{A}(\lambda_o)\right)\mathbf{u} = 0 \tag{3.5}$$

The stability analysis of nonlinear systems considered here deals with the study of Eq. (3.1) in the neighborhood of the value λ_o.

Assume that the matrix **A** is an analytic function of λ in the neighborhood of λ_o.

$$\mathbf{A} = \mathbf{A}_o + \mathbf{A}'(\lambda-\lambda_o) + \mathbf{A}''(\lambda-\lambda_o)^2 + \ldots \tag{3.6}$$

and the nonlinear terms vector is an analytic function of **x**

$$\mathbf{f} = \{f_n\} = \Big\{ \sum_{pq} b_{npq} x_p x_q + \sum_{pqr} c_{npqr} x_p x_q x_r + \sum_{pqrs} d_{npqrs} x_p x_q x_r x_s + \sum_{pqrst} e_{npqrst} x_p x_q x_r x_s x_t + \ldots \Big\} \tag{3.7}$$

3.2 Formulation of Problem

Assume for simplicity that the nonlinear terms are only of odd order*, namely

$$b_{npq} \equiv d_{npqrs} \equiv 0 \tag{3.8}$$

In this case, if ε is the order of magnitude of **x**, the solution is of the type

$$\mathbf{x} = \varepsilon \mathbf{x}_1 + \varepsilon^3 \mathbf{x}_3 + \varepsilon^5 \mathbf{x}_5 \tag{3.9}$$

where $\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_5, \ldots$ are functions of $t_0, t_2, t_4, \ldots$ (for, the odd-order scales do not appear in the solution[6]). Hence,

$$\frac{d}{dt} = \frac{\partial}{\partial t_o} + \varepsilon^2 \frac{\partial}{\partial t_2} + \varepsilon^4 \frac{\partial}{\partial t_4} \tag{3.10}$$

Furthermore, it is convenient to set

$$\lambda = \lambda_o + \varepsilon^2 \lambda_2 + \varepsilon^4 \lambda_4 + \ldots \tag{3.11}$$

where

$$\lambda_2 = \pm 1 \tag{3.12}$$

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*If the even-order nonlinear terms are different from zero, the analysis is only formally more complicated, but the results are essentially the same (Appendix B).

and λ_4 to be determined (see Subsection 4.3). Combining Eq. (3.1) with Eqs. (3.6) through (3.11), one obtains

$$\left(\frac{\partial}{\partial t_0} + \varepsilon^2 \frac{\partial}{\partial t_2} + \varepsilon^4 \frac{\partial}{\partial t_4} + \dots\right)\left(\varepsilon \mathbf{X}_1 + \varepsilon^3 \mathbf{X}_3 + \varepsilon^5 \mathbf{X}_5 + \dots\right)$$

$$+ \left(\mathbf{A}_0 + \varepsilon^2 \mathbf{A}_2 + \varepsilon^4 \mathbf{A}_4 + \dots\right)\left(\varepsilon \mathbf{X}_1 + \varepsilon^3 \mathbf{X}_3 + \varepsilon^5 \mathbf{X}_5 + \dots\right)$$

$$= \varepsilon^3 \mathbf{f}_3 + \varepsilon^5 \mathbf{f}_5 + \dots\dots \tag{3.13}$$

with

$$\mathbf{A}_2 = \lambda_2 \mathbf{A}' \tag{3.14a}$$

$$\mathbf{A}_4 = \lambda_4 \mathbf{A}' + \mathbf{A}'' \tag{3.14b}$$

and

$$\mathbf{f}_3 = \left\{f_{3,n}\right\} = \left\{\sum_{pqr} C_{npqr}\, X_{1,p}\, X_{1,q}\, X_{1,r}\right\} \tag{3.15a}$$

$$\begin{aligned}\mathbf{f}_5 = \left\{f_{5,n}\right\} = \Big\{ &\sum_{pqr} C_{npqr} \left(X_{1,p} X_{1,q} X_{3,r} + X_{1,p} X_{3,q} X_{1,r} + X_{3,p} X_{1,q} X_{1,r}\right) \\ &+ \sum_{pqrst} E_{npqrst} \left(X_{1,p} X_{1,q} X_{1,r} X_{1,s} X_{1,t}\right)\Big\}\end{aligned} \tag{3.15b}$$

$$\vdots$$

Equation (3.13) implies

$$\frac{\partial \mathbf{x}_1}{\partial t_0} + \mathbf{A}_0 \mathbf{x}_1 = 0 \tag{3.16a}$$

$$\frac{\partial \mathbf{x}_3}{\partial t_0} + \mathbf{A}_0 \mathbf{x}_3 = \mathbf{f}_3 - \frac{\partial \mathbf{x}_1}{\partial t_2} - \mathbf{A}_2 \mathbf{x}_1 \tag{3.16b}$$

$$\frac{\partial \mathbf{x}_5}{\partial t_0} + \mathbf{A}_0 \mathbf{x}_5 = \mathbf{f}_5 - \frac{\partial \mathbf{x}_3}{\partial t_2} - \frac{\partial \mathbf{x}_1}{\partial t_4} - \mathbf{A}_2 \mathbf{x}_3 - \mathbf{A}_4 \mathbf{x}_1 \tag{3.16c}$$

$$\dots\dots$$

3.3 Third Order Solution

The solution of Eqs. (3.16) including third order terms, is considered here. The effect of fifth order terms is considered in Section 4. The solution to Eq. (3.16) is given by

$$\mathbf{x}_1 = \mathbf{u} a e^{i\omega t_0} + \bar{\mathbf{u}} \bar{a} e^{-i\omega t_0} \tag{3.17}$$

In Eq. (3.17), the damped terms have been disregarded, since, as shown for instance in Ref. 6, these terms do not have any effect on the solution as $t \rightarrow \infty$.

By combining Eqs. (3.16b) and (3.17), one obtains

$$\begin{aligned} \frac{\partial \mathbf{x}_3}{\partial t_0} + \mathbf{A}_0 \mathbf{x}_3 = \mathbf{f}_3^{(3)} a^3 e^{i3\omega t} + \mathbf{f}_3^{(1)} a^2 \bar{a} e^{i\omega t} - \\ \mathbf{u} \frac{\partial a}{\partial t_2} e^{i\omega t} - \mathbf{A}_2 \mathbf{u}\, a e^{i\omega t} + \text{C.T.} \end{aligned} \tag{3.18}$$

with

$$\mathbf{f}_3^{(3)} = \left\{ \sum_{pqr} C_{npqr} u_p u_q u_r \right\} \tag{3.19a}$$

$$\mathbf{f}_3^{(1)} = \left\{ \sum_{pqr} C_{npqr} \bar{u}_p u_q u_r + u_p \bar{u}_q u_r + u_p u_q \bar{u}_r \right\} \tag{3.19b}$$

The condition for the elimination of the secular terms is that the component of the vector

$$\mathbf{z}_3^{(1)} = -\mathbf{u} \frac{\partial a}{\partial t_2} - \mathbf{A}_2 \mathbf{u} a + \mathbf{f}_3^{(1)} a^2 \bar{a} \tag{3.20a}$$

in the direction of the vector $\mathbf{u}$ is equal to zero.[6] In other words, the inner product between the vector $\mathbf{z}_3^{(1)}$ and the eigenvector $\mathbf{u}_A$ of the adjoint operator ($\bar{\mathbf{A}}^T$ is the conjugate of the transpose of $\mathbf{A}$), defined by

$$\left(-i\omega \mathbf{I} + \bar{\mathbf{A}}^T\right) \mathbf{u}_A = 0 \tag{3.20b}$$

must be equal to zero[6] or

$$\left(\mathbf{u}_A, \mathbf{z}_3^{(1)}\right) = \bar{\mathbf{u}}_A^T \mathbf{z}_3^{(1)} = 0 \tag{3.21}$$

Assuming for simplicity that the vector $\mathbf{u}_A$ is normalized by the condition

$$(\mathbf{u}_A, \mathbf{u}) = \bar{\mathbf{u}}_A^T \mathbf{u} = 1 \tag{3.22}$$

Equation (3.21) is equivalent to the condition

$$\frac{\partial a}{\partial t_2} + \beta a + \gamma a^2 \bar{a} = 0 \tag{3.23a}$$

with

$$\beta = (\mathbf{u}_A, \mathbf{A}_2 \mathbf{u}) = \bar{\mathbf{u}}_A^T \mathbf{A}_2 \mathbf{u} \tag{3.23b}$$

$$\gamma = -(\mathbf{u}_A, \mathbf{f}_3^{(1)}) = -\bar{\mathbf{u}}_A^T \mathbf{f}_3^{(1)} \tag{3.23c}$$

Equation (3.23a) is similar to Eq. (2.20) and the solution is given by

$$a = |a| e^{i(\omega_2 t_2 + \frac{\gamma_I}{\gamma_R} \ln|a| + \varphi_0)} \tag{3.24}$$

with

$$|a| = \sqrt{\frac{-\beta_R/\gamma_R}{1 + K e^{2\beta_R t_2}}} \tag{3.25a}$$

$$\omega_2 = \beta_R \frac{\gamma_I}{\gamma_R} - \beta_I \tag{3.25b}$$

where φ_0 and k are functions of t_4, t_6 ... and β_R, β_I, γ_R, and γ_I are, respectively, the real and imaginary parts of β and γ. Finally, the solution of Eq. (3.18), under the condition expressed by Eq. (3.23) is given by

$$\mathbf{x}_3 = \mathbf{p}_3^{(3)} e^{i3\omega t_0} + (b\mathbf{u} + \mathbf{p}_3^{(1)}) e^{i\omega t_0} \tag{3.26}$$

where b is a function of t_2, t_4 .... (see Section 4) and $\mathbf{p}_3^{(3)}$ and $\mathbf{p}_3^{(1)}$ are the solutions of the systems,

$$\left[i3\omega\mathbf{I}+\mathbf{A}_0\right]\mathbf{p}_3^{(3)} = \mathbf{f}_3^{(3)}a^3 \tag{3.27a}$$

$$\left[i\omega\mathbf{I}+\mathbf{A}_0\right]\mathbf{p}_3^{(1)} = \mathbf{z}_3^{(1)} = \mathbf{f}_3^{(1)}a^2\bar{a} - \mathbf{u}\frac{\partial a}{\partial t_2} - \mathbf{A}_2\mathbf{u}\,a \tag{3.27b}$$

respectively. The vector $\mathbf{p}_3{}^{(1)}$ contains an arbitrary constant which can be eliminated by setting, for convenience,

$$\left(\mathbf{u}_A, \mathbf{p}_3^{(1)}\right) = \bar{\mathbf{u}}_A{}^T\mathbf{p}_3^{(1)} = 0 \tag{3.28}$$

3.4 Comments

Note that the solution of Eq. (3.16b) is important not as much for obtaining the solution for $\mathbf{x}_3(t_0)$ (see Eq. 3.26), but for obtaining the functional dependence of $\mathbf{x}_1$ upon t_2. Eq. (3.26) contains the function $b(t_2\ldots.)$ which can be obtained only by studying Eq. (3.16c) for the unknown $\mathbf{x}_5(t_0)$. Hence, consider the solution which can be written as

$$\mathbf{x} = \varepsilon\left(\mathbf{u}ae^{i\omega t} + \bar{\mathbf{u}}\bar{a}\,e^{-i\omega t}\right) + O(\varepsilon^3) \tag{3.29}$$

where a is given by Eq. (3.24). In Eq. (3.29), the damped terms have been disregarded. Furthermore, still neglecting higher order terms, ε is related to λ by

$$\lambda = \lambda_0 + \varepsilon^2\lambda_2 \tag{3.30}$$

with $\lambda_2 = \pm 1$, or

$$\varepsilon = \sqrt{|\lambda-\lambda_0|} \tag{3.31}$$

$$\lambda_2 = \mathrm{sgn}\,(\lambda-\lambda_0) \tag{3.32}$$

It should be noted that by definition of λ, the linear terms are stabilizing (which implies $\beta_R > 0$) for $\lambda<\lambda_0$ $(\lambda_2 = -1)$ and vice versa for $\lambda > \lambda_o$. Combining Eqs. (3.23a) and (3.14a), one obtains

$$\beta = \lambda_2(\mathbf{u}_A, \mathbf{A}'\mathbf{u}) \tag{3.33}$$

Hence $\beta_R > 0$ for $\lambda_2 = -1$, implies that the real part of $(u_A, A'u)$ is negative. Hence, the discussion based on the results of Section 2, reduces to two cases only, $\gamma_R > 0$ and $\gamma_R < 0$, respectively. In the first case, $\gamma_R > 0$ (stabilizing nonlinear terms), the solution is unconditionally stable for $\lambda < \lambda_o$ (Fig. 2) while for $\lambda > \lambda_o$ there exists a stable limit cycle (Fig. 1) given by

$$\mathbf{x} = \left(\mathbf{u} e^{i(\omega t + \varphi_\infty)} + \bar{\mathbf{u}} e^{-i(\omega t + \varphi_\infty)}\right) \tag{3.34}$$

with φ_∞ given by Eq. (2.30c). This behavior is summarized in Fig. 5 where the amplitude of the limit cycle

$$h_{L.C.} = \sqrt{(\lambda - \lambda_o)\left|\frac{\beta_R}{\gamma_R}\right|} \tag{3.35}$$

is plotted and the different trends of the solution (for $\lambda > \lambda_o$ and $\lambda < \lambda_o$) are indicated.

In the second case, $\gamma_R < 0$ (destabilizing nonlinear terms), the solution is unconditionally stable (Fig. 3) for $\lambda > \lambda_o$, while, for $\lambda < \lambda_o$, there exists an unstable limit cycle (Fig. 4) given by

$$\mathbf{x} = \sqrt{(\lambda - \lambda_o)\frac{\beta_R}{\gamma_R}}\left(\mathbf{u} e^{i(\omega t + \varphi_{-\infty})} + \bar{\mathbf{u}} e^{-i(\omega t + \varphi_{-\infty})}\right) \tag{3.36}$$

with $\varphi_{-\infty}$ given by Eq. (2.35c). This behavior is summarized in

Fig. 6 where the amplitude of the limit cycle

$$h_{L.C.} = \sqrt{(\lambda-\lambda_o)\frac{\beta_R}{\gamma_R}} \tag{3.37}$$

is plotted and the different trends of the solution (for $\lambda > \lambda_o$ and $\lambda < \lambda_o$) are indicated.

3.5 Application to Aeroelasticity

The system of equations for panel flutter is of the type[7]

$$\ddot{\mathbf{y}} + \mathbf{G}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \lambda\mathbf{E}\mathbf{y} = \mathbf{g} \tag{3.38}$$

with

$$\mathbf{G} = \hat{\mathbf{G}} + \sqrt{\lambda}\,\tilde{\mathbf{G}} \tag{3.39a}$$

$$\mathbf{g} = \{g_n\} = \left\{ \sum_{pqr=1} C_{npqr}\, y_p\, y_q\, y_r \right\} \tag{3.39b}$$

Equation (3.38) is equivalent to the system

$$\dot{\mathbf{y}} = \mathbf{z} \tag{3.40a}$$

$$\dot{\mathbf{z}} + \mathbf{G}\mathbf{z} + \mathbf{K}\mathbf{y} + \lambda\mathbf{E}\mathbf{y} = \mathbf{g} \tag{3.40b}$$

which is of the type of Eq. (3.1)

$$\mathbf{x} = \left\{ \begin{array}{c} \mathbf{y} \\ \hline \mathbf{z} \end{array} \right\} \tag{3.41a}$$

$$\mathbf{f} = \left\{ \begin{array}{c} 0 \\ \hline \mathbf{g} \end{array} \right\} \tag{3.41b}$$

and finally

$$\mathbf{A} = \left[\begin{array}{c|c} 0 & -\mathbf{I} \\ \hline \mathbf{K}+\lambda\mathbf{E} & \hat{\mathbf{G}}+\sqrt{\lambda}\,\tilde{\mathbf{G}} \end{array} \right] = \mathbf{A}_o + \mathbf{A}'(\lambda-\lambda_o) + \ldots \tag{3.42}$$

with

$$\mathbf{A}_o = \left[\begin{array}{c|c} \mathbf{O} & -\mathbf{I} \\ \hline \mathbf{K}_o & \mathbf{G}_o \end{array}\right] \tag{3.43a}$$

$$\mathbf{A}' = \left[\begin{array}{c|c} \mathbf{O} & \mathbf{O} \\ \hline \mathbf{E} & \mathbf{G}' \end{array}\right] \tag{3.43b}$$

where $\mathbf{G}_o = \hat{\mathbf{G}} + \sqrt{\lambda_o}\,\tilde{\mathbf{G}}$, $\mathbf{G}' = \frac{1}{2\sqrt{\lambda_o}}\,\tilde{\mathbf{G}}$ and $\mathbf{K}_o = \mathbf{K} + \lambda_o \mathbf{E}_o$

Furthermore, the eigenvector $\mathbf{u}$ is given by

$$\mathbf{u} = \left\{\begin{array}{c} \mathbf{v} \\ \hline \mathbf{w} \end{array}\right\} \tag{3.44}$$

where $\mathbf{v}$ and $\mathbf{w}$ can be obtained from

$$i\omega\mathbf{v} - \mathbf{w} = 0 \tag{3.45a}$$

$$i\omega\mathbf{w} + (\mathbf{G}_o\mathbf{w} + \mathbf{K}_o\mathbf{v}) = 0 \tag{3.45b}$$

This implies that $\mathbf{v}$ is the eigenvector of the original system

$$\left[-\omega^2\mathbf{I} + i\omega\,\mathbf{G}_o + \mathbf{K}_o\right]\mathbf{v} = 0 \tag{3.46}$$

while $\mathbf{w}$ is given by Eq. (3.45a) and hence

$$\mathbf{u} = \left\{\begin{array}{c} \mathbf{v} \\ \hline i\omega\mathbf{v} \end{array}\right\} \tag{3.47}$$

Similarly, the eigenvector $\mathbf{u}_A^T$ is given by

$$\bar{\mathbf{u}}_A^T\left[i\omega\mathbf{I} + \mathbf{A}\right] = 0 \tag{3.48}$$

Setting

$$\bar{\mathbf{u}}_A^T = \left\lfloor \hat{\mathbf{w}}^L \,\middle|\, \hat{\mathbf{v}}^L \right\rfloor \tag{3.49}$$

one obtains

$$i\omega\hat{\mathbf{w}}^T + \hat{\mathbf{v}}^T\mathbf{K}_o = 0 \tag{3.50a}$$

$$i\omega\hat{\mathbf{v}}^T - \hat{\mathbf{w}}^T + \hat{\mathbf{w}}^T\mathbf{G}_o = 0 \tag{3.50b}$$

or, eliminating $\hat{\mathbf{w}}^T$

$$\hat{\mathbf{v}}^T\left[-\omega^2\mathbf{I}+i\omega\mathbf{G}_o+\mathbf{K}_o\right]=0 \tag{3.51}$$

In other words, the vector $\hat{\mathbf{v}}^T$ is the left-eigenvector of the original systems and the vector $\bar{\mathbf{u}}_A{}^T$ is given by

$$\bar{\mathbf{u}}_A{}^T=\lfloor\hat{\mathbf{v}}^T(i\omega\mathbf{I}+\mathbf{G}_o)\,|\,\hat{\mathbf{v}}^T\rfloor=\lfloor\tfrac{1}{i\omega}\hat{\mathbf{v}}^T\mathbf{K}_o\,|\,\hat{\mathbf{v}}^T\rfloor \tag{3.52}$$

Finally, by using the first expression for $\bar{\mathbf{u}}_A{}^T$, Eq. (3.22) is equivalent to

$$\begin{aligned}\bar{\mathbf{u}}_A{}^T\mathbf{u}&=\lfloor\hat{\mathbf{v}}^T(i\omega\mathbf{I}+\mathbf{G}_o)\,|\,\hat{\mathbf{v}}^T\rfloor\begin{Bmatrix}\mathbf{v}\\ i\omega\mathbf{v}\end{Bmatrix}\\ &=\hat{\mathbf{v}}^T(2i\omega\mathbf{I}+\mathbf{G})\mathbf{v}=1\end{aligned} \tag{3.53}$$

while Eqs. (3.23) are equivalent to

$$\begin{aligned}\beta=\lambda_2\bar{\mathbf{u}}_A{}^T\mathbf{A}'\mathbf{u}&=\lambda_2\lfloor\hat{\mathbf{w}}^T\,|\,\hat{\mathbf{v}}^T\rfloor\begin{bmatrix}0 & 0\\ \mathbf{E} & \mathbf{G}'\end{bmatrix}\begin{Bmatrix}\mathbf{v}\\ i\omega\mathbf{v}\end{Bmatrix}\\ &=\lambda_2\hat{\mathbf{v}}^T(\mathbf{E}+i\omega\mathbf{G}')\mathbf{v}\end{aligned} \tag{3.54a}$$

$$\gamma=\bar{\mathbf{u}}_A{}^T\mathbf{f}_3^{(1)}=\lfloor\hat{\mathbf{w}}^T\,|\,\hat{\mathbf{v}}^T\rfloor\begin{Bmatrix}0\\ \mathbf{h}_1\end{Bmatrix}=\hat{\mathbf{v}}^T\mathbf{h}_1 \tag{3.54b}$$

with

$$\mathbf{h}_1=\left\{\sum_{pqr}C_{npqr}\left(\bar{u}_p u_q u_r+u_p\bar{u}_q u_r+u_p u_q\bar{u}_r\right)\right\} \tag{3.55}$$

in agreement with the results of Refs. 6 and 7.

SECTION IV

FIFTH ORDER TERMS

4.1 Introduction

In this section, the effect of the fifth order nonlinear terms is analyzed. Combining Eqs. (3.16c), (3.17), and (3.25) yields

$$\frac{\partial \mathbf{x}_5}{\partial t_0} + \mathbf{A}_0 \mathbf{x}_5 = \mathbf{f}_5 - \frac{\partial}{\partial t_2}\left[\mathbf{p}_3^{(3)} e^{i3\omega t_0} + (\mathbf{p}_3^{(1)} + \mathbf{u} b) e^{i\omega t_0}\right]$$
$$- \mathbf{A}_2\left[\mathbf{p}_3^{(3)} e^{i3\omega t_0} + (\mathbf{p}_3^{(1)} + \mathbf{u} b) e^{i\omega t_0}\right]$$
$$- \frac{\partial a}{\partial t_4} \mathbf{u} e^{i\omega t_0} - \mathbf{A}_4 \mathbf{u} a e^{i\omega t_0} \tag{4.1}$$

Furthermore, according to Eq. (3.15b),

$$\mathbf{f}_5 = \left\{ \sum_{pqr} c_{npqr} \left(X_{1,p} X_{1,q} X_{1,r} + X_{1,p} X_{3,q} X_{1,r} + X_{3,p} X_{1,q} X_{1,r} \right) + \sum_{pqrst} e_{npqrst} \left(X_{1,p} X_{1,q} X_{1,r} X_{1,s} X_{1,t} \right) \right\} \tag{4.2}$$
$$= \left\{ \sum_{pqr} \hat{c}_{npqr} X_{1,p} X_{1,q} X_{3r} + \sum_{pqrst} e_{npqrst} X_{1,p} X_{1,q} X_{1,r} X_{1,s} X_{1,t} \right\}$$

where

$$\hat{c}_{npqr} = c_{npqr} + c_{nprq} + c_{nrqp} \tag{4.3}$$

By combining Eqs. (4.2), (3.17), and (3.26), one obtains

$$\mathbf{f}_5 = \{f_{5,n}\} = \Big\{ \sum_{pqr} \hat{c}_{npqr} \Big[a^2 e^{i2\omega t_o} u_p u_q + a\bar{a}(\bar{u}_p u_q + u_p \bar{u}_q) + \bar{a}^2 \bar{u}_p \bar{u}_q e^{-i2\omega t_o} \Big] \Big[\mathbf{p}_{3,r}^{(3)} e^{i3\omega t_o} + (b\mathbf{u}_r + \mathbf{p}_{3,r}^{(1)}) e^{i\omega t_o} + C.T. \Big]$$
$$+ \sum_{pqrst} e_{npqrst} \Big[(a^5 u_p u_q u_r u_s u_t) e^{i5\omega t_o} + a^4 \bar{a} (\bar{u}_p u_q u_r u_s u_t + u_p \bar{u}_q u_r u_s u_t + u_p u_q \bar{u}_r u_s u_t + \ldots + u_p u_q u_r u_s \bar{u}_t) e^{i3\omega t_o} + a^3 \bar{a}^2 (\bar{u}_p \bar{u}_q u_r u_s u_t + \bar{u}_p u_q \bar{u}_r u_s u_t + \ldots + u_p u_q u_r \bar{u}_s \bar{u}_t) e^{i\omega t_o} + C.T. \Big] \Big\} \quad (4.4)$$

$$= \mathbf{f}_5^{(5)} e^{i5\omega t_o} + \mathbf{f}_5^{(3)} e^{i3\omega t_o} + \mathbf{f}_5^{(1)} e^{i\omega t_o} + C.T.$$

where

$$\mathbf{f}_5^{(1)} = \Big\{ \sum_{pqr} \hat{c}_{npqr} \Big[a^2 u_p u_q (\bar{b}\bar{u}_r + \bar{\mathbf{p}}_{3,r}^{(1)}) + a\bar{a}(\bar{u}_p u_q + u_p \bar{u}_q) \times (b\mathbf{u}_r + \mathbf{p}_{3,r}^{(1)}) + \bar{a}^2 \bar{u}_p \bar{u}_q \mathbf{p}_{3,r}^{(3)} \Big] + \sum_{pqrst} e_{npqrst}\, a^3 \bar{a}^2 (\bar{u}_p \bar{u}_q u_r u_s u_t + \bar{u}_p u_q \bar{u}_r u_s u_t + \ldots + u_p u_q u_r \bar{u}_s \bar{u}_t) \Big\} \quad (4.5)$$

Similar expressions hold for $\mathbf{f}_5^{(3)}$ and $\mathbf{f}_5^{(5)}$. It may be noted that Eq. (3.27a) may be rewritten as

$$\mathbf{p}_3^{(3)} = a^3 \hat{\mathbf{p}}_3^{(3)} \quad (4.6)$$

with $\hat{\mathbf{p}}_3^{(3)}$ independent of a, while combining Eqs. (3.27b) and (3.23a), one may write

$$\mathbf{p}_3^{(1)} = a\hat{\mathbf{p}}_3^{(1)} + a^2\bar{a}\check{\mathbf{p}}_3^{(1)} \tag{4.7}$$

with $\hat{\mathbf{p}}_1^{(1)}$ and $\check{\mathbf{p}}_3^{(1)}$ independent of a. In summary, it is possible to rewrite Eq. (4.5) as

$$\mathbf{f}_5^{(1)} = (a^2\bar{b} + 2a\bar{a}b)\mathbf{f}_3^{(1)} + a\hat{\mathbf{f}}_5^{(1)} + a^2\bar{a}\check{\mathbf{f}}_5^{(1)} + a^3\bar{a}^2\tilde{\mathbf{f}}_5^{(1)} \tag{4.8}$$

where $\mathbf{f}_3^{(1)}$ is given by Eq. (3.19b) or, according to Eq. (4.3)

$$\begin{aligned}\mathbf{f}_3^{(1)} &= \left\{\sum_{pqr} C_{npqr}\left(\bar{u}_p u_q u_r + u_p \bar{u}_q u_r + u_p u_q \bar{u}_r\right)\right\} \\ &= \left\{\sum_{pqr} \hat{C}_{npqr}\, u_p u_q \bar{u}_r\right\}\end{aligned} \tag{4.9}$$

On the other hand, the explicit expressions for $\mathbf{f}_3$ and $\mathbf{f}_5$ (independent of a) can be obtained by combining Eqs. (4.5), (4.6), and (4.7). Finally, combining Eqs. (4.1) and (4.9) and noting that according to Eqs. (3.23) and (4.7), it is possible to write

$$\frac{\partial \mathbf{p}_3^{(1)}}{\partial t_2} = \hat{\hat{\mathbf{p}}}_3^{(1)} a + \check{\check{\mathbf{p}}}_3^{(1)} a^2\bar{a} + \tilde{\tilde{\mathbf{p}}}_3^{(1)} a^3\bar{a}^2 \tag{4.10}$$

(with $\hat{\hat{\mathbf{p}}}_3^{(1)}$, $\check{\check{\mathbf{p}}}_3^{(1)}$, and $\tilde{\tilde{\mathbf{p}}}_3^{(1)}$ independent of a), one obtains

$$\frac{\partial \mathbf{x}_5}{\partial t_o} + \mathbf{A}_o\mathbf{x}_5 = \mathbf{z}_5^{(5)} e^{i5\omega t_o} + \mathbf{z}_5^{(3)} e^{i3\omega t_o} + \mathbf{z}_5^{(1)} e^{i\omega t_o} + \mathrm{C.T.} \tag{4.11}$$

where

$$\mathbf{z}_5^{(4)} = -\Big[\mathbf{u}\frac{\partial b}{\partial t_2} + \mathbf{A}_2\mathbf{u}b - \mathbf{f}_3^{(1)}(a^2b + 2a\bar{a}b) + \mathbf{u}\frac{\partial a}{\partial t_4}$$

$$+ \big(\mathbf{A}_4\mathbf{u} + \hat{\hat{\mathbf{p}}}_3^{(1)} + \mathbf{A}_2\hat{\mathbf{p}}_3^{(1)} - \hat{\mathbf{f}}_5^{(1)}\big)a$$

$$+ \big(\check{\check{\mathbf{p}}}_3^{(1)} + \mathbf{A}_2\check{\mathbf{p}}_0^{(1)} - \check{\mathbf{f}}_5^{(1)}\big)a^2\bar{a} + \big(\tilde{\tilde{\mathbf{p}}}_3^{(1)} - \mathbf{f}_5^{(1)}\big)a^3\bar{a}^2\Big] \tag{4.12}$$

Similar expressions hold for $\mathbf{z}_5^{(5)}$ and $\mathbf{z}_5^{(3)}$.

The non-secular-terms condition is

$$(\mathbf{u}_A, \mathbf{z}_5^{(1)}) = \bar{\mathbf{u}}_A^T\mathbf{z}_5^{(1)} = 0 \tag{4.13}$$

or, according to Eqs. (3.22), (3.23a), and (3.23b),

$$\frac{\partial b}{\partial t_2} + \beta b + \gamma(a^2b + 2a\bar{a}b)$$

$$= \delta a + \delta' a^2\bar{a} + \delta'' a^3\bar{a}^2 - \frac{\partial a}{\partial t_4} \tag{4.14}$$

where

$$\delta = -\bar{\mathbf{u}}_A^T\big(\mathbf{A}_4\mathbf{u} + \hat{\hat{\mathbf{p}}}_3^{(1)} + \mathbf{A}_2\hat{\mathbf{p}}_3^{(1)} - \mathbf{f}_5^{(1)}\big) \tag{4.15a}$$

$$\delta' = -\bar{\mathbf{u}}_A^T\big(\check{\check{\mathbf{p}}}_3^{(1)} + \mathbf{A}_2\check{\mathbf{p}}_3^{(1)} - \check{\mathbf{f}}_5^{(1)}\big) \tag{4.15b}$$

$$\delta'' = -\bar{\mathbf{u}}_A^T\big(\tilde{\tilde{\mathbf{p}}}_3^{(1)} - \mathbf{f}_5^{(1)}\big) \tag{4.15c}$$

The solution of Eq. (4.14) is discussed in Subsection 4.2.

Finally, if Eq. (4.14) is satisfied, the solution of Eq. (4.11) is given by

$$\mathbf{X}_5 = \mathbf{p}_5^{(5)} e^{i5\omega t_0} + \mathbf{p}_5^{(3)} e^{i3\omega t_0} + \left(\mathbf{p}_5^{(1)} + c\,\mathbf{u}\right) e^{i\omega t_0} + \text{C.T.} \tag{4.16}$$

where $\mathbf{p}_5^{(K)}$ is the solution of the algebraic system

$$\left[iK\omega\mathbf{I} + \mathbf{A}_0\right]\mathbf{p}_5^{(K)} = \mathbf{z}_5^{(K)} \tag{4.17}$$

4.2 The Function $b(t_2)$

Consider Eq. (4.14) using Eq. (2.9)

$$a = |a| e^{i\varphi} \tag{4.18}$$

setting

$$b = b' e^{i\varphi} \tag{4.19}$$

and dividing by $e^{i\varphi}$, yields

$$\frac{\partial b'}{\partial t_2} + b' i \frac{\partial \varphi}{\partial t_2} + \beta b' + \gamma\left(|a|^2 \bar{b}' + 2|a|^2 b'\right)$$

$$= \delta|a| + \delta'|a|^3 + \delta''|a|^5 - \frac{\partial |a|}{\partial t_4} - i|a|\frac{\partial \varphi}{\partial t_4} \tag{4.20}$$

The imaginary part of Eq. (3.23) yields (see also Eq. 2.21b)

$$\frac{\partial \varphi}{\partial t_2} = -\beta_I - \gamma_I |a|^2 \tag{4.21}$$

Combining this equation with Eq. (4.20) yields

$$\frac{\partial b'}{\partial t_2} + \beta_R b' + \gamma_R |a|^2 b' + \gamma |a|^2 (b' + \bar{b}')$$

$$= \delta|a| + \delta'|a|^3 + \delta''|a|^5 - \frac{\partial |a|}{\partial t_4} - i|a|\frac{\partial \varphi}{\partial t_4} \tag{4.22}$$

Next, by separating real and imaginary parts, one obtains

$$\frac{\partial b_R'}{\partial t_2} + \beta_R b_R' + 3\gamma_R |a|^2 b_R'$$

$$= \delta_R |a| + \delta_R' |a|^3 + \delta_R'' |a|^5 - \frac{\partial |a|}{\partial t_4} \quad (4.23a)$$

$$\frac{\partial b_I'}{\partial t_2} + \beta_R b_I' + \gamma_R |a|^2 b_I' + 2\gamma_I |a|^2 b_R'$$

$$= \delta_I |a| + \delta_I' |a|^3 + \delta_I'' |a|^5 - |a| \frac{\partial \varphi}{\partial t_4} \quad (4.23b)$$

Finally, by setting —

$$b_R' = B_R \frac{\partial |a|}{\partial t_2} \quad (4.24a)$$

$$b_I' = B_I |a| \quad (4.24b)$$

and using the real part of Eq. (3.23) (see also Eq. 2.21a)

$$\frac{\partial |a|}{\partial t_2} + \beta_R |a| + \gamma_R |a|^3 = 0 \quad (4.25)$$

and its derivative with respect to t_2 —

$$\frac{\partial^2 |a|}{\partial t_2^2} + \beta_R \frac{\partial |a|}{\partial t_2} + 3\gamma_R |a|^2 \frac{\partial |a|}{\partial t_2} = 0 \quad (4.26)$$

one obtains

$$\frac{\partial B_R}{\partial t_2} = \frac{\delta_R |a| + \delta_R' |a|^3 + \delta_R'' |a|^5 - \partial |a| / \partial t_4}{\partial |a| / \partial t_2} \quad (4.27a)$$

$$\frac{\partial B_I}{\partial t_2} + 2\gamma_I |a| b_R' = \delta_I + \delta_I' |a|^2 + \delta_I'' |a|^4 - \frac{\partial \varphi}{\partial t_4} \quad (4.27b)$$

It may be noted that, according to Eq. (3.25a),

$$|a| = \sqrt{\frac{-\beta_R/\delta_R}{1+Ke^{\tau}}} = \sqrt{\frac{1}{-\frac{\gamma_R}{\beta_R}+K_o e^{\tau}}} \tag{4.28}$$

with

$$K_o = -K\,\gamma_R/\beta_R \tag{4.29}$$

$$\tau = 2\beta_R t_2 \tag{4.30}$$

which yields

$$\frac{\partial |a|}{\partial t_2} = -\beta_R |a|^3 K_o e^{\tau} \tag{4.31}$$

and

$$\frac{\partial |a|}{\partial t_4} = -\frac{1}{2}|a|^3 \frac{\partial K_o}{\partial t_4} e^{\tau} \tag{4.32}$$

Combining Eqs. (4.27a), (4.28), (4.31), and (4.32) yields

$$\frac{\partial B_R}{\partial \tau} = -\frac{1}{2\beta_R^2 K_o}\left[\delta_R\left(-\frac{\gamma_R}{\beta_R}e^{-\tau}+K_o\right)+\delta_R' e^{-\tau} + \delta_R''\left(\frac{e^{-\tau}}{-\frac{\gamma_R}{\beta_R}+K_o e^{-\tau}}\right)\right.$$
$$\left. - \frac{1}{4\beta_R^2 K_o}\frac{\partial K_o}{\partial t_4}\right. \tag{4.33}$$

and hence by integrating, one obtains*

$$B_R = -\frac{B_R^{(0)}}{\beta_R} - \frac{\gamma_R}{2\beta_R^3 K_o}\left[\delta_R - \delta_R'\frac{\beta_R}{\gamma_R} + \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2\right]e^{-\tau}$$
$$-\frac{1}{2\gamma_R^2}\delta_R'' \ln|a|^{-2} - \frac{1}{2\beta_R^2}\left[\delta_R - \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2 + \frac{1}{2K_o}\frac{\partial K_o}{\partial t_4}\right]\tau \tag{4.34}$$

*Note that

$$\int \frac{e^{-\tau}}{c+K_o e^{\tau}}\,d\tau = -\frac{1}{c}e^{-\tau} - \frac{K_o}{c^2}\tau + \frac{K_o}{c^2}\ln(c+K_o e^{\tau})$$

where $B_R^{(0)}$ is a function of t_2 .....

The condition for non-secular terms* is

$$\delta_R - \delta_R'' \left(\frac{\beta_R}{\gamma_R}\right)^2 + \frac{1}{2K_0}\frac{\partial K_0}{\partial t_4} = 0 \tag{4.35}$$

Combining Eqs. (4.34), (4.35), (4.30), (4.25), and (4.24a) yields

$$\begin{aligned} b_R' &= \frac{-\gamma_R}{2\beta_R^3}\left[\delta_R - \delta_R' \frac{\beta_R}{\gamma_R} + \delta''\left(\frac{\beta_R}{\gamma_R}\right)^2\right]\left(-\beta_R|a|^3\right) \\ &\quad + \left(\frac{B_R^{(0)}}{\beta_R} - \frac{1}{\gamma_R^2}\delta_R'' \ln|a|\right)\left(\beta_R|a| + \gamma_R|a|^3\right) \\ &= B_R^{(0)}|a| + B_R^{(1)}|a|^3 + \left(B_R^{(2)}|a| + B_R^{(3)}|a|^3\right)\ln|a| \end{aligned} \tag{4.36}$$

where as mentioned above, $B_R^{(0)}$ is a function of t_2 ...., while

$$B_R^{(1)} = \frac{\gamma_R}{2\beta_R^2}\left[\delta_R - \delta_R' \frac{\beta_R}{\gamma_R} + \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2\right] + B_R^{(0)}\frac{\gamma_R}{\beta_R} \tag{4.37a}$$

$$B_R^{(2)} = -\frac{\beta_R}{\gamma_R^2}\delta_R'' \tag{4.37b}$$

$$B_R^{(3)} = -\frac{1}{\gamma_R}\delta_R'' \tag{4.37c}$$

Similarly, by imposing the condition (Appendix C, Eqs. C.13 and C.14)

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*The secular terms must be considered for $t\to\infty$ in the case of stable limit cycle, and for $t\to-\infty$ in the case of unstable limit cycle.

$$\frac{\partial \varphi_0}{\partial t_4} = \delta_I - \delta_I' \frac{\beta_R}{\gamma_R} + \delta_I'' \left(\frac{\beta_R}{\gamma_R}\right)^2 - \frac{\gamma_I}{\gamma_R}\left(\delta_R - \delta_R' \frac{\beta_R}{\gamma_R} + \delta_R'' \left(\frac{\beta_R}{\gamma_R}\right)^2\right) = \chi \tag{4.38}$$

one obtains (Appendix C, Eq. C.18)

$$b_I' = B_I^{(0)} |a| + B_I^{(1)} |a|^3 + \left(B_I^{(2)} |a| + B_I^{(3)} |a|^3\right) \ln |a| \tag{4.39}$$

where $B^{(0)}$ is a function of t_2 ...., while (Appendix C, Eq. C.17)

$$b_I^{(1)} = -\frac{1}{2\gamma_R}\left(\delta_I'' - \frac{\gamma_I}{\gamma_R}\delta_R'\right) + \frac{\gamma_I}{\gamma_R} B_R^{(1)} \tag{4.40a}$$

$$b_I^{(2)} = -\frac{1}{\gamma_R}\left(\delta_I' - \frac{\beta_R}{\gamma_R}\delta_I''\right) + \frac{\gamma_I}{\gamma_R^2}\left(\delta_R' - 2\frac{\beta_R}{\gamma_R}\delta_R''\right) \tag{4.40b}$$

$$b_I^{(3)} = \frac{\gamma_I}{\gamma_R} b_R^{(3)} \tag{4.40c}$$

Finally, combining Eqs. (4.19), (4.36), and (4.39) yields

$$b = B^{(0)} a + B^{(1)} a^2 \bar{a} + \left(B^{(2)} a + B^{(3)} a^2 \bar{a}\right) \ln |a| \tag{4.41}$$

with

$$B^{(K)} = B_R^{(K)} + i\, B_I^{(K)} \tag{4.42}$$

4.3 The Function $a(t_2, t_4 \; ....)$

The solution of Eq. (4.35) is

$$K_0 = K_1 e^{\sigma t_4} \tag{4.43}$$

with

$$\sigma = -2\left[\delta_R - \delta_R'' \left(\frac{\beta_R}{\gamma_R}\right)^2\right] \tag{4.44}$$

while the solution of Eq. (4.35) is

$$\varphi_0 = \chi t_4 + \varphi_1 \tag{4.45}$$

Combining Eqs. (4.43), (4.45), (3.24), and (4.28) yields

$$a = |a| e^{i\varphi} \tag{4.46}$$

with

$$|a| = \sqrt{\frac{1}{-\frac{\gamma_R}{\beta_R} + K_1 e^{2\beta_R t_2 + \sigma t_4}}} \tag{4.47a}$$

$$\varphi = \omega_2 t_2 + \frac{\gamma_I}{\gamma_R} \ln|a| + \chi t_4 + \varphi_1 \tag{4.47b}$$

Finally, it is possible to determine the arbitrary constant λ_4 introduced in Eq. (3.11). Note that λ_4 appears only in A_4 which is given by Eq. (3.14b),

$$\mathbf{A}_4 = \lambda_4 \mathbf{A}' + \mathbf{A}'' = \lambda_4 \mathbf{A}_2 + \mathbf{A}'' \tag{4.48}$$

and that $\mathbf{A}_4$ appears only in δ, given by Eq. (4.15a),

$$\begin{aligned} \delta &= \bar{\mathbf{u}}_A^T \left(\frac{\lambda_4}{\lambda_2} \mathbf{A}_2 \bar{\mathbf{u}} + \mathbf{A}'' \bar{\mathbf{u}} + \hat{\mathbf{p}}_3^{(1)} + \mathbf{A}_2 \hat{\mathbf{p}}_3^{(1)} - \hat{\mathbf{f}}_5^{(1)} \right) \\ &= -\left(\frac{\lambda_4}{\lambda_2}\right)\beta + \delta_0 \end{aligned} \tag{4.49}$$

where δ_0 is the value for δ at $\lambda_4 = 0$ and β is given by Eq. (3.23a). Hence, Eq. (4.44) may be rewritten as

$$\begin{aligned} \sigma &= -2 \left[\frac{\lambda_4}{\lambda_2} \beta_R + \delta_{0_R} - \delta_R'' \left(\frac{\beta_R}{\gamma_R}\right)^2 \right] \\ &= 2\left(\frac{\lambda_4}{\lambda_2}\right)\beta_R + \sigma_0 \end{aligned} \tag{4.50}$$

and hence the exponent in Eq. (4.47a) is

$$\left(2\beta_R \left(1 + \varepsilon^2 \frac{\lambda_4}{\lambda_2}\right) + \varepsilon^2 \sigma_0\right) \varepsilon^2 t \tag{4.51}$$

If σ_0 has the opposite sign as β_R (which is positive for $\gamma_R < 0$ and negative for $\gamma_R > 0$) then the limit cycle switches from unstable to stable (or vice versa) for

$$\varepsilon_{CR}^2 = \frac{2\beta_R}{2\beta_R \frac{\lambda_4}{\lambda_2} + \sigma_0} \tag{4.52}$$

This value of ε must correspond to the knee of the curve $|a|$ as a function of λ, and hence must satisfy the condition

$$\frac{\partial \lambda}{\partial \varepsilon^2} = \lambda_2 + 2\lambda_4 \varepsilon_{CR}^2 = 0 \tag{4.53}$$

Combining Eqs. (4.52) and (4.53) yields

$$\lambda_4 = \frac{\sigma_0 \lambda_2}{2\beta_R} \tag{4.54}$$

Applications of these results to the panel-flutter problem are given in Ref. 8, where the equation

$$\ddot{x} + \left[\lambda + \mu(\dot{x}^2 + x^2) + \nu(\dot{x}^2 + x^2)^2\right]\dot{x} + x = 0 \tag{4.55}$$

(with $\lambda < 0$, $\mu > 0$, $\nu < 0$) is also considered. This equation yields two limit cycles (one stable and one unstable) given by (exact solution!)

$$x = X_k \cos(t + \phi_k) \qquad (k = 1, 2) \tag{4.56}$$

where X_k are the two positive roots of

$$\nu X^4 + \mu X^2 + \lambda = 0 \tag{4.57}$$

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APPENDIX A

COMPARISON WITH HARMONIC BALANCE METHOD

A.1 Introduction

In this Appendix, the multiple time scaling method is compared to the harmonic balance method. For simplicity, the analysis is limited to nonlinear terms of third order (see Eqs. 3.1, 3.7, and 3.8).

First, the system

$$\dot{x}_n + \sum_{p=1}^{N} A_{np}(\lambda) x_p = \sum_{pqr} e_{npqr} x_p x_q x_r + O(\varepsilon^5) \tag{A.1}$$

where ε is the order of magnitude of x_n, is analyzed by the harmonic balance method. The resulting system of algebraic equations is then solved by an algebraic perturbation method. Finally, it is shown that the results coincide with the ones obtained by using the multiple time scaling method.

Consider Eq. (A.1). According to the Harmonic Balance Method (which is equivalent to the Galerkin method), the solution of Eq. (A.1) can be obtained by setting

$$x_n = \sum_{p=1}^{P} \left(X_n^{[p]} e^{ip\omega t} + \bar{X}_n^{[p]} e^{-ip\omega t} \right) \tag{A.2}$$

(where $X_n^{[p]}$ is a complex number and $\bar{X}_n^{[p]}$ is its complex conjugate) and by "balancing" the first P harmonics of the resulting expression, i.e., by setting the coefficients of equal to zero for p = 1,2 ....P. For P = 1, and setting $X_n^{[1]} = X_n$

one obtains

$$i\omega X_n + \sum A_{np} X_p = \sum_{pqr} C_{npqr} \left(\bar{X}_p X_q X_r + X_p \bar{X}_q X_r + X_p X_q \bar{X}_r \right) + O(\varepsilon^5) \tag{A.3}$$

and the complex conjugate equation. This system of algebraic equations can be solved by a standard numerical technique such as the Newton-Raphson method, for instance, (Refs. 7 and 14). Here, Eq. (A.3) is solved by an algebraic perturbation method.

A.2 Algebraic Perturbation Method

Since x_n is of order ε, set

$$X_n = \varepsilon X_{n,1} + \varepsilon^3 X_{n,3} + O(\varepsilon^5) \tag{A.4}$$

By combining Eqs. (A.3) and (A.4) and neglecting higher order terms, one obtains

$$i\omega X_{n,1} + \sum_{p=1}^{N} A_{np}(\lambda) X_{p,1} = 0 \tag{A.5}$$

The solution of Eq. (A.5) is discussed in Section III (see Eq. 3.5) which shows that a solution exists for $\lambda = \lambda_o$ and $\omega = \omega_o$ and is given by

$$X_{n,1} = a\, u_n \tag{A.6}$$

where $\{u_n\}$ is the eigenvector of the matrix $[A_{np}(\lambda_o)]$, relative to the eigenvalue $i\omega_o$. Next set

$$\omega = \omega_o + \varepsilon^2 \omega_2 + O(\varepsilon^4) \tag{A.7}$$

$$\lambda = \lambda_o + \varepsilon^2 \lambda_2 + O(\varepsilon^4) \tag{A.8}$$

with $\lambda_2 = \pm 1$ (see Eqs. 3.11 and 3.12).

By using Eq. (A.8), one obtains (see Eqs. 3.6 and 3.14a),

$$A_{np}(\lambda) = A_{np}(\lambda_0) + \left(\frac{\partial A_{np}}{\partial \lambda}\right)_{\lambda=\lambda_0} \varepsilon^2 \lambda_2 + \ldots$$

$$= A_{np}^{(0)} + \varepsilon^2 A_{np}^{(2)} \tag{A.9}$$

with

$$\begin{aligned} A_{np}^{(0)} &= A_{np}(\lambda_0) \\ A_{np}^{(2)} &= \lambda_2 \left(\frac{\partial A_{np}}{\partial \lambda}\right)_{\lambda=\lambda_0} \end{aligned} \tag{A.10}$$

Combining Eqs. (A.3), (A.4), (A.7), and (A.9) and separating terms of the same order of magnitude yields

$$i\omega_0 X_{n,1} + \sum A_{np}^{(0)} X_{p,1} = 0 \tag{A.11a}$$

$$i\omega_0 X_{n,3} + \sum A_{np}^{(0)} X_{p,3} = Z_n \tag{A.11b}$$

with

$$Z_n = -i\omega_2 X_{n,1} - \sum A_{np}^{(2)} X_{p,1}$$
$$+ \sum_{pqr} C_{npqr} \left(\bar{X}_{p,1} X_{q,1} X_{r,1} + X_{p,1} \bar{X}_{q,1} X_{r,1} + X_{p,1} X_{q,1} \bar{X}_{r,1}\right) \tag{A.12}$$

The solution of Eq. (A.11a) is given by Eq. (A.6). Combining Eqs. (A.6) and (A.12) yields

$$Z_n = -i\omega_2 a u_n - a\sum A_{np}^{(2)} u_p$$
$$+ a^2 \bar{a} \sum_{pqr} C_{npqr} \left(\bar{u}_p u_q u_r + u_p \bar{u}_q u_r + u_p u_q \bar{u}_r\right) \tag{A.13}$$

Equation (A.11b) has a determinant equal to zero. Hence, the solution exists only if z_n is orthogonal to the left eigenvector $\{v_n\} = \bar{\mathbf{u}}_A$ (see Eq. 3.20b) of the matrix $\left[A_{np}^{(0)}\right]$

$$\sum_n v_n z_n = -i\omega_2 a \sum v_n u_n - a \sum_{np} v_n A_{np}^{(2)} u_p$$
$$+ a^2 \bar{a} \sum_{npqr} v_n C_{npqr} \left(\bar{u}_p u_q u_r + u_p \bar{u}_q u_r + u_p u_q \bar{u}_r \right) = 0 \tag{A.14}$$

or

$$i\omega_2 a + \beta a + \gamma a^2 \bar{a} = 0 \tag{A.15}$$

with

$$\beta = \frac{\sum_{np} v_n A_{np}^{(2)} u_p}{\sum v_n u_n} \tag{A.16}$$

$$\gamma = \frac{\sum_{npqr} v_n C_{npqr} \left(\bar{u}_p u_q u_r + u_p \bar{u}_q u_r + u_p u_q \bar{u}_r \right)}{\sum v_n u_n} \tag{A.17}$$

in agreement with Eqs. (3.22), (3.23a) and (3.23b). Separating the real and imaginary parts, Eq. (A.15) yields

$$\beta_R a + \gamma_R a^3 = 0 \tag{A.18}$$

$$\omega_2 a + \beta_I a + \gamma_I a^3 = 0 \tag{A.19}$$

Equation (A.18) yields

$$a = \sqrt{\frac{-\beta_R}{\gamma_R}} \tag{A.20}$$

(λ_2 , Eq. (3.12) must be chosen so that a is real) while Eq. (A.19) yields

$$\omega_2 = -\beta_I - \gamma_I a^2 = -\beta_I + \gamma_I \frac{\beta_R}{\gamma_R} \tag{A.21}$$

in agreement with Eqs. (2.30a) and (3.25b), respectively.

In conclusion, the harmonic balance method (combined with an algebraic perturbation technique) yields the same results as the multiple time scaling. It may be noted that the results are more easily derived with the harmonic balance method than with the multiple time scaling method. However, the harmonic balance method is limited to steady state (limit-cycle) solutions, while the multiple time scaling method yields (with no additional computational effort) also the envelope of the transient response and, in particular, the stability (or instability) or the limit cycle.

Finally, it may be worth noting a computational advantage that the algebraic-perturbation harmonic-balance method (as well as the multiple time scaling method) has with respect to the Newton-Raphson harmonic-balance method. Solving the linear system, Eq. (A.5), requires less time than solving the nonlinear system, Eq. (A.3). Moreover, the nonlinear system must be solved for different values of λ , while the linear system must be solved only once: the additional time to obtain the coefficients β and γ is negligible. Hence, the total time required by the Newton-Raphson harmonic balance method is N times larger than the one required by the algebraic-perturbation harmonic balance method (and multiple time scaling method) where N is the number of points used to describe the curve which gives the amplitude of the limit cycle versus λ . Note that the higher accuracy obtained by solving exactly Eq. (A.3) (Newton-Raphson harmonic balance) is only apparent since the difference between the two results is of the same order of magnitude, ε^5 , as the error in the original equation, Eq. (A.1).

APPENDIX B

SECOND ORDER NONLINEARITIES

B.1 Introduction

In the main body of this report Eq. (3.1) is analyzed under the assumption that the even order terms are equal to zero, Eq. (3.8). The solution of Eq. (3.16) including third order terms is considered in this Appendix without the assumptions $b_{npq} = 0$ and $d_{npqrs} = 0$. The solution is assumed to be given by

$$\mathbf{X} = \varepsilon \mathbf{X}_1 + \varepsilon^2 \mathbf{X}_2 + \varepsilon^3 \mathbf{X}_3 + O(\varepsilon^4) \tag{B.1}$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots$ are functions of $t_0, t_1, t_2, \ldots$ Hence

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \ldots \tag{B.2}$$

Combining Eqs. (3.1), (3.6), (3.7), (B.1), and (B.2), one obtains

$$\begin{aligned} &\left(\frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \ldots \right) \left(\varepsilon \mathbf{X}_1 + \varepsilon^2 \mathbf{X}_2 + \varepsilon^3 \mathbf{X}_3 + \ldots \right) \\ &+ \left(\mathbf{A}_0 + \varepsilon^2 \mathbf{A}_2 + \ldots \right) \left(\varepsilon \mathbf{X}_1 + \varepsilon^2 \mathbf{X}_2 + \varepsilon^3 \mathbf{X}_3 + \ldots \right) \\ &= \varepsilon^2 \mathbf{f}_2 + \varepsilon^3 \mathbf{f}_3 + \ldots \end{aligned} \tag{B.3}$$

with $\mathbf{A}_2$ and $\mathbf{A}_4$ given be Eq. (3.14), and

$$\mathbf{f}_2 = \{ f_{2,n} \} = \left\{ \sum_{pq} b_{npq} X_{1,p} X_{1,q} \right\} \tag{B.4a}$$

$$\mathbf{f}_3 = \{ f_{3,n} \} = \left\{ \sum_{pq} b_{npq} \left(X_{1,p} X_{2,q} + X_{2,p} X_{1,q} \right) + \sum_{pqr} c_{npqr} X_{1,p} X_{1,q} X_{1,r} \right\} \tag{B.4b}$$

Separating terms of the same order of magnitude yields

$$\frac{\partial \mathbf{X}_1}{\partial t_0} + \mathbf{A}_0 \mathbf{X}_1 = 0 \tag{B.5a}$$

$$\frac{\partial \mathbf{x}_2}{\partial t_0} + \mathbf{A}_0 \mathbf{x}_2 = \mathbf{f}_2 - \frac{\partial \mathbf{x}_1}{\partial t_1} \tag{B.5b}$$

$$\frac{\partial \mathbf{x}_3}{\partial t_0} + \mathbf{A}_0 \mathbf{x}_3 = \mathbf{f}_3 - \frac{\partial \mathbf{x}_1}{\partial t_2} - \frac{\partial \mathbf{x}_2}{\partial t_1} - \mathbf{A}_2 \mathbf{x}_1 \tag{B.5c}$$

B.2 Second Order Solution

The solution of Eq. (B.5a) is given by Eq. (3.17),

$$\mathbf{x}_1 = \mathbf{u} a e^{i\omega t_0} + \text{C.T.} \tag{B.6}$$

where a is now a function of $t_1, t_2, \ldots$ By combining Eqs. (B.5b) and (B.6), one obtains

$$\frac{\partial \mathbf{x}_2}{\partial t_0} + \mathbf{A}_0 \mathbf{x}_2 = a^2 \mathbf{f}_2^{(2)} e^{i2\omega t_0} + a\bar{a}\, \mathbf{f}_2^{(0)} - \frac{\partial a}{\partial t_1} \mathbf{u} e^{i\omega t_0} + \text{C.T.} \tag{B.7}$$

where

$$\mathbf{f}_2^{(2)} = \left\{ \sum_{pq} b_{npq} u_p u_q \right\} \tag{B.8a}$$

$$\mathbf{f}_2^{(0)} = \left\{ \frac{1}{2} \sum_{pq} b_{npq} \left(\bar{u}_p u_q + u_p \bar{u}_q \right) \right\} \tag{B.8b}$$

The condition to avoid secular terms is

$$\frac{\partial a}{\partial t_1} = 0 \tag{B.9}$$

and then the solution for $\mathbf{x}_2$ is given by*

$$\mathbf{x}_2 = a^2 \hat{\mathbf{p}}_2^{(2)} e^{i2\omega t_0} + a\bar{a}\, \hat{\mathbf{p}}_2^{(0)} + \text{C.T.} \tag{B.10}$$

where

$$\hat{\mathbf{p}}_2^{(2)} = \left[-i2\omega \mathbf{I} + \mathbf{A}_0 \right]^{-1} \mathbf{f}_2^{(2)} \tag{B.11a}$$

$$\hat{\mathbf{p}}_2^{(0)} = \mathbf{A}_0^{-1} \mathbf{f}_2^{(0)} \tag{B.11b}$$

It may be noted that the second order nonlinear terms do not yield any secular terms. For this reason the expression

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*Note that, as it will be clear from the third order analysis, the term of the type $\mathbf{u} e^{i\omega t_0}$ (solution of homogeneous equation) can be safely disregarded.

used for λ, Eq. (3.11), does not contain terms of order ε, since the perturbation of λ must be of the same order of magnitude as the nonlinear terms which yield secular terms.

B.3 Third Order Solution

Combining Eqs (B.4b), (B.6), and (B.10), one obtains

$$\begin{aligned}
\mathbf{f}_3 &= \left\{ \sum_{pq} (b_{npq} + b_{nqp})\, X_{1,p}\, X_{2,q} + \sum_{pqr} C_{npqr}\, X_{1,p}\, X_{1,q}\, X_{1,r} \right\} \\
&= \Big\{ \sum_{pq} (b_{npq} + b_{nqp}) \left(a\, u_p e^{i\omega t_0} + \bar{a}\, \bar{u}_p e^{-i\omega t_0} \right) \Big(a^2 \hat{P}^{(2)}_{2,q} e^{i2\omega t_0} \\
&\quad + 2 a \bar{a}\, \hat{P}^{(0)}_{2,q} + \bar{a}^2 \bar{\hat{P}}^{(2)}_{2,q} e^{-i2\omega t_0} \Big) + \sum_{pqr} C_{npqr} \Big(a\, u_p e^{i\omega t_0} + \\
&\quad \bar{a}\, \bar{u}_p e^{-i\omega t_0} \Big) \left(a\, u_q e^{i\omega t_0} + \bar{a}\, \bar{u}_q e^{-i\omega t_0} \right) \Big(a\, u_r e^{i\omega t_0} + \\
&\quad \bar{a}\, \bar{u}_r e^{-i\omega t_0} \Big) \Big\} \\
&= a^3 \mathbf{f}^{(3)}_3 e^{i3\omega t_0} + a^2 \bar{a}\, \mathbf{f}^{(1)}_3 e^{i\omega t_0} + \text{C.T.}
\end{aligned}$$

(B.12)

where

$$\mathbf{f}^{(3)}_3 = \left\{ \sum_{pq} (b_{npq} + b_{nqp})\, u_p\, \hat{P}^{(2)}_{2,q} + \sum_{pqr} C_{npqr}\, u_p u_q u_r \right\}$$

(B.13a)

$$\begin{aligned}
\mathbf{f}^{(1)}_3 = \Big\{ &\sum_{pq} (b_{npq} + b_{nqp}) \left(\bar{u}_p \hat{P}^{(2)}_{2,q} + 2 u_p \hat{P}^{(0)}_{2,q} \right) \\
&+ \sum_{pqr} C_{npqr} \left(\bar{u}_p u_q u_r + u_p \bar{u}_q u_r + u_p u_q \bar{u}_r \right) \Big\}
\end{aligned}$$

(B.13b)

Finally, combining Eqs. (B.5c), (B.6), (B.9), (B.10), and (B.12) one obtains

$$\frac{\partial \mathbf{x}_3}{\partial t_0} + \mathbf{A}_0 \mathbf{x}_3 = \mathbf{f}_3^{(3)} a^3 e^{i3\omega t_0} + \mathbf{f}_2^{(1)} a^2 \bar{a} e^{i\omega t_0}$$

$$- \frac{\partial a}{\partial t_2} \mathbf{u}\, e^{i\omega t_0} - \mathbf{A}_2 \mathbf{u} e^{i\omega t_0} + C.T. \qquad \text{(B.13)}$$

which is formally identical to Eq. (3.18). [The definitions of $\mathbf{f}_3^{(3)}$ and $\mathbf{f}_3^{(1)}$,however, are now given by Eq. (B.12) instead of Eq. (3.19)]. Hence, the solution of Eq. (B.13) is given in Section 3. The conclusions are the ones given in Section 3.4. The solution is given by

$$\mathbf{x} = \varepsilon(\mathbf{u} a e^{i\omega t_0} + \bar{\mathbf{u}} \bar{a} e^{-i\omega t_0}) + O(\varepsilon^2) \qquad \text{(B.14)}$$

where a is given in Eq. (3.24). Note that the only effect of the second order nonlinear terms on Eq. (B.14) is the order of magnitude of the "error" and the different definition of $\mathbf{f}_3$ [Eq. (B.12) instead of Eq. (3.19)] which is used in the definition of Υ [Eq. (3.23c)].

APPENDIX C

THE FUNCTION $\tilde{b}_I(t_2)$

C.1. Introduction

In Section 4, Eq. (4.23.b)

$$\frac{\partial \tilde{b}_I}{\partial t_2} + \beta_R \tilde{b}_I + \gamma_R |a|^2 \tilde{b}_I + 2\gamma_I |a|^2 \tilde{b}_R = \delta_I |a| + \delta_I' |a|^3 + \delta_I'' |a|^5 - |a| \frac{\partial \varphi}{\partial t_4} \tag{C.1}$$

was derived. By setting (see Eqs. (4.24.b))

$$\tilde{b}_I = B_I |a| \tag{C.2}$$

and using Eq. (4.25) one obtains Eq. (4.27.b)

$$\frac{\partial B_I}{\partial t_2} = -2\gamma_I |a| \tilde{b}_R + \delta_I + \delta_I' |a|^2 + \delta_I'' |a|^4 - \partial\varphi/\partial t_4 \tag{C.3}$$

On the other hand, according to Eqs. (4.36) and (4.26)

$$\tilde{b}_R = \frac{\gamma_R}{2\beta_R^2}\left[\delta_R - \delta_R' \frac{\beta_R}{\gamma_R} + \delta_R'' \left(\frac{\beta_R}{\gamma_R}\right)^2\right] |a|^3 - \left(\frac{B_R^{(0)}}{\beta_R} - \frac{1}{\gamma_R^2}\delta_R'' \ln|a|\right)\frac{\partial |a|}{\partial t_2} \tag{C.4}$$

while, according to Eqs. (2.9) and (3.24) (see also Eq. (2.23))

$$\varphi = \omega_2 t_2 + \frac{\gamma_I}{\gamma_R} \ln|a| + \varphi_0 \tag{C.5}$$

and thus

$$\frac{\partial\varphi}{\partial t_4} = \frac{\gamma_I}{\gamma_R}\frac{\partial}{\partial t_4}\left(\ln|a|\right) + \frac{\partial\varphi_0}{\partial t_4}$$

$$= -\frac{\gamma_I}{\gamma_R\beta_R}\left[\delta_R - \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2\right]\frac{1}{|a|}\frac{\partial|a|}{\partial t_2} + \frac{\partial\varphi_0}{\partial t_4} \qquad \text{(C.6)}$$

since (deriving $\frac{\partial|a|}{\partial t_4}$ from Eq. (4.32), e^{τ} from Eq. (4.31) and $\frac{\partial k}{\partial t_4}$ from Eq. (4.35))

$$\frac{\partial}{\partial t_4}\ln|a| = \frac{1}{|a|}\frac{\partial}{\partial t_4}|a| = -\frac{1}{2}|a|^2 e^{\tau}\frac{\partial k_0}{\partial t_4}$$

$$= -\frac{1}{2}|a|^2\frac{\partial|a|/\partial t_2}{-\beta_R|a|^3 k_0}\,2k_0\left[-\delta_R + \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2\right]$$

$$= -\frac{1}{\beta_R}\left[\delta_R - \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2\right]\frac{1}{|a|}\frac{\partial|a|}{\partial t_2} \qquad \text{(C.7)}$$

Finally combining Eqs. (C.3, C.4 and C.6) one obtains

$$\frac{\partial B_I}{\partial t_2} = -\frac{\gamma_I\gamma_R}{\beta_R^2}\left[\delta_R - \delta_R'\frac{\beta_R}{\gamma_R} + \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2\right]|a|^4$$

$$+ 2\gamma_I\left(\frac{B_R^{(0)}}{\beta_R} - \frac{1}{\gamma_R^2}\delta_R''\ln|a|\right)|a|\frac{\partial|a|}{\partial t_2}$$

$$+ \delta_I + \delta_I'|a|^2 + \delta_I''|a|^4$$

$$+ \frac{\gamma_I}{\gamma_R\beta_R}\left[\delta_R - \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2\right]\frac{\partial}{\partial t_2}\ln|a| - \frac{\partial\varphi_0}{\partial t_4} \qquad \text{(C.8)}$$

This is the desired differential for $B_I(t_2)$. This equation is analyzed in this Appendix.

C.2. The functions $B(t)_2$ and $b(t_2)$

Equation (C.8) may be rewritten as

$$\frac{\partial B_I}{\partial t_2} = \eta_0 + \eta_1 |a|^2 + \eta_2 |a|^4 + \eta_3 \frac{\partial}{\partial t_2} |a|^2 + \eta_4 \ln|a|^2 \frac{\partial |a|^2}{\partial t_2} + \eta_5 \frac{\partial}{\partial t_2} \ln|a| \tag{C.9}$$

where

$$\eta_0 = \delta_I - \frac{\partial \varphi_0}{\partial t_4}$$

$$\eta_1 = \delta_I'$$

$$\eta_2 = \delta_I'' - \frac{\gamma_I \gamma_R}{\beta_R^2} \left[\delta_R - \delta_R' \frac{\beta_R}{\gamma_R} + \delta_R'' \left(\frac{\beta_R}{\gamma_R} \right)^2 \right]$$

$$\eta_3 = \frac{\gamma_I}{\beta_R} B_R^{(0)}$$

$$\eta_4 = -\frac{1}{2} \frac{\gamma_I}{\gamma_R^2} \delta_R''$$

$$\eta_5 = \frac{\gamma_I}{\gamma_R \beta_R} \left[\delta_R - \delta_R'' \left(\frac{\beta_R}{\gamma_R} \right)^2 \right] \tag{C.10}$$

Integrating Eq. (C.9) one obtains*

$$B_I' = \eta_0 t_2 + \eta_1\left(-\frac{\beta_R}{\gamma_R} t_2 - \frac{1}{\gamma_R}\ln|a|\right)$$

$$+\eta_2\left[\frac{\beta_R^2}{\gamma_R^2} t_2 + \frac{\beta_R}{\gamma_R^2}\ln|a| - \frac{1}{2\gamma_R}|a|^2\right]$$

$$+\eta_3|a|^2 + \eta_4\left(|a|^2\ln|a|^2 - |a|^2\right) + \eta_5\ln|a| + B_I^{(0)}$$

$$= \left(\eta_0 - \eta_1\frac{\beta_R}{\gamma_R} + \eta_2\frac{\beta_R^2}{\gamma_R^2}\right)t_2 + \left(-\frac{\eta_1}{\gamma_R} + \eta_2\frac{\beta_R}{\gamma_R^2} + \eta_5\right)\ln|a|$$

$$+\left(-\eta_2\frac{1}{2\gamma_R} + \eta_3 - \eta_4\right)|a|^2 + 2\eta_4|a|^2\ln|a| + B_I^{(0)} \tag{C.11}$$

In order to avoid secular terms, set

$$\eta_0 - \eta_1\frac{\beta_R}{\gamma_R} + \eta_2\frac{\beta_R^2}{\gamma_R^2} = 0 \tag{C.12}$$

or (see Eq. (C.10))

$$-\frac{\partial\varphi_0}{\partial t_4} + \Phi_1 = 0 \tag{C.13}$$

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* Note that $\frac{\partial}{\partial t_2}\left(\frac{\beta_R}{\gamma_R} t_2 + \frac{1}{\gamma_R}\ln|a|\right) = \frac{\beta_R}{\gamma_R} - \frac{1}{\gamma_R}\frac{1}{|a|}\left(\beta_R|a| + \gamma_R|a|^3\right) = -|a|^2$

and $\frac{\partial}{\partial t_2}\left[\left(\frac{\beta_R}{\gamma_R}\right)^2 t_2 + \frac{\beta_R}{\gamma_R^2}\ln|a| - \frac{1}{2\gamma_R}|a|^2\right] = \left(\frac{\beta_R}{\gamma_R}\right)^2 - \frac{\beta_R}{\gamma_R^2}\frac{1}{|a|}\left(\beta_R|a| + \gamma_R|a|^3\right)$

$+\frac{1}{2\gamma_R}2|a|\left(\beta_R|a| + \gamma_R|a|^3\right) = |a|^4$

with

$$\Phi_1 = \left[\delta_I - \delta_I' \frac{\beta_R}{\gamma_R} + \delta_I'' \left(\frac{\beta_R}{\gamma_R}\right)^2 \right]$$

$$- \frac{\gamma_I}{\gamma_R} \left[\delta_R - \delta_R' \frac{\beta_R}{\gamma_R} + \delta_R'' \left(\frac{\beta_R}{\gamma_R}\right)^2 \right] \qquad \text{(C.14)}$$

Equation (C.13) yields

$$\varphi_0 = \Phi_1 t_4 + \Phi_0 \qquad \text{(C.15)}$$

and combined with Eq. (C.11), yields

$$B_I = B_I^{(0)} + B_I^{(1)} |a|^2 + B_I^{(2)} \ln|a| + B_I^{(3)} |a|^2 \ln|a| \qquad \text{(C.16)}$$

with (see Eqs. (C.10) and (4.37))

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$$B_I^{(1)} = - \frac{\eta_2}{2\gamma_R} + \eta_3 - \eta_4$$

$$= - \frac{1}{2\gamma_R} \left\{ \delta_I'' - \frac{\gamma_I \gamma_R}{\beta_R^2} \left[\delta_R - \delta_R' \frac{\beta_R}{\gamma_R} - \delta_R'' \left(\frac{\beta_R}{\gamma_R}\right)^2 \right] \right\}$$

$$+ \frac{\gamma_I}{\beta_R} B_R^{(0)} + \frac{1}{2} \frac{\gamma_I}{\gamma_R^2} \delta_R''$$

$$= \frac{\gamma_I}{\gamma_R} B_R^{(1)} - \frac{1}{2\gamma_R} \left(\delta_I'' - \frac{\gamma_I}{\gamma_R} \delta_R'' \right)$$

$$B_I^{(2)} = -\frac{\eta_1}{\gamma_R} + \eta_2 \frac{\beta_R}{\gamma_R^2} + \eta_5$$

$$= -\frac{\delta_I'}{\gamma_R} + \frac{\beta_R}{\gamma_R^2}\left\{\delta_I'' - \frac{\gamma_I\gamma_R}{\beta_R^2}\left[\delta_R - \delta_R'\frac{\beta_R}{\gamma_R} + \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2\right]\right\}$$

$$+ \frac{\gamma_I}{\gamma_R\beta_R}\left[\delta_R - \delta_R''\left(\frac{\beta_R}{\gamma_R}\right)^2\right]$$

$$= -\frac{1}{\gamma_R}\left[\delta_I' - \frac{\beta_R}{\gamma_R}\delta_I''\right] + \frac{\gamma_I}{\gamma_R^2}\left[\delta_R' - 2\frac{\beta_R}{\gamma_R}\delta_R''\right]$$

$$B_I^{(3)} = 2\eta_4 = -\frac{\gamma_I}{\gamma_R^2}\delta_R'' = \frac{\gamma_I}{\gamma_R}B_R^{(3)} \qquad (C.17)$$

Finally combining Eqs. (C.2) and (C.16) yields

$$\tilde{b}_I = B_I^{(0)}|a| + B_I^{(1)}|a|^3 + \left(B_I^{(2)} + B_I^{(3)}|a|^2\right)|a| \ln|a| \qquad (C.18)$$