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March 1975

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by

David Parry Rubincam

ABSTRACT

The general relativistic correction to the position of a satellite is found by retaining Newtonian physics for an observer on the satellite and introducing a r^{-3} potential. The potential is expanded in terms of the Keplerian elements of the orbit and substituted in Lagrange's equations. Integration of the equations shows that a typical earth satellite with small orbital eccentricity is displaced by about 17 cm from its unperturbed position after a single orbit, while the periodic displacement over the orbit reaches a maximum of about 3 cm. The moon is displaced by about the same amounts. Application of the equations to Mercury gives a total displacement of about 58 km after one orbit and a maximum periodic displacement of about 12 km.

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INTRODUCTION

The primary purpose of this work is to investigate the effect of general relativity on the orbits of artificial satellites; but the results may be applied to any body of negligible mass orbiting about a massive, spherically symmetric object. In particular, we will discuss the moon orbiting around the earth and the planet Mercury orbiting around the sun.

Past attacks on the problem have centered around solving the equation (see Ghaffari, 1970 and references contained therein):

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

Our technique will be to back off from this equation a little to a point where we may interpret the equations of motion in the following way: the geometry of space is Euclidian and the physics is Newtonian. The price we pay for this approach is that we must modify the law of gravity and introduce an extra (relativistic) potential. This poses no particular problems, however, since the relativistic potential now becomes a disturbing function susceptible to the methods of celestial mechanics. In particular, the potential may be expressed in terms of the Keplerian elements of the orbit and substituted in Lagrange's equations. The equations may then be integrated to give the osculating elements of the orbit.

The technique has a sound philosophical basis. Even though Einstein (and others) developed general relativity within the framework of non-Euclidian geometry, we can, however, obtain an equivalent description of the world by retaining Euclidian geometry and modifying the laws of physics. This was discussed by Poincaré (1905) and clearly explained by Carnap (1966). Convenience dictates the point of view we choose. Poincaré felt mankind was so accustomed to Euclidian geometry that it might never abandon it in favor of non-Euclidian geometry, even though the latter point of view might represent a simpler picture of the world. Einstein and physicists in general, however, adopted the non-Euclidian approach for reasons of conceptual clarity and mathematical elegance. Indeed, it is doubtful general relativity could have been developed without it. But we will follow Poincaré and introduce an extra Newtonian force, since it results in an elegant description of the motion of a satellite.

(Elegant for satellites but perhaps for not much else. For instance, if we measure the circumference and radius of a circle about the earth we discover that their ratio is not π. Hence we would need some laws about the expansion and contraction of meter sticks. This particular problem is ignored here since the displacements we are concerned with are so small that this effect may be neglected.)

DERIVATION OF THE EQUATIONS OF MOTION*

Let us consider the motion of a body with negligible mass about a massive central object. We will call the two bodies satellite and earth, respectively, since we are primarily concerned with the motion of artificial satellites about the earth.

The geometry of spacetime in the neighborhood of a spherically symmetric earth is given by the Schwarzschild line element (Tolman, eq. 82.9):

$$ds^2 = -\frac{dr^2}{\left(1 - \frac{2GM_E}{c^2 r}\right)} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \left(1 - \frac{2GM_E}{c^2 r}\right) c^2 dt^2 \quad (1)$$

Here ds is the interval of proper distance, (r, θ, ϕ) are polar coordinates, t is the coordinate time, and M_E is the mass of the earth. The speed of light c and universal constant of gravitation G are explicitly retained, in contrast to the usual procedure of setting $G = 1$ and $c = 1$.

The satellite will follow a geodesic in the Schwarzschild geometry according to the geodesic equation (Tolman, eq. 83.1):

$$\frac{d^2x^\sigma}{ds^2} + \{\mu\nu, \sigma\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \quad (2)$$

where $r = x^1$, $\theta = x^2$, $\phi = x^3$, $t = x^4$, and $\{\mu\nu, \sigma\}$ is the Christoffel symbol.

One may derive from equations (1) and (2) the equations of motion along with two constants of motion, k and h (Tolman, eqs. 83.10-11):

$$\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 - \frac{2GM_E}{r}\left(1 + \frac{r^2}{c^2}\frac{d\phi^2}{d\tau^2}\right) = (k^2 - 1)\, c^2 \quad (3)$$

* Our treatment summarizes that of Tolman (1934).

$$r^2 \frac{d\phi}{d\tau} = h. \tag{4}$$

Here ds = $cd\tau$, so that $d\tau$ is an element of proper time as measured by a clock on the satellite (Tolman, pg. 207). Angle θ does not appear in equations (3) and (4) since θ has been set equal to $\pi/2$ without loss of generality. Hence the satellite remains in a fixed plane passing through the center of the earth.

NEWTONIAN FORMULATION

Substitution of (4) into (3) and dividing by 2 yields

$$T + V_N + V_{GR} = \text{constant} \tag{5}$$

where

$$T = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2} r^2 \left(\frac{d\phi}{d\tau}\right)^2$$

$$V_N = -\frac{GM_E}{r} \tag{6}$$

and

$$V_{GR} = -\frac{GM_E h^2}{c^2 r^3}. \tag{7}$$

Consider an observer on the satellite. His space coordinates are (r, θ, ϕ) and he measures time τ with his clock. If the observer assumes his space is Euclidian and his physics is Newtonian, then T is the kinetic energy per unit mass of the satellite and V_N is the ordinary Newtonian potential. V_{GR} is the general relativistic potential which we are now forced to introduce.

Equation (5) now represents conservation of energy, while equation (4) represents conservation of angular momentum. (That angular momentum is conserved is easily seen from equations (6) and (7); both potentials represent central forces.) Hence, from the point of view of the satellite, it moves through Euclidian space under the action of the total potential $V_N + V_{GR}$, with conserved energy and angular momentum.

EQUATIONS OF CELESTIAL MECHANICS

With our Newtonian approach in hand, we are now ready to apply the methods of celestial mechanics.

A satellite moving under the influence of V_N only will describe an ellipse with constant orbital elements (except for M_0) a_0, e_0, i_0, M_0, ω_0, Ω_0. A satellite moving under the added influence of a disturbing function R will have osculating elements a, e, i, M, ω, Ω which change in time according to Lagrange's equations (Brouwer and Clemence, 1961; Blanco and McCuskey, 1961):

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\left\{\sqrt{1-e^2}\,\frac{\partial R}{\partial M} - \frac{\partial R}{\partial \omega}\right\}$$

$$\frac{di}{dt} = -\frac{1}{na^2\sqrt{1-e^2}\,\sin i}\left\{\frac{\partial R}{\partial \Omega} - \cos i\,\frac{\partial R}{\partial \omega}\right\}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\,\sin i}\left\{\frac{\partial R}{\partial i}\right\}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\,\sin i}\frac{\partial R}{\partial i}$$

$$\frac{dM}{dt} = n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{(1-e^2)}{na^2 e}\frac{\partial R}{\partial e}$$

where

$$n = \frac{\sqrt{GM_E}}{a^{3/2}}.$$

For the case under discussion we set $\tau = t$ and $-V_{GR} = R$.

Our task now is to express V_{GR} in terms of the orbital elements and substitute in Lagrange's equations. This may be elegantly done by noting that (Kaula, 1966; Caputo, 1967):

$$\frac{1}{r^{\ell+1}} \cos\{(\ell - 2p)(\omega + f) + m(\Omega - \theta)\}$$

$$= \frac{1}{a^{\ell+1}} \sum_{q=-\infty}^{\infty} G_{\ell pq}(e) \cos\{(\ell - 2p)\omega + (\ell - 2p + q) M + m(\Omega - \theta)\}$$

Here f is the true anomaly, θ is the Greenwich sidereal time, and the $G_{\ell pq}(e)$ are the eccentricity functions. Tables of $G_{\ell pq}(e)$ may be found in Kaula (1966), Caputo (1967), and Cayley (1861); see also Table 1.

Table 1
Eccentricity functions. From Kaula (1966), Caputo (1967), and Cayley (1861).

$G_{210}(e)$	$(1 - e^2)^{-3/2}$
$G_{211}(e)$, $G_{21-1}(e)$	$\frac{3}{2} e + \frac{27}{16} e^3 + \ldots$
$G_{212}(e)$, $G_{21-2}(e)$	$\frac{9}{4} e^2 + \ldots$
$G_{213}(e)$, $G_{21-3}(e)$	$\frac{53}{16} e^3 + \ldots$

Setting $\ell = 2$; $p = 1$; and $m = 0$, we obtain

$$\frac{1}{r^3} = \frac{1}{a^3} \sum_{q=-\infty}^{\infty} G_{21q}(e) \cos(qM)$$

so that

$$R = \frac{GM_E h^2}{c^2} \frac{1}{a^3} \sum_{q=-\infty}^{\infty} G_{21q}(e) \cos(qM).$$

The areal-velocity constant h may be evaluated in terms of the orbital elements. From considerations of the osculating ellipse we find (Blanco and McCuskey, pg. 133):

$$h^2 = GM_E a(1 - e^2).$$

Substitution of the disturbing function into Lagrange's equations yield

$$\frac{di}{dt} = 0 \tag{8}$$

$$\frac{d\Omega}{dt} = 0 \tag{9}$$

$$\frac{d\omega}{dt} = \frac{(GM_E)^{1/2} h^2}{c^2} \frac{(1 - e^2)^{1/2}}{ea^{7/2}} \sum_{q=-\infty}^{\infty} G'_{21q}(e) \cos(qM) \tag{10}$$

$$\frac{dM}{dt} = -\frac{(GM_E)^{1/2} h^2}{c^2} a^{1/2} \left[\frac{1 - e^2}{ea^4} \sum_{q=-\infty}^{\infty} G'_{21q}(e) \cos(qM) - \frac{6}{a^4} \sum_{q=-\infty}^{\infty} G_{21q}(e) \cos(qM)\right] + \frac{(GM_E)^{1/2}}{a^{3/2}} \tag{11}$$

$$\frac{da}{dt} = -\frac{2h^2 (GM_E)^{1/2}}{c^2} \frac{1}{a^{5/2}} \sum_{q=-\infty}^{\infty} qG_{21q}(e) \sin(qM) \tag{12}$$

$$\frac{de}{dt} = -\frac{(GM_E)^{1/2} h^2}{c^2} \frac{1 - e^2}{ea^{7/2}} \sum_{q=-\infty}^{\infty} qG_{21q}(e) \sin(qM) \tag{13}$$

The prime on G_{21q} (e) denotes differentiation with respect to e.

We see from equations (8) and (9) that the inclination i and node Ω remain constant.

We obtain the secular rate of change of the elements by examining the terms for which q = 0:

$$\left[\frac{da}{dt}\right]_S = \left[\frac{de}{dt}\right]_S = 0$$

$$\left[\frac{d\omega}{dt}\right]_S = \frac{(GM_E)^{1/2}\, h^2 (1-e^2)^{1/2}}{c^2 e a^{7/2}} G'_{210}(e) \tag{14}$$

$$\left[\frac{dM}{dt}\right]_S = \frac{(GM_E)^{1/2}\, h^2}{c^2 a^{7/2}} \left[6G_{210}(e) - G'_{210}(e)\,\frac{(1-e^2)}{e}\right] + \frac{(GM_E)^{1/2}}{a^{3/2}} = \left[\frac{dM}{dt}\right]_{S,GR} + \left[\frac{dM}{dt}\right]_{S,N} \tag{15}$$

Here the subscripts S, GR, and N mean "secular", "general relativity" and "Newtonian", respectively.

There is no secular change in the semimajor axis a or eccentricity e.

The well-known expression for the rate of rotation of the argument of perigee ω may be obtained by noting that

$$G_{210}(e) = (1-e^2)^{-3/2}, \qquad G'_{210}(e) = \frac{3e}{(1-e^2)^{5/2}}$$

and $h^2 = GM_E\, a(1-e^2)$. Substituting these expressions into equation (14), we get

$$\left[\frac{d\omega}{dt}\right]_S = \frac{3(GM_E)^{3/2}}{c^2} \frac{1}{a^{5/2}(1-e^2)}\,.$$

This agrees with the usual expression obtained by other means (Tolman, 1934; Bergmann, 1942). Table 2 contains the secular rates of change of the argument of perigee for the satellite Beacon Explorer C, the moon, and the planet Mercury.

Table 2
Secular rates of change of the argument of perigee ω and mean anomaly M for Beacon Explorer C, the moon, and Mercury

Orbiting body	Central object	$\frac{GM_E}{c^2}$	a	e	$\left[\frac{d\omega}{dt}\right]_S$	$\left[\frac{dM}{dt}\right]_{S,GR}$
Beacon Explorer C	Earth	0.443 cm	7.502×10^8 cm	0.025	0.0307″/day	0.0307″/day
Moon	Earth	0.443 cm	3.844×10^{10} cm	0.055	1.63×10^{-6} ″/day	1.63×10^{-6} ″/day
Mercury	Sun	1.477 km	5.791×10^7 km	0.206	43″/century	42″/century

From equation (15) we may write

$$\left[\frac{dM}{dt}\right]_{S,GR} = \frac{(GM_E)^{1/2}\, h^2}{c^2 a^{7/2}} \left[6G_{210}(e) - G'_{210}(e)\left(\frac{1 - e^2}{e}\right)\right]$$

$$= \frac{3(GM_E)^{1/2}\, h^2}{c^2 a^{7/2}(1 - e^2)^{3/2}} = \frac{3(GM_E)^{3/2}}{c^2 a^{5/2}(1 - e^2)^{1/2}}$$

Values for the secular rate of change of the mean anomaly M for the above mentioned bodies may be found in Table 2.

Table 3

Summary of displacement data for Beacon Explorer C, the moon, and Mercury

Orbiting body	Displacement after one orbit	Maximum periodic displacement
Beacon Explorer C	16.7 cm	3.1 cm
Moon	16.8 cm	3.1 cm
Mercury	58.1 km	12.4 km

INTEGRATION OF THE EQUATIONS

If we substitute the elements of the unperturbed orbit a_0, e_0, i_0, M_0, ω_0 into the right side of equations (10)-(13) and set

$$dt = \frac{dM_0}{\dfrac{(GM_E)^{1/2}}{a_0^{3/2}}},$$

then the equations may be integrated to give

$$\omega = \omega_0 + \left(\frac{GM_E}{c^2}\right)\frac{3}{(1 - e_0^2)\, a_0} M_0 + \left(\frac{GM_E}{c^2}\right)\frac{2(1 - e_0^2)^{3/2}}{e_0 a_0}\left\{\sum_{q=1}^{\infty} G'_{21}(e_0)\frac{\sin qM_0}{q}\right\} \qquad (16)$$

$$M = M_0 + \left(\frac{GM_E}{c^2}\right) \frac{3}{(1 - e_0)^{1/2}\, a_0} M_0$$

$$- \left(\frac{GM_E}{c^2}\right) \frac{2(1 - e_0)^2}{e_0 a_0} \left\{ \sum_{q=1}^{\infty} G'_{21q}(e_0) \frac{\sin qM_0}{q} \right\}$$

$$+ \left(\frac{GM_E}{c^2}\right) \frac{12(1 - e_0^2)}{a_0} \left\{ \sum_{q=1}^{\infty} G_{21q}(e_0) \frac{\sin qM_0}{q} \right\} \qquad (17)$$

$$a = a_0 + \left(\frac{GM_E}{c^2}\right) 4(1 - e_0^2) \left\{ \sum_{q=1}^{\infty} G_{21q}(e_0) \cos qM_0 - \sum_{q=1}^{\infty} G_{21q}(e) \right\} \qquad (18)$$

$$e = e_0 + \left(\frac{GM_E}{c^2}\right) \frac{2(1 - e_0^2)^2}{e_0 a_0} \left\{ \sum_{q=1}^{\infty} G_{21q}(e_0) \cos qM_0 - \sum_{q=1}^{\infty} G_{21q}(e_0) \right\} \qquad (19)$$

as the approximate expressions for the elements of the perturbed orbit. The constants of integration have been adjusted so that the perturbed and unperturbed elements are identical at perigee.

The displacement in the position of the satellite due to the relativistic potential may now be found. This was accomplished with the computer program given in the Appendix. The program takes the Keplerian elements of the perturbed orbit given by equations (16)-(19) and converts them to Cartesian coordinates and velocities $x, y, z, \dot{x}, \dot{y}, \dot{z}$. The vector $\vec{r} = (x, y, z)$ gives the perturbed position of the satellite. The process is repeated for the unperturbed orbit, obtaining the position vector $\vec{r}_0 = (x_0, y_0, z_0)$. The difference $\Delta \vec{r} = \vec{r} - \vec{r}_0$ gives the displacement of the satellite due to the relativistic potential.

The orbits are taken to lie in the xy plane as shown in Figure 1, so that $z = z_0 = 0$ and $\dot{z} = \dot{z}_0 = 0$. The x axis lies along the line of perigee of the unperturbed orbit, and the mean anomaly increases in the positive (counterclockwise) sense.

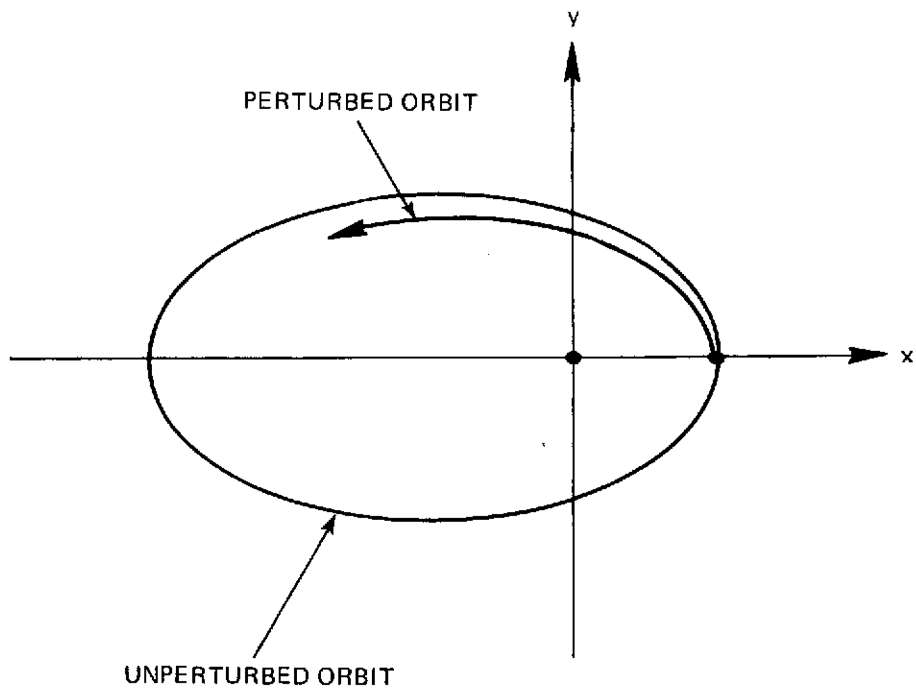

Figure 1. Orientation of the Orbit.

The program uses double precision variables and can consider powers of $e \leq 2$ or ≤ 1 in the eccentricity functions and their derivatives, depending upon the choice of the programmer.

Figure 2 plots $\Delta\vec{r}$ for a typical earth satellite (Beacon Explorer C). The unperturbed position is at the origin (point O in the figure). The perturbed and unperturbed positions are coincident at perigee (point O) and the perturbed position moves away from the unperturbed position in a spiral as time progresses. After one revolution the perturbed position is at point A, about 16.7 cm from the unperturbed position. The displacement of the moon is given in Figure 3. Here the displacement is about 16.8 cm at the end of one revolution.

Figure 4 gives $\Delta\vec{r}$ for Mercury. The rather distorted shape of the spiral is due to the large eccentricity of the orbit. Mercury is displaced by about 58.1 km at the end of one revolution.

The secular displacement of Mercury is shown in Figure 5. Here the periodic terms have been omitted.

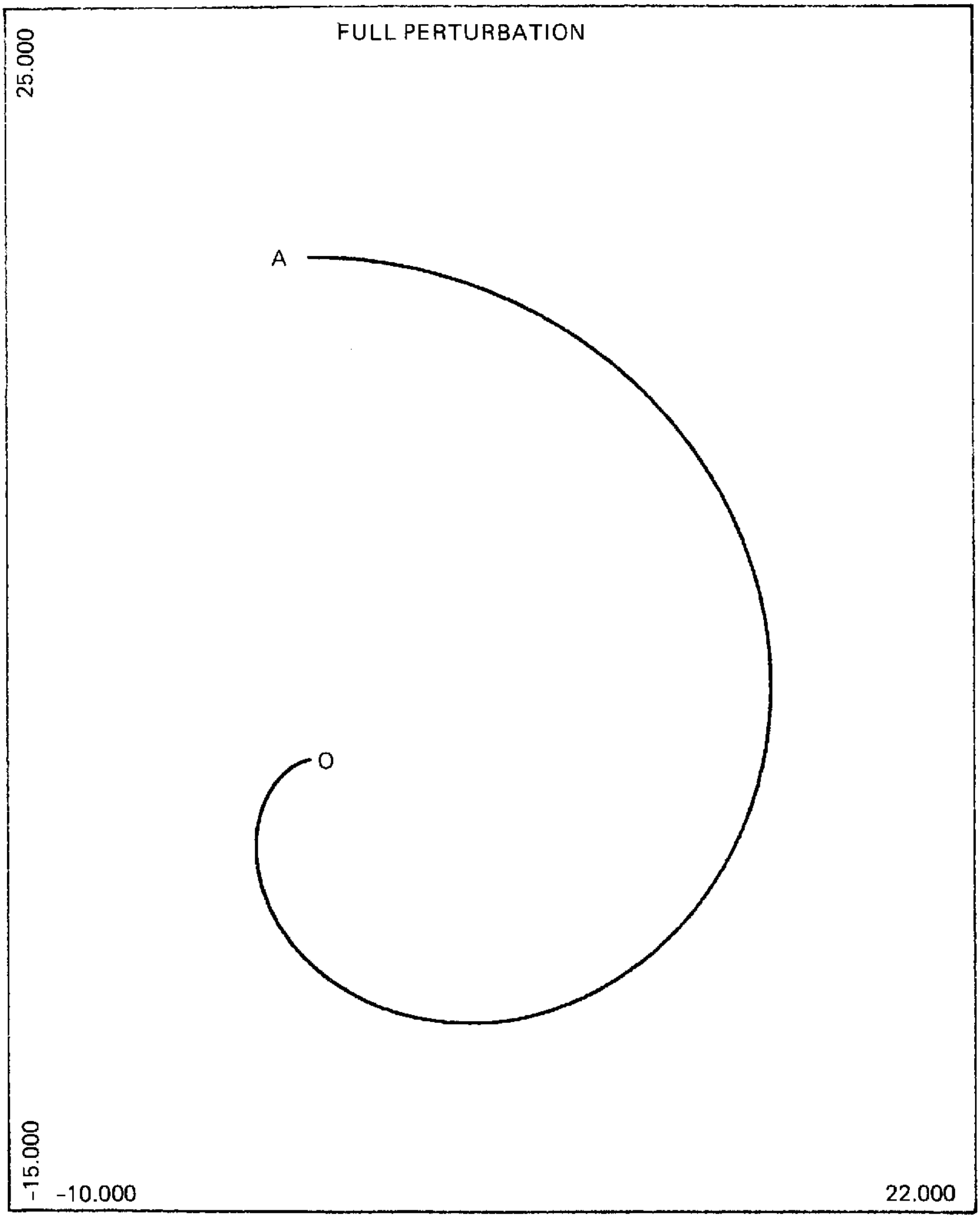

Figure 2. The total displacement of Beacon Explorer C due to the relativistic potential. The unperturbed position is at point O. The perturbed position is at point A after one revolution of the unperturbed orbit. The numbers in the diagram refer to centimeters. The diagram itself is one-half actual size.

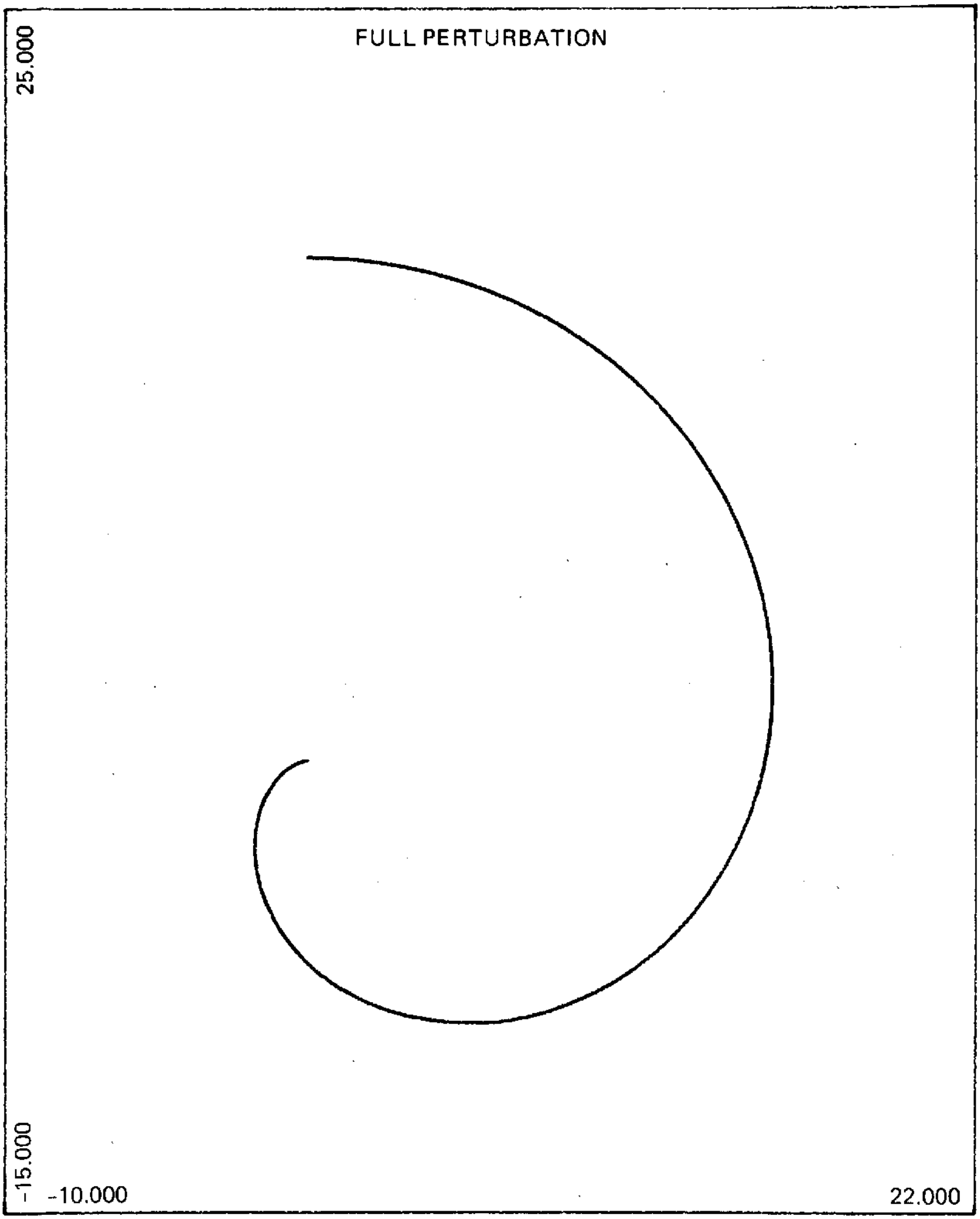

Figure 3. Total displacement of the moon over one orbit. The numbers in the diagram refer to centimeters. One-half actual size.

The periodic displacement for Beacon Exploer C (top) and the moon (bottom) are given in Figure 6. In each case the maximum displacement (maximum $|\Delta \vec{r}|$) is about 3.1 cm. The sense of rotation is counterclockwise.

The periodic displacement for Mercury is shown in Figure 7 (top). The maximum distance from the unperturbed position is about 12.4 km.

A summary of the numerical data for Figure 2-7 is given in Table 3.

Powers of $e \leq 2$ were retained in the eccentricity functions and their derivatives for the computations for Figures 2, 3, 4, 6, and 7 (top). The periodic displacement for Mercury with powers of $e \leq 1$ retained is given in Figure 7 (bottom). The great difference in the shapes of the curves emphasizes the importance of keeping many terms in the eccentricity functions and their derivatives when the eccentricity is large.

It is interesting to note that the displacements are the same for any two bodies orbiting the same massive central object, regardless of the values of the semi-major axes, so long as the eccentricities are the same. This is apparently due to the appearance of a_0 in the denominators of the corrections to ω_0, M_0, and e_0 (equations 16, 17, and 19). As a_0 changes scale, ω_0, M_0, and e_0 change in such a manner as to compensate for it. This explain why the displacement of Beacon Explorer C and the moon are very nearly the same.

THE ORBIT AS SEEN FROM THE GROUND

So far we have proceeded from the point of view of an observer on the satellite. An observer on the ground sees a somewhat more complicated set of forces acting than does the observer on the satellite; this may be verified by examining the Schwarzschild metric from the point of view of the ground observer. We will not pursue this very far, except to say that both observers will agree on the track of the satellite across the sky. That this is so may be seen by imaging space to be laced with coordinate lines. The two observers must agree that the satellite arrives at various points in the coordinate system as the satellite moves around the earth. This line of reasoning is implicit in the equation given in the Introduction; the solution of the equation gives r $(= u^{-1})$ as a function of ϕ, which is the same regardless of who is looking at the satellite. Hence when both the observer on the ground and the satellite plot out the orbit, they will get the same answer.

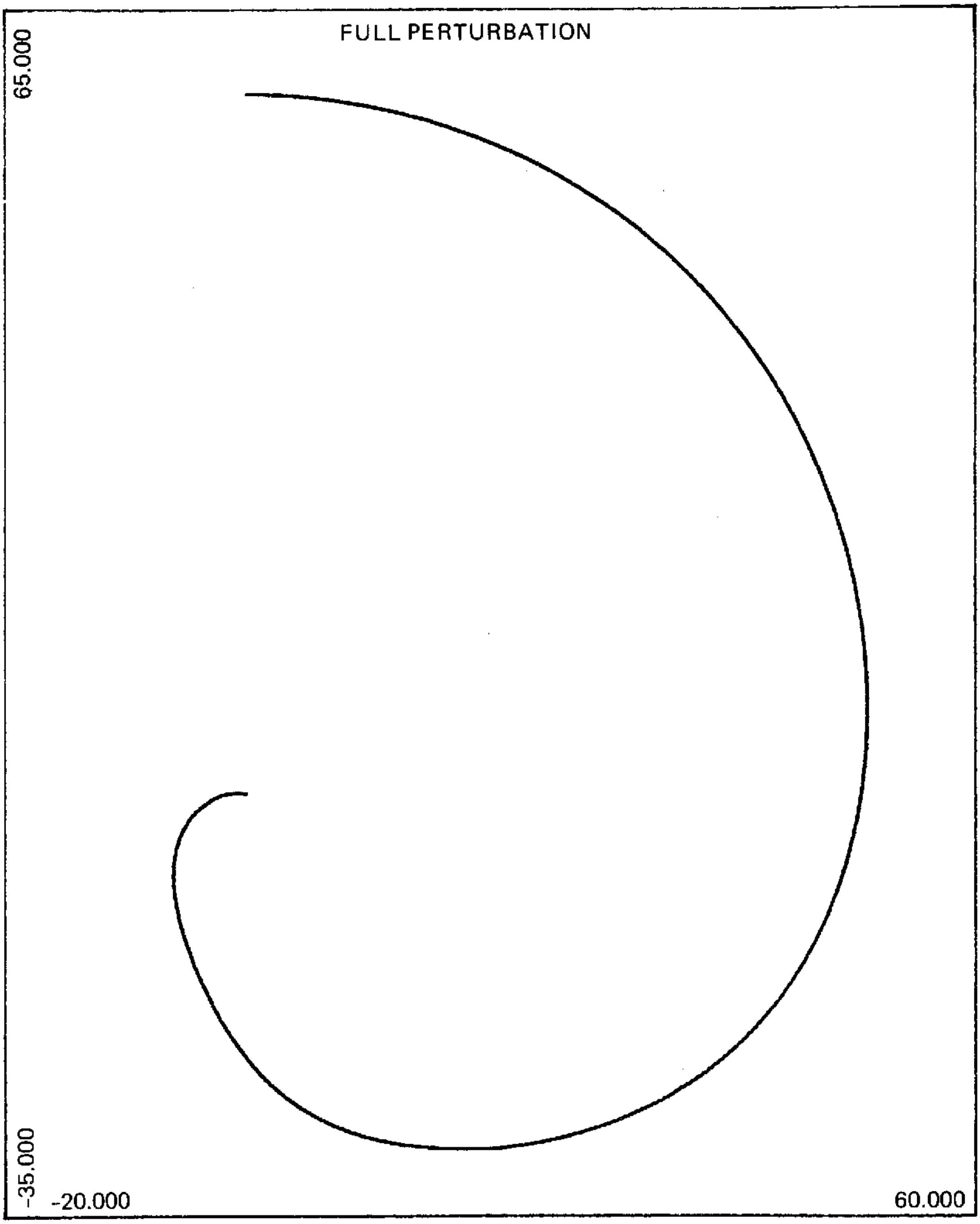

Figure 4. Total displacement of Mercury over one orbit. The distorted shape of the spiral is due to the large eccentricity of the orbit. The numbers in the diagram refer to kilometers.

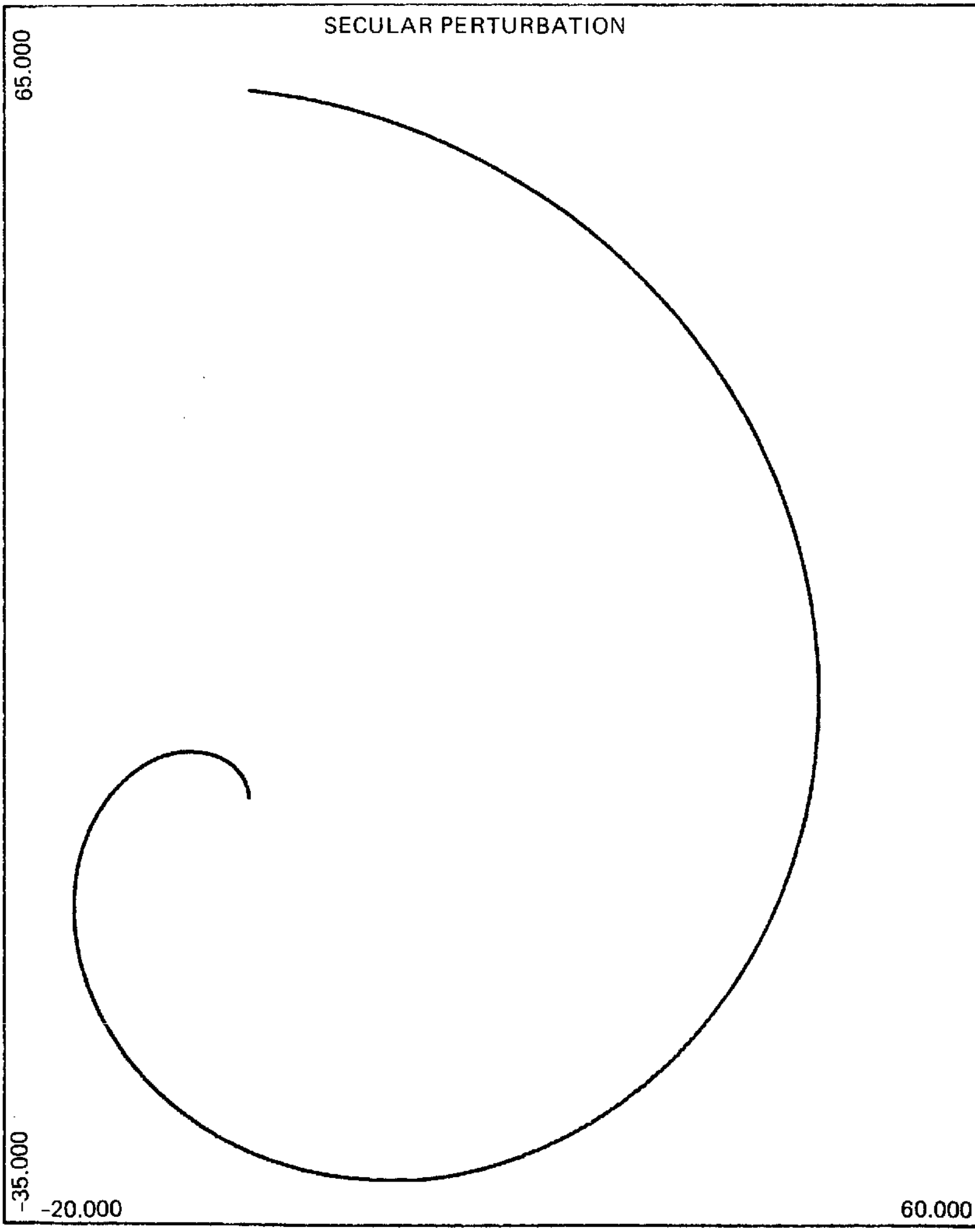

Figure 5. Secular displacement of Mercury. The numbers in the diagram refer to kilometers.

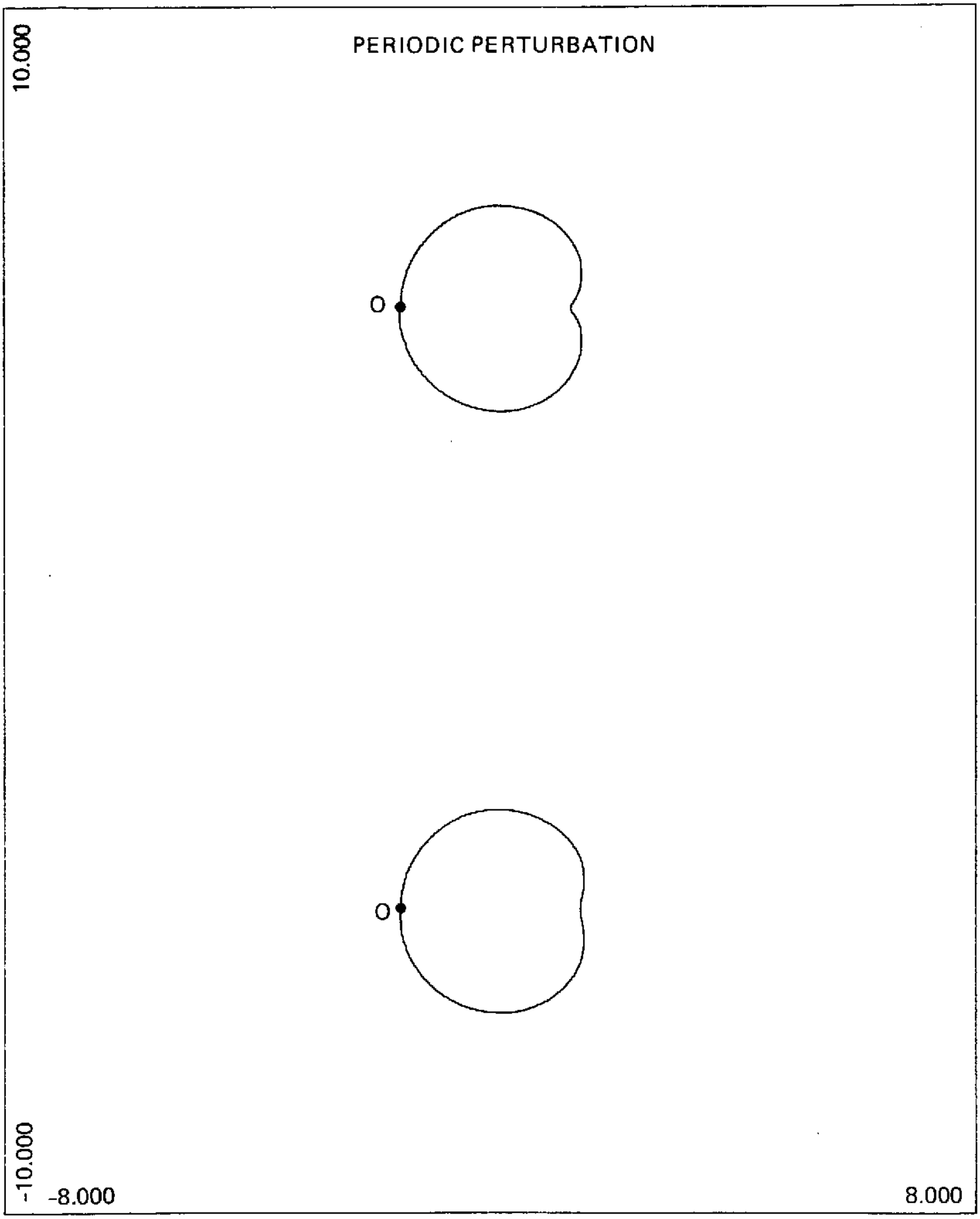

Figure 6. The periodic displacement of Beacon Explorer C (top) and the moon (bottom). The sense of rotation is counterclockwise. The numbers refer to centimeters. Actual size.

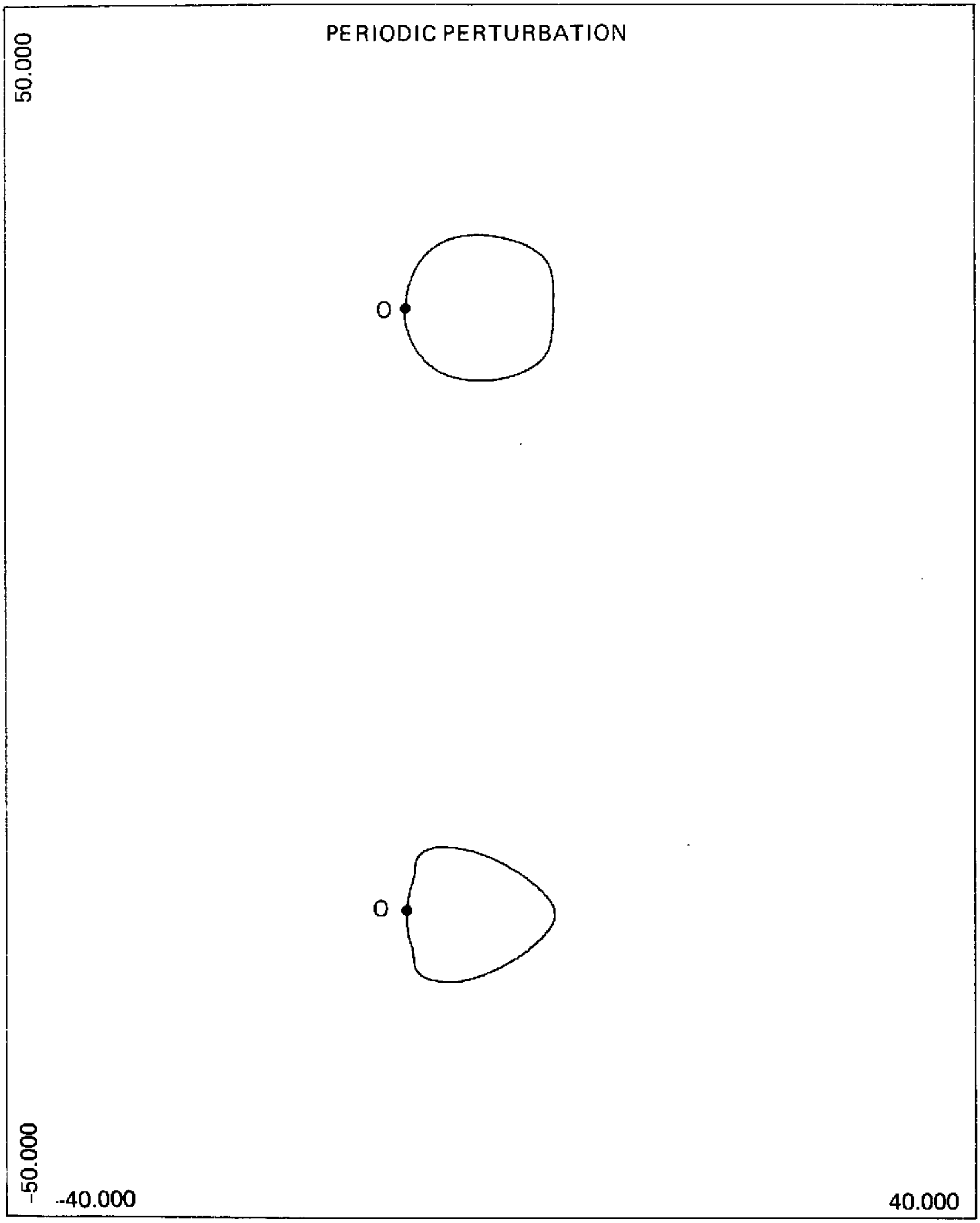

Figure 7. The periodic displacement of Mercury keeping powers of e ⩽ 2 (top) and ⩽1 (bottom). The top diagram gives a more accurate picture of the periodic displacement than the bottom diagram. The numbers refer to kilometers.

The two observers will, however, disagree on the times of arrival of the satellite at various points in the coordinate system. Their clocks run at different rates, since they are in relative motion and in different parts of the gravitational field. But the time intervals recorded by the clocks will differ from each other by a factor of about GM_E/c^2r, or about one part in 10^8 for the earth. Hence when the two observers compute the perigee shift, say, they will disagree on when the shift was α radians after a day's time by about a millisecond. This demonstrates that the timing problem is not important when tracking artificial satellites.

ALTERNATIVE VIEWPOINT

We can look at what we have done from the more usual non-Euclidian viewpoint. To do so requires a few words about coordinates. Let us confine our remarks to the (r, ϕ) plane, i.e. $\theta = \pi/2$.

Now r measures radial distance from the center of the earth; in fact, $r = (\text{proper area of sphere}/4\pi)^{1/2}$ (Misner, Thorne, and Wheeler, 1973; pg. 596). ϕ measures the angle on a sphere. So to get to the point (r, ϕ) in physical space we go out the appropriate distance r and swing through the angle ϕ.

We of course wish to find the set of values $\{r, \phi\}$ which gives us the path of the satellite through physical space. We can accomplish this by doing the following: take a point (r, ϕ) in physical space and assign it a point (r_E, ϕ_E) in a Euclidian plane with numerical values $r = r_E$ and $\phi = \phi_E$. Do this for all the points in physical space. Solve the equations of motion of the satellite by the methods of celestial mechanics to get its path in this Euclidian plane. In particular, we wind up with a set of x and y values $\{x_E, y_E\}$ for the track of the satellite.

To find its path in physical space, what do we do? Take each (x_E, y_E) in the set and put $r_E = \sqrt{x_E^2 + y_E^2}$, $\phi_E = \text{Arctan}\, y_E/x_E$. Then set $r = r_E$ and $\phi = \phi_E$. Go out distance r from the center of the earth and swing through angle ϕ. Do this for all the points in the set to get the track of the satellite through physical space.

Now if we let the speed of light approach infinity, then the trajectory of the satellite becomes the unperturbed ellipse. We say that the difference between the perturbed and unperturbed positions is the displacement of the satellite. Computing the distances between the positions and getting so many centimeters displacement (for the case of the earth) may be done in the ordinary Euclidian sense, since the effect of the curvature of space is so small over such short distances that it may be neglected.

CONCLUSION

The general relativistic correction to the position of an earth satellite with small orbital eccentricity has been shown to be about 17 cm per revolution. This effect is at present too small to be separated from other perturbing influences, such as radiation from the earth and atmospheric drag. However, improved knowledge of satellite perturbations and small atmospheric drag may allow the relativistic effect to be measured from observations of the proposed new geodynamic satellite LAGEOS. If it hoped that such observations will provide tests of the Einstein and Brans-Dicke theories of relativity.

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REFERENCES

1. Bergmann, Peter Gabriel, Introduction to the Theory of Relativity, Prentice-Hall, Englewood Cliffs, N.J., 1942.

2. Blanco, V. M., and S. W. McCuskey, Basic Physics of the Solar System, Addison-Wesley Publishing Company, Reading, Massachusetts, 1961.

3. Brouwer, Dirk, and Gerald M. Clemence, Methods of Celestial Mechanics, Academic Press, New York, 1961.

4. Caputo, Michele, The Gravity Field of the Earth, Academic Press, New York, 1967.

5. Carnap, Rudolf, Philosophical Foundations of Physics, edited by Martin Gardner, Basic Books, New York, 1966.

6. Cayley, A., Tables of the developments of functions in the theory of elliptic motion, Memoirs of the Royal Astronomical Society, 29, 191-306, 1861.

7. Ghaffari, Abolghassem, "Integration of the Relativistic Equations of Motion of An Artificial Earth Satellite," NASA Technical Report R-341, National Aeronautics and Space Administration, Washington, D.C., July 1970.

8. Kaula, William M., Theory of Satellite Geodesy, Blaisdell Publishing Company, Waltham, Massachusetts, 1966.

9. Misner, Charles W., Kip S. Thorne, and John Archibald Wheeler, Gravitation, W. H. Freeman and Company, San Francisco, 1973.

10. Poincaré, Henri, Science and Hypothesis, 1905; reprinted by Dover Publications, New York, 1952.

11. Tolman, Richard C., Relativity, Thermodynamics, and Cosmology, Oxford University Press, London, 1934.

APPENDIX

This program takes the Keplerian elements of the perturbed and unperturbed orbits and calculates displacements. Explanations of its operation are given in the program itself. Some sample output is given.

```
C
C*****MAIN PROGRAM
C
C
C*****THIS PROGRAM COMPUTES A SEQUENCE OF DIFFERENCE VECTORS BETWEEN THE
C      POSITION IN AN UNPERTURBED ORBIT AND THE POSITION IN THE ORBIT AS
C      PERTURBED BY THE RELATIVISTIC POTENTIAL AS THE MEAN ANOMALY OF
C      THE UNPERTURBED ORBIT INCREASES WITH TIME. THE NUMBER OF SUCH
C      VECTORS IS NPOINT AND THE STEP SIZE IN THE MEAN ANOMALY IS DEL.
C      THE COORDINATES OF THE DISTURBED POSITION RELATIVE TO THE
C      UNDISTURBED POSITION ARE LISTED AND PLOTTED. THE SAME IS DONE FOR
C      THE VELOCITY VECTORS, EXCEPT THAT NO PLOT IS GIVEN.
C      THE KEPLERIAN ELEMENTS OF BOTH PERTURBED AND UNPERTURBED ORBITS
C      ARE INITIALLY CHOSEN TO BE THE SAME AT PERIGEE.
C
C
C*****NOTATION FOR INPUT DATA.
C
C.....INDEX TELLS WHAT OBJECT THE SATELLITE IS ORBITING ABOUT. INDEX=1
C      IS THE SUN, 2 THE EARTH, AND 3 ANY OTHER GIVEN OBJECT.
C.....ACM IS THE SEMIMAJOR AXIS OF THE ORBIT IN CENTIMETERS.
C.....E1 IS THE ECCENTRICITY.
C.....XMASS3 IS THE MASS OF THE OBJECT IN GRAMS IF INDEX=3.
C.....DIS3 IS THE DISTANCE FACTOR IN CENTIMETERS CHOSEN SUCH THAT
C      ACM/DIS3=1 APPROXIMATELY, IF INDEX=3.
C.....NTERM TELLS WHAT TERMS ARE TO BE RETAINED IN THE PERTURBATION.
C      NTERM =0 GIVES THE FULL PERTURBATION, NTERM=1 GIVES PERIODIC TERMS
C      ONLY, WHILE NTERM=2 GIVES SECULAR TERMS ONLY.
C.....NPOINT IS THE NUMBER OF POINTS IN THE SEQUENCE.
C.....XMSTRT IS THE INITIAL VALUE OF THE MEAN ANOMALY IN DEGREES.
C.....DEL IS THE STEP SIZE OF THE MEAN ANOMALY IN DEGREES.
C.....NPERT TELLS WHAT POWERS OF ECCENTRICITY ARE RETAINED IN THE
C      ECCENTRICITY FUNCTIONS AND THEIR DERIVATIVES. IF NPERT=1, THEN
C      POWERS OF ECCENTRICITY GREATER THAN 1 ARE NEGLECTED. IF NPERT=0,
C      THEN POWERS GREATER THAN 2 ARE NEGLECTED.
C
C*****NOTATION FOR OUTPUT DATA.
C
C.....X AND Y ARE THE CARTESIAN COORDINATES IN CENTIMETERS OF THE
C      PERTURBED POSITION RELATIVE TO THE UNPERTURBED POSITION (WHICH IS
C      AT X=0 AND Y=0.)
C.....R=DSQRT(X**2 + Y**2) IS THE DISTANCE FROM THE ORIGIN TO (X,Y).
C.....XDT AND YDT ARE THE CARTESIAN VELOCITY DIFFERENCES IN CM/SEC
C      BETWEEN THE PERTURBED AND UNPERTURBED VELOCITIES.
C.....V=DSQRT(XDT**2 + YDT**2) IS THE MAGNITUDE OF THE VELOCITY
C      DIFFERENCE.
C
       IMPLICIT REAL*8 (A-H,O-Z)
       DIMENSION X1(400),Y1(400)
       DATA RAD/0.01745329251994300/
       DATA BIGG/6.67D-8/,C/2.998D10/
       DATA XMASS1/1.987D33/,XMASS2/5.976D27/,DIS1/1.496D13/
       DATA DIS2/6.37815115408/
       DATA P1,P2,P3/5H  SUN,5HEARTH,5HOTHER/
C*****READ IN THE INPUT DATA.
 1     READ (5,2,END=12) INDEX,ACM,E1,XMASS3,DIS3
       READ (5,16) NTERM,NPOINT,XMSTRT,DEL,NPERT
 2     FORMAT (I5,D15.5,F10.5,2D15.5)
 16    FORMAT (2I5,2F10.5,I5)
C*****SET THE VALUES OF THE INCLINATION,LONGITUDE OF NODE,AND ARGUMENT
C      OF PERIGEE TO ZERO, SO THAT THE ORBIT LIES IN THE X Y PLANE AND
C      THE SEMIMAJOR AXIS OF THE UNPERTURBED ORBIT LIES ALONG THE X AXIS
C      WITH THE POINT OF PERIGEE ON THE POSITIVE SIDE OF THE AXIS.
       FI1=0.0D0
       OMEG1=0.0D0
       FNODE1=0.0D0
C*****CHECK TO SEE WHAT MASS IS BEING USED.
       IF (INDEX - 2) 3,4,5
 3     XMASS=XMASS1
       DISFAC=DIS1
       P=P1
       GO TO 6
 4     XMASS=XMASS2
       DISFAC=DIS2
       P=P2
       GO TO 6
 5     XMASS=XMASS3
       DISFAC=DIS3
       P=P3
 6     CONTINUE
```

```
C*****GMC2=BIG G * MASS/(SPEED OF LIGHT)**2 IN CGS UNITS.
      GMC2=BIGG*XMASS/(C**2)
      SQ=(BIGG*XMASS)/DISFAC
C*****CONV IS A CONVERSION FACTOR FOR THE VELOCITIES.
      CONV=DSQRT(SQ)
      A1=ACM/DISFAC
C*****WRITE OUT THE HEADINGS.
      WRITE (6,7)
    7 FORMAT (1H1)
      WRITE (6,8) P,XMASS,DISFAC
    8 FORMAT (///,10X,16HDISTURBING BODY=,1X,A5,10X,5HMASS=,D15.5,1X,2HG
     1M,10X,16HDISTANCE FACTOR=,1X,D15.5,1X,2HCM)
      WRITE (6,9) ACM,E1
    9 FORMAT (/////,10X,2HA=,1X,D15.5,1X,2HCM,20X,2HE=,1X,F10.5)
      WRITE (6,14) NPOINT,DEL
   14 FORMAT (///,10X,7HNPOINT=,1X,I5,20X,4HDEL=,1X,F10.5,1X,7HDEGREES)
      IF (NTERM - 1) 17,18,19
   17 WRITE (6,20)
   20 FORMAT (///,10X,17HFULL PERTURBATION)
      GO TO 23
   18 WRITE (6,21)
   21 FORMAT (///,10X,19HPERIODIC TERMS ONLY)
      GO TO 23
   19 WRITE (6,22)
   22 FORMAT (///,10X,18HSECULAR TERMS ONLY)
   23 CONTINUE
      IF (NPERT - 1) 24,26,26
   24 WRITE (6,25)
   25 FORMAT (///,10X,77HTERMS UP TO AND INCLUDING SECOND ORDER RETAINED
     1 IN THE ECCENTRICITY FUNCTIONS)
      GO TO 28
   26 WRITE (6,27)
   27 FORMAT (///,10X,77HTERMS UP TO AND INCLUDING FIRST  ORDER RETAINED
     1 IN THE ECCENTRICITY FUNCTIONS)
   28 CONTINUE
      WRITE (6,29)
   29 FORMAT (10X,21HAND THEIR DERIVATIVES)
      WRITE (6,7)
      WRITE (6,30)
   30 FORMAT (///,10X)
      WRITE (6,13)
   13 FORMAT (3X,1HN,4X,12HMEAN ANOMALY,7X,1HX,14X,1HY,14X,1HR,13X,3HXDOT
     1,12X,3HYDOT,13X,1HV)
      WRITE (6,15)
   15 FORMAT (10X,9H(DEGREES),7X,4H(CM),11X,4H(CM),11X,4H(CM),9X,8H(CM/S
     1EC),7X,8H(CM/SEC),7X,8H(CM/SEC),//)
C*****DO THE ITERATION.
      DO 10 N=1,NPOINT
      XMEAN=(N-1)*DEL + XMSTRT
      XM1=XMEAN*RAD
C*****A1,E1,XM1,OMEG1 ARE THE UNPERTURBED ELEMENTS.
C*****A2,E2,XM2,OMEG2 ARE THE PERTURBED ELEMENTS.
C*****CALL DELTEL TO GET THE RELATIVISTIC CORRECTIONS TO THE ELEMENTS.
      CALL DELTEL (NTERM,NPERT,GMC2,ACM,E1,XM1,DELA,DELE,DELM,DELOMG)
      A2=A1 + DELA/DISFAC
      E2=E1 + DELE
      XM2=XM1 + DELM
      OMEG2=OMEG1 + DELOMG
C*****CALL OETRV TO CONVERT THE KEPLERIAN ELEMENTS TO CARTESIAN
C     COORDINATES AND VELOCITIES.
      CALL OETRV(A1,E1,FI1,XM1,OMEG1,FNODE1,XA,YA,ZA,XDTA,YDTA,ZDTA,RA,V
     1A)
      CALL OETRV(A2,E2,FI1,XM2,OMEG2,FNODE1,XB,YB,ZB,XDTB,YDTB,ZDTB,RB,V
     1B)
C*****COMPUTE THE DIFFERENCES IN COORDINATES AND VELOCITIES.
      X=(XB-XA)*DISFAC
      Y=(YB-YA)*DISFAC
      Z=(ZB-ZA)*DISFAC
      R2=(X**2) + (Y**2) + (Z**2)
      R=DSQRT(R2)
      XDT=(XDTB-XDTA)*CONV
      YDT=(YDTB-YDTA)*CONV
      ZDT=(ZDTB-ZDTA)*CONV
      V2=(XDT**2) + (YDT**2) + (ZDT**2)
      V=DSQRT(V2)
C*****PUT THE POINTS IN THE ARRAY FOR PLOT1.
      X1(N)=X
      Y1(N)=Y
C*****WRITE OUT THE MEAN ANOMALY, DISTANCES, AND VELOCITIES.
      WRITE (6,11) N,XMEAN,X,Y,R,XDT,YDT,V
```

```
   10   CONTINUE
C*****CALL PLCT1 TO PLOT THE POINTS.
        CALL PLCT1(X1,Y1,NPOINT)
        GO TO 1
   12   CONTINUE
   11   FORMAT (1X,I4,2X,F10.4,3X,6D15.5)
        STOP
        END
```

```
      SUBROUTINE DELTEL (NTERM,NPERT,GMC2,A0,E0,XM0,DELA,DELE,DELM,DELOM
     1G)
C
C
C*****THIS SUBROUTINE FINDS THE RELATIVISTIC CORRECTIONS TO THE
C     KEPLERIAN ELEMENTS, WHICH ARE TO BE ADDED TO THE UNPERTURBED
C     ELEMENTS TO GIVE THE ELEMENTS OF THE RELATIVISTICALLY PERTURBED
C     ORBIT.
C
C*****UNPERTURBED ELEMENTS
C
C*****A0 IS THE SEMIMAJOR AXIS IN CENTIMETERS.
C*****E0 IS THE ECCENTRICITY.
C*****XM0 IS THE MEAN ANOMALY IN RADIANS.
C
C*****PERTURBED ELEMENTS
C
C*****DELA IS THE CORRECTION TO THE SEMIMAJOR AXIS IN CENTIMETERS.
C*****DELE IS THE CORRECTION TO THE ECCENTRICITY.
C*****DELM IS THE CORRECTION TO THE MEAN ANOMALY IN RADIANS.
C*****DELOMG IS THE CORRECTION TO THE ARGUMENT OF PERIGEE IN RADIANS.
C*****THE PERTURBATIONS IN INCLINATION AND LONGITUDE OF NODE ARE
C     ZERO AND HENCE NOT INCLUDED IN THE SUBROUTINE.
C*****NTERM TELLS WHAT TERMS TO RETAIN IN THE PERTURBATION. NTERM=0
C     GIVES THE FULL PERTURBATION. NTERM=1 GIVES PERIODIC TERMS ONLY,
C     WHILE NTERM=2 GIVES SECULAR TERMS ONLY.
C*****GMC2 = BIG G * MASS/(SPEED OF LIGHT)**2 IN CGS UNITS.
C*****NPERT TELLS WHAT POWERS OF ECCENTRICITY ARE RETAINED IN THE
C     ECCENTRICITY FUNCTIONS AND THEIR DERIVATIVES. IF NPERT=1, THEN
C     POWERS OF ECCENTRICITY GREATER THAN 1 ARE NEGLECTED. IF NPERT=0,
C     THEN POWERS GREATER THAN 2 ARE NEGLECTED.
C
      IMPLICIT REAL*8(A-H,O-Z)
      F1=1.0D0 - (E0**2)
      F2=DSQRT(F1)
      SM=DSIN(XM0)
      CM=DCOS(XM0)
      XM2=2.0D0*XM0
      XM3=3.0D0*XM0
      SM2=DSIN(XM2)
      CM2=DCOS(XM2)
      SM3=DSIN(XM3)
C*****G211 AND G212 ARE ECCENTRICITY FUNCTIONS.
C*****GP211, GP212, AND GP213 ARE DERIVATIVES OF ECCENTRICITY FUNCTIONS.
      IF (NPERT .EQ. 1) GO TO 4
      G211=1.5D0*E0
      G212=(9.0D0/4.0D0)*(E0**2)
      GP211=1.5D0 + (81.0D0/16.0D0)*(E0**2)
      GP212=4.5D0*E0
      GP213=(153.0D0/16.0D0)*(E0**2)
      GO TO 5
    4 G211=(1.5D0)*E0
      G212=0.0D0
      GP211=1.5D0
      GP212=(4.5D0)*E0
      GP213=0.0D0
    5 CONTINUE
      B1=G211*CM
      B2=GP211*SM
      B3=G211*SM
      C1=G212*CM2
      C2=GP212*SM2/2.0D0
      C3=G212*SM2/2.0D0
      P2=GP213*SM3/3.0D0
      FA=GMC2*(4.0D0)*F1
      DELA=FA*B1 - FA*G211 + FA*C1 - FA*G212
      FE=GMC2*(2.0D0)*(F1**2)/(E0*A0)
      DELE=FE*B1 - FE*G211 + FE*C1 - FE*G212
      FMS=GMC2*(3.0D0)/(F2*A0)
      XMS=FMS*XM0
      FM1=GMC2*(2.0D0)*(F1**2)/(E0*A0)
      FM2=GMC2*(12.0D0)*F1/A0
      DELM=XMS - FM1*(B2 + C2 + P2) + FM2*(B3 + C3)
      FOMGS=GMC2*(3.0D0)/(F1*A0)
      FOMG1=GMC2*(2.0D0)*F1*F2/(E0*A0)
      DELOMG=FOMGS*XM0 + FOMG1*(B2 + C2 + P2)
      IF (NTERM - 1) 3,2,1
    1 DELM=XMS
      DELOMG=FOMGS*XM0
      DELA=0.0D0
```

```
      DELE=0.0D0
      GO TO 3
    2 DELM=-FM1*(B2 + C2 + P2) + FM2*(B3 + C3)
      DELOMG=FCMG1*(E2 + C2 + P2)
    3 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE DETRV(DEA,DEE,DEINC,DEMA,DEOMEG,DECAP,DEX,DEY,DEZ,
     1DEXDT,DEYDT,DEZDT,DERMAG,DEVMAG)
C
C
C*****THIS SUBROUTINE CONVERTS KEPLERIAN ELEMENTS TO CARTESIAN
C      COORDINATES AND VELOCITIES.
C
C*****INPUT
C
C*****DEA IS THE DIMENSIONLESS SEMIMAJOR AXIS (USUALLY DEA=1
C      APPROXIMATELY.)
C*****DEE IS THE ECCENTRICITY.
C*****DEINC IS THE INCLINATION IN RADIANS.
C*****DEMA IS THE MEAN ANOMALY IN RADIANS.
C*****DEOMEG IS THE ARGUMENT OF PERIGEE IN RADIANS.
C*****DECAP IS THE LONGITUDE OF NODE IN RADIANS.
C
C*****OUTPUT
C
C*****DEX,DEY,AND DEZ ARE THE CARTESIAN X Y Z COORDINATES. THE ORBITED
C      OBJECT IS AT THE ORIGIN.
C*****DERMAG IS THE DISTANCE FROM THE ORIGIN TO THE SATELLITE.
C*****DEXDT,DEYDT, AND DEZDT ARE THE VELOCITIES IN THE X,Y, AND Z
C      DIRECTIONS.
C*****DEVMAG IS THE MAGNITUDE OF THE VELOCITY.
C*****THE DISTANCES AND THE VELOCITIES MUST EACH BE MULTIPLIED BY
C      CONVERSION FACTORS IN THE MAIN PROGRAM TO CONVERT TO CGS UNITS.
C      IF THE CONVERSION FACTOR FOR THE DISTANCES IS DISFAC (IN
C      CENTIMETERS), THEN THE CONVERSION FACTOR FOR THE VELOCITIES IS
C      CONV=DSQRT(BIG G * MASS/DISFAC) IN CGS UNITS.
C
      IMPLICIT REAL*8(A-H,O-Z,$)
      TWOPI=6.2831853071795860D0
      TAMEAN = DEMA
      IF (TAMEAN) 10,11,10
   10 TAMEAN = DMOD(TAMEAN + TWOPI,TWOPI)
      ECCA1 = TAMEAN + DEE*DSIN(TAMEAN) + .5D0*DEE**2*DSIN(2.D0*TAMEAN)
      DO 13  I2 = 1,100
      DIFF = (DEE*DSIN(ECCA1) - ECCA1+TAMEAN)/(1.D0-DEE*DCOS(ECCA1))
      ECCA2 = ECCA1 + DIFF
      SECCA2 = DSIN(ECCA2)
      DIFF = DABS(ECCA2 - TAMEAN - DEE*SECCA2)
      IF (DIFF - 0.1D-13) 16,16,13
   13 ECCA1 = ECCA2
      WRITE      (6,2)DEMA,DMA,DIFF
    2 FORMAT(5X,53HNO CONVERGENCE IN KEPLERS EQUATION - SUBROUTINE DETRV
     1      /3D16.8)
      STOP
   11 ECCA2 = TAMEAN
      SECCA2= DSIN(ECCA2)
   16 X = DCOS(ECCA2) - DEE
      SP= 1.D0 - DEE**2
      SQ = DSQRT(SP)
      Y  = SQ*SECCA2
      TRUEA = ARCTAN(Y,X)
      CTRUEA = DCOS(TRUEA)
      STRUEA = DSIN(TRUEA)
      DERMAG = DEA*SP/(1.D0+DEE*CTRUEA)
      X = DERMAG*CTRUEA
      Y = DERMAG*STRUEA
      SNW   = DSIN(DEOMEG)
      CSW   = DCOS(DEOMEG)
      SNCPW = DSIN(DECAP)
      CSCPW = DCOS(DECAP)
      SNI   = DSIN(DEINC)
      CSI   = DCOS(DEINC)
      AC = CSW*CSCPW - SNW*SNCPW*CSI
      BC = CSW*SNCPW + SNW*CSCPW*CSI
      CC = SNW*SNI
      FC =-SNW*CSCPW - CSW*SNCPW*CSI
      GC =-SNW*SNCPW + CSW*CSCPW*CSI
      HC = CSW*SNI
      DEX = AC*X + FC*Y
      DEY = BC*X + GC*Y
      DEZ = CC*X + HC*Y
      XXD1 = -STRUEA
      YYD1 = DEE + CTRUEA
      DEXDT= AC*XXD1 + FC*YYD1
      DEYDT= BC*XXD1 + GC*YYD1
      DEZDT= CC*XXD1 + HC*YYD1
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```
      OEVMAG = DSQRT(2.D0/OERMAG - 1.D0/OEA)
      SQSMSQ = DSQRT(OEXDT**2 + OEYDT**2 + OEZDT**2)
      FMULT  = OEVMAG/SQSMSQ
      OEXDT = OEXDT * FMULT
      OEYDT = OEYDT * FMULT
      OEZDT = OEZDT * FMULT
      RETURN
      END
```

```
      FUNCTION ARCTAN(S,C)
      IMPLICIT REAL*8(A-H,O-Z,$)
      Y=S
      X=C
      IF (X) 108,100,116
  100 IF (Y) 102,104,106
  102 ARCTAN=4.712388980384689D0
      GO TO 138
  104 ARCTAN=0.0D0
      RETURN
  106 ARCTAN=1.570796326794896D0
      GO TO 138
  108 IF (Y) 110,112,114
  110 ADD=3.141592653589793D0
      GO TO 124
  112 ARCTAN=3.141592653589793D0
      GO TO 138
  114 ADD=3.141592653589793D0
      GO TO 132
  116 IF (Y) 118,120,122
  118 ADD=6.283185307179586D0
      GO TO 132
  120 ARCTAN=0.0D0
      GO TO 138
  122 ADD=0.0D0
  124 IF (DABS(Y)-DABS(X)) 126,128,130
  126 ARCTAN=DATAN(Y/X) + ADD
      GO TO 138
  128 ARCTAN=0.7853981633974482D0 + ADD
      GO TO 138
  130 ARCTAN=1.570796326794896D0 - DATAN(X/Y) + ADD
      GO TO 138
  132 IF (DABS(Y) - DABS(X)) 126,134,136
  134 ARCTAN=-0.7853981633974482D0 + ADD
      GO TO 138
  136 ARCTAN=-1.570796326794896D0 - DATAN(X/Y) + ADD
  138 RETURN
      END
```

```
      SUBROUTINE PLOT1(X,Y,N)
C
C*****THIS SUBROUTINE PLOTS A GRAPH CONSISTING OF N POINTS IN ARRAYS
C     X AND Y.
C     THE X AXIS IS HORIZONTAL AND THE Y AXIS IS VERTICAL. THE AXES ARE
C     PRINTED AS DOTS. THE SCALE OF THE DIAGRAM IS ALWAYS ADJUSTED TO
C     FIT ONTO A SINGLE SHEET OF COMPUTER PAPER, WITH 61 SPACES UP AND
C     101 ACROSS. THE DISTORTION DUE TO THE DIFFERENT SPACING IN THE X
C     AND Y DIRECTIONS IS REMOVED, SO THAT A SQUARE LOOKS LIKE A SQUARE.
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(N),Y(N)
      DIMENSION AP(101,61)
      DATA BLANK,DOT,D,OH/1H ,1H.,1HO,1H0/
C*****XSCALE TAKES CARE OF THE DIFFERENT SPACING IN THE VERTICAL AND
C     HORIZONTAL DIRECTIONS.
      XSCALE=6.25D0/5.0D0
      YMIN=Y(1)
      YMAX=Y(1)
      XMIN=X(1)
      XMAX=X(1)
      SCALE1=100.0D0
      SCALE2=60.0D0
      NSC2=SCALE2 + 0.1D0
      DO 24 I=2,N
      Q=Y(I)
      IF (Q-YMAX) 12,13,13
   13 YMAX=Q
      GO TO 14
   12 CONTINUE
      IF (Q-YMIN) 15,14,14
   15 YMIN=Q
   14 CONTINUE
      R=X(I)
      IF (R-XMAX) 22,23,23
   23 XMAX=R
      GO TO 24
   22 CONTINUE
      IF (R-XMIN) 25,24,24
   25 XMIN=R
   24 CONTINUE
      YLGTH=YMAX-YMIN
      XLGTH=XMAX-XMIN
      FAC=SCALE1/(XLGTH*XSCALE)
      YF=YLGTH*FAC
      IF (YF - SCALE2) 28,28,27
   27 FAC=SCALE2/YLGTH
   28 CONTINUE
      WRITE (6,41)
      WRITE (6,70)
   70 FORMAT (///,10X,33HINFORMATION FROM SUBROUTINE PLOT1,///)
      WRITE (6,71)
   71 FORMAT  (10X,37HYMAX IS THE MAXIMUM Y VALUE ATTAINED.)
      WRITE (6,72)
   72 FORMAT  (10X,37HYMIN IS THE MINIMUM Y VALUE ATTAINED.)
      WRITE (6,73)
   73 FORMAT (10X,19HYLGTH IS YMAX-YMIN.)
      WRITE (6,74)
   74 FORMAT (10X,70HXMAX,XMIN AND XLGTH ARE THE ANALOGOUS EXPRESSIONS F
     1OR THE X DIRECTION.)
      WRITE (6,75)
   75 FORMAT (10X,41HFAC ADJUSTS THE GRAPH TO THE PROPER SIZE.,///)
      WRITE (6,39)
   39 FORMAT (////,5X,4HYMAX,12X,4HYMIN,10X,4HXMAX,12X,4HXMIN,10X,5HYLGT
     1H,10X,5HXLGTH,11X,3HFAC,/)
      WRITE (6,60) YMAX,YMIN,XMAX,XMIN,YLGTH,XLGTH,FAC
   60 FORMAT (8D15.5)
      DO 57 I=1,N
   57 X(I)=X(I)*XSCALE
      XMIN=XMIN*XSCALE
      XMAX=XMAX*XSCALE
      DO 29 I=1,101
      DO 29 J=1,61
   29 AP(I,J)=BLANK
      J1=(-XMIN)*FAC + 1.5D0
      J2=(-YMIN)*FAC + 1.5D0
      IF (J1 .GT. 0 .AND. J2 .GT. 0) GO TO 30
      GO TO 31
   30 CONTINUE
      DO 32 I=1,101
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```
   32   AP(I,J2)=DOT
        DO 33 J=1,61
   33   AP(J1,J)=DOT
   31   CONTINUE
        DO 34 I=1,N
        J1=(X(I) - XMIN)*FAC + 1.5D0
        J2=(Y(I) - YMIN)*FAC + 1.5D0
        AP(J1,J2)=D
   34   CONTINUE
        WRITE (6,41)
   41   FORMAT (1H1)
        WRITE (6,76)
   76   FORMAT (///,10X)
        DO 35 J=1,61
        J3=61 - (J-1)
        WRITE (6,40) (AP(I,J3), I=1,101)
   35   CONTINUE
   40   FORMAT (10X,101A1)
        RETURN
        END
```

DISTURBING BODY= EARTH MASS= 0.559760D 28 GM DISTANCE FACTOR= 0.637820 09 CM

A= 0.384400 11 CM E= 0.05500

NPOINT= 73 DEL= 5.00000 DEGREES

FULL PERTURBATION

TERMS UP TO AND INCLUDING SECOND ORDER RETAINED IN THE ECCENTRICITY FUNCTIONS
AND THEIR DERIVATIVES

SAMPLE OUTPUT SAMPLE OUTPUT

	MEAN ANOMALY (DEGREES)	X (CM)	Y (CM)	R (CM)	XDT (CM/SEC)	YDT (CM/SEC)	V (CM/SEC)
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	5.0000	-0.66688D-02	0.10084D-03	0.66695D-02	-0.38487D-06	-0.17959D-07	0.38529D-06
3	10.0000	-0.26666D-01	-0.11064D-02	0.26689D-01	-0.77248D-06	-0.73000D-07	0.77593D-06
4	15.0000	-0.60044D-01	-0.50078D-02	0.60252D-01	-0.11648D-05	-0.16841D-06	0.11769D-05
5	20.0000	-0.10675D 00	-0.13143D-01	0.10756D 00	-0.15621D-05	-0.30939D-06	0.15924D-05
6	25.0000	-0.16670D 00	-0.27239D-01	0.16891D 00	-0.19625D-05	-0.50214D-06	0.20257D-05
7	30.0000	-0.23958D 00	-0.49267D-01	0.24459D 00	-0.23616D-05	-0.75329D-06	0.24791D-05
8	35.0000	-0.32485D 00	-0.81346D-01	0.33488D 00	-0.27533D-05	-0.10691D-05	0.29536D-05
9	40.0000	-0.42168D 00	-0.12577D 00	0.44004D 00	-0.31280D-05	-0.14545D-05	0.34497D-05
10	45.0000	-0.52885D 00	-0.18491D 00	0.56024D 00	-0.34751D-05	-0.19128D-05	0.39667D-05
11	50.0000	-0.64479D 00	-0.26108D 00	0.69564D 00	-0.37822D-05	-0.24454D-05	0.45039D-05
12	55.0000	-0.76754D 00	-0.35662D 00	0.84635D 00	-0.40365D-05	-0.30537D-05	0.50597D-05
13	60.0000	-0.89481D 00	-0.47358D 00	0.10124D 01	-0.42248D-05	-0.37252D-05	0.56325D-05
14	65.0000	-0.10240D 01	-0.61382D 00	0.11939D 01	-0.43342D-05	-0.44626D-05	0.62210D-05
15	70.0000	-0.11522D 01	-0.77890D 00	0.13908D 01	-0.43528D-05	-0.52548D-05	0.68234D-05
16	75.0000	-0.12764D 01	-0.97005D 00	0.16032D 01	-0.42694D-05	-0.60912D-05	0.74384D-05
17	80.0000	-0.13932D 01	-0.11881D 01	0.18311D 01	-0.40746D-05	-0.69597D-05	0.80648D-05
18	85.0000	-0.14993D 01	-0.14335D 01	0.20744D 01	-0.37601D-05	-0.78466D-05	0.87010D-05
19	90.0000	-0.15911D 01	-0.17064D 01	0.23331D 01	-0.33196D-05	-0.87369D-05	0.93463D-05
20	95.0000	-0.16648D 01	-0.20062D 01	0.26070D 01	-0.27466D-05	-0.96142D-05	0.99994D-05
21	100.0000	-0.17166D 01	-0.23322D 01	0.28959D 01	-0.20447D-05	-0.10461D-04	0.10659D-04
22	105.0000	-0.17432D 01	-0.26828D 01	0.31995D 01	-0.12078D-05	-0.11261D-04	0.11325D-04
23	110.0000	-0.17403D 01	-0.30563D 01	0.35170D 01	-0.23997D-06	-0.11994D-04	0.11996D-04
24	115.0000	-0.17041D 01	-0.34501D 01	0.38480D 01	0.85418D-06	-0.12642D-04	0.12671D-04
25	120.0000	-0.16311D 01	-0.38612D 01	0.41916D 01	0.20675D-05	-0.13188D-04	0.13350D-04
26	125.0000	-0.15177D 01	-0.42860D 01	0.45467D 01	0.33903D-05	-0.13615D-04	0.14030D-04
27	130.0000	-0.13607D 01	-0.47202D 01	0.49124D 01	0.48106D-05	-0.13904D-04	0.14713D-04
28	135.0000	-0.11573D 01	-0.51592D 01	0.52874D 01	0.63140D-05	-0.14043D-04	0.15397D-04
29	140.0000	-0.90508D 00	-0.55978D 01	0.56705D 01	0.78843D-05	-0.14017D-04	0.16082D-04
30	145.0000	-0.60226D 00	-0.60305D 01	0.60605D 01	0.95035D-05	-0.13816D-04	0.16769D-04
31	150.0000	-0.24757D 00	-0.64513D 01	0.64561D 01	0.11152D-04	-0.13430D-04	0.17456D-04
32	155.0000	0.15956D 00	-0.68544D 01	0.68562D 01	0.12809D-04	-0.12852D-04	0.18146D-04
33	160.0000	0.61900D 00	-0.72334D 01	0.72598D 01	0.14454D-04	-0.12079D-04	0.18837D-04
34	165.0000	0.11299D 01	-0.75820D 01	0.76658D 01	0.16065D-04	-0.11108D-04	0.19532D-04
35	170.0000	0.16908D 01	-0.78942D 01	0.80733D 01	0.17620D-04	-0.99388D-05	0.20230D-04
36	175.0000	0.22994D 01	-0.81639D 01	0.84814D 01	0.19097D-04	-0.85737D-05	0.20934D-04
37	180.0000	0.29527D 01	-0.83847D 01	0.88894D 01	0.20474D-04	-0.70169D-05	0.21643D-04
38	185.0000	0.36470D 01	-0.85514D 01	0.92966D 01	0.21729D-04	-0.52747D-05	0.22360D-04
39	190.0000	0.43779D 01	-0.86581D 01	0.97020D 01	0.22839D-04	-0.33594D-05	0.23084D-04
40	195.0000	0.51405D 01	-0.86997D 01	0.10105D 02	0.23784D-04	-0.12694D-05	0.23818D-04
41	200.0000	0.59289D 01	-0.86712D 01	0.10504D 02	0.24542D-04	0.97053D-06	0.24562D-04
42	205.0000	0.67366D 01	-0.85682D 01	0.10859D 02	0.25094D-04	0.33492D-05	0.25316D-04
43	210.0000	0.75570D 01	-0.83866D 01	0.11289D 02	0.25417D-04	0.58491D-05	0.26081D-04
44	215.0000	0.83817D 01	-0.81229D 01	0.11672D 02	0.25493D-04	0.84496D-05	0.26857D-04
45	220.0000	0.92026D 01	-0.77744D 01	0.12047D 02	0.25305D-04	0.11128D-04	0.27644D-04
46	225.0000	0.10011D 02	-0.73389D 01	0.12413D 02	0.24837D-04	0.13857D-04	0.28441D-04
47	230.0000	0.10797D 02	-0.68153D 01	0.12768D 02	0.24075D-04	0.16609D-04	0.29249D-04
48	235.0000	0.11551D 02	-0.62035D 01	0.13112D 02	0.23010D-04	0.19352D-04	0.30066D-04
49	240.0000	0.12264D 02	-0.55043D 01	0.13442D 02	0.21632D-04	0.22052D-04	0.30891D-04
50	245.0000	0.12924D 02	-0.47197D 01	0.13759D 02	0.19941D-04	0.24675D-04	0.31725D-04
51	250.0000	0.13522D 02	-0.38530D 01	0.14060D 02	0.17935D-04	0.27181D-04	0.32565D-04
52	255.0000	0.14049D 02	-0.29086D 01	0.14346D 02	0.15620D-04	0.29534D-04	0.33410D-04
53	260.0000	0.14493D 02	-0.18922D 01	0.14616D 02	0.13006D-04	0.31695D-04	0.34260D-04
54	265.0000	0.14848D 02	-0.81080D 00	0.14870D 02	0.10109D-04	0.33625D-04	0.35112D-04
55	270.0000	0.15103D 02	0.32753D 00	0.15106D 02	0.69485D-05	0.35287D-04	0.35964D-04
56	275.0000	0.15251D 02	0.15135D 01	0.15326D 02	0.35490D-05	0.36644D-04	0.36816D-04
57	280.0000	0.15286D 02	0.27369D 01	0.15529D 02	-0.58705D-07	0.37664D-04	0.37664D-04
58	285.0000	0.15201D 02	0.39862D 01	0.15715D 02	-0.38395D-05	0.38314D-04	0.38506D-04
59	290.0000	0.14992D 02	0.52493D 01	0.15885D 02	-0.77534D-05	0.38568D-04	0.39340D-04
60	295.0000	0.14656D 02	0.65131D 01	0.16038D 02	-0.11756D-04	0.38402D-04	0.40161D-04
61	300.0000	0.14190D 02	0.77639D 01	0.16175D 02	-0.15798D-04	0.37800D-04	0.40968D-04
62	305.0000	0.13554D 02	0.89875D 01	0.16297D 02	-0.19827D-04	0.36749D-04	0.41756D-04
63	310.0000	0.12869D 02	0.10169D 02	0.16402D 02	-0.23789D-04	0.35245D-04	0.42522D-04
64	315.0000	0.12018D 02	0.11295D 02	0.16493D 02	-0.27627D-04	0.33292D-04	0.43262D-04
65	320.0000	0.11046D 02	0.12350D 02	0.16569D 02	-0.31284D-04	0.30901D-04	0.43972D-04
66	325.0000	0.99580D 01	0.13319D 02	0.16630D 02	-0.34704D-04	0.28092D-04	0.44650D-04
67	330.0000	0.87633D 01	0.14191D 02	0.16679D 02	-0.37835D-04	0.24895D-04	0.45291D-04
68	335.0000	0.74715D 01	0.14952D 02	0.16715D 02	-0.40628D-04	0.21344D-04	0.45894D-04
69	340.0000	0.60942D 01	0.15591D 02	0.16740D 02	-0.43041D-04	0.17485D-04	0.46457D-04
70	345.0000	0.46441D 01	0.16100D 02	0.16756D 02	-0.45037D-04	0.13366D-04	0.46979D-04
71	350.0000	0.31351D 01	0.16469D 02	0.16765D 02	-0.46591D-04	0.90408D-05	0.47460D-04
72	355.0000	0.15820D 01	0.16694D 02	0.16769D 02	-0.47663D-04	0.45665D-05	0.47901D-04
73	360.0000	0.0	0.16770D 02	0.16770D 02	-0.48304D-04	0.0	0.48304D-04

```
INFORMATION FROM SUBROUTINE PLOT1

YMAX IS THE MAXIMUM Y VALUE ATTAINED.
YMIN IS THE MINIMUM Y VALUE ATTAINED.
YLGTH IS YMAX-YMIN.
XMAX,XMIN AND XLGTH ARE THE ANALOGOUS EXPRESSIONS FOR THE X DIRECTION.
FAC ADJUSTS THE GRAPH TO THE PROPER SIZE.
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YMAX	YMIN	XMAX	XMIN	YLGTH	XLGTH	FAC
0.16770D 02	-0.86997D 01	0.15286D 02	-0.17432D 01	0.25469D 02	0.17029D 02	0.23558D 01

ORIGINAL PAGE IS
OF POOR QUALITY