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ON THE OSCILLATION OF THE LATERALLY HETEROGENEOUS EARTH, I

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ON THE OSCILLATION OF THE LATERALLY HETEROGENEOUS EARTH, I

Peter Musen

Abstract

The perturbative effects, as caused by lateral inhomogeneities in the Earth structure and by Coriolis force, contaminate the originally toroidal and spheroidal Earth's oscillations, making them of mixed type. For this reason, in order to make the computation of the perturbations more uniform and homogeneous, we suggest the expansion of Earth's free oscillations into a series in terms of generalized harmonics familiar from the theory of angular momentum in quantum mechanics. Making use of Gibbsian symbolism and of some operators from the theory of angular momentum, we deduced the explicit expressions, in terms of generalized harmonics, of the perturbative terms in the differential equation of Earth's free oscillations. We also obtained decomposition of the strain tensor in terms of canonical vectors. The integration problem for the cases of geophysical interest will be discussed in subsequent reports.

ON THE OSCILLATION OF THE LATERALLY HETEROGENEOUS EARTH, I

INTRODUCTION

In the present and forthcoming article we suggest an apparatus and scheme for the computation of the perturbed elastic oscillations of the laterally inhomogeneous non-spherical Earth. The perturbative effects treated in this work are additive to the solution of the "main problem" as obtained by Takeuchi (1950) and by Alterman, Jarosch and Pekeris (1959), which represents the foundation of all modern theories of the long period oscillations of the elastic Earth. Methods being used to obtain the solution of the zero order are, more or less, a paraphrase of the Pekeris method. These authors have assumed that the density of the Earth's material and elastic parameters depend only upon the distance of the particle from the Earth's center and that the perturbative effect of the geostrophic force can be neglected. Under these suppositions the oscillation can be conveniently decomposed into a sum of spheroidal and toroidal oscillations with the coefficients depending only upon the radius-vector.

The initial differential equation of the elastic oscillation can be split into disjoint final systems of ordinary differential equations for the coefficients. Each system must be integrated numerically using an adapted model of the Earth. It has now become evident, however, that the results of modern seismic observations warrant the introduction of perturbative forces into the original Pekeris' theory.

The influence of lateral variations in density and elastic parameters (Toksöz and Anderson, 1966), (Dziewonski, 1970), (Lopatina and Ryaboy, 1971), and the perturbative effect of the geostrophic force, which causes the splitting of frequencies (Backus and Gilbert, 1961), (Pekeris, Alterman and Jarosch, 1961) are now being recognized in seismic records. The ellipticity correction, together with numerous coupling effects, became important and shall be considered. On occasion the Coriolis coupling between spheroidal and toroidal oscillations can exceed all other couplings (Luh, 1974). We must also consider the degeneracy and quasi-degeneracy of frequencies, familiar from quantum-mechanics. With these considerations the expansion of the perturbed elastic vibrations in terms of spheroidal and toroidal vector-harmonics can still be achieved (Arkani-Hamed, 1972), (Madariaga, 1972), (Luh, 1973), (Dahlen, 1968, 1969).

It is doubtful, however, that the splitting of the displacement into toroidal and spheroidal components provides the best possible theoretical and computational approach in the case of a rotating, laterally non-homogeneous non-spherical Earth when numerous couplings, degeneracies and quasi-degeneracies do occur. It seems that decomposition of oscillations into a sum of so-called "generalized spherical harmonics" can provide a better service, because it removes the discrimination between different kinds of oscillations and thus paves the way for more direct and homogeneous computational procedures. The generalized spherical harmonics are familiar from the theory of angular momentum in quantum-mechanics. There are numerous recursive relations between them, and their products, which appear in the perturbing forces, can be conveniently expanded into Clebsch-Gordon series.

It seems that Petrashen (1949) was the first to recognize the usefulness of the generalized spherical harmonics in the theory of elasticity. In recent years several works on the spectroscopy of the Earth were published which make use of the generalized spherical harmonics (Burridge, 1969), (Phinney and Burridge, 1973), (Smith, 1974) and the theory of group representations from quantum-mechanics. Formation of the scalar differential equations of Earth's perturbed oscillations is usually based on the application of covariant differentiation of vectors and tensors involved, using Einstein summation convention, sometimes partially combined with Gibbsian symbolism (Backus, 1967). In the present article we suggest the use of Gibbsian vectorial and dyadic symbolism, and of operators familiar from the angular momentum theory in quantum mechanics to establish the differential equations of the perturbed elastic-gravitational vibrations of the Earth. The oscillations are decomposed along Petrashen's vectorial harmonics.

The combined symbolism represents a fast and expedient geometrical way to establish the vectorial differential equations governing the oscillations and to convert them into the scalar equations. In the case of perturbed oscillations of the Earth these differential equations are no longer disjoint. The resulting scalar equations can be considered as a generalization of Pekeris equations and, like them, can be integrated only numerically using an adapted Earth model. After the solution is obtained the oscillations can again be easily split into spheroidal and toroidal parts.

BASIC RELATIONS

In this section we summarize some basic formulas to be used in the exposition. By applying the del-operator

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\left(\mathbf{e}_\theta \frac{\partial}{\partial \theta} + \frac{\mathbf{e}_\phi}{\sin\theta} \frac{\partial}{\partial \phi}\right) \tag{1}$$

to the local unit vectors $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, $\mathbf{e}_r$ of the spherical system we deduce

$$\begin{aligned} \nabla \mathbf{e}_\theta &= +\frac{1}{r}\left(-\,\mathbf{e}_\theta \mathbf{e}_r + \mathbf{e}_\phi \mathbf{e}_\phi \cot\theta\right) , \\ \nabla \mathbf{e}_\phi &= -\frac{1}{r}\left(+\,\mathbf{e}_\phi \mathbf{e}_r + \mathbf{e}_\phi \mathbf{e}_\theta \cot\theta\right), \\ \nabla \mathbf{e}_r &= +\frac{1}{r}\left(\mathbf{e}_\theta \mathbf{e}_\theta + \mathbf{e}_\phi \mathbf{e}_\phi\right), \end{aligned} \tag{2}$$

and, as a consequence, by forming the scalars and vectors of the symbolic dyadics (2), we obtain:

$$\begin{aligned} \nabla \cdot \mathbf{e}_\theta &= +\frac{\cot\theta}{r} \\ \nabla \cdot \mathbf{e}_\phi &= 0 \\ \nabla \cdot \mathbf{e}_r &= +\frac{2}{r} \end{aligned} \tag{3}$$

$$\nabla \times \mathbf{e}_\theta = +\frac{1}{r}\,\mathbf{e}_\phi,$$

$$\nabla \times \mathbf{e}_\phi = +\frac{1}{r}\,(-\mathbf{e}_\theta + \mathbf{e}_r \cot\theta), \tag{4}$$

$$\nabla \times \mathbf{e}_r = 0.$$

Repeating the procedure, we have:

$$\nabla^2 \mathbf{e}_\theta = +\frac{1}{r^2 \sin^2\theta}\,(-\mathbf{e}_\theta - 2\mathbf{e}_r \sin\theta\cos\theta),$$

$$\nabla^2 \mathbf{e}_\phi = -\frac{1}{r^2 \sin^2\theta}\,\mathbf{e}_\phi, \tag{5}$$

$$\nabla^2 \mathbf{e}_r = -\frac{2}{r^2}\,\mathbf{e}_r,$$

and

$$\nabla\nabla\cdot \mathbf{e}_\theta = -\frac{1}{r^2 \sin^2\theta}\,(\mathbf{e}_\theta + \mathbf{e}_r \sin\theta\cos\theta),$$

$$\nabla\nabla\cdot \mathbf{e}_\phi = 0 \tag{6}$$

$$\nabla\nabla\cdot \mathbf{e}_r = -\frac{2}{r^2}\,\mathbf{r}_r$$

We obtain a more compact system of formulas if we decompose vectors and tensors which appear in the present exposition not along $\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}_r$, but along the isotropic canonical vectors

$$\mathbf{e}_- = + \frac{1}{\sqrt{2}} (+ \mathbf{e}_\theta - i\mathbf{e}_\phi) \tag{7}$$

$$\mathbf{e}_+ = - \frac{1}{\sqrt{2}} (+ \mathbf{e}_\theta + i\mathbf{e}_\phi) \tag{8}$$

and along

$$\mathbf{e}_0 = \mathbf{e}_r \tag{9}$$

It is convenient in the frame work of the present exposition to define the the products of (7) - (9), scalar, vectorial and dyadic, in accordance with the standard rules of classical vector algebra, considering the complex coordinates as scalars.

From the orthonormality of $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, $\mathbf{e}_r$ we deduce:

$$\begin{aligned}
&\mathbf{e}_- \cdot \mathbf{e}_- = \mathbf{e}_+ \cdot \mathbf{e}_+ = \mathbf{e}_- \cdot \mathbf{e}_0 = \mathbf{e}_+ \cdot \mathbf{e}_0 = 0, \\
&\mathbf{e}_- \cdot \mathbf{e}_+ = -1, \quad \mathbf{e}_0 \cdot \mathbf{e}_0 = +1, \\
&\mathbf{e}_+ \times \mathbf{e}_- = +i\mathbf{e}_0, \quad \mathbf{e}_0 \times \mathbf{e}_- = +i\mathbf{e}_-, \quad \mathbf{e}_+ \times \mathbf{e}_0 = +i\mathbf{e}_+.
\end{aligned} \tag{10}$$

Elimination of $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, $\mathbf{e}_r$ from the idemfactor

$$\mathbf{I} = \mathbf{e}_\theta\mathbf{e}_\theta + \mathbf{e}_\phi\mathbf{e}_\phi + \mathbf{e}_r\mathbf{e}_r$$

in favor of canonical vectors leads to the representation

$$\mathbf{I} = \mathbf{e}_0\mathbf{e}_0 - \mathbf{e}_+\mathbf{e}_- - \mathbf{e}_-\mathbf{e}_+$$

which is useful in performing the rotational transformation of vectors and tensors.

From (2) - (6) we deduce a set of basic relations:

$$\nabla \mathbf{e}_- = +\frac{1}{r}\left[-\mathbf{e}_-\mathbf{e}_0 - \frac{1}{\sqrt{2}}(\mathbf{e}_- + \mathbf{e}_+)\,\mathbf{e}_- \cot\theta\right],$$

$$\nabla \mathbf{e}_+ = +\frac{1}{r}\left[-\mathbf{e}_+\mathbf{e}_0 + \frac{1}{\sqrt{2}}(\mathbf{e}_- + \mathbf{e}_+)\,\mathbf{e}_+ \cot\theta\right], \tag{11}$$

$$\nabla \mathbf{e}_0 = -\frac{1}{r}(\mathbf{e}_-\mathbf{e}_+ + \mathbf{e}_+\mathbf{e}_-),$$

and, by forming the scalars and the vectors of the symbolic dyadics (11) and taking (10) into account,

$$\nabla\cdot\mathbf{e}_- = +\frac{\cot\theta}{r\sqrt{2}},$$

$$\nabla\cdot\mathbf{e}_+ = -\frac{\cot\theta}{r\sqrt{2}}, \tag{12}$$

$$\nabla\cdot\mathbf{e}_0 = 0,$$

$$\nabla\times\mathbf{e}_- = +\frac{i}{r}\left(+\mathbf{e}_- - \frac{\mathbf{e}_0}{\sqrt{2}}\cot\theta\right),$$

$$\nabla\times\mathbf{e}_+ = +\frac{i}{r}\left(-\mathbf{e}_+ - \frac{\mathbf{e}_0}{\sqrt{2}}\cot\theta\right), \tag{13}$$

$$\nabla\times\mathbf{e}_0 = 0,$$

$$\nabla^2 \mathbf{e}_- = -\frac{1}{r^2 \sin^2\theta}\,(\mathbf{e}_- + \mathbf{e}_0\sqrt{2}\sin\theta\cos\theta),$$

$$\nabla^2 \mathbf{e}_+ = -\frac{1}{r^2 \sin^2\theta}\,(\mathbf{e}_+ - \mathbf{e}_0\sqrt{2}\sin\theta\cos\theta), \tag{14}$$

$$\nabla^2 \mathbf{e}_0 = -\frac{2\mathbf{e}_0}{r^2}$$

$$\nabla\nabla\cdot\mathbf{e}_- = -\frac{1}{r^2\sin^2\theta}\left[+\frac{1}{2}(\mathbf{e}_- - \mathbf{e}_+) + \frac{\mathbf{e}_0}{\sqrt{2}}\sin\theta\cos\theta\right],$$

$$\nabla\nabla\cdot\mathbf{e}_+ = +\frac{1}{r^2\sin^2\theta}\left[+\frac{1}{2}(\mathbf{e}_- - \mathbf{e}_+) + \frac{\mathbf{e}_0}{\sqrt{2}}\sin\theta\cos\theta\right], \tag{15}$$

$$\nabla\nabla\cdot\mathbf{e}_0 = -\frac{2}{r^2}\mathbf{e}_0.$$

The set

$$\begin{aligned} \mathbf{Y}^{+}_{\ell m} &= \mathbf{e}_+ Y^{\ell}_{m,-1(\theta,\phi)}, \\ \mathbf{Y}^{-}_{\ell m} &= \mathbf{e}_- Y^{\ell}_{m,+1(\theta,\phi)}, \\ \mathbf{Y}^{0}_{\ell m} &= \mathbf{e}_0 Y^{\ell}_{m,0(\theta,\phi)}, \end{aligned} \tag{16}$$

where $Y^{\ell}_{m,-1}$, $Y^{\ell}_{m,+1}$, $Y^{\ell}_{m,0}$ are particular elements of the set of generalized spherical harmonics $Y^{\ell}_{m,n}$, familiar from angular momentum theory, and they constitute the basis of the expansion of the elastic displacement of the Earth (Phinney and Burridge, 1973), (Petrashen, 1949). We have

$$Y^{\ell}_{m,n} = e^{-im\phi}P^{\ell}_{mn}(\cos\theta),$$

where P^{ℓ}_{mn} ($\cos\theta$) are the generalized associate Legendre functions (Edmonds, 1960), (Vilenkin, 1965).

The selection of (16) as a basis for the expansion removes the discrimination between spheroidal and toroidal oscillations and makes the computation of coupling effects more uniform and homogeneous. After the solution is completed we can again decompose it into spheroidal and toroidal modes. When no ambiguity results we will omit, for the sake of brevity, the indices ℓ and m in (16).

The reduction of the vectorial differential equation governing the oscillations to the scalar equations can be simplified by making use of the operators

$$H_+ = e^{-i\psi}\left(+\cot\theta\,\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi} + i\,\frac{\partial}{\partial\theta}\right),$$

$$H_- = e^{+i\psi}\left(-\cot\theta\,\frac{\partial}{\partial\psi} + \frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi} + i\,\frac{\partial}{\partial\theta}\right) \tag{17}$$

of infinitesimal rotations and by introducing the generalized del-operator:

$$\nabla = \mathbf{e}_0\,\frac{\partial}{\partial r} + \frac{i}{r\sqrt{2}}\,(e^{-i\psi}\mathbf{e}_+H_- - e^{+i\psi}\mathbf{e}_-H_+), \tag{18}$$

where ψ, ϕ and θ are Euler angles.

By applying the operator (18) to the elements

$$T^{\ell}_{m,n} = e^{-in\psi}Y^{\ell}_{m,n}$$

of the irreducible representation of the rotation group and taking into account (Edmonds, 1960), (Vilenkin, 1965)

$$H_+ T^{\ell}_{mn} = -\alpha_{\ell,n+1} T^{\ell}_{m,n+1}$$
$$H_- T^{\ell}_{mn} = -\alpha_{\ell,n} T^{\ell}_{m,n-1},$$

where

$$\alpha_{\ell,n} = [(\ell + n)(\ell - n + 1)]^{1/2},$$

and

$$\nabla e^{-in\psi} = -\frac{n}{r\sqrt{2}} (\mathbf{e}_+ + \mathbf{e}_-) \cot\theta e^{-in\psi},$$

we deduce after easy transformations:

$$\nabla \mathbf{Y}^{\ell}_{m,n} = +\frac{i}{r\sqrt{2}} (\alpha_{\ell,n+1} \mathbf{e}_- \mathbf{Y}^{\ell}_{m,n+1} - \alpha_{\ell,n} \mathbf{e}_+ \mathbf{Y}^{\ell}_{m,n-1})$$
$$+ \frac{n}{r\sqrt{2}} (\mathbf{e}_- + \mathbf{e}_+) \mathbf{Y}^{\ell}_{m,n} \cot\theta. \tag{19}$$

Taking (10) into account, we have

$$\mathbf{e}_0 \times \nabla \mathbf{Y}^{\ell}_{m,n} = -\frac{1}{r\sqrt{2}} (\alpha_{\ell,n+1} \mathbf{e}_- \mathbf{Y}^{\ell}_{m,n+1} + \alpha_{\ell,n} \mathbf{e}_+ \mathbf{Y}^{\ell}_{m,n-1}) + \frac{in}{r\sqrt{2}} (\mathbf{e}_- - \mathbf{e}_+) \mathbf{Y}^{\ell}_{m,n} \cot\theta \tag{20}$$

In particular

$$\nabla \mathbf{Y}^{\ell}_{m,0} = +\frac{i\alpha_{\ell,0}}{r\sqrt{2}} (\mathbf{Y}^-_{\ell m} - \mathbf{Y}^+_{\ell m}), \tag{21}$$

and

$$\mathbf{e}_0 \times \nabla \mathbf{Y}^{\ell}_{m,0} = -\frac{\alpha_{\ell,0}}{r\sqrt{2}} (\mathbf{Y}^-_{\ell m} + \mathbf{Y}^+_{\ell m}). \tag{22}$$

The left-hand sides of (21)-(22) are the spheroidal and toroidal harmonics, respectively. Thus, the generalized spherical harmonics $\mathbf{Y}^-$, $\mathbf{Y}^+$ represent the linear combinations of the standard spheroidal and toroidal harmonics.

Making use of (19) and (11) we deduce:

$$\nabla \mathbf{Y}^{+}_{\ell m} = + \frac{i}{r\sqrt{2}} (\alpha_{\ell,0} \mathbf{e}_{-} Y^{\ell}_{m,0} - \alpha_{\ell,-1} \mathbf{e}_{+} Y^{\ell}_{m,-2}) \, \mathbf{e}_{+}$$

$$- \frac{1}{r} Y^{\ell}_{m,-1} \mathbf{e}_{+} \mathbf{e}_{0},$$

$$\nabla \mathbf{Y}^{-}_{\ell m} = + \frac{i}{r\sqrt{2}} (\alpha_{\ell,2} \mathbf{e}_{-} Y^{\ell}_{m,+2} - \alpha_{\ell,1} \mathbf{e}_{+} Y^{\ell}_{m,0}) \, \mathbf{e}_{-} \tag{23}$$

$$- \frac{1}{r} Y^{\ell}_{m,+1} \mathbf{e}_{-} \mathbf{e}_{0},$$

$$\nabla \mathbf{Y}^{0}_{\ell m} = - \frac{1}{r} Y^{\ell}_{m,0} (\mathbf{e}_{-} \mathbf{e}_{+} + \mathbf{e}_{+} \mathbf{e}_{-})$$

$$+ \frac{i}{r\sqrt{2}} (\alpha_{\ell,1} \mathbf{e}_{-} Y^{\ell}_{m,+1} - \alpha_{\ell,0} \mathbf{e}_{+} Y^{\ell}_{m,-1}) \, \mathbf{e}_{0}$$

and, as a consequence, taking (10) into account:

$$\nabla \times \mathbf{Y}^{+}_{\ell m} = + \frac{\alpha_{\ell,0}}{r\sqrt{2}} \mathbf{Y}^{0}_{\ell m} - \frac{i}{r} \mathbf{Y}^{+}_{\ell m},$$

$$\nabla \times \mathbf{Y}^{-}_{\ell m} = + \frac{\alpha_{\ell,0}}{r\sqrt{2}} \mathbf{Y}^{0}_{\ell m} + \frac{i}{r} \mathbf{Y}^{-}_{\ell m}, \tag{24}$$

$$\nabla \times \mathbf{Y}^{0}_{\ell m} = + \frac{\alpha_{\ell,0}}{r\sqrt{2}} (\mathbf{Y}^{+}_{\ell m} + \mathbf{Y}^{-}_{\ell m}),$$

and

$$\nabla\cdot\mathbf{Y}^{+}_{\ell_m} = -\nabla\cdot\mathbf{Y}^{-}_{\ell_m} = -\frac{i\alpha_{\ell,0}}{r\sqrt{2}}\,\mathbf{Y}^{\ell}_{m,0}, \qquad \nabla\cdot\mathbf{Y}^{0}_{\ell_m} = +\frac{2}{r}\mathbf{Y}^{\ell}_{m,0}, \tag{25}$$

DECOMPOSITION OF THE STRAIN TENSOR

In this section we deduce the canonical dyadic decomposition of the strain tensor

$$\epsilon = \frac{1}{2}\,(\mathbf{u}\nabla + \nabla\mathbf{u}) \tag{26}$$

assuming the canonical decomposition

$$\mathbf{u} = U^{0}(r)\,\mathbf{Y}^{0} + U^{+}(r)\,\mathbf{Y}^{+} + U^{-}(r)\,\mathbf{Y}^{-} \tag{27}$$

of the displacement.

We have

$$\begin{aligned}\nabla\mathbf{u} = \mathbf{e}_0 &\left(\frac{dU^0}{dr}\,\mathbf{Y}^0 + \frac{dU^+}{dr}\,\mathbf{Y}^+ + \frac{dU^-}{dr}\,\mathbf{Y}^-\right)\\ &+ (U^0\nabla\mathbf{Y}^0 + U^+\nabla\mathbf{Y}^+ + U^-\nabla\mathbf{Y}^-)\end{aligned} \tag{28}$$

Taking (23) into account and rearranging the terms in two different manners, we obtain the following two canonical dyadic representations of the gradient of the displacement:

$$\begin{aligned}\nabla\mathbf{u} = &\left[\frac{dU^0}{dr}\,\mathbf{e}_0 Y_0 + \frac{i\alpha_0}{r\sqrt{2}}\,U^0(\mathbf{e}_- Y_{+1} - \mathbf{e}_+ Y_{-1})\right.\\ &\left. - \frac{U^+}{r}\,\mathbf{e}_+ Y_{-1} - \frac{U^-}{r}\,\mathbf{e}_- Y_{+1}\right]\,\mathbf{e}_0\end{aligned} \tag{29}$$

$$+ \left[\frac{dU^+}{dr} e_0 Y_{-1} - \frac{U^0}{r} e_- Y_0\right.$$

$$\left. + \frac{i}{r\sqrt{2}} U^+ (\alpha_0 e_- Y_0 - \alpha_{-1} e_+ Y_{-2})\right] e_+$$

$$+ \left[\frac{dU^-}{dr} e_0 Y_{+1} - \frac{U^0}{r} e_+ Y_0\right.$$

$$\left. + \frac{i}{r\sqrt{2}} U^- (\alpha_2 e_- Y_{+2} - \alpha_1 e_+ Y_0)\right] e_-,$$

and

$$\begin{aligned}
\nabla \mathbf{u} = &+ e_0 \left(\frac{dU^0}{dr} e_0 Y_0 + \frac{dU^+}{dr} e_+ Y_{-1} + \frac{dU^-}{dr} e_- Y_{+1}\right) \\
&+ e_+ \left[- U^0 \left(+ \frac{1}{r} e_- Y_0 + \frac{i\alpha_0}{r\sqrt{2}} e_0 Y_{-1}\right)\right. \\
&- U^+ \left(+ \frac{i\alpha_{-1}}{r\sqrt{2}} e_+ Y_{-2} + \frac{1}{r} e_0 Y_{-1}\right) \\
&\left. - \frac{i\alpha_0}{r\sqrt{2}} U^- e_- Y_0\right] \\
&+ e_- \left[+ U^0 \left(- \frac{1}{r} e_+ Y_0 + \frac{i\alpha_0}{r\sqrt{2}} e_0 Y_{+1}\right)\right. \\
&+ U^+ \frac{i\alpha_0}{r\sqrt{2}} e_+ Y_0 \\
&\left. + U^- \left(+ \frac{i\alpha_2}{r\sqrt{2}} e_- Y_{+2} - \frac{1}{r} e_0 Y_{+1}\right)\right]
\end{aligned} \tag{30}$$

Combining (29) with (30) transposed, we obtain the canonical decomposition of the strain tensor:

$$\begin{aligned}\epsilon = {} & (\epsilon^{00}\mathbf{e}_0 Y_0 + \epsilon^{+0}\mathbf{e}_+ Y_{-1} + \epsilon^{-0}\mathbf{e}_- Y_{+1})\,\mathbf{e}_0 \\ & + (\epsilon^{0+}\mathbf{e}_0 Y_{-1} + \epsilon^{++}\mathbf{e}_+ Y_{-2} + \epsilon^{-+}\mathbf{e}_- Y_0)\,\mathbf{e}_+ \\ & + (\epsilon^{0-}\mathbf{e}_0 Y_{+1} + \epsilon^{+-}\mathbf{e}_+ Y_0 + \epsilon^{--}\mathbf{e}_- Y_{+2})\,\mathbf{e}_-,\end{aligned} \tag{31}$$

where

$$\begin{aligned}\epsilon^{00} &= \frac{dU^0}{dr}, \\ \epsilon^{++} &= -\frac{i\alpha_{-1}}{r\sqrt{2}}\,U^+, \\ \epsilon^{--} &= +\frac{i\alpha_2}{r\sqrt{2}}\,U^-, \\ \epsilon^{+0} = \epsilon^{0+} &= \frac{1}{2}\left(\frac{dU^+}{dr} - \frac{U^+}{r} - \frac{i\alpha_0}{r\sqrt{2}}\,U^0\right) \\ \epsilon^{-0} = \epsilon^{0-} &= \frac{1}{2}\left(\frac{dU^-}{dr} - \frac{U^-}{r} + \frac{i\alpha_0}{r\sqrt{2}}\,U^0\right) \\ \epsilon^{-+} = \epsilon^{+-} &= -\frac{U^0}{r} + \frac{i\alpha_0}{2r\sqrt{2}}\,(U^+ - U^-)\end{aligned} \tag{32}$$

By forming the vector of the dyadic (29), i.e. replacing all dyadic products by the vectorial ones, and taking (10) into account, we deduce

$$\nabla \times \mathbf{u} = Z^0\mathbf{Y}^0 + Z^+\mathbf{Y}^+ + Z^-\mathbf{Y}^-,$$

where

$$Z^0 = + \frac{\alpha_0}{r\sqrt{2}}(U^+ + U^-)$$

$$Z^+ = + \frac{\alpha_0}{r\sqrt{2}} U^0 - i \left(\frac{dU^+}{dr} + \frac{U^+}{r}\right) \tag{33}$$

$$Z^- = + \frac{\alpha_0}{r\sqrt{2}} U^0 + i \left(\frac{dU^-}{dr} + \frac{U^-}{r}\right).$$

Similarly, by forming the scalar of (29), i.e. replacing the dyadic products by the scalar ones, we have

$$\nabla \cdot \mathbf{u} = X^0 Y_0, \tag{34}$$

where

$$X^0(\mathbf{u}) = \frac{dU^0}{dr} + \frac{2U^0}{r} + \frac{i\alpha_0}{r\sqrt{2}} (U^- - U^+). \tag{35}$$

Repeating the process for $\nabla \times \mathbf{u}$, we obtain:

$$\nabla \times \nabla \times \mathbf{u} = W^0\mathbf{Y}^0 + W^+\mathbf{Y}^+ + W^-\mathbf{Y}^-, \tag{36}$$

where

$$W^0 = + \frac{\alpha_0}{r\sqrt{2}}(Z^+ + Z^-),$$

$$W^+ = + \frac{\alpha_0}{r\sqrt{2}} Z^c - i \left(\frac{dZ^+}{dr} + \frac{Z^+}{r}\right), \tag{37}$$

$$W^- = + \frac{\alpha_0}{r\sqrt{2}} Z^0 + i \left(\frac{dZ^-}{dr} + \frac{Z^-}{r}\right). \tag{37}$$

From (34) and (21) we deduce

$$\nabla\nabla \cdot \mathbf{u} = \frac{dX^0}{dr} \mathbf{Y}^0 + \frac{i\alpha_0}{r\sqrt{2}} X^0(\mathbf{Y}^- - \mathbf{Y}^+), \tag{38}$$

We shall need (29) - (38) in the transformation of the differential equation of oscillations of the Earth. In this process we shall also make use of the Clebsch-Gordon expansion

$$Y^{\ell}_{mn} Y^{p}_{qs} = \sum_{\nu = |\ell - p|}^{|\ell + p|} C(\ell,\ p,\ \nu;\ m,\ q)\ C(\ell,\ p,\ \nu:\ n,\ s)\ Y^{\nu}_{m+q,\, n+s} \tag{39}$$

where the symbols $C(\ell, p, \nu; \alpha, \beta)$ designate Clebsch-Gordan coefficients. They can be computed directly on an electronic computer or can be obtained from existing tables.

Every scalar in the present theory can be represented as a series in harmonics $Y^{\ell}_{m,0}$ and every vector as a series in $\mathbf{Y}^{+}_{\ell m}$, $\mathbf{Y}^{-}_{\ell m}$, $\mathbf{Y}^{0}_{\ell m}$ ($\ell = 1, 2, 3, \ldots$; $m = -\ell, \ldots, +\ell$). The coefficients in the expansion depend only on r.

DIFFERENTIAL EQUATION OF THE ELASTIC OSCILLATIONS OF THE EARTH

If we assume hydrostatic equilibrium, but permit lateral inhomogeneities in density and in elastic parameters, then the differential equation of the elastic oscillations of the Earth takes the form:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} + 2\rho\Omega\mathbf{k} \times \frac{\partial \mathbf{u}}{\partial t} = \nabla\cdot\sigma - \nabla\cdot(\rho\mathbf{u})\,\nabla\mathbf{V}$$

$$+ \nabla(\rho\mathbf{u}\cdot\nabla\mathbf{V}) + \rho\nabla\psi, \tag{40}$$

where ρ is the undisturbed density, -V - the interior gravitational potential (it includes the potential of the centrifugal force), Ω - the angular velocity of rotation of the Earth, $\mathbf{k}$ - the unit vector along the polar axis, $-\psi$ - the increment in internal geopotential due to redistribution of masses, and σ - the stress tensor,

$$\sigma = \lambda\mathbf{I}\nabla\cdot\mathbf{u} + \mu(\mathbf{u}\nabla + \nabla\mathbf{u}). \tag{41}$$

We assume that the density ρ and the elastic parameters λ, μ can be represented in the form:

$$\begin{aligned} \rho &= \rho_0(\mathbf{r}) + \delta\rho, \\ \lambda &= \lambda_0(\mathbf{r}) + \delta\lambda, \\ \mu &= \mu_0(\mathbf{r}) + \delta\mu, \end{aligned} \tag{42}$$

where the perturbative terms $\delta\lambda$, $\delta\lambda$, $\delta\mu$ represent the lateral deviations from the spherically symmetric mean values ρ_0, λ_0, μ_0. We assume the existence of the expansions

$$\delta\rho = \sum_{p=1}^{+\infty} \sum_{q=-p}^{q=+p} \rho_{pq}(\mathbf{r})\, Y^p_{q,0},$$

$$\delta\lambda = \sum_{p=1}^{+\infty} \sum_{q=-p}^{q=+p} \lambda_{pq}(\mathbf{r})\, Y^p_{q,0},$$

$$\delta\mu = \sum_{p=1}^{+\infty} \sum_{q=-p}^{q=+p} \mu_{pq}(r)\, Y_{q,0}^{p} \tag{43}$$

$$(p = 1, 2, \ldots; \quad q = -p, \ldots, +p)$$

in terms of associated Legendre functions . For the geopotential inside a given domain of integration we assume an expansion of the form

$$V = V_0(r) + \delta V,$$

$$\delta V = \sum_{\delta, k} V_{\delta k}(r)\, Y_{k,0}^{\delta}. \tag{43'}$$

With the domains of convergence determined by the shape and position of the surfaces of discontinuity inside the Earth. In addition to (42) we found it useful to introduce the combination:

$$\beta = \beta_0 + \delta\beta = \lambda + 2\mu = \beta_0(r) + \sum_{p=1}^{+\infty} \sum_{q=-p}^{q=+p} \beta_{pq}(r)\, Y_{q,0}^{p}, \tag{44}$$

$$\beta_0 = \lambda_0 + 2\mu_0, \qquad \beta_{pq} = \lambda_{pq} + 2\mu_{pq}.$$

Substituting (41) into (40) and taking the Fourier transform (or searching for a periodic solution) we obtain

$$\begin{aligned} \rho(-\omega^2 \mathrm{I} + 2i\Omega\omega\mathbf{k} \times \mathrm{I}) \cdot \mathbf{u} &= \beta\nabla\nabla \cdot \mathbf{u} - \mu\nabla \times \nabla \times \mathbf{u} \\ &+ 2\epsilon \cdot \nabla\mu + (\nabla \cdot \mathbf{u})\nabla\lambda - \nabla \cdot (\rho\mathbf{u})\nabla V + \nabla(\rho\mathbf{u} \cdot \nabla V) + \rho\nabla\psi, \end{aligned} \tag{45}$$

where $\mathbf{u}$ now designates the <u>transform</u> of the displacement.

We assume that the higher order terms

$$\delta\rho\nabla\delta V \text{ and } (\nabla\delta\rho)\,(\nabla\delta V)$$

are negligible. Under this assumption the equation (45) takes the form:

$$\begin{aligned}\rho(-\omega^2 I + 2i\omega\Omega\mathbf{k}\times I)\cdot\mathbf{u} &= \beta\nabla\nabla\cdot\mathbf{u} - \mu\nabla\times\nabla\times\mathbf{u} \\ &+ 2\epsilon\cdot\nabla\mu + (\nabla\cdot\mathbf{u})\,\nabla\lambda - \nabla\cdot(\rho_0\mathbf{u})\,\nabla V_0 + \nabla(\rho_0\mathbf{u}\cdot\nabla V_0) + \rho\nabla\psi \\ &- [\nabla\cdot(\mathbf{u}\,\delta\rho)\,\nabla V_0 + \nabla\cdot(\rho_0\mathbf{u})\,\nabla\delta V - \nabla(\delta\rho\,\mathbf{u}\cdot\nabla V_0) - \nabla(\rho_0\mathbf{u}\cdot\nabla\delta V)]. \quad (45')\end{aligned}$$

The last four terms are perturbative. They represent the effects of the lateral inhomogeneities in density and of the deviation of the figure of the Earth from a sphere.

UNPERTURBED PROBLEM

The solution of zeroth order is obtained under the assumptions that the density, elastic parameters and the interior geopotential are spherically symmetric functions, that the perturbative effect of the geostrophic force is negligible and the oscillations are purely elastic. For the solution of the zeroth order we assume an expansion of the form:

$$\mathbf{u} = \sum_{\ell=1}^{+\infty}\sum_{m=-\ell}^{m=+\ell}\mathbf{u}_{\ell m}, \quad (46)$$

where

$$\mathbf{u}_{\ell m} = U^0_{\ell m}(r,\omega)\,\mathbf{Y}^0_{\ell m} + U^+_{\ell m}(r,\omega)\,\mathbf{Y}^+_{\ell m} + U^-_{\ell m}(r,\omega)\,\mathbf{Y}^-_{\ell m}, \quad (47)$$

and the eigenfrequency ω is determined from the boundary conditions on the surfaces of discontinuity, assumed to be welded together (if the materials are solid). The radial functions $U^0_{\ell_m}$, $U^+_{\ell_m}$, $U^-_{\ell_m}$ satisfy a set of disjoint ordinary differential equations. For each $\mathbf{u}_{\ell_m}$ they can be integrated separately. For the zeroth order solution of the form (47) we have for the particular terms in the differential equation (45), taking (31) - (38) into consideration (we repeat here the complete set of formulas given previously),

$$\beta\nabla\nabla\cdot\mathbf{u} = \beta_0\left[\frac{dX^0}{dr}\mathbf{Y}^0 + \frac{i\alpha_0}{r\sqrt{2}}X^0(\mathbf{Y}^- - \mathbf{Y}^+)\right], \tag{48}$$

where

$$X^0 = \frac{dU^0}{dr} + \frac{2}{r}U^0 + \frac{i\alpha_0}{r\sqrt{2}}(U^- - U^+), \tag{49}$$

$$\begin{aligned}\mu\nabla\times\nabla\times\mathbf{u} = \mu_0 \Bigg\{ &+\frac{\alpha_0}{r\sqrt{2}}(Z^+ + Z^-)\,\mathbf{Y}^0 \\ &+ \left[+\frac{\alpha_0}{r\sqrt{2}}Z^0 - i\left(\frac{dZ^+}{dr} + \frac{Z^+}{r}\right)\right]\mathbf{Y}^+ \\ &+ \left[+\frac{\alpha_0}{r\sqrt{2}}Z^0 + i\left(\frac{dZ^-}{dr} + \frac{Z^-}{r}\right)\right]\mathbf{Y}^- \Bigg\},\end{aligned} \tag{50}$$

where

$$\begin{aligned} Z^0 &= +\frac{\alpha_0}{r\sqrt{2}}(U^+ + U^-), \\ Z^+ &= +\frac{\alpha_0}{r\sqrt{2}}U^0 - i\left(\frac{dU^+}{dr} + \frac{U^+}{r}\right), \\ Z^- &= +\frac{\alpha_0}{r\sqrt{2}}U_0 + i\left(\frac{dU^-}{dr} + \frac{U^-}{r}\right), \end{aligned} \tag{51}$$

$$2\epsilon \cdot \nabla\mu = 2\,\frac{d\mu_0}{dr}\,(\epsilon^{00}\mathbf{Y}^0 + \epsilon^{0+}\mathbf{Y}^+ + \epsilon^{0-}\mathbf{Y}^-), \tag{52}$$

where

$$\epsilon^{00} = \frac{dU^0}{dr},$$

$$\epsilon^{0+} = \frac{1}{2}\left(\frac{dU^+}{dr} - \frac{U^+}{r} + \frac{i\alpha_0}{r\sqrt{2}}\,U^0\right), \tag{53}$$

$$\epsilon^{0-} = \frac{1}{2}\left(\frac{dU^-}{dr} - \frac{U^-}{r} + \frac{i\alpha_0}{r\sqrt{2}}\,U^0\right),$$

$$(\nabla\cdot\mathbf{u})\,\nabla\lambda = \frac{d\lambda_0}{dr}\,X^0\mathbf{Y}^0, \tag{54}$$

$$\nabla\cdot(\rho\mathbf{u})\,\nabla V = \frac{dV_0}{dr}\left(\frac{d\rho_0}{dr}\,U^0 + \rho_0 X^0\right)\mathbf{Y}^0, \tag{55}$$

and, taking (21) into consideration, we have:

$$\nabla(\rho_0\mathbf{u}\cdot\nabla V) = \frac{d}{dr}\left(\rho_0 U^0\,\frac{dV_0}{dr}\right)\mathbf{Y}^0 + \frac{i\alpha_0}{r\sqrt{2}}\left(\rho_0 U^0\,\frac{dV_0}{dr}\right)(\mathbf{Y}^- - \mathbf{Y}^+). \tag{56}$$

For the increment of the interior geopotential we have

$$\psi = P(r)\,Y_0, \tag{57}$$

assuming the Earth to be a selfgravitating system. From (57) we have:

$$\nabla\psi = \frac{dP}{dr}\,\mathbf{Y}^0 + \frac{i\alpha_0}{r\sqrt{2}}\,P(\mathbf{Y}^- - \mathbf{Y}^+), \tag{58}$$

and, taking

$$\nabla^2 Y_0 = -\frac{\alpha_0^2}{r^2} Y_0$$

and (25) into account, the Poisson equation

$$\nabla^2 \psi = + 4\pi G \nabla \cdot (\rho \mathbf{u}) \tag{59}$$

becomes

$$\frac{d^2 P}{dr^2} + \frac{2}{r}\frac{dP}{dr} - \frac{\alpha_0^2}{r^2} P = 4\pi G \left(U^0 \frac{d\rho_0}{dr} + \rho_0 X^0 \right), \tag{60}$$

the same as in the Pekeris work. For the sake of brevity we systematically omitted the indices ℓ and m in the equations (48) - (60).

Substituting (48), (50) or (36), (52), (55), (56) and (58) into (45') and keeping only the terms of the zeroth order we obtain the differential equation:

$$\begin{aligned} -\rho_0 \omega^2 \mathbf{u} = &\left[\beta_0 \frac{dX^0}{dr} - \mu_0 W^0 + 2\frac{d\mu_0}{dr}\epsilon^{00} + X^0 \frac{d\lambda_0}{dr} \right. \\ &\left. - \rho_0 \left(\frac{2}{r}\frac{dV_0}{dr} - \frac{d^2 V_0}{dr^2} \right) U^0 - \frac{i\alpha_0}{r\sqrt{2}} \frac{dV_0}{dr} \rho_0 (U^- - U^+) + \rho_0 \frac{dP}{dr} \right] \mathbf{Y}^0 \\ &+ \frac{i\alpha_0}{r\sqrt{2}} (\beta_0 X^0 + \rho_0 P)(\mathbf{Y}^- - \mathbf{Y}^+) \\ &+ 2\frac{d\mu_0}{dr}\epsilon^{0-} - \mu_0 W^- \quad \mathbf{Y}^- + \left(2\frac{d\mu_0}{dr}\epsilon^{0+} - \mu_0 W^+ \right) \mathbf{Y}^+, \end{aligned} \tag{61}$$

or, in scalar form

$$- \rho_0 \omega^2 U^0 = \beta_0 \frac{dX^0}{dr} - \mu_0 W^0 + 2 \frac{d\mu_0}{dr} \epsilon^{00} + X^0 \frac{d\lambda_0}{dr} \tag{62}$$

$$- \rho_0 \left(\frac{2}{r} \frac{dV_0}{dr} - \frac{d^2 V_0}{dr^2} \right) U^0 - \frac{i\alpha_0}{r\sqrt{2}} \rho_0 \frac{dV_0}{dr} (U^- - U^+) + \rho_0 \frac{dP}{dr},$$

$$\begin{aligned} - \rho_0 \omega^2 U^- &= + K^0 - \mu_0 W^- + 2 \frac{d\mu_0}{dr} \epsilon^{0-}, \\ - \rho_0 \omega^2 U^+ &= - K^0 - \mu_0 W^+ + 2 \frac{d\mu_0}{dr} \epsilon^{0+}, \end{aligned} \tag{63}$$

where

$$K^0 = + \frac{i\alpha_0}{r\sqrt{2}} \left(\beta_0 X^0 + \rho_0 U^0 \frac{dV_0}{dr} + \rho_0 P \right) \tag{64}$$

and X^0 is given by (35), W^0, W^-, W^+ by (37) and (33), and ϵ^{00}, ϵ^{0-} , ϵ^{0+} by (32).

PERTURBATIVE TERMS

In this section we give the typical perturbative terms in the differential equation of the elastic oscillations (45'). These typical terms carry the influence on $\mathbf{u}_{\ell m}$ of the coupling effects between ρ_{pq}, λ_{pq}, μ_{pq} and $\mathbf{u}_{sk}$ and of the geostrophic force. In developing the formulas we assume that the square of the displacement (and frequency) can be obtained by means of the Schrodinger-Rayleigh technique as a linear combination of Petrashen's harmonics.

From (38) and (44):

$$\mathbf{F}_1 = \beta \nabla \nabla \cdot \mathbf{u}$$
$$= + \left(1;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ 0\right) \mathbf{Y}^0_{\ell m} + \left(1;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ -\right) \mathbf{Y}^-_{\ell m} + \left(1;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ +\right) \mathbf{Y}^+_{\ell m}, \tag{65}$$

where

$$\left(1;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ 0\right) = + C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ 0,\ 0)\, \delta_{k+q,m} \beta_{pq} \frac{dX^0_{sk}}{dr},$$

$$\left(1;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ -\right) = + C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ +1,\ 0)\, \delta_{k+q,m} \beta_{pq} \frac{i\alpha_{s0}}{r\sqrt{2}} X^0_{sk}, \tag{66}$$

$$\left(1;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ +\right) = - C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ -1,\ 0)\, \delta_{k+1,m} \beta_{pq} \frac{i\alpha_{s0}}{r\sqrt{2}} X^0_{sk},$$

and from (36) and (43):

$$\mathbf{F}_2 = \mu \nabla \times \nabla \times \mathbf{u}$$
$$= + \left(2;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ 0\right) \mathbf{Y}^0_{\ell m} + \left(2;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ -\right) \mathbf{Y}^-_{\ell m} + \left(2;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ +\right) \mathbf{Y}^+_{\ell m}, \tag{67}$$

where

$$\begin{aligned}
\left(2;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ 0\right) &= + C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ 0,\ 0)\, \delta_{k+q,m} \mu_{pq} W^0_{sk}, \\
\left(2;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ -\right) &= + C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ +1,\ 0)\, \delta_{k+q,m} \mu_{pq} W^-_{sk}, \\
\left(2;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ +\right) &= + C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ -1,\ 0)\, \delta_{k+q,m} \mu_{pq} W^+_{sk}.
\end{aligned} \tag{68}$$

From (21), (31) and (43):

$$\mathbf{F}_3 = 2\epsilon \cdot \nabla \mu$$
$$= \left(3;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ 0\right) \mathbf{Y}^0_{\ell m} + \left(3;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ -\right) \mathbf{Y}^-_{\ell m} + \left(3;\ \begin{matrix} \ell & s & p \\ m & k & q \end{matrix};\ +\right) \mathbf{Y}^+_{\ell m}, \tag{69}$$

where

$$\left(3;\ \begin{matrix}\ell\ s\ p\\ m\ k\ q\end{matrix};\ 0\right) = +\ 2C(s,\ p,\ \ell;\ k,\ q)\ \delta_{k+q,m}$$

$$\left\{+\ \frac{d\mu_{pq}}{dr}\ \epsilon_{sk}^{00}\ C(s,\ p,\ \ell;\ 0,\ 0)\right.$$

$$\left.-\ \frac{i\alpha_{p0}}{r\sqrt{2}}\ \mu_{pq}\ [\epsilon_{sk}^{0+}C(s,\ p,\ \ell;\ -1,\ +1) - \epsilon_{sk}^{0-}C(s,\ p,\ \ell;\ +1,\ -1)]\right\}$$

$$\left(3;\ \begin{matrix}\ell\ s\ p\\ m\ k\ q\end{matrix};\ -\right) = +\ 2C(s,\ p,\ \ell;\ k,\ q)\ \delta_{k+q,m}$$

$$\left\{+\ \frac{d\mu_{pq}}{dr}\ \epsilon_{sk}^{-0}C(s,\ p,\ \ell;\ +\ 1,\ 0)\right. \tag{70}$$

$$\left.-\ \frac{i\alpha_{p0}}{r\sqrt{2}}\ \mu_{pq}\cdot[\epsilon_{sk}^{-+}C(s,\ p,\ \ell;\ 0,\ +\ 1) - \epsilon_{sk}^{--}C(s,\ p,\ \ell;\ +\ 2,\ -\ 1)]\right\},$$

$$\left(3;\ \begin{matrix}\ell\ s\ p\\ m\ k\ q\end{matrix};\ +\right) = +\ 2C(s,\ p,\ \ell;\ k,\ q)\ \delta_{k+q,m}$$

$$\left\{+\ \frac{d\mu_{pq}}{dr}\ \epsilon_{sk}^{+0}C(s,\ p,\ \ell;\ -\ 1,\ 0)\right.$$

$$\left.-\ \frac{i\alpha_{p0}}{r\sqrt{2}}\ \mu_{pq}\ [\epsilon_{sk}^{++}C(s,\ p,\ \ell;\ -2,\ +\ 1) - \epsilon_{sk}^{+-}C(s,\ p,\ \ell;\ 0,\ -1)]\right\},$$

where, in accordance with (32):

$$\epsilon_{sk}^{00} = \frac{dU_{sk}^{0}}{dr},$$

$$\epsilon_{sk}^{++} = -\ \frac{i\alpha_{s,-1}}{r\sqrt{2}}\ U_{sk}^{+}, \tag{71}$$

$$\epsilon_{sk}^{--} = +\ \frac{i\alpha_{s,+2}}{r\sqrt{2}}\ U_{sk}^{-},$$

$$\epsilon_{sk}^{+0} = \epsilon_{sk}^{0+} = +\frac{1}{2}\left(\frac{dU_{sk}^{+}}{dr} - \frac{U_{sk}^{+}}{r} - \frac{i\alpha_{s0}}{r\sqrt{2}} U_{sk}^{0}\right),$$

$$\epsilon_{sk}^{-0} = \epsilon_{sk}^{0-} = +\frac{1}{2}\left(\frac{dU_{sk}^{-}}{dr} - \frac{U_{sk}^{-}}{r} + \frac{i\alpha_{s0}}{r\sqrt{2}} U_{sk}^{0}\right), \tag{71}$$

$$\epsilon_{sk}^{-+} = \epsilon_{sk}^{+-} = -\frac{U_{sk}^{0}}{r} + \frac{i\alpha_{s0}}{2r\sqrt{2}} (U_{sk}^{+} - U_{sk}^{-}),$$

From (21), (34) and (43):

$$\begin{aligned} \mathbf{F}_4 &= (\nabla\cdot\mathbf{u})\nabla\lambda \\ &= \left(4; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; 0\right) \mathbf{Y}_{\ell m}^{0} + \left(4; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; -\right) \mathbf{Y}_{\ell m}^{-} + \left(4; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; +\right) \mathbf{Y}_{\ell m}^{+}, \end{aligned} \tag{72}$$

where

$$\left(4; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; 0\right) = +C(s, p, \ell; k, q)\, C(s, p, \ell; 0, 0)\, \delta_{k+q,m} X_{sk}^{0} \cdot \frac{d\lambda_{pq}}{dr},$$

$$\left(4; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; -\right) = +C(s, p, \ell; k, q)\, C(s, p, \ell; 0, +1)\, \delta_{k+q,m} \frac{i\alpha_{p0}}{r\sqrt{2}} \lambda_{pq} X_{sk}^{0} \tag{73}$$

$$\left(4; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; +\right) = -C(s, p, \ell; k, q)\, C(s, p, \ell; 0, -1)\, \delta_{k+q,m} \frac{i\alpha_{p0}}{r\sqrt{2}} \lambda_{pq} X_{sk}^{0},$$

and again from (21), (34) and (43):

$$\mathbf{F}_s = \nabla\cdot(\mathbf{u}\delta\rho)\nabla V_0 = \left(s; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; 0\right) \mathbf{Y}_{\ell m}^{0}, \tag{74}$$

where

$$\begin{aligned} \left(s; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; 0\right) &= C(s, p, \ell; k, q)\, \delta_{k+q,m} \frac{dV_0}{dr} \\ &\left\{\left(\frac{d\rho_{pq}}{dr} U_{sk}^{0} + \rho_{pq} X_{sk}^{0}\right) C(s, p, \ell; 0, 0)\right. \\ &\left. + \frac{i\alpha_{p0}}{r\sqrt{2}} \rho_{pq} [U_{sk}^{-} C(s, p, \ell; +1, -1) - U_{sk}^{+} C(s, p, \ell; -1, +1)]\right\}, \end{aligned} \tag{75}$$

$$\mathbf{F}_6 = \nabla\cdot(\rho_0 \mathbf{u})\,\nabla\delta V$$
$$= \begin{pmatrix}6; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; 0\end{pmatrix} \mathbf{Y}^0_{\ell m} + \begin{pmatrix}6; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; -\end{pmatrix} \mathbf{Y}^-_{\ell m} + \begin{pmatrix}6; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; +\end{pmatrix} \mathbf{Y}^+_{\ell m},$$

where

$$\begin{pmatrix}6; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; 0\end{pmatrix} = + C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ 0,\ 0)\ \delta_{k+q,m}$$
$$\cdot \left(\frac{d\rho_0}{dr} U^0_{sk} + \rho_0 X^0_{sk}\right) \frac{dV_{pq}}{dr},$$

$$\begin{pmatrix}6; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; -\end{pmatrix} = + C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ 0,\ +1)\ \delta_{k+q,m} \left(\frac{d\rho_0}{dr} U^0_{sk} + \rho_0 X^0_{sk}\right) \tag{76}$$
$$\cdot \frac{i\alpha_{p0}}{r\sqrt{2}} V_{pq},$$

$$\begin{pmatrix}6; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; +\end{pmatrix} = - C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ 0,\ -1)\ \delta_{k+1} \left(\frac{d\rho_0}{dr} U^0_{sk} + \rho_0 X^0_{sk}\right)$$
$$\cdot \frac{i\alpha_{p0}}{r\sqrt{2}} V_{pq},$$

$$\mathbf{F}_7 = \nabla(\delta\rho\mathbf{u}\cdot\nabla V_0)$$
$$= \begin{pmatrix}7; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; 0\end{pmatrix} \mathbf{Y}^0_{\ell m} + \begin{pmatrix}7; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; -\end{pmatrix} \mathbf{Y}^-_{\ell m} + \begin{pmatrix}7; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; +\end{pmatrix} \mathbf{Y}^+_{\ell m},$$

where

$$\begin{pmatrix}7; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; 0\end{pmatrix} = + C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ 0,\ 0)\ \delta_{k+q,m}$$
$$\cdot \frac{d}{dr}\left(\rho_{pq} U^0_{sk} \frac{dV_0}{dr}\right),$$

$$\begin{pmatrix}7; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; -\end{pmatrix} = - \begin{pmatrix}7; \begin{matrix}\ell & s & p\\ m & k & q\end{matrix}; +\end{pmatrix} = + C(s,\ p,\ \ell;\ k,\ q)\, C(s,\ p,\ \ell;\ 0,\ 0)\ \delta_{k+q,m} \tag{77}$$
$$\cdot \frac{i\alpha_{p,0}}{r\sqrt{2}} \left(\rho_{pq} U^0_{sk} \frac{dV_0}{dr}\right)$$

$$\mathbf{F}_8 = \nabla(\rho_0 \mathbf{u} \cdot \nabla \delta V)$$
$$= \left(8; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; 0\right) \mathbf{Y}^0_{\ell m} + \left(8; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; -\right) \mathbf{Y}^-_{\ell m} + \left(8; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; +\right) \mathbf{Y}^+_{\ell m},$$

where

$$\left(8; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; 0\right) = \frac{d}{dr} \left[8; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}\right],$$

$$\left(8; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; -\right) = -\left(8; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; +\right) = + \frac{i\alpha_{\ell,0}}{r\sqrt{2}} \left[8; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}\right], \tag{78}$$

and

$$\left[8; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}\right] = \rho_0 C(s, p, \ell; k, q)\, \delta_{k+q,m} \Big\{ C(s, p, \ell; 0, 0)\, U^0_{sk} \frac{dV_{pq}}{dr} + \\ + \frac{i\alpha_{p0}}{r\sqrt{2}} \rho_0 V_{pq} [C(s, p, \ell; +1, -1)\, U^-_{sk} - C(s, p, \ell; -1, +1)\, U^+_{sk}] \Big\} \tag{79}$$

In order to obtain the representation of a typical term in the expansion of the geostrophic force

$$i\mathbf{F}_6 = 2\rho_0\, i\omega\Omega\, \mathbf{k} \times \mathbf{u}$$

We substitute into the last expression

$$\mathbf{k} = -\,\mathbf{e}_\theta \sin\theta + \mathbf{e}_r \cos\theta$$

or

$$\mathbf{k} = +\frac{1}{\sqrt{2}} (\mathbf{e}_+ - \mathbf{e}_-) \sin\theta + \mathbf{e}_0 \cos\theta$$

and

$$\mathbf{u} = \mathbf{u}_{\ell-1,m} + \mathbf{u}_{\ell,m} + \mathbf{u}_{\ell+1,m}.$$

Taking

$$P^1_{0,+1} = P^1_{+1,0} = P^1_{0,-1} = P^1_{-1,0} = \frac{i \sin\theta}{\sqrt{2}}$$

$$P^1_{00} = + \cos\theta$$

into consideration and making use of the Clebsch-Gordan expansion, we obtain for the typical term in F_9:

$$\mathbf{F}_9 = \left(9; {\ell \atop m}; 0\right) \mathbf{Y}^0_{\ell m} + \left(9; {\ell \atop m}; -\right) \mathbf{Y}^-_{\ell m} + \left(9; {\ell \atop m}; +\right) \mathbf{Y}^+_{\ell m}, \tag{80}$$

where

$$\begin{aligned}
\left(9; {\ell \atop m}; 0\right) = {} & C(\ell-1, 1, \ell; m, 0)\, [C(\ell-1, 1, \ell; -1, +1)\, U^+_{\ell-1,m} + C(\ell-1, 1, \ell; +1, -1)\, U^-_{\ell-1,m}] \\
& + C(\ell, 1, \ell; m. 0)\, [C(\ell, 1, \ell; -1, +1)\, U^+_{\ell,m} + C(\ell, 1, \ell; +1, -1)\, U^-_{\ell,m}] \\
& + C(\ell+1, 1, \ell; m, 0)\, [C(\ell+1, 1, \ell; -1, +1)\, U^+_{\ell+1,m} + C(\ell+1, 1, \ell; +1, -1)\, U^-_{\ell+1,m}],
\end{aligned}$$

$$\begin{aligned}
\left(9; {\ell \atop m}; -\right) = {} & C(\ell-1, 1, \ell; m, 0)\, [C(\ell-1, 1, \ell; 0, +1)\, U^0_{\ell-1,m} + iC(\ell-1, 1, \ell; +1, 0)\, U^-_{\ell-1,m}] \\
& + C(\ell, 1, \ell; m, 0)\, [C(\ell, 1, \ell; 0, +1)\, U^0_{\ell,m} + iC(\ell, 1, \ell; +1, 0)\, U^-_{\ell,m}] \qquad (81) \\
& + C(\ell+1, 1, \ell; m, 0)\, [C(\ell+1, 1, \ell; 0, +1)\, U^0_{\ell+1,m} + iC(\ell+1, 1, \ell; +1, 0)\, U^-_{\ell+1,m}]
\end{aligned}$$

$$\begin{aligned}
\left(9; {\ell \atop m}; +\right) = {} & C(\ell-1, 1, \ell; m, 0)\, [C(\ell-1, 1, \ell; 0, -1)\, U^0_{\ell-1,m} - iC(\ell-1, 1, \ell; -1, 0)\, U^+_{\ell-1,m}] \\
& + C(\ell, 1, \ell; m, 0)\, [C(\ell, 1, \ell; 0, -1)\, U^0_{\ell m} - iC(\ell, 1, \ell; -1, 0)\, U^+_{\ell,m}] \\
& + C(\ell+1, 1, \ell; m, 0)\, [C(\ell+1, 1, \ell; 0, -1)\, U^0_{\ell+1,m} - iC(\ell+1, 1, \ell; -1, 0)\, U^+_{\ell+1,m}]
\end{aligned}$$

Finally, for the perturbative term

$$\mathbf{F}_{10} = \omega^2 \delta\rho\mathbf{u}$$

we obtain:

$$\mathbf{F}_{10} = \left(10; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; 0\right) \mathbf{Y}^0_{\ell m} + \left(10; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; -\right) \mathbf{Y}^-_{\ell m} + \left(10; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; +\right) \mathbf{Y}^+_{\ell m},$$

where

$$\begin{aligned} \left(10; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; 0\right) &= \omega_0^2 \rho_{pq} C(s, p, \ell; k, q) C(s, p, \ell; 0, 0) \delta_{k+q, m}, \\ \left(10; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; -\right) &= \omega_0^2 \rho_{pq} C(s, p, \ell; k, q) C(s, p, \ell; +1, 0) \delta_{k+q, m}, \\ \left(10; \begin{matrix} \ell & s & p \\ m & k & q \end{matrix}; +\right) &= \omega_0^2 \rho_{pq} C(s, p, \ell; k, q) C(s, p, \ell; -1, 0) \delta_{k+q, m} \end{aligned} \tag{82}$$

In the group of formulas (66), (68), (70), (73), (75), (77), (79), (83) the selection rules are

$$|s - p| \leqq \ell \leqq s + p$$

$$k + q = m.$$

and

$$|k| \leqq p, \qquad |m| \leqq \ell.$$

CONCLUSION

The proper understanding of the mechanics of free oscillations of the Earth permits us to improve our knowledge about the internal structure of the Earth. Free oscillations are normally expanded into a series of vectorial and toroidal harmonics, assuming spherical symmetry of the Earth. However, the perturbative effects of lateral inhomogeneities, of the Coriolis force, and of couplings between the oscillations are now also being considered in seismology. They cause the splitting of frequencies and "contaminate" the originally pure toroidal or pure spherical oscillations. The originally toroidal oscillations aquire perturbative spheroidal components and vice versa. For example, the Coriolis force adds radial components to the toroidal ones (Mac Donald and Ness, 1961). Thus we can talk only about oscillations predominantly spheroidal or predominantly toroidal. For this reason, and to make to computations of the perturbative effects more uniform and homogeneous, we suggest in the present work the expansion of perturbative effects in the free oscillations of the Earth in terms of vectorial generalized harmonics familiar from quantum mechanics. We give the explicite form of those perturbative terms in the differential equation of the Earth's oscillations which are caused by the lateral inhomogeneities and by the Coriolis force. The problem of integration for the cases of geophysical interest and of degeneracy we shall treat in subsequent reports.

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