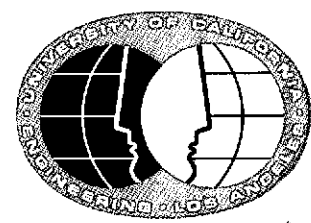
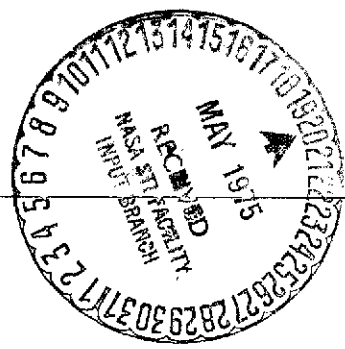


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THE MOTION OF INTERCONNECTED FLEXIBLE BODIES

Final Report
Volume 3

A. S. HOPKINS

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THE MOTION OF INTERCONNECTED FLEXIBLE BODIES

Arthur Stewart Hopkins

School of Engineering and Applied Science
University of California
Los Angeles, California

PREFACE

The research described in this report forms the basis for a Ph.D. dissertation, "The Motion of Interconnected Flexible Bodies," by Arthur Stewart Hopkins. This dissertation serves as Volume 3 of the Final Report for Contract NAS8-28358, Mod. 6, sponsored by the NASA George C. Marshall Space Flight Center, and completed under the direction of Peter Likins, Principal Investigator.

ABSTRACT

In this report the equations of motion for an arbitrarily interconnected collection of substructures are derived. The substructures are elastic bodies which may be idealized as finite element assemblies. The substructures are subject to small deformations relative to a nominal state. Interconnections between the elastic substructures permit large relative translations and rotations between substructures, governed by Pfaffian constraints describing the connections. In the special case of screw connections (permitting rotation about and translation along a single axis), constraint forces are eliminated and modal coupling is incorporated.

This work is a generalization and extension of the recent literature directed at the problem of flexible spacecraft simulation. It is an extension of Hurty's "component mode" approach in that interconnected elastic substructures are permitted large motions relative to each other and relative to inertial space. It is more general than the "hybrid coordinate" methods advanced by Likins and others, in permitting all substructures to be flexible (rather than only the terminal members of a topological tree of substructures). The presentation here is further distinguished from the spacecraft literature by its development of the basic relationships of continuum mechanics, on which spacecraft simulation models are based.

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CONTENTS

	Page
SYMBOLS IN TEXT	ix
INTRODUCTION	1
SECTION 1 MATERIAL CONTINUA	5
1.1 Space and Time	6
1.2 Coordinates	8
1.3 Matter	9
1.4 Velocity	13
1.5 Reference Frames	15
1.6 Newton's Laws	26
1.7 Stress	31
1.8 Virtual Work	32
1.9 Virtual Work for an Arbitrary Reference Frame	39
SECTION 2 SUBSTRUCTURES	45
2.1 Finite Elements	45
2.2 Finite Element Dynamics	49
2.3 Finite Element Assembly	60
2.4 Quadratic Expansions	65
2.5 Substructure Modes	66
2.6 Extended Modal Analysis	73
SECTION 3 STRUCTURES	75
3.1 Constraints	75
3.2 Interconnections	76
3.3 Rotational Constraints	79
3.4 Translational Constraints	90
3.5 Elimination of Constraint Equations and Forces	101
3.6 Modal Synthesis	104
SECTION 4 SUMMARY	113
4.1 Isolation of Structural Elements	113
4.2 Selection of Substructures	114
4.3 Substructure Dynamics	115
4.4 Substructure Modes	116
4.5 Substructure Coupling	117
4.6 Point Connections	119
4.7 Pfaffian Constraints	120
4.8 Equations for Structure	120
4.9 System Synthesis	121
BIBLIOGRAPHY	123

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CONTENTS (Continued)

		Page
APPENDIX A	MATHEMATICS	129
A.1	Set Theory	130
A.2	Metric Spaces	133
A.3	Topological Spaces	133
A.4	Manifolds	134
A.5	Tangents	136
A.6	Vector Spaces	140
A.7	Tensor Spaces	142
A.8	Tensor Fields	149
A.9	The Hodge Star Operator	151
A.10	Integration	155
A.11	Variations	157
A.12	Nabla	159
A.13	Dot and Cross Product Identities	160
A.14	The Variation of a Vector	162
A.15	Modes	164
APPENDIX B	STRESS	169
B.1	Stress Transformations	170
B.2	Generalized Stress Transformations	171
APPENDIX C	BEAMS	175
C.1	Beam Finite Element	176
C.2	Beam Dynamic Parameters	189
C.3	Substructure Modes	200
APPENDIX D	QUADRATIC TERMS	205
D.1	Quadratic Terms in Lagrangian	206
D.2	Quadratic Terms in Gimbal System.	210

SYMBOLS IN TEXT

<u>SYMBOL</u>	<u>FIRST EQUATION</u>	<u>BRIEF DESCRIPTION</u>
$\overline{(\)}$	1.3-1	Material Space
$\overline{(\)}$	2.2-5	Mean
∂	1.4-1	Partial Derivative
∂	1.6-10	Boundary
$\frac{\partial}{\partial x}$	1.4-1	Basis Vectors
$\frac{\partial}{\partial y}$	3.3-1	Local Constraint Basis
$(\)_*$	1.4-1	Differential
\circ	1.3-1	Composition
\otimes	1.5-3	Tensor Product
\cup	1.5-27	Spatial Vector
$*$	1.5-38	Hodge Star Operator
$\tilde{(\)}$	1.5-40	Associated Skew Tensor
\times	1.5-46	Spatial Vector Cross Product
\cdot	1.6-5	Inner Product
$\hat{(\)}$	1.6-8	Referred to Undeformed Volume
\circ	1.6-11	Little \circ
∇	1.7-5	Nabla
$:$	1.8-6	Double Dot
\emptyset	3.6-10	Null
(Presuperscript)	2.3-1	Substructure Index
$^0(\)$	2.5-1	Zeroth Order
$^1(\)$	2.5-4	First Order
(Presubscript)	2.2-4	Generalized Coordinate Index

SYMBOLS IN TEXT (Continued)

<u>SYMBOL</u>	<u>FIRST EQUATION</u>	<u>BRIEF DESCRIPTION</u>
(Postsuperscript)	1.2-1	Contravariant Index
()'	1.5-1	Noninertial Frame
(Postsubscript)	1.5-11	Covariant Index
(Greek postscript)	1.5-3	Range 1,2,3,4
(Latin Postscript)	1.5-19	Range 1,2,3
(Lower Case)	1.5-2	Coordinate Expression for Associated Upper Case
(Upper Case)	1.5-2	Function whose Coordinate Expression is given by Associated Lower Case
A (Alpha)	3.2-1	Constraint Coefficient
B (Beta)	3.2-1	Constraint Coefficient
Γ	3.2-1	Constraint Coefficient
δ	1.5-3	Kronecker Delta
δ	1.6-5	Variation
Δ	3.2-1	Constraint Coefficient
ε	1.1-1	Metric
ε	1.5-39	Levi-Civita Epsilon
E (Epsilon)	1.8-17	Strain
Θ	3.3-1	Rotation
Λ	1.5-1	Reference Frame Transform
λ	2.3-2	Lagrange Multiplier
μ	1.4-2	Chart
ξ	3.4-1	Translation
π	1.3-1	Traceability

SYMBOLS IN TEXT (Continued)

<u>SYMBOL</u>	<u>FIRST EQUATION</u>	<u>BRIEF DESCRIPTION</u>
ρ	1.6-1	Mass Density
σ	1.1-1	Distance
Σ	1.7-1	Stress
τ	1.1-1	Duration
ϕ	1.6-14	Interaction Force
ψ	3.3-23	Deformational Rotation
Ω	1.5-38	Angular Velocity
A	1.6-1	Acceleration
A	1.8-24	Strain Energy
B	3.6-10	Basis Space
c	1.5-11	Direction Cosine
C	2.2-8	Damping
d	1.5-3	Differential
$d()$	3.3-3	Deformation
$E()$	2.3-3	Relative Position
E	2.3-2	Number of Finite Elements
F	1.6-1	Body Force Density
F	2.3-3	Force
$\mathcal{G}()$	3.3-2	Geometric
$\mathcal{G}()$	3.6-1	In \mathcal{G}_x Local Basis
H	1.9-10	Angular Momentum
i^I, ij^I	2.2-12	Inertia Terms
I	2.2-14	Moment of Inertia

SYMBOLS IN TEXT (Continued)

<u>SYMBOL</u>	<u>FIRST EQUATION</u>	<u>BRIEF DESCRIPTION</u>
K	2.2-8	Stiffness
L	1.6-12	Lagrange Multiplier
m	1.4-1	Event
M	2.2-5	Mass
M	2.3-2	Number of Nodes
N	1.7-2	Normal
$N()$	1.8-3	Nonconservative
N	3.6-10	Null Space
i^0, ij^0	2.2-12	Inertia Terms
$P()$	1.8-10	Prescribed
q	2.2-4	Generalized Displacement
Q	2.2-8	Generalized Force
r	1.5-11	Position of Primed Origin
s	1.1-1	Spatial Point
S	1.6-15	Short Range Force
$S()$	2.3-3	Local Origin Position
t	1.1-1	Temporal Point
T	2.3-1	Connectivity Transform
T	2.3-4	Torque
U	1.9-2	Unity
V	1.4-1	Velocity
W	1.8-3	Potential Energy

SYMBOLS (Continued)

<u>SYMBOL</u>	<u>FIRST EQUATION</u>	<u>BRIEF DESCRIPTION</u>
x	1.2-1	Coordinate
X	1.6-15	Long Range Force

INTRODUCTION

Prompted by increasing economic incentives for weight reduction and, in application to spacecraft, increasingly stringent pointing requirements of scientific instruments, there has been a growing interest in the dynamical behavior of flexible structures. Of course, the basic equations governing dynamical behavior were developed by scientists in the seventeenth and eighteenth centuries, principally by Newton, Euler and Lagrange. However, direct application of these equations to elastic continuum models is neither economically nor computationally feasible. Research in recent years has been directed toward obtaining efficient approximate techniques. Essentially two independent approaches were developed.

The first approach was directed toward increasing computational efficiency for the detailed models required to accurately portray the small motion behavior of discretized models of flexible structures. This method is known as component modal analysis, and is generally attributed to Hurty (1965). As originally developed, the method applies to small motions of nonrotating structures.

The second approach, in recognition of the need to include large relative motions between portions of the structure, was directed toward efficient computation of collections of point connected rigid bodies. Early research in this area was conducted by Hooker and Margulies (1965), followed by Roberson and Wittenberg (1966) using what may be classified as augmented body methods. A somewhat different approach to the same problem was developed by Velman (1967) and Russell (1969), who used what may be classified as nested body methods. As presented, both of

these methods apply only to point connected rigid bodies assembled in a configuration described as a topological tree.

The failure of the rigid body methods to provide adequate definition of flexible behavior, and the failure of the component modes methods to include large relative motions, led Likins (1967, 1968, 1970) to a marriage of the two approaches. Using what he terms a hybrid coordinate approach, he has developed a procedure for treating the extremal bodies of a topological tree of point connected rigid bodies as flexible. As currently conceived, the method employs the Hooker-Margulies equations (as subsequently modified by Hooker (1970) to reflect the constraint force elimination techniques developed by Russell (1969)) for the central rigid bodies of the topological tree, and techniques similar to Hurty's for adding flexible extremal bodies.

The present study consummates the union of the two approaches. In this study all of the bodies may be flexible, the configuration is arbitrary (topologically) and general differential (Pfaffian) constraints are permitted. In addition to the general development, the special case of screw connections is treated and the resulting simplifications to the governing equations are detailed.

The study is presented in three sections. The first section presents the theoretical background and the derivation of the variational equations governing the behavior of a material continuum in a rotating frame of reference. The second section applies the finite element method to obtain the governing ordinary differential equations for a substructure. These equations describe the small deformational motions of a structure idealized as a collection of finite elements.

This section also includes a demonstration of the adequacy of linear assumed displacement fields in the inherently quadratic Lagrangian formulation. A procedure for linearizing the equations about arbitrary frame motions is also presented, in conjunction with modal analysis procedures. The third section details procedures for connecting the substructures to form a structure, with the possibility of large relative motions between substructures. In addition to the equations for general Pfaffian constraints, the results are specialized to the case of screw connections (a rotation about and a translation along an axis). In this special case (which includes most connections of practical use), constraint forces are eliminated and the system equations are reduced by modal coupling. The summarizing fourth section presents the analysis procedure for an arbitrary dynamical system, as a means of abstracting the salient features of the study.

Section 1

MATERIAL CONTINUA

Newtonian mechanics provides an approximate description of a portion of the physical phenomena within man's experience. Specifically it accurately describes some of those phenomena for which relativistic effects are negligible; and whose spatial and temporal scales are large compared to atomic phenomena and small compared to cosmological phenomena.

Physical phenomena are generally analyzed by means of the following procedure. First, a mathematical model is associated with the physical phenomenon being studied. Next, results are obtained from the model by purely mathematical methods. Finally, the mathematical deductions are extrapolated to the physical phenomenon. Experience has shown that these extrapolations accurately portray the physical phenomenon, if the model is developed with sufficient care.

Since the second step in the procedure is mathematical, some understanding of mathematics is necessary. Mathematics consists of "mathematical objects" called "sets" (e.g. functions, numbers, geometrical objects) and "relations" between these objects (A.0). (To preserve the continuity of the presentation, the definitions of mathematical terms have been placed in Appendix A. When a term with a specific mathematical meaning is first used, it is set in quotation marks and followed by a reference to the appropriate section in the appendix.) The most fundamental mathematical concepts cannot be defined and so are introduced, a priori, as primitive notions. The most fundamental rules governing

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mathematics cannot be deduced, and so are taken, a priori, as true "axioms" (A.O).

Analogously, certain physical concepts will be accepted as primitive; and certain relationships will be taken as axiomatic. Physical concepts are, ultimately, high order abstraction of information obtained through the senses; therefore description of primitives is most conveniently carried out in anthropomorphic terms. The purpose of this section is to identify the relevant physical concepts, and to associate a mathematical model with them. Although a reasonably coherent development is intended, a mathematically rigorous treatment lies outside the scope of this study.

1.1 Space and Time

The concept of a physical system implicitly involves the notions of space and time. Both will be taken as undefined primitives. For the range of phenomena being considered, space and time may be identified with the usual intuitive concepts. Some of the more relevant characteristics of space and time will be listed.

Space is infinite, homogeneous and isotropic. It is independent of time and physical phenomena. The concept of a point in space will also be primitive: A distance can be associated with any two points in space. An arbitrary reference distance (e.g. a meter) may be selected as a unit. The number of reference distances comprising the distance will be associated with the distance and the selected unit.

Time is infinite, homogeneous and independent of space and physical phenomena. The concept of a point in time will be primitive. Time is anisotropic; for any two points, one is before and the other

after. A duration can be associated with any two points. An arbitrary reference duration (e.g. a second) may be selected as a unit. The number of reference durations comprising the duration will be associated with the duration and the selected unit.

The determination of absolute position in space or time is precluded by the infinite and homogeneous characteristics. Similarly, isotropy precludes determination of absolute direction in space. An anthropomorphic primitive, the observer, will be introduced to allow the unique identification of points in space and time. The observer, as a spatially anisotropic collection of inhomogeneous points in space and time of finite extent, is able to uniquely identify the points of space and time by relating them to himself.

The mathematical model associated with any specific, uniquely identified point in space will be a unique "mathematical object" (A.0), generically denoted by "s"; similarly, for any point in time, "t". The distinction between the physical and mathematical is customarily left implicit and the mathematical objects, s and t, are also called a point in space and a point in time. The model of space is the "set" (A.0) of all spatial points, which will be denoted "S", = {s}, and called physical space. The model of time is the set of all temporal points, which will be denoted "T", = {t}, and called temporal space. It is convenient to introduce the "ordered pairs" (A.1) of spatial and temporal points which will be called events and denoted "e", = (s,t). The set of all events, denoted "E", = {e}, is the "cartesian product" (A.1) of spatial and temporal points ($E = S \times T$) and will be called event space.

The mathematical models associated with distance and duration will be "functions" (A.1), " σ ": $S \times S \rightarrow R$ and " τ ": $T \times T \rightarrow R$, assigning non-negative real numbers to pairs of points in physical space or temporal space. The functions, σ and τ , define "metrics" (A.2) on S and T . The function, " ϵ_p ": $E \times E \rightarrow R$, $p \geq 1$, defined by

$$\epsilon_p[(1s, 1t), (2s, 2t)] = \left([\sigma(1s, 2s)]^p + [\tau(1t, 2t)]^p \right)^{1/p}, \quad (1.1-1)$$

is a metric for E . Similarly, in the limit as $p \rightarrow \infty$,

$$\epsilon_\infty[(1s, 1t), (2s, 2t)] = \max[\sigma(1s, 2s), \tau(1t, 2t)] \quad (1.1-2)$$

is also a metric. All of the ϵ_p -metrics are "strongly equivalent" (A.2). Physical space, S , with σ , temporal space, T , with τ , and event space, E , with any of the ϵ_p , are all "metric spaces" (A.2).

If the "open sets" (A.3) of S, T or E are defined as arbitrary unions of "open balls" (A.2) the collections of all open sets are "topologies" (A.3), specifically they are "metric topologies" (A.3); and the sets, along with the topologies, define "topological spaces" (A.3). The topology on E obtained from any of the ϵ_p -metrics is the same as the "topological product" (A.3) of S and T .

1.2 Coordinates

The mathematical models of space and time will be related to the real numbers to facilitate manipulation. The observer can establish this relationship by constructing a coordinate system. He may specialize an event as the origin, and certain spatial points as dextral, orthogonal, coordinate axes. The duration between an event and the origin, and the distances between the event's geometric projections on the axes and the origin, along with appropriate sign conventions, establish the usual coordinate system.

The mathematical model of a "coordinate system" (A.4) is a function from the topological space onto a cartesian product of the real numbers (e.g. $S^\mu: S \rightarrow R^3$, $T^\mu: T \rightarrow R$, or $E^\mu: E \rightarrow R^4$). Taking this function as the only "chart" (A.4) in an "atlas" (A.4), physical, temporal and event space become " C^∞ manifolds" (A.4). The product topology, E , with the chart, " E^μ ", = (S^μ, T^μ) , is called the "product manifold" (A.4).

The manifold provides a mathematical model that transcends the coordinate system. Since a manifold, by definition, includes all "admissible charts" (A.4), a change in coordinates does not change the manifold. For physical space, origin translations and axes rotations do not change the manifold; polar, spherical and the other customary curvilinear coordinates (with minor restrictions) are all just special cases of the admissible coordinates.

The metrics, $\sigma: S \times S \rightarrow R$ and $\tau: T \times T \rightarrow R$, have a simple form in terms of the usual coordinates, $S^\mu: S \rightarrow R^3$, = (x^1, x^2, x^3) , and $T^\mu: T \rightarrow R$, = (x^4) ; namely

$$\sigma(1s, 2s) = \left([x^1(1s) - x^1(2s)]^2 + [x^2(1s) - x^2(2s)]^2 + [x^3(1s) - x^3(2s)]^2 \right)^{1/2} \quad (1.2-1)$$

and

$$\tau(1t, 2t) = |x^4(1t) - x^4(2t)| \quad (1.2-2)$$

1.3 Matter

Certain events, called material events, are characterized by the presence of matter. Matter will be taken as an undefined primitive notion, some of the more important characteristics of which will be listed.

Matter occurs as aggregates of spatial and temporal points, never as an isolated point in space or time. Mathematically, material events occur in "open sets" (A.3). Only aggregates of finite spatial and temporal extent will be treated, so the open sets will be "bounded" (A.2) (and their closures will be compact). An aggregate of matter, over which the physical characteristics vary smoothly, will be called a material continuum and will be denoted "M". The characteristics are called material properties and are determined by the constituent atoms. Mathematically, the material properties are functions, and varying smoothly corresponds to being a " C^∞ Map" (A.4).

Inertia is an important characteristic of matter that will be taken as primitive. It can be measured by the material property mass density. An arbitrary reference mass density (e.g. a kilogram/meter³) may be selected as a unit. The number of reference mass densities comprising the mass density will be associated with the mass density and the selected unit. Mathematically, the mass density is the C^∞ map " ρ ": $M \rightarrow R$ assigning a non-negative real number to each point in a material continuum.

Another important characteristic of matter is that it interacts with other matter. This interaction is called force, and may be associated with the anthropomorphic notion of a push or pull. Force will be taken as a primitive, even though it is intimately related to inertia. Forces may generally be categorized as either long range (remote or field) forces or as short range (contact) forces; depending on whether the distance between the interacting matter is very large or very small in comparison to the dimension of the material continuum.

Long range forces can be measured by body force densities. An arbitrary reference body force density (e.g. a Newton/meter³) may be selected as a unit. The number of reference body force densities comprising the body force density will be associated with the body force density, and the selected unit. If the material properties are constant over the material continuum, the long range forces vary slowly over the continuum. Gravity is a primary example of a long range force.

Conversely, short range forces vary extremely rapidly over the material continuum. If the interaction between matter exterior to and interior to a material continuum is considered, the forces are extremely high in the region of the "boundary" (A.3) and drop to negligible values at an extremely short distance into the "interior" (A.3). It is customary to treat the integral of these forces, normal to the surface, as a surface force density defined on the boundary. An arbitrary reference surface force density (e.g. a Newton/meter²) may be selected as a unit. The number of reference surface force densities comprising the surface force density will be associated with the surface force density and the selected unit. Forces at the points of contact between bodies are primary examples of short range forces.

Forces are characterized by direction as well as magnitude. The mathematical model for the surface or body force density at a point in a material continuum is a "vector" (A.6) which is a member of a "vector space" (A.6) at the point. The "tangent space" (A.5) at a point in a manifold is a natural vector space to associate with a manifold. The forces at each event of a material continuum collectively form a "vector field" (A.5).

The final characteristic of matter to be described here is traceability. Matter persists through time; exactly the same matter is present at any point in time in a material continuum (i.e., matter neither appears in nor disappears from a material continuum). For any specified event in a material continuum, and any (possibly different) given time, there is exactly one event in the material continuum which occurs at the given time and which may be identified as being the identical matter (i.e., consisting of the same atoms) as the previously specified event.

The subset of the events in M for which the time "projection" (A.4) is some reference time " t_0 " will be called the time slice at t_0 . It is a "submanifold" (A.4) of M (which is an "open submanifold" (A.4) in E). A "diffeomorphism" (A.4) may be selected from the time slice onto the subset " \bar{S} " of physical space. The mapping is usually just the spatial projection, but it may be used to establish a reference spatial state of the material which does not occur in the time period being treated (e.g., an undeformed state). The image, \bar{S} , is an open submanifold of S , and will be referred to as material space. The temporal projection of M is an open submanifold of T , which will be denoted " \bar{T} ," and may be referred to as the time range of the material continuum. The product manifold, " \bar{M} ", = $\bar{S} \times \bar{T}$, will be called material event space. It may be noted that \bar{M} and M are different and provide two different means of identifying matter. The first, \bar{M} , by its reference position and the time (referred to as Lagrangian) and the second, M , by its actual position at the given time and the time (referred to as Eulerian).

The property of traceability can now be formulated as the diffeomorphism, " Π ": $\bar{M} \rightarrow M$, which assigns to each material point (i.e., the reference position of the matter), " \bar{s} " $\in \bar{S}$, and to each time, " \bar{t} " $\in \bar{T}$, that is to each material event, " \bar{m} " $\in \bar{M}$, the event in the material continuum, " m " $\in M$, which represents the location at time \bar{t} of the matter occupying \bar{s} in the reference state. The event, m , will be called the position of the matter. It may be noted that this implicitly limits the material continua to those subsets of event space for which there is a 1-1 correspondence between the spatial points in any time slice and the spatial points in the reference state.

The manner in which \bar{M} is constructed implies certain relationships for Π . The times are the same (e.g., if $(s,t) = \Pi(\bar{s}, \bar{t})$, then $t = \bar{t}$). The diffeomorphism from the time slice to \bar{S} is just the restriction to $t = t_0$ of the inverse of Π (e.g., $(\bar{s}, \bar{t}_0) = \Pi^{-1}(s, t_0)$). The usual coordinate system, $\{\bar{x}^i\}$, on \bar{M} is the product of the restrictions of μ_s and μ_T to \bar{S} and \bar{T} . In terms of the usual coordinates, the "coordinate expression" (A.4) for Π is

$$x^i \circ \Pi = \pi^i(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4) . \quad (1.3-1)$$

1.4 Velocity

For any specific $\bar{s} \in \bar{S}$, the position function assigns a point in M to each time, \bar{t} , $\in \bar{T}$. These points in M or \bar{M} form a "differentiable curve" (A.5) parameterized by the usual time coordinate. These curves represent the trajectory of the matter which occupies \bar{s} in the reference state. The "tangent vector" (A.5) to these curves at any event has a physical interpretation as the velocity at the event. The collection of all such tangents is modeled as a "vector field" (A.5)

called the velocity field and denoted "V" or " \bar{V} ". Each of the curves is the "integral curve" (A.5) that passes through $\Pi(\bar{s}, \bar{t}_0)$ or (\bar{s}, \bar{t}_0) , respectively.

An integral curve in \bar{M} has the form " $\gamma(u)$ " = $\mu^{-1}(\bar{x}^1(\bar{s}), \bar{x}^2(\bar{s}), \bar{x}^3(\bar{s}), u)$; that is it is the fourth "coordinate curve" (A.5). Its tangent is " $\gamma_{*}(\bar{m})$ ", = $\frac{\partial}{\partial x^4}(\bar{m})$. The velocity field, \bar{V} , in \bar{M} is thus $\bar{V} = \frac{\partial}{\partial x^4}$. The tangent to the integral curve in M, $\Pi \circ \gamma(u)$, is

obtained from the "differential" of Π (A.5) as

$$V[\Pi(\bar{m})] = \Pi_{*}[\gamma_{*}(\bar{m})] = \frac{\partial[x^i \circ \Pi]}{\partial x^4}(\bar{m}) \frac{\partial}{\partial x^i} \circ \Pi(\bar{m}). \quad (1.4-1)$$

Recalling the coordinate expression for Π , (1.3-1), the velocity field in M is

$$v = \frac{\partial(x^i \circ \Pi)}{\partial x^4} \circ \Pi^{-1} \frac{\partial}{\partial x^i} = \frac{\partial \pi^i}{\partial u^4} \circ \bar{\mu} \circ \Pi^{-1} \frac{\partial}{\partial x^i}. \quad (1.4-2)$$

It may be noted that since $x^4 \circ \Pi = \bar{x}^4$ the component of $\frac{\partial}{\partial x^4}$ is 1. (In the usual notation with $x^4 = t$, $x^1 \circ \Pi = r_x$, $x^2 \circ \Pi = r_y$, $x^3 \circ \Pi = r_z$,

$\frac{\partial}{\partial x^1} = \hat{i}$, $\frac{\partial}{\partial x^2} = \hat{j}$, $\frac{\partial}{\partial x^3} = \hat{k}$, this becomes

$$v = \frac{\partial r_x}{\partial t} \hat{i} + \frac{\partial r_y}{\partial t} \hat{j} + \frac{\partial r_z}{\partial t} \hat{k} + \frac{\partial}{\partial t}; \quad (1.4-3)$$

the spatial part of which is the usual definition of velocity.)

The velocity at any point, m, is algebraically a vector in the vector space " M_m ," the "tangent space" (A.5) to the material continuum. The "coordinate vector fields," (A.5) " $\left\{ \frac{\partial}{\partial x^i} \right\}$ ", form a "basis" (A.6) for the space. Among all the bases, the coordinate vector fields

satisfy the additional relationship of the "usual tensor transform"

(A.5). If " $\{y^i\}$ " are also coordinates, then $V = a^i \frac{\partial}{\partial x^i} (m) = b^i \frac{\partial}{\partial y^i} (m)$ implies $a^i = b^j \frac{\partial x^i}{\partial y^j} (m)$. The velocity at a point is, thus, a "tensor" of "type (1,0)", a "contravariant vector" (A.7), belonging to the "tensor space" M_m (A.7).

1.5 Reference Frames

The notion of a reference frame is equivalent to the concept of the event space of an observer. If there are two observers, they will be, in general, in motion relative to one another. The spatial points that appear fixed to one observer appear to be moving relative to the other observer (i.e., the observers' event spaces or frames of reference are different). If at some point in time the two observers mutually agree upon the identification of points in space and time, and upon a coordinate system, their temporal spaces and coordinates will be the same. However, at any time other than the reference, their physical spaces may differ (e.g. a point in space that they agreed was the same at t_0 , they will, in general, interpret as different points at any other time). If they both select the reference time for the time slice and the same reference state, material event space will be the same for both observers.

The mathematical model of this relationship is a diffeomorphism,

$$A : E' \rightarrow E, \quad (1.5-1)$$

which assigns to any event in the second observer's space, the event in the first observer's space that they would mutually agree is the identical event physically. Let the $\mu' - \mu$ coordinate expression for this relationship (1.5-1) be

$$x^i \circ \Lambda = \lambda^i(x'^1, x'^2, x'^3, x'^4) \quad (1.5-2)$$

for the usual coordinates, $\{x^i\}$ and $\{x'^i\}$, on E and E'.

Since the distance and duration functions are associated with space and time, and are not dependent on the observer, they should appear the same in both observers' physical and temporal spaces. The metrics are equivalent to "bilinear forms" (A.7) on the manifold. In fact, σ and τ may be obtained as the "distance" (A.7) between points with respect to the bilinear forms (in terms of the usual coordinates)

$$\delta_{\alpha\beta} dx^\alpha \otimes dx^\beta \quad (1.5-3)$$

and

$$dx^4 \otimes dx^4. \quad (1.5-4)$$

(The Greek postscripts indicate a range of 3, $\alpha, \beta = 1, 2, 3$. Latin postscripts will be used when the range is 4.) These forms are " Λ -related" (A.8) to the bilinear forms

$$\delta_{\alpha\beta} \frac{\partial(\lambda^\alpha \circ \mu')}{\partial x'^\gamma} \frac{\partial(\lambda^\beta \circ \mu')}{\partial x'^\delta} dx'^\gamma \otimes dx'^\delta \quad (1.5-5)$$

and

$$\frac{\partial(\lambda^4 \circ \mu')}{\partial x'^4} \frac{\partial(\lambda^4 \circ \mu')}{\partial x'^4} dx'^4 \otimes dx'^4. \quad (1.5-6)$$

For the bilinear forms to have the expressions on E',

$$\delta_{\alpha\beta} dx'^\alpha \otimes dx'^\beta \quad (1.5-7)$$

and

$$dx'^4 \otimes dx'^4, \quad (1.5-8)$$

the λ^i must satisfy

$$\frac{\partial(\lambda^\alpha \circ \mu')}{\partial x'^\beta} \frac{\partial(\lambda^\alpha \circ \mu')}{\partial x'^\gamma} = \delta_{\beta\gamma} \quad (1.5-9)$$

and

$$\frac{\partial(\lambda^4 \circ \mu')}{\partial x'^4} \frac{\partial(\lambda^4 \circ \mu')}{\partial x'^4} = 1 \quad (1.5-10)$$

throughout E' . For these relationships to hold at every event in E' , the indicated partial derivatives must be constants, and consequently the associated λ^1 must be linear in the appropriate coordinates. The $\mu' - \mu$ coordinate expression must, therefore, have the form

$$\begin{aligned} x^1 \circ \Lambda &= r^1(x'^4) + c'^1_1(x'^4) x'^1 + c'^1_2(x'^4) x'^2 + c'^1_3(x'^4) x'^3, \\ x^2 \circ \Lambda &= r^2(x'^4) + c'^2_1(x'^4) x'^1 + c'^2_2(x'^4) x'^2 + c'^2_3(x'^4) x'^3, \\ x^3 \circ \Lambda &= r^3(x'^4) + c'^3_1(x'^4) x'^1 + c'^3_2(x'^4) x'^2 + c'^3_3(x'^4) x'^3, \\ x^4 \circ \Lambda &= x'^4 \end{aligned} \quad (1.5-11)$$

and satisfy the constraints

$$c'^\alpha_\beta(x'^4) c'^\alpha_\gamma(x'^4) = \delta_{\beta\gamma}. \quad (1.5-12)$$

The arbitrary translation in time coordinates has been set to zero, so the observers share the same time coordinate.

The coefficients of this expression have a physical interpretation. The r terms represent the unprimed components of the displacement from the unprimed to primed origin. The c'^α_β terms represent the direction cosines between the x^α and the x'^β axes.

The transform is apparently determined by the twelve real valued functions of time, r and c' ; however, the constraints limit the number of independent c' to three. There are several alternative parameterizations of the c' part of the transform. The Euler-Rodrigues parameters are an example of a four parameter set, and the Euler angles are an

example of a three parameter set. All three parameter sets have singularities (i.e., transforms that cannot be represented by the set).

The inverse map, $\Lambda^{-1} : E \rightarrow E'$, has a $\mu - \mu'$ coordinate expression

$$x'^1 \circ \Lambda^{-1} = -r'^1(x^4) + c_1^1(x^4) x^1 + c_2^1(x^4) x^2 + c_3^1(x^4) x^3,$$

$$x'^2 \circ \Lambda^{-1} = -r'^2(x^4) + c_1^2(x^4) x^1 + c_2^2(x^4) x^2 + c_3^2(x^4) x^3,$$

$$x'^3 \circ \Lambda^{-1} = -r'^3(x^4) + c_1^3(x^4) x^1 + c_2^3(x^4) x^2 + c_3^3(x^4) x^3$$

and

$$x'^4 \circ \Lambda^{-1} = x^4, \quad (1.5-13)$$

subject to the constraints

$$c_{\beta}^{\alpha}(x^4) c_{\gamma}^{\alpha}(x^4) = \delta_{\beta\gamma}. \quad (1.5-14)$$

Since the composition of Λ and Λ^{-1} must be the identity map,

$$\begin{aligned} x^{\alpha} &= r^{\alpha}(x^4) + c'^{\alpha}_{\beta}(x^4) [-r'^{\beta}(x^4) + c^{\beta}_{\gamma}(x^4) x^{\gamma}] \\ &= r^{\alpha}(x^4) - c'^{\alpha}_{\beta}(x^4) r'^{\beta}(x^4) + c'^{\alpha}_{\beta}(x^4) c^{\beta}_{\gamma}(x^4) x^{\gamma}. \end{aligned} \quad (1.5-15)$$

From this it may be concluded that r , r' , c , and c' satisfy the relations

$$r^{\alpha}(x^4) = c'^{\alpha}_{\beta}(x^4) r'^{\beta}(x^4) \quad (1.5-16)$$

and

$$c'^{\alpha}_{\beta}(x^4) c^{\beta}_{\gamma}(x^4) = \delta_{\gamma}^{\alpha}. \quad (1.5-17)$$

Recalling the constraints (1.5-12 and -14), it may be seen that

$$c^{\beta}_{\alpha}(x^4) = c'^{\alpha}_{\beta}(x^4); \quad (1.5-18)$$

that is, that C is the transpose of C' . Since its transpose is its inverse, C is an orthogonal matrix.

A contravariant vector field, $V' : M' \rightarrow TM'$, (e.g. velocity) has a Λ -related vector field, $V : M \rightarrow TM$, defined by

$$V = \Lambda_{*}(V') = v'^i \circ \Lambda^{-1} \frac{\partial(x^j \circ \Lambda)}{\partial x'^i} \circ \Lambda^{-1} \frac{\partial}{\partial x^j} \quad (1.5-19)$$

(for the usual coordinates). Recalling the coordinate expression for Λ (1.5-11), this is

$$V = \left(v'^\beta c'^\alpha_{\beta}(x'^4) + v'^4 \left[\frac{\partial r^\alpha(x'^4)}{\partial x'^4} + \frac{\partial c'^\alpha_{\beta}(x'^4)}{\partial x'^4} x'^\beta \right] \right) \circ \Lambda^{-1} \frac{\partial}{\partial x^\alpha} + v'^4 \circ \Lambda^{-1} \frac{\partial}{\partial x^4} \quad (1.5-20)$$

where the v'^i are the components of V' with respect to the usual coordinates. Note that $v'^4 = v^4 = 1$ for velocity.

As an alternative procedure, the real valued functions on M , $x'^i \circ \Lambda^{-1}$, may be taken as coordinates, $\{y^i\}$, on M . The expression for V under a change of coordinates is provided by the usual tensor transformation $V = v'^i \frac{\partial}{\partial y^i} = v'^i \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$. So

$$V = \left(v'^\alpha c'^\beta_{\alpha}(y^4) + v'^4 \left[\frac{\partial r^\beta(y^4)}{\partial y^4} + \frac{\partial c'^\beta_{\alpha}(y^4)}{\partial y^4} y^\alpha \right] \right) \frac{\partial}{\partial x^\beta} + v'^4 \frac{\partial}{\partial x^4} \quad (1.5-21)$$

Thus reference frames may be considered either as special classes of diffeomorphic manifolds, or as special classes of coordinatizations of a manifold.

It remains to be shown that the transform of the primed observer's velocity is actually the velocity that the unprimed observer would assign. If $\Pi' : \bar{M} \rightarrow M'$ is the mapping from material event space to the primed material continuum, then $\Pi = \Lambda \circ \Pi'$, since \bar{M} is the same for both observers. If the coordinate expression for Π' is

$$x'^i \circ \Pi' = \pi'^i (\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4), \quad (1.5-22)$$

then the velocity in M' is

$$v' = \frac{\partial(x'^i \circ \Pi')}{\partial \bar{x}^4} \circ \Pi'^{-1} \frac{\partial}{\partial x'^i} = \frac{\partial \pi'^i}{\partial \bar{u}^4} \circ \bar{\mu} \circ \Pi'^{-1} \frac{\partial}{\partial x'^i}. \quad (1.5-22b)$$

Substituting $\Pi = \Lambda \circ \Pi'$ in $x^i \circ \Pi = \pi^i \circ \bar{\mu}$ yields $x^i \circ \Lambda \circ \Pi' = \pi^i \circ \bar{\mu}$.

Inserting the coordinate expression for Λ (1.5-11) yields

$$\begin{aligned} \pi^\alpha \circ \bar{\mu} &= [r^\alpha(x'^4) + c'^\alpha_\beta(x'^4) x'^\beta] \circ \Pi' \quad \text{and} \\ \pi^4 \circ \bar{\mu} &= x'^4 \circ \Pi'. \end{aligned} \quad (1.5-23)$$

The usual velocity in M is

$$v = \frac{\partial(x^i \circ \Pi)}{\partial \bar{x}^4} \circ \Pi^{-1} \frac{\partial}{\partial x^i} = \frac{\partial(\pi^i \circ \bar{\mu})}{\partial \bar{x}^4} \circ \Pi^{-1} \frac{\partial}{\partial x^i}. \quad (1.5-24)$$

Substituting (1.5-23) in (1.5-24) yields

$$\begin{aligned} v = & \left[\frac{\partial r^\alpha(\bar{x}^4)}{\partial \bar{x}^4} + \frac{\partial c'^\alpha_\beta(\bar{x}^4)}{\partial \bar{x}^4} (x'^\beta \circ \Pi') + c'^\alpha_\beta(\bar{x}^4) \frac{\partial(x'^\beta \circ \Pi')}{\partial \bar{x}^4} \right] \circ \Pi^{-1} \frac{\partial}{\partial x^\alpha} \\ & + \frac{\partial}{\partial \bar{x}^4} \end{aligned} \quad (1.5-25)$$

since $x'^4 \circ \Pi' = \bar{x}^4$. Noting that $\bar{x}^4 \circ \Pi^{-1} = x^4$ and $\Pi' \circ \Pi^{-1} = \Lambda^{-1}$,

$$\begin{aligned} v = & \left([c'^\alpha_\beta(x^4)] [v'^\beta \circ \Lambda^{-1}] + \frac{\partial r^\alpha(x^4)}{\partial x^4} + \left[\frac{\partial c'^\alpha_\beta(x^4)}{\partial x^4} \right] [x'^\beta \circ \Lambda^{-1}] \right) \frac{\partial}{\partial x^\alpha} \\ & + \frac{\partial}{\partial x^4}, \end{aligned} \quad (1.5-26)$$

as was obtained before (1.5-20 or -21).

There is a class of reference frames which are preferred since the laws of mechanics, when expressed in these frames, have a simple form. They are called inertial reference frames. In practice no such frame

can be found; instead a frame which is sufficiently inertial to provide the degree of accuracy required by the problem is selected. Examples of sufficiently inertial frames of increasing accuracy are: 1) a frame fixed on the surface of the earth; 2) a frame fixed at the earth's center with an axis pointed at the sun; 3) a frame fixed at the sun's center with an axis pointed at a nearby star; 4) a frame fixed in the "fixed stars." In what follows, a reference to an inertial frame will mean a frame selected by a user as sufficiently inertial for the problem being solved. Unless otherwise noted, the unprimed reference frame will be taken as inertial.

A physical interpretation of the transformation of velocity between reference frames may be obtained by considering just the components in physical space. Time is then viewed as a parameter and is not explicitly in the manifold structure. Physical space is a submanifold of event space, and the tangent space to physical space is a submanifold of the tangent space to event space. Specifically, if $M(t)$ is the submanifold of the material continuum at time t (i.e., the time slice at t), and $V = a^i \frac{\partial}{\partial x^i}$, $i = 1, \dots, 4$ is a tangent to M at $m \in M(t)$, then the associated tangent to $M(t)$ is $\underline{V} = a^\alpha(t) \frac{\partial}{\partial x^\alpha}(t)$, $\alpha = 1, 2, 3$ and will be called a spatial vector, or the spatial part of V . The spatial vectors are generally easier to work with and are more familiar, however a contravariant vector (e.g. velocity) does not transform properly when treated as spatial, and an artificial means must be developed for treating them.

The spatial position relative to the primed origin may be defined as

$$\underline{P}' = \pi'^{\alpha}(\bar{x}^1, \bar{x}^2, \bar{x}^3) \circ \Pi'^{-1} \frac{\partial}{\partial x'^{\alpha}} = x'^{\alpha} \frac{\partial}{\partial x'^{\alpha}} , \quad (1.5-27)$$

noting the coordinate expression for Π' (1.5-22). Similarly

$$\underline{P} = \pi^{\alpha}(\bar{x}^1, \bar{x}^2, \bar{x}^3) \circ \Pi^{-1} \frac{\partial}{\partial x^{\alpha}} = x^{\alpha} \frac{\partial}{\partial x^{\alpha}} \quad (1.5-28)$$

is the spatial position relative to the unprimed origin. Adopting the convention of using the same symbol for spatially related fields,

$$\begin{aligned} \underline{P}' &= \Lambda_{*}(\underline{P}') = p'^{\alpha}(\bar{x}^1, \bar{x}^2, \bar{x}^3) \circ \Pi'^{-1} \circ \Lambda^{-1} \frac{\partial(x^{\beta} \circ \Lambda)}{\partial x'^{\alpha}} \circ \Lambda^{-1} \frac{\partial}{\partial x^{\beta}} \\ &= p'^{\alpha}(\bar{x}^1, \bar{x}^2, \bar{x}^3) \circ \Pi'^{-1} \circ \Lambda^{-1} c'_{\alpha}{}^{\beta} \circ \Lambda^{-1} \frac{\partial}{\partial x^{\beta}} \\ &= x'^{\alpha} \circ \Lambda^{-1} c'_{\alpha}{}^{\beta} \circ \Lambda^{-1} \frac{\partial}{\partial x^{\beta}} . \end{aligned} \quad (1.5-29)$$

Recalling the coordinate expression for Λ (1.5-11) yields

$$\begin{aligned} \underline{P}' &= (x^{\beta} - r^{\beta} \circ \Lambda^{-1}) \frac{\partial}{\partial x^{\beta}} = \underline{P} - r^{\beta} \circ \Lambda^{-1} \frac{\partial}{\partial x^{\beta}} \\ &= \underline{P} - \underline{R} , \end{aligned} \quad (1.5-30)$$

where \underline{R} is defined by

$$\underline{R} = r^{\beta} \circ \Lambda^{-1} \frac{\partial}{\partial x^{\beta}} . \quad (1.5-31)$$

The spatial field \underline{R} may be recognized as the position of the primed origin relative to the unprimed. In summary

$$\underline{P} = \underline{R} + \underline{P}' . \quad (1.5-32)$$

As noted, velocity does not transform properly as a spatial vector field. To circumvent this, differentiation with respect to a reference frame will be defined. Differentiation is conceived of as acting on the

vector field as a whole; so if $\underline{U} = u^\alpha \frac{\partial}{\partial x^\alpha}$, then $\frac{\partial}{\partial t}(\underline{U}) = \frac{\partial}{\partial t}(u^\alpha) \frac{\partial}{\partial x^\alpha}$
 $+ u^\alpha \frac{\partial}{\partial t} \left(\frac{\partial}{\partial x^\alpha} \right)$. The derivative of a basis in its own frame is

defined to be zero (e.g. $\frac{\partial}{\partial t} \left(\frac{\partial}{\partial x^\alpha} \right) = 0 = \frac{\partial'}{\partial t} \left(\frac{\partial}{\partial x'^\alpha} \right)$ but $\frac{\partial}{\partial t} \left(\frac{\partial}{\partial x'^\alpha} \right) \neq 0$).

With this notation, the definitions of velocity become

$$\underline{v} = \frac{\partial}{\partial t} (\underline{p}) = \frac{\partial p^\alpha(x^1, x^2, x^3)}{\partial t} \circ \Pi^{-1} \frac{\partial}{\partial x^\alpha} \quad (1.5-33)$$

and

$$\underline{v}' = \frac{\partial'}{\partial t} (\underline{p}') = \frac{\partial' p'^\alpha(x^1, x^2, x^3)}{\partial t} \circ \Pi'^{-1} \frac{\partial}{\partial x'^\alpha} \quad (1.5-34)$$

Inserting the Λ^{-1} - related basis,

$$\Lambda_*^{-1} \left(\frac{\partial}{\partial x^\alpha} \right) = \frac{\partial(x'^\gamma \circ \Lambda^{-1})}{\partial x^\alpha} \circ \Lambda \frac{\partial}{\partial x'^\gamma} = c_\alpha^\gamma \frac{\partial}{\partial x'^\gamma}, \quad (1.5-35)$$

into the spatial part of the velocity (1.5-26), yields

$$\begin{aligned} \underline{v} &= \frac{\partial r^\alpha}{\partial t} \frac{\partial}{\partial x^\alpha} + c_\alpha^\gamma c'^\alpha_\beta v'^\beta \frac{\partial}{\partial x'^\gamma} + c_\alpha^\gamma \frac{\partial c'^\alpha_\beta}{\partial t} x'^\beta \frac{\partial}{\partial x'^\gamma} \\ &= \frac{\partial}{\partial t} (\underline{R}) + \underline{v}' + c_\alpha^\gamma \frac{\partial}{\partial t} \left(c'^\alpha_\beta \right) x'^\beta \frac{\partial}{\partial x'^\gamma}, \end{aligned} \quad (1.5-36)$$

where it has been noted that

$$c'^\alpha_\beta c_\alpha^\gamma = \delta_\beta^\gamma \quad \text{and} \quad \underline{v}' = \Lambda_* (\underline{v}')$$

The term $c_\alpha^\gamma \frac{\partial}{\partial t} c'^\alpha_\beta = c_\alpha^\gamma \frac{\partial}{\partial t} c_\alpha^\beta$ is the component matrix of a "skew-symmetric" tensor (A.7) since

$$\begin{aligned} c_\alpha^\gamma \frac{\partial}{\partial t} (c_\alpha^\beta) &= \frac{\partial}{\partial t} \left(c_\alpha^\gamma c_\alpha^\beta \right) - \frac{\partial}{\partial t} \left(c_\alpha^\gamma \right) c_\alpha^\beta \\ &= \frac{\partial}{\partial t} (\delta^{\gamma\beta}) - c_\alpha^\beta \frac{\partial}{\partial t} \left(c_\alpha^\gamma \right) \\ &= - c_\alpha^\beta \frac{\partial}{\partial t} \left(c_\alpha^\gamma \right). \end{aligned} \quad (1.5-37)$$

A skew-symmetric tensor of type (2,0) is naturally isomorphic to one of type (1,0) (in a three dimensional vector space). If this isomorphism is denoted by the "Hodge star operator" (A.9), then the associated contravariant vector field is

$$* \left[c_{\alpha}^{\gamma} \frac{\partial}{\partial t} \left(c_{\alpha}^{\beta} \right) \frac{\partial}{\partial x^{\gamma}} \otimes \frac{\partial}{\partial x^{\beta}} \right] = - \omega^{\prime \delta} \frac{\partial}{\partial x^{\delta}} = - \underline{\underline{\Omega}}^{\prime} . \quad (1.5-38)$$

This vector field may be physically interpreted as, and will be called, the angular velocity of the primed frame relative to the unprimed frame. The components, $\omega^{\prime \delta}$, are determined by

$$\omega^{\prime \delta} = - \frac{1}{2} \epsilon_{\gamma \alpha}^{\delta} c_{\beta}^{\gamma} \frac{\partial}{\partial t} \left(c_{\beta}^{\alpha} \right) , \quad (1.5-39)$$

where $\epsilon_{\gamma \alpha}^{\delta}$ is the "Levi-Civita epsilon" (A.9). Conversely, a tensor of type (1,0) is isomorphic to a skew symmetric tensor of type (2,0), customarily denoted by a tilde, " \sim ", over the symbol for the vector field. Thus

$$\begin{aligned} * (\underline{\underline{\Omega}}^{\prime}) &= * \left(\omega^{\prime \delta} \frac{\partial}{\partial x^{\delta}} \right) = - \tilde{\omega}^{\prime \alpha \beta} \frac{\partial}{\partial x^{\alpha}} \otimes \frac{\partial}{\partial x^{\beta}} \\ &= - \underline{\underline{\tilde{\Omega}}^{\prime}} , \end{aligned} \quad (1.5-40)$$

where

$$\tilde{\omega}^{\prime \alpha \beta} = - \epsilon_{\delta}^{\alpha \beta} \omega^{\prime \delta} .$$

The double underline denotes a tensor of type (2,0). Combining the two definitions yields

$$\tilde{\omega}^{\prime \alpha \beta} = - \epsilon_{\gamma}^{\alpha \beta} \left[- \frac{1}{2} \epsilon_{\epsilon \delta}^{\gamma} c_{\eta}^{\epsilon} \frac{\partial}{\partial t} \left(c_{\eta}^{\delta} \right) \right] . \quad (1.5-41)$$

Noting the identity,

$$\epsilon_{\gamma}^{\alpha \beta} \epsilon_{\epsilon \delta}^{\gamma} = \left(\delta_{\epsilon}^{\alpha} \delta_{\delta}^{\beta} - \delta_{\delta}^{\alpha} \delta_{\epsilon}^{\beta} \right) , \quad (1.5-42)$$

(1.5-41) becomes

$$\begin{aligned}\tilde{\omega}'^{\alpha\beta} &= \frac{1}{2} \left[\delta_{\epsilon}^{\alpha} \delta_{\delta}^{\beta} - \delta_{\delta}^{\alpha} \delta_{\epsilon}^{\beta} \right] \left[c_{\eta}^{\epsilon} \frac{\partial}{\partial t} \left(c_{\eta}^{\delta} \right) \right] \\ &= \frac{1}{2} \left[c_{\eta}^{\alpha} \frac{\partial}{\partial t} \left(c_{\eta}^{\beta} \right) - c_{\eta}^{\beta} \frac{\partial}{\partial t} \left(c_{\eta}^{\alpha} \right) \right].\end{aligned}\quad (1.5-43)$$

Recalling the skew-symmetry, (1.5-37),

$$\tilde{\omega}'^{\alpha\beta} = c_{\eta}^{\alpha} \frac{\partial}{\partial t} \left(c_{\eta}^{\beta} \right) = -\epsilon_{\gamma}^{\alpha\beta} \omega'^{\gamma}, \quad (1.5-44)$$

in accord with the definition (1.5-40).

With this notation, the previously obtained expression for velocity, (1.5-36), becomes

$$\begin{aligned}\underline{V} &= \frac{\partial}{\partial t} (\underline{R}) + \underline{V}' + \tilde{\omega}'^{\alpha\beta} x'^{\beta} \frac{\partial}{\partial x'^{\alpha}} \\ &= \frac{\partial}{\partial t} (\underline{R}) + \underline{V}' - \epsilon_{\gamma}^{\alpha\beta} \omega'^{\gamma} x'^{\beta} \frac{\partial}{\partial x'^{\alpha}}.\end{aligned}\quad (1.5-45)$$

The last term may be recognized as the "cross product" (A.9) of the vectors $\underline{\Omega}'$ and \underline{P}' at any point in the field. So

$$\underline{V} = \frac{\partial}{\partial t} \underline{R} + \underline{V}' + \underline{\Omega}' \times \underline{P}'. \quad (1.5-46)$$

This may be written in the more suggestive form

$$\frac{\partial}{\partial t} (\underline{P}) = \frac{\partial}{\partial t} (\underline{R}) + \frac{\partial'}{\partial t} (\underline{P}') + \underline{\Omega}' \times \underline{P}'. \quad (1.5-47)$$

On the other hand, (1.5-32) implies

$$\frac{\partial}{\partial t} (\underline{P}) = \frac{\partial}{\partial t} (\underline{R}) + \frac{\partial}{\partial t} (\underline{P}'). \quad (1.5-48)$$

Thus the correct transform for velocity is obtained by setting

$$\frac{\partial}{\partial t} (\underline{P}') = \frac{\partial'}{\partial t} (\underline{P}') + \underline{\Omega}' \times \underline{P}'. \quad (1.5-49)$$

Actually no redefinition is required since (1.5-49) holds for any vector field. To demonstrate this, let $\underline{U}' = u'^{\alpha} \frac{\partial}{\partial x'^{\alpha}}$. Then in terms of the Λ -related basis,

$$\begin{aligned}
\frac{\partial}{\partial t} (\underline{U}') &= \frac{\partial}{\partial t} (u'^{\alpha}) \frac{\partial}{\partial x'^{\alpha}} + u'^{\alpha} \frac{\partial}{\partial t} \left(\frac{\partial}{\partial x'^{\alpha}} \right) \\
&= \frac{\partial'}{\partial t} (u'^{\alpha}) \frac{\partial}{\partial x'^{\alpha}} + u'^{\alpha} \frac{\partial}{\partial t} \left(c'_{\alpha}{}^{\beta} \frac{\partial}{\partial x'^{\beta}} \right) \\
&= \frac{\partial'}{\partial t} (\underline{U}') + u'^{\alpha} \frac{\partial}{\partial t} \left(c'_{\alpha}{}^{\beta} \right) \frac{\partial}{\partial x'^{\beta}} .
\end{aligned} \tag{1.5-50}$$

Returning to the primed basis yields

$$\begin{aligned}
\frac{\partial}{\partial t} (\underline{U}') &= \frac{\partial'}{\partial t} (\underline{U}') + u'^{\alpha} c'_{\beta}{}^{\gamma} \frac{\partial}{\partial t} \left(c'_{\alpha}{}^{\beta} \right) \frac{\partial}{\partial x'^{\gamma}} \\
&= \frac{\partial'}{\partial t} (\underline{U}') + \tilde{\omega}'^{\gamma\alpha} u'^{\alpha} \frac{\partial}{\partial x'^{\gamma}} \\
&= \frac{\partial'}{\partial t} (\underline{U}') + \underline{\Omega}' \times \underline{U}' ,
\end{aligned} \tag{1.5-51}$$

recalling the previous definitions.

In summary, the proper transformation of velocity,

$$\underline{V} = \frac{\partial}{\partial t} (\underline{R}) + \underline{V}' + \underline{\Omega}' \times \underline{P}' \tag{1.5-52}$$

or

$$\frac{\partial}{\partial t} (\underline{P}) = \frac{\partial}{\partial t} (\underline{R}) + \frac{\partial'}{\partial t} (\underline{P}') + \underline{\Omega}' \times \underline{P}' , \tag{1.5-53}$$

is obtained by the definition of differentiation,

$$\frac{\partial}{\partial t} (\underline{U}') = \frac{\partial'}{\partial t} (\underline{U}') + \underline{\Omega}' \times \underline{U}' . \tag{1.5-54}$$

1.6 Newton's Laws

Newton's laws form the basis for much of structural mechanics.

Although originally stated for a particle, the first and second laws have been generalized to a continuum. For any point, m , in a material continuum, M , which is in an inertial reference frame, E ,

$$\rho \underline{A} - \underline{F} = \underline{0} , \quad (1.6-1)$$

where \underline{F} is a spatially contravariant body force density representing the sum of all long and short range forces at the point; ρ is the mass density; and \underline{A} is the acceleration.

The acceleration will be defined as the spatially contravariant vector

$$\underline{A} = \frac{\partial}{\partial t} (\underline{V}) = \frac{\partial^2}{\partial t^2} \left[p^\alpha (\bar{x}^1, \bar{x}^2, \bar{x}^3) \right] \circ \Pi^{-1} \frac{\partial}{\partial x^\alpha} . \quad (1.6-2)$$

The acceleration in the primed reference frame,

$$\underline{A}' = \frac{\partial'}{\partial t} (\underline{V}') = \frac{\partial'^2}{\partial t'^2} \left[p'^\alpha (\bar{x}^1, \bar{x}^2, \bar{x}^3) \right] \circ \Pi'^{-1} \frac{\partial'}{\partial x'^\alpha} , \quad (1.6-3)$$

is related to \underline{A} by

$$\begin{aligned} \underline{A} &= \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} (\underline{R}) + \underline{V}' + \underline{\Omega}' \times \underline{P}' \right] \\ &= \frac{\partial^2}{\partial t^2} (\underline{R}) + \frac{\partial'}{\partial t} (\underline{V}') + \underline{\Omega}' \times \underline{V}' + \frac{\partial'}{\partial t} (\underline{\Omega}') \times \underline{P}' + \underline{\Omega}' \times \frac{\partial'}{\partial t} (\underline{P}') \\ &\quad + \underline{\Omega}' \times (\underline{\Omega}' \times \underline{P}') \\ &= \frac{\partial^2}{\partial t^2} (\underline{R}) + \underline{A}' + 2\underline{\Omega}' \times \underline{V}' + \frac{\partial'}{\partial t} (\underline{\Omega}') \times \underline{P}' + \underline{\Omega}' \times (\underline{\Omega}' \times \underline{P}') . \end{aligned} \quad (1.6-4)$$

Since Newton's laws hold at any event in a material continuum, taking the "inner product" (A.7) with arbitrary spatial vector field, $\underline{\delta P}$, and "integrating" (A.10) does not affect the identity,

$$\int_{\bar{T}} \int_{M(t)} *(\rho \underline{A} \cdot \underline{\delta P}) - \int_{\bar{T}} \int_{M(t)} *(\underline{F} \cdot \underline{\delta P}) = 0 . \quad (1.6-5)$$

The integration may be considered as an integral over a time slice followed by integration with respect to the time parameter, or by returning to a four dimensional model, as the integral over the

material continuum expressed in iterated form in accord with Fubini's theorem. This distinction will not be pursued since the "associated Riemann integrals", (A.10), are essentially the same,

$$\int_{\mu_T(\bar{T})} \int_{\mu_S(M(t))} (\rho a^\alpha \delta p^\alpha) \circ \mu^{-1} d\mu_3 dt - \int_{\mu_T(\bar{T})} \int_{\mu_S(M(t))} (f^\alpha \delta p^\alpha) \circ \mu^{-1} d\mu_3 dt = 0 . \quad (1.6-6)$$

The first term is transformed to material event space to remove the time dependence of the spatial region of integration,

$$\int_{\mu_T(\bar{T})} \int_{\mu_S(\bar{M})} (\rho a^\alpha \delta p^\alpha) \circ \Pi \circ \bar{\mu}^{-1} \left| \det \left(\frac{\partial p^\beta}{\partial u^\gamma} \right) \right| d\mu_3 dt . \quad (1.6-7)$$

The Jacobian determinant, $\det (\partial p^\beta / \partial u^\gamma)$, may be interpreted as the ratio of an infinitesimal volume in M to one in \bar{M} . Since the mass density, ρ , may be interpreted as the ratio of the infinitesimal mass to its volume in M , the product $\rho \circ \Pi \left| \det \left(\frac{\partial p^\beta}{\partial u^\gamma} \right) \right|$ may be interpreted as the mass density in \bar{M} and will be denoted " $\hat{\rho}$ ". The first term in (1.6-5) becomes

$$\int_{\bar{T}} \int_{\bar{S}} * (\hat{\rho} \underline{A} \cdot \underline{\delta P}) . \quad (1.6-8)$$

Recalling that $\underline{A} = \frac{\partial}{\partial t} \underline{V}$, noting the identity

$$\frac{\partial}{\partial t} (\underline{V}) \cdot \underline{\delta P} = \frac{\partial}{\partial t} (\underline{V} \cdot \underline{\delta P}) - \underline{V} \cdot \frac{\partial}{\partial t} (\underline{\delta P}) \quad (1.6-9)$$

and performing the temporal integration (since the spatial region is now independent of time) yields for the first term of (1.6-5)

$$\int_{\partial \bar{T}} \int_{\bar{S}} * (\hat{\rho} \underline{V} \cdot \underline{\delta P}) - \int_{\bar{T}} \int_{\bar{S}} * \left[\hat{\rho} \underline{V} \cdot \frac{\partial}{\partial t} (\underline{\delta P}) \right] , \quad (1.6-10)$$

where $\partial \bar{T}$ may be taken as meaning $\left|_{t_1}^{t_2}$. Recalling $\underline{V} = \partial / \partial t (\underline{P})$, and

noting that

$$2 \left[\frac{\partial}{\partial t} (\underline{P}) \cdot \frac{\partial}{\partial t} (\underline{\delta P}) \right] = \frac{\partial}{\partial t} (\underline{P} + \underline{\delta P}) \cdot \frac{\partial}{\partial t} (\underline{P} + \underline{\delta P}) - \frac{\partial}{\partial t} (\underline{P}) \cdot \frac{\partial}{\partial t} (\underline{P}) + o(\|\underline{\delta P}\|), \quad (1.6-11)$$

the last term of (1.6-10) may be recognized as the "variation of a functional" (A.11). The first term of (1.6-5) may be written as

$$\int_{\partial T} \int_S * (\hat{\rho} \underline{v} \cdot \underline{\delta P}) - \delta \int_T \int_S * \left(\frac{1}{2} \hat{\rho} \underline{v} \cdot \underline{v} \right) + \delta \int_T \int_S * \left[\underline{L} \cdot \left(\underline{v} - \frac{\partial}{\partial t} \underline{P} \right) \right]. \quad (1.6-12)$$

The term in the inner parentheses of the last integral is constrained to be zero, so the term \underline{L} may be thought of as Lagrange multipliers for the constraint. The term $\frac{1}{2} \hat{\rho} \underline{v} \cdot \underline{v}$ is called the kinetic energy density and will be denoted " \hat{T} ". The process which yields the second two terms of (1.6-12) from the last term of (1.6-10) may be recognized as a Legendre transformation.

The stationary conditions for the terms in (1.6-12) include requiring the coefficients of the independent variables, $\delta \underline{v}$ and $\delta \underline{L}$, be equal to zero (e.g. $\underline{v} = \frac{\partial}{\partial t} \underline{P}$ and $\underline{L} = \hat{\rho} \underline{v}$). Eliminating \underline{L} as an independent variable by substituting $\hat{\rho} \underline{v}$ yields

$$\int_{\partial T} \int_S * (\hat{\rho} \underline{v} \cdot \underline{\delta P}) - \delta \int_T \int_S * \left(\frac{1}{2} \hat{\rho} \underline{v} \cdot \underline{v} \right) + \delta \int_T \int_S * \left[\hat{\rho} \underline{v} \cdot \left(\underline{v} - \frac{\partial}{\partial t} \underline{P} \right) \right] \quad (1.6-13)$$

for the first term in (1.6-5).

The previous results are associated with Newton's first and second laws. Newton's third law states that the force on matter, due to its

interaction with some other matter, is equal but opposite in sign to the force on the other matter due to the first. If the force on the matter at m_1 due to interaction with matter at m_2 is denoted " $\underline{\phi}(m_1, m_2)$ ", then

$$\underline{\phi}(m_2, m_1) = - \underline{\phi}(m_1, m_2) . \quad (1.6-14)$$

At any point, m , in a spatial region, R , let the sum of all long range forces be denoted " \underline{X} ", let the sum of all short range forces on the matter at the point due to matter outside the region be denoted " $\underline{\hat{S}}$ ", then the total force, \underline{F} , at the point has components

$$f^\alpha = x^\alpha + \hat{s}^\alpha + \int_{\mu(R)} \phi^\alpha(m, \hat{\mu}^{-1}) dx dy dz . \quad (1.6-15)$$

Integrating these component fields with respect to the usual coordinates, find

$$\begin{aligned} \int_{\mu(R)} f^\alpha \circ \mu^{-1} dx dy dz &= \int_{\mu(R)} x^\alpha \circ \mu^{-1} dx dy dz + \int_{\mu(R)} \hat{s}^\alpha \circ \mu^{-1} dx dy dz \\ &+ \int_{\mu(R)} \int_{\hat{\mu}(R)} \phi^\alpha(\mu^{-1}, \hat{\mu}^{-1}) d\hat{x} d\hat{y} d\hat{z} dx dy dz . \end{aligned} \quad (1.6-16)$$

The last integral vanishes; and the integral of short range external forces is customarily replaced by the associated boundary integral, yielding

$$\int_{\mu(R)} f^\alpha \circ \mu^{-1} dx dy dz = \int_{\mu(R)} x^\alpha \circ \mu^{-1} dx dy dz + \int_{\mu(\partial R)} s^\alpha du^1 du^2 , \quad (1.6-17)$$

or on the manifold

$$\int_R * \left(\underline{F} \cdot \frac{\partial}{\partial x^\alpha} \right) = \int_R * \left(\underline{X} \cdot \frac{\partial}{\partial x^\alpha} \right) + \int_{\partial R} * \left(\underline{S} \cdot \frac{\partial}{\partial x^\alpha} \right) . \quad (1.6-18)$$

1.7 Stress

In the previous section (1.6) the integrals of the components of the total force and the body force vector fields over a region were related to the integral of the components of the surface force over the boundary. In an appendix (B.1) this result is applied to a tetrahedron with an apex at an arbitrary point, m , base normal to an arbitrary vector, $\underline{\epsilon N}$, and sides defined by the usual coordinate planes.

The form of the result suggests the definition of a stress tensor field of type (2,0),

$$\underline{\underline{\Sigma}} = \sigma^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} \quad (1.7-1)$$

Then a surface force on a surface with an outward pointing normal, $n^\alpha \frac{\partial}{\partial x^\alpha}$, is determined as

$$\underline{\underline{S}} = \underline{\underline{N}} \cdot \underline{\underline{\Sigma}}, \quad (1.7-2)$$

where the "dot product" (A.7) in the usual coordinates is

$$n^\alpha \sigma^{\alpha\beta} \frac{\partial}{\partial x^\beta} \quad (1.7-3)$$

It may be noted that the surface force on the face with an outward pointing normal, $-\frac{\partial}{\partial x^\alpha}$, is $-\sigma^{\alpha\beta} \frac{\partial}{\partial x^\beta}$ in accord with the definition in the appendix.

It has been shown that there is a tensor field called stress, that when dot multiplied by the unit outward pointing normal to a surface, yields the surface force density; and that the force density so obtained is consistent with (1.6-18). However, this has only been shown for the usual coordinates. In Appendix B.2 it is demonstrated that the relations are true for arbitrary coordinate systems. Thus

the relation, (1.6-18), may be written

$$\int_R * \left(\underline{F} \cdot \frac{\partial}{\partial \underline{x}^\alpha} \right) = \int_R * \left(\underline{X} \cdot \frac{\partial}{\partial \underline{x}^\alpha} \right) + \int_{\partial R} * \left(\underline{N} \cdot \underline{\Sigma} \cdot \frac{\partial}{\partial \underline{x}^\alpha} \right). \quad (1.7-4)$$

The last integral can be converted to a volume integral by means of "Stokes' theorem" (A.10),

$$\begin{aligned} \int_{\partial R} * \left(\underline{N} \cdot \underline{\Sigma} \cdot \frac{\partial}{\partial \underline{x}^\alpha} \right) &= \int_R d * \left(\underline{N} \cdot \underline{\Sigma} \cdot \frac{\partial}{\partial \underline{x}^\alpha} \right) \\ &= \int_R * \left[*d * \left(\underline{N} \cdot \underline{\Sigma} \cdot \frac{\partial}{\partial \underline{x}^\alpha} \right) \right] \\ &= \int_R * \left(\underline{\nabla} \cdot \underline{\Sigma} \cdot \frac{\partial}{\partial \underline{x}^\alpha} \right). \end{aligned} \quad (1.7-5)$$

Since the region of integration is arbitrary, the components, and thus the vector fields, are related by

$$\underline{F} = \underline{X} + \underline{\nabla} \cdot \underline{\Sigma}, \quad (1.7-6)$$

where "nabla, ∇ ," with a dot denotes the "divergence" (A.12). In the usual coordinates, this is

$$f^\alpha \frac{\partial}{\partial x^\alpha} = x^\alpha \frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\beta} (\sigma^{\beta\alpha}) \frac{\partial}{\partial x^\alpha}. \quad (1.7-7)$$

1.8 Virtual Work

Substituting (1.6-13) for the first term and (1.7-6) for the second term of (1.6-5), yields as a consequence of Newton's Laws

$$\begin{aligned} & - \delta \int_{\bar{T}} \int_{\bar{S}} * \left(\hat{T} \right) + \delta \int_{\bar{T}} \int_{\bar{S}} * \left[\hat{\rho} \underline{V} \cdot \left(\underline{V} - \frac{\partial}{\partial t} \underline{P} \right) \right] \\ & + \int_{\partial \bar{T}} \int_{\bar{S}} * \left(\hat{\rho} \underline{V} \cdot \delta \underline{P} \right) \\ & - \int_{\bar{T}} \int_{M(t)} * \left(\underline{X} \cdot \delta \underline{P} \right) - \int_{\bar{T}} \int_{M(t)} * \left[\left(\underline{\nabla} \cdot \underline{\Sigma} \right) \cdot \delta \underline{P} \right] = 0. \end{aligned} \quad (1.8-1)$$

Body forces generally depend on the amount of matter present, so it is convenient to transform to material event space. The product

$\underline{X} \left| \text{Det} \left(\frac{\partial p^\alpha}{\partial u^\beta} \right) \right|$ is called the body force referred to the undeformed

volume and is denoted " \hat{X} ." The body force integral may then be written

$$-\int_{\bar{T}} \int_{M(t)} * (\underline{X} \cdot \delta \underline{P}) = -\int_{\bar{T}} \int_{\bar{S}} * (\hat{X} \cdot \delta \underline{P}) . \quad (1.8-2)$$

The body force may be separated into two parts, a conservative part which may be obtained as a "gradient" (A.12), $C \hat{X} = -\underline{\nabla} \hat{W}$, and a non-conservative part, $N \hat{X}$. The body force integral (1.8-2) becomes

$$-\int_{\bar{T}} \int_{M(t)} * (\underline{X} \cdot \delta \underline{P}) = -\int_{\bar{T}} \int_{\bar{S}} * (-\underline{\nabla} \hat{W} \cdot \delta \underline{P}) - \int_{\bar{T}} \int_{\bar{S}} * (N \hat{X} \cdot \delta \underline{P}) . \quad (1.8-3)$$

Noting that

$$-\underline{\nabla} \hat{W} \cdot \delta \underline{P} = -\hat{W}(\underline{P} + \delta \underline{P}) + \hat{W}(\underline{P}) + o(\|\delta \underline{P}\|), \quad (1.8-4)$$

the body force integral may be written

$$-\int_{\bar{T}} \int_{M(t)} * (\underline{X} \cdot \delta \underline{P}) = \delta \int_{\bar{T}} \int_{\bar{S}} * (\hat{W}) - \int_{\bar{T}} \int_{\bar{S}} * (N \hat{X} \cdot \delta \underline{P}) . \quad (1.8-5)$$

The function \hat{W} is called the potential energy density field. This result may be generalized slightly by allowing \hat{W} to be a non-conservative explicit function of time, in which case \hat{W} is said to be lamellar or irrotational.

The integral involving the stress may be rewritten by noting the identity

$$\left(\underline{\nabla} \cdot \underline{\Sigma} \right) \cdot \delta \underline{P} = \underline{\nabla} \cdot \left(\underline{\Sigma} \cdot \delta \underline{P} \right) - \underline{\Sigma} : \underline{\nabla} \delta \underline{P} . \quad (1.8-6)$$

The double dot ":" means contractions are to be performed on the first indicies of both tensors and then on the second. In terms of the usual coordinates this identity has the form

$$\frac{\partial}{\partial x^\alpha} (\sigma^{\alpha\beta}) \delta p^\beta = \frac{\partial}{\partial x^\alpha} (\sigma^{\alpha\beta} \delta p^\beta) - \sigma^{\alpha\beta} \frac{\partial}{\partial x^\alpha} (\delta p^\beta) . \quad (1.8-7)$$

The stress integral in (1.8-1) may thus be written in two parts,

$$\begin{aligned}
 -\int_{\bar{T}} \int_{M(t)} * \left[\left(\underline{\nabla} \cdot \underline{\underline{\Sigma}} \right) \cdot \delta \underline{P} \right] &= -\int_{\bar{T}} \int_{M(t)} * \left[\underline{\nabla} \cdot \left(\underline{\underline{\Sigma}} \cdot \delta \underline{P} \right) \right] \\
 + \int_{\bar{T}} \int_{M(t)} * \left(\underline{\underline{\Sigma}} : \underline{\nabla} \delta \underline{P} \right) &. \quad (1.8-8)
 \end{aligned}$$

The first stress integral in (1.8-8) may be converted to a boundary integral via the generalized Stokes' theorem (A.10), yielding

$$-\int_{\bar{T}} \int_{M(t)} * \left[\underline{\nabla} \cdot \left(\underline{\underline{\Sigma}} \cdot \delta \underline{P} \right) \right] = -\int_{\bar{T}} \int_{\partial M(t)} * \left(\underline{N} \cdot \underline{\underline{\Sigma}} \cdot \delta \underline{P} \right) . \quad (1.8-9)$$

The boundary integral may be separated into two parts, the spatial surface, $\partial M(t)_S$, over which surface forces are prescribed, \underline{P}_S , and the surface, $\partial M(t)_P$, over which positions are prescribed, \underline{P}_P . The condition that actual positions equal the prescribed can be introduced with \underline{S} as a Lagrange multiplier. With these definitions (1.8-9) may be written

$$\begin{aligned}
 -\int_{\bar{T}} \int_{M(t)} * \left[\underline{\nabla} \cdot \left(\underline{\underline{\Sigma}} \cdot \delta \underline{P} \right) \right] &= -\int_{\bar{T}} \int_{\partial M(t)_S} * \left(\underline{P}_S \cdot \delta \underline{P} \right) \\
 -\int_{\bar{T}} \int_{\partial M(t)_P} * \left(\underline{S} \cdot \delta \underline{P}_P \right) - \int_{\bar{T}} \int_{\partial M(t)_P} * \delta \left[\underline{S} \cdot \left(\underline{P} - \underline{P}_P \right) \right] &. \quad (1.8-10)
 \end{aligned}$$

The second stress integral in (1.8-8) may be transformed to material event space yielding

$$\int_{\bar{T}} \int_{M(t)} * \left(\underline{\underline{\Sigma}} : \underline{\nabla} \delta \underline{P} \right) = \int_{\bar{T}} \int_{\bar{S}} * \left[\left(\underline{\underline{\Sigma}} : \underline{\nabla} \delta \underline{P} \right) \circ \Pi \left| \det \left(\frac{\partial \underline{P}}{\partial \underline{u}} \right) \right| \right] . \quad (1.8-11)$$

The stress tensor may be expressed in terms of the usual basis for material event space. However, the stress represents the magnitude of the force on a unit area perpendicular to a unit normal in \bar{S} , not $M(t)$. The rectangular parallelepiped with a base defining a unit area and with a height determined by the unit normal becomes, in \bar{S} , an oblique parallelepiped with volume equal to the Jacobian determinant. To

obtain the force for a unit normal and unit area in \bar{S} , $\underline{\Sigma}$ can be multiplied by the Jacobian determinant $|\det(\frac{\partial p}{\partial u})|$. The result is called the stress referred to the dimensions of an element before deformation, and is denoted " $\underline{\Sigma}^\wedge$ ".

The differentiation in the gradient may be converted to material event space coordinates. In terms of the usual coordinates, this becomes

$$\begin{aligned}
 \left[\overset{\wedge}{\sigma}^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \left(\delta p^\beta \circ \bar{\mu} \circ \Pi^{-1} \right) \right] \circ \Pi &= \overset{\wedge}{\sigma}^{\alpha\beta} \circ \Pi \left[\frac{\partial}{\partial u^\alpha} \left(\delta p^\beta \circ \bar{\mu} \circ \Pi^{-1} \circ \mu^{-1} \right) \circ \mu \circ \Pi \right] \\
 &= \overset{\wedge}{\sigma}^{\alpha\beta} \circ \Pi \left[\frac{\partial}{\partial u^\alpha} \left(\bar{\mu}^\gamma \circ \Pi^{-1} \circ \mu^{-1} \right) \circ \mu \circ \Pi \right] \left[\frac{\partial}{\partial u^\gamma} \left(\delta p^\beta \right) \circ \bar{\mu} \right] \\
 &= \overset{\wedge}{\sigma}^{\alpha\beta} \circ \Pi \left[\frac{\partial}{\partial x^\alpha} \left(\bar{x}^\gamma \circ \Pi^{-1} \right) \circ \Pi \right] \left[\frac{\partial}{\partial \bar{x}^\gamma} \left(\delta p^\beta \circ \bar{\mu} \right) \right] \\
 &= \left[\frac{\partial \left(\bar{x}^\gamma \circ \Pi^{-1} \right)}{\partial x^\alpha} \circ \Pi \right] \overset{\wedge}{\sigma}^{\alpha\beta} \circ \Pi \left[\frac{\partial \left(\delta p^\beta \circ \bar{\mu} \right)}{\partial \bar{x}^\gamma} \right] . \quad (1.8-12)
 \end{aligned}$$

Noting the identity

$$\begin{aligned}
 \delta_{\beta\delta} &= \frac{\partial (x^\delta)}{\partial x^\beta} = \frac{\partial}{\partial u^\beta} \left(x^\delta \circ \Pi \circ \mu^{-1} \circ \mu \circ \Pi^{-1} \circ \mu^{-1} \right) \circ \mu \\
 &= \frac{\partial}{\partial u^\beta} \left(\bar{\mu}^\epsilon \circ \Pi^{-1} \circ \mu^{-1} \right) \circ \mu \frac{\partial}{\partial \bar{\mu}^\epsilon} \left(x^\delta \circ \Pi \circ \mu^{-1} \right) \circ \mu \circ \Pi^{-1} \\
 &= \frac{\partial}{\partial x^\beta} \left(\bar{x}^\epsilon \circ \Pi^{-1} \right) \frac{\partial}{\partial \bar{x}^\epsilon} \left(x^\delta \circ \Pi \right) \circ \Pi^{-1} , \quad (1.8-13)
 \end{aligned}$$

(1.8-12) may be written in the form

$$\left(\left[\frac{\partial(\bar{x}^\gamma \circ \Pi^{-1})}{\partial \bar{x}^\alpha} \circ \Pi \right] \circ \sigma^{\alpha\beta} \circ \Pi \left[\frac{\partial(\bar{x}^\epsilon \circ \Pi^{-1})}{\partial \bar{x}^\beta} \circ \Pi \right] \right) \left(\left[\frac{\partial(\bar{x}^\delta \circ \Pi)}{\partial \bar{x}^\epsilon} \right] \left[\frac{\partial(\delta p^\delta \circ \bar{\mu})}{\partial \bar{x}^\gamma} \right] \right), \quad (1.8-14)$$

where the first term in the product may be recognized as the related stress in \bar{S} , $\sigma^{\alpha\gamma\epsilon}$. Alternatively, the expression for the related stress in $M(t)$,

$$\begin{aligned} \Pi_* \left(\sigma^{\alpha\gamma\epsilon} \frac{\partial}{\partial \bar{x}^\gamma} \otimes \frac{\partial}{\partial \bar{x}^\epsilon} \right) &= \\ &= \left[\frac{\partial(\bar{x}^\alpha \circ \Pi)}{\partial \bar{x}^\gamma} \circ \Pi^{-1} \right] \sigma^{\alpha\gamma\epsilon} \circ \Pi^{-1} \left[\frac{\partial(\bar{x}^\beta \circ \Pi)}{\partial \bar{x}^\epsilon} \circ \Pi^{-1} \right] \frac{\partial}{\partial \bar{x}^\alpha} \otimes \frac{\partial}{\partial \bar{x}^\beta}, \end{aligned} \quad (1.8-15)$$

could be directly substituted in (1.8-12) with the same result.

The second term in the product in (1.8-14) may be recognized as the variation of the strains. The strains are determined from the fundamental bilinear form (1.5-3). The Π -related form in \bar{S} is

$$\delta_{\alpha\beta} \frac{\partial(\bar{x}^\alpha \circ \Pi)}{\partial \bar{x}^\gamma} \frac{\partial(\bar{x}^\beta \circ \Pi)}{\partial \bar{x}^\delta} d\bar{x}^\gamma \otimes d\bar{x}^\delta. \quad (1.8-16)$$

The strain tensor, " $\underline{\underline{E}}$ ", is defined to be 1/2 the difference between the bilinear forms,

$$\begin{aligned} \underline{\underline{E}} &= \frac{1}{2} [\Pi^*(b) - \bar{b}] \\ &= \frac{1}{2} \left[\frac{\partial(\bar{x}^\alpha \circ \Pi)}{\partial \bar{x}^\gamma} \frac{\partial(\bar{x}^\alpha \circ \Pi)}{\partial \bar{x}^\delta} - \delta_{\gamma\delta} \right] d\bar{x}^\gamma \otimes d\bar{x}^\delta \end{aligned} \quad (1.8-17)$$

the usual form can be obtained by noting

$$\delta_{\gamma\delta} = \frac{\partial \bar{x}^\alpha}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial \bar{x}^\delta}. \quad (1.8-18)$$

The strain can then be written in the form

$$\begin{aligned}
 \underline{\underline{E}} &= \frac{1}{2} \left[\frac{\partial(x^\alpha \circ \Pi - \bar{x}^\alpha)}{\partial \bar{x}^\gamma} \frac{\partial(x^\alpha \circ \Pi - \bar{x}^\alpha)}{\partial \bar{x}^\delta} + \delta_\alpha^\gamma \frac{\partial(x^\alpha \circ \Pi - \bar{x}^\alpha)}{\partial \bar{x}^\delta} \right. \\
 &\quad \left. + \delta_\alpha^\delta \frac{\partial(x^\alpha \circ \Pi - \bar{x}^\alpha)}{\partial \bar{x}^\gamma} \right] d\bar{x}^\gamma \otimes d\bar{x}^\delta \\
 &= \frac{1}{2} \left[\frac{\partial(x^\gamma \circ \Pi - \bar{x}^\gamma)}{\partial \bar{x}^\delta} + \frac{\partial(x^\delta \circ \Pi - \bar{x}^\delta)}{\partial \bar{x}^\gamma} \right. \\
 &\quad \left. + \frac{\partial(x^\alpha \circ \Pi - \bar{x}^\alpha)}{\partial \bar{x}^\gamma} \frac{\partial(x^\alpha \circ \Pi - \bar{x}^\alpha)}{\partial \bar{x}^\delta} \right] d\bar{x}^\gamma \otimes d\bar{x}^\delta . \tag{1.8-19}
 \end{aligned}$$

In the customary notation, the difference in coordinates, $x^\delta \circ \Pi - \bar{x}^\delta$, is called the displacement, u^δ , the form $\frac{1}{2} \left[\frac{\partial u^\delta}{\partial \bar{x}^\gamma} + \frac{\partial u^\gamma}{\partial \bar{x}^\delta} \right]$ is just the usual definition of the linear strains, and the term $\frac{1}{2} \left[\frac{\partial u^\alpha}{\partial \bar{x}^\delta} \frac{\partial u^\alpha}{\partial \bar{x}^\gamma} \right]$ is just the correction to provide the geometrically nonlinear definition of the strain. Now, noting that

$$\begin{aligned}
 &\frac{\partial(p^\delta \circ \bar{\mu})}{\partial \bar{x}^\epsilon} \frac{\partial(\delta p^\delta \circ \bar{\mu})}{\partial \bar{x}^\gamma} + \frac{\partial(\delta p^\delta \circ \bar{\mu})}{\partial \bar{x}^\epsilon} \frac{\partial(p^\delta \circ \bar{\mu})}{\partial \bar{x}^\gamma} \\
 &= \frac{\partial(p^\delta + \delta p^\delta \circ \bar{\mu})}{\partial \bar{x}^\epsilon} \frac{\partial(p^\delta + \delta p^\delta \circ \bar{\mu})}{\partial \bar{x}^\gamma} - \frac{\partial(p^\delta \circ \bar{\mu})}{\partial \bar{x}^\epsilon} \frac{\partial(p^\delta \circ \bar{\mu})}{\partial \bar{x}^\gamma} + o(\|\delta p\|) \\
 &= \delta \left[\frac{\partial(p^\delta \circ \bar{\mu})}{\partial \bar{x}^\epsilon} \frac{\partial(p^\delta \circ \bar{\mu})}{\partial \bar{x}^\gamma} - \delta_{\epsilon\gamma} \right] \\
 &= \delta \left[2\epsilon_{\epsilon\gamma} \right] , \tag{1.8-20}
 \end{aligned}$$

(1.8-14) may be written as

$$\frac{1}{2} \left[\frac{\Lambda^\gamma \epsilon}{\sigma} + \frac{\Lambda^\epsilon \gamma}{\sigma} \right] \left[\frac{\partial(p^\delta \circ \bar{\mu})}{\partial \bar{x}^\epsilon} \frac{\partial(\delta p^\delta \circ \bar{\mu})}{\partial \bar{x}^\gamma} \right] . \tag{1.8-21}$$

Since the stress tensor is symmetric, interchanging indices gives

$$\begin{aligned} & \frac{1}{2} \left[\overset{\wedge}{\sigma}^{\gamma\epsilon} \right] \left[\frac{\partial(p \overset{\delta}{\sigma} \bar{\mu})}{\partial \bar{x}^\epsilon} \frac{\partial(\delta p \overset{\delta}{\sigma} \bar{\mu})}{\partial \bar{x}^\gamma} + \frac{\partial(p \overset{\delta}{\sigma} \bar{\mu})}{\partial \bar{x}^\gamma} \frac{\partial(\delta p \overset{\delta}{\sigma} \bar{\mu})}{\partial \bar{x}^\epsilon} \right] \\ &= \frac{1}{2} \overset{\wedge}{\sigma}^{\gamma\epsilon} \delta [2 \epsilon_{\epsilon\gamma}] = \overset{\wedge}{\sigma}^{\gamma\epsilon} \delta \epsilon_{\gamma\epsilon} . \end{aligned} \quad (1.8-22)$$

Thus the second stress integral (1.8-11) may be written

$$\int_{\bar{T}} \int_{\bar{M}(t)} * \left(\underline{\underline{\Sigma}} : \nabla \delta \underline{\underline{P}} \right) = \int_{\bar{T}} \int_{\bar{S}} * \left(\overset{\wedge}{\underline{\underline{\Sigma}}} : \delta \underline{\underline{E}} \right) + \delta \int_{\bar{T}} \int_{\bar{S}} * \left[\overset{\wedge}{\underline{\underline{\Sigma}}} : (\bar{\nabla} \underline{\underline{P}} - \underline{\underline{E}}) \right], \quad (1.8-23)$$

where the term in the inner parentheses of the last term is constrained to be zero, so the term $\overset{\wedge}{\underline{\underline{\Sigma}}}$ may be thought of as a Lagrange multiplier. The bar over nabla implies the associated expression in material event space.

The stresses may be represented as the sum of non-conservative and conservative parts. If the deformation is adiabatic or isothermal, then the process is conservative and $\overset{\wedge}{\sigma}^{\gamma\epsilon} d\epsilon_{\gamma\epsilon}$ is a perfect differential (Washizu, 1968). This assures the existence of a strain energy density function \hat{A} . The second stress integral (1.8-23) may thus be written as

$$\begin{aligned} \int_{\bar{T}} \int_{\bar{M}(t)} * \left(\underline{\underline{\Sigma}} : \nabla \delta \underline{\underline{P}} \right) &= \delta \int_{\bar{T}} \int_{\bar{S}} * (\hat{A}) + \delta \int_{\bar{T}} \int_{\bar{S}} * \left[\overset{\wedge}{\underline{\underline{\Sigma}}} : (\bar{\nabla} \underline{\underline{P}} - \underline{\underline{E}}) \right] \\ &+ \int_{\bar{T}} \int_{\bar{S}} * \left(\overset{N}{\underline{\underline{\Sigma}}} : \delta \underline{\underline{E}} \right), \end{aligned} \quad (1.8-24)$$

where the prescript "N" denotes the non-conservative part of the stress.

In summary, Newton's laws have as a consequence (1.8-1), which may be rewritten, inserting Equations (1.8-5, -8, -10 and -24), as

$$\begin{aligned}
& - \delta \int_{\bar{T}} \int_{\bar{S}} * (\hat{T}) + \delta \int_{\bar{T}} \int_{\bar{S}} * (\hat{W}) + \delta \int_{\bar{T}} \int_{\bar{S}} * (\hat{A}) \\
& + \delta \int_{\bar{T}} \int_{\bar{S}} * \left[\hat{\rho} \underline{v} \cdot \left(\underline{v} - \frac{\partial}{\partial t} \underline{p} \right) \right] + \delta \int_{\bar{T}} \int_{\bar{S}} * \left[\hat{\underline{\Sigma}} : \left(\bar{\underline{v}} \underline{p} - \underline{\underline{E}} \right) \right] \\
& - \int_{\bar{T}} \int_{\partial M(t)_P} * \left\{ \delta \left[\underline{s} \cdot \left(\underline{p} - \underline{p}_P \right) \right] \right\} + \int_{\partial \bar{T}} \int_{\bar{S}} * (\hat{\rho} \underline{v} \cdot \delta \underline{p}) \\
& - \int_{\bar{T}} \int_{\partial M(t)_S} * (\underline{p}_S \cdot \delta \underline{p}) - \int_{\bar{T}} \int_{\partial M(t)_P} * (\underline{s} \cdot \delta \underline{p}_P) \\
& - \int_{\bar{T}} \int_{\bar{S}} * (\underline{N}^{\hat{X}} \cdot \delta \underline{p}) + \int_{\bar{T}} \int_{\bar{S}} * (\underline{N}^{\hat{\underline{\Sigma}}} : \delta \underline{\underline{E}}) = 0. \tag{1.8-25}
\end{aligned}$$

In the absence of inertial terms and non-linearities, and assuming the prescribed surface forces and displacements are constant, this becomes

$$\begin{aligned}
& \delta \int_{\bar{T}} \left\{ \int_{\bar{S}} * (\hat{A}) + \int_{\bar{S}} * (\hat{W}) + \int_{\bar{S}} * \left[\hat{\underline{\Sigma}} : \left(\bar{\underline{v}} \underline{p} - \underline{\underline{E}} \right) \right] \right\} \\
& + \int_{\bar{T}} \left\{ - \int_{\partial M(t)_P} * \delta \left[\underline{s} \cdot \left(\underline{p} - \underline{p}_P \right) \right] - \int_{\partial M(t)_S} * \delta \left(\underline{p}_S \cdot \underline{p} \right) \right\} = 0, \tag{1.8-26}
\end{aligned}$$

which may be recognized as the Hu-Washizu variational form.

1.9 Virtual Work for an Arbitrary Reference Frame

The expression of the principle of virtual work in Equation (1.8-25) was based on Newton's laws in an inertial reference frame (1.6-1). The result may be transformed to an arbitrary reference frame via the results of Section 1.5 (Equations (1.5-32, -46 and -54)). It may be noted that since the strain is defined in terms of bilinear forms, and since bilinear forms are independent of reference frames, the strains do not change. Proceeding term by term,

$$\begin{aligned}
-\int_{\underline{T}} \int_{\underline{S}} * \langle \hat{\underline{T}} \rangle &= -\int_{\underline{T}} \int_{\underline{S}} * \left(\frac{1}{2} \hat{\rho} \underline{V} \cdot \underline{V} \right) \\
&= -\int_{\underline{T}} \int_{\underline{S}} * \left[\frac{1}{2} \hat{\rho} \left(\frac{\partial}{\partial t} \underline{R} + \underline{V}' + \underline{\Omega}' \times \underline{P}' \right) \cdot \left(\frac{\partial}{\partial t} \underline{R} + \underline{V}' + \underline{\Omega}' \times \underline{P}' \right) \right] \\
&= -\int_{\underline{T}} \int_{\underline{S}} * \left\{ \frac{1}{2} \hat{\rho} \left(\frac{\partial}{\partial t} \underline{R} \cdot \frac{\partial}{\partial t} \underline{R} + \underline{V}' \cdot \underline{V}' + [\underline{\Omega}' \times \underline{P}'] \cdot [\underline{\Omega}' \times \underline{P}'] \right) \right. \\
&\quad \left. + \hat{\rho} \left(\frac{\partial}{\partial t} \underline{R} \cdot \underline{V}' + \left[\frac{\partial}{\partial t} \underline{R} \right] \cdot [\underline{\Omega}' \times \underline{P}'] + [\underline{V}'] \cdot [\underline{\Omega}' \times \underline{P}'] \right) \right\}. \quad (1.9-1)
\end{aligned}$$

Noting the identities in Appendix A.13, this may be written

$$\begin{aligned}
-\int_{\underline{T}} \int_{\underline{S}} * \langle \hat{\underline{T}} \rangle &= -\int_{\underline{T}} \left[\frac{1}{2} \frac{\partial}{\partial t} \underline{R} \cdot \frac{\partial}{\partial t} \underline{R} \int_{\underline{S}} * \langle \hat{\rho} \rangle + \frac{1}{2} \int_{\underline{S}} * \langle \hat{\rho} \underline{V}' \cdot \underline{V}' \rangle \right. \\
&\quad \left. + \frac{1}{2} \underline{\Omega}' \cdot \int_{\underline{S}} * \langle \hat{\rho} [(\underline{P}' \cdot \underline{P}') \underline{U} - \underline{P}' \underline{P}'] \rangle \cdot \underline{\Omega}' \right. \\
&\quad \left. + \frac{\partial}{\partial t} \underline{R} \cdot \int_{\underline{S}} * \langle \hat{\rho} \underline{V}' \rangle + \frac{\partial}{\partial t} \underline{R} \times \underline{\Omega}' \cdot \int_{\underline{S}} * \langle \hat{\rho} \underline{P}' \rangle + \underline{\Omega}' \cdot \int_{\underline{S}} * \langle \hat{\rho} \underline{P}' \times \underline{V}' \rangle \right], \quad (1.9-2)
\end{aligned}$$

where \underline{U} is an identity tensor (i.e., $\underline{U} \cdot \underline{x} = \underline{x}$). The potential energy becomes

$$\begin{aligned}
\int_{\underline{T}} \int_{\underline{S}} * [\hat{\underline{W}}(\underline{P})] &= \int_{\underline{T}} \int_{\underline{S}} * [\hat{\underline{W}}(\underline{R} + \underline{P}')] \\
&= \int_{\underline{T}} \int_{\underline{S}} * [\hat{\underline{W}}'(\underline{R}, \underline{P}')]. \quad (1.9-3)
\end{aligned}$$

The strain energy, as previously noted is independent of frame. The constraint between velocity and position becomes

$$\begin{aligned}
& \int_{\bar{T}} \int_{\bar{S}} * \left[\hat{\rho} \underline{v} \cdot \left(\underline{v} - \frac{\partial}{\partial t} \underline{p} \right) \right] \\
&= \int_{\bar{T}} \int_{\bar{S}} * \left[\hat{\rho} \left(\frac{\partial}{\partial t} \underline{R} + \underline{v}' + \underline{\Omega}' \times \underline{p}' \right) \cdot \left(\frac{\partial}{\partial t} \underline{R} + \underline{v}' + \underline{\Omega}' \times \underline{p}' \right. \right. \\
&\quad \left. \left. - \frac{\partial}{\partial t} \underline{R} - \frac{\partial'}{\partial t} \underline{p}' - \underline{\Omega}' \times \underline{p}' \right) \right] \\
&= \int_{\bar{T}} \left\{ \int_{\bar{S}} * (\hat{\rho} \underline{v}' \cdot \underline{v}') + \frac{\partial}{\partial t} \underline{R} \cdot \int_{\bar{S}} * (\hat{\rho} \underline{v}') + \underline{\Omega}' \cdot \int_{\bar{S}} * (\hat{\rho} \underline{p}' \times \underline{v}') \right. \\
&\quad \left. - \frac{\partial}{\partial t} \underline{R} \cdot \frac{\partial'}{\partial t} \int_{\bar{S}} * (\hat{\rho} \underline{p}') - \int_{\bar{S}} * (\hat{\rho} \underline{v}' \cdot \frac{\partial'}{\partial t} \underline{p}') - \underline{\Omega}' \cdot \int_{\bar{S}} * (\hat{\rho} \underline{p}' \times \frac{\partial'}{\partial t} \underline{p}') \right\}.
\end{aligned} \tag{1.9-4}$$

The next integral becomes

$$\int_{\bar{T}} \int_{\bar{S}} * \left[\underline{\Sigma} : (\underline{\nabla} \underline{p} - \underline{E}) \right] = \int_{\bar{T}} \int_{\bar{S}} * \left[\underline{\Sigma} : (\underline{\nabla} \underline{p}' - \underline{E}) \right], \tag{1.9-5}$$

since \underline{R} is constant spatially, and since the stress and strain are independent of reference frame.

The integral on the temporal boundary becomes

$$\begin{aligned}
|_{\partial \bar{T}} \int_{\bar{S}} * (\hat{\rho} \underline{v} \cdot \delta \underline{p}) &= |_{\partial \bar{T}} \int_{\bar{S}} * \left[\hat{\rho} \left(\frac{\partial}{\partial t} \underline{R} + \underline{v}' + \underline{\Omega}' \times \underline{p}' \right) \cdot (\delta \underline{R} + \delta \underline{p}') \right] \\
&= |_{\partial \bar{T}} \left\{ \frac{\partial}{\partial t} \underline{R} \cdot \delta \underline{R} \int_{\bar{S}} * (\hat{\rho}) + \underline{\Omega}' \cdot \int_{\bar{S}} * (\hat{\rho} \underline{p}' \times \delta \underline{p}') \right. \\
&\quad + \frac{\partial}{\partial t} \underline{R} \cdot \int_{\bar{S}} * (\hat{\rho} \delta \underline{p}') - \underline{\Omega}' \times \delta \underline{R} \cdot \int_{\bar{S}} * (\hat{\rho} \underline{p}') \\
&\quad \left. + \delta \underline{R} \cdot \int_{\bar{S}} * (\hat{\rho} \underline{v}') + \int_{\bar{S}} * (\hat{\rho} \underline{v}' \cdot \delta \underline{p}') \right\}.
\end{aligned} \tag{1.9-6}$$

The prescribed positions on the spatial boundary become

$$\begin{aligned}
-\int_{\bar{T}} \int_{\partial M(t)_P} * \left[\underline{S} \cdot (\underline{p} - \underline{p}_P) \right] &= -\int_{\bar{T}} \int_{\partial M(t)_P} * \left[\underline{S}' \cdot (\underline{R} + \underline{p}' - \underline{R} - \underline{p}_P') \right] \\
&= -\int_{\bar{T}} \left\{ \int_{\partial M(t)_P} * (\underline{S}' \cdot \underline{p}') \right. \\
&\quad \left. - \int_{\partial M(T)_P} * (\underline{S}' \cdot \underline{p}_P') \right\}.
\end{aligned} \tag{1.9-7}$$

The other remains unchanged. The prescribed forces on the spatial boundary become

$$\begin{aligned}
 -\int_{\bar{T}} \int_{\partial M(t)} \underline{S} * \left(\underline{P}_S \cdot \delta \underline{P} \right) &= -\int_{\bar{T}} \int_{\partial M(t)} \underline{S} * \left[\underline{P}_{S'} \cdot (\delta \underline{R} + \delta \underline{P}') \right] \\
 &= -\int_{\bar{T}} \left\{ \delta \underline{R} \cdot \int_{\partial M(t)} \underline{S} * \left(\underline{P}_{S'} \right) + \int_{\partial M(t)} \underline{S} * \left(\underline{P}_{S'} \cdot \delta \underline{P}' \right) \right\} .
 \end{aligned} \tag{1.9-8}$$

The non-conservative body forces become

$$\begin{aligned}
 -\int_{\bar{T}} \int_S * \left(\underline{N}_X^\Lambda \cdot \delta \underline{P} \right) &= -\int_{\bar{T}} \int_S * \left[\underline{N}_{X'}^\Lambda \cdot (\delta \underline{R} + \delta \underline{P}') \right] \\
 &= -\int_{\bar{T}} \left\{ \delta \underline{R} \cdot \int_S * \left(\underline{N}_{X'}^\Lambda \right) + \int_S * \left(\underline{N}_{X'}^\Lambda \cdot \delta \underline{P}' \right) \right\} .
 \end{aligned} \tag{1.9-9}$$

The non-conservative stress integral remains unchanged.

The dot products of \underline{R} with vectors defined in the primed basis introduces the direction cosines into the variational form. The relationship between the direction cosines and the angular velocity (1.5-39) may be introduced as a constraint

$$\delta \int_{\bar{T}} * \left\{ \underline{H}' \cdot \left[\left(\omega'^\delta + \frac{1}{2} \epsilon_{\gamma\alpha}^\delta c_\beta^\gamma \frac{\partial}{\partial t} c_\beta^\alpha \right) \frac{\partial}{\partial x'^\delta} \right] \right\} . \tag{1.9-10}$$

The Lagrange multiplier, \underline{H}' , may be interpreted as the angular momentum.

Inserting Equations (1.9-2) through Equations (1.9-10), yields for Equation (1.8-25)

$$\begin{aligned}
& \delta \int_{\mathbb{T}} \left\{ -\frac{1}{2} \frac{\partial}{\partial t} \underline{R} \cdot \frac{\partial}{\partial t} \underline{R} \int_{\mathbb{S}} * (\hat{\rho}) - \frac{1}{2} \int_{\mathbb{S}} * (\hat{\rho} \underline{V}' \cdot \underline{V}') \right. \\
& \quad \left. - \frac{1}{2} \underline{\Omega}' \cdot \int_{\mathbb{S}} * \left(\hat{\rho} [(\underline{P}' \cdot \underline{P}') \underline{U} - \underline{P}' \underline{P}' \cdot \underline{I}] \right) \cdot \underline{\Omega}' \right. \\
& \quad \left. - \frac{\partial}{\partial t} \underline{R} \cdot \int_{\mathbb{S}} * (\hat{\rho} \underline{V}') + \underline{\Omega}' \times \frac{\partial}{\partial t} \underline{R} \cdot \int_{\mathbb{S}} * (\hat{\rho} \underline{P}') - \underline{\Omega}' \cdot \int_{\mathbb{S}} * (\hat{\rho} \underline{P}' \times \underline{V}') \right. \\
& \quad \left. + \int_{\mathbb{S}} * (\hat{W}') + \int_{\mathbb{S}} * (\hat{A}) + \frac{\partial}{\partial t} \underline{R} \cdot \int_{\mathbb{S}} * \left[\hat{\rho} \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \right] \right. \\
& \quad \left. + \int_{\mathbb{S}} * \left[\hat{\rho} \underline{V}' \cdot \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \right] + \underline{\Omega}' \cdot \int_{\mathbb{S}} * \left[\hat{\rho} \underline{P}' \times \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \right] \right. \\
& \quad \left. + \int_{\mathbb{S}} * \left[\hat{\underline{\Sigma}} : (\underline{V} \underline{P}' - \underline{E}) \right] + \left[h'_{\delta} \left(\omega'^{\delta} + \frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} c_{\beta}^{\gamma} \frac{\partial}{\partial t} c_{\beta}^{\alpha} \right) \right] \right\} \\
& + \int_{\mathbb{T}} \left\{ - \int_{\partial M(t)_P} * [\underline{S}' \cdot \delta \underline{R}] - \int_{\partial M(t)_P} * [\delta(\underline{S}' \cdot \underline{P}')] + \int_{\partial M(t)_P} * [\delta(\underline{S}' \cdot \underline{P}'_P)] \right\} \\
& + \int_{\partial \mathbb{T}} \left\{ \frac{\partial}{\partial t} \underline{R} \cdot \delta \underline{R} \int_{\mathbb{S}} * (\hat{\rho}) + \frac{\partial}{\partial t} \underline{R} \cdot \int_{\mathbb{S}} (\hat{\rho} \delta \underline{P}') \right. \\
& \quad \left. + \delta \underline{R} \cdot \int_{\mathbb{S}} * (\hat{\rho} \underline{V}') + \int_{\mathbb{S}} * (\hat{\rho} \underline{V}' \cdot \delta \underline{P}') - \underline{\Omega}' \cdot \left[\delta \underline{R} \times \int_{\mathbb{S}} * (\hat{\rho} \underline{P}') \right. \right. \\
& \quad \left. \left. - \int_{\mathbb{S}} * (\hat{\rho} \underline{P}' \times \delta \underline{P}') \right] \right\} \\
& + \int_{\mathbb{T}} \left\{ - \delta \underline{R} \cdot \int_{\partial M(t)_S} * (\underline{P}'_S) - \int_{\partial M(t)_S} * (\underline{P}'_S \cdot \delta \underline{P}') - \int_{\partial M(t)_P} * (\underline{S}' \cdot \delta \underline{P}'_P) \right. \\
& \quad \left. - \delta \underline{R} \cdot \int_{\mathbb{S}} * (\underline{N}'_{\underline{X}}) - \int_{\mathbb{S}} * (\underline{N}'_{\underline{X}} \cdot \delta \underline{P}') + \int_{\mathbb{S}} * (\underline{N}'_{\underline{\Sigma}} : \delta \underline{E}) \right\} = 0. \quad (1.9-11)
\end{aligned}$$

SECTION 2

SUBSTRUCTURES

A substructure will be defined as a finite collection of material continua satisfying certain properties. The continua must all coexist in the same region of time. They must have no interior points in common, however, each must have part of its boundary in common with another. The shared boundaries of the continua will be called the internal boundaries of the substructure, and any parts of the boundaries of the continua which are not shared will be called external boundaries of the substructure. The union of the closures of all the continua must be connected, and the boundary of the union is the union of all of the external boundaries. The position and stress are generally required to be continuous at all internal boundaries.

The final requirement is that the positions deviate from a reference spatial state by a small amount for some reference frame. This last requirement forms the basis for an approximate solution, which is the subject of this section.

2.1 Finite Elements

The variational expression (1.9-11) has as known functions of the coordinates and the other variables the potential energy, \hat{W} , the strain energy, \hat{A} , the non-conservative body forces and stresses, $\frac{N\hat{A}}{X}$ and $N \frac{\hat{\Sigma}}{\underline{\Sigma}}$, and on the external boundaries, the prescribed positions, and surface forces, \underline{P}_P and \underline{P}_S . The prescribed positions and surface forces on the internal boundaries, \underline{P}_P and \underline{P}_S , the velocity, \underline{V} , the position, \underline{P} , the stress, $\hat{\Sigma}$, the strain, \underline{E} , and the surface force, \underline{S} are all unknown functions of the coordinates.

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Direct solution of the governing equations is seldom possible. Instead, the unknowns are represented as a series, and the least significant terms are truncated. The domain of the expansion may be the whole substructure, the material continuum or a mixed form involving both. Moreover, the domains for the various fields need not be the same. When the domain of the expansion is the material continuum, it is called a finite element. Since use of expansions for the other domains is rare, only finite elements will be treated.

It may be noted that the series expansion could be in the temporal as well as spatial coordinates. In fact this has the advantage of reducing the partial differential equations to algebraic equations, as opposed to the ordinary differential equations obtained by expanding only in the spatial coordinates. However, the temporal approach will not be pursued here, and the conventional spatial expansion will be used.

The development of a finite element seems, at present, to be as much art as science. The developer may arbitrarily select the geometry of the elements, the fields to be approximated, the approximations to be used, and the variational form to be applied. Aside from certain mathematical and practical considerations, the worth of an element is determined by the accuracy it provides for a given number of degrees of freedom, and by its convenience in use. Although the development of finite elements lies outside the scope of this presentation, some of the more pertinent aspects of the method will be included.

Geometrically regular material continua are generally selected because of their analytical convenience, and the ease with which they are assembled. Parametric elements are a notable exception, where an irregular spatial region is mapped into a regular region in a parametric space.

The choice of which fields to approximate can be divided into three categories: first assumed compatible displacement fields, second assumed equilibrium stress fields, and third mixed partial assumptions on the stresses and displacements. There are further choices within each category. For instance, with a displacement assumed model, the strain could either be separately assumed, or determined from the strain displacement relations. In fact, the displacement fields could even be different depending on application (e.g., one for strains, another for the potential energy associated with body forces).

The fact that any complete series expansion provides an exact solution to the equations, makes them desirable choices. (However, any series expansion may be acceptable.) Polynomials are frequently used because of their convenience and since P_∞ (the set of all polynomials to order ∞) is a complete function space for the class of continuous functions (Weierstrass theorem). Trigonometric expansions are also used, especially for the circumferential direction in axisymmetric elements.

The variational form used may be some variant of (1.9-11), which is essentially of the Hu-Washizu form, or some variant of the dual form obtained by a Legendre transformation of the strain energy,

yielding a complementary energy formulation. Generally, finite elements are developed ignoring the kinetic energy terms, considering only the strain energy and sometimes the potential energy. When the finite element developers do include kinetic energy, it is usually in the simpler form of Equation (1.8-25) rather than Equation (1.9-11), with the velocity field explicitly assumed to be the time derivative of the position field. In this special case, the associated terms are usually referred to as "consistent mass."

To be of practical value, the finite element field approximations must converge to the correct values for the field as the element size decreases. A finite element generally must satisfy the following criteria to be convergent. First, the approximation should be capable of representing a constant value of the field or its derivatives up to the highest order appearing in the functional. Secondly, the approximation should be continuous to one order less than the highest derivative appearing in the function. Generally this is met by requiring that the highest order derivative be continuous almost everywhere (i.e., except on a set of measure zero, e.g., piecewise continuous).

The finite elements are assembled by applying the variational form to the whole substructure. The conditions at the interfaces are just the previously mentioned continuity requirements. If these are met, the internal boundary integrals cancel. However, the requirement may be relaxed, if the discontinuity is at the boundary, by explicitly retaining the internal boundary integrals. In practice, when using an assumed displacement field, the satisfaction of compatibility requirements is usually made automatic by transforming from coefficients of

an approximate expansion to the displacements at nodes on the boundary. If the displacement on a boundary surface depends only on the displacements of the nodes defining the surface, compatibility is established by requiring the displacements of those nodes be identical for adjacent elements. This is the basis of the "direct stiffness" method.

As an illustrative example, the derivation of a beam finite element based on an assumed displacement field is presented in Appendix C.1. The derivation includes the effects of geometric nonlinearities.

2.2 Finite Element Dynamics

In this section, and in the material that follows, it will be assumed that the assumed displacement finite element method is being used and the assumed displacement field is known. Alternatively, a mixed or stress assumed formulation may be used, provided that the displacement field is known, or can be obtained in closed form. To incorporate dynamics, the variational expression in Equation (1.9-11) must be used instead of Equation (1.8-25). Taking the indicated variations in Equation (1.9-11) and noting the definitions of variations of vectors in Appendix (A.14),

$$\begin{aligned}
& \int_{\bar{T}} \int_{\bar{S}} * \left\{ - \hat{\rho} \frac{\partial}{\partial t} \underline{R} \cdot \delta \frac{\partial}{\partial t} \underline{R} - \hat{\rho} \underline{V}' \cdot \delta' \underline{V}' - \hat{\rho} \underline{\Omega}' \cdot [(\underline{P}' \cdot \underline{P}') \underline{U} - \underline{P}' \underline{P}'] \cdot \delta' \underline{\Omega}' \right. \\
& - \hat{\rho} \underline{\Omega}' \cdot [(\underline{P}' \cdot \delta' \underline{P}') \underline{U} - \underline{P}' \delta' \underline{P}'] \cdot \underline{\Omega}' - \hat{\rho} \underline{V}' \cdot \delta \frac{\partial}{\partial t} \underline{R} - \hat{\rho} \frac{\partial}{\partial t} \underline{R} \cdot \delta' \underline{V}' \\
& - \hat{\rho} \frac{\partial}{\partial t} r^\alpha_{\underline{V}'} \delta c^\beta_\alpha + \hat{\rho} \frac{\partial}{\partial t} \underline{R} \times \underline{P}' \cdot \delta' \underline{\Omega}' - \hat{\rho} \underline{\Omega}' \times \underline{P}' \cdot \delta \frac{\partial}{\partial t} \underline{R} \\
& + \hat{\rho} \underline{\Omega}' \times \frac{\partial}{\partial t} \underline{R} \cdot \delta' \underline{P}' - \hat{\rho} \frac{\partial}{\partial t} r^\alpha_{(\underline{\Omega}' \times \underline{P}')} \delta c^\beta_\alpha - \hat{\rho} \underline{P}' \times \underline{V}' \cdot \delta' \underline{\Omega}' \\
& + \hat{\rho} \underline{\Omega}' \times \underline{V}' \cdot \delta' \underline{P}' - \hat{\rho} \underline{\Omega}' \times \underline{P}' \cdot \delta' \underline{V}' + \underline{\nabla} \hat{W}' \cdot \delta \underline{R} + \underline{\nabla} \hat{W}' \cdot \delta' \underline{P}' \\
& + \underline{\nabla} \hat{W}' \cdot \left(p' \gamma c^\beta_\alpha \delta c^\gamma_\alpha \frac{\partial}{\partial x' \beta} \right) + \left(a^{\alpha\beta} + a^{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta} \right) \delta \epsilon^{\alpha\beta} \\
& + \hat{\rho} \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \cdot \delta \frac{\partial}{\partial t} \underline{R} + \hat{\rho} \frac{\partial}{\partial t} \underline{R} \cdot \left(\delta' \underline{V}' - \delta' \frac{\partial'}{\partial t} \underline{P}' \right) \\
& \quad + \hat{\rho} \frac{\partial}{\partial t} r^\alpha_{(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}')} \delta c^\beta_\alpha \\
& + \hat{\rho} \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \cdot \delta' \underline{V}' + \hat{\rho} \underline{V}' \cdot \left(\delta' \underline{V}' - \delta' \frac{\partial'}{\partial t} \underline{P}' \right) + \hat{\rho} \underline{P}' \times \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \cdot \delta' \underline{\Omega}' \\
& - \hat{\rho} \underline{\Omega}' \times \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \cdot \delta' \underline{P}' + \hat{\rho} \underline{\Omega}' \times \underline{P}' \cdot \left(\delta' \underline{V}' - \delta' \frac{\partial'}{\partial t} \underline{P}' \right) + (\underline{\nabla} \underline{P}' - \underline{E}) : \delta \underline{\underline{\Sigma}} \\
& + \underline{\underline{\Sigma}} : (\delta \underline{\nabla} \underline{P}' - \delta \underline{E}) \left. \right\} + \int_{\bar{T}} * \left\{ \left(\omega' \delta + \frac{1}{2} \epsilon_{\gamma\alpha}^\delta c^\gamma_\beta \frac{\partial}{\partial t} c^\alpha_\beta \right) \delta' h'_\delta + h'_\delta \delta' \omega' \delta \right. \\
& + h'_\delta \frac{1}{2} \epsilon_{\gamma\alpha}^\delta \left(\frac{\partial}{\partial t} c^\alpha_\beta \delta c^\gamma_\beta + c^\gamma_\beta \delta \frac{\partial}{\partial t} c^\alpha_\beta \right) \left. \right\} + \int_{\bar{T}} \int_{\bar{S}} * \left\{ - \frac{N \hat{X}}{\underline{X}} \cdot \delta \underline{R} - \frac{N \hat{X}}{\underline{X}} \cdot \delta' \underline{P}' \right. \\
& \left. - \frac{N \hat{X}}{\underline{X}} \cdot \left(p' \gamma c^\beta_\alpha \delta c^\gamma_\alpha \frac{\partial}{\partial x' \beta} \right) + N \hat{\underline{\Sigma}} : \delta \underline{E} \right\} \\
& + \int_{\bar{T}} \int_{\partial M(t)_P} * \left\{ - \underline{S}' \cdot \delta \underline{R} - \underline{P}' \cdot \delta' \underline{S}' - \underline{S}' \cdot \delta' \underline{P}' + \underline{P}' \cdot \delta' \underline{S}' - \underline{S}' \cdot \left(p' \gamma c^\beta_\alpha \delta c^\gamma_\alpha \frac{\partial}{\partial x' \beta} \right) \right\} \\
& + \int_{\bar{T}} \int_{\partial M(t)_S} * \left\{ - \underline{P}' \cdot \delta \underline{R} - \underline{S}' \cdot \delta' \underline{P}' - \underline{P}' \cdot \left(p' \gamma c^\beta_\alpha \delta c^\gamma_\alpha \frac{\partial}{\partial x' \beta} \right) \right\} \\
& + \int_{\partial \bar{T}} \int_{\bar{S}} * \left\{ \hat{\rho} \frac{\partial}{\partial t} \underline{R} \cdot \delta \underline{R} + \hat{\rho} \frac{\partial}{\partial t} \underline{R} \cdot \delta' \underline{P}' + \hat{\rho} \frac{\partial}{\partial t} \underline{R} \cdot \left(p' \gamma c^\beta_\alpha \delta c^\gamma_\alpha \frac{\partial}{\partial x' \beta} \right) + \hat{\rho} \underline{V}' \cdot \delta \underline{R} \right. \\
& + \hat{\rho} \underline{V}' \cdot \delta' \underline{P}' + \hat{\rho} \underline{V}' \cdot \left(p' \gamma c^\beta_\alpha \delta c^\gamma_\alpha \frac{\partial}{\partial x' \beta} \right) + \hat{\rho} \underline{\Omega}' \times \underline{P}' \cdot \delta \underline{R} + \hat{\rho} \underline{\Omega}' \times \underline{P}' \cdot \delta' \underline{P}' \\
& \left. + \hat{\rho} \underline{\Omega}' \times \underline{P}' \cdot \left(p' \gamma c^\beta_\alpha \delta c^\gamma_\alpha \frac{\partial}{\partial x' \beta} \right) \right\} \quad (2.2-1)
\end{aligned}$$

Noting the derivative relationships,

$$\underline{A} \cdot \delta \frac{\partial}{\partial t} \underline{R} = \frac{\partial}{\partial t} (\underline{A} \cdot \delta \underline{R}) - \frac{\partial}{\partial t} \underline{A} \cdot \delta \underline{R},$$

$$\underline{A}' \cdot \delta' \frac{\partial'}{\partial t} \underline{P}' = \frac{\partial'}{\partial t} (\underline{A}' \cdot \delta \underline{P}') - \frac{\partial'}{\partial t} \underline{A}' \cdot \delta \underline{P}' \quad (2.2-2)$$

and integrating by parts, yields

$$\begin{aligned} & \int_{\bar{T}} \left(\int_{\bar{S}} * \left\{ \frac{\partial}{\partial t} (\hat{\rho} \frac{\partial}{\partial t} \underline{R}) + \frac{\partial}{\partial t} (\hat{\rho} \underline{V}') + \frac{\partial}{\partial t} (\hat{\rho} \underline{\Omega}' \times \underline{P}') + \underline{\nabla} \hat{W}' - \frac{\partial}{\partial t} \left[\hat{\rho} \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \right] \right. \right. \\ & \quad \left. \left. - \frac{N \hat{X}'}{\underline{X}'} \right\} + \int_{\partial M(t)} * \left\{ - \underline{S}' \right\} + \int_{\partial M(t)} * \left\{ - \underline{P}' \underline{S}' \right\} \right) \cdot \delta \underline{R} \\ & + \int_{\partial \bar{T}} \left(\int_{\bar{S}} * \left\{ - \hat{\rho} \frac{\partial}{\partial t} \underline{R} - \hat{\rho} \underline{V}' - \hat{\rho} \underline{\Omega}' \times \underline{P}' + \hat{\rho} \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) + \hat{\rho} \frac{\partial}{\partial t} \underline{R} \right. \right. \\ & \quad \left. \left. + \hat{\rho} \underline{V}' + \hat{\rho} \underline{\Omega}' \times \underline{P}' \right\} \cdot \delta \underline{R} \right. \\ & + \int_{\bar{T}} \int_{\bar{S}} * \left\{ \left(- \hat{\rho} \underline{V}' - \hat{\rho} \frac{\partial}{\partial t} \underline{R} - \hat{\rho} \underline{\Omega}' \times \underline{P}' + \hat{\rho} \frac{\partial}{\partial t} \underline{R} + \hat{\rho} \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \right. \right. \\ & \quad \left. \left. + \hat{\rho} \underline{V}' + \hat{\rho} \underline{\Omega}' \times \underline{P}' \right) \cdot \delta' \underline{V}' \right\} \\ & + \int_{\bar{T}} \int_{\bar{S}} * \left\{ \left(a^{\alpha\beta} + a^{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta} - \frac{N \alpha\beta}{\sigma} + \frac{N \alpha\beta}{\sigma} \right) \delta \epsilon^{\alpha\beta} \right\} + \int_{\bar{T}} \int_{\bar{S}} * \left\{ (\underline{\nabla} \underline{P}' - \underline{E}) : \delta \hat{\underline{\Sigma}} \right\} \\ & + \int_{\bar{T}} * \left\{ \left(\hat{\omega}^{\delta} + \frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} c_{\beta}^{\gamma} \frac{\partial}{\partial t} c_{\beta}^{\alpha} \right) \delta h^{\delta} \right\} + \int_{\bar{T}} \int_{\partial M(t)} * \left\{ \underline{P}' - \underline{P}' \right\} \cdot \delta' \underline{S}' \\ & + \int_{\bar{T}} \int_{\bar{S}} * \left\{ \left(\hat{\rho} \underline{\Omega}' \times [\underline{\Omega}' \times \underline{P}'] + \hat{\rho} \underline{\Omega}' \times \frac{\partial}{\partial t} \underline{R} + \hat{\rho} \underline{\Omega}' \times \underline{V}' + \underline{\nabla} \hat{W}' + \frac{\partial'}{\partial t} \left[\hat{\rho} \frac{\partial}{\partial t} \underline{R} \right] \right. \right. \\ & \quad \left. \left. + \frac{\partial'}{\partial t} [\hat{\rho} \underline{V}'] - \hat{\rho} \underline{\Omega}' \times \left[\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right] + \frac{\partial'}{\partial t} [\hat{\rho} \underline{\Omega}' \times \underline{P}'] - \frac{N \hat{X}'}{\underline{X}'} \right) \cdot \delta' \underline{P}' \right\} \\ & + \int_{\partial \bar{T}} \int_{\bar{S}} * \left\{ \left(- \hat{\rho} \frac{\partial}{\partial t} \underline{R} - \hat{\rho} \underline{V}' - \hat{\rho} \underline{\Omega}' \times \underline{P}' + \hat{\rho} \frac{\partial}{\partial t} \underline{R} + \hat{\rho} \underline{V}' + \hat{\rho} \underline{\Omega}' \times \underline{P}' \right) \cdot \delta' \underline{P}' \right\} \\ & + \int_{\bar{T}} \int_{\bar{S}} * \left\{ \hat{\underline{\Sigma}} : \delta \underline{\nabla} \underline{P}' \right\} + \int_{\bar{T}} \int_{\partial M(t)} * \left\{ - \underline{S}' \cdot \delta' \underline{P}' \right\} + \int_{\bar{T}} \int_{\partial M(t)} * \left\{ - \underline{P}' \underline{S}' \cdot \delta' \underline{P}' \right\} \\ & + \int_{\bar{T}} \left(\int_{\bar{S}} * \left\{ \left(- \hat{\rho} \underline{\Omega}' \cdot [(\underline{P}' \cdot \underline{P}') \underline{U} - \underline{P}' \underline{P}'] + \hat{\rho} \frac{\partial}{\partial t} \underline{R} \times \underline{P}' - \hat{\rho} \underline{P}' \times \underline{V}' \right. \right. \right. \\ & \quad \left. \left. + \hat{\rho} \underline{P}' \times \left(\underline{V}' - \frac{\partial'}{\partial t} \underline{P}' \right) \right\} + \underline{H}' \right) \cdot \delta' \underline{\Omega}' \\ & + \int_{\bar{T}} \left(\int_{\bar{S}} * \left\{ - \hat{\rho} \frac{\partial}{\partial t} r^{\alpha} v^{\beta} - \hat{\rho} \frac{\partial}{\partial t} r^{\alpha} [\underline{\Omega}' \times \underline{P}']^{\beta} + [\underline{\nabla} \hat{W}']^{\gamma} P'^{\beta} c_{\alpha}^{\gamma} \right. \right. \end{aligned}$$

Equation (2.2-3) Continued

$$\begin{aligned}
& + \rho \frac{\partial}{\partial t} r^\alpha \left[\underline{v}' - \frac{\partial'}{\partial t} \underline{p}' \right]^\beta - \frac{N \Lambda^\gamma}{X^\gamma} p'^\beta c_\alpha^\gamma \\
& - h'_\delta \frac{1}{2} \epsilon_{\gamma\beta}^\delta \frac{\partial}{\partial t} c_\alpha^\gamma - \frac{\partial}{\partial t} \left[h'_\delta \frac{1}{2} \epsilon_{\gamma\beta}^\delta c_\alpha^\gamma \right] \delta c_\alpha^\beta \\
& + \int_{\partial M(t)_P} * \left\{ - s'^\gamma p'^\beta c_\alpha^\gamma \right\} + \int_{\partial M(t)_S} * \left\{ - p'^\gamma p'^\beta c_\alpha^\gamma \right\} \delta c_\alpha^\beta \\
& + \int_{\partial T} \left(h'_\delta \frac{1}{2} \epsilon_{\gamma\beta}^\delta c_\alpha^\gamma + \int_S * \left\{ \rho \frac{\partial}{\partial t} r^\alpha p'^\beta + \rho v'^\gamma p'^\beta c_\alpha^\gamma \right. \right. \\
& \quad \left. \left. + \rho (\underline{\Omega}' \times \underline{p}')^\gamma p'^\beta c_\alpha^\gamma \right\} \right) \delta c_\alpha^\beta. \tag{2.2-3}
\end{aligned}$$

The stationary conditions of this form provide, in essence, the Lagrange equations for the element. It has been assumed that all vectors except \underline{R} are expressed in the primed (body fixed) vector basis, and that \underline{R} is expressed in the unprimed (inertially fixed) vector basis.

If the displacement and velocity fields are assumed to have the form

$$\begin{aligned}
\underline{p}' &= \bar{x} \frac{\partial}{\partial x'} + \bar{y} \frac{\partial}{\partial y'} + \bar{z} \frac{\partial}{\partial z'} + \sum_{i=1}^M i \underline{p}'_i q \quad \text{and} \\
\underline{v}' &= \sum_{j=1}^N j \underline{v}'_j p^j, \tag{2.2-4}
\end{aligned}$$

some of the stationary conditions may be immediately identified. The coefficient of $\delta \underline{R}$ must vanish over the time domain, \bar{T} ,

$$\begin{aligned}
& M \frac{\partial^2}{\partial t^2} \underline{R} + M \frac{\partial'}{\partial t} \underline{\Omega}' \times (\underline{0}' + i \underline{p}'_i q) + M \underline{\Omega}' \times i \underline{p}'_i \dot{q} \\
& + M \underline{\Omega}' \times [\underline{\Omega}' \times (\underline{0}' + i \underline{p}'_i q)] + M i \underline{p}'_i \ddot{q} + M \underline{\Omega}' \times i \underline{p}'_i \dot{q} \\
& = \int_S * \left\{ - \underline{v} \hat{w} + \frac{N \Lambda^\gamma}{X^\gamma} \right\} + \int_{\partial M(t)_P} * [\underline{S}'] + \int_{\partial M(t)_S} * \left\{ \underline{P} \underline{S}' \right\}, \tag{2.2-5}
\end{aligned}$$

where the repeated Latin subscript implies summation over the M q's,
and

$$M = \int_S * \{\hat{\rho}\}, \quad \bar{P}' = \int_S * \left\{ \hat{\rho} \left(\bar{x} \frac{\partial}{\partial x'} + \bar{y} \frac{\partial}{\partial y'} + \bar{z} \frac{\partial}{\partial z'} \right) \right\} / M \quad \text{and}$$

$$\bar{P}'_i = \int_S * \{\hat{\rho}_{iP'}\} / M .$$

In addition, the coefficient of δR must vanish on the time boundary,
 $\partial \bar{T}$,

$$\sum_{j=1}^N j \bar{V}'_j P' = \bar{P}'_i \dot{q} \quad (2.2-6)$$

where

$$j \bar{V}'_j = \int_S * \{\hat{\rho}_{jV'}\} / M .$$

The coefficients of the independent variations in $\delta' V'$ (i.e., the δ_{jp}) must vanish for all time,

$$\sum_{i=1}^N i j^V i^P = k_j^G k^{\dot{q}} \quad (2.2-7)$$

where

$$i j^V = \int_S * \{\hat{\rho}_{iV'} \cdot j^V\}, \quad i j^G = \int_S * \{\hat{\rho}_{iP'} \cdot j^V\} .$$

The coefficients of $\delta \varepsilon_{\alpha\beta}$, $\delta \hat{\underline{S}}'$, $\delta' \underline{S}'$, and the terms $\underline{\nabla W}' \cdot \delta' \underline{P}'$, $\underline{X}'^N \delta' \underline{P}'$, $\hat{\underline{S}}' : \delta \bar{\underline{V}} \underline{P}'$, $\underline{S}' \cdot \delta' \underline{P}'$, and $\underline{P}'_{\underline{S}'} \cdot \delta' \underline{P}'$ are the terms associated with the elasticity problem. It will now be assumed that the elasticity problem has already been solved by the methods noted in Section 2.1.

Moreover, it will be assumed that the results are of the form

$$i^Q = i^{\sigma Q} + i j^K j^q + i j^C j^{\dot{q}} , \quad (2.2-8)$$

where the i^Q are the generalized forces associated with the surface and body forces, and the j^q are the generalized (generally nodal)

displacements with any constrained additional coordinates eliminated. The coefficients are presumed to include the effects of preload (geometric non-linearity), and any other effects, linearized for small j^q , so that they are constant or at most known functions of time.

If additional displacement degrees of freedom were eliminated during the solution of the elastic problem, those constraints are modified by dynamics and the static coefficients are improper for the dynamics problem. Properly, those extra coordinates should be retained in the problem; however, the problem will be disregarded here by selecting the field resulting from eliminating the constraints as the initially assumed field. In this case, the previously noted terms can be replaced

$$\begin{aligned}
& \int_{\underline{T}} \int_{\underline{S}} * \left\{ \left(a^{\alpha\beta} + a^{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta} - \overset{\wedge}{\sigma}^{\alpha\beta} + \overset{N\wedge}{\sigma}^{\alpha\beta} \right) \delta \epsilon_{\alpha\beta} + (\underline{\bar{V}} \underline{P}' - \underline{E}) : \delta \underline{\overset{\wedge}{\Sigma}} + \underline{\overset{\wedge}{\Sigma}} : \delta \underline{\bar{V}} \underline{P}' \right. \\
& \quad \left. + \left(\underline{\bar{V}} \underline{W}' - \underline{N} \underline{X}' \right) \cdot \delta \underline{P}' \right\} + \int_{\underline{T}} \int_{\underline{M}(t)} * \left\{ \left(\underline{P} \underline{P}' - \underline{P}' \right) \cdot \delta \underline{S}' - \underline{S}' \cdot \delta \underline{P}' \right\} \\
& \quad + \int_{\underline{T}} \int_{\underline{M}(t)} * \left\{ -\underline{P} \underline{S}' \cdot \delta \underline{P}' \right\} \\
& = \left[\overset{\sigma}{i} \overset{Q}{Q} + \overset{K}{ij} \overset{K}{j} \overset{q}{q} + \overset{C}{ij} \overset{C}{j} \overset{\dot{q}}{\dot{q}} - \overset{i}{i} \overset{Q}{Q} \right] \delta \overset{i}{i} \overset{q}{q} . \tag{2.2-9}
\end{aligned}$$

Returning now to the stationary conditions for Equation (2.2-3), the coefficients of the independent variations in $\delta \underline{P}'$ (i.e., the δ_j^q) must vanish for all time. In the simplest dynamic case, $\underline{\Omega}'$ and \underline{R} both zero, this gives

$$\overset{i}{i} \overset{Q}{Q} = \overset{\sigma}{i} \overset{Q}{Q} + \overset{K}{ij} \overset{K}{j} \overset{q}{q} + \overset{C}{ij} \overset{C}{j} \overset{\dot{q}}{\dot{q}} + \overset{G}{ik} \overset{G}{k} \overset{\dot{p}}{\dot{p}} . \tag{2.2-10}$$

Solving Equation (2.2-7), assuming $\overset{V}{ij}$ is non-singular, or taking a pseudo-inverse if it is, the stationary conditions are

$$i_j^M \ddot{q} + i_j^C \dot{q} + i_j^K q = i^Q - i^{\sigma} Q, \quad (2.2-11)$$

where

$$i_j^M = \sum_{k=1}^N \sum_{\ell=1}^N i_k^G k_{\ell} V^{-1} j_{\ell}^G.$$

This result is the usual dynamic model. If $j_{\underline{V}'} = j_{\underline{P}'}$, then the mass is called consistent. If a piecewise continuous linear field is used for the velocity field, the result is nodal masses. However, just the assumption of this simplified field does not eliminate coupling between the nodes. To eliminate coupling the constraints must be violated. This procedure is justified if the resultant energy error vanishes as the element size tends to zero. It may be noted that the assumed velocity field appears nowhere else. Thus for what follows, the question of proper velocity field is immaterial and will not be further treated. Instead, it will be assumed that the array i_j^M is available.

The mass terms associated with the displacement field could conceivably be derived based on a different assumed displacement field from that assumed for the elastic solution, with or without violation of the constraints. This possibility will not be pursued here. Returning to the general case, the stationary conditions of Equation (2.2.-3) provide, instead of Equation (2.2.-1), as coefficients of the independent variations in $\delta' \underline{P}'$ (i.e., the $\delta_i q$)

$$\begin{aligned} & - \underline{\Omega}' \cdot \underline{i}_{\underline{I}} \cdot \underline{\Omega}' - \underline{\Omega}' \cdot j_{\underline{I}} \underline{i} \cdot \underline{\Omega}' j^q + M \underline{i}^{\underline{P}'} \cdot \frac{\partial^2}{\partial t^2} \underline{R} + 2 \underline{\Omega}' \cdot j_{\underline{i}^0} j^{\dot{q}} + \frac{\partial'}{\partial t} \underline{\Omega}' \cdot \underline{i}^0 \\ & + \frac{\partial'}{\partial t} \underline{\Omega}' \cdot j_{\underline{i}^0} j^q + i_j^M \ddot{q} + i_j^C \dot{q} + i_j^K q = i^Q - i^{\sigma} Q, \end{aligned} \quad (2.2-12)$$

$$\begin{aligned}
\underline{i}_I &= \int_S * \{ \hat{\rho} [(\underline{0}^{P'} \cdot \underline{i}^{P'}) \underline{U} - \underline{0}^{P'} \underline{i}^{P'}] \} , \\
\underline{i}_{jI} &= \int_S * \{ \hat{\rho} [(\underline{i}^{P'} \cdot \underline{j}^{P'}) \underline{U} - \underline{i}^{P'} \underline{j}^{P'}] \} , \\
\underline{i}_O &= \int_S * \{ \hat{\rho} \underline{0}^{P'} \times \underline{i}^{P'} \} \quad \text{and} \\
\underline{i}_{jO} &= \int_S * \{ \hat{\rho} \underline{i}^{P'} \times \underline{j}^{P'} \} .
\end{aligned}$$

The remaining terms govern the rotational behavior of the element.

The coefficients of $\delta H'_\delta$ must be zero so

$$\omega'^{\delta} = - \frac{1}{2} \varepsilon_{\gamma\alpha}^{\delta} c_{\beta}^{\gamma} \frac{\partial}{\partial t} c_{\beta}^{\alpha} . \quad (2.2-13)$$

This may be recognized as the nonlinear differential definition of $\underline{\Omega}'$.

The coefficients of $\delta' \underline{\Omega}'$ must also vanish,

$$\begin{aligned}
\underline{H}' &= \underline{\Omega}' \cdot \underline{I} + \underline{\Omega}' \cdot \underline{i}_I \underline{i}^q + \underline{i}_I \cdot \underline{\Omega}' \underline{i}^q + \underline{\Omega}' \cdot \underline{i}_{jI} \underline{i}^q \underline{j}^q \\
&\quad + M \underline{0}^{\bar{P}'} \times \frac{\partial}{\partial t} \underline{R} + M \underline{i}^{\bar{P}'} \times \frac{\partial}{\partial t} \underline{R} \underline{i}^q + \underline{i}_O \underline{i}^{\dot{q}} + \underline{i}_{jO} \underline{i}^q \underline{j}^{\dot{q}}
\end{aligned}$$

where

$$\underline{I} = \int_S * \{ \hat{\rho} [(\underline{0}^{P'} \cdot \underline{0}^{P'}) \underline{U} - \underline{0}^{P'} \underline{0}^{P'}] \} . \quad (2.2-14)$$

The Lagrange multiplier, \underline{H}' , may be recognized as the angular momentum about the primed origin (i.e., $\int_S * \{ \hat{\rho} \underline{P}' \times \underline{V} \}$).

The variations, $\delta c_{\alpha}^{\beta}$ are not independent; only three are. Instead of reducing the generality by selecting a specific three parameter set (e.g. a set of Euler angles), the three independent equations can be obtained by noting the skew symmetry of $c_{\alpha}^{\gamma} \delta c_{\alpha}^{\beta}$. The difference with its transpose will contain only the three independent linear combinations of variations,

$$\begin{aligned}
a_{\beta}^{\alpha} \delta c_{\alpha}^{\beta} &= a_{\beta}^{\alpha} c_{\alpha}^{\gamma} c_{\delta}^{\gamma} \delta c_{\delta}^{\beta} \\
&= a_{\beta}^{\alpha} c_{\alpha}^{\gamma} \delta_{\epsilon}^{\beta} \delta_{\gamma}^{\zeta} c_{\delta}^{\zeta} \delta c_{\delta}^{\epsilon} \\
&= - a_{\beta}^{\alpha} c_{\alpha}^{\gamma} \delta_{\epsilon}^{\beta} \delta_{\gamma}^{\zeta} c_{\delta}^{\epsilon} \delta c_{\delta}^{\zeta},
\end{aligned}
\tag{2.2-15}$$

where the skew-symmetry has been noted. Adding these expressions, noting appendix Equation (A.9-14) and interchanging indices,

$$\begin{aligned}
a_{\beta}^{\alpha} \delta c_{\alpha}^{\beta} &= \frac{1}{2} \left[\left(a_{\beta}^{\alpha} c_{\alpha}^{\gamma} \epsilon_{\gamma}^{\eta\beta} \right) \left(\epsilon_{\epsilon\zeta}^{\eta} c_{\delta}^{\zeta} \delta c_{\delta}^{\epsilon} \right) \right] \\
&= b^{\eta} \left(\frac{1}{2} \epsilon_{\epsilon\zeta}^{\eta} c_{\delta}^{\zeta} \delta c_{\delta}^{\epsilon} \right).
\end{aligned}
\tag{2.2-16}$$

The quantity in the parentheses represents the three independent variations ($\eta = 1,2,3$), so the stationary conditions $b^{\eta} = 0$, can be obtained by multiplying the coefficient of $\delta c_{\alpha}^{\beta}$ by $\epsilon_{\gamma}^{\eta\beta} c_{\alpha}^{\gamma}$.

The coefficient of $\delta c_{\alpha}^{\beta}$ on the time boundary, after multiplication by $\epsilon_{\epsilon}^{\eta\beta} c_{\alpha}^{\epsilon}$ is

$$\begin{aligned}
0 &= \epsilon_{\epsilon}^{\eta\beta} c_{\alpha}^{\epsilon} \left(h'_{\delta} \frac{1}{2} \epsilon_{\gamma\beta}^{\delta} c_{\alpha}^{\gamma} + \int_S * \left\{ \hat{\rho}_P^{\alpha,\beta} \left[\frac{\partial}{\partial t} r^{\alpha} + c_{\alpha}^{\gamma} v^{\gamma} + c_{\alpha}^{\gamma} (\underline{\Omega}' \times \underline{P}')^{\gamma} \right] \right\} \right) \\
&= \left(\delta_{\epsilon}^{\delta} \delta_{\gamma}^{\eta} - \delta_{\epsilon\gamma}^{\delta\eta} \right) \delta^{\epsilon\gamma} h'_{\delta} \frac{1}{2} + \int_S * \left\{ \epsilon_{\epsilon}^{\eta\beta} \hat{\rho}_P^{\alpha,\beta} \left[c_{\alpha}^{\epsilon} \frac{\partial}{\partial t} r^{\alpha} + \delta^{\epsilon\gamma} v^{\gamma} \right. \right. \\
&\quad \left. \left. + \delta^{\epsilon\gamma} (\underline{\Omega}' \times \underline{P}')^{\gamma} \right] \right\} \\
&= h'_{\eta} \left(\frac{1}{2} - \frac{3}{2} \right) + \epsilon_{\epsilon}^{\eta\beta} \int_S * \left\{ \hat{\rho}_P^{\alpha,\beta} \left[c_{\alpha}^{\epsilon} \frac{\partial}{\partial t} r^{\alpha} + v^{\epsilon} + (\underline{\Omega}' \times \underline{P}')^{\epsilon} \right] \right\} \\
\Rightarrow \underline{H}' &= \int_S * \left\{ \hat{\rho}_P^{\alpha,\beta} \underline{P}' \times \left[\frac{\partial}{\partial t} \underline{R} + \underline{V}' + \underline{\Omega}' \times \underline{P}' \right] \right\},
\end{aligned}
\tag{2.2-17}$$

which is the same as Equation (2.2-14) (i.e., \underline{H}' is the angular momentum about the primed origin on the time boundary as well as the interior, $\bar{t} \in \bar{T}$). The final stationary condition is that the coefficient of $\delta c_{\alpha}^{\beta}$, after multiplication by $\epsilon_{\epsilon}^{\eta\beta} c_{\alpha}^{\epsilon}$, must vanish in

the interior of the time domain,

$$\begin{aligned}
0 &= \epsilon^{\eta\beta} c_\alpha^\epsilon \left(-h'_\delta \frac{1}{2} \epsilon_{\gamma\beta}^\delta \frac{\partial}{\partial t} c_\alpha^\delta - \frac{\partial}{\partial t} h'_\delta \frac{1}{2} \epsilon_{\gamma\beta}^\delta c_\alpha^\gamma - h'_\delta \frac{1}{2} \epsilon_{\gamma\beta}^\delta \frac{\partial}{\partial t} c_\alpha^\gamma \right. \\
&\quad + \int_S * \left\{ -\hat{\rho} \frac{\partial}{\partial t} r^\alpha [\underline{\Omega}' \times \underline{P}']^\beta - \hat{\rho} \frac{\partial}{\partial t} r^\alpha \frac{\partial}{\partial t} p'^{\beta} \right. \\
&\quad \left. + [\underline{\nabla} \hat{W}']^\gamma p'^{\beta} c_\alpha^\gamma - N_{\underline{X}}^{\Lambda, \gamma} p'^{\beta} c_\alpha^\gamma \right\} \\
&\quad + \int_{\partial M(t)_P} * \left\{ -s'^{\gamma} p'^{\beta} c_\alpha^\gamma \right\} + \int_{\partial M(t)_S} * \left\{ -p'^{\gamma} p'^{\beta} c_\alpha^\gamma \right\} \\
&= -\frac{\partial}{\partial t} h'_\delta \frac{1}{2} \left(\delta_\epsilon^\delta \delta_\gamma^\eta - \delta_{\epsilon\gamma} \delta^{\eta\delta} \right) \delta^{\epsilon\gamma} + h'_\delta \left(\delta_\epsilon^\delta \delta_\gamma^\eta - \delta_{\epsilon\gamma} \delta^{\eta\delta} \right) \epsilon_\zeta^{\epsilon\gamma} \omega'^\zeta \\
&\quad + \int_S * \left\{ -\epsilon^{\eta\beta} \hat{\rho} [\underline{\Omega}' \times \underline{P}']^\beta c_\alpha^\epsilon \frac{\partial}{\partial t} r^\alpha - \epsilon^{\eta\beta} \hat{\rho} \frac{\partial}{\partial t} p'^{\beta} c_\alpha^\epsilon \frac{\partial}{\partial t} r^\alpha \right. \\
&\quad \left. + \epsilon^{\eta\beta} p'^{\beta} (\underline{\nabla} \hat{W}'^\epsilon - N_{\underline{X}}^{\Lambda \epsilon}) \right\} \\
&\quad + \int_{\partial M(t)_P} * \left\{ -\epsilon^{\eta\beta} p'^{\beta} s'^{\epsilon} \right\} + \int_{\partial M(t)_S} * \left\{ -\epsilon^{\eta\beta} p'^{\beta} s'^{\epsilon} \right\} \\
&= \left(\frac{\partial}{\partial t} \underline{H}' \right)^\eta + (\underline{\Omega}' \times \underline{H}')^\eta + \int_S * \left\{ -\hat{\rho} [(\underline{\Omega}' \times \underline{P}') \times \frac{\partial}{\partial t} \underline{R}]^\eta - \rho \left(\frac{\partial}{\partial t} \underline{P}' \times \frac{\partial}{\partial t} \underline{R} \right)^\eta \right. \\
&\quad \left. + [\underline{P}' \times (\underline{\nabla} \hat{W}' - N_{\underline{X}}^{\Lambda})]^\eta \right\} \\
&\quad + \int_{\partial M(t)_P} * \left\{ -\underline{P}' \times \underline{S}' \right\}^\eta + \int_{\partial M(t)_S} * \left\{ -\underline{P}' \times \underline{S}' \right\} . \quad (2.2-18)
\end{aligned}$$

Rearranging and multiplying by $\frac{\partial}{\partial x'^\eta}$ gives

$$\begin{aligned}
\frac{\partial}{\partial t} \underline{H}' + \underline{\Omega}' \times \underline{H}' &= \int_S * \left\{ \hat{\rho} \left[\frac{\partial}{\partial t} \underline{P}' + \underline{\Omega}' \times \underline{P}' \right] \times \frac{\partial}{\partial t} \underline{R} + \underline{P}' \times \left[-\underline{\nabla} \hat{W}' + N_{\underline{X}}^{\Lambda} \right] \right\} \\
&\quad + \int_{\partial M(t)_P} * \left\{ \underline{P}' \times \underline{S}' \right\} + \int_{\partial M(t)_S} * \left\{ \underline{P}' \times \underline{S}' \right\} . \quad (2.2-19)
\end{aligned}$$

This may be recognized as the identity

$$\frac{\partial}{\partial t} \int_S * \{ \underline{P}' \times \hat{\rho} \underline{V} \} = \int_S * \left\{ \hat{\rho} \frac{\partial}{\partial t} \underline{P}' \times \left[\frac{\partial}{\partial t} \underline{R} + \frac{\partial}{\partial t} \underline{P}' \right] + \underline{P}' \times \hat{\rho} \underline{A} \right\} ,$$

where it may be noted that $\hat{\rho A} = \underline{F}$. The expression can be converted to a function of $\underline{\Omega}'$, instead of \underline{H}' , by inserting Equation (2.2-14). To linear order in q this is

$$\begin{aligned}
& \frac{\partial'}{\partial t} \underline{\Omega}' \cdot \underline{I} + \underline{\Omega}' \times (\underline{I} \cdot \underline{\Omega}') + \frac{\partial'}{\partial t} \underline{\Omega}' \cdot \underline{I}_{i1} q + \underline{\Omega}' \cdot \underline{I}_{i1} \dot{q} + \underline{\Omega}' \times (\underline{\Omega}' \cdot \underline{I}_{i1})_{i1} q \\
& + \underline{I}_{i1} \cdot \frac{\partial'}{\partial t} \underline{\Omega}'_{i1} q + \underline{I}_{i1} \cdot \underline{\Omega}'_{i1} \dot{q} + \underline{\Omega}' \times \underline{I}_{i1} \cdot \underline{\Omega}'_{i1} q + M_{0\underline{P}'} \times \frac{\partial^2}{\partial t^2} \underline{R} \\
& \quad + M(\underline{\Omega}' \times \underline{0\underline{P}'}) \times \frac{\partial}{\partial t} \underline{R} \\
& + M_{i\underline{P}'} \times \frac{\partial^2}{\partial t^2} \underline{R}_{i1} q + M_{i\underline{P}'} \times \frac{\partial}{\partial t} \underline{R}_{i1} \dot{q} + M(\underline{\Omega}' \times \underline{i\underline{P}'}) \times \frac{\partial}{\partial t} \underline{R}_{i1} q \\
& \quad + \underline{i\underline{0}}_{i1} \ddot{q} + \underline{\Omega}' \times \underline{i\underline{0}}_{i1} \dot{q} \\
& = M_{i\underline{P}'} \times \frac{\partial}{\partial t} \underline{R}_{i1} \dot{q} + M(\underline{\Omega}' \times \underline{0\underline{P}'}) \times \frac{\partial}{\partial t} \underline{R} + M(\underline{\Omega}' \times \underline{i\underline{P}'}) \times \frac{\partial}{\partial t} \underline{R}_{i1} q \\
& + \int_{\underline{S}} * \{ \underline{P}' \times [-\underline{\nabla} \hat{W}' + \underline{N}_{\underline{X}'}^{\Lambda}] \} + \int_{\partial M(t)_P} * \{ \underline{P}' \times \underline{S}' \} + \int_{\partial M(t)_S} * \{ \underline{P}' \times \underline{P}_{\underline{S}'} \} . \\
& \hspace{15em} (2.2-20)
\end{aligned}$$

or rearranging,

$$\begin{aligned}
& \frac{\partial'}{\partial t} \underline{\Omega}' \cdot \underline{I} + \underline{\Omega}' \times \underline{I} \cdot \underline{\Omega}' + \frac{\partial'}{\partial t} \underline{\Omega}' \cdot \underline{I}_{i1} q + \underline{\Omega}' \cdot \underline{I}_{i1} \dot{q} + \underline{\Omega}' \times (\underline{\Omega}' \cdot \underline{I}_{i1})_{i1} q \\
& \quad + \underline{I}_{i1} \cdot \frac{\partial'}{\partial t} \underline{\Omega}'_{i1} q \\
& + \underline{I}_{i1} \cdot \underline{\Omega}'_{i1} \dot{q} + \underline{\Omega}' \times \underline{I}_{i1} \cdot \underline{\Omega}'_{i1} q + M_{0\underline{P}'} \times \frac{\partial^2}{\partial t^2} \underline{R} + M_{i\underline{P}'} \times \frac{\partial^2}{\partial t^2} \underline{R}_{i1} q \\
& \quad + \underline{i\underline{0}}_{i1} \ddot{q} + \underline{\Omega}' \times \underline{i\underline{0}}_{i1} \dot{q} \\
& = \int_{\underline{S}} * \{ \underline{P}' \times [-\underline{\nabla} \hat{W}' + \underline{N}_{\underline{X}'}^{\Lambda}] \} + \int_{\partial M(t)_P} * \{ \underline{P}' \times \underline{S}' \} + \int_{\partial M(t)_S} * \{ \underline{P}' \times \underline{P}_{\underline{S}'} \} . \\
& \hspace{15em} (2.2-21)
\end{aligned}$$

2.3 Finite Element Assembly

A substructure, as a collection of material continua, is itself a material continuum. Therefore it must satisfy the same variational principle as the finite elements. Each finite element deformation field may be thought of as a piecewise continuous deformation field for the substructure, defined on the element and zero elsewhere. The variational form for the entire substructure differs from the sum of the forms for the finite elements by the presence of pairs of internal boundary integrals in the latter. The displacements must be continuous (compatible), and the stresses must be continuous (in equilibrium) throughout the substructure. In particular these conditions must be satisfied on the element boundaries. In this case, the pairs of integrals cancel. Since the finite element method is only approximate, there are generally violations (usually of equilibrium for the displacement method) and the integrals do not vanish. However, under appropriate conditions, it can be shown that the contribution of these integrals goes to zero as the mesh size approaches zero (i.e., as the number of elements in the model approaches infinity). The details of these effects belong to the study of finite elements, and will not be treated here. It will be assumed that an appropriate model has been developed, and that the mesh (number of elements) is sufficiently refined so that the contribution of the integrals is small enough to be ignored.

The result of the application of the compatibility requirements to the collection of finite elements is to assemble them into the substructure by eliminating the dependent generalized displacements.

In the case of nodal displacements, the assembly of the stiffness terms is quite automatic and is called the direct stiffness method. To generalize this procedure slightly, all the parameters associated with a given finite element will be denoted by a pre-superscript index associated with the element. For an element, α , the vector $\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix} P'$ will be divided into two parts, a constant portion, $\begin{smallmatrix} \alpha \\ S_0 \end{smallmatrix} P'$, defining the position of a local origin (used in developing element properties) relative to the substructure, plus the position relative to the local origin for the element, $\begin{smallmatrix} \alpha \\ E_0 \end{smallmatrix} P'$. Now if the variational forms (e.g. (2.2-3)) are summed for all of the elements, assuming the internal boundary integrals are negligible, the stationary conditions for the frame variables are the sums of the element conditions. If the compatibility requirements between elements are expressed in terms of N independent \hat{q}_j in the form

$$\begin{smallmatrix} \alpha \\ i \end{smallmatrix} q = \sum_{j=1}^N \begin{smallmatrix} \alpha \\ ij \end{smallmatrix} T \hat{q}_j, \quad (2.3-1)$$

they may be incorporated in the form with Lagrange multipliers,

$$\sum_{\alpha=1}^E \sum_{i=1}^{M_\alpha} \alpha_\lambda \left(\begin{smallmatrix} \alpha \\ i \end{smallmatrix} q - \begin{smallmatrix} \alpha \\ ij \end{smallmatrix} T \hat{q}_j \right), \quad (2.3-2)$$

where E is the number of elements. (In the case of direct stiffness, the \hat{q}_j are the nodal displacements in global coordinates, the $\begin{smallmatrix} \alpha \\ i \end{smallmatrix} q$ are the nodal displacements of the α element in local coordinates, and $\begin{smallmatrix} \alpha \\ ij \end{smallmatrix} T$ consists of direction cosines.) The stationary conditions on the deformation variables are the same as for the elements individually, except for the additional term α_λ . The coefficients of $\delta \begin{smallmatrix} \alpha \\ i \end{smallmatrix} \lambda$ just give

Equation (2.3-1), and the coefficients of the variations of the global coordinates, $\delta_j \hat{q}$, are just $\sum_{\alpha=1}^E \sum_{i=1}^{M_\alpha} \alpha_{i\lambda} \alpha_{ij}^T$.

Summing Equation (2.2-5) over all elements provides the translational equation,

$$S_M \left\{ \frac{\partial^2}{\partial t^2} \underline{R} + \frac{\partial'}{\partial t} \underline{\Omega}' \times \frac{S_{\underline{P}'}}{0_{\underline{P}'}} + \frac{\partial'}{\partial t} \underline{\Omega}' \times \frac{S_{\underline{P}'}}{j_{\underline{P}'}} \hat{q} + 2 \underline{\Omega}' \times \frac{S_{\underline{P}'}}{j_{\underline{P}'}} \dot{\hat{q}} + \underline{\Omega}' \times (\underline{\Omega}' \times \frac{S_{\underline{P}'}}{0_{\underline{P}'}}) + \underline{\Omega}' \times (\underline{\Omega}' \times \frac{S_{\underline{P}'}}{j_{\underline{P}'}}) \hat{q} + \frac{S_{\underline{P}'}}{j_{\underline{P}'}} \ddot{\hat{q}} \right\} = \underline{S}_{\underline{F}}, \quad (2.3-3)$$

where double Latin subscripts imply summation $i=1, \dots, N$ and

$$S_M = \sum_{\alpha=1}^E \alpha_M,$$

$$\frac{S_{\underline{P}'}}{0_{\underline{P}'}} = \sum_{\alpha=1}^E \alpha_M \left(\frac{\alpha_{\underline{P}'}}{S_{0_{\underline{P}'}}} + \frac{\alpha_{\underline{P}'}}{E_{0_{\underline{P}'}}} \right) / S_M,$$

$$\frac{S_{\underline{P}'}}{j_{\underline{P}'}} = \sum_{\alpha=1}^E \sum_{i=1}^{M_\alpha} \alpha_M \frac{\alpha_{\underline{P}'}}{i_{\underline{P}'}} \alpha_{ij}^T / S_M \quad \text{and}$$

$$\underline{S}_{\underline{F}} = \sum_{\alpha=1}^E \left(\int_{\alpha_{\underline{S}}} * \{ - \underline{V} \hat{W}' + \frac{N \hat{X}'}{\underline{X}'} \} + \int_{\partial \alpha_{M(t)}^P} * \{ \underline{S}' \} + \int_{\partial \alpha_{M(t)}^S} * \{ \underline{S}'^P \} \right),$$

noting, however, that only the external boundary integrals are retained in the force calculation. Similarly, the sum of the nominal rotational Equations (2.2-21) over all elements gives

$$\begin{aligned} & S_{\underline{I}} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' + \underline{\Omega}' \times S_{\underline{I}} \cdot \underline{\Omega}' + \frac{\partial'}{\partial t} \underline{\Omega}' \cdot \frac{S_{\underline{I}}}{j_{\underline{I}}} \hat{q} + \frac{S_{\underline{I}}}{j_{\underline{I}}} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' \cdot \hat{q} \\ & + \underline{\Omega}' \cdot \frac{S_{\underline{I}}}{j_{\underline{I}}} \dot{\hat{q}} + \frac{S_{\underline{I}}}{j_{\underline{I}}} \cdot \underline{\Omega}' \cdot \dot{\hat{q}} + \underline{\Omega}' \times (\underline{\Omega}' \cdot \frac{S_{\underline{I}}}{j_{\underline{I}}}) \hat{q} + \underline{\Omega}' \times \frac{S_{\underline{I}}}{j_{\underline{I}}} \cdot \underline{\Omega}' \cdot \hat{q} \\ & + S_M \frac{S_{\underline{P}'}}{0_{\underline{P}'}} \times \frac{\partial^2}{\partial t^2} \underline{R} + S_M \frac{S_{\underline{P}'}}{j_{\underline{P}'}} \times \frac{\partial^2}{\partial t^2} \underline{R} \hat{q} + \frac{S_{\underline{O}}}{j_{\underline{O}}} \ddot{\hat{q}} + \underline{\Omega}' \times \frac{S_{\underline{O}}}{j_{\underline{O}}} \dot{\hat{q}} = \underline{S}_{\underline{T}}, \end{aligned} \quad (2.3-4)$$

where

$$\begin{aligned}
S_{\underline{I}} &= \sum_{\alpha=1}^E \left\{ \alpha_{\underline{I}} + \alpha_M \left[\left(2 \alpha_{SO_{\underline{I}}}^{\alpha} \cdot \alpha_{EO_{\underline{I}}}^{\alpha} + \alpha_{SO_{\underline{I}}}^{\alpha} \cdot \alpha_{SO_{\underline{I}}}^{\alpha} \right) \underline{U} \right. \right. \\
&\quad \left. \left. - \left(\alpha_{SO_{\underline{I}}}^{\alpha} \alpha_{EO_{\underline{I}}}^{\alpha} + \alpha_{EO_{\underline{I}}}^{\alpha} \alpha_{SO_{\underline{I}}}^{\alpha} + \alpha_{SO_{\underline{I}}}^{\alpha} \alpha_{SO_{\underline{I}}}^{\alpha} \right) \right] \right\} , \\
S_{j\underline{I}} &= \sum_{\alpha=1}^E \sum_{i=1}^M \alpha_i^{\alpha} \left\{ \alpha_{i\underline{I}}^{\alpha} \alpha_{ij}^{\alpha} T + \alpha_M \left[\left(\alpha_{SO_{\underline{I}}}^{\alpha} \cdot \alpha_{i\underline{I}}^{\alpha} \right) \underline{U} - \alpha_{SO_{\underline{I}}}^{\alpha} \alpha_{i\underline{I}}^{\alpha} \right] \alpha_{ij}^{\alpha} T \right\} , \\
S_{j\underline{O}} &= \sum_{\alpha=1}^E \sum_{i=1}^M \alpha_i^{\alpha} \left\{ \alpha_{i\underline{O}}^{\alpha} \alpha_{ij}^{\alpha} T + \alpha_M \left[\alpha_{SO_{\underline{I}}}^{\alpha} \times \alpha_{i\underline{I}}^{\alpha} \right] \alpha_{ij}^{\alpha} T \right\} \quad \text{and} \\
S_{\underline{T}} &= \sum_{\alpha=1}^E \sum_{i=1}^M \left(\int_{\underline{S}}^{\alpha} * \left\{ \left[\alpha_{SO_{\underline{I}}}^{\alpha} + \alpha_{EO_{\underline{I}}}^{\alpha} + \alpha_{i\underline{I}}^{\alpha} \alpha_{ij}^{\alpha} T \right] \hat{q}_j \right\} \times \left[-\nabla \hat{W} + \frac{N \hat{X}}{X} \right] \right) \\
&\quad + \int_{\partial M(t)_P}^{\alpha} * \left\{ \left[\alpha_{SO_{\underline{I}}}^{\alpha} + \alpha_{EO_{\underline{I}}}^{\alpha} + \alpha_{i\underline{I}}^{\alpha} \alpha_{ij}^{\alpha} T \right] \hat{q}_j \right\} \times [S'] \Big\} \\
&\quad + \int_{\partial M(t)_S}^{\alpha} * \left[\alpha_{SO_{\underline{I}}}^{\alpha} + \alpha_{EO_{\underline{I}}}^{\alpha} + \alpha_{i\underline{I}}^{\alpha} \alpha_{ij}^{\alpha} T \right] \hat{q}_j \times [S'] \Big) ,
\end{aligned}$$

again noting that only external boundary integrals are retained in the torque calculation.

As previously noted, the incorporation of the constraints via Lagrange multipliers results in the appearance of the additional term α_i^{λ} in the i^{th} equation for the α^{th} element. Solving these equations for the α_i^{λ} , the stationary conditions for the $\delta_j \hat{q}$ are the sum

$$\sum_{\alpha=1}^E \sum_{i=1}^M \alpha_i^{\lambda} \alpha_{ij}^{\alpha} T = 0. \quad \text{In other words, the new stationary conditions are}$$

obtained by premultiplying by $\alpha_{ij}^{\alpha} T$ and summing over all i in all elements α . The elastic equations (2.2-12) become

$$\begin{aligned}
& - \underline{\Omega}' \cdot \underline{S}_{j\underline{I}} \cdot \underline{\Omega}' - \underline{\Omega}' \cdot \underline{S}_{lj\underline{I}} \cdot \underline{\Omega}' \hat{\underline{q}} + \underline{S}_M \underline{S}_{j\underline{P}'} \cdot \frac{\partial^2}{\partial \tau^2} \underline{R} \\
& + 2 \underline{\Omega}' \cdot \underline{S}_{lj\underline{O}} \hat{\underline{q}} + \underline{S}_{j\underline{O}} \cdot \frac{\partial'}{\partial \tau} \underline{\Omega}' + \underline{S}_{lj\underline{O}} \cdot \frac{\partial'}{\partial \tau} \underline{\Omega}' \hat{\underline{q}} \\
& + \underline{S}_{jl}^M \ddot{\underline{q}} + \underline{S}_{jl}^C \dot{\underline{q}} + \underline{S}_{jl}^K \underline{q} = \underline{S}_Q
\end{aligned} \tag{2.3-5}$$

where

$$\underline{S}_{jl}^I = \sum_{\alpha=1}^E \sum_{i=1}^{M_\alpha} \sum_{k=1}^{M_\alpha} \begin{pmatrix} \alpha_T & \alpha_I & \alpha \\ ij^T & ik^I & kl^T \end{pmatrix},$$

$$\underline{S}_{jl}^O = \sum_{\alpha=1}^E \sum_{i=1}^{M_\alpha} \sum_{k=1}^{M_\alpha} \begin{pmatrix} \alpha_T & \alpha_O & \alpha \\ ij^T & ik^O & kl^T \end{pmatrix},$$

$$\underline{S}_{jl}^M = \sum_{\alpha=1}^E \sum_{i=1}^{M_\alpha} \sum_{k=1}^{M_\alpha} \begin{pmatrix} \alpha_T & \alpha_M & \alpha \\ ij^T & ik^M & kl^T \end{pmatrix},$$

$$\underline{S}_{jl}^C = \sum_{\alpha=1}^E \sum_{i=1}^{M_\alpha} \sum_{k=1}^{M_\alpha} \begin{pmatrix} \alpha_T & \alpha_C & \alpha \\ ij^T & ik^C & kl^T \end{pmatrix},$$

$$\underline{S}_{jl}^K = \sum_{\alpha=1}^E \sum_{i=1}^{M_\alpha} \sum_{k=1}^{M_\alpha} \begin{pmatrix} \alpha_T & \alpha_K & \alpha \\ ij^T & ik^K & kl^T \end{pmatrix} \text{ and}$$

$$\begin{aligned}
\underline{S}_Q = & \sum_{\alpha=1}^E \sum_{i=1}^{M_\alpha} \left(\alpha_{ij}^T \left[\int_{\alpha_S} * \left\{ \alpha_{i\underline{P}'} \cdot \left(-\underline{\nabla} \hat{\underline{w}}' + \frac{N \underline{\lambda}}{X} \right) \right\} + \int_{\partial \alpha_{M(t)}^P} * \left\{ \alpha_{i\underline{P}'}^P \cdot \underline{S}' \right\} \right. \right. \\
& \left. \left. + \int_{\partial \alpha_{M(t)}^S} * \left\{ \alpha_{i\underline{P}'} \cdot \underline{P}_{S'} \right\} - \sigma_{iQ} \right] \right),
\end{aligned}$$

again including only external boundaries.

It may be seen from the above equations that the following parameters are required for each finite element:

$$\alpha_M, \alpha_{E0} \bar{P}', \alpha_{i\bar{P}'}', \alpha_{i\bar{I}}', \alpha_{i\bar{I}}', \alpha_{i\bar{O}}', \alpha_{ik\bar{I}}', \alpha_{ik\bar{O}}', \alpha_{ik}^M, \alpha_{ik}^C, \alpha_{ik}^K, \alpha_{iQ}^{\sigma}.$$

The last two parameters, α_{ik}^K and α_{iQ}^{σ} , were calculated for a beam element in Appendix C.1. The remaining terms are calculated for the same element in Appendix C.2, as an illustrative example. The parameters, $\alpha_{S0} \bar{P}'$, α_{ij}^T , appearing in the preceding equation are specifically associated with the assembly procedure. It should be noted that the positions are defined for a deformed state which may require a geometrically nonlinear static analysis for their computation. Finally, the generalized force parameters must be computed from an assumed environmental model.

2.4 Quadratic Expansions

The development of the finite element as presented here, uses the unknown coefficients of a series expansion as generalized coordinates. However, an expansion in the form

$$\bar{P}' = \alpha_0 \bar{P}' + \sum_{i=1}^M \alpha_i \bar{P}'_i q \quad (2.4-1)$$

suggests the possibility of extending the expansion in q 's to include higher orders (i.e., terms involving the square of q). Although the approach adopted here does not prohibit such expansions, it does make them seem rather unnatural. However, such is not the case when the problem is approached from a physical rather than a mathematical viewpoint. It is quite natural to interpret a linear displacement field on a surface as a rotation of the surface. However, if rotations are taken as generalized coordinates, the displacement is not linear.

All of the remarks above would be largely academic if the linearized equations resulting from a quadratic approximation were the same as

those resulting from a linear expansion. However, the essentially Lagrangian approach adopted here results in a different system of equations when quadratic terms are included. The best that can be hoped for is that the systems of equations resulting from the two methods are mathematically equivalent, although non-identical. The two systems are equivalent (i.e., have identical solutions) if each equation of one set is an independent linear combination of equations in the other set. If the equations are equivalent, then inclusion of quadratic terms is optional, and, for convenience, they may be omitted. That the resulting equations are equivalent, has been demonstrated both analytically and by example. Appendix D.1 presents an analytical demonstration of the equivalence of the equations. Appendix D.2 presents a particularly challenging example of a gimbaled mass for which the nonlinear field is more natural, and for which the nonlinear system of equations is more easily derived. It is also noted that the same result holds for a sequence of rotations about inertially fixed axes, as for the body fixed sequence in the example. Based on these results, it may be concluded that quadratic terms may be omitted. Therefore only linear terms will be retained in the rest of this presentation.

2.5 Substructure Modes

The motion of a substructure, isolated from all other substructures, is governed by the three scalar second order equations in \underline{R} (2.3-3), by the three scalar first order equations in $\underline{\Omega}'$ (2.3-4), by the three scalar first order equations in C_{β}^{α} (2.2-13), and by "N" scalar second order equations in \hat{q}_j (2.3-5). Unfortunately, "N" is frequently

large, sometimes hundreds or even thousands. Moreover, much of the motion governed by these equations is of the uninteresting, small, high frequency type. It is desirable to reduce the dimension of the problem by eliminating the uninteresting motions and retaining only the larger, usually lower frequency, motions. Modal analysis provides a basis for this reduction by uncoupling the equations and providing the frequencies associated with the degrees of freedom. The frequencies provide one of several possible criteria for truncation. The uncoupled nature of the equations guarantees that the solution of the remaining equations is unaffected by assumptions on the truncated modes, and allows formal establishment of bounds on the truncation error.

Calculation of modes is possible only for constant coefficient linear equations. The substructure equations are inherently nonlinear, and even if linearization about a nominal motion is possible, the coefficients are not generally constant. However, the need to reduce the problem is sufficiently overriding that the method is frequently applied to problems for which it is not formally correct. In the presence of good engineering judgment, this has been quite successful. Before pursuing this tack (in 2.6), however, it will be helpful to identify the class of problems for which the method is formally correct.

The equations will be linearized by assuming that the ${}_j\hat{q}$ represent a small disturbance about the nominal motion (the rigid body motion solution when all ${}_j\hat{q} = 0$). (It should be noted that there are cases for which the nominal solution is grossly different from the frame motion in the actual solution.) Now, arbitrarily partitioning the

force into a nominal force, ${}^0\mathbf{F}$, and a disturbance force, ${}^1\mathbf{F}$, the nominal motion equations corresponding to Equations (2.3-3, -4 and -5) are

$$S_M \left\{ \frac{\partial^2}{\partial t^2} \underline{R} + \frac{\partial'}{\partial t} \underline{\Omega}' \times \frac{S_{P'}}{O'} + \underline{\Omega}' \times \left(\underline{\Omega}' \times \frac{S_{P'}}{O'} \right) \right\} = {}^0S_{\underline{F}}, \quad (2.5-1)$$

$$S_{\underline{I}} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' + \underline{\Omega}' \times S_{\underline{I}} \cdot \underline{\Omega}' + S_M \frac{S_{P'}}{O'} \times \frac{\partial^2}{\partial t^2} \underline{R} = {}^0S_{\underline{T}} \quad \text{and} \quad (2.5-2)$$

$$- \underline{\Omega}' \cdot S_{\underline{I}} \cdot \underline{\Omega}' + S_M \frac{S_{P'}}{j} \cdot \frac{\partial^2}{\partial t^2} \underline{R} + \frac{S_O}{j} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' = {}^0S_{jQ}. \quad (2.5-3)$$

Similarly, the disturbed motion equations are

$$S_M \left\{ \frac{S_{P'}}{j} \ddot{\hat{q}} + 2\underline{\Omega}' \times \frac{S_{P'}}{j} \dot{\hat{q}} + \left[\frac{\partial'}{\partial t} \underline{\Omega}' \times \frac{S_{P'}}{j} + \underline{\Omega}' \times \left(\underline{\Omega}' \times \frac{S_{P'}}{j} \right) \right] \hat{q} \right\} = {}^1S_{\underline{F}}, \quad (2.5-4)$$

$$\begin{aligned} & \frac{S_O}{j} \ddot{\hat{q}} + \left(\underline{\Omega}' \times \frac{S_O}{j} + \underline{\Omega}' \cdot S_{\underline{I}} + S_{\underline{I}} \cdot \underline{\Omega}' \right) \dot{\hat{q}} + \left[S_M \frac{S_{P'}}{j} \times \frac{\partial^2}{\partial t^2} \underline{R} + \frac{\partial'}{\partial t} \underline{\Omega}' \cdot S_{\underline{I}} \right. \\ & \left. + S_{\underline{I}} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' + \underline{\Omega}' \times \left(\underline{\Omega}' \cdot S_{\underline{I}} \right) + \underline{\Omega}' \times S_{\underline{I}} \cdot \underline{\Omega}' \right] \hat{q} = {}^1S_{\underline{T}} \quad \text{and} \quad (2.5-5) \end{aligned}$$

$$\begin{aligned} & S_{j\ell}^M \ddot{\hat{q}} + \left(S_{j\ell}^C + 2\underline{\Omega}' \cdot \frac{S_O}{j\ell} \right) \dot{\hat{q}} + \left(S_{j\ell}^K - \underline{\Omega}' \cdot \frac{S_{\underline{I}}}{j\ell} \cdot \underline{\Omega}' + \frac{S_{j\ell}^O}{j\ell} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' \right) \hat{q} \\ & = {}^1S_{jQ}. \quad (2.5-6) \end{aligned}$$

Equation (2.5-3) may be written in integral form as

$$\begin{aligned} & \int_S * \left\{ \frac{S_{P'}}{j} \cdot \hat{p} \left[\frac{\partial^2}{\partial t^2} \underline{R} + \frac{\partial'}{\partial t} \underline{\Omega}' \times \frac{S_{P'}}{O'} + \underline{\Omega}' \times \left(\underline{\Omega}' \times \frac{S_{P'}}{O'} \right) \right] \right\} \\ & = \int_S * \left\{ \frac{S_{P'}}{j} \cdot \underline{X} \right\} + jQ. \quad (2.5-7) \end{aligned}$$

Noting that in the limit as the mesh size approaches zero, the $\frac{S_{P'}}{j}$ become a complete function space, it may be concluded that the body

force must satisfy

$$\underline{X} = \hat{\rho} \left[\frac{\partial^2}{\partial t^2} \underline{R} + \frac{\partial'}{\partial t} \underline{\Omega}' \times \underline{S}_{0\underline{P}'} + \underline{\Omega}' \times \left(\underline{\Omega}' \times \underline{S}_{0\underline{P}'} \right) \right] + {}^0\underline{X} \quad , \quad (2.5-8)$$

where ${}^0\underline{X}$ is constant in time and equilibrates the initial and boundary stresses. It may be noted that this body force also satisfies Equations (2.5-1 and -2).

This leads to the perhaps surprising conclusion that linearization may be about an arbitrary time varying nominal motion, if the nominal motion is driven by body forces in the form of Equation (2.5-8). It may be noted that the prospects of finding a physical source for body forces of the indicated form are rather dim. However, this result suggests an analytical technique. Any forces applied to the sub-structure may be represented as the sum of the equivalent body force and a disturbance composed of the applied force and the negative of the equivalent body force. This technique provides a formally correct solution and reduces the structural equations to linear form (2.5-6).

There is one potential difficulty with the method. The linear solution is predicated on the \hat{q}_j remaining small. For those problems in which flexibility grossly alters the nominal motion, the gross difference between the nominal and actual motion would appear in the rigid body modes inherent in the \hat{q}_j . (The rigid body modes are a consequence of the unconstrained, displacement method, finite element formulation adopted here.) There are two ways in which this difficulty might be overcome. If this behavior is anticipated (e.g., by "energy sink" analysis) or is detected during integration, a pair of body

forces, a nominal force, and an equal but opposite disturbance force (summing to no net force on the system), could be applied to transfer the rigid body motion in the ${}_j\hat{q}$ to the reference frame variables and maintain the smallness of the ${}_j\hat{q}$. Alternatively, when the motion in the rigid body modes exceeds some predetermined bound during integration, the motion in the rigid body modes could be transferred to the reference frame, and the rigid body motion in the ${}_j\hat{q}$ reset to zero.

The presence of rigid body degrees of freedom in the ${}_j\hat{q}$, is a consequence of the displacement method. The method also results in redundant Equations (2.5-4 and -5). Equation (2.5-6) is sufficient to determine the ${}_j\hat{q}$. Qualitatively, this is a system of ordinary differential equations of the form

$$M\ddot{q} + C\dot{q} + G\dot{q} + Kq + Jq + Aq = Q . \quad (2.5-9)$$

The terms have the following characteristics. The "mass" matrix, M , is constant, real, symmetric, and positive definite (in some special cases positive semi-definite). The "damping" matrix, C , is constant, real, symmetric, and positive semi-definite. In the special case of proportional damping, it is assumed that C can be expressed in a Caughey series. The "coriolis" matrix, G , is constant only if ${}^0\Omega'$ is, otherwise it is a function of time. It is real but skew-symmetric. The "stiffness" matrix, K , is constant, real, symmetric, positive semi-definite and consists of two parts, the elastic stiffness and the "geometric" stiffness associated with the preload. (If the preload depends on spin, then K is not constant unless ${}^0\Omega'$ is.) It may be

noted that in the case of structural instability (buckling) K becomes indefinite. Such cases are excluded here. Sometimes K is made complex to incorporate "structural" damping. The "centripetal" matrix, J , is constant only if ${}^0\Omega'$ is, otherwise it is a function of time. It is real, symmetric, and negative semi-definite. The "spin-up" matrix, A , is null if the spin is constant, constant if $\frac{\partial}{\partial t} {}^0\Omega'$ is constant, and otherwise is a function of time. (In some formulations A is a function of spin; see 2.5 and related appendices. The spin independence is a result of the Lagrangian formulation.) It is real but skew-symmetric.

The equations, in general, must be integrated numerically. However, if the coefficients are almost constant, certain stability results may be derived, and if they are periodic, Floquet theory may be used. However, in order to calculate modes, they must be constant. Consequently, the nominal angular velocity must be constant. In addition to systems for which the spin is constant, it is customary engineering practice to approximate a slowly varying system by time slices (i.e., assuming that the deviation from constant coefficients is negligible over some period of time). This can be formalized by selecting the nominal body forces to provide constant spin, and absorbing the change in the disturbance equation for some period of time. Although a large constant spin and a small constant rate of change of spin might be assumed, for formally constant spin A vanishes.

In general, second order equations must be converted to first order to obtain modes. Appendix A.15 outlines the method. The

special case of zero spin and proportional damping is an exception for which eigen solutions can be obtained directly from the second order equations.

When a substructure must ultimately be connected to others, a number of researchers have noted that "free" modes of the substructure do not always provide the most rapid convergence during a process of modal synthesis. A number of alternative procedures have been suggested which may speed up convergence. These include constraining certain degrees of freedom, adding masses and adding stiffnesses. A comparative analysis is forgone here; it is merely noted that all such methods result in a transformation of the variables, ${}_j\hat{q}$ and $\dot{{}_j\hat{q}}$, to a smaller dimension set of coordinates, η ,

$$\begin{Bmatrix} \hat{q} \\ \dot{\hat{q}} \end{Bmatrix} = \begin{bmatrix} \phi \end{bmatrix} \begin{Bmatrix} \eta \end{Bmatrix}, \quad (2.5-10)$$

and a reduced order set of equations

$$[\alpha] \{\dot{\eta}\} + [\beta] \{\eta\} = \{\psi\} . \quad (2.5-11)$$

The analyst's choice of method is based upon a number of factors including: ease of calculating the modes, storage requirements for the reduced equation, rate of convergence, and ease of coupling sub-assemblies.

As an example of the modal analysis procedure, the component mode method, generally attributed to Hurty (i.e., constraining certain degrees of freedom) is applied to a very simple substructure in Appendix C.3.

2.6 Extended Modal Analysis

As noted in the previous Section (2.5), the benefits of modal truncation are sufficiently important to justify application to problems for which it is not formally correct. In such cases, even though linearization is not formally possible, a nominal state or perhaps several nominal states are chosen to linearize about anyhow. Similarly, constant coefficients are selected, and modal analysis proceeds as before. The extended procedure now departs from the method previously outlined by relaxing all the assumptions and substituting back into the governing equations. The Equations (2.3-3 and -4) assume the form

$$S_M \left\{ \frac{\partial^2}{\partial t^2} \underline{R} + \frac{\partial'}{\partial t} \underline{\Omega}' \times \underline{S}_{\underline{P}'} + \underline{\Omega}' \times (\underline{\Omega}' \times \underline{S}_{\underline{P}'}) \right\} + \frac{T_\alpha}{j} \dot{\eta} + \frac{T_\beta}{j} \eta = \underline{S}_F$$

$$\underline{S}_I \cdot \frac{\partial'}{\partial t} \underline{\Omega}' + \underline{\Omega}' \times \underline{S}_I \cdot \underline{\Omega}' + S_M \underline{S}_{\underline{P}'} \times \frac{\partial^2}{\partial t^2} \underline{R} + \frac{R_\alpha}{j} \dot{\eta} + \frac{R_\beta}{j} \eta = \underline{S}_T, \quad (2.6-1)$$

and after premultiplying by the transpose of the adjoint matrix, Equation (2.3-5) becomes

$$- \underline{\Omega}' \cdot \frac{S_I}{j} \cdot \underline{\Omega}' + S_M \frac{S_{\underline{P}'}}{j} \cdot \frac{\partial^2}{\partial t^2} \underline{R} + \frac{S_Q}{j} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' + \frac{M}{jk} \alpha_k \dot{\eta} + \frac{M}{jk} \beta_k \eta = \underline{S}_Q. \quad (2.6-2)$$

If the complete set of modes is retained, there is no compromise in the above equations. They contain exactly the same information as their non-modal counterparts. The α and β coefficients are functions of the variables \underline{R} , C_β^α , and $\underline{\Omega}'$, and the equations are no easier to solve than their predecessors.

At this point engineering judgment is invoked. It is assumed that the effect of the modes selected for truncation on Equations (2.6-1) is negligible. It is assumed (as in normal modal analysis) that the response in the truncated modes is insignificant in the result. And, it is assumed that the coupling in Equation (2.6-2) between the truncated and retained modes has no significant effect on the retained modes (and conversely that the retained modes do not make the truncated modes important in the result). Under these assumptions, the modes may be truncated and the reduced set of nonlinear equations integrated. It should be noted that these assumptions are frequently reasonable and lead to good results. Some truncation can almost certainly be justified since the highest modes are usually more a reflection of the details of the finite element model than of the substructure itself. The degree of truncation suitable for a given problem is sometimes determined by repeating the simulation several times, with differing degrees of truncation, and observing the rate of convergence of the solution.

Section 3

STRUCTURES

A structure will be defined as a finite collection of interconnected substructures. The interconnection mechanisms will be classified in three categories: those producing known forces, those producing known relative motions, and those with a known relationship between forces and motions. The interconnections producing known forces will be treated as external forces on the system. The interconnections producing known relative motions constrain not only the nominal motion of the structure but the disturbed motion as well. The interconnections yielding relationships will be assumed to be in the special form of the sum of a large known motion (i.e., constrained for nominal motion) plus a linear force-displacement relationship for small disturbed motion. Although arbitrary connections can be treated by direct numerical integration of the connection relations and equations of motion, a reduction of the equations is possible in certain cases. These reductions are the primary subject of this section.

3.1 Constraints

An interconnection relationship that prescribes some aspect of the relative motion between two substructures is called a constraint. Constraints may be broadly categorized as either equality or inequality constraints. Inequality constraints will not be specifically treated; rather it is recommended that they be treated as two separate problems, unconstrained, and equality constrained, with the integrator constructed to detect the passage from one regime to another.

There are some discrepancies in the literature as to whether or not an inequality constraint may be classified as holonomic. However, among the equality constraints, a constraint is classified as holonomic if it may be expressed in integrated form

$$\phi({}^1\xi, {}^2\xi, \dots, {}^N\xi, t) = 0, \quad (3.1-1)$$

and non-holonomic if it must be expressed in differential form

$$\sum_{i=1}^N {}^i\alpha d{}^i\xi + \alpha dt = 0, \quad (3.1-2)$$

where the ${}^i\xi$ are generalized coordinates, and the coefficients $\alpha, {}^i\alpha$ are functions of ${}^i\xi$ and t . The differential form of constraint (Equation 3.1-2) is known as Pfaffian form. It should be noted that a holonomic constraint in the form of Equation (3.1-1) can always be written in Pfaffian form by setting ${}^i\alpha = \frac{\partial\phi}{\partial{}^i\xi}$, $\alpha = \frac{\partial\phi}{\partial t}$; but the converse is not true.

Constraints may be further classified as catatastatic if $\alpha = 0$ and acatastatic if $\alpha \neq 0$. A constraint is scleronomic if it is independent of time, and rheonomic if it is explicitly dependent on time. Again there are discrepancies in the literature as to whether this classification can be applied to non-holonomic systems.

3.2 Interconnections

The position of any point in a substructure is defined by the variables of C_{β}^{α} , \underline{R} , and ${}_j^A \underline{q}$. Thus a non-holonomic (Pfaffian) constraint between the s -th and t -th substructure may be expressed in the form

$$\begin{aligned} & {}^s A_{\alpha\beta} {}^s C_{\gamma}^{\alpha} d {}^s C_{\gamma}^{\beta} + {}^t A_{\alpha\beta} {}^t C_{\gamma}^{\alpha} d {}^t C_{\gamma}^{\beta} + {}^s \underline{B} \cdot d {}^s \underline{R} + {}^t \underline{B} \cdot d {}^t \underline{R} + {}^s \Gamma_j d {}^s {}_j^A \underline{q} \\ & + {}^t \Gamma_j d {}^t {}_j^A \underline{q} + \Delta dt = 0 \end{aligned} \quad (3.2-1)$$

The variational form governing the structure is the sum of the forms for the substructures and terms governing the interconnections. The constraints may be incorporated via Lagrange multipliers. If there are "C" such constraints among the "S" substructures, then there are "C" subsidiary conditions of the form

$$\sum_{s=1}^S \left(c_s \underline{A} \cdot \underline{s}_{\Omega'} + c_s \underline{B} \cdot \frac{d}{dt} \underline{s}_R + c_s \underline{\Gamma} \frac{d}{dt} \underline{s}_j^{\Lambda} + c_{\Delta} = 0 \right) \quad (3.2-2)$$

where $c = 1, \dots, C$ and

$$\begin{aligned} c_s \underline{A} &= \left(\varepsilon^{\varepsilon\beta} \delta^{\delta} s_{C\gamma}^{\alpha} \left[s_{C\gamma}^{\alpha} c_s A_{\alpha\beta} \right] \right) \frac{\partial}{\partial s_{x'}^{\varepsilon}} \\ &= \varepsilon^{\gamma\beta\alpha} c_s A_{\alpha\beta} \frac{\partial}{\partial s_{x'}^{\gamma}} \end{aligned}$$

In addition, the term

$$-\int_{\bar{T}} * \left\{ \sum_{c=1}^C \sum_{s=1}^S c_{\lambda} \left(c_s A_{\alpha\beta} s_{C\gamma}^{\alpha} \delta^{\beta} s_{C\gamma}^{\beta} + c_s \underline{B} \cdot \delta \underline{s}_R + c_s \underline{\Gamma} \delta \underline{s}_j^{\Lambda} \right) \right\} \quad (3.2-3)$$

is incorporated in the variational form. This results in an additional term in the translational Equations (2.3-3) for each of the substructures

$$\begin{aligned} s_M \left\{ \frac{\partial^2}{\partial t^2} \underline{s}_R + \frac{\partial}{\partial t} \underline{s}_{\Omega'} \times \underline{s}_{0P'} + \frac{\partial}{\partial t} \underline{s}_{\Omega'} \times \underline{s}_{jP'} \frac{s_{\Lambda}}{j^q} + 2 \underline{s}_{\Omega'} \times \underline{s}_{jP'} \frac{s_{\Lambda}}{j^q} \right. \\ \left. + \underline{s}_{\Omega'} \times \left(\underline{s}_{\Omega'} \times \underline{s}_{0P'} \right) + \underline{s}_{\Omega'} \times \left(\underline{s}_{\Omega'} \times \underline{s}_{jP'} \right) \frac{s_{\Lambda}}{j^q} + \underline{s}_{jP'} \frac{s_{\Lambda}}{j^q} \right\} = \underline{s}_F + \sum_{c=1}^C c_{\lambda} c_s \underline{B} \end{aligned} \quad (3.2-4)$$

The substructure rotational Equations (2.3-4) also gain an additional term

$$\begin{aligned}
& \underline{s}_I \cdot \frac{\partial'}{\partial t} \underline{s}_\Omega' + \underline{s}_\Omega' \times \underline{s}_I \cdot \underline{s}_\Omega' + \frac{\partial'}{\partial t} \underline{s}_\Omega' \cdot \underline{s}_I \underline{s}_j^{\Lambda} + \underline{s}_I \cdot \frac{\partial'}{\partial t} \underline{s}_\Omega' \underline{s}_j^{\Lambda} \\
& + \underline{s}_\Omega' \cdot \underline{s}_I \underline{s}_j^{\dot{\Lambda}} + \underline{s}_I \cdot \underline{s}_\Omega' \underline{s}_j^{\dot{\Lambda}} + \underline{s}_\Omega' \times (\underline{s}_\Omega' \cdot \underline{s}_I) \underline{s}_j^{\Lambda} + \underline{s}_\Omega' \times \underline{s}_I \cdot \underline{s}_\Omega' \underline{s}_j^{\Lambda} \\
& + \underline{s}_M \underline{s}_{0P}' \times \frac{\partial^2}{\partial t^2} \underline{s}_R + \underline{s}_M \underline{s}_{jP}' \times \frac{\partial^2}{\partial t^2} \underline{s}_R \underline{s}_j^{\Lambda} + \underline{s}_0 \underline{s}_j^{\ddot{\Lambda}} + \underline{s}_\Omega' \times \underline{s}_0 \underline{s}_j^{\dot{\Lambda}} \\
& = \underline{s}_I + \sum_{c=1}^C c_\lambda \underline{cs}_A
\end{aligned} \tag{3.2-5}$$

Finally, the deformation Equations (2.3-5) acquire an additional term

$$\begin{aligned}
& - \underline{s}_\Omega' \cdot \underline{s}_I \cdot \underline{s}_\Omega' - \underline{s}_\Omega' \cdot \underline{s}_{lj} \underline{s}_\Omega' \underline{s}_l^{\Lambda} + \underline{s}_M \underline{s}_{jP}' \cdot \frac{\partial^2}{\partial t^2} \underline{s}_R + 2 \underline{s}_\Omega' \cdot \underline{s}_{lj} \underline{s}_l^{\dot{\Lambda}} \\
& + \underline{s}_0 \cdot \frac{\partial'}{\partial t} \underline{s}_\Omega' + \underline{s}_{j0} \cdot \frac{\partial'}{\partial t} \underline{s}_\Omega' \underline{s}_l^{\Lambda} + \underline{s}_{jL} \underline{s}_{lq}^{\dot{\Lambda}} + \underline{s}_{jL} \underline{s}_l^{\dot{\Lambda}} + \underline{s}_{jL} \underline{s}_l^{\Lambda} \\
& = \underline{s}_j + \sum_{c=1}^C c_\lambda \underline{cs}_j
\end{aligned} \tag{3.2-6}$$

In general, these equations or their counterparts in modal coordinates (i.e., with η 's instead of q 's as in 2.6) with Equation (2.2-13) after some action to eliminate the redundant rigid body modes, must be numerically integrated for the c_λ as well as the $\underline{s}_{C\beta}^\alpha$, \underline{s}_R , \underline{s}_Ω' , $\underline{s}_j^{\Lambda}$, since \underline{cs}_A , \underline{cs}_B , \underline{cs}_j are, in general, functions of $\underline{s}_{C\beta}^\alpha$, \underline{s}_R , $\underline{s}_j^{\Lambda}$, and time. If there are force deflection relationships, the forces become additional parameters and the relationships become additional equations.

The redundancy between the rigid body modes inherent in the \hat{q} 's and the frame variables permits the partitioning of each constraint into a nominal and a disturbance constraint. The nominal constraint is determined by setting all ${}_j \hat{q} = 0$ (i.e., the zeroth order term in ${}_j \hat{q}$) and the disturbance constraint is what remains (i.e., the first order terms in ${}_j \hat{q}$). Although doubling the number of constraints is not usually possible, the redundancy permits it here as long as the nominal

constraints are not contradictory (e.g., no rotation between two substructures about an axis and a prescribed rotation about a parallel axis is not permitted). This is equivalent to not allowing the interconnection to enforce deformation. This may be relaxed, if the enforced deformation is sufficiently small, by including the incompatible term in the disturbance constraint instead of the nominal constraint. If the \hat{q}_j are thought of as the primary variables, and the frame variables as redundant, and the disturbance constraints are thought of as primary, the redundant nominal constraints can do no more than eliminate some of the redundant frame variables. If the constraints are not contradictory, any excess of nominal constraints can be at most redundant.

The doubled set of constraints may be introduced via Lagrange multipliers, as before, except there are now twice as many, and their coefficients are either zeroth or first order in the \hat{q}_j . The analysis proceeds by eliminating constraint forces, excess coordinates, and redundant equations wherever possible, recomputing eigenvalues and vectors for any constant coefficient portions of the equation, and integrating, taking some action on any remaining redundancy between frame variables and rigid body modes. The following sections outline the procedure in more detail for a special class of constraints.

3.3 Rotational Constraints

The equations governing the behavior of a structure may be simplified for special classes of interconnections. The remainder of this study will illustrate the procedure for one such class. Each interconnection will allow only one translational and one rotational

degree of relative motion along a single axis passing through and connected to a single point, σ , on a substructure, s , and a single point, τ , on a second substructure, t . (Any substructure may have several such interconnections at several points.) Thus four degrees of freedom are completely constrained and the remaining degrees of freedom may be either unconstrained, a known function of time, or a small motion satisfying a known force-deflection relationship. The restriction to a translation and a rotation about a single axis (known as a screw displacement) is not as restrictive as it might appear, since trivial substructures (i.e., massless rigid bodies) may be introduced to represent more complex interconnections.

Two substructures, point connected with one degree of rotational freedom must have bases related by

$$\left[\frac{\partial}{\partial s_y^1}, \frac{\partial}{\partial s_y^2}, \frac{\partial}{\partial s_y^3} \right] = \left[\frac{\partial}{\partial t_y^1}, \frac{\partial}{\partial t_y^2}, \frac{\partial}{\partial t_y^3} \right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^{st}\theta & \sin^{st}\theta \\ 0 & -\sin^{st}\theta & \cos^{st}\theta \end{bmatrix} \quad (3.3-1)$$

whereby $\frac{\partial}{\partial s_y}$ is a dextral orthogonal basis in substructure s , and $\frac{\partial}{\partial t_y}$, in substructure t , with both $\frac{\partial}{\partial s_y^1}$ and $\frac{\partial}{\partial t_y^1}$ directed along the axis of rotation from s to t . " $^{st}\theta$ " is then the rotation of t relative to s in the sense indicated. Such a set of axes can always be defined in terms of the deformed body axes by a set of direction cosines depending only on the substructure geometry

$$\frac{\partial}{\partial s_d^i \alpha} = s g_{C \alpha}^{\beta} \frac{\partial}{\partial s_y^{\beta}} \quad (3.3-2)$$

(The image of the material event space axes under the position mapping provides such a set of coordinate axes which participate in the deformation.) The direction cosines between these axes, $\frac{\partial}{\partial s_x^{\prime\alpha}}$, and the coordinate axes of the substructure, $\frac{\partial}{\partial s_x^\alpha}$, are given by

$$\frac{\partial}{\partial s_x^{\prime\alpha}} = s_{C\alpha}^{\beta} \frac{\partial}{\partial s_x^{\prime\beta}} \quad (3.3-3)$$

where

$$s_{C\alpha}^{\beta} = \delta_{\beta}^{\alpha} + \frac{\partial s_{P'}^{\alpha}(\sigma)}{\partial \bar{x}^{\beta}} \frac{s^{\Lambda}}{j^q}$$

when the images of the material axes coincide with the physical axes in the reference state (otherwise there is a constant matrix which may be absorbed in the geometric, $sg_{C\alpha}^{\beta}$). And where the direction cosines have this simple form because the elongations and shears have been assumed small in comparison to one. Finally the substructure basis is related to the inertial basis by

$$\frac{\partial}{\partial x^{\alpha}} = s_{C\alpha}^{\beta} \frac{\partial}{\partial s_x^{\prime\beta}} \quad (3.3-4)$$

Combining these relationships provides

$$sg_{C\gamma}^{\beta} s_{C\delta}^{\gamma} s_{C\epsilon}^{\delta} = st_{C\beta}^{\alpha} tg_{C\zeta}^{\alpha} td_{C\eta}^{\zeta} t_{C\theta}^{\eta} \delta_{\epsilon}^{\theta} \quad (3.3-5)$$

or rearranging

$$st_{C\beta}^{\alpha} = sg_{C\gamma}^{\beta} s_{C\delta}^{\gamma} s_{C\epsilon}^{\delta} t_{C\epsilon}^{\zeta} td_{C\zeta}^{\eta} tg_{C\eta}^{\alpha} \quad (3.3-6)$$

Expanding the deformations gives

$$st_{C\beta}^{\alpha} = sg_{C\gamma}^{\beta} \left(\delta_{\gamma}^{\delta} + \frac{\partial s_{P'}^{\delta}(\sigma)}{\partial s_x^{\gamma}} \frac{s^{\Lambda}}{j^q} \right) s_{C\epsilon}^{\delta} t_{C\epsilon}^{\zeta} \left(\delta_{\eta}^{\zeta} + \frac{\partial t_{P'}^{\zeta}(\tau)}{\partial t_x^{\eta}} \frac{t^{\Lambda}}{j^q} \right) tg_{C\eta}^{\alpha} \quad (3.3-7)$$

The nominal constraint (i.e., with $\hat{q}_j = 0$) may be obtained from this as

$$st0_{C_\beta}^\alpha = sg_{C_\gamma}^\beta s_{C_\delta}^\gamma t_{C_\delta}^\epsilon tg_{C_\epsilon}^\alpha \quad (3.3-8)$$

This relationship may be used to determine either $t_{C_\delta}^\epsilon$ or $s_{C_\delta}^\gamma$ in terms of the other and 0_θ . The equation is not suitable for direct inclusion as a constraint because the direction cosines have only three independent terms (i.e., Equation (3.3-8) represents three constraints) but Equation (3.3-8) represents nine equations. The number of equations may be reduced to three by writing the Pfaffian form

$$-\frac{1}{2} \epsilon_{\gamma\alpha}^\delta st0_{C_\beta}^\gamma \frac{\partial}{\partial t} st0_{C_\beta}^\alpha. \quad (3.3-9)$$

Inserting Equation (3.3-1), it may be seen that the values of Equation (3.3-9) are $\delta = 1, 0_\theta^\circ$; $\delta = 2, 0$; $\delta = 3, 0$. Defining

$$st0_{\Omega}^{\prime} = st0_{\theta}^{\circ} \frac{\partial}{\partial t0_y^{\prime} 1} \quad (3.3-10)$$

where

$$\frac{\partial}{\partial t_x^{\prime} \alpha} = tg_{C_\alpha}^\beta \frac{\partial}{\partial t0_y^{\prime} \beta}$$

and inserting Equation (3.3-8) in (3.3-9) gives

$$st0_{\Omega}^{\prime \delta} = -\frac{1}{2} \epsilon_{\gamma\alpha}^\delta \left[sg_{C_\epsilon}^\beta s_{C_\zeta}^\epsilon t_{C_\zeta}^\eta tg_{C_\eta}^\gamma \right] \left[sg_{C_\theta}^\beta \frac{\partial}{\partial t} s_{C_1}^\theta t_{C_1}^K tg_{C_K}^\alpha + sg_{C_\theta}^\beta s_{C_1}^\theta \frac{\partial}{\partial t} t_{C_1}^K tg_{C_K}^\alpha \right] \quad (3.3-11)$$

Noting the identity, $C_\beta^\alpha C_\beta^\gamma = \delta^{\alpha\gamma}$, and regrouping terms gives

$$\begin{aligned}
st0_{\Omega, \delta} &= \left(s_{C\lambda}^{\mu} \frac{\partial}{\partial t} s_{C\lambda}^{\theta} \right) \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} tg_{C\kappa}^{\alpha} t_{C\mathbf{i}}^{\kappa} s_{C\mathbf{i}}^{\mu} s_{C\zeta}^{\theta} t_{C\zeta}^{\eta} tg_{C\eta}^{\gamma} \right) \\
&+ \left(t_{C\mathbf{i}}^{\eta} \frac{\partial}{\partial t} t_{C\mathbf{i}}^{\kappa} \right) \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} tg_{C\eta}^{\gamma} tg_{C\kappa}^{\alpha} \right) \\
&= \left(s_{C\lambda}^{\mu} \frac{\partial}{\partial t} s_{C\lambda}^{\theta} \right) s_{A\mu\theta}^{\delta} + \left(t_{C\mathbf{i}}^{\eta} \frac{\partial}{\partial t} t_{C\mathbf{i}}^{\kappa} \right) t_{A\eta\kappa}^{\delta} \tag{3.3-12}
\end{aligned}$$

which is in the form of Equation (3.2-1). The associated vector coefficients (3.2-2) may be calculated as

$$\begin{aligned}
s_{\underline{A}}^{\delta} &= \epsilon_{\mu\theta}^{\beta} s_{A\mu\theta}^{\delta} \frac{\partial}{\partial s_x^{\prime\beta}} \\
t_{\underline{A}}^{\delta} &= \epsilon_{\eta\kappa}^{\beta} t_{A\eta\kappa}^{\delta} \frac{\partial}{\partial t_x^{\prime\beta}} \tag{3.3-13}
\end{aligned}$$

Noting the identities in Equations (A.9-15 and -17) of Appendix A, and using Equation (3.3-8) these formulae become

$$\begin{aligned}
s_{\underline{A}}^{\delta} &= \epsilon_{\mu\theta}^{\beta} \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} tg_{C\kappa}^{\alpha} t_{C\mathbf{i}}^{\kappa} s_{C\mathbf{i}}^{\mu} s_{C\zeta}^{\theta} t_{C\zeta}^{\eta} tg_{C\eta}^{\gamma} \right) \frac{\partial}{\partial s_x^{\prime\beta}} \\
&= \epsilon_{\mu\theta}^{\beta} \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} sg_{C\mu}^{\epsilon} st0_{C\epsilon}^{\alpha} st0_{C\nu}^{\gamma} sg_{C\theta}^{\nu} \right) \frac{\partial}{\partial s_x^{\prime\beta}} \\
&= \epsilon_{\mu\theta}^{\beta} sg_{C\mu}^{\epsilon} sg_{C\theta}^{\nu} \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} st0_{C\epsilon}^{\alpha} st0_{C\nu}^{\gamma} \right) sg_{C\beta}^{\zeta} \frac{\partial}{\partial st0_y^{\prime\zeta}} \\
&= \epsilon_{\epsilon\nu}^{\zeta} st0_{C\epsilon}^{\alpha} st0_{C\nu}^{\gamma} \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \right) st0_{C\zeta}^{\eta} \frac{\partial}{\partial t0_y^{\prime\eta}} \\
&= \epsilon_{\alpha\gamma}^{\eta} \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \right) \frac{\partial}{\partial t0_y^{\prime\eta}} \\
&= \frac{\partial}{\partial t0_y^{\prime\delta}} \tag{3.3-14}
\end{aligned}$$

and

$$\begin{aligned}
t_{\underline{A}}^{\delta} &= \epsilon_{\eta\kappa}^{\beta} \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} t_{C\eta}^{\gamma} t_{C\kappa}^{\alpha} \right) \frac{\partial}{\partial t_x^{\gamma\beta}} \\
&= \epsilon_{\eta\kappa}^{\beta} t_{C\eta}^{\gamma} t_{C\kappa}^{\alpha} \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \right) t_{C\beta}^{\epsilon} \frac{\partial}{\partial t_y^{\epsilon\delta}} \\
&= \epsilon_{\gamma\alpha}^{\epsilon} \left(-\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \right) \frac{\partial}{\partial t_y^{\epsilon\delta}} \\
&= -\frac{\partial}{\partial t_y^{\epsilon\delta}}
\end{aligned} \tag{3.3-15}$$

The computation may be reversed

$$\begin{aligned}
\frac{1}{2} \epsilon_{\beta\delta\gamma}^{\alpha} A^{\beta\gamma} &= \frac{1}{2} \epsilon_{\beta\delta\gamma}^{\alpha} \epsilon_{\mu\theta}^{\beta} A_{\mu\theta} \\
&= \frac{1}{2} (\delta_{\delta\mu} \delta_{\gamma\theta} - \delta_{\delta\theta} \delta_{\gamma\mu}) A_{\mu\theta} \\
&= A_{\delta\gamma}
\end{aligned} \tag{3.3-16}$$

since the A are clearly skew-symmetric. Inserting this expression in Equation (3.3-12) provides

$$st_{\underline{\Omega}}^0{}^{\delta} = -s_{\underline{\Omega}} \cdot s_{\underline{A}}^{\delta} - t_{\underline{\Omega}} \cdot t_{\underline{A}}^{\delta} \tag{3.3-17}$$

since $\epsilon_{\mu\theta}^{\beta} s_{C\lambda}^{\mu} \frac{\partial}{\partial t} s_{C\lambda}^{\theta}$ and $s_{\underline{A}}^{\delta}{}^{\beta}$ are both the betath component of vectors expressed in the $\frac{\partial}{\partial t_x^{\delta}}$ basis. Noting Equations (3.3-14 and -15) and rearranging

$$\begin{aligned}
t_{\underline{\Omega}} \cdot \frac{\partial}{\partial t_y^{\epsilon\delta}} &= st_{\underline{\Omega}}^0 \cdot \frac{\partial}{\partial t_y^{\epsilon\delta}} + s_{\underline{\Omega}} \cdot \frac{\partial}{\partial t_y^{\epsilon\delta}} \\
t_{\underline{\Omega}} &= s_{\underline{\Omega}} + st_{\underline{\Omega}}^0
\end{aligned} \tag{3.3-18}$$

This provides either $t_{\underline{\Omega}}$ or $s_{\underline{\Omega}}$ in terms of the other and $st_{\underline{\Omega}}^0$ (much as Equation (3.3-8) did for direction cosines).

Returning to Equation (3.3-7), the disturbance constraint obtained from the Pfaffian form analogous to Equation (3.3-9) is (to linear order)

$$\begin{aligned}
& -\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \left(st_{C\beta}^{\gamma} \frac{\partial}{\partial t} st_{C\beta}^{\alpha} \right) + \frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \left(st_0^{\gamma} C_{\beta}^{\gamma} \frac{\partial}{\partial t} st_0^{\alpha} C_{\beta}^{\alpha} \right) \\
& = -\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \left\{ \left[sg_{C\epsilon}^{\beta} sd_{C\zeta}^{\epsilon} s_{C\eta}^{\zeta} t_{C\eta}^{\theta} td_{C\theta}^{\lambda} tg_{C\lambda}^{\gamma} \right] \right. \\
& \quad \left[sg_{C\kappa}^{\beta} \frac{\partial s_{C\lambda}^{\mu} (\sigma)^{\lambda}}{\partial s_{C\kappa}^{\mu}} \frac{s_{C\mu}^{\lambda}}{1} s_{C\mu}^{\lambda} t_{C\mu}^{\nu} td_{C\nu}^{\xi} tg_{C\xi}^{\alpha} \right. \\
& \quad + sg_{C\kappa}^{\beta} sd_{C\lambda}^{\kappa} \frac{\partial}{\partial t} s_{C\mu}^{\lambda} t_{C\mu}^{\nu} td_{C\nu}^{\xi} tg_{C\xi}^{\alpha} \\
& \quad + sg_{C\kappa}^{\beta} sd_{C\lambda}^{\kappa} s_{C\mu}^{\lambda} \frac{\partial}{\partial t} t_{C\mu}^{\nu} td_{C\nu}^{\xi} tg_{C\xi}^{\alpha} \\
& \quad \left. \left. + sg_{C\kappa}^{\beta} sd_{C\lambda}^{\kappa} s_{C\mu}^{\lambda} t_{C\mu}^{\nu} \frac{\partial t_{C\mu}^{\nu} (\tau)^{\nu}}{\partial t_{C\lambda}^{\mu}} \frac{t_{C\mu}^{\nu}}{j} tg_{C\xi}^{\alpha} \right] \right\} \\
& + \frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \left\{ \left[sg_{C\epsilon}^{\beta} \delta_{\epsilon}^{\zeta} s_{C\eta}^{\zeta} t_{C\eta}^{\theta} \delta_{\lambda}^{\theta} tg_{C\lambda}^{\gamma} \right] \right. \\
& \quad \left[sg_{C\kappa}^{\beta} \delta_{\kappa}^{\lambda} \frac{\partial}{\partial t} s_{C\mu}^{\lambda} t_{C\mu}^{\nu} \delta_{\xi}^{\nu} tg_{C\xi}^{\alpha} \right. \\
& \quad \left. \left. + sg_{C\kappa}^{\beta} \delta_{\kappa}^{\lambda} s_{C\mu}^{\lambda} \frac{\partial}{\partial t} t_{C\mu}^{\nu} \delta_{\xi}^{\nu} tg_{C\xi}^{\alpha} \right] \right\}. \tag{3.3-19}
\end{aligned}$$

Since the nominal constraints must be satisfied, any linear combination of products of the constraints with terms of linear or higher order in $\frac{s_{C\lambda}^{\mu}}{j}$ may be added to the disturbance constraint. In particular the second terms may be replaced by

$$\begin{aligned}
& \frac{1}{2} \epsilon_{\gamma\alpha}^{\rho} \left(st_0^{\gamma} C_{\beta}^{\gamma} \frac{\partial}{\partial t} st_0^{\alpha} C_{\beta}^{\alpha} \right) tg_{C_0}^{\rho} td_{C_0}^{\pi} tg_{C_{\pi}}^{\delta} \\
& = \frac{1}{2} \epsilon_{\gamma\alpha}^{\rho} \left\{ \left[sg_{C\epsilon}^{\beta} \delta_{\epsilon}^{\zeta} s_{C\eta}^{\zeta} t_{C\eta}^{\theta} \delta_{\lambda}^{\theta} tg_{C\lambda}^{\gamma} \right] \right. \\
& \quad \left[sg_{C\kappa}^{\beta} \delta_{\kappa}^{\lambda} \frac{\partial}{\partial t} s_{C\mu}^{\lambda} t_{C\mu}^{\nu} \delta_{\xi}^{\nu} tg_{C\xi}^{\alpha} \right. \\
& \quad \left. \left. + sg_{C\kappa}^{\beta} \delta_{\kappa}^{\lambda} s_{C\mu}^{\lambda} \frac{\partial}{\partial t} t_{C\mu}^{\nu} \delta_{\xi}^{\nu} tg_{C\xi}^{\alpha} \right] \right\} tg_{C_0}^{\rho} td_{C_0}^{\pi} tg_{C_{\pi}}^{\delta} \\
& = \frac{1}{2} \epsilon_{\theta\nu}^{\rho} \left\{ \left[s_{C\eta}^{\lambda} t_{C\eta}^{\theta} \right] \left[\frac{\partial}{\partial t} s_{C\mu}^{\lambda} t_{C\mu}^{\nu} + s_{C\mu}^{\lambda} \frac{\partial}{\partial t} t_{C\mu}^{\nu} \right] td_{C_0}^{\pi} tg_{C_{\pi}}^{\delta} \right\} \tag{3.3-20}
\end{aligned}$$

Using the reverse of the identities, this may be re-expanded to

$$\begin{aligned}
&= \frac{1}{2} \epsilon_{\gamma\alpha}^{\tau} \left\{ \left[sg_{C_{\epsilon}}^{\beta} sd_{C_{\zeta}}^{\epsilon} s_{C_{\eta}}^{\zeta} t_{C_{\eta}}^{\theta} td_{C_{\theta}}^{\iota} tg_{C_{\iota}}^{\gamma} \right] \right. \\
&\quad \left[sg_{C_{\kappa}}^{\beta} sd_{C_{\lambda}}^{\kappa} \frac{\partial}{\partial t} s_{C_{\mu}}^{\lambda} t_{C_{\mu}}^{\nu} td_{C_{\nu}}^{\xi} tg_{C_{\xi}}^{\alpha} \right. \\
&\quad \left. + sg_{C_{\kappa}}^{\beta} sd_{C_{\lambda}}^{\kappa} \frac{\partial}{\partial t} s_{C_{\mu}}^{\lambda} t_{C_{\mu}}^{\nu} td_{C_{\nu}}^{\xi} tg_{C_{\xi}}^{\alpha} \right] \\
&\quad \left. td_{C_{\sigma}}^{\sigma} tg_{C_{\sigma}}^{\tau} td_{C_{\pi}}^{\pi} tg_{C_{\pi}}^{\delta} \right\} \\
&= \frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \left\{ \left[sg_{C_{\epsilon}}^{\beta} sd_{C_{\zeta}}^{\epsilon} s_{C_{\eta}}^{\zeta} t_{C_{\eta}}^{\theta} td_{C_{\theta}}^{\iota} tg_{C_{\iota}}^{\gamma} \right] \right. \\
&\quad \left[sg_{C_{\kappa}}^{\beta} sd_{C_{\lambda}}^{\kappa} \frac{\partial}{\partial t} s_{C_{\mu}}^{\lambda} t_{C_{\mu}}^{\nu} td_{C_{\nu}}^{\xi} tg_{C_{\xi}}^{\alpha} \right. \\
&\quad \left. + sg_{C_{\kappa}}^{\beta} sd_{C_{\lambda}}^{\kappa} \frac{\partial}{\partial t} s_{C_{\mu}}^{\lambda} t_{C_{\mu}}^{\nu} td_{C_{\nu}}^{\xi} tg_{C_{\xi}}^{\alpha} \right] \left. \right\}. \tag{3.3-21}
\end{aligned}$$

This may be recognized as the negative of the middle part of the first term in Equation (3.3-19). Defining

$$st_{\underline{\Omega}}^{\prime} = st_{\theta}^{\prime} \frac{\partial}{\partial t_{y^1}} \tag{3.3-22}$$

Equation (3.3-19), as modified by Equation (3.3-20) may be written as

$$\begin{aligned}
&st_{\underline{\Omega}}^{\prime\delta} - st_{\underline{\Omega}}^{\prime\rho} tg_{C_{\rho}}^{\rho} td_{C_{\pi}}^{\pi} tg_{C_{\pi}}^{\delta} \\
&= -\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} \left[sg_{C_{\epsilon}}^{\beta} sd_{C_{\zeta}}^{\epsilon} s_{C_{\eta}}^{\zeta} t_{C_{\eta}}^{\theta} td_{C_{\theta}}^{\iota} tg_{C_{\iota}}^{\gamma} \right] \\
&\quad \left[sg_{C_{\kappa}}^{\beta} s_{i\psi(\sigma)}^{\lambda} s_{i^q}^{\lambda} s_{C_{\mu}}^{\lambda} t_{C_{\mu}}^{\nu} td_{C_{\nu}}^{\xi} tg_{C_{\xi}}^{\alpha} \right. \\
&\quad \left. + sg_{C_{\kappa}}^{\beta} sd_{C_{\lambda}}^{\kappa} s_{C_{\mu}}^{\lambda} t_{C_{\mu}}^{\nu} t_{j\psi(\tau)}^{\nu} t_{j^q}^{\lambda} tg_{C_{\xi}}^{\alpha} \right] \\
&= s_{i\Gamma}^{\delta} s_{i^q}^{\lambda} + t_{j\Gamma}^{\delta} t_{j^q}^{\lambda}. \tag{3.3-23}
\end{aligned}$$

where

$$s_i \psi(\sigma)^\alpha_\beta = \frac{\partial s_{P'}^\alpha(\sigma)}{\partial s_{-x}^\beta}$$

which is in the form of Equation (3.2-1). This may be expressed in vector form by multiplying by $\frac{\partial}{\partial t_y^\delta}$ and summing

$$\begin{aligned} & st_{\underline{\Omega}'} - st_0 \quad \Omega' \rho \quad tg_{C_o}^\rho \quad td_{C_o}^\pi \quad tg_{C_\pi}^\delta \quad \frac{\partial}{\partial t_y^\delta} \\ &= \underline{st}_\Omega' - st_0 \quad \underline{\Omega}' \\ &= -\frac{1}{2} \epsilon_{\gamma\alpha}^\delta \left[sg_{C_\epsilon}^\beta \quad sd_{C_\zeta}^\epsilon \quad s_{C_\eta}^\zeta \quad t_{C_\eta}^\theta \quad td_{C_\theta}^l \quad tg_{C_l}^\gamma \right] \\ &\quad \left[sg_{C_k}^\beta \quad s_i \psi(\sigma)^\lambda \quad s_{i^q}^\lambda \quad s_{C_\mu}^\lambda \quad t_{C_\mu}^\nu \quad td_{C_\nu}^\xi \quad tg_{C_\xi}^\alpha \right. \\ &\quad \left. + sg_{C_k}^\beta \quad sd_{C_\lambda}^\kappa \quad s_{C_\mu}^\lambda \quad t_{C_\mu}^\nu \quad t_{\psi(\tau)}^\nu \quad t_{\xi j^q}^\lambda \quad tg_{C_\xi}^\alpha \right] \frac{\partial}{\partial t_y^\delta} \\ &= sd_{C_\zeta}^\kappa \quad s_i \psi(\sigma)^\lambda \quad s_{i^q}^\lambda \left(-\frac{1}{2} \epsilon_{\zeta\lambda}^\sigma \right) \frac{\partial}{\partial s_{-x}^\sigma} \\ &\quad + td_{C_\nu}^l \quad t_{\psi(\tau)}^\nu \quad t_{j^q}^\lambda \left(-\frac{1}{2} \epsilon_{l\xi}^o \right) \frac{\partial}{\partial t_{-x}^o} \\ &= -\frac{1}{2} \epsilon_{\zeta\lambda}^\sigma \quad s_i \psi(\sigma)^\lambda \quad s_{i^q}^\lambda - \frac{1}{2} \epsilon_{l\xi}^o \quad t_{\psi(\tau)}^l \quad t_{j^q}^\lambda \left. \right\}. \end{aligned} \quad (3.3-24)$$

The product $C\psi_{i^q}^\lambda$ may be recognized as $\tilde{\Omega}$ for that direction cosine matrix. However, since the rotation is zero to zeroth order in q , the angular velocities are defined by

$$\begin{aligned} s_i \psi(\sigma)^\sigma &= +\frac{1}{2} \epsilon_{\zeta\lambda}^\sigma \quad s_i \psi(\sigma)^\lambda = +\frac{1}{2} \epsilon_{\zeta\lambda}^\sigma \left(\frac{\partial s_{P'}^\lambda(\sigma)}{\partial s_{-x}^\zeta} \right) \\ t_j \psi(\tau)^o &= -\frac{1}{2} \epsilon_{l\xi}^o \quad t_{\psi(\tau)}^l = -\frac{1}{2} \epsilon_{l\xi}^o \left(\frac{\partial t_{P(\tau)}^l}{\partial t_{-x}^\xi} \right); \end{aligned} \quad (3.3-25)$$

which are just the theory of elasticity definitions of the rotation due to deformation. It may be noted that these are known constants once the assumed deformation fields for the finite elements are established (in the case where nodal rotations are selected as generalized coordinates, and if the points of connection between substructures are nodes, this is just a Boolean array, one for the 1 = the rotation at the point, and otherwise zero.) Defining the vectors

$$\begin{aligned} s_{1\underline{\psi}}(\sigma) &= s_{1\underline{\psi}}(\sigma) \frac{\partial}{\partial s_{\underline{x}} \tau \sigma} \\ t_{j\underline{\psi}}(\tau) &= t_{j\underline{\psi}}(\tau) \frac{\partial}{\partial t_{\underline{x}} \tau \sigma} \end{aligned} \quad (3.3-26)$$

Equation (3.3-24) becomes

$$st_{\underline{\Omega}}' - st_{\underline{\Omega}}^0' = -s_{1\underline{\psi}}(\sigma) \dot{s}_{1\underline{q}}^{\Lambda} + t_{j\underline{\psi}}(\tau) \dot{t}_{j\underline{q}}^{\Lambda} \quad (3.3-27)$$

The terms Γ in Equations (3.3-23 and 3.2-1) may now be recognized as

$$\begin{aligned} s_{1\underline{\Gamma}}^{\delta} &= -s_{1\underline{\psi}}(\sigma) \cdot \frac{\partial}{\partial t_{\underline{y}} \delta} \\ t_{j\underline{\Gamma}}^{\delta} &= t_{j\underline{\psi}}(\tau) \cdot \frac{\partial}{\partial t_{\underline{y}} \delta} \end{aligned} \quad (3.3-28)$$

This disturbance constraint may be thought of as providing either some of the $\dot{t}_{j\underline{q}}^{\Lambda}$ or the $\dot{s}_{1\underline{q}}^{\Lambda}$ in terms of the other and $st_{\underline{\Omega}}'$ and $st_{\underline{\Omega}}^0'$.

In summary, a one axis rotational constraint is incorporated via Lagrange multipliers with constraint Equations (3.3-12 and -23). The additional Equations (3.3-18 and -27) are incorporated in the system of equations with the definition in Equations (3.3-10 and -22) implicit. The nominal rotation, st_{θ}^0 , is either a known function of time

(including zero) or unconstrained. In the latter case, the first line of the constraint equation may be thought of as introducing the new variable, ${}^{st0}\theta$. In this case the variations ${}^0\lambda^1\delta {}^{st0}\theta$, $-{}^1\lambda^1\delta {}^{st0}\theta$, and ${}^1\lambda^1\delta {}^{st0}\theta$, must be included. When ${}^{st0}\theta$ is unconstrained, ${}^{st}\theta$ may be selected in two ways. First, if it is desirable to include the total (nominal and deformation) rotation in the frame state variable, ${}^{st0}\theta$, set ${}^{st}\theta = {}^{st0}\theta$. In this case the first line of Equation (3.3-23) constrains the deformation rotations to be the same in each substructure. In this case, the variations $-{}^1\lambda^1\delta {}^{st0}\theta$ and ${}^1\lambda^1\delta {}^{st}\theta$ cancel, providing the stationary condition ${}^0\lambda^1 = 0$. Second, if the frame state variable, ${}^{st0}\theta$, only represents the nominal part of the rotation, ${}^{st}\theta$, is arbitrary. In this case the first line defines the new variable, ${}^{st}\theta$, and there are stationary conditions ${}^0\lambda^1 = {}^1\lambda^1$ and ${}^1\lambda^1 = 0$. The second method is preferable because the associated free boundary is consistent with the physical system, and hence modal convergence may be improved. (The first method does, however provide an opportunity for modal synthesis).

When the nominal rotation, ${}^{st0}\theta$, is a known function of time (including zero), the rotation due to deformation may be either known or arbitrary. A known function may result from several situations. If the nominal motion is a non-zero function of time then the deformation may be the correction for the deformed axis ${}^{st}\theta - {}^{st0}\theta$, or the sum of such a term and a small known motion not included in the nominal. If the nominal motion is zero, the deformation may result from a small known prescribed motion (possibly incompatible with other nominal

constraints) or may be zero. Such cases can be directly included. However, the case of arbitrary motion requires special treatment.

Any arbitrary disturbance motion (in the presence of a known nominal motion) must be small. To maintain smallness there must be, in general, a restoring force. If the force lies outside the system being treated (e.g., a controller), the disturbance rotation, ${}^{st1}\theta = {}^{st}\theta - {}^{st0}\theta$, can be introduced via the first line of the constraint and it may again be noted that ${}^1\lambda^1 = 0$. The restoring force is treated as an external force on the system. If, on the other hand, the restoring force mechanism is included in the system, there is a known force deflection relationship governing ${}^{st1}\theta$ and the Lagrange multiplier does not vanish. If $F = f({}^{st1}\theta)$ is the force on the mechanism then the term $F\delta{}^{st1}\theta$ must be included in the system variational expression to reflect the virtual work done on the mechanism. If $F\delta{}^{st1}\theta$ is a perfect variation, the form δV may be included instead (e.g., if $F = K{}^{st1}\theta$, $V = \frac{1}{2} K({}^{st1}\theta)^2$). In either case, the coefficient of the variation, $\delta{}^{st1}\theta$, is now ${}^1\lambda^1 + F$ with the stationary condition ${}^1\lambda^1 = -F$. The substitution $-f({}^{st1}\theta)$ may be made for ${}^1\lambda^1$ in the equations, or ${}^1\lambda^1$ may be retained as a variable and the equation ${}^1\lambda^1 = -f({}^{st1}\theta)$ included in the system of equations. (With $\frac{\partial}{\partial y^1}$ directed from s to t, ${}^1\lambda^1$ is a clockwise torque applied to t and ${}^{st1}\theta$ is a clockwise deformational rotation as viewed from s.)

3.4 Translational Constraints

This section continues the subject of single degree of freedom point constraints, introduced in the previous section. In this section, however, the single degree of freedom will be taken to be

translation along the axis $\frac{\partial}{\partial s_{y1}} = \frac{\partial}{\partial t_{y1}}$. If st_{ξ} is the magnitude of the translation, then defining

$$st_{\xi} = st_{\xi} \frac{\partial}{\partial t_{y1}} \quad (3.4-1)$$

The position of the connection point of t may be expressed in two ways

$$t_{\underline{R}} + t_{\underline{P}}'(\tau) = s_{\underline{R}} + s_{\underline{P}}'(\sigma) + st_{\xi} \quad (3.4-2)$$

or rearranging

$$st_{\xi} = t_{\underline{R}} + t_{\underline{P}}'(\tau) + t_{\underline{P}}'(\tau) \frac{t_{\underline{P}}^{\Lambda}}{j_{\underline{P}}^{\Lambda}} - s_{\underline{R}} - s_{\underline{P}}'(\sigma) - s_{\underline{P}}'(\sigma) \frac{s_{\underline{P}}^{\Lambda}}{i_{\underline{P}}^{\Lambda}} \quad (3.4-3)$$

The nominal part of this is

$$st_{\xi}^0 = t_{\underline{R}} + t_{\underline{P}}'(\tau) - s_{\underline{R}} - s_{\underline{P}}'(\sigma) \quad (3.4-4)$$

or rearranging

$$t_{\underline{R}} = st_{\xi}^0 + s_{\underline{R}} + s_{\underline{P}}'(\sigma) - t_{\underline{P}}'(\tau) \quad (3.4-5)$$

This relationship may be used to determine either $t_{\underline{R}}$ or $s_{\underline{R}}$ in terms of the other and st_{ξ}^0 . This is not in Pfaffian form. The Pfaffian form will now be obtained for inclusion in the system equations.

Taking the dot product of Equation (3.4-3) with $\frac{\partial}{\partial t_{y1}}$ gives

$$\begin{aligned} st_{\xi}^{\alpha} &= tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{C_{\delta}}^{\gamma} t_{\underline{R}}^{\delta} + tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{\underline{P}}'(\tau)^{\gamma} \\ &- st_{C_{\beta}}^{\alpha} sg_{C_{\gamma}}^{\beta} sd_{C_{\delta}}^{\gamma} s_{C_{\epsilon}}^{\delta} s_{\underline{R}}^{\epsilon} - st_{C_{\beta}}^{\alpha} sg_{C_{\gamma}}^{\beta} sd_{C_{\delta}}^{\gamma} s_{\underline{P}}'(\sigma)^{\delta} \\ &= tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{C_{\delta}}^{\gamma} t_{\underline{R}}^{\delta} + tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{\underline{P}}'(\tau)^{\gamma} \\ &- tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{C_{\delta}}^{\gamma} s_{\underline{R}}^{\delta} - tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{C_{\delta}}^{\gamma} s_{C_{\epsilon}}^{\delta} s_{\underline{P}}'(\sigma)^{\epsilon} \end{aligned} \quad (3.4-6)$$

where Equation (3.3-6) has been used.

Differentiating this gives

$$\begin{aligned}
\frac{\partial}{\partial t} st_{\xi}^{\alpha} &= tg_{C_{\beta}}^{\alpha} t_{\psi(\tau)}^{\gamma} t_{\beta}^{\dot{\lambda}} t_{C_{\delta}}^{\gamma} t_{R}^{\delta} + tg_{C_{\beta}}^{\alpha} t_{\psi(\tau)}^{\gamma} t_{\beta}^{\dot{\lambda}} t_{P'}^{\gamma}(\tau)^{\gamma} \\
&- tg_{C_{\beta}}^{\alpha} t_{\psi(\tau)}^{\gamma} t_{\beta}^{\dot{\lambda}} t_{C_{\delta}}^{\gamma} s_{R}^{\delta} - tg_{C_{\beta}}^{\alpha} t_{\psi(\tau)}^{\gamma} t_{\beta}^{\dot{\lambda}} t_{C_{\delta}}^{\gamma} s_{C_{\delta}}^{\epsilon} s_{P'}^{\epsilon}(\sigma)^{\epsilon} \\
&+ tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} \frac{\partial}{\partial t} t_{C_{\delta}}^{\gamma} t_{R}^{\delta} - tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} \frac{\partial}{\partial t} t_{C_{\delta}}^{\gamma} s_{R}^{\delta} \\
&- tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} \frac{\partial}{\partial t} t_{C_{\delta}}^{\gamma} s_{C_{\delta}}^{\epsilon} s_{P'}^{\epsilon}(\sigma)^{\epsilon} - tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{C_{\delta}}^{\gamma} \frac{\partial}{\partial t} s_{C_{\delta}}^{\epsilon} s_{P'}^{\epsilon}(\sigma)^{\epsilon} \\
&+ tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{C_{\delta}}^{\gamma} \frac{\partial}{\partial t} t_{R}^{\delta} - tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{C_{\delta}}^{\gamma} \frac{\partial}{\partial t} s_{R}^{\delta} \\
&+ tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{P'}^{\gamma}(\tau)^{\gamma} t_{\beta}^{\dot{\lambda}} - tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{C_{\delta}}^{\gamma} s_{C_{\delta}}^{\epsilon} s_{P'}^{\epsilon}(\sigma)^{\epsilon} t_{\beta}^{\dot{\lambda}}.
\end{aligned}$$

(3.4-7)

The nominal constraint may be determined from this, or by differentiating the dot product of Equation (3.4-4) with $\frac{\partial}{\partial t} t_{0}^{\alpha}$, α as

$$\begin{aligned}
\frac{\partial}{\partial t} st_{0}^{\alpha} &= tg_{C_{\beta}}^{\alpha} \frac{\partial}{\partial t} t_{C_{\gamma}}^{\beta} t_{C_{\gamma}}^{\delta} t_{C_{\epsilon}}^{\delta} t_{R}^{\epsilon} - tg_{C_{\beta}}^{\alpha} \frac{\partial}{\partial t} t_{C_{\gamma}}^{\beta} t_{C_{\gamma}}^{\delta} t_{C_{\epsilon}}^{\delta} s_{R}^{\epsilon} \\
&- tg_{C_{\beta}}^{\alpha} \frac{\partial}{\partial t} t_{C_{\gamma}}^{\beta} t_{C_{\gamma}}^{\delta} t_{C_{\epsilon}}^{\delta} s_{C_{\epsilon}}^{\zeta} s_{P'}^{\zeta}(\sigma)^{\zeta} - tg_{C_{\beta}}^{\alpha} t_{C_{\gamma}}^{\beta} s_{C_{\gamma}}^{\delta} s_{C_{\epsilon}}^{\delta} \frac{\partial}{\partial t} s_{C_{\epsilon}}^{\zeta} s_{P'}^{\zeta}(\sigma)^{\zeta} \\
&+ tg_{C_{\beta}}^{\alpha} t_{C_{\gamma}}^{\beta} \frac{\partial}{\partial t} t_{R}^{\gamma} - tg_{C_{\beta}}^{\alpha} t_{C_{\gamma}}^{\beta} \frac{\partial}{\partial t} s_{R}^{\gamma} \\
&= s_{A_{\delta\zeta}}^{\alpha} \left(s_{C_{\epsilon}}^{\delta} \frac{\partial}{\partial t} s_{C_{\epsilon}}^{\zeta} \right) + t_{A_{\delta\beta}}^{\alpha} \left(t_{C_{\gamma}}^{\delta} \frac{\partial}{\partial t} t_{C_{\gamma}}^{\beta} \right) \\
&+ s_{B_{\gamma}}^{\alpha} \left(\frac{\partial}{\partial t} s_{R}^{\gamma} \right) + t_{B_{\gamma}}^{\alpha} \left(\frac{\partial}{\partial t} t_{R}^{\gamma} \right)
\end{aligned}$$

(3.4-8)

This is in the form of Equation (3.2-1) the associated vector coefficients (3.2-2) may be calculated as

$$\begin{aligned}
s_{\underline{A}}^{\alpha} &= \varepsilon \eta^{\delta \zeta} s_{A \delta \zeta}^{\alpha} \frac{\partial}{\partial s_x^{\prime \eta}} \\
&= \varepsilon \eta^{\delta \zeta} \left(- t_{C \beta}^{\alpha} t_{C \gamma}^{\beta} s_{C \gamma}^{\delta} s_{0 P}^{\prime}(\sigma) \zeta \right) \frac{\partial}{\partial s_x^{\prime \zeta}} \\
&= - \left(t_{C \beta}^{\alpha} t_{C \gamma}^{\beta} s_{C \gamma}^{\delta} \frac{\partial}{\partial s_x^{\prime \delta}} \right) \times \left(s_{0 P}^{\prime}(\sigma) \zeta \frac{\partial}{\partial s_x^{\prime \zeta}} \right) \\
&= - \frac{\partial}{\partial t_{0 y}^{\prime \alpha}} \times s_{0 P}^{\prime}(\sigma)
\end{aligned} \tag{3.4-9}$$

and

$$\begin{aligned}
t_{\underline{A}}^{\alpha} &= \varepsilon \eta^{\delta \beta} t_{A \delta \beta}^{\alpha} \frac{\partial}{\partial t_x^{\prime \eta}} \\
&= \varepsilon \eta^{\delta \beta} \left(t_{C \beta}^{\alpha} t_{C \varepsilon}^{\delta} t_{R \varepsilon} - t_{C \beta}^{\alpha} t_{C \varepsilon}^{\delta} s_{R \varepsilon} \right. \\
&\quad \left. - t_{C \beta}^{\alpha} t_{C \varepsilon}^{\delta} s_{C \zeta}^{\zeta} s_{0 P}^{\prime}(\sigma) \zeta \right) \frac{\partial}{\partial t_x^{\prime \eta}} \\
&= - \left(t_{C \beta}^{\alpha} \frac{\partial}{\partial t_x^{\prime \beta}} \right) \times \left(t_{C \varepsilon}^{\delta} t_{R \varepsilon} \frac{\partial}{\partial t_x^{\prime \delta}} \right) + \left(t_{C \beta}^{\alpha} \frac{\partial}{\partial t_x^{\prime \beta}} \right) \times \left(t_{C \varepsilon}^{\delta} s_{R \varepsilon} \frac{\partial}{\partial t_x^{\prime \delta}} \right) \\
&\quad + \left(t_{C \beta}^{\alpha} \frac{\partial}{\partial t_x^{\prime \beta}} \right) \times \left(t_{C \varepsilon}^{\delta} s_{C \zeta}^{\zeta} s_{0 P}^{\prime}(\sigma) \zeta \frac{\partial}{\partial t_x^{\prime \delta}} \right) \\
&= - \frac{\partial}{\partial t_{0 y}^{\prime \alpha}} \times t_{\underline{R}} + \frac{\partial}{\partial t_{0 y}^{\prime \alpha}} \times s_{\underline{R}} + \frac{\partial}{\partial t_{0 y}^{\prime \alpha}} \times s_{0 P}^{\prime}(\sigma)
\end{aligned} \tag{3.4-10}$$

Similarly, the other coefficients are

$$\begin{aligned}
s_{\underline{B}} &= - t_{C \beta}^{\alpha} t_{C \gamma}^{\beta} \frac{\partial}{\partial x^{\gamma}} \\
&= - \frac{\partial}{\partial t_{0 y}^{\prime \alpha}}
\end{aligned} \tag{3.4-11}$$

and

$$\begin{aligned}
t_{\underline{B}} &= t_{C \beta}^{\alpha} t_{C \gamma}^{\beta} \frac{\partial}{\partial x^{\gamma}} \\
&= \frac{\partial}{\partial t_{0 y}^{\prime \alpha}}
\end{aligned} \tag{3.4-12}$$

The nominal constraint in vectorial form may be obtained by multiplying Equation (3.4-8) by $\frac{\partial}{\partial t^0_y} r_\alpha$

$$\begin{aligned}
\frac{\partial}{\partial t} st^0_{\xi\alpha} \frac{\partial}{\partial t^0_y} r_\alpha &= tg_{C\beta}^\alpha \frac{\partial}{\partial t} t_{C\gamma}^\beta t_{C\gamma}^\delta t_{C\epsilon}^\delta t_{R\epsilon} \frac{\partial}{\partial t^0_y} r_\alpha \\
&- tg_{C\beta}^\alpha \frac{\partial}{\partial t} t_{C\gamma}^\beta t_{C\gamma}^\delta t_{C\epsilon}^\delta s_{R\epsilon} \frac{\partial}{\partial t^0_y} r_\alpha \\
&- tg_{C\beta}^\alpha \frac{\partial}{\partial t} t_{C\gamma}^\beta t_{C\gamma}^\delta t_{C\epsilon}^\delta s_{C\epsilon}^\zeta s_{0P'}^\zeta(\sigma) \frac{\partial}{\partial t^0_y} r_\alpha \\
&- tg_{C\beta}^\alpha t_{C\gamma}^\beta s_{C\gamma}^\delta s_{C\epsilon}^\delta \frac{\partial}{\partial t} s_{C\epsilon}^\zeta s_{0P'}^\zeta(\sigma) \frac{\partial}{\partial t^0_y} r_\alpha \\
&+ tg_{C\beta}^\alpha t_{C\gamma}^\beta \frac{\partial}{\partial t} t_{R\gamma} \frac{\partial}{\partial t^0_y} r_\alpha - tg_{C\beta}^\alpha t_{C\gamma}^\beta \frac{\partial}{\partial t} s_{R\gamma} \frac{\partial}{\partial t^0_y} r_\alpha
\end{aligned}
\tag{3.4-13}$$

or

$$\begin{aligned}
st^0_{\xi} &= - t_{\Omega} \times t_{\underline{R}} + t_{\Omega} \times s_{\underline{R}} + t_{\Omega} \times s_{0P'}(\sigma) \\
&- s_{\Omega} \times s_{0P'}(\sigma) + \dot{t}_{\underline{R}} - \dot{s}_{\underline{R}}
\end{aligned}
\tag{3.4-14}$$

which may be recognized as the nominal part of Equation (3.4-2), differentiated in the $\frac{\partial}{\partial t^0_y}$ reference frame. This may be rearranged slightly to

$$\dot{t}_{\underline{R}} = t_{\Omega} \times t_{\underline{R}} - t_{\Omega} \times s_{\underline{R}} - t_{\Omega} \times s_{0P'}(\sigma) + \dot{s}_{\underline{R}} + s_{\Omega} \times s_{0P'}(\sigma) + st^0_{\xi}
\tag{3.4-15}$$

Noting Equations (3.3-18) and (3.4-5), this may also be written as

$$\begin{aligned}
\dot{t}_{\underline{R}} &= \left(s_{\underline{\Omega}} + st0_{\underline{\Omega}}' \right) \times \left(st0_{\underline{\xi}} + s_{\underline{R}} + s_{0\underline{P}}'(\sigma) - t_{0\underline{P}}'(\tau) - s_{\underline{R}} - s_{0\underline{P}}'(\sigma) \right) \\
&\quad + s_{\underline{R}} + s_{\underline{\Omega}} \times s_{0\underline{P}}'(\sigma) + st0_{\underline{\xi}} \\
&= s_{\underline{\Omega}} \times st0_{\underline{\xi}} - s_{\underline{\Omega}} \times t_{0\underline{P}}'(\tau) + s_{\underline{\Omega}} \times s_{0\underline{P}}'(\sigma) + s_{\underline{R}} + st0_{\underline{\Omega}}' \times st0_{\underline{\xi}} \\
&\quad - st0_{\underline{\Omega}}' \times t_{0\underline{P}}'(\tau) + st0_{\underline{\xi}} \quad (3.4-16)
\end{aligned}$$

This may be thought of as providing either $\dot{t}_{\underline{R}}$ or $s_{\underline{R}}$ in terms of the other and $st0_{\underline{\xi}}$.

Returning to Equation (3.4-7), the disturbance constraint may be obtained by first multiplying Equation (3.4-8) by $tg_{C_{\theta}}^1 td_{C_{\eta}}^{\theta} tg_{C_{\eta}}^{\alpha}$.

$$\begin{aligned}
&tg_{C_{\theta}}^1 td_{C_{\eta}}^{\theta} tg_{C_{\eta}}^{\alpha} st0_{\underline{\xi}}^{\alpha} \\
&= tg_{C_{\theta}}^1 td_{C_{\eta}}^{\theta} \frac{\partial}{\partial t} t_{C_{\gamma}}^{\eta} t_{C_{\gamma}}^{\delta} t_{C_{\epsilon}}^{\delta} t_{R}^{\epsilon} \\
&\quad - tg_{C_{\theta}}^1 td_{C_{\eta}}^{\theta} \frac{\partial}{\partial t} t_{C_{\gamma}}^{\eta} t_{C_{\gamma}}^{\delta} t_{C_{\epsilon}}^{\delta} s_{R}^{\epsilon} \\
&\quad - tg_{C_{\theta}}^1 td_{C_{\eta}}^{\theta} \frac{\partial}{\partial t} t_{C_{\gamma}}^{\eta} t_{C_{\gamma}}^{\delta} t_{C_{\epsilon}}^{\delta} s_{C_{\epsilon}}^{\zeta} s_{0\underline{P}}'(\sigma)^{\zeta} \\
&\quad - tg_{C_{\theta}}^1 td_{C_{\eta}}^{\theta} t_{C_{\gamma}}^{\eta} s_{C_{\gamma}}^{\delta} s_{C_{\epsilon}}^{\delta} \frac{\partial}{\partial t} s_{C_{\epsilon}}^{\zeta} s_{0\underline{P}}'(\sigma)^{\zeta} \\
&\quad + tg_{C_{\theta}}^1 td_{C_{\eta}}^{\theta} t_{C_{\gamma}}^{\eta} \frac{\partial}{\partial t} t_{R}^{\gamma} - tg_{C_{\theta}}^1 td_{C_{\eta}}^{\theta} t_{C_{\gamma}}^{\eta} \frac{\partial}{\partial t} s_{R}^{\gamma} \quad (3.4-17)
\end{aligned}$$

The first, second, fifth and sixth terms of (3.4-17) are the same as the fifth, sixth, ninth and tenth terms of Equation (3.4-7), so subtracting (3.4-17) from (3.4-7) provides the disturbance constraint.

$$\begin{aligned}
\frac{\partial}{\partial t} st_{\xi}^{\alpha} &= tg_{C\beta}^{\alpha} td_{C\gamma}^{\beta} tg_{C\delta}^{\gamma} \frac{\partial}{\partial t} st0_{\xi}^{\delta} \\
&= tg_{C\beta}^{\alpha} t_{j\psi(\tau)}^{\gamma} t_{\beta j^q}^{\dot{\lambda}} t_{C\delta}^{\gamma} t_R^{\delta} + tg_{C\beta}^{\alpha} t_{j\psi(\tau)}^{\gamma} t_{\beta j^q}^{\dot{\lambda}} t_{P'}^{\gamma}(\tau)^{\gamma} \\
&- tg_{C\beta}^{\alpha} t_{j\psi(\tau)}^{\gamma} t_{\beta j^q}^{\dot{\lambda}} t_{C\delta}^{\gamma} s_R^{\delta} - tg_{C\beta}^{\alpha} t_{j\psi(\tau)}^{\gamma} t_{\beta j^q}^{\dot{\lambda}} t_{C\delta}^{\gamma} s_{C\delta}^{\epsilon} s_{P'}^{\epsilon}(\sigma)^{\epsilon} \\
&- tg_{C\beta}^{\alpha} td_{C\gamma}^{\beta} \frac{\partial}{\partial t} t_{C\delta}^{\gamma} t_{C\delta}^{\epsilon} t_{C\zeta}^{\epsilon} s_{C\zeta}^{\eta} s_{P'}^{\eta}(\sigma)^{\eta} s_{i^q}^{\lambda} \\
&- tg_{C\beta}^{\alpha} td_{C\gamma}^{\beta} t_{C\delta}^{\gamma} s_{C\delta}^{\epsilon} s_{C\zeta}^{\epsilon} \frac{\partial}{\partial t} s_{C\zeta}^{\eta} s_{P'}^{\eta}(\sigma)^{\eta} s_{i^q}^{\lambda} \\
&+ tg_{C\beta}^{\alpha} td_{C\gamma}^{\beta} t_{jP'}^{\gamma}(\tau)^{\gamma} t_{j^q}^{\dot{\lambda}} - tg_{C\beta}^{\alpha} td_{C\gamma}^{\beta} t_{C\delta}^{\gamma} s_{C\delta}^{\epsilon} s_{P'}^{\epsilon}(\sigma)^{\epsilon} s_{i^q}^{\lambda} \\
&= s_{A_{\epsilon\eta}}^{\alpha} \left(s_{C\zeta}^{\epsilon} \frac{\partial}{\partial t} s_{C\zeta}^{\eta} \right) + t_{A_{\epsilon\gamma}}^{\alpha} \left(t_{C\delta}^{\epsilon} \frac{\partial}{\partial t} t_{C\delta}^{\gamma} \right) \\
&+ s_{i^q}^{\alpha} \Gamma \left(s_{i^q}^{\dot{\lambda}} \right) + t_{j^q}^{\alpha} \Gamma \left(t_{j^q}^{\dot{\lambda}} \right) \tag{3.4-18}
\end{aligned}$$

This is in the form of Equation (3.2-1). The associated vector coefficients (3.2-2) may be calculated as

$$\begin{aligned}
s_{\underline{A}}^{\alpha} &= \epsilon^{\theta\epsilon\eta} s_{A_{\epsilon\eta}}^{\alpha} \frac{\partial}{\partial s_x^{\theta}} \\
&= \epsilon^{\theta\epsilon\eta} \left(-tg_{C\beta}^{\alpha} td_{C\gamma}^{\beta} t_{C\delta}^{\gamma} s_{C\delta}^{\epsilon} s_{P'}^{\epsilon}(\sigma)^{\eta} s_{i^q}^{\lambda} \right) \frac{\partial}{\partial s_x^{\theta}} \\
&= - \left(tg_{C\beta}^{\alpha} td_{C\gamma}^{\beta} t_{C\delta}^{\gamma} s_{C\delta}^{\epsilon} \frac{\partial}{\partial s_x^{\epsilon}} \right) \times \left(s_{P'}^{\epsilon}(\sigma)^{\eta} s_{i^q}^{\lambda} \frac{\partial}{\partial s_x^{\eta}} \right) \\
&= - \frac{\partial}{\partial t_y^{\alpha}} \times s_{i^q}^{\epsilon}(\sigma)^{\eta} s_{i^q}^{\lambda} \tag{3.4-19}
\end{aligned}$$

and

$$\begin{aligned}
t_{\underline{A}}^{\alpha} &= \varepsilon^{\theta \varepsilon \gamma} t_{\underline{A} \varepsilon \gamma} \frac{\partial}{\partial t_x^{\theta}} \\
&= \varepsilon^{\theta \varepsilon \gamma} \left(-t_{\underline{C} \beta}^{\alpha} t_{\underline{C} \gamma}^{\beta} t_{\underline{C} \zeta}^{\varepsilon} s_{\underline{C} \zeta}^{\eta} s_{\underline{I} \eta}^{\prime}(\sigma) \right) \frac{\partial}{\partial t_x^{\theta}} \\
&= \left(t_{\underline{C} \beta}^{\alpha} t_{\underline{C} \gamma}^{\beta} \frac{\partial}{\partial t_x^{\gamma}} \right) \times \left(t_{\underline{C} \zeta}^{\varepsilon} s_{\underline{C} \zeta}^{\eta} s_{\underline{I} \eta}^{\prime}(\sigma) \right) \frac{\partial}{\partial t_x^{\varepsilon}} \\
&= \frac{\partial}{\partial t_y^{\alpha}} \times s_{\underline{I} \eta}^{\prime}(\sigma) s_{\underline{I} \eta}^{\Lambda}
\end{aligned} \tag{3.4-20}$$

Similarly the other coefficients are

$$\begin{aligned}
s_{\underline{I} \eta}^{\alpha} &= -t_{\underline{C} \beta}^{\alpha} t_{\underline{C} \gamma}^{\beta} t_{\underline{C} \delta}^{\gamma} s_{\underline{C} \delta}^{\varepsilon} s_{\underline{I} \eta}^{\prime}(\sigma) \varepsilon \\
&= -\frac{\partial}{\partial t_y^{\alpha}} \cdot s_{\underline{I} \eta}^{\prime}(\sigma)
\end{aligned} \tag{3.4-21}$$

and

$$\begin{aligned}
t_{\underline{j} \eta}^{\alpha} &= t_{\underline{C} \beta}^{\alpha} t_{\underline{j} \psi(\tau)}^{\beta} t_{\underline{C} \delta}^{\gamma} t_{\underline{R} \delta}^{\varepsilon} + t_{\underline{C} \beta}^{\alpha} t_{\underline{j} \psi(\tau)}^{\beta} t_{\underline{P} \eta}^{\prime}(\tau) \gamma \\
&\quad - t_{\underline{C} \beta}^{\alpha} t_{\underline{j} \psi(\tau)}^{\beta} t_{\underline{C} \delta}^{\gamma} s_{\underline{R} \delta}^{\varepsilon} - t_{\underline{C} \beta}^{\alpha} t_{\underline{j} \psi(\tau)}^{\beta} t_{\underline{C} \delta}^{\gamma} s_{\underline{C} \delta}^{\varepsilon} s_{\underline{P} \eta}^{\prime}(\sigma) \varepsilon \\
&\quad + t_{\underline{C} \beta}^{\alpha} t_{\underline{C} \gamma}^{\beta} t_{\underline{j} \eta}^{\prime}(\tau) \gamma \\
&= \left[t_{\underline{j} \psi(\tau)}^{\alpha} \times \left(t_{\underline{R} \eta} - t_{\underline{P} \eta}^{\prime}(\tau) + s_{\underline{R} \eta} + s_{\underline{P} \eta}^{\prime}(\sigma) \right) + t_{\underline{j} \eta}^{\prime}(\tau) \right] \cdot \frac{\partial}{\partial t_y^{\alpha}}
\end{aligned} \tag{3.4-22}$$

This can be expressed in vectorial form by multiplying by $\frac{\partial}{\partial t_y^{\alpha}}$

$$\begin{aligned}
st_{\underline{I} \eta}^{\alpha} - st_{\underline{0} \eta}^{\alpha} &= -t_{\underline{j} \psi(\tau)}^{\alpha} t_{\underline{j} \eta}^{\Lambda} \times t_{\underline{R} \eta} - t_{\underline{j} \psi(\tau)}^{\alpha} t_{\underline{j} \eta}^{\Lambda} \times t_{\underline{0} \eta}^{\prime}(\tau) + t_{\underline{j} \psi(\tau)}^{\alpha} t_{\underline{j} \eta}^{\Lambda} \times s_{\underline{R} \eta} \\
&\quad + t_{\underline{j} \psi(\tau)}^{\alpha} t_{\underline{j} \eta}^{\Lambda} \times s_{\underline{0} \eta}^{\prime}(\sigma) + t_{\underline{0} \eta}^{\alpha} \times s_{\underline{I} \eta}^{\prime}(\sigma) s_{\underline{I} \eta}^{\Lambda} - s_{\underline{0} \eta}^{\alpha} \times s_{\underline{I} \eta}^{\prime}(\sigma) s_{\underline{I} \eta}^{\Lambda} \\
&\quad + t_{\underline{j} \eta}^{\prime}(\tau) t_{\underline{j} \eta}^{\Lambda} - s_{\underline{I} \eta}^{\prime}(\sigma) s_{\underline{I} \eta}^{\Lambda}
\end{aligned} \tag{3.4.23}$$

where it has been noted that $s_i^{\psi(\sigma)} \dot{\gamma}_\beta$ is skew-symmetric for small deformations. This result may also be obtained by differentiating Equation (3.4-3) in the $\frac{\partial}{\partial t_y}$ reference frame.

$$\begin{aligned}
\frac{\partial t_y'}{\partial t} st_{\underline{\xi}} &= t_{\underline{R}} \dot{\cdot} - t_{\underline{\Omega}} \times t_{\underline{R}} - t_{j\underline{\psi}(\tau)} t_{j\underline{q}} \dot{\cdot} \times t_{\underline{R}} - t_{j\underline{\psi}(\tau)} t_{j\underline{q}} \dot{\cdot} \times t_{0\underline{P}'}(\tau) \\
&+ t_{j\underline{P}'}(\tau) t_{j\underline{q}} \dot{\cdot} - s_{\underline{R}} \dot{\cdot} + t_{\underline{\Omega}} \times s_{\underline{R}} + t_{j\underline{\psi}(\tau)} t_{j\underline{q}} \dot{\cdot} \times s_{\underline{R}} - s_{\underline{\Omega}} \times s_{0\underline{P}'}(\sigma) \\
&+ t_{\underline{\Omega}} \times s_{0\underline{P}'}(\sigma) + t_{j\underline{\psi}(\tau)} t_{j\underline{q}} \dot{\cdot} \times s_{0\underline{P}'}(\sigma) - s_{i\underline{P}'}(\sigma) s_{i\underline{q}} \dot{\cdot} - s_{\underline{\Omega}} \\
&\times s_{i\underline{P}'}(\sigma) s_{i\underline{q}} \dot{\cdot} + t_{\underline{\Omega}} \times s_{i\underline{P}'}(\sigma) s_{i\underline{q}} \dot{\cdot}
\end{aligned} \tag{3.4-24}$$

and subtracting Equation (3.4-14). Noting Equations (3.3-18, -27 and 3.4-5) , Equation (3.4-23) may be written as

$$\begin{aligned}
st_{\underline{\xi}} \dot{\cdot} - st_{0\underline{\xi}} \dot{\cdot} &= - t_{j\underline{\psi}(\tau)} t_{j\underline{q}} \dot{\cdot} \times \left(st_{0\underline{\xi}} + s_{\underline{R}} + s_{0\underline{P}'}(\sigma) - t_{0\underline{P}'}(\tau) \right) - t_{j\underline{\psi}(\tau)} t_{j\underline{q}} \dot{\cdot} \times t_{0\underline{P}'}(\tau) \\
&+ t_{j\underline{\psi}(\tau)} t_{j\underline{q}} \dot{\cdot} \times s_{\underline{R}} + t_{j\underline{\psi}(\tau)} t_{j\underline{q}} \dot{\cdot} \times s_{0\underline{P}'}(\sigma) + \left(s_{\underline{\Omega}} + st_{0\underline{\Omega}} \dot{\cdot} \right) \\
&\times s_{i\underline{P}'}(\sigma) s_{i\underline{q}} \dot{\cdot} - s_{\underline{\Omega}} \times s_{i\underline{P}'}(\sigma) s_{i\underline{q}} \dot{\cdot} + t_{j\underline{P}'}(\tau) t_{j\underline{q}} \dot{\cdot} - s_{i\underline{P}'}(\sigma) s_{i\underline{q}} \dot{\cdot} \\
&= - t_{j\underline{\psi}(\tau)} t_{j\underline{q}} \dot{\cdot} \times st_{0\underline{\xi}} + st_{0\underline{\Omega}} \dot{\cdot} \times s_{i\underline{P}'}(\sigma) s_{i\underline{q}} \dot{\cdot} + t_{j\underline{P}'}(\tau) t_{j\underline{q}} \dot{\cdot} \\
&- s_{i\underline{P}'}(\sigma) s_{i\underline{q}} \dot{\cdot}
\end{aligned} \tag{3.4-25}$$

The terms in this expression may be identified by noting that the identity

$$st_{\underline{\xi}} = st_{0\underline{\xi}} - s_{i\underline{P}'}(\sigma) s_{i\underline{q}} \dot{\cdot} + t_{j\underline{P}'}(\tau) t_{j\underline{q}} \dot{\cdot} \tag{3.4-26}$$

differentiated in the $\frac{\partial}{\partial t_y}$ reference frame gives

$$\begin{aligned} \underline{st}_{\xi}^{\dot{\xi}} = & \left(\underline{st0}_{\xi}^{\dot{\xi}} - \underline{t}_{j\underline{q}}(\tau) \underline{t}_{jq}^{\dot{\lambda}} \times \underline{st0}_{\xi}^{\dot{\xi}} \right) - \left(\underline{s}_{i\underline{p}}'(\sigma) \underline{s}_{iq}^{\dot{\lambda}} - \underline{st0}_{\Omega}' \times \underline{s}_{i\underline{p}}'(\sigma) \underline{s}_{iq}^{\dot{\lambda}} \right) \\ & + \underline{t}_{j\underline{p}}'(\tau) \underline{t}_{jq}^{\dot{\lambda}} \end{aligned} \quad (3.4-27)$$

This disturbance constraint may be thought of as providing some of the $\underline{t}_{jq}^{\dot{\lambda}}$ in terms of the $\underline{s}_{iq}^{\dot{\lambda}}$ and $\underline{st}_{\xi}^{\dot{\xi}}$.

In summary, a single axis translational constraint is incorporated via Lagrange multipliers with constraint Equations (3.4-8 and -18). The additional Equations (3.4-5 and -26 or -25) are incorporated in the system of equations. The nominal translation, $\underline{st0}_{\xi}^{\dot{\xi}}$ is either a known function of time (including zero) or unconstrained. In the latter case, the first line of the constraint equation may be thought of as introducing the new variable $\underline{st0}_{\xi}^{\dot{\xi}}$. In such cases, the variations ${}^0_{\lambda}1_{\delta} \underline{st0}_{\xi}^{\dot{\xi}}$, ${}^{-1}_{\lambda}1_{\delta} \underline{st0}_{\xi}^{\dot{\xi}}$ and ${}^1_{\lambda}1_{\delta} \underline{st}_{\xi}^{\dot{\xi}}$ must also be included in the variational form. When $\underline{st0}_{\xi}^{\dot{\xi}}$ is unconstrained, there are two ways to select $\underline{st}_{\xi}^{\dot{\xi}}$. First, if it is desirable to include the total (nominal and deformational) translation in the frame state variable $\underline{st0}_{\xi}^{\dot{\xi}}$, set $\underline{st}_{\xi}^{\dot{\xi}} = \underline{st0}_{\xi}^{\dot{\xi}}$. In this case, the first line of Equation (3.4-18) constrains the deformational translations to be the same in each substructure. The variations ${}^{-1}_{\lambda}1_{\delta} \underline{st0}_{\xi}^{\dot{\xi}}$ and ${}^1_{\lambda}1_{\delta} \underline{st}_{\xi}^{\dot{\xi}}$ cancel giving the stationary condition ${}^0_{\lambda}1 = 0$. The second option is to let the frame state variable, $\underline{st0}_{\xi}^{\dot{\xi}}$, represent only the nominal part of the translation by making $\underline{st}_{\xi}^{\dot{\xi}}$ arbitrary. In this case the first line defines the new variable, $\underline{st}_{\xi}^{\dot{\xi}}$. There are stationary conditions ${}^1_{\lambda}1 = 0$, and ${}^0_{\lambda}1 = {}^1_{\lambda}1$ so that ${}^0_{\lambda}1 = 0$ also. The second method is preferable because the associated free boundary is consistent with the physical system, and hence modal

convergence may be improved. (The first method, however, does provide an opportunity for modal synthesis.)

When the nominal translation, st^0_ξ , is a known function of time (including zero), the translation due to deformation may be either known or arbitrary. A known function may result from several situations. If the nominal motion is a non-zero function of time, then the deformation may be the correction for the deformed axis, $st_\xi - st^0_\xi$, or the sum of such a term and a small known motion not included in the nominal. If the nominal motion is zero, the deformation may either result from a small known prescribed motion (possibly incompatible with other nominal constraints) or be zero. Such cases can be directly included. However, the case of arbitrary motion requires special treatment.

Any arbitrary disturbance motion (in the presence of a known nominal motion) must be small. To maintain smallness, there must, in general, be a restoring force. If the force lies outside the system being treated (e.g., a controller), the disturbance translation, $st^1_\xi = st_\xi - st^0_\xi$ can be introduced via the first line of the constraint; and it may again be noted that ${}^1_\lambda{}^1 = 0$. The restoring force would be treated as an external force on the system. If, on the other hand, the restoring force mechanism is included in the system, there is a known force deflection relationship governing st^1_ξ and the Lagrange multiplier does not vanish. If $F = f(st^1_\xi)$ is the force on the mechanism, then the term $F\delta^{st^1}_\xi$ must be included in the system variational expression to reflect the virtual work done in the mechanism. If $F\delta^{st^1}_\xi$ is a perfect variation, the form δV may be included instead (e.g., if $F = K^{st^1}_\xi$, $V = \frac{1}{2} K (st^1_\xi)^2$). In either case the

coefficient of the variation, $\delta^{stl}\xi$, becomes ${}^1\lambda^1 + F$ with the stationary condition ${}^1\lambda^1 = -F$. The substitution $-f({}^{stl}\xi)$ may be made for ${}^1\lambda^1$ in the equations, or ${}^1\lambda^1$ may be retained as a variable, and the equation ${}^1\lambda^1 = -f({}^{stl}\xi)$ included in the system of equations. (With $\frac{\partial}{\partial y^1}$ directed from s to t, ${}^1\lambda^1$ is a positive force on t, and ${}^{stl}\xi$ is a positive deformation displacement of t away from s.)

3.5 Elimination of Constraint Equations and Forces

Thus far, constraints between substructures have been incorporated via Lagrange multipliers. They may be interpreted as forces (and torques) on the substructures, due to the constraints. The number of system equations has been augmented by the C constraint equations, and the number of variables has been augmented by the C Lagrange multipliers. In some cases, it is desirable to retain the λ as explicit variables. This practice may even be computationally more efficient. However, if the integration forces are not of interest, there is frequently a computational advantage in reducing the dimension of the system equations by eliminating them (and the associated equations). Even if the forces are desired, it may still be advantageous to eliminate them and subsequently recover them.

The nominal translational constraint (3.4-2) results in constraint forces (see Equation (3.2-4)) on the s substructure (3.4-11)

$$- {}^0\lambda^1 \frac{\partial}{\partial t^0 y^1} - {}^0\lambda^2 \frac{\partial}{\partial t^0 y^2} - {}^0\lambda^3 \frac{\partial}{\partial t^0 y^3} \quad (3.5-1)$$

and on the t substructure (3.4-12)

$$+ {}^0\lambda^1 \frac{\partial}{\partial t^0 y^1} + {}^0\lambda^2 \frac{\partial}{\partial t^0 y^2} + {}^0\lambda^3 \frac{\partial}{\partial t^0 y^3} \quad (3.5-2)$$

and, in addition in constraint torques (see Equation (3.2-5)) on the s substructure (3.4-9).

$$- {}^0\lambda^1 \frac{\partial}{\partial t^0 y^1} \times \underline{s}_P'(\sigma) - {}^0\lambda^2 \frac{\partial}{\partial t^0 y^2} \times \underline{s}_P'(\sigma) - {}^0\lambda^3 \frac{\partial}{\partial t^0 y^3} \times \underline{s}_P'(\sigma) \quad (3.5-3)$$

and on the t substructure (3.4-10)

$$\left({}^0\lambda^1 \frac{\partial}{\partial t^0 y^1} + {}^0\lambda^2 \frac{\partial}{\partial t^0 y^2} + {}^0\lambda^3 \frac{\partial}{\partial t^0 y^3} \right) \times \left(\underline{s}_R + \underline{s}_P'(\sigma) - \underline{t}_R \right) \quad (3.5-4)$$

The torques can be eliminated from the rotational equations by subtracting the cross product of the s translational equation with $\underline{s}_P'(\sigma)$ from the s rotational equation. And, similarly, subtracting the cross product of the t translational equation with $\underline{s}_R + \underline{s}_P'(\sigma) - \underline{t}_R$ from the t rotational equation. The resulting equations replace the rotational equations, but the translational equations are retained for no net change in the number of equations. (This effectively translates the origins to the nominal attachment point on the s substructure.)

The constraint forces can be eliminated from the translational equations by summing the equations for the two substructures. The sum replaces the individual equations resulting in a reduction by three equations (and three λ 's). If ${}^{st0}\xi$ is arbitrary, the constraint has added ${}^{st0}\xi$ to the system variables, so another equation is needed. The required equation may be obtained as the dot product of the t substructure translational equation with $\frac{\partial}{\partial t^0 y^1}$. This contains no constraint force since only ${}^0\lambda^1$ is involved, and it is zero for this case.

The nominal rotational constraint (3.3-12) results in constraint torques (see Equation (3.2-5)) on the s substructure (Equation (3.3-14))

$${}^0_{\lambda}{}^1 \frac{\partial}{\partial t^0 y^1} + {}^0_{\lambda}{}^2 \frac{\partial}{\partial t^0 y^2} + {}^0_{\lambda}{}^3 \frac{\partial}{\partial t^0 y^3} \quad (3.5-5)$$

and on the t substructure (3.3-15)

$$- {}^0_{\lambda}{}^1 \frac{\partial}{\partial t^0 y^1} - {}^0_{\lambda}{}^2 \frac{\partial}{\partial t^0 y^2} - {}^0_{\lambda}{}^3 \frac{\partial}{\partial t^0 y^3} \quad (3.5-6)$$

Summing the rotational equations for the s and t substructures results in the cancelling of the constraint torques. The sum replaces the individual equations in the system equations resulting in a reduction by three equations (and three λ 's). If ${}^{st}{}^0\theta$ is arbitrary, the constraint has added ${}^{st}{}^0\theta$ to the system variables, so another equation is needed. The required equation may be obtained as the dot product of the rotation equation of the t substructure with $\frac{\partial}{\partial t^0 y^1}$. This equation contains no constraint torque since only ${}^0_{\lambda}{}^1$ is involved and it is zero for this case. It may be noted that since the rotational equations for s and t are in different bases, constant coefficients for either may not remain so after transforming to a common basis. In such cases, it may not be computationally advantageous to sacrifice constant coefficients for reduced dimension. However, if ${}^{st}{}^0\theta$ is constant in time, or if one of the substructures is symmetric about the $\frac{\partial}{\partial t^0 y^1}$ axis, constant coefficients can be preserved.

The disturbance translational constraint (3.4-18), in addition to forces on the disturbance equations, results in constraint torques (see Equation (3.2-5)) on the s substructure (3.4-19)

$$-\left(+ {}^1\lambda^1 \frac{\partial}{\partial t_y^1} + {}^1\lambda^2 \frac{\partial}{\partial t_y^2} + {}^1\lambda^3 \frac{\partial}{\partial t_y^3} \right) \times s_{i-}^p{}^t(\sigma) \frac{s^\wedge}{i^q} \quad (3.5-7)$$

and on the t substructure (3.4-20)

$$\left({}^1\lambda^1 \frac{\partial}{\partial t_y^1} + {}^1\lambda^2 \frac{\partial}{\partial t_y^2} + {}^1\lambda^3 \frac{\partial}{\partial t_y^3} \right) \times s_{i-}^p{}^t(\sigma) \frac{s^\wedge}{i^q} \quad (3.5-8)$$

Since the ${}^1\lambda^1$ are forces on the deformational equations, they are first order in q. Consequently, the constraint torques (3.5-7 and -8) are second order in q and may be dropped from the rotational equations.

The elimination of constraint forces for the deformation equations will be postponed to the following section on modal coupling. The remainder of this section will be directed to eliminating the constraint equations. The constraint equations may be eliminated by substituting the right hand side of Equation (3.3-18) for each appearance of t_Ω , and the right hand side of Equation (3.4-5) for each appearance of t_R . In addition, the direction cosine relation (2.2-13) for the t substructure may be deleted in favor of the implicit definition of $st0_\Omega^t$ (3.3-10) and the fact that resolving the rotational equations in a common basis introduces $st0_\theta$, and makes the equation second order.

3.6 Modal Synthesis

The partitioning of constraints into nominal and disturbance parts, leads to constraints between disturbance variables, \hat{q} , in addition to those between frame variables, discussed in the previous section. As in the previous section, it may, or may not, be computationally advantageous to eliminate the constraint forces. In some cases, the structure may be divided into substructures, primarily to obtain

smaller, more tractable models. The constraints, in this case, merely reassemble the pieces, eliminating all six relative motions. In such cases, it is generally computationally advantageous to eliminate the constraint forces. If the dimension of the substructure equations has been reduced by modal truncation, the procedure is known as modal coupling or modal synthesis.

Since the single degree of freedom point connected constraints are holonomic, it is more convenient to deal with their integrated form. Noting Equations (3.3-7 and -8)

$$\begin{aligned} {}^{st}C_{\beta}^{\alpha} &= {}^{sg}C_{\gamma}^{\beta} \left(\delta_{\gamma}^{\delta} + s_{i\psi(\sigma)} \delta_{\gamma}^{\delta} s_{i\psi}^{\Delta} \right) \left[{}^{sg}C_{\delta}^{\epsilon} {}^{st}C_{\epsilon}^{\theta} {}^{tg}C_{\zeta}^{\theta} \right] \left(\delta_{\eta}^{\zeta} + t_{j\psi(\tau)} \zeta_{\eta} t_{j\psi}^{\Delta} \right) {}^{tg}C_{\eta}^{\alpha} \\ &= {}^{st}C_{\beta}^{\alpha} + s_{i\psi(\sigma)} \delta_{\beta}^{\epsilon} s_{i\psi}^{\Delta} {}^{st}C_{\epsilon}^{\alpha} + {}^{st}C_{\beta}^{\theta} {}^{tg}C_{\psi(\tau)}^{\theta} t_{\alpha j\psi}^{\Delta} \end{aligned} \quad (3.6-1)$$

where

$$g_{\psi\beta}^{\alpha} = g_{C_{\gamma}^{\beta}}^{\beta} \psi_{\gamma}^{\delta} g_{C_{\delta}^{\alpha}}$$

(i.e., the rotations resolved in the $\frac{\partial}{\partial s_0^y}$ or $\frac{\partial}{\partial t_0^y}$ basis). Recalling Equation (3.3-1), this may be written as nine explicit equations

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^0\theta & s^0\theta \\ 0 & -s^0\theta & c^0\theta \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^0\theta & s^0\theta \\ 0 & -s^0\theta & c^0\theta \end{bmatrix} \begin{bmatrix} 0 & -s_{\psi}^3 & s_{\psi}^2 \\ s_{\psi}^3 & 0 & -s_{\psi}^1 \\ -s_{\psi}^2 & s_{\psi}^1 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & +t_{\psi}^3 & t_{\psi}^2 \\ -t_{\psi}^3 & 0 & +t_{\psi}^1 \\ +t_{\psi}^2 & -t_{\psi}^1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^0\theta & s^0\theta \\ 0 & -s^0\theta & c^0\theta \end{bmatrix} \quad (3.6-2) \\ &= \begin{bmatrix} 1, & -s_{\psi}^3 + c^0\theta t_{\psi}^3 + s^0\theta t_{\psi}^2, & s_{\psi}^2 + s^0\theta t_{\psi}^3 - c^0\theta t_{\psi}^2 \\ c^0\theta s_{\psi}^3 - s^0\theta s_{\psi}^2 - t_{\psi}^3, & c^0\theta + s^0\theta s_{\psi}^1 - s^0\theta t_{\psi}^1, & s^0\theta - c^0\theta s_{\psi}^1 + c^0\theta t_{\psi}^1 \\ -s^0\theta s_{\psi}^3 - c^0\theta s_{\psi}^2 + t_{\psi}^2, & -s^0\theta + c^0\theta s_{\psi}^1 - c^0\theta t_{\psi}^1, & c^0\theta + s^0\theta s_{\psi}^1 - s^0\theta t_{\psi}^1 \end{bmatrix} \end{aligned}$$

where the abbreviations are defined as

$$c\theta = \cos(st\theta), \quad s\theta = \sin(st\theta), \quad c^0\theta = \cos(st^0\theta), \quad s^0\theta = \sin(st^0\theta),$$

$$s_{\psi}^{\alpha} = \frac{1}{2} \epsilon_{\gamma}^{\alpha\beta} s g_{i\psi(\sigma)}^{\gamma} s_{i^q}^{\alpha}, \quad t_{\psi}^{\alpha} = \frac{1}{2} \epsilon_{\gamma}^{\alpha\beta} t g_{j\psi(\tau)}^{\gamma} t_{j^q}^{\alpha}.$$

Noting the small angle approximations, $\cos \epsilon \cong 1$, $\sin \epsilon \cong \epsilon$, and the sum of angles formulae $\sin(x+y) = \sin x \cos y + \cos x \sin y$, and $\cos(x+y) = \cos x \cos y - \sin x \sin y$, the (2,2) (2,3), (3,2) and (3,3) elements all yield

$$st_{\theta} - st^0_{\theta} = -s g_{i\psi(\sigma)}^{\gamma} s_{i^q}^{\alpha} + t g_{j\psi(\tau)}^{\gamma} t_{j^q}^{\alpha} \quad (3.6-3)$$

where

$$g_{\psi}^{\alpha} = \frac{1}{2} \epsilon_{\beta\gamma}^{\alpha} g_{\psi\beta}^{\gamma}$$

$$= \frac{1}{2} \epsilon_{\beta\gamma}^{\zeta} g_{\psi\delta}^{\beta} \psi_{\delta}^{\epsilon} g_{\epsilon}^{\gamma} g_{\eta}^{\zeta} g_{\eta}^{\alpha}$$

$$= \frac{1}{2} \epsilon_{\delta\epsilon}^{\eta} \psi_{\delta}^{\epsilon} g_{\eta}^{\alpha}$$

$$= g_{\eta}^{\alpha} \psi^{\eta}$$

The (1,2) and (1,3) equations may seem to be the linear combinations

$$-\cos(st^0\theta) [(2,1)] + \sin(st^0\theta) [(3,1)] \text{ and}$$

$$-\sin(st^0\theta) [(2,1)] - \cos(st^0\theta) [(3,1)] \text{ of the (2,1) and (3,1)}$$

equations. Selecting these as the two independent equations provides

$$t g_{j\psi(\tau)}^{\gamma} t_{j^q}^{\alpha} = \cos(st^0\theta) s g_{i\psi(\sigma)}^{\gamma} s_{i^q}^{\alpha} - \sin(st^0\theta) s g_{i\psi(\sigma)}^{\gamma} s_{i^q}^{\alpha}$$

$$t g_{j\psi(\tau)}^{\gamma} t_{j^q}^{\alpha} = \cos(st^0\theta) s g_{i\psi(\sigma)}^{\gamma} s_{i^q}^{\alpha} + \sin(st^0\theta) s g_{i\psi(\sigma)}^{\gamma} s_{i^q}^{\alpha} \quad (3.6-4)$$

The translational constraint may be obtained by dot multiplying

Equation (3.4-26) by $\frac{\partial}{\partial t_y^{\alpha}}$

$$\begin{aligned}
st_{\xi}^{\alpha} &= tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} tg_{C_{\gamma}}^{\delta} st0_{\xi}^{\delta} - tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} tg_{C_{\gamma}}^{\delta} st0_{C_{\epsilon}}^{\delta} sg_{C_{\zeta}}^{\epsilon} s_{iP'}^{\zeta} s_{iQ}^{\Lambda} \\
&+ tg_{C_{\beta}}^{\alpha} td_{C_{\gamma}}^{\beta} t_{jP'}^{\gamma} t_{jQ}^{\Lambda} \\
&= st0_{\xi}^{\alpha} + tg_{j\psi(\tau)}^{\beta} t_{jQ}^{\Lambda} st0_{\xi}^{\beta} - st0_{C_{\beta}}^{\alpha} sg_{iP'}^{\beta} s_{iQ}^{\Lambda} \\
&+ tg_{jP'}^{\gamma} t_{jQ}^{\Lambda}
\end{aligned} \tag{3.6-5}$$

where

$$g_{P'}^{\alpha} = g_{C_{\beta}}^{\alpha} P'^{\beta}$$

This may be written as the three equations

$$\begin{aligned}
\begin{Bmatrix} st_{\xi} \\ 0 \\ 0 \end{Bmatrix} &= \begin{Bmatrix} st0_{\xi} \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & t_{\psi}^3 & -t_{\psi}^2 \\ -t_{\psi}^3 & 0 & t_{\psi}^1 \\ t_{\psi}^2 & -t_{\psi}^1 & 0 \end{bmatrix} \begin{Bmatrix} st0_{\xi} \\ 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & c^0_{\theta} & s^0_{\theta} \\ 0 & -s^0_{\theta} & c^0_{\theta} \end{bmatrix} \begin{Bmatrix} s_{P'}^1 \\ s_{P'}^2 \\ s_{P'}^3 \end{Bmatrix} + \begin{Bmatrix} t_{P'}^1 \\ t_{P'}^2 \\ t_{P'}^3 \end{Bmatrix} \\
&= \begin{Bmatrix} st0_{\xi} & -s_{P'}^1 + t_{P'}^1 \\ -st0_{\xi} & t_{\psi}^3 - c^0_{\theta} s_{P'}^2 - s^0_{\theta} s_{P'}^3 + t_{P'}^2 \\ st0_{\xi} & t_{\psi}^2 + s^0_{\theta} s_{P'}^2 - c^0_{\theta} s_{P'}^3 + t_{P'}^3 \end{Bmatrix}
\end{aligned} \tag{3.6-6}$$

where the abbreviations are defined by

$$s_{P'}^{\alpha} = sg_{iP'}^{\alpha} s_{iQ}^{\Lambda} \text{ and } t_{P'}^{\alpha} = tg_{jP'}^{\alpha} t_{jQ}^{\Lambda}. \text{ In explicit form, the}$$

constraints are

$$\begin{aligned}
st_{\xi} - st0_{\xi} &= -sg_{iP'}^1 s_{iQ}^{\Lambda} + tg_{jP'}^1 t_{jQ}^{\Lambda} \\
\left(tg_{jP'}^2 - st0_{\xi} tg_{j\psi(\tau)}^3 \right) t_{jQ}^{\Lambda} &= \left[\cos(st0_{\theta}) sg_{iP'}^2 + \sin(st0_{\theta}) sg_{iP'}^3 \right] s_{iQ}^{\Lambda} \\
\left(tg_{jP'}^3 + st0_{\xi} tg_{j\psi(\tau)}^2 \right) t_{jQ}^{\Lambda} &= \left[\cos(st0_{\theta}) sg_{iP'}^3 - \sin(st0_{\theta}) sg_{iP'}^2 \right] s_{iQ}^{\Lambda}
\end{aligned} \tag{3.6-7}$$

The constraints (3.6-3, -4 and -7) are complicated by the presence of the nonlinear nominal relative translation and rotation terms. In general, the nonlinear constraint might not be eliminated, or might be eliminated each integration step. If the nominal relative translation and rotation are known functions of time, a non-autonomous solution to the constraints might be found. In either case the calculation can be simplified by selecting the rotation and translation at the connection point as explicit degrees of freedom (e.g., boundary points in a Hurty component mode analysis). One possible simplification is to apply a stronger constraint requiring the deformational rotations and translations (normal to the axis) to be zero in one substructure. This would be formally acceptable to the extent that it eliminated still present redundant rigid body modes. However, it would generally be expected that there would be insufficient redundant variables to allow this for every constraint. This might be an acceptable approximation if the constraint point was not expected to participate significantly in the deformation. If the connection point is constrained on one substructure the constraint equations yield a similar constraint for the other substructure.

The greatest simplification occurs when there is no nominal relative translation or rotation. In this case the nonlinearities are gone, and explicit solution is possible. It may be noted that the constraints in Equation (3.6-3) and the first of Equations (3.6-7) are always linear (although inhomogenous) as noted earlier in this section. The case of no nominal relative motion frequently occurs in modal

coupling. The remainder of this section will be dedicated to the treatment of this special case.

Without loss of generality, it may be assumed that st^0_θ and st^0_ξ are zero (either by changing the geometrical direction cosines or linearly extrapolating the displacement field). In this case the constraints assume the simpler form.

$$\begin{aligned}
 st_\theta &= -sg_{i1}\psi(\sigma)^1 s_{i1}^\wedge + tg_{j1}\psi(\tau)^1 t_{j1}^\wedge \\
 0 &= -sg_{i1}\psi(\sigma)^2 s_{i1}^\wedge + tg_{j1}\psi(\tau)^2 t_{j1}^\wedge \\
 0 &= -sg_{i1}\psi(\sigma)^3 s_{i1}^\wedge + tg_{j1}\psi(\tau)^3 t_{j1}^\wedge \\
 st_\xi &= -sg_{i1}^p(\sigma)^1 s_{i1}^\wedge + tg_{j1}^p(\tau)^1 t_{j1}^\wedge \\
 0 &= -sg_{i1}^p(\sigma)^2 s_{i1}^\wedge + tg_{j1}^p(\tau)^2 t_{j1}^\wedge \\
 0 &= -sg_{i1}^p(\sigma)^3 s_{i1}^\wedge + tg_{j1}^p(\tau)^3 t_{j1}^\wedge
 \end{aligned} \tag{3.6-8}$$

The union of the sets of s_{i1}^\wedge and t_{j1}^\wedge may be thought of as a vector space. Similarly the relative motions on the left of Equations (3.6-8) may also be thought of as a vector space. The Equations (3.6-8) may then be recognized as a linear function. This may be written in abbreviated form as

$$\{s\} = [T] \{q\} \tag{3.6-9}$$

Since the number of q is generally much greater than the number of s , and prohibiting contradictory constraints, the linear function, T , generally has a non-trivial null space (and the matrix has a rank smaller than its larger dimension). The q can be represented in a

new basis consisting of a basis for the null space and a set of vectors whose images under T span the space of s.

$$\{q\} = [N] \{\xi\} + [B]\{\zeta\} \quad (3.6-10)$$

where

$$[T][N] = [\emptyset]$$

and

$$[T][B] = [U]$$

B is sometimes referred to as a pseudo-inverse. (Redundant constraints could lead to zeros in the identity matrix, however, this is of no concern here.) Substituting Equation (3.6-10 in -9) yields

$$\{\zeta\} = \{s\} \quad (3.6-11)$$

It may be noted that a "B" basis vector added to any linear combination of "N" basis vectors is still a "B" basis vector. The disturbance equations have the generic form

$$[D]\{q\} = \{Q\} + [\Gamma]\{\lambda\} \quad (3.6-12)$$

where equations for both the s and t substructures have been included (and D is thought of as an operator including differentiation in time). Forming the linear combinations N^T and B^T , and noting Equation (3.6-10)

$$\begin{aligned} [N]^T [D] [N] \{\xi\} + [N]^T [D] [B] \{\zeta\} &= [N]^T \{Q\} + [N]^T [\Gamma] \{\lambda\} \\ [B]^T [D] [N] \{\xi\} + [B]^T [D] [B] \{\zeta\} &= [B]^T \{Q\} + [B]^T [\Gamma] \{\lambda\} \end{aligned} \quad (3.6-13)$$

The term $N^T \Gamma \lambda$ may be recognized as the virtual work done by the constraint force on a virtual displacement compatible with the constraint, and is therefore zero. Thus the constraint forces do not appear in the first equations. This is not true of the term $B^T \Gamma \lambda$. In fact,

the second equations may be thought of as determining the λ . If the constraint forces are of no interest, the second set of equations may be dropped, or if they are of interest, retained for subsequent recovery of the constraint forces. Only the first equations are retained in the system equations. Noting Equation (3.6-11), they may now be written

$$[N]^T [D][N]\{\xi\} = [N]^T \{Q - [D][B]\{s\}\} \quad (3.6-14)$$

Since these equations already reflect the constraint (3.6-9), it may be dropped from the system equations and if desired, Equation (3.6-10) can be used for subsequent data recovery. Thus for each such constraint, there is typically a reduction by twelve equations, six λ 's, and six q 's (in favor of ξ 's).

The use of one fully constrained connection above is illustrative, not restrictive. For instance, three substructures with four such connections (as long as they are not contradictory) can be treated in exactly the same manner by merely increasing the dimensions of the indicated matrices. And, even if the connections are not fully constrained the two constraints, Equation (3.6-3) and the first of Equations (3.6-7), can always be treated in this manner. If either ${}^{st}\xi$ or ${}^{st}\theta$ is arbitrary, an additional equation is required. This is obtained by adding the appropriate (first or third) member of the second set of Equations (3.6-13) to the system equations as previously noted, the associated λ is zero for this case, so the equations has no constraint forces.

At this point the methods outlined in Section 2.5 can be applied to the coupled substructures. So if constant coefficients can be assumed, modes can be generated for the coupled system. If constrained modes are to be calculated for further coupling (e.g., Hurty's method) it may be desirable to defer the calculation of the null space until after the static shapes have been calculated, since the calculation of static shapes is simplified by having the boundary nodes as explicit degrees of freedom, and transforming to the null space could result in variables involving linear combinations of the boundary degrees of freedom.

Section 4

SUMMARY

The purpose of this last section is to summarize the concepts and procedures outlined in the previous sections. As a vehicle for presenting this summary, the procedure for analyzing a dynamical system will be outlined.

4.1 Isolation of Structural Elements

The first step in the analysis of any dynamic system is to define the system under consideration. The process involves two procedures: defining the boundary, and defining the interaction across the boundary. For instance, for a train on a track, the track could be treated as external to the system, providing a rolling constraint; or in the system, with the road bed external to the system providing an elastic foundation.

The second step is the division of the system into dynamical subsystems of separate analytical types. For instance, in the train example, some of the dynamical subsystems might be: the hydraulic coupling system, the pneumatic brake system, the turbulent boundary layer, the biological engineer, the electrical power system, in addition to the mechanical structure of the train itself. Each subsystem might be separately analyzed, and in some cases equations governing the behavior of the subsystem might be obtained. In some cases a trivial solution might be selected (e.g. constant temperature for a thermodynamic subsystem if it is not particularly relevant to the system), or the subsystem might be excluded from the system. (This latter case really falls under the first step, defining the system.)

As in the first step, the second step involves two procedures: isolating the subsystem, and defining the interaction. For instance, the acoustic subsystem loads the structure, and the structure displaces the gas. The interactions now include not only interactions with things outside the system, but also with different subsystems within the system.

This study treats a subsystem identified as a structure. The primary interactions between structure and other subsystems are displacement and force (including body and surface forces). Although other interactions are possible (e.g. thermal and chemical), they will not be explicitly treated. It will now be assumed that all of the dynamical subsystems except the structure have been treated, and that their interactions with the structure result in either displacements or forces. Although the discussion in Section 1 has applicability to general material systems, the primary thrust of this study is solid materials. Therefore fluids and gases are included in the dynamical subsystems assumed to be already analyzed; not in the structural subsystem.

4.2 Selection of Substructures

A structure may be divided into a number of substructures for several reasons. Some subdivision may have already been made by the removal of other dynamical subsystems (i.e. if the dynamical subsystem joined the two substructures). If there are any joints in the structure which allow large relative motion, a subdivision should be made across the joint so that no substructure includes large relative motions internally. It may be desirable to select substructures to

isolate portions of the structure of different types (e.g. to isolate a visco-elastic material, or an orthotropic shell). The structure may also be subdivided to distribute the burden of analysis among individuals, or in some cases companies. Finally, and frequently most importantly, the structure may be subdivided to realize the substantial reductions in computational costs available through modal synthesis techniques. Since the savings is dependent on the amount of interface a substructure has with others, when there is no other reason for picking a given division, it should be selected to minimize the amount of interface (e.g. a long slender structure should be divided into two substructures by a cut perpendicular to the long axis rather than parallel to it).

4.3 Substructure Dynamics

The next step after dividing the structure into substructures is to obtain the equations governing the behavior of each substructure. Although there are numerous ways of obtaining these equations (e.g. finite difference, closed form, etc.), one of the most popular and powerful methods is the displacement formulation of the finite element. This method is developed in detail in Sections 2.1 - 2.3. Whatever method is used, the result should be a set of equations governing the behavior of the substructure in the form of Equations (2.2-13), (2.3-3 -4, and -5) which are reproduced here for reference

$$\omega^{,\delta} = -\frac{1}{2} \epsilon_{\gamma\alpha}^{\delta} c_{\beta}^{\gamma} \frac{\partial}{\partial t} c_{\beta}^{\alpha} \quad (4.3-1)$$

$$\begin{aligned} {}^s_M \frac{\partial^2}{\partial t^2} \underline{R} = & {}^s_F - {}^s_M \left\{ \frac{\partial'}{\partial t} \underline{\Omega}' \times {}^s_0 \underline{\bar{P}}' + \underline{\Omega}' \times (\underline{\Omega}' \times {}^s_0 \underline{\bar{P}}') + {}^s_j \underline{\bar{P}}' \ddot{q} \right. \\ & \left. + 2\underline{\Omega}' \times {}^s_j \underline{\bar{P}}' \dot{q} + \frac{\partial'}{\partial t} \underline{\Omega}' \times {}^s_j \underline{\bar{P}}' \dot{q} + \underline{\Omega}' \times (\underline{\Omega}' \times {}^s_j \underline{\bar{P}}') \dot{q} \right\} \quad (4.3-2) \end{aligned}$$

$$\begin{aligned}
& \underline{s}_I \cdot \frac{\partial'}{\partial t} \underline{\Omega}' + \frac{\partial'}{\partial t} \underline{\Omega}' \cdot \underline{s}_j \underline{I} \underline{j} \underline{q}^\wedge + \underline{s}_j \underline{I} \underline{j} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' \underline{j} \underline{q}^\wedge \\
& + \underline{\Omega}' \times \underline{s}_j \underline{O} \underline{j} \underline{q}^\wedge + \underline{\Omega}' \cdot \underline{s}_j \underline{I} \underline{j} \underline{q}^\wedge + \underline{s}_j \underline{I} \underline{j} \cdot \underline{\Omega}' \underline{j} \underline{q}^\wedge \\
& + \underline{\Omega}' \times \underline{s}_j \underline{I} \underline{j} \cdot \underline{\Omega}' \underline{j} \underline{q}^\wedge + \underline{\Omega}' \times (\underline{\Omega}' \cdot \underline{s}_j \underline{I} \underline{j}) \underline{j} \underline{q}^\wedge + \underline{\Omega}' \times \underline{s}_I \underline{j} \cdot \underline{\Omega}' \\
& = \underline{s}_I \underline{j} - \underline{s}_j \underline{O} \underline{j} \underline{q}^\wedge - \underline{s}_M \underline{s}_j \underline{P}' \underline{j} \times \frac{\partial^2}{\partial t^2} \underline{R} \underline{j} \underline{q}^\wedge - \underline{s}_M \underline{s}_O \underline{P}' \underline{j} \times \frac{\partial^2}{\partial t^2} \underline{R} \quad (4.3-3)
\end{aligned}$$

$$\begin{aligned}
& \underline{s}_{j\ell} M \underline{\ell} \underline{q}^\wedge + \underline{s}_{j\ell} C \underline{\ell} \underline{q}^\wedge + 2\underline{\Omega}' \cdot \underline{s}_{\ell j} \underline{O} \underline{\ell} \underline{q}^\wedge \\
& + \underline{s}_{j\ell} K \underline{\ell} \underline{q}^\wedge - \underline{\Omega}' \cdot \underline{s}_{\ell j} \underline{I} \underline{j} \cdot \underline{\Omega}' \underline{\ell} \underline{q}^\wedge + \underline{s}_{\ell j} \underline{O} \underline{j} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' \underline{\ell} \underline{q}^\wedge \\
& = \underline{s}_j Q + \underline{\Omega}' \cdot \underline{s}_j \underline{I} \underline{j} \cdot \underline{\Omega}' - \underline{s}_M \underline{s}_j \underline{P}' \underline{j} \cdot \frac{\partial^2}{\partial t^2} \underline{R} - \underline{s}_j \underline{O} \underline{j} \cdot \frac{\partial'}{\partial t} \underline{\Omega}' \quad (4.3-4)
\end{aligned}$$

Along with the governing equations, the equation for recovering displacements on the interfaces (e.g. Equation (2.4-1)) must be available

$$\underline{P}' = \underline{O} \underline{P}' + \underline{i} \underline{P}' \underline{i} \underline{q}^\wedge \quad (4.3-5)$$

The class of substructures or dynamical subsystems treated may now be expanded to include any resulting in equations of the above form.

4.4 Substructure Modes

In many cases, substantial reductions in the dimensions can be obtained by transforming from the generalized coordinates to modal coordinates. The circumstances under which this is permissible and the procedure for obtaining modes are outlined in Sections 2.5 and 2.6. Because of the substantial savings resulting from this procedure, it should be carried out for all substructures for which it is possible.

The result of the transformation is to reduce the number of equations and to modify the numerical values in the governing equations. However, the basic form is unaltered. (At least when rewritten in

first order differential equation format.) It will therefore be assumed that modal analysis has been performed for all substructures for which it is possible, the governing equations now involve modal variables, and the number of elastic equations has been appropriately reduced. Conversely, if modal analysis is not possible for the substructure, the equations remain unaltered.

If the interconnection points are retained as degrees of freedom (e.g. Hurty's method), then no back transformation is required to apply the constraints as developed. If they are not retained, and the substructure was treated as first order, the constraints must be modified to reflect the contact transformation. It will be assumed that the displacements at the interaction boundary are available as a linear combinations of generalized displacements (not velocities).

4.5 Substructure Coupling

At this point, there are at least two distinct paths which might be followed. It has been assumed, thus far, that the deformational coordinates contain all the rigid body modes, so there is a redundancy between them and the frame variables. The first path eliminates the redundancy at this point in the development by constraining the deformation and frame variables. For instance, a mean motion frame, or a principal axis frame, or a frame attached to the material at some point of the substructure. If this path is chosen, a constraint requiring that the motion of some degree of freedom be the same on two substructures, would be reflected by a single equation involving both frame and deformational variables.

The second path defers the elimination of the redundancy between deformational and frame variables. Consequently the physical interpretation of the frame motion (e.g. as mean motion) may be lost. Since each constraint can be thought of as constraining frame as well as deformational motion, each constraint is partitioned into a nominal (zero order in deformation) and deformational (first order in deformations) part. Each part is treated as an independent constraint. Focusing on the nominal constraints for a moment, it is clear that for substructures with numerous, redundant, connections, the number of nominal constraints could easily outnumber the frame variables. As long as the constraints are prohibited from being contradictory, the excess constraints are at most redundant. This procedure leaves as many frame variables as there are rigid body degrees of freedom for the structure.

In this study the second path has been selected because it tends to minimize the coupling between frame and deformational variables. Since this method tends to place motion in the deformational variables that the first path would associate with the frame, the chances of obtaining deformations large enough to compromise the linearization are increased. In such cases the model should allow relative frame motion at some joint to reduce the deformations.

If the first path is elected, the appropriate constraints may be written, (including those defining the frame) and the resulting equations integrated. The following sections, however, are based on selecting the second path.

4.6 Point Connections

As noted in Section 3.6, rigid connections provide six linear constraints, and single axis point connections provide at least two linear constraints. All of the constraints of this type and all of the associated deformational equations may be gathered together. The techniques of Section 3.6 may then be applied to any subset (including all) of those constraints and reiterated as required. As noted, when the coefficient matrices can be approximated as constant, the techniques outlined in Sections 2.5 and 2.6 may be applied to obtain modes of the coupled substructures.

The procedures outlined in Section 3.3 - 3.5 may be applied to incorporate the nominal constraints for all of the point connections. As noted, it is generally possible to eliminate the constraint forces, at the expense of complicating the equations. However for rigid connections or for the variables along the axis in single axis connections, it is generally desirable to eliminate them. So, in general, some of the constraint forces will have been eliminated. In this event the number of equations is reduced, the number of variables is reduced, the constraint equations need not be retained, and the constraint forces do not appear. Those constraint forces which are not eliminated are treated in the next section (4.7).

In the event the deformational constraint motion is unknown and governed by a force-deflection relationship compatible with the form of the equations, the relative motion variable can be included in the deformation variables, and the governing relationship in the assembled equations. Analysis may then proceed as above.

4.7 Pfaffian Constraints

Those single axis connections for which an election was made not to eliminate constraint forces, may be treated in the same way as Pfaffian constraints. The procedure is outlined in Section 3.2. The effect of this class of constraint is to augment the system equations by the constraint equations, and to augment the system variables by the constraint forces (Lagrange multipliers). As noted, unknown motions may be treated in a similar fashion.

4.8 Equations for Structure

After applying all these constraints, the number of frame variables is equal to the number of rigid body degrees of freedom. Similarly the number of rigid body modes remaining in the deformation equations is the number of rigid body degrees of freedom unless the option to reflect total relative motion in a frame variable (Sections 3.3 and 3.4) was exercised. For each of these cases where the nominal motion is arbitrary, but the relative deformation is constrained, the number of rigid body deformational modes is reduced by one. The rigid body deformational modes can now be eliminated by selecting a mean motion or principal axis frame for the composite substructure, by fixing some degrees of freedom, or they can be retained. The method outlined in Section 2.5 can be used to integrate the coupled system in appropriate cases, if all of the rigid body degrees of freedom are retained in the deformations.

At this point, any structural elements which do not fit the procedure outlined above (e.g. non-Pfaffian constraints) can be incorporated in the equations governing the structure.

4.9 System Synthesis

The complete set of structural equations may now be combined with the equations governing the behavior of the other dynamical subsystems. To whatever extent the equations for the other subsystems share the form of the structural equations, the same techniques may apply. In particular, any subset of constant coefficient linear equations may be subjected to modal analysis. The complete set of system equations is now available for the intended analysis (e.g. numerical integration to determining system behavior).

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APPENDIX A

MATHEMATICS

To preserve the continuity of the main body of this presentation, the mathematical definitions have been placed in this appendix. The material in this appendix is not new, and has not been originated by the author; rather it has been abstracted from the sources noted in the bibliography, particularly from Bishop and Goldberg's text "Tensor Analysis on Manifolds." The material is reproduced here primarily to establish the nomenclature, and secondarily, to review some of the more important results. For a more complete study, the noted sources are recommended.

Throughout this appendix, the most fundamental aspects of mathematics are applied implicitly. Some of these fundamental concepts and signs are noted below. Mathematics consists of mathematical objects and relations. The term set is synonymous with mathematical object. Mathematics is written in terms of complicated assemblies of fundamental signs and letters. A letter represents a totally indeterminate mathematical object. The fundamental rules governing the use of the fundamental signs and letters are called axioms. The true relations which may be logically deduced from the axioms are called theorems. Additional signs are introduced to abbreviate complex assemblies.

The fundamental logical signs may be taken as "or" and "not". If R and S are relations, the assemblies "R or S" and "not R" are relations called the logical disjunction and negation. If at least one of R and S is true, then "R or S" is true. The negation of a true relation

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is false. The derived logical signs: "and", " \Rightarrow " and " \Leftrightarrow " yield relations: "R and S", " $R \Rightarrow S$ " and " $R \Leftrightarrow S$ ", called logical conjunction, logical implication and logical equivalence; and defined to be "not [(not R) or (not S)]", "S or (not R)" and " $(R \Rightarrow S)$ and $(S \Rightarrow R)$ ", respectively. If R is a relation, A is a mathematical object and x is a letter, then " $(A|x)R$ " is the relation obtained by substituting A for x in R. If " $(A|x)R$ " is true then A satisfies the relation R. The derived logical signs: " \exists " and " \forall " are called the existential and universal quantifiers and are read "there exists" and "for all". They yield relations: " $(\exists x)R$ " and " $(\forall x)R$ "; the first is defined by the axiom

$$(A|x)R \Rightarrow (\exists x)R. \quad (\text{A.0-1})$$

The second abbreviates "not[($\exists x$)(not R)]".

The fundamental mathematical signs may be taken as: "=" and " \in " called the sign of equality and the sign of membership. The rules governing the sign of equality are

$$\begin{aligned} &(\forall x)(x = x), \\ &(\forall x)(\forall y)[(x=y) \Leftrightarrow (y=x)], \\ &(\forall x)(\forall y)(\forall z)[(x=y \text{ and } y=z) \Rightarrow (x=z)] \text{ and} \\ &(u=v) \Rightarrow [(u|x)R \Leftrightarrow (v|x)R]. \end{aligned} \quad (\text{A.0-2})$$

The rule governing the sign of membership is

$$(A=B) \Leftrightarrow [(x \in A) \Leftrightarrow (x \in B)]. \quad (\text{A.0-3})$$

A.1 Set Theory

The fundamental concepts and symbols used in set theory are outlined in the introduction to this appendix. The symbol " \subset ", called the inclusion sign, yields a relation " $A \subset B$ " which is an abbreviation

for " $(\forall x)[(x \in A) \Rightarrow (x \in B)]$ ". The relation is read "A is a subset of B" and is also written " $B \supset A$ ". The relation satisfies

$$[(A \subset B) \text{ and } (B \subset C)] \Rightarrow (A \subset C), \text{ and} \\ (A = B) \Leftrightarrow [(A \subset B) \text{ and } (B \subset A)]. \quad (\text{A.1-1})$$

If $R\{x\}$ is a relation with x a letter (variable), then for every set, B , \exists a unique $A \subset B$ where

$$x \in A \Leftrightarrow (x \in B \text{ and } R\{x\} \text{ is true}). \quad (\text{A.1-2})$$

It is conventional to indicate this set by " $\{x \mid R\{x\}\}$ ". The braces conventionally indicate a set and the vertical bar may be read "such that". In other uses, "such that" may be denoted " \ni ". The complement of A in B (where $A \subset B$) or the set theoretic difference of B and A is denoted

$$B - A = \{x \mid (x \in B) \text{ and } (x \notin A)\}. \quad (\text{A.1-3})$$

The set " \emptyset " = $A - A$ is called the null or empty set. For any two sets, A and B , the intersection and union of A and B are denoted " $A \cap B$ " and " $A \cup B$ " and are defined as

$$A \cap B = \{x \mid (x \in A) \text{ and } (x \in B)\} \quad \text{and} \\ A \cup B = \{x \mid (x \in A) \text{ or } (x \in B)\}. \quad (\text{A.1-4})$$

The notations " $\bigcup_{i=1}^N A_i$ ", " $\bigcap_{i=1}^N A_i$ ", " $\bigcup_{\alpha \in J} A_\alpha$ " and " $\bigcap_{\alpha \in J} A_\alpha$ " are used to indicate unions and intersections of families of sets.

If x and y are mathematical objects, the mathematical object " (x, y) " is called an ordered pair. More generally, (x_1, \dots, x_n) is called an ordered n-tuple. The ordered pair satisfies

$$[(w, x) = (y, z)] \Leftrightarrow [(w=y) \text{ and } (x=z)]. \quad (\text{A.1-5})$$

The cartesian product of two sets, A and B , is denoted " $A \times B$ " and defined as

$$A \times B = \{(x,y) | (x \in A) \text{ and } (y \in B)\} . \quad (\text{A.1-6})$$

More generally, " $A_1 \times A_2 \times \dots \times A_N$ " is the set of ordered n-tuples. If two adjacent sets are the same, exponential notation may be used (e.g. $A \times A \times B = A^2 \times B$).

A set consisting only of ordered pairs (i.e. a subset of a cartesian product) is called a graph. A function, " $f: A \rightarrow B$ ", is a set, A, called the domain, a set, B, called the range and a graph, $G, \subset A \times B$, satisfying $\forall x \in A, \exists$ exactly one $y \in B \ni (x,y) \in G$. The unique y associated with any x is denoted " $f(x)$ ". The image, " $f(A)$ ", is defined by

$$f(A) = \{f(x) | x \in A\} . \quad (\text{A.1-7})$$

When $f(A) = B$, the function is called onto, otherwise it is into. If $\forall y \in f(A), \exists$ only one $(x \in A) \ni (f(x) = y)$, the function is one to one; which is abbreviated "1-1". The function $f': C \rightarrow B, C \subset A, \ni f'(x) = f(x), \forall x \in C$ is called a restriction of f and is denoted " $f|_C$ ". If $f: A \rightarrow B$ and $g: C \rightarrow D$, then " $g \circ f: E \rightarrow D$ " is the function obtained by following f by g (i.e. $g[f(x)]$). The domain is $E = \{x | (x \in A) \text{ and } (f(x) \in C)\}$. The function " $g \circ f$ " is called the composition of g and f. In the event $E = \emptyset$, the function is the null function $\emptyset: \emptyset \rightarrow D$. If $f: A \rightarrow B$ is 1-1 and onto, \exists a unique function called the inverse of f, denoted " $f^{-1}: B \rightarrow A$ ", $\ni f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(y) = y$.

The algebraic development of the number system and its properties will not be pursued here. Notationally, the natural numbers will be denoted "N", the integer "I", the rational "Z", the real "R" and the complex "C".

A.2 Metric Spaces

A function, $\rho: A \times A \rightarrow R$, is a metric if $\forall x, y, z \in A$,

$$0 \leq \rho(x, y) < \infty,$$

$$\rho(x, y) = 0 \Leftrightarrow x = y,$$

$$\rho(x, y) = \rho(y, x) \text{ and}$$

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y). \quad (\text{A.2-1})$$

Two metrics, $\rho_1, \rho_2: A \times A \rightarrow R$, are strongly equivalent if $\exists c_1, c_2 > 0$,
 $\exists \forall x, y \in A$,

$$\left[\rho_1(x, y) \leq c_2 \rho_2(x, y) \right] \text{ and } \left[\rho_2(x, y) \leq c_1 \rho_1(x, y) \right]. \quad (\text{A.2-2})$$

Two metrics are equivalent if they give rise to the same "topology"

(A.3). A strongly equivalent metric is equivalent. A set with a metric is called a metric space. The open ball centered at x with radius r , is defined by

$$B(x, r) = \{y \mid (y \in A) \text{ and } [\rho(x, y) < r]\}. \quad (\text{A.2-3})$$

A bounded set is one which is contained in some open ball.

A.3 Topological Spaces

A collection, τ , of subsets of a set, A , is a topology in A if

$$\emptyset \in \tau, \quad A \in \tau,$$

$$B_i \in \tau, \quad i=1, \dots, N \Rightarrow \bigcap_{i=1}^N B_i \in \tau \text{ and}$$

$$B_\alpha \in \tau, \quad \alpha \in J \Rightarrow \bigcup_{\alpha \in J} B_\alpha \in \tau \quad (\text{A.3-1})$$

where J may be finite, countable, or uncountable. Any set, $B \in \tau$, is called an open set. A set with a topology is called a topological space. A topology constructed from a metric is called a metric topology.

If A and B are topological spaces, the topology whose open sets are cartesian products of open sets in A and B is called the product topology. The formal statement " A with τ is a topological space" is customarily abbreviated as " A is a topology."

A neighborhood of a point is defined as an open set containing the point. A closed set is the complement of an open set. For any subset, B , the union of all open sets, $C_\alpha \subset B$, is called the interior of B and is denoted " B° "; the intersection of all closed sets, $C_\alpha \supset B$ is called the closure of B and is denoted " B^- "; the set $B^- - B^\circ$ is called the boundary of B and is denoted " ∂B ". A topological space, A , is connected if \emptyset and A are the only sets which are both open and closed. An open connected set is called a region. A function, $f: A \rightarrow B$, from a topology, A , into a topology, B , is continuous if for every open $C \subset B$, $f^{-1}(C)$ is open in A .

Two topologies, A and B , are homeomorphic, and a function, $f: A \rightarrow B$, is a homeomorphism, if f is 1-1, onto and both $f: A \rightarrow B$ and $f^{-1}: B \rightarrow A$ are continuous. This is the natural notion of topological equivalence.

A.4 Manifolds

A chart at a point, $p \in A$, a topological space, is a homeomorphism, $\mu: U \rightarrow R^d$, mapping U , a neighborhood of p , onto an open subset of R^d . The dimension of the chart is d . The standard Cartesian chart on R^d is the identity map, $\mu_I: R^d \rightarrow R^d$, which is defined by $(u^1(p), u^2(p), \dots, u^d(p))$, where $p = (a^1, a^2, \dots, a^d)$ and $u^i(a^1, a^2, \dots, a^d) = a^i$. The function $u^i: R^d \rightarrow R$ is called the i -th projection. A chart is also called a coordinate map. The entries in the chart, μ , are defined by

$u^i \circ \mu$ and are called the coordinate functions, usually denoted by " x^i ". Collectively the coordinate functions " (x^1, x^2, \dots, x^d) " are called the coordinates or coordinate system at the point.

A function, $f : V \rightarrow \mathbb{R}$, V open in \mathbb{R}^d , is continuous to order infinity, denoted " C^∞ ", if f has continuous partial derivatives of all orders and types. A function, $\varphi : V \rightarrow \mathbb{R}^e$, is a C^∞ map if its entries, $u^i \circ \varphi$, $i = 1, \dots, e$ are C^∞ . Two charts, $\mu : U \rightarrow \mathbb{R}^d$, and $\tau : V \rightarrow \mathbb{R}^e$, on a topological space, are C^∞ -related if $d = e$ and either $U \cap V = \emptyset$ or both $\mu \circ \tau^{-1}$ and $\tau \circ \mu^{-1}$ are C^∞ maps. A collection of charts, $\{\mu_\alpha : U_\alpha \rightarrow \mathbb{R}^d \mid \alpha \in I, U_\alpha \subset A\}$, such that $\{U_\alpha \mid \alpha \in I\}$ is a covering of the topological space, A , (i.e., $A \subset \bigcup_{\alpha \in I} U_\alpha$) is called an atlas. A C^∞ atlas is one for which every pair of charts is C^∞ -related. A chart is admissible to a C^∞ atlas if it is C^∞ -related to every chart in the atlas. A topological manifold is a separable Hausdorff space \exists a d -dimensional chart at every point. The dimension of the manifold is d . (A metric topology is a Hausdorff space, and the metric topologies in this presentation are separable.) A C^∞ manifold is a topological manifold with all of the admissible charts to some C^∞ atlas.

If M and N are manifolds of dimension d and e , then the product manifold is the manifold of dimension $d+e$ obtained from the product topology with an atlas obtained from the products of charts from the atlases of M and N . The product of charts, $\mu : U \rightarrow \mathbb{R}^d$ and $\tau : V \rightarrow \mathbb{R}^e$, is $(\mu, \tau) : U \times V \rightarrow \mathbb{R}^{d+e}$, where

$$[(\mu, \tau)](m, n) = [\mu(m), \tau(n)]. \quad (\text{A.4-1})$$

The definition of a C^∞ map may be extended to include a function, $F : M \rightarrow N$, where M and N are C^∞ manifolds. Let $\mu_1 : U \rightarrow R^d$ and $\mu_2 : V \rightarrow R^e$ be C^∞ charts on M and N . The $\mu_1 - \mu_2$ coordinate expression for F is

$$\mu_2 \circ F \circ \mu_1^{-1} : \mu_1 \circ F^{-1}(V) \rightarrow R^e. \quad (A.4-2)$$

If all such expressions, for all admissible charts, μ_1 and μ_2 , are C^∞ Cartesian maps, the map F is called a C^∞ map. (It is sufficient that the coordinate expressions for one atlas in M and one in N are C^∞ .) A diffeomorphism is a 1-1, onto, C^∞ map, $F : M \rightarrow N$, such that the inverse map, $F^{-1} : N \rightarrow M$, is also C^∞ ; M and N are called diffeomorphic. This is the natural notion of equivalence between manifolds.

If there is a 1-1, C^∞ map, $F : M \rightarrow N$, \ni at every $m \in M \ni$ a neighborhood, U of m , and \exists a chart of N at $F(m)$, $\mu : V \rightarrow R^e$, $\mu = (y^1, \dots, y^e)$ $\ni x^i = y^i \circ F|_U$, $i = 1, \dots, d$ are coordinates on U for M , the map, F , is called an imbedding. A submanifold of N is a subset, $F(M)$, where $F : M \rightarrow N$ is an imbedding, provided with the manifold structure for which $F : M \rightarrow F(M)$ is a diffeomorphism. An open submanifold is an open subset of a manifold, with the manifold structure obtained by restricting the topology and the coordinate maps to the subset.

A.5 Tangents

A differentiable curve, $\gamma : [a, b] \rightarrow M$, is a map of an interval of real numbers, $[a, b]$, into a manifold, M , $\ni \exists$ an extension to an open interval, $\bar{\gamma} : (a-c, b+c) \rightarrow M$ ($c > 0$ and $\gamma = \bar{\gamma}$ on $[a, b]$), which is a C^∞ map.

The set of all real valued C^∞ functions, $F : U \rightarrow R$, where U is a neighborhood of a point, $m, \in M$, is denoted " $F^\infty(m)$ ". A tangent at m is

a derivation of $F^\infty(m)$, which is a function, $t : F^\infty(m) \rightarrow R$, \exists

$\forall f, g \in F^\infty(m)$ and $\forall a, b \in R$

$$t(af + bg) = at(f) + bt(g) \text{ and}$$

$$t(fg) = t(f)g(m) + f(m)t(g). \quad (\text{A.5-1})$$

If γ is a C^∞ curve in M $\exists \gamma(c) = m$, then the tangent to γ at c , " $\gamma_*(c)$ ", is defined by, $\forall f \in F^\infty(m)$,

$$[\gamma_*(c)](f) = \frac{df \circ \gamma}{du}(c). \quad (\text{A.5-2})$$

The set of all tangents at a point, m , is called the tangent space at m and is denoted " M_m ".

A real valued function at m , $f \in F^\infty(m)$, has a coordinate expression, $f = g(x^1, \dots, x^d)$, where $g = f \circ \mu^{-1} : U \rightarrow R$, U open in R^d . The partial derivate of f with respect to a coordinate, x^i , is defined as

$$\frac{\partial}{\partial x^i}(f) = \frac{\partial}{\partial u^i}(g) \circ \mu = \frac{\partial(f \circ \mu^{-1})}{\partial u^i} \circ \mu. \quad (\text{A.5-3})$$

The operators " $\partial/\partial x^i$ " are called the coordinate vector fields. If application of $\frac{\partial}{\partial x^i}$ is followed by evaluation at m , the result is a tangent at m , which is denoted " $\frac{\partial}{\partial x^i}(m)$ ", defined by

$$\left[\frac{\partial}{\partial x^i}(m) \right](f) = \left[\frac{\partial}{\partial x^i}(f) \right](m). \quad (\text{A.5-4})$$

Specifically, it is the tangent to γ_i at m , where γ_i is the i -th coordinate curve through m defined by

$$\gamma_i(u) = \mu^{-1}[x^1(m), \dots, x^{i-1}(m), u, x^{i+1}(m), \dots, x^d(m)]. \quad (\text{A.5-5})$$

These tangents form a "basis" (A.6) so that for any $a^i \in R$, $i=1, \dots, d$

$$t = \sum_{i=1}^d a^i \frac{\partial}{\partial x^i} (m) \quad (\text{A.5-6})$$

is a tangent. Conversely, for any tangent, $t, \in M_m, \exists$ unique a^i equation (A.5-6) holds. These a^i are called the components of t with respect to the coordinates x^i .

It is convenient to introduce an abbreviation for sums called the Einstein summation convention. The appearance of an index exactly twice in a product (usually once up and once down) implies summation over the range of the index. For example, if a^i and b_i are defined for $i=1, \dots, N$, then

$$a^i b_i \Rightarrow \sum_{i=1}^N a^i b_i. \quad (\text{A.5-7})$$

Thus the expression for any tangent may be written as

$$t = a^i \frac{\partial}{\partial x^i} (m). \quad (\text{A.5-8})$$

The a^i may be determined by noting that

$$\begin{aligned} t(x^j) &= a^i \left[\frac{\partial}{\partial x^i} (m) \right] (x^j) \\ &= a^i \delta_i^j \\ &= a^j. \end{aligned} \quad (\text{A.5-9})$$

The symbol " δ_i^j " is called the Kronecker delta and is defined to be one if $i=j$ and zero if $i \neq j$. The contraction is

$$\delta_i^i = \delta_1^1 + \delta_2^2 + \delta_3^3 = 3. \quad (\text{A.5-10})$$

The tangents are algebraically "vectors" (A.6) and the set of tangents at a point forms a "vector space" (A.6). In addition, the components with respect to the coordinates satisfy the usual tensor

transformation relations. If (x^i) and (y^i) are coordinates then the tangent $\frac{\partial}{\partial y^i}(m)$ has an expression in terms of the x^i ,

$$\frac{\partial}{\partial y^i}(m) = a^j \frac{\partial}{\partial x^j}(m), \quad (\text{A.5-11})$$

where

$$a^j = t(x^j) = \frac{\partial x^j}{\partial y^i}(m). \quad (\text{A.5-12})$$

So

$$\frac{\partial}{\partial y^i}(m) = \frac{\partial x^j}{\partial y^i}(m) \frac{\partial}{\partial x^j}(m). \quad (\text{A.5-13})$$

Thus for any tangent,

$$t = a^i \frac{\partial}{\partial x^i}(m) = b^j \frac{\partial}{\partial y^j}(m) \Rightarrow a^i = b^j \frac{\partial x^i}{\partial y^j}(m), \quad (\text{A.5-14})$$

which is the customary tensor transform.

The set of all tangent spaces at all points of a manifold, M , is called the tangent bundle, and is denoted "TM". If $\mu: M \rightarrow N$ is a C^∞ map, then the tangents in M are mapped to the tangents of N by $\mu_*: TM \rightarrow TN$, the differential of μ . In terms of coordinates, $x^i, i=1, \dots, d$, at m and $y^\alpha, \alpha=1, \dots, e$ at $n = \mu(m)$, if μ has a coordinate expression, $y^\alpha \circ \mu = f^\alpha(x^1, \dots, x^d)$, then a tangent, $t \in M_m$, $t = a^i \frac{\partial}{\partial x^i}(m)$ is mapped to the tangent $\mu_*(t) \in N_n$, ($n = \mu(m)$), where

$$\mu_*(t) = b^\alpha \frac{\partial}{\partial y^\alpha}(n) \text{ and } b^\alpha = a^i \left[\frac{\partial}{\partial x^i} (y^\alpha \circ \mu) \right] (m). \quad (\text{A.5-15})$$

The array $\frac{\partial}{\partial x^i} (y^\alpha \circ \mu)$ is called the Jacobian of μ with respect to the coordinates $\{x^i\}$ and $\{y^\alpha\}$. In the special case of a real valued function, $f: M \rightarrow R$,

$$f_*(t) = t(f) \frac{d}{du}(c), \text{ where } c = f(m). \quad (\text{A.5-16})$$

The component, $t(f)$, is redefined as the differential of f and the notation is changed to

$$[df](t) = t(f). \quad (\text{A.5-17})$$

If $\{x^i\}$ are the coordinates on M ,

$$df = \frac{\partial}{\partial x^i} (f) dx^i \quad (\text{A.5-18})$$

and

$$[dx^i] \left(\frac{\partial}{\partial x^j} \right) = \delta_j^i. \quad (\text{A.5-19})$$

A vector field, X , on $E \subset M$ is a function assigning to each $m \in E$ a vector, $X(m)$, $\in M_m$. It is C^∞ if the components of X with respect to every coordinate system, x^i , are C^∞ functions. A curve, γ , is an integral curve of a vector field, X , defined on ECM if the range of γ is in E and for every s in the domain of γ

$$\gamma_*(s) = X[\gamma(s)]. \quad (\text{A.5-20})$$

A.6 Vector Spaces

A commutative group is a set, X , with a law of composition,

$$(x, y) \rightarrow x+y, \quad \forall x, y \in X \text{ satisfying, } \forall x, y, z \in X,$$

$$x + (y + z) = (x + y) + z,$$

$$x + y = y + x,$$

$$\exists 0 \ni x + 0 = 0 + x = x \text{ and}$$

$$\exists (-x) \ni x + (-x) = (-x) + x = 0. \quad (\text{A.6-1})$$

A ring is a commutating group with a law of composition, $(x, y) \rightarrow xy$,

$$\forall x, y \in X, \text{ satisfying, } \forall x, y, z \in X,$$

$$x(yz) = (xy)z,$$

$$\exists 1 \ni 1x = x1 = x,$$

$$x(y + z) = xy + xz \text{ and } (x + y)z = xz + yz. \quad (\text{A.6-2})$$

A vector space is a set, V , whose elements are called vectors, a ring, S , whose elements are called scalars, and two operations, vector addition, $(x, y) \rightarrow x + y$, $\forall x, y \in V$ and scalar multiplication, $(\lambda, x) \rightarrow \lambda x$, $\forall x \in V$, $\lambda \in S$ such that V with addition is a commutative group and $\forall x, y \in V$ and $\lambda, \mu \in S$

$$\lambda(\mu x) = (\lambda\mu)x,$$

$$1x = x,$$

$$(\lambda + \mu)x = \lambda x + \mu x \text{ and}$$

$$\lambda(x + y) = \lambda x + \lambda y. \quad (\text{A.6-3})$$

The ring of scalars will be taken to be R , the real numbers, unless explicitly noted otherwise.

A sum of products of scalars, a^i , and vectors, v_i , i.e., $\sum a^i v_i$ is called a linear combination. If all of the $a^i = 0$ it is trivial, otherwise it is nontrivial. A finite set of vectors $\{v_i\}$ is linearly dependent $\Leftrightarrow \exists$ a nontrivial, null linear combination, $\sum a^i v_i = 0$; otherwise the set is linearly independent. A non-empty subset, U , of vector space is a subspace if it is closed under addition and scalar multiplication (e.g., $\forall u, v \in U, a \in S, u + v \in U$ and $au \in U$). The minimal subspace containing a subset, T , is called the subspace spanned by T . A linearly independent set of vectors, T , spanning a space, V , is called a basis for V . It can be shown that every basis for a space has the same number of elements, d , the dimension of the space.

Any vector, $v \in V$ may be represented as a unique linear combination of basis vectors for any basis $\{v_i\}$, $v = a^i v_i$. If $\{e_i\}$ and $\{f_i\}$ are two bases of V then each basis vector, e_i , has an expression in terms of the $\{f_j\}$,

$$e_i = a_i^j f_j. \quad (\text{A.6-4})$$

Similarly,

$$f_i = b_i^j e_j. \quad (\text{A.6-5})$$

If the scalars, a_i^j , are arranged in an array with j constant on rows and i constant on columns, the array is called a matrix and is denoted " (a_i^j) ". Specifically, it is the matrix of change of basis from $\{e_i\}$ to $\{f_j\}$. Since

$$e_i = a_i^j (b_j^k e_k), \quad (\text{A.6-6})$$

it may be concluded that

$$a_i^j b_j^k = \delta_i^k. \quad (\text{A.6-7})$$

Similarly,

$$b_i^j a_j^k = \delta_i^k. \quad (\text{A.6-8})$$

Matrices, (a_i^j) and (b_i^j) , satisfying these relationships are called inverses.

A.7 Tensor Spaces

If V and W are vector spaces, $f: V \rightarrow W$ and $\forall v_1, v_2 \in V$ and $a \in R$,

$$f(v_1 + v_2) = f(v_1) + f(v_2) \text{ and}$$

$$f(a v_1) = a f(v_1), \quad (\text{A.7-1})$$

then f is a linear function. The vector spaces V and W are isomorphic and the function f is an isomorphism if f is a 1-1, onto linear function. The set of points mapping into the zero of W , $f^{-1}(0)$, is called the null space of f . The set of all linear $f: V \rightarrow W$, $\{f | f: V \rightarrow W \text{ and } f \text{ linear}\}$, is called the space of linear functions from V into W and is denoted " $L(V,W)$ ". Defining $[f + g](v) = f(v) + g(v)$ and $[af](v) = a f(v)$ as vector addition and scalar multiplication, $L(V,W)$ is a vector space.

Just as the selection of a basis provides a coordinatization of a vector space, selection of bases, $\{e_i\}$, $i = 1, \dots, d_1$ on V and $\{\bar{e}_\alpha\}$, $\alpha = 1, \dots, d_2$ on W (where d_1 and d_2 are the dimensions of V and W) provides a coordinatization of linear functions as $d_1 \times d_2$ matrices,

$$\begin{pmatrix} f \\ i \end{pmatrix}^\alpha, \quad f(e_i) = f_i^\alpha (\bar{e}_\alpha). \quad (\text{A.7-2})$$

A basis for $L(V,W)$ is $\{E_\beta^j\}$, where

$$E_\beta^j e_i = \delta_i^j \bar{e}_\beta. \quad (\text{A.7-3})$$

For any f ,

$$f = f_j^\beta E_\beta^j \quad (\text{A.7-4})$$

The matrix of E_β^j is $\begin{pmatrix} \delta_i^j & \delta_\beta^\alpha \end{pmatrix}$.

The set of scalar-valued linear functions, $L(V,R)$, is called the dual space of V and is denoted " V^* ". For any given basis of V , $\{e_i\}$, there is a unique basis of V^* , $\{\epsilon^i\}$, such that $\epsilon^i e_j = \delta_j^i$. $\{\epsilon^i\}$ is called the dual basis to $\{e_i\}$. The dual basis to the coordinate vector field, $\left\{ \frac{\partial}{\partial x^i} (m) \right\}$, is a basis for the space of differentials,

$\{dx^i(m)\}$, (i.e. $dx^i \partial_j = \delta_j^i$). If $\{f_i\}$ is another basis of V and $\{\varphi^i\}$ its dual, where $f_i = a_i^j e_j$, and $e_i = b_i^j f_j$ and (b_i^j) is the inverse of (a_i^j) , then $\varepsilon^i = a_j^i \varphi^j$ and $\varphi^i = b_j^i \varepsilon^j$. The dual space to V^* (i.e. $(V^*)^*$) is naturally isomorphic to V . The two will not be distinguished and $(V^*)^*$ will be denoted V .

If $V_i, i=1, \dots, r$, and W are vector spaces, $f: V_1 \times \dots \times V_r \rightarrow W$, and $\forall v_i, \bar{v}_i \in V_i$, and $\forall a, \bar{a} \in R$,

$$f(v_1, \dots, av_i + \bar{a} \bar{v}_i, \dots, v_r) = a f(v_1, \dots, v_i, \dots, v_r) + \bar{a} f(v_1, \dots, \bar{v}_i, \dots, v_r), \quad (\text{A.7-5})$$

then f is a multi-linear, specifically an r-linear function. The space of r -linear functions is denoted " $L(V_1, \dots, V_r; W)$ ". The scalar valued multi-linear functions with variables all in V or V^* are called tensors over V and the vector spaces they form are called tensor spaces over V . The number of variables taken from V^* is called the contra-variant degree, and from V the covariant degree. The functions, $f: V^* \times \dots \times V^* \times V \times \dots \times V \rightarrow R$ (V^* r times, V s times) form a tensor space denoted " $T_s^r(V)$ ",

$$T_s^r(V) = V \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^* \quad (\text{A.7-6})$$

(V r times, V^* s times). They are called tensors of type (r,s) and have contravariant degree r and covariant degree s .

A tensor of type (0,0) is called a scalar, $T_0^0 = R$; of type (1, 0) is called a contravariant vector, $T_0^1 = V = L(V^*, R)$; of type (0,1) is called a covariant vector, $T_1^0 = V^* = L(V, R)$. The symbol, " \otimes ", denotes the tensor product which assigns to tensors $A \in T_s^r$ and $B \in T_u^t$

a tensor $A \otimes B \in T_{s+u}^{r+t}$. The tensor, $A \otimes B : (V^*)^{r+t} \times (V)^{s+u} \rightarrow R$, is defined by

$$\begin{aligned} A \otimes B (T^1, \dots, T^{r+t}, v_1, \dots, v_{s+u}) \\ = A(T^1, \dots, T^r, v_1, \dots, v_s) B(T^{r+1}, \dots, T^{r+t}, v_{s+1}, \dots, v_{s+u}). \end{aligned} \quad (A.7-7)$$

Selection of a basis, $\{e_i\}$, and its dual, $\{\varepsilon^i\}$, provides a coordinatization of a tensor, $A \in T_s^r$, as a multi-dimensional array;

$$A = A_{j_1 \dots j_s}^{i_1 \dots i_r} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \varepsilon_{j_1} \otimes \dots \otimes \varepsilon_{j_s} \quad (A.7-8)$$

where

$$A_{j_1 \dots j_s}^{i_1 \dots i_r} = A(\varepsilon^{i_1}, \dots, \varepsilon^{i_r}, e_{j_1}, \dots, e_{j_s}). \quad (A.7-9)$$

If the basis is changed to $\{f_i\}$ with an associated dual $\{\varphi^i\}$, where $f_i = a_i^j e_j$ and $\varphi^i = b_j^i \varepsilon^j$, then the coordinate expression for A in the $\{f_i\}$ basis is

$$f_{n_1 \dots n_s}^{m_1 \dots m_r} = b_{i_1}^{m_1} \dots b_{i_r}^{m_r} a_{n_1}^{j_1} \dots a_{n_s}^{j_s} e_{j_1 \dots j_s}^{i_1 \dots i_r} \quad (A.7-10)$$

A tensor is called symmetric if it is both covariant and contravariant symmetric. a tensor is (co- or contra-) variant symmetric if it is symmetric in every pair of (co- or contra-) variant indices. It is symmetric in the p-th and q-th (co- or contra-) variant indices if the components with respect to every basis are unchanged when the indices are interchanged. If the interchanging changes only the sign, the tensor is skew-symmetric in the p-th and q-th (co- or contra-) variant indices. It can be shown that symmetry with respect to one basis implies symmetry with respect to every basis.

A tensor valued function of tensors is frequently described in terms of the components with respect to a certain basis. If the result is independent of the basis, the function is called an invariant. A contraction is an example of an invariant. A contraction assigns to a tensor of type (r,s) another tensor of type (r-1, s-1). Formally, a contraction in the p-th contravariant and q-th covariant index is defined by

$$B_{j_1 \dots j_{s-1}}^{i_1 \dots i_{r-1}} = A_{j_1 \dots j_{q-1} k}^{i_1 \dots i_{p-1} k} j_q^{i_p \dots i_{r-1}} \quad (\text{A.7-11})$$

The scalar product of a vector and a dual vector is an example of a contraction. If $v = v^i e_i$ and $\tau = \tau_j \epsilon^j$ then

$$v \otimes \tau = v^i \tau_j e_i \otimes \epsilon^j \quad (\text{A.7-12})$$

and the contraction is $v^k \tau_k$. There are several alternative interpretations. Recalling $V^* = L(V, R)$ the scalar product may be thought of as a function, $\tau(V)$, or with $V = L(V^*, R)$ as a function V of τ , or as a bilinear function $\langle , \rangle : V \times V^* \rightarrow R$ defined by

$$\langle v, \tau \rangle = \tau(v). \quad (\text{A.7-13})$$

An invariant may be linear or multi-linear. A linear invariant of the p-fold tensor product of a variable with itself is called an invariant of degree p.

A quadratic form on V is an invariant of degree 2 with variable in V . A tensor of type (0,2) is called a bilinear form, $b : V \times V \rightarrow R$, and has a coordinate expression

$$b = b_{ij} \epsilon^i \otimes \epsilon^j. \quad (\text{A.7-14})$$

For every quadratic form, $q(v)$, there is an associated symmetric bilinear form, b , defined by

$$b(v, w) = \frac{1}{2} [q(v+w) - q(v) - q(w)]. \quad (\text{A.7-15})$$

Conversely for every symmetric bilinear form, b , there is an associated quadratic form, q , defined by

$$q(v) = b(v, v). \quad (\text{A.7-16})$$

A bilinear form is non-degenerate if for every $v \in V$, $v \neq 0$, $\exists w \in V \ni b(v, w) \neq 0$, or equivalently, if the matrix (b_{ij}) is non-singular. A quadratic form is positive or negative semidefinite if $\forall v \neq 0$, $q(v) \geq 0$ or $q(v) \leq 0$, respectively; if $q(v) \neq 0$ for $v \neq 0$ the form is (positive or negative) definite. A definite form is non-degenerate.

A non-degenerate symmetric form (usually positive definite) is called an inner product. A vector space with a positive definite inner product is called a Hilbert space and implies a topological structure. Two vectors, v, w , are orthogonal with respect to b if $b(v, w) = 0$. If $b(v, v) = 0$ then v is called a null vector of b . A basis, $\{e_i\}$, of V , is orthonormal with respect to b if for $i \neq j$, $b(e_i, e_j) = 0$ and $b(e_a, e_a)$, no sum, is one of the values $+1, 0, -1$. The bases, $\partial_i(m)$, in general, may not even be orthogonal. However, an orthonormal basis always exists. If the form is definite, the Gram-Schmidt process yields an orthonormal basis. If, as in the usual orthogonal curvilinear coordinates, the basis is orthogonal (and definite) but not normal, the change of basis, $e_i = h_i^{-1} \sum_j \partial_j(m)$, yields an orthonormal basis. The h_i^j are the Lamé coefficients defined by

$$h_{ij}^k = \sqrt{\frac{\partial x^k}{\partial y^i} \frac{\partial x^k}{\partial y^j}} \text{ and } h^{-1} \frac{k_{hj}}{j^i} = \delta_i^k \quad (\text{A.7-17})$$

A metric on a manifold is a bilinear form, b , which is defined and non-degenerate at every point of the manifold. For the usual coordinates on E , S or T ,

$$b = \delta_{ij} dx^i \otimes dx^j \quad (\text{A.7-18})$$

($i, j = 1, 2, 3, 4$ for E , $i, j = 1, 2, 3$ for S , and $i, j = 1$ for T).

This metric is Reimannian (positive definite) and the pair (M, b) is called a Reimannian manifold. The inner product or dot product is frequently denoted by " \langle , \rangle " or " \cdot ";

$$b(v, w) = \langle v, w \rangle = v \cdot w. \quad (\text{A.7-19})$$

The use of the dot product is extended to tensors (e.g. $A \cdot B$) by interpreting it as acting on the last contravariant index of the preceding tensor, and the first of the following tensor. There are three interpretations of the action of the fundamental bilinear form represented by the dot: 1) lowering the last contravariant index of the preceding tensor and contracting, 2) lowering the first contravariant index of the following tensor and contracting, or 3) directly assigning a tensor of contravariant degree two less. As an example,

$$\begin{aligned} \underline{N} \cdot \underline{S} &= \left(n^\gamma \frac{\partial}{\partial x^\gamma} \right) \cdot \left(s^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} \right) \\ &= n^\gamma \delta_{\delta\epsilon} s^{\alpha\beta} \frac{\partial}{\partial x^\gamma} \otimes \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} \otimes dx^\delta \otimes dx^\epsilon. \end{aligned} \quad (\text{A.7-20})$$

Using the first interpretation,

$$\begin{aligned} \underline{N} \cdot \underline{S} &= n_{\epsilon} s^{\alpha\beta} \frac{\partial}{\partial x^{\alpha}} \otimes \frac{\partial}{\partial x^{\beta}} \otimes dx^{\epsilon} \\ &= n_{\alpha} s^{\alpha\beta} \frac{\partial}{\partial x^{\beta}} . \end{aligned} \tag{A.7-21}$$

The length of a tangent on a Riemannian manifold is

$$||v|| = \langle v, v \rangle^{\frac{1}{2}} . \tag{A.7-22}$$

The angle between vectors is the value of θ , $0 \leq \theta \leq \pi$, such that

$$\cos(\theta) = \frac{\langle v, w \rangle}{||v|| ||w||} . \tag{A.7-23}$$

The length of a curve, $\gamma : [a, b] \rightarrow M$, denoted " $|\gamma|$ ", is the integral of the lengths of the tangent vectors,

$$|\gamma| = \int_a^b ||\gamma_*(t)|| dt . \tag{A.7-24}$$

The distance between points is the greatest lower bound of the lengths of all curves. The Riemannian metric, b , is consistent with the previous definitions of distance and duration, and straight lines are minimum distance curves.

The usual coordinates make E, S and T essentially equivalent to R^4, R^3 and R . If R^d had been taken as the models of the physical system, or if the metric is carried over to R^d , then b would be the standard flat metric and (R^d, b) would be ordinary Euclidean d-space.

A.8 Tensor Fields

For each type, (r, s) , of tensor, and each $m \in M, M$ a manifold, there is a corresponding tensor space " M_{ms}^r " over M_m , the tangent space. For any (r, s) the union of these tensor spaces $\forall m \in M$ is called the

bundle of tensors of type (r,s) over M , and is denoted " $T_S^r M$." In particular the tangent bundle is $TM = T_0^1 M$, and the bundle of differentials is $T_1^0 M$. A tensor field, T , of type (r,s) is a function, $T : E \rightarrow T_S^r M$ where $E \subset M$. In particular, $X : E \rightarrow TM$ is a vector field, and the differential of a C^∞ function, f , $df : E \rightarrow T_1^0 M$ is a tensor field of type $(0,1)$.

A vector field is " C^∞ " \Leftrightarrow for every coordinate system $\{x^i\}$, the components of X with respect to the x^i , $X^i = X(x^i)$, are C^∞ functions. The components of $T \in T_S^r M$, with respect to the coordinates $\{x^i\}$, are the d^{r+s} real valued functions.

$$T_{j_1 \dots j_s}^{i_1 \dots i_r} = T \left(dx^{i_1}, \dots, dx^{i_r}, \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_s}} \right). \quad (A.8-1)$$

The tensor field, T , is " C^∞ " if its components are C^∞ functions. A tensor field of type $(0,1)$ that is C^∞ is called a 1-form (or Pfaffian form).

If $\varphi : M \rightarrow N$ is a C^∞ map, then vector fields, X and Y , on M and N are φ -related $\Leftrightarrow \forall m$ in the domain of X

$$\varphi_* [X(m)] = Y[\varphi(m)]. \quad (A.8-2)$$

The definition is extended to tensor products by

$$\varphi_{*m} (A \otimes B) = [\varphi_{*m} (A)] \otimes [\varphi_{*m} (B)]. \quad (A.8-3)$$

Then two contravariant tensor fields, S , on M , and T , on N , are φ -related if $\forall m \in \text{domain of } S$.

$$\varphi_{*m} [S(m)] = T[\varphi(m)]. \quad (A.8-4)$$

For covariant tensors,

$$\varphi^*_m (T[\varphi(m)]) = S(m), \quad (A.8-5)$$

where $\varphi_m^* : N^*(\varphi(m)) \rightarrow M_m^*$ is defined by

$$\langle \varphi_{*m}(\nu), \tau \rangle = \langle \nu, \varphi_m^*(\tau) \rangle. \quad (\text{A.8-6})$$

In terms of coordinates if

$$y^\alpha \circ \varphi = F^\alpha(x^1, \dots, x^d)$$

and

$$T = T_{\alpha\beta}(y^1, \dots, y^e) dy^\alpha \otimes dy^\beta,$$

Then

$$\varphi^*(T) = T_{\alpha\beta}(F^1(x), \dots, F^e(x)) \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} dx^i \otimes dx^j. \quad (\text{A.8-7})$$

A.9 The Hodge Star Operator

The space of skew-symmetric tensors of type $(r, 0)$ is denoted by a wedge, " $\wedge^r V$," where V is the associated vector space. The dimension of $\wedge^r V$ is the binomial coefficient " $\binom{d}{r}$ " where d is the dimension of V . If j_1, \dots, j_r is a permutation of i_1, \dots, i_r , it may be obtained in any of a number of ways by transposing pairs of indices. The number of transpositions required for a permutation is odd or even and the corresponding sign of the permutation is -1 or 1 , respectively. For a permutation, π , this is denoted " $\text{SGN}(\pi)$." The component of a tensor, $A^{j_1 \dots j_r}$, is the same as, or the negative of $A^{i_1 \dots i_r}$ if $\text{SGN}(j_1 \dots j_r)$ is $+1$ or -1 , respectively. The alternating operator, $A: T_O^r \rightarrow \wedge^r V$, assigns to each tensor, B , its skew-symmetric part, denoted " B_a ". For $\theta_1 \dots \theta_r \in V^*$ B_a is defined as

$$B_a(\theta_1, \dots, \theta_r) = \frac{1}{r!} \sum_{(i_1, \dots, i_r)} \text{SGN}(i_1, \dots, i_r) B(\theta_{i_1}, \dots, \theta_{i_r}), \quad (\text{A.9-1})$$

where the sum runs over all $r!$ permutations of $(1, \dots, r)$. The

exterior product, denoted by " \wedge ", is defined by

$$A \wedge B = (A \otimes B)_a, \quad (\text{A.9-2})$$

where A and B are skew-symmetric. The treatment of covariant tensors is analogous.

An orientation of V is given by a non-zero element, θ , of $\Lambda^d V$. For the usual coordinates on R^3

$$\theta = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \quad (\text{A.9-3})$$

is taken. An ordered basis, (e_1, \dots, e_d) , is in the orientation given by θ if

$$e_1 \wedge \dots \wedge e_d = \alpha \theta, \quad \alpha > 0. \quad (\text{A.9-4})$$

Since the binomial coefficients, $\binom{d}{r}$ and $\binom{d}{d-r}$ are the same, the spaces of skew-symmetric tensors, $\Lambda^r V$ and $\Lambda^{d-r} V$, have the same dimension. The Hodge star operator is an isomorphism between the two spaces. Let (e_1, \dots, e_d) be an ordered orthonormal basis in the orientation. A typical basis element of $\Lambda^r V$ is $e_{i_1} \wedge \dots \wedge e_{i_r}$. Let j_1, \dots, j_{d-r} be chosen such that $(i_1, \dots, i_r, j_1, \dots, j_{d-r})$ is an even permutation of $(1, \dots, d)$. Then $*$ is the linear transformation such that

$$*(e_{i_1} \wedge \dots \wedge e_{i_r}) = e_{j_1} \wedge \dots \wedge e_{j_r}. \quad (\text{A.9-5})$$

It may be noted that for an odd dimensional space (e.g., 3), $* \circ *$ is the identity. The dot product may also be expressed in terms of the $*$ operator,

$$\underline{a} \cdot \underline{b} = * (*a \wedge b). \quad (\text{A.9-6})$$

If a and b are vectors of dimension three, the vector or cross product is defined to be

$$a \times b = *(a \wedge b). \quad (\text{A.9-7})$$

By definition, if $a = a^\alpha \frac{\partial}{\partial x^\alpha}$ and $b = b^\alpha \frac{\partial}{\partial x^\alpha}$,

$$\begin{aligned} a \wedge b &= (a \otimes b)_a \\ &= \left(a^\alpha b^\beta \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} \right)_a \\ &= \frac{1}{2} \left(a^\alpha b^\beta - a^\beta b^\alpha \right) \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} \\ &= \left(a^2 b^3 - a^3 b^2 \right) \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} + \left(a^3 b^1 - a^1 b^3 \right) \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1} + \left(a^1 b^2 - a^2 b^1 \right) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}. \end{aligned} \quad (\text{A.9-8})$$

So in terms of components, $a \times b$ is

$$*(a \wedge b) = \left(a^2 b^3 - a^3 b^2 \right) \frac{\partial}{\partial x^1} + \left(a^3 b^1 - a^1 b^3 \right) \frac{\partial}{\partial x^2} + \left(a^1 b^2 - a^2 b^1 \right) \frac{\partial}{\partial x^3}. \quad (\text{A.9-9})$$

This may be abbreviated by use of the Levi-Civita epsilon, $\epsilon_{\alpha\beta\gamma}$, defined as: +1 if (α, β, γ) is an even permutation of (1,2,3); -1 if it is an odd permutation; and zero otherwise.

$$a \times b = \epsilon_{\beta\gamma}^\alpha a^\beta b^\gamma \frac{\partial}{\partial x^\alpha}. \quad (\text{A.9-10})$$

The Levi-Civita epsilon may be used to express the determinant of a matrix of order three.

$$\begin{aligned} \text{DET } (A_{ij}) &= \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \\ &= \epsilon_{ijk} a_{i1} a_{j2} a_{k3}. \end{aligned} \quad (\text{A.9-11})$$

The rule for row or column interchange may be expressed as

$$\begin{vmatrix} a_{ip} & a_{iq} & a_{ir} \\ a_{jp} & a_{jq} & a_{jr} \\ a_{kp} & a_{kq} & a_{kr} \end{vmatrix} = \epsilon_{ijk} \epsilon_{pqr} |A|. \quad (\text{A.9-12})$$

The contraction on the first indices may be obtained by setting

$$a_{MN} = \delta_{MN},$$

$$\epsilon_{ijk} \epsilon_{iqr} = \begin{vmatrix} \delta_{11} & \delta_{1q} & \delta_{1r} \\ \delta_{j1} & \delta_{jq} & \delta_{jr} \\ \delta_{k1} & \delta_{kq} & \delta_{kr} \end{vmatrix} + \begin{vmatrix} \delta_{22} & \delta_{2q} & \delta_{2r} \\ \delta_{j2} & \delta_{jq} & \delta_{jr} \\ \delta_{k2} & \delta_{kq} & \delta_{kr} \end{vmatrix} + \begin{vmatrix} \delta_{33} & \delta_{3q} & \delta_{3r} \\ \delta_{j3} & \delta_{jq} & \delta_{jr} \\ \delta_{k3} & \delta_{kq} & \delta_{kr} \end{vmatrix}. \quad (\text{A.9-13})$$

For non-trivial cases, $q \neq r$ and the first column will be identical to the second or third in two of the arrays. The third reduces to

$$\begin{aligned} \epsilon_{ijk} \epsilon_{iqr} &= \begin{vmatrix} \delta_{jq} & \delta_{jr} \\ \delta_{kq} & \delta_{kr} \end{vmatrix} \\ &= \delta_{jq} \delta_{kr} - \delta_{jr} \delta_{kq}. \end{aligned} \quad (\text{A.9-14})$$

A further contraction gives

$$\begin{aligned} \epsilon_{ijk} \epsilon_{ijr} &= \delta_{jj} \delta_{kr} - \delta_{jr} \delta_{kj} \\ &= 3\delta_{kr} - \delta_{kr} \\ &= 2\delta_{kr}. \end{aligned} \quad (\text{A.9-15})$$

A final contraction gives

$$\begin{aligned} \epsilon_{ijk} \epsilon_{ijk} &= 2\delta_{kk} \\ &= 6. \end{aligned} \quad (\text{A.9-16})$$

The results in Equations (A.9-11, - 12) yield some useful identities for direction cosine matrices (whose determinants are one), namely

$$\begin{aligned}\epsilon_{ijk} &= \epsilon_{pqr} c_{ip} c_{jq} c_{kr} \\ &= \epsilon_{pqr} c_{pi} c_{qj} c_{kr}.\end{aligned}\tag{A.9-17}$$

A.10 Integration

A differential p-form is a C^∞ , covariant, skew-symmetric tensor field of degree p (type (0,p)). For the case of R^3 , the bases for forms are:

- 0 - forms: {1} (i.e. the real valued functions)
- 1 - forms: {dx, dy, dz} (i.e. the space of differentials (A.7))
- 2 - forms: {dy \wedge dz, dz \wedge dx, dx \wedge dy}
- 3 - forms: {dx \wedge dy \wedge dz}.

The Hodge star operator (A.9) may be used as it was for contravariant tensor fields.

The exterior derivative of a p-form, θ , is the p+1 form

$$d\theta = d\theta_{(i_1 \dots i_p)} \wedge dx^{i_1} \dots \wedge dx^{i_p},\tag{A.10-1}$$

summed over all increasing sets of indices.

As an example, on R^3 , if

$$\theta = f dx + g dy + h dz,$$

then

$$\begin{aligned}
d\theta &= df \wedge dx + dg \wedge dy + dh \wedge dz \\
&= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) \wedge dx \\
&\quad + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz \right) \wedge dy \\
&\quad + \left(\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy + \frac{\partial h}{\partial z} dz \right) \wedge dz \\
&= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dy \wedge dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dz \wedge dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy.
\end{aligned}
\tag{A.10-2}$$

Integration on a manifold is defined only for forms. Forms are integrated over sums of oriented C^∞ p-cubes. A C^∞ p-cube is a mapping, $\alpha : U \rightarrow M$ from a rectilinear p-cube in R^p , U , into the manifold, M . A rectilinear p-cube in R^p , U , is the set

$$U = \{(u^1, \dots, u^p) \mid b^i \leq u^i \leq b^i + c^i, \quad i=1, \dots, p\}. \tag{A.10-3}$$

The integration is performed by pulling back the differential form to R^p (i.e., $\alpha^*(\theta)$). A scalar valued function on R^p is obtained by defining an inner product on p-forms such that the orientation, ω , is unitary,

$$\langle \omega, \omega \rangle_p = 1, \tag{A.10-4}$$

and taking the inner product with the orientation,

$$\langle \alpha^*(\theta), \omega \rangle_p. \tag{A.10-5}$$

The integral of a p-form, θ , on an oriented (ω) C^∞ p-cube, $\alpha: U \rightarrow M$, is denoted

$$\int_{(\alpha, \omega)} \theta, \tag{A.10-6}$$

and is defined to be the associated Riemannian integral

$$\int_{(\alpha, \omega)} \theta = \int_U \langle \alpha^*(\theta), \omega \rangle_p \, d\mu_p. \quad (\text{A.10-7})$$

The generalization of Stokes' theorem to forms is as follows:

let θ be a $(p-1)$ -form defined on a chain of p -cubes, C , ($p > 0$), then

$$\int_C d\theta = \int_{\partial C} \theta. \quad (\text{A.10-8})$$

A.11 Variations

A real valued function of functions is called a functional.

(e.g., $\int_M \mu \langle \underline{v}, \underline{v} \rangle \, d\mu_M$ is a real valued function of the functions

\underline{v} .) The variation of a functional, $J[y_1, \dots, y_N]$, is denoted

" $\delta J[h_1, \dots, h_N]$ " and is defined as the principal linear part of the increment.

$$\Delta J [h_1, \dots, h_N] = J[y_1 + h_1, \dots, y_N + h_N] - J[y_1, \dots, y_N] \quad (\text{A.11-1})$$

is the increment of the functional, J , corresponding to the increments

$h_i(x^1, \dots, x^d)$ of the functions $y_i(x^1, \dots, x^d)$.

The increments are arbitrary members of the normed function space

appropriate to the associated function y . (e.g., $\int_{\bar{R}} F\left(\frac{\partial^3 y^1}{\partial x^1 \partial x^2}\right)$,

etc., suggests the space of 3-times continuously differentiable functions over \bar{R} , $D_3(\bar{R})$.) If the increment can be expressed as

$$\Delta J[h_1, \dots, h_n] = \phi[h_1, \dots, h_n] + \epsilon(||h||) \quad (\text{A.11-2})$$

$\epsilon \rightarrow 0$ as $||h|| \rightarrow 0$; and ϕ is a linear functional, then ϕ is called the principal linear part and is thus, by definition, the variation of the functional. A functional, ϕ , is linear if ϕ is continuous and

$$\begin{aligned} \phi[h_1, \dots, \alpha h_1 + \beta \bar{h}_1, \dots, h_N] &= \alpha \phi[h_1, \dots, h_1, \dots, h_N] + \\ &+ \beta \phi[h_1, \dots, \bar{h}_1, \dots, h_N] \end{aligned} \quad (\text{A.11-3})$$

A natural norm for h is

$$||h|| = \sum_{i=1}^N ||h_i|| \quad (\text{A.11-4})$$

and for an $h_i \in D_M(\bar{R})$ is

$$||h_i|| = \frac{\text{MAX}}{\bar{R}} |h_i| + \dots + \sum_{j_1=1}^d \dots \sum_{j_M=1}^d \frac{\text{MAX}}{\bar{R}} \left| \frac{\partial^M h_i}{\partial x_{j_1} \dots \partial x_{j_M}} \right|. \quad (\text{A.11-5})$$

Alternatively maximums could be used in place of sums. The term $\epsilon(||h||)$, if $\epsilon \rightarrow 0$ as $||h|| \rightarrow 0$, can be written using little - 0 notation as " $o(||h||)$," meaning

$$\lim_{||h|| \rightarrow 0} \frac{\epsilon(||h||)}{||h||} = 0, \quad (\text{A.11-6})$$

or with big - 0 notation as " $O(||h||^2)$," meaning $\exists M \ni$

$$\epsilon ||h|| \leq M ||h||^2. \quad (\text{A.11-7})$$

A.12 Nabla

The symbol " ∇ ", called nabla or del, is used to represent the operation of the exterior derivative on differential forms (A.10). The gradient is denoted " ∇f " and is defined as df . The divergence is denoted " $\nabla \cdot \underline{V}$ " and is defined as $* d * \underline{V}$, where "*" is the Hodge star operator (A.9). The curl is denoted " $\nabla \times \underline{V}$ " and is defined as $* d \underline{V}$.

In terms of the usual coordinates on three dimensional space, these have the usual forms

$$d(f) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz , \quad (\text{A.12-1})$$

$$\begin{aligned} \nabla \cdot \underline{V} &= * d * (v_1 dx^1 + v_2 dx^2 + v_3 dx^3) \\ &= * d (v_1 dx^2 \wedge dx^3 + v_2 dx^3 \wedge dx^1 + v_3 dx^1 \wedge dx^2) \\ &= * \left(\frac{\partial v_1}{\partial x^1} dx^1 \wedge dx^2 \wedge dx^3 + \frac{\partial v_2}{\partial x^2} dx^2 \wedge dx^3 \wedge dx^1 \right. \\ &\quad \left. + \frac{\partial v_3}{\partial x^3} dx^3 \wedge dx^1 \wedge dx^2 \right) \\ &= * \left(\left[\frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3} \right] dx^1 \wedge dx^2 \wedge dx^3 \right) \\ &= \left[\frac{\partial v_1}{\partial x^1} + \frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3} \right] \text{ and} \end{aligned} \quad (\text{A.12-2})$$

$$\begin{aligned} \nabla \times \underline{V} &= * d (v_1 dx^1 + v_2 dx^2 + v_3 dx^3) \\ &= * \left(\frac{\partial v_1}{\partial x^2} dx^2 \wedge dx^1 + \frac{\partial v_1}{\partial x^3} dx^3 \wedge dx^1 + \frac{\partial v_2}{\partial x^1} dx^1 \wedge dx^2 \right. \end{aligned}$$

Equation (A.12-3) continued

$$\begin{aligned}
& + \frac{\partial v_2}{\partial x^3} dx^3 \wedge dx^2 + \frac{\partial v_3}{\partial x^1} dx^1 \wedge dx^3 + \frac{\partial v_3}{\partial x^2} dx^2 \wedge dx^3 \Big) \\
= & * \left(\left[\frac{\partial v_2}{\partial x^1} - \frac{\partial v_1}{\partial x^2} \right] dx^1 \wedge dx^2 + \left[\frac{\partial v_3}{\partial x^2} - \frac{\partial v_2}{\partial x^3} \right] dx^2 \wedge dx^3 \right. \\
& \left. + \left[\frac{\partial v_1}{\partial x^3} - \frac{\partial v_3}{\partial x^1} \right] dx^3 \wedge dx^1 \right) \\
= & \left[\frac{\partial v_3}{\partial x^2} - \frac{\partial v_2}{\partial x^3} \right] dx^1 + \left[\frac{\partial v_1}{\partial x^3} - \frac{\partial v_3}{\partial x^1} \right] dx^2 + \left[\frac{\partial v_2}{\partial x^1} - \frac{\partial v_1}{\partial x^2} \right] dx^3 .
\end{aligned} \tag{A.12-3}$$

A.13 Dot and Cross Product Identities

Some of the identities involving dot and cross products will be listed here with proofs as needed. Since the dot product is defined by a symmetric bilinear form,

$$\underline{A} \cdot \underline{B} = \underline{B} \cdot \underline{A} . \tag{A.13-1}$$

Similarly, since the cross product is defined by the exterior product,

$$\underline{A} \times \underline{B} = - \underline{B} \times \underline{A} . \tag{A.13-2}$$

Using the tilde symbol introduced in (1.5-40), the cross product may also be written

$$\begin{aligned}
\underline{A} \times \underline{B} &= \epsilon_{\beta\gamma}^{\alpha} a^{\beta} b^{\gamma} e_{\alpha} \\
&= \left[\epsilon_{\beta}^{\delta\alpha} a^{\beta} e_{\alpha} \otimes e_{\delta} \right] \cdot \left[b^{\gamma} e_{\gamma} \right] \\
&= \tilde{\underline{A}} \cdot \underline{B} .
\end{aligned} \tag{A.13-3}$$

Similarly,

$$\underline{A} \times \underline{B} = \tilde{\underline{A}} \cdot \underline{B} = \underline{A} \cdot \tilde{\underline{B}} = - \underline{B} \cdot \tilde{\underline{A}} = - \tilde{\underline{B}} \cdot \underline{A} = - \underline{B} \times \underline{A} . \tag{A.13-4}$$

The vector triple product may be rewritten by applying Equation (A.13-3) as

$$\underline{A} \cdot (\underline{B} \times \underline{C}) = \underline{A} \cdot \tilde{\underline{B}} \cdot \underline{C} = (\underline{A} \times \underline{B}) \cdot \underline{C} . \quad (\text{A.13-5})$$

The twelve permutations of $(\underline{A}, \underline{B}, \underline{C})$ may be obtained from the symmetry properties, Equations (A.13-1 and -2) and from (A.13-5). All even permutations are equal, and all odd permutations are equal in magnitude but of opposite sign. In terms of the tilde notation,

$$\underline{A} \cdot \underline{B} \times \underline{C} = \tilde{\underline{A}} \cdot \underline{B} \cdot \underline{C} = \underline{A} \cdot \tilde{\underline{B}} \cdot \underline{C} = \underline{A} \cdot \underline{B} \cdot \tilde{\underline{C}} , \quad (\text{A.13-6})$$

where the parentheses have been dropped since there is only one interpretation possible.

The parentheses must be retained in the iterated vector product since there are two different forms. The iterated vector product may be written

$$\underline{A} \times (\underline{B} \times \underline{C}) = \epsilon_{\beta\gamma}^{\alpha} a^{\beta} (\epsilon_{\delta\epsilon}^{\gamma} b^{\delta} c^{\epsilon}) e_{\alpha} . \quad (\text{A.13-7})$$

Noting the expression for the contraction of Levi-Civita epsilons (A.9-14), this may be written as

$$\begin{aligned} \underline{A} \times (\underline{B} \times \underline{C}) &= (\delta_{\delta}^{\alpha} \delta_{\beta\epsilon} - \delta_{\epsilon}^{\alpha} \delta_{\beta\delta}) a^{\beta} b^{\delta} c^{\epsilon} e_{\alpha} \\ &= (a^{\beta} \delta_{\beta\epsilon} c^{\epsilon}) b^{\alpha} e_{\alpha} - (a^{\beta} \delta_{\beta\delta} b^{\delta}) c^{\alpha} e_{\alpha} \\ &= (\underline{A} \cdot \underline{C}) \underline{B} - (\underline{A} \cdot \underline{B}) \underline{C} . \end{aligned} \quad (\text{A.13-8})$$

From the anti-symmetry of the cross product

$$\underline{A} \times (\underline{B} \times \underline{C}) = (\underline{C} \times \underline{B}) \times \underline{A} . \quad (\text{A.13-9})$$

Finally it may be noted that

$$\begin{aligned} &\underline{A} \times (\underline{B} \times \underline{C}) + \underline{B} \times (\underline{C} \times \underline{A}) + \underline{C} \times (\underline{A} \times \underline{B}) \\ &= (\underline{A} \cdot \underline{C}) \underline{B} - (\underline{A} \cdot \underline{B}) \underline{C} + (\underline{B} \cdot \underline{A}) \underline{C} - (\underline{B} \cdot \underline{C}) \underline{A} + (\underline{C} \cdot \underline{B}) \underline{A} - (\underline{C} \cdot \underline{A}) \underline{B} \\ &= \underline{0} . \end{aligned} \quad (\text{A.13-10})$$

In terms of the tilde notation, this last result may be written

$$(\underline{\tilde{C}} \cdot \underline{B})^{\sim} \cdot \underline{A} = - \underline{\tilde{B}} \cdot (\underline{\tilde{C}} \cdot \underline{A}) + \underline{\tilde{C}} \cdot (\underline{\tilde{B}} \cdot \underline{A}) , \quad (\text{A.13-11})$$

where the tilde outside the parentheses implies the quantity inside is to be operated on. This may be written in an operational form as

$$(\underline{\tilde{C}} \underline{B})^{\sim} = \underline{\tilde{C}} \underline{\tilde{B}} - \underline{\tilde{B}} \underline{\tilde{C}} . \quad (\text{A.13-12})$$

The four fold iterated vector product may be expressed in either of two forms,

$$\begin{aligned} \underline{A} \times [\underline{B} \times (\underline{C} \times \underline{D})] &= \underline{A} \times [(\underline{B} \cdot \underline{D})\underline{C} - (\underline{B} \cdot \underline{C})\underline{D}] \\ &= (\underline{B} \cdot \underline{D})\underline{A} \times \underline{C} - (\underline{B} \cdot \underline{C})\underline{A} \times \underline{D} \end{aligned} \quad (\text{A.13-13})$$

or as

$$\underline{A} \times (\underline{B} \times [\underline{C} \times \underline{D}]) = (\underline{A} \cdot \underline{C} \times \underline{D})\underline{B} - (\underline{A} \cdot \underline{B}) \underline{C} \times \underline{D} . \quad (\text{A.13-14})$$

A.14 The Variation of a Vector

The use of the variational operator, δ , in conjunction with vector notation, in an inertial reference frame, has the interpretation

$$\delta \underline{A} = (\delta a^{\alpha}) \frac{\partial}{\partial x^{\alpha}} . \quad (\text{A.14-1})$$

This may be expressed in terms of the Λ^{-1} related basis, (1.5-35), as

$$\begin{aligned} \delta \underline{A} &= (\delta a^{\alpha}) c_{\alpha}^{\beta} \frac{\partial}{\partial x'^{\beta}} \\ &= \delta (a^{\alpha} c_{\alpha}^{\beta}) \frac{\partial}{\partial x'^{\beta}} - (a^{\alpha} \delta c_{\alpha}^{\beta}) \frac{\partial}{\partial x'^{\beta}} . \end{aligned} \quad (\text{A.14-2})$$

If the combination, $a^{\alpha} c_{\alpha}^{\beta}$, is denoted $'a^{\beta}$ (noting that $a^{\gamma} = c^{\gamma}_{\beta} 'a^{\beta} = c^{\beta}_{\gamma} 'a^{\beta}$), then Equation (A.14-2) becomes

$$\begin{aligned}\delta \underline{A} &= (\delta 'a^\beta) \frac{\partial}{\partial x'^\beta} - ('a^\gamma c_\alpha^\gamma \delta c_\alpha^\beta) \frac{\partial}{\partial x'^\beta} \\ &= \delta' \underline{A} - ('a^\gamma c_\alpha^\gamma \delta c_\alpha^\beta) \frac{\partial}{\partial x'^\beta},\end{aligned}\quad (\text{A.14-3})$$

where the symbol δ' is defined by

$$\delta' \underline{B} = (\delta b^\alpha) \frac{\partial}{\partial x'^\alpha} . \quad (\text{A.14-4})$$

It may be noted that the array, $c_\alpha^\gamma \delta c_\alpha^\beta$, is skew symmetric since

$$\begin{aligned}0 &= \delta(\delta^{\gamma\beta}) = \delta(c_\alpha^\gamma c_\alpha^\beta) = \delta c_\alpha^\gamma c_\alpha^\beta + c_\alpha^\gamma \delta c_\alpha^\beta \\ \Rightarrow c_\alpha^\beta \delta c_\alpha^\gamma &= -c_\alpha^\gamma \delta c_\alpha^\beta .\end{aligned}\quad (\text{A.14-5})$$

Thus in a dot product,

$$\begin{aligned}\delta(\underline{A} \cdot \underline{B}) &= \delta a^\alpha b^\alpha + a^\alpha \delta b^\alpha \\ &= (\delta 'a^\beta - 'a^\gamma c_\alpha^\gamma \delta c_\alpha^\beta) 'b^\beta + 'a^\beta (\delta 'b^\beta - 'b^\gamma c_\alpha^\gamma \delta c_\alpha^\beta) \\ &= \delta 'a^\beta 'b^\beta + 'a^\beta \delta 'b^\beta - 'a^\gamma 'b^\beta c_\alpha^\gamma \delta c_\alpha^\beta - 'a^\beta 'b^\gamma c_\alpha^\gamma \delta c_\alpha^\beta \\ &= \delta 'a^\beta 'b^\beta + 'a^\beta \delta 'b^\beta \\ &= \delta'(\underline{A} \cdot \underline{B}) .\end{aligned}\quad (\text{A.14-6})$$

In the case of mixed bases,

$$\begin{aligned}\delta \left[a^\alpha \frac{\partial}{\partial x^\alpha} \cdot 'b^\beta \frac{\partial}{\partial x'^\beta} \right] &= \delta (a^\alpha c_\alpha^\beta 'b^\beta) \\ &= \delta a^\alpha c_\alpha^\beta 'b^\beta + a^\alpha \delta c_\alpha^\beta 'b^\beta + a^\alpha c_\alpha^\beta \delta 'b^\beta \\ &= \delta \underline{A} \cdot \underline{B} + a^\alpha 'b^\beta \delta c_\alpha^\beta + \underline{A} \cdot \delta' \underline{B} .\end{aligned}\quad (\text{A.14-7})$$

A.15 Modes

An ordinary, second order constant coefficient system of differential equations in the form

$$M\ddot{q} + C\dot{q} + Gq + (1 + \alpha i)Kq = 0 \quad (\text{A.15-1})$$

must be converted to first order before modes can be calculated. The matrices M, C, K, are real positive semi-definite, G is real skew-symmetric, and $i = \sqrt{-1}$. First the q must be transformed into those spanning $M, {}^1q$, and those spanning the null space of M, 2q . Pre and post multiplying by the transform gives

$$\begin{bmatrix} 11_M & 0 \\ - & - \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 1 \cdot \dot{q} \\ - \\ 2 \cdot \dot{q} \end{Bmatrix} + \begin{bmatrix} 11_C & 12_C \\ - & - \\ 21_C & 22_C \end{bmatrix} \begin{Bmatrix} 1 \cdot \dot{q} \\ - \\ 2 \cdot \dot{q} \end{Bmatrix} + \begin{bmatrix} 11_G & 12_G \\ - & - \\ 21_G & 22_G \end{bmatrix} \begin{Bmatrix} 1 \cdot \dot{q} \\ - \\ 2 \cdot \dot{q} \end{Bmatrix} + (1 + \alpha i) \begin{bmatrix} 11_K & 12_K \\ - & - \\ 21_K & 22_K \end{bmatrix} \begin{Bmatrix} 1 \cdot q \\ - \\ 2 \cdot q \end{Bmatrix} = \begin{Bmatrix} 0 \\ - \\ 0 \end{Bmatrix} \quad (\text{A.15-2})$$

The equation may be written in first order form as

$$\begin{bmatrix} 0 & 11_M & 0 \\ - & - & - \\ 0 & 21_G & 22_C \end{bmatrix} \begin{Bmatrix} 1 \cdot \dot{q} \\ - \\ 2 \cdot q \end{Bmatrix} + \begin{bmatrix} 11_M & 0 & 0 \\ - & - & - \\ 0 & (1 + \alpha i) 11_K & (1 + \alpha i) 12_K \\ - & - & - \\ 0 & (1 + \alpha i) 21_K & (1 + \alpha i) 22_K \end{bmatrix} \begin{Bmatrix} 1 \cdot \dot{q} \\ - \\ 1 \cdot q \\ - \\ 2 \cdot q \end{Bmatrix} = \begin{Bmatrix} 0 \\ - \\ 0 \\ - \\ 0 \end{Bmatrix} \quad (\text{A.15-3})$$

If C is zero, then the first matrix is skew-symmetric. If G is zero (and C is not), then reversing the sign on the first equation would result in both coefficients being symmetric.

A first order differential equation of the form

$$A \dot{q} + Bq = 0 \quad (\text{A.15-4})$$

may be subjected to further reduction if A is of less than full rank. Transforming q into those spanning A, 1q , and those spanning A's null space, 2q , and then pre- and post multiplying by the transform gives

$$\begin{bmatrix} 11_A & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} {}^1q \\ {}^2q \end{Bmatrix} + \begin{bmatrix} 11_B & 12_B \\ \vdots & \vdots \\ 21_B & 22_B \end{bmatrix} \begin{Bmatrix} {}^1q \\ {}^2q \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{A.15-5})$$

The last of these equations is algebraic and can be solved for the space spanning the partition ${}^{21}_B$ ${}^{22}_B$. If ${}^{22}_B$ is non-singular this is simply

$$\begin{Bmatrix} {}^2q \end{Bmatrix} = - \left[{}^{22}_B \right]^{-1} \left[{}^{21}_B \right] \begin{Bmatrix} {}^1q \end{Bmatrix}, \quad (\text{A.15-6})$$

which when substituted in give

$$\left[{}^{11}_A \right] \begin{Bmatrix} {}^1q \end{Bmatrix} + \left[{}^{11}_B - {}^{12}_B \left({}^{22}_B \right)^{-1} {}^{21}_B \right] \begin{Bmatrix} {}^1q \end{Bmatrix} = 0. \quad (\text{A.15-7})$$

The singular case yields similar results. In either case, we are returned to an equation of the form

$$D\dot{q} + Eq = 0 \quad (\text{A.15-8})$$

where D has full rank.

The procedure for reducing the dimension of the equations generalizes a method applied in structural analysis known as static condensation or Guyan reduction. Those methods were developed for the special case where moment of inertia terms are considered negligible and dropped from the mass matrix, reducing its rank to about half. In those applications the mass matrix is frequently diagonal, there is usually no damping or coriolis terms, and the resulting ${}^{22}_K$ is

non-singular, allowing immediate identification of the null space and solution in the form of Equation (A.15-6). In the more general procedure, finding a null space is not trivial. Eigen routines may be used (if they can treat multiple zero roots), some Choleski decomposition routines, based on maximal pivots, provide the spaces, and there are a class of elimination methods sometimes known as "structure cutters" which do the job.

Returning to the Equation (A.15-8), if E has less than full rank, transforming gives

$$\begin{bmatrix} 11_D & 12_D \\ 21_D & 22_D \end{bmatrix} \begin{Bmatrix} 1_q \\ 2_q \end{Bmatrix} + \begin{bmatrix} 11_E & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 1_q \\ 2_q \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (\text{A.15-9})$$

and solving as in Equation (A.15-6) provides the rigid body modes and leaves

$$F\dot{q} + Hq = 0 \quad (\text{A.15-10})$$

The matrix H is symmetric and may be factored into lower, L, and upper, L^T , triangular matrices (which is a by-product of using a Choleski routine on E). Pre- and post multiplying by L^{-1} and L^{-1T} provides

$$J\dot{q} + Iq = 0, \quad (\text{A.15-11})$$

where I is an identity, and $J = L^{-1} F L^{-1T}$. Assuming an exponential solution $q = \hat{q} e^{\tau/\lambda}$ provides the standard eigenvalue problem on the dynamic matrix J

$$[J] + \lambda[I] \{\hat{q}\} = \{0\}. \quad (\text{A.15-12})$$

The frequencies of the system are $1/\lambda$ so the usual procedure of calculating highest eigenvalues first provides the desired lowest frequencies first (or, with a shift, in some low range).

It may be noted that in the absence of damping ($C = 0$, $\alpha = 0$), $i[J]$ is hermitian, so the eigenvalues, $i\lambda$, are real and the resulting motion is sinusoidal. In the general case, the adjoint problem must also be solved

$$\left[[J]^T + \lambda [I] \right] \{^a q\} = \{0\} . \quad (\text{A.15-13})$$

The eigenvalues are the same, and occur in conjugate pairs or are real, but the eigenvectors are different. After transforming the vectors back to the original q 's, premultiplying by the transpose of the adjoint vectors and substituting the vectors times modal amplitudes, $[\phi] \{\eta\}$, into the first order equations (e.g., A.15-3) diagonalizes the system of equations.

APPENDIX B

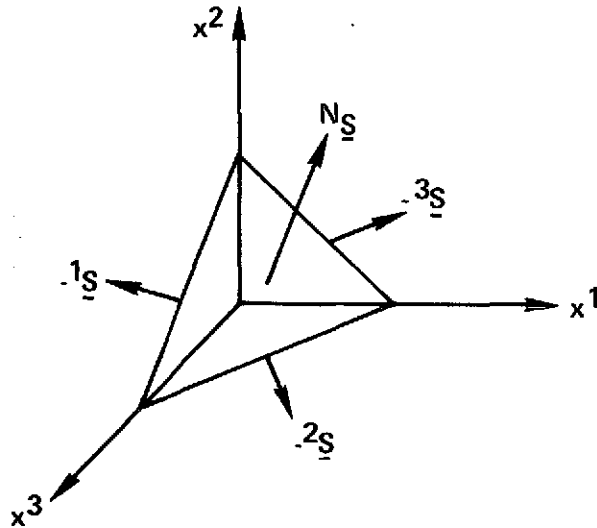
STRESS

The presentation of the relationship between Newton's Laws and stress has been removed to this appendix to preserve the continuity of the text.

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B.1 Stress Transformations

Let m be an arbitrary point in a material continuum, \underline{N} an arbitrary spatially contravariant vector of unit length. Let (x^1, x^2, x^3) be Cartesian coordinates with origin at m , and such that the point defined by $\epsilon \underline{N}$ lies in the first quadrant. The plane normal to \underline{N} through $\epsilon \underline{N}$, and the coordinate planes define a tetrahedron with vertices $(0,0,0)$, $(\frac{\epsilon}{n_1}, 0, 0)$, $(0, \frac{\epsilon}{n_2}, 0)$, $(0, 0, \frac{\epsilon}{n_3})$ where $\underline{N} = n^\alpha \frac{\partial}{\partial x^\alpha}$.



The surface force densities on the planes normal to \underline{N} and the coordinate axes x^α will be denoted \underline{N}_S and $-\alpha_S$ respectively. The volume of the tetrahedron is

$$V = \frac{1}{6} \frac{\epsilon^3}{n_1 n_2 n_3} \quad (\text{B.1-1})$$

and the areas of the faces are

$$N^A = \frac{1}{2} \frac{\epsilon^2}{n_1 n_2 n_3} \quad (\text{B.1-2})$$

and

$$\alpha^A = \frac{1}{2} \frac{n^\alpha \epsilon^2}{n_1 n_2 n_3} \quad (\text{B.1-3})$$

From Section 1.6, Newton's laws imply

$$\begin{aligned} \int_R * \left(\underline{\rho A} \cdot \frac{\partial}{\partial \mathbf{x}^\alpha} \right) &= \int_R * \left(\underline{F} \cdot \frac{\partial}{\partial \mathbf{x}^\alpha} \right) \\ &= \int_R * \left(\underline{X} \cdot \frac{\partial}{\partial \mathbf{x}^\alpha} \right) + \int_{\partial R} * \left(\underline{S} \cdot \frac{\partial}{\partial \mathbf{x}^\alpha} \right) . \end{aligned} \quad (\text{B.1-4})$$

Since the terms $\underline{\rho}$, \underline{A} , \underline{X} remain constant as ϵ varies ($O(1)$) and the integral varies as the cube ($O(\epsilon^3)$), and noting

$$\underline{S}(\mathbf{m}') = \underline{S}(\mathbf{m}) + O(\epsilon) , \quad (\text{B.1-5})$$

the equation becomes

$$- \underline{\alpha}_S \frac{1}{2} \frac{n^\alpha \epsilon^2}{n_1 n_2 n_3} + \underline{N}_S \frac{1}{2} \frac{\epsilon^2}{n_1 n_2 n_3} = O(\epsilon^3) . \quad (\text{B.1-6})$$

Thus

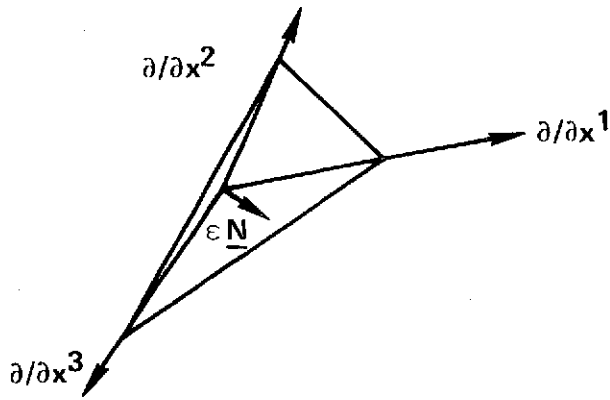
$$\underline{N}_S(\mathbf{m}') = n^\alpha \underline{\alpha}_S(\mathbf{m}) + O(\epsilon) , \quad (\text{B.1-7})$$

In the limit as $\epsilon \rightarrow 0$ this gives

$$\underline{N}_S(\mathbf{m}) = n^\alpha \underline{\alpha}_S(\mathbf{m}) . \quad (\text{B.1-8})$$

B.2 Generalized Stress Transformations

As in Appendix B.1, let \mathbf{m} be an arbitrary point in a material continuum, \underline{N} an arbitrary spatially contravariant vector with unit length. Let (x^1, x^2, x^3) be an arbitrary coordinate system such that \underline{N} lies in the first quadrant. The plane normal to \underline{N} and at distance ϵ from \mathbf{m} , and the planes defined by the tangents to the coordinate curves at \mathbf{m} describe a tetrahedron.



The vectors from m to the apexes may be denoted

$$\underline{i}_U = U^i \frac{\partial}{\partial x^i} \quad (\text{No Sum}) . \quad (\text{B.2-1})$$

The product of the area of the triangle and a unit outward normal vector is given by the cross products

$$\begin{aligned} {}^1_A \underline{1}_N &= \frac{1}{2} \underline{3}_U \times \underline{2}_U \\ {}^2_A \underline{2}_N &= \frac{1}{2} \underline{1}_U \times \underline{3}_U \\ {}^3_A \underline{3}_N &= \frac{1}{2} \underline{2}_U \times \underline{1}_U . \end{aligned} \quad (\text{B.2-2})$$

For the face normal to \underline{N} ,

$$\begin{aligned} {}^N_A \underline{N} &= \frac{1}{2} (\underline{2}_U - \underline{1}_U) \times (\underline{3}_U - \underline{1}_U) \\ &= \frac{1}{2} (\underline{2}_U \times \underline{3}_U + \underline{3}_U \times \underline{1}_U + \underline{1}_U \times \underline{2}_U) . \end{aligned} \quad (\text{B.2-3})$$

As in Appendix B.1, in the limit as $\epsilon \rightarrow 0$

$${}^N_A \underline{N}_S^\beta + {}^1_A \underline{1}_S^\beta = 0 . \quad (\text{B.2-4})$$

To demonstrate that this holds for

$$\underline{N}_S = \underline{N} \cdot \underline{\Sigma} , \quad (\text{B.2-5})$$

Equations (B.2-2 and -3) may be substituted yielding

$$\frac{1}{2}(\underline{2}_U \times \underline{3}_U + \underline{3}_U \times \underline{1}_U + \underline{1}_U \times \underline{2}_U) \cdot \underline{\Sigma} + \frac{1}{2}(\underline{3}_U \times \underline{2}_U \cdot \underline{\Sigma} + \underline{1}_U \times \underline{3}_U \cdot \underline{\Sigma} + \underline{2}_U \times \underline{1}_U \cdot \underline{\Sigma}) = \underline{0} , \quad (\text{B.2-6})$$

which is satisfied identically.

APPENDIX C

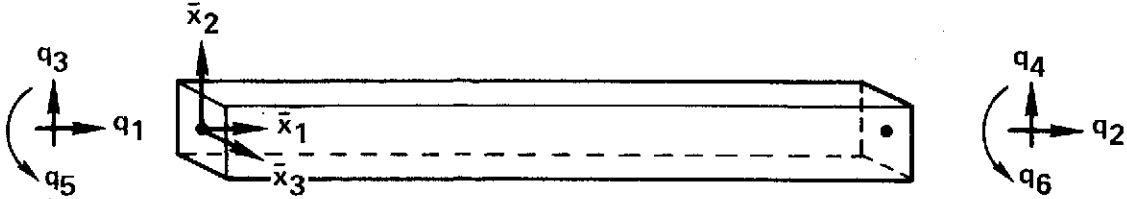
BEAMS

The calculation of numerical results for a beam finite element are presented in this appendix. This material is placed in this appendix to preserve continuity in the text.

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C.1 Beam Finite Element

The material continuum is a beam of length ℓ , depth h , and width w .



The position vector will be approximated by

$$\underline{P} = [\bar{x} + u(\bar{x}, \bar{y})] \frac{\partial}{\partial x} + [\bar{y} + v(\bar{x})] \frac{\partial}{\partial y} + [\bar{z}] \frac{\partial}{\partial z} ,$$

where the displacements are

$$\begin{aligned} u &= 0^u + 10^u \bar{x} + 01^u \bar{y} + 11^u \bar{x} \bar{y} + 21^u \frac{1}{2} \bar{x}^2 \bar{y} \\ v &= 0^v + 1^v \bar{x} + 2^v \frac{1}{2} \bar{x}^2 + 3^v \frac{1}{6} \bar{x}^3 . \end{aligned} \quad (C.1-1)$$

Surface forces will be prescribed as zero on the lateral surfaces of the beam. Body forces will also be prescribed as zero. The prescribed positions will be expressed in terms of nodes located at the center of each end. They will be assumed to have the form

$$\begin{aligned} \underline{P}|_{\bar{x}=0} &= (1^q - 5^q \bar{y}) \frac{\partial}{\partial x} + (\bar{y} + 3^q) \frac{\partial}{\partial y} + (\bar{z}) \frac{\partial}{\partial z} , \\ \underline{P}|_{\bar{x}=\ell} &= [\ell + (2^q - 6^q \bar{y})] \frac{\partial}{\partial x} + (\bar{y} + 4^q) \frac{\partial}{\partial y} + (\bar{z}) \frac{\partial}{\partial z} . \end{aligned} \quad (C.1-2)$$

The element will be restricted to infinitesimal (or at least sufficiently small) strains so that the strain energy density can be approximated by a quadratic in the strains

$$\hat{A} = \hat{A}^0 + \hat{A}^{\alpha\beta} E_{\alpha\beta} + \frac{1}{2} \hat{A}^{\alpha\beta\gamma\delta} E_{\alpha\beta} E_{\gamma\delta} . \quad (C.1-3)$$

The material is required to be homogeneous, so that the coefficients are independent of the coordinates. And, the material is required to be isotropic so that the coefficients are determined by Young's modulus, E, and Poisson's ratio, ν

$$\hat{A}^{IIIII} = \frac{E}{1 + \nu} + \lambda, \quad \hat{A}^{IILJJ} = \lambda,$$

$$\hat{A}^{IJJJI} = \hat{A}^{IJJJI} = G \quad (\text{No Sums}), \quad (\text{C.1-4})$$

where $I \neq J$, all other coefficients are zero, and λ and G are Lamé's constant and the shear modulus defined by

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \text{and } G = \frac{E}{2(1 + \nu)}.$$

Taking the variation indicated in Equation (1.8-26) gives

$$0 = \int_{\bar{S}} * \left[\left(\hat{A}^{\alpha\beta} + \hat{A}^{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta} - \hat{\Sigma}^{\alpha\beta} \right) \delta \epsilon_{\alpha\beta} \right]$$

$$+ \int_{\bar{S}} * \left(\left[(\bar{\nabla} P)_{\alpha\beta} - \epsilon_{\alpha\beta} \right] \delta \hat{\Sigma}^{\alpha\beta} \right)$$

$$\left\{ \int_{\bar{S}} * \left[\hat{\Sigma}^{\alpha\beta} \delta (\bar{\nabla} P)_{\alpha\beta} \right] - \int_{\partial S_P} * \left[S^{\alpha} \delta P_{\alpha} \right] \right\}$$

$$- \int_{\partial S_P} * \left[\left(P_{\alpha} - P_{P\alpha} \right) \delta S^{\alpha} \right]. \quad (\text{C.1-5})$$

The coefficient of $\delta \epsilon_{\alpha\beta}$ must be zero, so

$$\hat{\Sigma}^{\alpha\beta} - \hat{A}^{\alpha\beta} = \hat{A}^{\alpha\beta\gamma\delta} \epsilon_{\gamma\delta}. \quad (\text{C.1-6})$$

The beam will be taken to be in a state of uniaxial tension along the \bar{x} -axis in the reference state. Thus $\hat{A}^{\bar{x}\bar{x}} = \hat{\sigma}$, and all others are zero.

The term " $\bar{\nabla} P$ " may be calculated from Equation (1.8-19) as

$$\begin{aligned}
(\bar{\nabla}P)_{11} &= \frac{\partial u}{\partial \bar{x}} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial \bar{x}} \right)^2 + \left(\frac{\partial v}{\partial \bar{x}} \right)^2 + \left(\frac{\partial w}{\partial \bar{x}} \right)^2 \right] \\
(\bar{\nabla}P)_{12} &= \frac{1}{2} \left[\frac{\partial u}{\partial \bar{y}} + \frac{\partial v}{\partial \bar{x}} + \frac{\partial u}{\partial \bar{x}} \frac{\partial u}{\partial \bar{y}} + \frac{\partial v}{\partial \bar{x}} \frac{\partial v}{\partial \bar{y}} + \frac{\partial w}{\partial \bar{x}} \frac{\partial w}{\partial \bar{y}} \right] \\
(\bar{\nabla}P)_{22} &= \frac{\partial v}{\partial \bar{y}} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial \bar{y}} \right)^2 + \left(\frac{\partial v}{\partial \bar{y}} \right)^2 + \left(\frac{\partial w}{\partial \bar{y}} \right)^2 \right]
\end{aligned} \tag{C.1-7}$$

Inserting Equation (C.1-1) yields

$$\begin{aligned}
(\bar{\nabla}P)_{11} &= 10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} + \frac{1}{2} \left[\left(10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \right)^2 \right. \\
&\quad \left. + \left(1^v + 2^v \bar{x} + 3^v \frac{1}{2} \bar{x}^2 \right)^2 \right] \\
(\bar{\nabla}P)_{12} &= \frac{1}{2} \left[\left(01^u + 11^u \bar{x} + 21^u \frac{1}{2} \bar{x}^2 \right) + \left(1^v + 2^v \bar{x} + 3^v \frac{1}{2} \bar{x}^2 \right) \right. \\
&\quad \left. + \left(10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \right) \left(01^u + 11^u \bar{x} + 21^u \frac{1}{2} \bar{x}^2 \right) \right] \\
(\bar{\nabla}P)_{22} &= \frac{1}{2} \left[\left(01^u + 11^u \bar{x} + 21^u \frac{1}{2} \bar{x}^2 \right)^2 \right]
\end{aligned} \tag{C.1-8}$$

The coefficient of $\delta \hat{\Sigma}^{\alpha\beta}$ in Equation (C.1-5) must be zero. Retaining only linear terms, this gives

$$\begin{aligned}
\epsilon_{11} &= 10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \\
\epsilon_{12} &= \frac{1}{2} \left[01^u + 11^u \bar{x} + 21^u \frac{1}{2} \bar{x}^2 + 1^v + 2^v \bar{x} + 3^v \frac{1}{2} \bar{x}^2 \right]
\end{aligned} \tag{C.1-9}$$

Inserting this result in Equation (C.1-6) yields

$$\begin{aligned}
\hat{\Sigma}^{11} &= \hat{\sigma} + \left(\frac{E}{1+v} + \lambda \right) \left(10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \right) \\
\hat{\Sigma}^{22} &= \lambda \left(10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \right) \\
\hat{\Sigma}^{33} &= \lambda \left(10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \right) \\
\hat{\Sigma}^{12} &= G \left(01^u + 11^u \bar{x} + 21^u \frac{1}{2} \bar{x}^2 + 1^v + 2^v \bar{x} + 3^v \frac{1}{2} \bar{x}^2 \right)
\end{aligned} \tag{C.1-10}$$

The third integral in Equation (C.1-5) may now be written as

$$\begin{aligned}
& \int_S^* \left\{ \left[\delta + \left(\frac{E}{1+\nu} + \lambda \right) \left(10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \right) \right] \left[\delta 10^u + \delta 11^u \bar{y} + \delta 21^u \bar{x} \bar{y} \right] \right. \\
& + \left(10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \right) \left(\delta 10^u + \delta 11^u \bar{y} + \delta 21^u \bar{x} \bar{y} \right) \\
& \quad \left. + \left(1^v + 2^v \bar{x} + 3^v \frac{1}{2} \bar{x}^2 \right) \right. \\
& \quad \left. \left(\delta 1^v + \delta 2^v \bar{x} + \delta 3^v \frac{1}{2} \bar{x}^2 \right) \right] + \left[G \left(01^u + 11^u \bar{x} + 21^u \frac{1}{2} \bar{x}^2 + 1^v \right. \right. \\
& + \left. \left. 2^v \bar{x} + 3^v \frac{1}{2} \bar{x}^2 \right) \right] \left[\delta 01^u + \delta 11^u \bar{x} + \delta 21^u \frac{1}{2} \bar{x}^2 + \delta 1^v + \delta 2^v \bar{x} \right. \\
& \quad \left. + \delta 3^v \frac{1}{2} \bar{x}^2 \right] \\
& + \left(\delta 10^u + \delta 11^u \bar{y} + \delta 21^u \bar{x} \bar{y} \right) \left(01^u + 11^u \bar{x} + 21^u \frac{1}{2} \bar{x}^2 \right) \\
& \quad + \left(10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \right) \\
& \quad \left(\delta 01^u + \delta 11^u \bar{x} + \delta 21^u \frac{1}{2} \bar{x}^2 \right) \right] + \left[\lambda \left(10^u + 11^u \bar{y} + 21^u \bar{x} \bar{y} \right) \right] \\
& \left. \left[\left(01^u + 11^u \bar{x} + 21^u \frac{1}{2} \bar{x}^2 \right) \left(\delta 01^u + \delta 11^u \bar{x} + \delta 21^u \frac{1}{2} \bar{x}^2 \right) \right] \right\} \quad (C.1-11)
\end{aligned}$$

Collecting coefficients of the independent variations (i.e., $\delta 10^u$, $\delta 01^u$, $\delta 11^u$, $\delta 21^u$, $\delta 1^v$, $\delta 2^v$, $\delta 3^v$), retaining only linear terms, and defining

$$\begin{aligned}
\int \bar{y} \bar{d}y \bar{d}z &= A, \int \bar{y} \bar{d}y \bar{d}z = 0, \int \bar{y}^{-2} \bar{d}y \bar{d}z = I, \int \bar{y}^{-4} \bar{d}y \bar{d}z = \frac{20}{3} h^2 I \\
\int \bar{d}x &= \ell, \int \bar{x} \bar{d}x = \frac{\ell^2}{2}, \int \bar{x}^{-2} \bar{d}x = \frac{\ell^3}{3}
\end{aligned}$$

Equation (C.1-11) becomes

$$\begin{aligned}
& \left[\hat{\sigma} A \ell + \left(\frac{E}{1+\nu} + \lambda \right) A \ell_{10}^u + \hat{\sigma} A \ell_{10}^u \right] \delta_{10}^u \\
& + \left[G A \left(\ell_{01}^u + \frac{\ell^2}{2} \ell_{11}^u + \frac{\ell^3}{6} \ell_{21}^u + \ell_{1^v} + \frac{\ell^2}{2} \ell_{2^v} + \frac{\ell^3}{6} \ell_{3^v} \right) \right] \delta_{01}^u \\
& + \left[\left(\frac{E}{1+\nu} + \lambda \right) I \left(\ell_{11}^u + \frac{\ell^2}{2} \ell_{21}^u \right) + \hat{\sigma} I \left(\ell_{11}^u + \frac{\ell^2}{2} \ell_{21}^u \right) + G A \left(\frac{\ell^2}{2} \ell_{01}^u + \frac{\ell^3}{3} \ell_{11}^u \right. \right. \\
& \quad \left. \left. + \frac{\ell^4}{8} \ell_{21}^u + \frac{\ell^2}{2} \ell_{1^v} + \frac{\ell^3}{3} \ell_{2^v} + \frac{\ell^4}{8} \ell_{3^v} \right) \right] \delta_{11}^u \\
& + \left[\left(\frac{E}{1+\nu} + \lambda \right) I \left(\frac{\ell^2}{2} \ell_{11}^u + \frac{\ell^3}{3} \ell_{21}^u \right) + \hat{\sigma} I \left(\frac{\ell^2}{2} \ell_{11}^u + \frac{\ell^3}{3} \ell_{21}^u \right) + G A \left(\frac{\ell^3}{6} \ell_{01}^u \right. \right. \\
& \quad \left. \left. + \frac{\ell^4}{8} \ell_{11}^u + \frac{\ell^5}{20} \ell_{21}^u + \frac{\ell^3}{6} \ell_{1^v} + \frac{\ell^4}{8} \ell_{2^v} + \frac{\ell^5}{20} \ell_{3^v} \right) \right] \delta_{21}^u \\
& + \left[\hat{\sigma} A \left(\ell_{1^v} + \frac{\ell^2}{2} \ell_{2^v} + \frac{\ell^3}{6} \ell_{3^v} \right) + G A \left(\ell_{01}^u + \frac{\ell^2}{2} \ell_{11}^u + \frac{\ell^3}{6} \ell_{21}^u + \ell_{1^v} \right. \right. \\
& \quad \left. \left. + \frac{\ell^2}{2} \ell_{2^v} + \frac{\ell^3}{6} \ell_{3^v} \right) \right] \delta_{1^v} \\
& + \left[\hat{\sigma} A \left(\frac{\ell^2}{2} \ell_{1^v} + \frac{\ell^3}{3} \ell_{2^v} + \frac{\ell^4}{8} \ell_{3^v} \right) + G A \left(\frac{\ell^2}{2} \ell_{01}^u + \frac{\ell^2}{3} \ell_{11}^u + \frac{\ell^4}{8} \ell_{21}^u \right. \right. \\
& \quad \left. \left. + \frac{\ell^2}{2} \ell_{1^v} + \frac{\ell^3}{3} \ell_{2^v} + \frac{\ell^4}{8} \ell_{3^v} \right) \right] \delta_{2^v} \\
& + \left[\hat{\sigma} A \left(\frac{\ell^3}{6} \ell_{1^v} + \frac{\ell^4}{8} \ell_{2^v} + \frac{\ell^5}{20} \ell_{3^v} \right) + G A \left(\frac{\ell^3}{6} \ell_{01}^u + \frac{\ell^4}{8} \ell_{11}^u + \frac{\ell^5}{20} \ell_{21}^u \right. \right. \\
& \quad \left. \left. + \frac{\ell^3}{6} \ell_{1^v} + \frac{\ell^4}{8} \ell_{2^v} + \frac{\ell^5}{20} \ell_{3^v} \right) \right] \delta_{3^v} \tag{C.1-12}
\end{aligned}$$

The two remaining integrals in Equation (C.1-5) are over the deformed (physical space) ends of the beams. A rectangular area perpendicular to the \bar{x} -axis, with sides $d\bar{y}$, $d\bar{z}$ in material space, becomes, as a result of deformation, a parallelogram with sides $(1 + E_2)d\bar{y} \hat{\underline{i}}_2$, $(1 + E_3)d\bar{z} \hat{\underline{i}}_3$, where $1 + E_2 = \sqrt{1 + 2\epsilon_{22}}$, and $\hat{\underline{i}}_3$ is a unit vector tangent to the deformed \bar{x}_3 axis. The ratio of

the area after deformation to the area before is $(1+E_2)(1+E_3) \sin(\hat{\underline{i}}_2, \hat{\underline{i}}_3)$, where $\cos(\hat{\underline{i}}_2, \hat{\underline{i}}_3) = \epsilon_{23}/(1+E_2)(1+E_3)$. The Jacobian of the transformation is equal to the ratio of the areas

$$\begin{aligned}
 J &= (1+E_2)(1+E_3) \sqrt{1 - \frac{\epsilon_{23}^2}{(1+E_2)^2(1+E_3)^2}} \\
 &= \sqrt{(1+2\epsilon_{22})(1+2\epsilon_{33}) - \epsilon_{23}^2} \quad (C.1-13)
 \end{aligned}$$

Noting Equation (C.1-9), to linear order $J = 1$. Transforming the fourth integral in Equation (C.1-5) to material coordinates gives

$$\begin{aligned}
 \int_{\partial \bar{S}} * \left\{ \left[S^x \left(\delta_0^u + \delta_{10}^u \bar{x} + \delta_{01}^u \bar{y} + \delta_{11}^u \bar{x} \bar{y} + \delta_{21}^u \frac{1}{2} \bar{x}^2 \bar{y} \right) + \right. \right. \\
 \left. \left. + S^y \left(\delta_0^v + \delta_1^v \bar{x} + \delta_2^v \frac{1}{2} \bar{x}^2 + \delta_3^v \frac{1}{6} \bar{x}^3 \right) \right] \right\} \cdot [J] \quad (C.1-14)
 \end{aligned}$$

Evaluating this integral at the two ends (i.e. $x = 0$ and $x = \ell$) and introducing the symbols

$$\begin{aligned}
 \int_{\partial \bar{S}} \Big|_{x=0} * \{S^x [J]\} &= {}_1Q \\
 \int_{\partial \bar{S}} \Big|_{x=0} * \{S^x \bar{y} [J]\} &= - {}_5Q \\
 \int_{\partial \bar{S}} \Big|_{x=0} * \{S^y [J]\} &= {}_3Q \\
 \int_{\partial \bar{S}} \Big|_{x=\ell} * \{S^x [J]\} &= {}_2Q \\
 \int_{\partial \bar{S}} \Big|_{x=\ell} * \{S^x \bar{y} [J]\} &= - {}_6Q \\
 \int_{\partial \bar{S}} \Big|_{x=\ell} * \{S^y [J]\} &= {}_4Q \quad (C.1-15)
 \end{aligned}$$

yields for Equation (C.1-14)

$$\begin{aligned}
& ({}_1^Q \delta_0^u - {}_5^Q \delta_{01}^u + {}_3^Q \delta_0^v) + ({}_2^Q \delta_0^u + {}_2^Q \ell \delta_{10}^u - {}_6^Q \delta_{01}^u \\
& - {}_6^Q \ell \delta_{11}^u - {}_6^Q \frac{\ell^2}{2} \delta_{21}^u + {}_4^Q \delta_0^v + {}_4^Q \ell \delta_1^v + {}_4^Q \frac{\ell^2}{2} \delta_2^v \\
& + {}_4^Q \frac{\ell^3}{6} \delta_3^v) \tag{C.1-16}
\end{aligned}$$

Summing Equation (C.1-12) and the negative of (C.1-16), and setting the coefficients of the independent variations equal to zero provides the stationary conditions

$$\begin{aligned}
\delta_0^u & : - {}_1^Q - {}_2^Q = 0 \\
\delta_{10}^u & : \hat{\sigma} A \ell + \hat{\sigma} A \ell_{10}^u + \left(\frac{E}{1+\nu} + \lambda \right) A \ell_{10}^u - {}_2^Q \ell = 0 \\
\delta_{01}^u & : GA \left(\ell_{01}^u + \frac{\ell^2}{2} \ell_{11}^u + \frac{\ell^3}{6} \ell_{21}^u + \ell_1^v + \frac{\ell^2}{2} \ell_2^v + \frac{\ell^3}{6} \ell_3^v \right) + {}_5^Q + {}_6^Q = 0 \\
\delta_{11}^u & : \hat{\sigma} I \left(\ell_{11}^u + \frac{\ell^2}{2} \ell_{21}^u \right) + \left(\frac{E}{1+\nu} + \lambda \right) I \left(\ell_{11}^u + \frac{\ell^2}{2} \ell_{21}^u \right) \\
& + AG \left(\frac{\ell^2}{2} \ell_{01}^u + \frac{\ell^3}{3} \ell_{11}^u + \frac{\ell^4}{8} \ell_{21}^u + \frac{\ell^2}{2} \ell_1^v + \frac{\ell^3}{3} \ell_2^v + \frac{\ell^4}{8} \ell_3^v \right) + {}_6^Q \ell = 0 \\
\delta_{21}^u & : \hat{\sigma} I \left(\frac{\ell^2}{2} \ell_{11}^u + \frac{\ell^3}{3} \ell_{21}^u \right) + \left(\frac{E}{1+\nu} + \lambda \right) I \left(\frac{\ell^2}{2} \ell_{11}^u + \frac{\ell^3}{3} \ell_{21}^u \right) \\
& + AG \left(\frac{\ell^3}{6} \ell_{01}^u + \frac{\ell^4}{8} \ell_{11}^u + \frac{\ell^5}{20} \ell_{21}^u + \frac{\ell^3}{6} \ell_1^v + \frac{\ell^4}{8} \ell_2^v + \frac{\ell^5}{20} \ell_3^v \right) \\
& + {}_6^Q \frac{\ell^2}{2} = 0 \\
\delta_0^v & : - {}_3^Q - {}_4^Q = 0 \\
\delta_1^v & : \hat{\sigma} A \left(\ell_1^v + \frac{\ell^2}{2} \ell_2^v + \frac{\ell^3}{6} \ell_3^v \right) \\
& + AG \left(\ell_{01}^u + \frac{\ell^2}{2} \ell_{11}^u + \frac{\ell^3}{6} \ell_{21}^u + \ell_1^v + \frac{\ell^2}{2} \ell_2^v + \frac{\ell^3}{6} \ell_3^v \right) - {}_4^Q \ell = 0
\end{aligned}$$

Equation (C.1-17) continued

$$\begin{aligned}
\delta_{2^v} &: \hat{\sigma}_A \left(\frac{\ell^2}{2} 1^v + \frac{\ell^3}{3} 2^v + \frac{\ell^4}{8} 3^v \right) \\
&+ AG \left(\frac{\ell^2}{2} 01^u + \frac{\ell^3}{3} 11^u + \frac{\ell^4}{8} 21^u + \frac{\ell^2}{2} 1^v + \frac{\ell^3}{3} 2^v + \frac{\ell^4}{8} 3^v \right) - 4^Q \frac{\ell^2}{2} = 0 \\
\delta_{3^v} &: \hat{\sigma}_A \left(\frac{\ell^3}{6} 1^v + \frac{\ell^4}{8} 2^v + \frac{\ell^5}{20} 3^v \right) \\
&+ AG \left(\frac{\ell^3}{6} 01^u + \frac{\ell^4}{8} 11^u + \frac{\ell^5}{20} 21^u + \frac{\ell^3}{6} 1^v + \frac{\ell^4}{8} 2^v + \frac{\ell^5}{20} 3^v \right) - 4^Q \frac{\ell^3}{6} = 0
\end{aligned}$$

(C.1-17)

These nine relations may be solved for the six Q's and three constraints.

$$\begin{aligned}
1^Q &= - \left[A \hat{\sigma} + A \hat{\sigma} 10^u + A \left(\frac{E}{1+v} + \lambda \right) 10^u \right] \\
2^Q &= A \hat{\sigma} + A \hat{\sigma} 10^u + A \left(\frac{E}{1+v} + \lambda \right) 10^u \\
3^Q &= - \left\{ \frac{A}{\ell} \hat{\sigma} \left[\ell 1^v + \frac{\ell^2}{2} 2^v + \frac{\ell^3}{6} 3^v \right] + \frac{AG}{\ell} \left[\ell 01^u + \frac{\ell^2}{2} 11^u + \frac{\ell^3}{6} 21^u \right. \right. \\
&\quad \left. \left. + \left(\ell 1^v + \frac{\ell^2}{2} 2^v + \frac{\ell^3}{6} 3^v \right) \right] \right\} \\
4^Q &= \frac{A}{\ell} \hat{\sigma} \left(\ell 1^v + \frac{\ell^2}{2} 2^v + \frac{\ell^3}{6} 3^v \right) + \frac{AG}{\ell} \left[\ell 01^u + \frac{\ell^2}{2} 11^u + \frac{\ell^3}{6} 21^u \right. \\
&\quad \left. + \left(\ell 1^v + \frac{\ell^2}{2} 2^v + \frac{\ell^3}{6} 3^v \right) \right] \\
5^Q &= \frac{I \hat{\sigma}}{\ell} \left(\ell 11^u + \frac{\ell^2}{2} 21^u \right) + \frac{I}{\ell} \left(\frac{E}{1+v} + \lambda \right) \left(\ell 11^u + \frac{\ell^2}{2} 21^u \right) \\
&\quad - \frac{AG}{\ell} \left(\frac{\ell^2}{2} 01^u + \frac{\ell^3}{6} 11^u + \frac{\ell^4}{24} 21^u + \frac{\ell^2}{2} 1^v + \frac{\ell^3}{6} 2^v + \frac{\ell^4}{24} 3^v \right) \\
6^Q &= - \frac{I \hat{\sigma}}{\ell} \left(\ell 11^u + \frac{\ell^2}{2} 21^u \right) - \frac{I}{\ell} \left(\frac{E}{1+v} + \lambda \right) \left(\ell 11^u + \frac{\ell^2}{2} 21^u \right) \\
&\quad - \frac{AG}{\ell} \left(\frac{\ell^2}{2} 01^u + \frac{\ell^3}{3} 11^u + \frac{\ell^4}{8} 21^u + \frac{\ell^2}{2} 1^v + \frac{\ell^3}{3} 2^v + \frac{\ell^4}{8} 3^v \right)
\end{aligned}$$

Equation (C.1-18) continued

with constraints

$$\begin{aligned}
\hat{\sigma} I_{21}^u + \left(\frac{E}{1+\nu} + \lambda \right) I_{21}^u &= AG \left(01^u + \frac{\ell}{2} 11^u + \frac{3\ell^2}{20} 21^u \right. \\
&\quad \left. + 1^v + \frac{\ell}{2} 2^v + \frac{3\ell^2}{20} 3^v \right) \\
\hat{\sigma} A \left(2^v + \frac{\ell}{2} 3^v \right) + AG \left(11^u + \frac{\ell}{2} 21^u + 2^v + \frac{\ell}{2} 3^v \right) &= 0 \\
\hat{\sigma} A \left(2^v + \frac{8}{15} \ell 3^v \right) + AG \left(11^u + \frac{8}{15} \ell 21^u + 2^v + \frac{8}{15} \ell 3^v \right) &= 0 \quad (C.1-18)
\end{aligned}$$

the last two of these imply

$$\begin{aligned}
\hat{\sigma} 2^v + G(11^u + 2^v) &= 0, \text{ or } 2^v = -\frac{G}{\hat{\sigma}+G} 11^u \\
\hat{\sigma} 3^v + G(21^u + 3^v) &= 0, \text{ or } 3^v = -\frac{G}{\hat{\sigma}+G} 21^u \quad (C.1-19)
\end{aligned}$$

and inserting these in the first of Equations (C.1-18) gives

$$\begin{aligned}
v_1 &= -01^u + \frac{1}{AG} \left[-\frac{A\hat{\sigma}G}{\hat{\sigma}+G} \frac{\ell}{2} 11^u - \frac{A\hat{\sigma}G}{\hat{\sigma}+G} \frac{3\ell^2}{20} 21^u + \hat{\sigma} I_{21}^u \right. \\
&\quad \left. + \left(\frac{E}{1+\nu} + \lambda \right) I_{21}^u \right] \\
&= -01^u - \frac{\hat{\sigma}}{\hat{\sigma}+G} \frac{\ell}{2} 11^u - \frac{\hat{\sigma}}{\hat{\sigma}+G} \frac{3\ell^2}{20} 21^u + \frac{\hat{\sigma}}{G} \frac{I}{A} 21^u \\
&\quad + \left(\frac{2[1-\nu]}{[1-2\nu]} \right) \frac{I}{A} 21^u \quad (C.1-20)
\end{aligned}$$

Substituting back in Equation (C.1-18) yields

$$\begin{aligned}
{}_1Q &= -A\hat{\sigma} - A\hat{\sigma} 10^u - A \left(\frac{E}{1+\nu} + \lambda \right) 10^u \\
{}_2Q &= A\hat{\sigma} + A\hat{\sigma} 10^u + A \left(\frac{E}{1+\nu} + \lambda \right) 10^u
\end{aligned}$$

Equation (C.1-21) continued

$$\begin{aligned}
3^Q &= A \frac{\Delta}{\sigma} \hat{\sigma} \sigma_{01} u + A \frac{\Delta^2}{\hat{\sigma} \sigma + G} \frac{\ell}{2} 11^u + A \frac{\Delta^2}{\hat{\sigma} \sigma + G} \frac{3\ell^2}{20} 21^u - A \frac{\Delta^2}{\hat{\sigma}} \frac{I}{A} 21^u \\
&\quad - A \hat{\sigma} \left[\frac{2(1-\nu)}{(1-2\nu)} \right] \frac{I}{A} 21^u \\
&\quad + A \frac{\Delta}{\hat{\sigma} \sigma + G} \frac{\ell}{2} 11^u + \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{3\ell^2}{20} 21^u - \hat{\sigma} I 21^u - \left(\frac{E}{1+\nu} + \lambda \right) I 21^u \\
4^Q &= - A \hat{\sigma} \sigma_{01} u - A \frac{\Delta^2}{\hat{\sigma} \sigma + G} \frac{\ell}{2} 11^u - A \frac{\Delta^2}{\hat{\sigma} \sigma + G} \frac{3\ell^2}{20} 21^u + A \frac{\Delta^2}{\hat{\sigma}} \frac{I}{A} 21^u \\
&\quad + A \hat{\sigma} \left[\frac{2(1-\nu)}{1-2\nu} \right] \frac{I}{A} 21^u \\
&\quad - \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{\ell}{2} 11^u - \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{3\ell^2}{20} 21^u + \hat{\sigma} I 21^u + \left(\frac{E}{1+\nu} + \lambda \right) I 21^u \\
5^Q &= \frac{I \hat{\sigma}}{\ell} \left(\ell 11^u + \frac{\ell^2}{2} 21^u \right) + \frac{I}{\ell} \left(\frac{E}{1+\nu} + \lambda \right) \left(\ell 11^u + \frac{\ell^2}{2} 21^u \right) \\
&\quad + \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{\ell^2}{4} 11^u + \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{3\ell^3}{40} 21^u - \hat{\sigma} I \frac{\ell}{2} 21^u - \left(\frac{E}{1+\nu} + \lambda \right) I \frac{\ell}{2} 21^u \\
&\quad - \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{\ell^2}{6} 11^u - \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{\ell^3}{24} 21^u \\
6^Q &= - \frac{I \hat{\sigma}}{\ell} \left(\ell 11^u + \frac{\ell^2}{2} 21^u \right) - \frac{I}{\ell} \left(\frac{E}{1+\nu} + \lambda \right) \left(\ell 11^u + \frac{\ell^2}{2} 21^u \right) \\
&\quad + \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{\ell^2}{4} 11^u + \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{3\ell^3}{40} 21^u - \hat{\sigma} I \frac{\ell}{2} 21^u - \left(\frac{E}{1+\nu} + \lambda \right) I \frac{\ell}{2} 21^u \\
&\quad - \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{\ell^2}{3} 11^u - \frac{A \Delta}{\hat{\sigma} \sigma + G} \frac{\ell^3}{8} 21^u \tag{C.1-21}
\end{aligned}$$

It will be assumed that the preload is very small compared to the shear or elastic moduli (e.g., for steel $G = 11000$ ksi, $E = 29000$ ksi, and the yield stress is 36 ksi; so $\frac{\Delta}{\hat{\sigma}} < .0033G$ to be in the elastic

range). It will also be assumed that the beam is slender (e.g. $h \ll \ell$) so that $\frac{12I}{A\ell^2} \ll 1$ (recall $I = wh^3/12$, $A = wh$, so $\frac{12I}{A\ell^2} = \left(\frac{h}{\ell}\right)^2$). Equation (C.1-21) can then be written

$$\begin{aligned}
 1^Q &= -A\sigma - A\sigma_{10}^u - AE \left[\frac{1-\nu}{(1+\nu)(1-2\nu)} \right]_{10} u \\
 2^Q &= A\sigma + A\sigma_{10}^u + AE \left[\frac{1-\nu}{(1+\nu)(1-2\nu)} \right]_{10} u \\
 3^Q &= A\sigma_{01}^u + A\sigma \frac{\ell}{2} 11^u + A\sigma \frac{3\ell^2}{20} 21^u - EI \left[\frac{1-\nu}{(1+\nu)(1-2\nu)} \right]_{21} u \\
 4^Q &= -A\sigma_{01}^u - A\sigma \frac{\ell}{2} 11^u - A\sigma \frac{3\ell^2}{20} 21^u + EI \left[\frac{1-\nu}{(1+\nu)(1-2\nu)} \right]_{21} u \\
 5^Q &= +A\sigma \frac{\ell^2}{12} 11^u + A\sigma \frac{\ell^3}{30} 21^u + EI \left[\frac{1-\nu}{(1+\nu)(1-2\nu)} \right]_{11} u \\
 6^Q &= -A\sigma \frac{\ell^2}{12} 11^u - A\sigma \frac{\ell^3}{20} 21^u - EI \left[\frac{1-\nu}{(1+\nu)(1-2\nu)} \right]_{11} u \\
 &\quad - EI \left[\frac{1-\nu}{(1+\nu)(1-2\nu)} \right]_{\ell} 21^u
 \end{aligned} \tag{C.1-22}$$

The above noted approximations along with the constraints (C.1-19 and-20) may be used to re-express the second of Equations (C.1-1)

$$v = 0^v - 01^u \bar{x} - 11^u \frac{A}{G} \frac{\ell}{2} \bar{x} - 21^u \frac{A}{G} \frac{3\ell^2}{20} \bar{x} - 11^u \frac{1}{2} \bar{x}^2 - 21^u \frac{1}{6} \bar{x}^3 \tag{C.1-23}$$

The final integral in Equation (C.1-5) when converted to material event space assumes the form

$$\begin{aligned}
& \int_{\partial \bar{S}} \Big|_{x=0}^* \left\{ \left[\left(0^u + 01^u \bar{y} - 1^q + \bar{y} 5^q \right) \delta S^x + \left(0^v - 3^q \right) \delta S^y \right] J \right\} \\
& + \int_{\partial \bar{S}} \Big|_{x=l}^* \left\{ \left[\left(\ell + 0^u + 10^{u\ell} + 01^{u\bar{y}} + 11^{u\ell\bar{y}} + 21^u \frac{\ell^2}{2} \bar{y} \right. \right. \right. \\
& \qquad \qquad \qquad \left. \left. \left. - \ell - 2^q + \bar{y} 6^q \right) \delta S^x \right. \right. \\
& \qquad \qquad \qquad \left. \left. \left. + \left(0^v - 01^{u\ell} - 11^u \frac{\ell}{G} \frac{\ell^2}{2} - 21^u \frac{\ell}{G} \frac{3\ell^3}{20} - 11^u \frac{\ell^2}{2} \right. \right. \right. \\
& \qquad \qquad \qquad \left. \left. \left. - 21^u \frac{\ell^3}{6} - 4^q \right) \delta S^y \right] J \right\} \tag{C.1-24}
\end{aligned}$$

The stationary conditions are

$$\begin{aligned}
\delta S^x \Big|_{x=0}^- & : 0^u - 1^q = 0 \\
\delta S^y \Big|_{x=0}^- & : 01^u + 5^q = 0 \\
\delta S^y \Big|_{x=0}^- & : 0^v - 3^q = 0 \\
\delta S^x \Big|_{x=l}^- & : 0^u + 10^{u\ell} - 2^q = 0 \\
\delta S^y \Big|_{x=l}^- & : 01^u + 11^{u\ell} + 21^u \frac{\ell^2}{2} + 6^q = 0 \\
\delta S^y \Big|_{x=l}^- & : 0^v - 01^{u\ell} - 11^u \frac{\ell^2}{2} - 21^u \frac{\ell^3}{6} - 4^q = 0 \tag{C.1-25}
\end{aligned}$$

Solving for the displacement coefficients

$$\begin{aligned}
0^u & = 1^q \\
10^u & = -\frac{1}{\ell} 1^q + \frac{1}{\ell} 2^q \\
01^u & = -5^q \\
11^u & = \frac{6}{\ell^2} 3^q - \frac{6}{\ell^2} 4^q + \frac{4}{\ell} 5^q + \frac{2}{\ell} 6^q \\
21^u & = -\frac{12}{\ell^3} 3^q + \frac{12}{\ell^3} 4^q - \frac{6}{\ell^2} 5^q - \frac{6}{\ell^2} 6^q \\
0^v & = 3^q \tag{C.1-26}
\end{aligned}$$

Substituting this into Equation (C.1-22) the result may be expressed in matrix form as

$$Q = Q_{\sigma} + K_G q + K q \quad (C.1-27)$$

where

$$Q = [Q_1, Q_2, Q_3, Q_4, Q_5, Q_6]^T$$

$$Q_{\sigma} = [-A\delta, A\delta, 0, 0, 0, 0]^T$$

$$q = [q_1, q_2, q_3, q_4, q_5, q_6]^T$$

$$K_G = A\delta \begin{bmatrix} \frac{1}{\ell} & -\frac{1}{\ell} & 0 & 0 & 0 & 0 \\ -\frac{1}{\ell} & \frac{1}{\ell} & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/\ell - 9/5\ell & -3/\ell + 9/5\ell & -1+2 & -1 \\ 0 & 0 & = \frac{6}{5\ell} & = -\frac{6}{5\ell} & = \frac{1}{10} & = \frac{1}{10} \\ 0 & 0 & -3/\ell + 9/5\ell & 3/\ell - 9/5\ell & 1-2 & -1 \\ 0 & 0 & = -\frac{6}{5\ell} & = \frac{6}{5\ell} & = -\frac{1}{10} & = -\frac{1}{10} \\ 0 & 0 & 1/2 - 2/5 & -1/2 + 2/5 & \ell/3 - \ell/5 & \ell/6 - \ell/5 \\ 0 & 0 & = \frac{1}{10} & = -\frac{1}{10} & = \frac{2\ell}{15} & = -\frac{\ell}{30} \\ 0 & 0 & 1/2 + 3/5 & -1/2 - 3/5 & -\ell/3 + 3\ell/10 & -\ell/6 + 3\ell/10 \\ 0 & 0 & = \frac{1}{10} & = -\frac{1}{10} & = -\frac{\ell}{30} & = \frac{2\ell}{15} \end{bmatrix}$$

$$K = \frac{\bar{E}I}{\ell} \begin{bmatrix} \frac{A\ell}{I} & -\frac{A\ell}{I} & 0 & 0 & 0 & 0 \\ -\frac{A\ell}{I} & \frac{A\ell}{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 12/\ell & -12/\ell & 6 & 6 \\ 0 & 0 & -12/\ell & 12/\ell & -6 & -6 \\ 0 & 0 & 6 & -6 & 4\ell & 2\ell \\ 0 & 0 & -6 & 6 & -4\ell + 6\ell & -2\ell + 6\ell \\ 0 & 0 & = \underline{6} & = -\underline{6} & = \underline{2\ell} & = \underline{4\ell} \end{bmatrix}$$

where $\bar{E} = E \left(\frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} \right)$.

This result is identical to the geometrically nonlinear beam finite element usually obtained by assuming a stress distribution, except for \bar{E} replacing E . This is due to the fact that the simplified displacement field used in this example allows no extension in the lateral direction, and the associated stresses fail to vanish on the lateral surfaces. This deficiency may be corrected by selecting a more complete displacement field, or merely by replacing \bar{E} by E , noting that \bar{E} is the axial modulus for laterally restrained material, and E for laterally free material. That is, \bar{E} is the elastic coefficient A^{IIII} (C.1-4) giving the stress for unit strain with the other normal strains zero. The inverse set of elastic constants $B^{IIII} = \frac{1}{E}$, $B^{IIJJ} = -\frac{\nu}{E}$ provide the strains given the stresses. Thus for a unit stress with the other normal stresses zero, the strain is $\frac{1}{E}$, B^{IIII} .

C.2 Beam Dynamic Parameters

The purpose of this section is to develop the dynamic parameters for the finite element described in the previous appendix (C.1). The assumed position field may be obtained by substituting the results of Equations (C.1-19,-20,-26) in Equation (C.1-1)

$$\begin{aligned}
 \underline{P} = & \left\{ \bar{x} + \frac{1}{\ell} q + \bar{x} \left(-\frac{1}{\ell} \frac{1}{\ell} q + \frac{1}{\ell} \frac{2}{\ell} q \right) + \bar{y} \left(-\frac{5}{\ell} q \right) \right. \\
 & + \bar{x} \bar{y} \left(\frac{6}{\ell^2} \frac{3}{\ell} q - \frac{6}{\ell^2} \frac{4}{\ell} q + \frac{4}{\ell} \frac{5}{\ell} q + \frac{2}{\ell} \frac{6}{\ell} q \right) \\
 & \left. + \frac{1}{2} \bar{x}^2 \bar{y} \left(-\frac{12}{\ell^3} \frac{3}{\ell} q + \frac{12}{\ell^3} \frac{4}{\ell} q - \frac{6}{\ell^2} \frac{5}{\ell} q - \frac{6}{\ell^2} \frac{6}{\ell} q \right) \right\} \frac{\partial}{\partial x} \\
 & + \left\{ \bar{y} + \frac{3}{\ell} q + \bar{x} \left(\frac{5}{\ell} q \right) + \frac{1}{2} \bar{x}^2 \left(-\frac{6}{\ell^2} \frac{3}{\ell} q + \frac{6}{\ell^2} \frac{4}{\ell} q - \frac{4}{\ell} \frac{5}{\ell} q - \frac{2}{\ell} \frac{6}{\ell} q \right) \right. \\
 & \left. + \frac{1}{6} \bar{x}^3 \left(\frac{12}{\ell^3} \frac{3}{\ell} q - \frac{12}{\ell^3} \frac{4}{\ell} q + \frac{6}{\ell^2} \frac{5}{\ell} q + \frac{6}{\ell^2} \frac{6}{\ell} q \right) \right\} \frac{\partial}{\partial y} \\
 & + \{ \bar{z} \} \frac{\partial}{\partial z}
 \end{aligned} \tag{C.2-1}$$

From this, the functions $\underline{EO}^{P'}$, $\underline{1}^{P'}$ may be determined

$$\begin{aligned}
 \underline{EO}^{P'} &= \bar{x} \frac{\partial}{\partial x} + \bar{y} \frac{\partial}{\partial y} + \bar{z} \frac{\partial}{\partial z} \\
 \underline{1}^{P'} &= \left(1 - \frac{\bar{x}}{\ell}\right) \frac{\partial}{\partial x} \\
 \underline{2}^{P'} &= \left(\frac{\bar{x}}{\ell}\right) \frac{\partial}{\partial x} \\
 \underline{3}^{P'} &= \left(\frac{6}{\ell^2} \bar{x} \bar{y} - \frac{6}{\ell^3} \bar{x}^2 \bar{y}\right) \frac{\partial}{\partial x} + \left(1 - 3 \frac{\bar{x}^2}{\ell^2} + 2 \frac{\bar{x}^3}{\ell^3}\right) \frac{\partial}{\partial y} \\
 \underline{4}^{P'} &= \left(-\frac{6}{\ell^2} \bar{x} \bar{y} + \frac{6}{\ell^3} \bar{x}^2 \bar{y}\right) \frac{\partial}{\partial x} + \left(3 \frac{\bar{x}^2}{\ell^2} - 2 \frac{\bar{x}^3}{\ell^3}\right) \frac{\partial}{\partial y} \\
 \underline{5}^{P'} &= \left(-\bar{y} + \frac{4}{\ell} \bar{x} \bar{y} - \frac{3}{\ell^2} \bar{x}^2 \bar{y}\right) \frac{\partial}{\partial x} + \left(\bar{x} - \frac{2}{\ell} \bar{x}^2 + \frac{1}{\ell^2} \bar{x}^3\right) \frac{\partial}{\partial y} \\
 \underline{6}^{P'} &= \left(\frac{2}{\ell} \bar{x} \bar{y} - \frac{3}{\ell^2} \bar{x}^2 \bar{y}\right) \frac{\partial}{\partial x} + \left(-\frac{1}{\ell} \bar{x}^2 + \frac{1}{\ell^2} \bar{x}^3\right) \frac{\partial}{\partial y} \quad (C.2-2)
 \end{aligned}$$

Assuming a uniform density $\hat{\rho}$ the dynamic parameters may be computed as

$$\begin{aligned}
 M &= \int_S * \{\hat{\rho}\} \\
 &= \hat{\rho} w h \ell \quad (C.2-3)
 \end{aligned}$$

The center of mass is

$$\begin{aligned}
 \underline{EO}^{\bar{P}'} &= \int_S * \left\{ \hat{\rho} \underline{EO}^{P'} \right\} / M \\
 &= \left(\hat{\rho} h w \frac{\ell^2}{2} \right) \frac{\partial}{\partial x} / \hat{\rho} w h \ell \\
 &= \left(\frac{\ell}{2} \right) \frac{\partial}{\partial x}
 \end{aligned}$$

The shift in mass center due to generalized displacements are

$$\begin{aligned}
1\bar{P}' &= \int_S * \left\{ \hat{\rho} \ 1P' \right\} / M \\
&= \left[\hat{\rho} \ w \ h \ \left(\ell - \frac{\ell^2}{2\ell} \right) \right] \frac{\partial}{\partial x} / \hat{\rho} \ w \ h \ \ell \\
&= \left(\frac{1}{2} \right) \frac{\partial}{\partial x}
\end{aligned}$$

$$\begin{aligned}
2\bar{P}' &= \int_S * \left\{ \hat{\rho} \ 2P' \right\} / M \\
&= \left[\hat{\rho} \ w \ h \ \left(\frac{\ell^2}{2\ell} \right) \right] \frac{\partial}{\partial x} / \hat{\rho} \ w \ h \ \ell \\
&= \left(\frac{1}{2} \right) \frac{\partial}{\partial x}
\end{aligned}$$

$$\begin{aligned}
3\bar{P}' &= \int_S * \left\{ \hat{\rho} \ 3P' \right\} / M \\
&= \left[\hat{\rho} \ w \ h \ \left(\ell - 3 \frac{\ell^3}{3\ell^2} + 2 \frac{\ell^4}{4\ell^3} \right) \right] \frac{\partial}{\partial y} / \hat{\rho} \ w \ h \ \ell \\
&= \left(\frac{1}{2} \right) \frac{\partial}{\partial y}
\end{aligned}$$

$$\begin{aligned}
4\bar{P}' &= \int_S * \left\{ \hat{\rho} \ 4P' \right\} / M \\
&= \left[\hat{\rho} \ w \ h \ \left(3 \frac{\ell^3}{3\ell^2} - 2 \frac{\ell^4}{4\ell^3} \right) \right] \frac{\partial}{\partial y} / \hat{\rho} \ w \ h \ \ell \\
&= \left(\frac{1}{2} \right) \frac{\partial}{\partial y}
\end{aligned}$$

$$\begin{aligned}
5\bar{P}' &= \int_S * \left\{ \hat{\rho} \ 5P' \right\} / M \\
&= \left[\hat{\rho} \ w \ h \ \left(\frac{\ell^2}{2} - 2 \frac{\ell^3}{3\ell} + \frac{\ell^4}{4\ell^2} \right) \right] \frac{\partial}{\partial y} / \hat{\rho} \ w \ h \ \ell \\
&= \left(\frac{\ell}{12} \right) \frac{\partial}{\partial y}
\end{aligned}$$

$$\begin{aligned}
6\bar{P}' &= \int_S * \left\{ \hat{\rho} \ 6P' \right\} / M \\
&= \left[\hat{\rho} \ w \ h \ \left(- \frac{\ell^3}{3\ell} + \frac{\ell^4}{4\ell^2} \right) \right] \frac{\partial}{\partial y} / \hat{\rho} \ w \ h \ \ell \\
&= \left(- \frac{\ell}{12} \right) \frac{\partial}{\partial y}
\end{aligned}$$

(C.2-4)

The moment of inertia is

$$\begin{aligned}
\mathbb{I} &= \int_S * \left\{ \hat{\rho} \left[\left(\mathbf{EO}^{\mathbf{P}'} \cdot \mathbf{EO}^{\mathbf{P}'} \right) \mathbb{U} - \mathbf{EO}^{\mathbf{P}'} \mathbf{EO}^{\mathbf{P}'} \right] \right\} \\
&= \hat{\rho} \left[\left(wh \frac{\ell^3}{3} + w \left\{ \frac{h^3}{24} - \frac{-h^3}{24} \right\} \ell + \left\{ \frac{w^3}{24} - \frac{-w^3}{24} \right\} h\ell \right) \left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} \right. \right. \\
&\quad \left. \left. + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right) \right. \\
&\quad \left. - wh \frac{\ell^3}{3} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} - w \left\{ \frac{h^3}{24} - \frac{-h^3}{24} \right\} \ell \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \right. \\
&\quad \left. - \left\{ \frac{w^3}{24} - \frac{-w^3}{24} \right\} h\ell \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right] \\
&= M \left\{ \left[\frac{h^2}{12} + \frac{w^2}{12} \right] \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \left[\frac{\ell^2}{3} + \frac{w^2}{12} \right] \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} \right. \\
&\quad \left. + \left[\frac{\ell^2}{3} + \frac{h^2}{12} \right] \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right\} \tag{C.2-5}
\end{aligned}$$

The change in moment of inertia due to generalized displacement is

$$\begin{aligned}
{}_1\mathbb{I} &= \int_S * \left\{ \hat{\rho} \left[\left(\mathbf{EO}^{\mathbf{P}'} \cdot \mathbf{1}^{\mathbf{P}'} \right) \mathbb{U} - \mathbf{EO}^{\mathbf{P}'} \mathbf{1}^{\mathbf{P}'} \right] \right\} \\
&= \hat{\rho} \left[\left(wh \left\{ \frac{\ell^2}{2} - \frac{\ell^3}{3\ell} \right\} \right) \left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right) \right. \\
&\quad \left. - \left(wh \left\{ \frac{\ell^2}{2} - \frac{\ell^3}{3\ell} \right\} \right) \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} \right] \\
&= M \left\{ \frac{\ell}{6} \left(\frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right) \right\} \\
{}_2\mathbb{I} &= \int_S * \left\{ \hat{\rho} \left[\left(\mathbf{EO}^{\mathbf{P}'} \cdot \mathbf{2}^{\mathbf{P}'} \right) \mathbb{U} - \mathbf{EO}^{\mathbf{P}'} \mathbf{2}^{\mathbf{P}'} \right] \right\} \\
&= \hat{\rho} \left[wh \frac{\ell^3}{3\ell} \left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right) - \left(wh \frac{\ell^3}{3\ell} \right) \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} \right] \\
&= M \left\{ \frac{\ell}{3} \left(\frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right) \right\}
\end{aligned}$$

Equation (C.2-6) continued

$$\begin{aligned}
3^{\text{I}} &= \int_S^{\text{A}} * \left\{ \left[\left(\text{EO}^{\text{P}'} \cdot 3^{\text{P}'} \right) \underline{\text{U}} - \text{EO}^{\text{P}'} 3^{\text{P}'} \right] \right\} \\
&= \frac{\text{A}}{\rho} \left[-wh \left(\frac{\ell^2}{2} - 3 \frac{\ell^4}{4\ell^2} + 2 \frac{\ell^5}{5\ell^3} \right) \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - w \left(\frac{h^3}{24} - \frac{-h^3}{24} \right) \left(\frac{6}{\ell^2} \frac{\ell^2}{2} \right. \right. \\
&\quad \left. \left. - \frac{6}{\ell^3} \frac{\ell^3}{3} \right) \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right] \\
&= \text{M} \left\{ - \frac{3\ell}{20} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{h^2}{12\ell} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right\} \\
4^{\text{I}} &= \int_S^{\text{A}} * \left\{ \left[\left(\text{EO}^{\text{P}'} \cdot 4^{\text{P}'} \right) \underline{\text{U}} - \text{EO}^{\text{P}'} 4^{\text{P}'} \right] \right\} \\
&= \frac{\text{A}}{\rho} \left[-wh \left(3 \frac{\ell^4}{4\ell^2} - 2 \frac{\ell^5}{5\ell^3} \right) \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - w \left(\frac{h^3}{24} - \frac{-h^3}{24} \right) \left(- \frac{6}{\ell^2} \frac{\ell^2}{2} + \frac{6}{\ell^3} \frac{\ell^3}{3} \right) \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right] \\
&= \text{M} \left\{ - \frac{7\ell}{20} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \frac{h^2}{12\ell} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right\} \\
5^{\text{I}} &= \int_S^{\text{A}} * \left\{ \left[\left(\text{EO}^{\text{P}'} \cdot 5^{\text{P}'} \right) \underline{\text{U}} - \text{EO}^{\text{P}'} 5^{\text{P}'} \right] \right\} \\
&= \frac{\text{A}}{\rho} \left[-wh \left(\frac{\ell^3}{3} - \frac{2}{\ell} \frac{\ell^4}{4} + \frac{1}{\ell^2} \frac{\ell^5}{5} \right) \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \right. \\
&\quad \left. - w \left(\frac{h^3}{24} - \frac{-h^3}{24} \right) \left(-\ell + \frac{4}{\ell} \frac{\ell^2}{2} - \frac{3}{\ell^2} \frac{\ell^3}{3} \right) \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right] \\
&= \text{M} \left\{ - \frac{\ell^2}{30} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \right\} \\
6^{\text{I}} &= \int_S^{\text{A}} * \left\{ \left[\left(\text{EO}^{\text{P}'} \cdot 6^{\text{P}'} \right) \underline{\text{U}} - \text{EO}^{\text{P}'} 6^{\text{P}'} \right] \right\} \\
&= \frac{\text{A}}{\rho} \left[-wh \left(- \frac{1}{\ell} \frac{\ell^4}{4} + \frac{1}{\ell^2} \frac{\ell^5}{5} \right) \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \right. \\
&\quad \left. - w \left(\frac{h^3}{24} - \frac{-h^3}{24} \right) \left(\frac{2}{\ell} \frac{\ell^2}{2} - \frac{3}{\ell^2} \frac{\ell^3}{3} \right) \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \right] \\
&= \text{M} \left\{ + \frac{\ell^2}{20} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \right\}
\end{aligned}$$

(C.2-6)

The first order moment terms are

$$1^0 = \int_S * \left\{ \hat{\rho} \text{EO}^{\underline{P}'} \times 1^{\underline{P}'} \right\}$$

$$= \underline{\emptyset}$$

$$2^0 = \int_S * \left\{ \hat{\rho} \text{EO}^{\underline{P}'} \times 2^{\underline{P}'} \right\}$$

$$= \underline{\emptyset}$$

$$3^0 = \int_S * \left\{ \hat{\rho} \text{EO}^{\underline{P}'} \times 3^{\underline{P}'} \right\}$$

$$= \hat{\rho} \left[\text{wh} \left(\frac{\ell^2}{2} - 3 \frac{\ell^4}{4\ell^2} + 2 \frac{\ell^5}{5\ell^3} \right) - w \left(\frac{h^3}{24} - \frac{-h^3}{24} \right) \left(\frac{6}{\ell^2} \frac{\ell^2}{2} - \frac{6}{\ell^3} \frac{\ell^3}{3} \right) \right] \frac{\partial}{\partial z}$$

$$= M \left\{ \frac{3\ell}{20} - \frac{h^2}{12\ell} \right\} \frac{\partial}{\partial z}$$

$$4^0 = \int_S * \left\{ \hat{\rho} \text{EO}^{\underline{P}'} \times 4^{\underline{P}'} \right\}$$

$$= \hat{\rho} \left[\text{wh} \left(3 \frac{\ell^4}{4\ell^2} - 2 \frac{\ell^5}{5\ell^3} \right) - w \left(\frac{h^3}{24} - \frac{-h^3}{24} \right) \left(-\frac{6}{\ell^2} \frac{\ell^2}{2} + \frac{6}{\ell^3} \frac{\ell^3}{3} \right) \right] \frac{\partial}{\partial z}$$

$$= M \left\{ \frac{7\ell}{20} + \frac{h^2}{12\ell} \right\} \frac{\partial}{\partial z}$$

$$5^0 = \int_S * \left\{ \hat{\rho} \text{EO}^{\underline{P}'} \times 5^{\underline{P}'} \right\}$$

$$= \hat{\rho} \left[\text{wh} \left(\frac{\ell^3}{3} - \frac{2}{\ell} \frac{\ell^4}{4} + \frac{1}{2} \frac{\ell^5}{5} \right) - w \left(\frac{h^3}{24} - \frac{-h^3}{24} \right) \left(-\ell + \frac{4}{\ell} \frac{\ell^2}{2} - \frac{3}{\ell^2} \frac{\ell^3}{3} \right) \right] \frac{\partial}{\partial z}$$

$$= M \left\{ \frac{\ell^2}{30} \right\} \frac{\partial}{\partial z}$$

$$6^0 = \int_S * \left\{ \hat{\rho} \text{EO}^{\underline{P}'} \times 6^{\underline{P}'} \right\}$$

$$= \hat{\rho} \left[\text{wh} \left(-\frac{1}{\ell} \frac{\ell^4}{4} + \frac{1}{\ell^2} \frac{\ell^5}{5} \right) - w \left(\frac{h^3}{24} - \frac{-h^3}{24} \right) \left(\frac{2}{\ell} \frac{\ell^2}{2} - \frac{3}{\ell^2} \frac{\ell^3}{3} \right) \right] \frac{\partial}{\partial z}$$

$$= M \left\{ -\frac{\ell^2}{20} \right\} \frac{\partial}{\partial z}$$

(C.2-7)

It may be noted that these calculations are of the same form as some of the $i^{\underline{I}}$ terms and need not be separately computed. The second order changes in moment of inertia are

$$\begin{aligned}
11^{\underline{I}} &= \int_{\underline{S}} * \left\{ \hat{\rho} \left[\left(\underline{1}^{\underline{P}'} \cdot \underline{1}^{\underline{P}'} \right) \underline{U} - \underline{1}^{\underline{P}'} \underline{1}^{\underline{P}'} \right] \right\} \\
&= \hat{\rho} \left[\left(\text{wh} \left\{ \ell - \frac{2}{\ell} \frac{\ell^2}{2} + \frac{1}{\ell^2} \frac{\ell^3}{3} \right\} \right) \left(\frac{\partial}{\partial \underline{x}} \otimes \frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{y}} \otimes \frac{\partial}{\partial \underline{y}} + \frac{\partial}{\partial \underline{z}} \otimes \frac{\partial}{\partial \underline{z}} \right) \right. \\
&\quad \left. - \left(\text{wh} \left\{ \ell - \frac{2}{\ell} \frac{\ell^2}{2} + \frac{1}{\ell^2} \frac{\ell^3}{3} \right\} \right) \frac{\partial}{\partial \underline{x}} \otimes \frac{\partial}{\partial \underline{x}} \right] \\
&= M \left\{ \frac{1}{3} \right\} \left(\frac{\partial}{\partial \underline{y}} \otimes \frac{\partial}{\partial \underline{y}} + \frac{\partial}{\partial \underline{z}} \otimes \frac{\partial}{\partial \underline{z}} \right) \\
12^{\underline{I}} &= \int_{\underline{S}} * \left\{ \hat{\rho} \left[\left(\underline{1}^{\underline{P}'} \cdot \underline{2}^{\underline{P}'} \right) \underline{U} - \underline{1}^{\underline{P}'} \underline{2}^{\underline{P}'} \right] \right\} \\
&= \hat{\rho} \left[\left(\text{wh} \left\{ \frac{1}{\ell} \frac{\ell^2}{2} - \frac{1}{\ell^2} \frac{\ell^3}{3} \right\} \right) \left(\frac{\partial}{\partial \underline{x}} \otimes \frac{\partial}{\partial \underline{x}} + \frac{\partial}{\partial \underline{y}} \otimes \frac{\partial}{\partial \underline{y}} + \frac{\partial}{\partial \underline{z}} \otimes \frac{\partial}{\partial \underline{z}} \right) \right. \\
&\quad \left. - \left(\text{wh} \left\{ \frac{1}{\ell} \frac{\ell^2}{2} - \frac{1}{\ell^2} \frac{\ell^3}{3} \right\} \right) \frac{\partial}{\partial \underline{x}} \otimes \frac{\partial}{\partial \underline{x}} \right] \\
&= M \left\{ \frac{1}{6} \right\} \left(\frac{\partial}{\partial \underline{y}} \otimes \frac{\partial}{\partial \underline{y}} + \frac{\partial}{\partial \underline{z}} \otimes \frac{\partial}{\partial \underline{z}} \right) \\
13^{\underline{I}} &= \int_{\underline{S}} * \left\{ \hat{\rho} \left[\left(\underline{1}^{\underline{P}'} \cdot \underline{3}^{\underline{P}'} \right) \underline{U} - \underline{1}^{\underline{P}'} \underline{3}^{\underline{P}'} \right] \right\} \\
&= \hat{\rho} \left[- \text{wh} \left(\ell - \frac{3}{\ell^2} \frac{\ell^3}{3} + \frac{2}{\ell^3} \frac{\ell^4}{4} - \frac{1}{\ell} \frac{\ell^2}{2} + \frac{3}{\ell^3} \frac{\ell^4}{4} - \frac{2}{\ell^4} \frac{\ell^5}{5} \right) \frac{\partial}{\partial \underline{x}} \otimes \frac{\partial}{\partial \underline{y}} \right] \\
&= - M \left\{ \frac{7}{20} \right\} \frac{\partial}{\partial \underline{x}} \otimes \frac{\partial}{\partial \underline{y}} \\
14^{\underline{I}} &= \int_{\underline{S}} * \left\{ \hat{\rho} \left[\left(\underline{1}^{\underline{P}'} \cdot \underline{4}^{\underline{P}'} \right) \underline{U} - \underline{1}^{\underline{P}'} \underline{4}^{\underline{P}'} \right] \right\} \\
&= - M \left\{ \frac{3}{20} \right\} \frac{\partial}{\partial \underline{x}} \otimes \frac{\partial}{\partial \underline{y}} \\
15^{\underline{I}} &= \int_{\underline{S}} * \left\{ \hat{\rho} \left[\left(\underline{1}^{\underline{P}'} \cdot \underline{5}^{\underline{P}'} \right) \underline{U} - \underline{1}^{\underline{P}'} \underline{5}^{\underline{P}'} \right] \right\} \\
&= - M \ell \left\{ \frac{1}{20} \right\} \frac{\partial}{\partial \underline{x}} \otimes \frac{\partial}{\partial \underline{y}} \\
16^{\underline{I}} &= \int_{\underline{S}} * \left\{ \hat{\rho} \left[\left(\underline{1}^{\underline{P}'} \cdot \underline{6}^{\underline{P}'} \right) \underline{U} - \underline{1}^{\underline{P}'} \underline{6}^{\underline{P}'} \right] \right\} \\
&= + M \ell \left\{ \frac{1}{30} \right\} \frac{\partial}{\partial \underline{x}} \otimes \frac{\partial}{\partial \underline{y}}
\end{aligned}$$

Equation (C.2-8) continued.

$$\begin{aligned}
22^{\text{I}} &= \int_{\text{S}} * \left\{ \rho^{\wedge} \left[(2^{\text{P}'} \cdot 2^{\text{P}'})_{\text{U}} - 2^{\text{P}'} 2^{\text{P}'} \right] \right\} \\
&= M \left\{ \frac{1}{3} \right\} \left(\frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right) \\
23^{\text{I}} &= \int_{\text{S}} * \left\{ \rho^{\wedge} \left[(2^{\text{P}'} \cdot 3^{\text{P}'})_{\text{U}} - 2^{\text{P}'} 3^{\text{P}'} \right] \right\} \\
&= - M \left\{ \frac{3}{20} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \\
24^{\text{I}} &= \int_{\text{S}} * \left\{ \rho^{\wedge} \left[(2^{\text{P}'} \cdot 4^{\text{P}'})_{\text{U}} - 2^{\text{P}'} 4^{\text{P}'} \right] \right\} \\
&= - M \left\{ \frac{7}{20} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \\
25^{\text{I}} &= \int_{\text{S}} * \left\{ \rho^{\wedge} \left[(2^{\text{P}'} \cdot 5^{\text{P}'})_{\text{U}} - 2^{\text{P}'} 5^{\text{P}'} \right] \right\} \\
&= - M \ell \left\{ \frac{1}{30} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \\
26^{\text{I}} &= \int_{\text{S}} * \left\{ \rho^{\wedge} \left[(2^{\text{P}'} \cdot 6^{\text{P}'})_{\text{U}} - 2^{\text{P}'} 6^{\text{P}'} \right] \right\} \\
&= + M \ell \left\{ \frac{1}{20} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} \\
33^{\text{I}} &= \int_{\text{S}} * \left\{ \rho^{\wedge} \left[(3^{\text{P}'} \cdot 3^{\text{P}'})_{\text{U}} - 3^{\text{P}'} 3^{\text{P}'} \right] \right\} \\
&= M \left[\left\{ \frac{13}{35} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \left\{ \frac{1}{10} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ \frac{1}{10} \frac{h^2}{\ell^2} + \frac{13}{35} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right] \\
34^{\text{I}} &= \int_{\text{S}} * \left\{ \rho^{\wedge} \left[(3^{\text{P}'} \cdot 4^{\text{P}'})_{\text{U}} - 3^{\text{P}'} 4^{\text{P}'} \right] \right\} \\
&= M \left[\left\{ \frac{9}{70} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} - \left\{ \frac{1}{10} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ -\frac{1}{10} \frac{h^2}{\ell^2} + \frac{9}{70} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right] \\
35^{\text{I}} &= \int_{\text{S}} * \left\{ \rho^{\wedge} \left[(3^{\text{P}'} \cdot 5^{\text{P}'})_{\text{U}} - 3^{\text{P}'} 5^{\text{P}'} \right] \right\} \\
&= M \ell \left[\left\{ \frac{11}{210} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \left\{ \frac{1}{120} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ \frac{1}{120} \frac{h^2}{\ell^2} + \frac{11}{210} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right]
\end{aligned}$$

Equation (C.2-8) continued

$$\begin{aligned}
36\bar{I} &= \int_S * \left\{ \hat{\rho} \left[\left(3^{P'} \cdot 6^{P'} \right) \underline{U} - 3^{P'} 6^{P'} \right] \right\} \\
&= M\ell \left[- \left\{ \frac{13}{420} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \left\{ \frac{1}{120} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ \frac{1}{120} \frac{h^2}{\ell^2} - \frac{13}{420} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right] \\
44\bar{I} &= \int_S * \left\{ \hat{\rho} \left[\left(4^{P'} \cdot 4^{P'} \right) \underline{U} - 4^{P'} 4^{P'} \right] \right\} \\
&= M \left[\left\{ \frac{13}{35} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \left\{ \frac{1}{10} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ \frac{1}{10} \frac{h^2}{\ell^2} + \frac{13}{35} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right] \\
45\bar{I} &= \int_S * \left\{ \hat{\rho} \left[\left(4^{P'} \cdot 5^{P'} \right) \underline{U} - 4^{P'} 5^{P'} \right] \right\} \\
&= M\ell \left[\left\{ \frac{13}{420} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} - \left\{ \frac{1}{120} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ - \frac{1}{120} \frac{h^2}{\ell^2} + \frac{13}{420} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right] \\
46\bar{I} &= \int_S * \left\{ \hat{\rho} \left[\left(4^{P'} \cdot 6^{P'} \right) \underline{U} - 4^{P'} 6^{P'} \right] \right\} \\
&= - M\ell \left[\left\{ \frac{11}{210} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \left\{ \frac{1}{120} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ \frac{1}{120} \frac{h^2}{\ell^2} + \frac{11}{120} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right] \\
55\bar{I} &= \int_S * \left\{ \hat{\rho} \left[\left(5^{P'} \cdot 5^{P'} \right) \underline{U} - 5^{P'} 5^{P'} \right] \right\} \\
&= M\ell^2 \left[\left\{ \frac{1}{105} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \left\{ \frac{1}{90} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ \frac{1}{90} \frac{h^2}{\ell^2} + \frac{1}{105} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right] \\
56\bar{I} &= \int_S * \left\{ \hat{\rho} \left[\left(5^{P'} \cdot 6^{P'} \right) \underline{U} - 5^{P'} 6^{P'} \right] \right\} \\
&= - M\ell^2 \left[\left\{ \frac{1}{140} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \left\{ \frac{1}{360} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ \frac{1}{360} \frac{h^2}{\ell^2} + \frac{1}{140} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right] \\
66\bar{I} &= \int_S * \left\{ \hat{\rho} \left[\left(6^{P'} \cdot 6^{P'} \right) \underline{U} - 6^{P'} 6^{P'} \right] \right\} \\
&= M\ell^2 \left[\left\{ \frac{1}{105} \right\} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} + \left\{ \frac{1}{90} \frac{h^2}{\ell^2} \right\} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial y} + \left\{ \frac{1}{90} \frac{h^2}{\ell^2} + \frac{1}{105} \right\} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} \right]
\end{aligned} \tag{C.2-8}$$

where the remaining terms may be determined by the symmetry relation:

$$ab\bar{I} = ab I^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} = ba I^{\beta\alpha} \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} \tag{C.2-9}$$

(e.g., $35^{\underline{I}} = 53^{\underline{I}}$ but $16^{\underline{I}} = \frac{M\ell}{30} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}$ and $61^{\underline{I}} = \frac{M\ell}{30} \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x}$).

Similarly, the second order moment terms are

$$\begin{aligned} 13^{\underline{O}} &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} 1^{\underline{P}'} \times 3^{\underline{P}'} \right\} \\ &= M \left\{ \frac{7}{20} \right\} \frac{\partial}{\partial z} \end{aligned}$$

$$\begin{aligned} 14^{\underline{O}} &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} 1^{\underline{P}'} \times 4^{\underline{P}'} \right\} \\ &= M \left\{ \frac{3}{20} \right\} \frac{\partial}{\partial z} \end{aligned}$$

$$\begin{aligned} 15^{\underline{O}} &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} 1^{\underline{P}'} \times 5^{\underline{P}'} \right\} \\ &= M\ell \left\{ \frac{1}{20} \right\} \frac{\partial}{\partial z} \end{aligned}$$

$$\begin{aligned} 16^{\underline{O}} &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} 1^{\underline{P}'} \times 6^{\underline{P}'} \right\} \\ &= - M\ell \left\{ \frac{1}{30} \right\} \frac{\partial}{\partial z} \end{aligned}$$

$$\begin{aligned} 23^{\underline{O}} &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} 2^{\underline{P}'} \times 3^{\underline{P}'} \right\} \\ &= M \left\{ \frac{3}{20} \right\} \frac{\partial}{\partial z} \end{aligned}$$

$$\begin{aligned} 24^{\underline{O}} &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} 2^{\underline{P}'} \times 4^{\underline{P}'} \right\} \\ &= M \left\{ \frac{7}{20} \right\} \frac{\partial}{\partial z} \end{aligned}$$

$$\begin{aligned} 25^{\underline{O}} &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} 2^{\underline{P}'} \times 5^{\underline{P}'} \right\} \\ &= M\ell \left\{ \frac{1}{30} \right\} \frac{\partial}{\partial z} \end{aligned}$$

$$\begin{aligned} 26^{\underline{O}} &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} 2^{\underline{P}'} \times 6^{\underline{P}'} \right\} \\ &= - M\ell \left\{ \frac{1}{20} \right\} \frac{\partial}{\partial z} \end{aligned}$$

(C.2-10)

The elements with interchanged subscripts are obtained from antisymmetry ($\underline{a}_b^0 = -\underline{a}_a^b$). All other elements are zero. The mass matrix elements for a consistent formulation (i.e., assumed velocity field the same as the assumed displacement field) are

$$\begin{aligned}
 11^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{1}^{\underline{P}'} \cdot \underline{1}^{\underline{P}'} \right\} = M \left\{ \frac{1}{3} \right\} \\
 12^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{1}^{\underline{P}'} \cdot \underline{2}^{\underline{P}'} \right\} = M \left\{ \frac{1}{6} \right\} \\
 13^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{1}^{\underline{P}'} \cdot \underline{3}^{\underline{P}'} \right\} = 0 \\
 14^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{1}^{\underline{P}'} \cdot \underline{4}^{\underline{P}'} \right\} = 0 \\
 15^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{1}^{\underline{P}'} \cdot \underline{5}^{\underline{P}'} \right\} = 0 \\
 16^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{1}^{\underline{P}'} \cdot \underline{6}^{\underline{P}'} \right\} = 0 \\
 22^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{2}^{\underline{P}'} \cdot \underline{2}^{\underline{P}'} \right\} = M \left\{ \frac{1}{3} \right\} \\
 23^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{2}^{\underline{P}'} \cdot \underline{3}^{\underline{P}'} \right\} = 0 \\
 24^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{2}^{\underline{P}'} \cdot \underline{4}^{\underline{P}'} \right\} = 0 \\
 25^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{2}^{\underline{P}'} \cdot \underline{5}^{\underline{P}'} \right\} = 0 \\
 26^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{2}^{\underline{P}'} \cdot \underline{6}^{\underline{P}'} \right\} = 0 \\
 33^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{3}^{\underline{P}'} \cdot \underline{3}^{\underline{P}'} \right\} = M \left\{ \frac{1}{10} \frac{h^2}{\ell^2} + \frac{13}{35} \right\} \\
 34^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{3}^{\underline{P}'} \cdot \underline{4}^{\underline{P}'} \right\} = M \left\{ -\frac{1}{10} \frac{h^2}{\ell^2} + \frac{9}{70} \right\} \\
 35^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{3}^{\underline{P}'} \cdot \underline{5}^{\underline{P}'} \right\} = M\ell \left\{ \frac{1}{120} \frac{h^2}{\ell^2} + \frac{11}{210} \right\} \\
 36^M &= \int_{\underline{S}} * \left\{ \overset{\wedge}{\rho} \underline{3}^{\underline{P}'} \cdot \underline{6}^{\underline{P}'} \right\} = M\ell \left\{ \frac{1}{120} \frac{h^2}{\ell^2} - \frac{13}{420} \right\}
 \end{aligned}$$

Equation (C.2-11) continued

$$\begin{aligned}
44^M &= \int_S * \left\{ \hat{\rho} \left[4^{P'} \cdot 4^{P'} \right] \right\} = M \left\{ \frac{1}{10} \frac{h^2}{\ell^2} + \frac{13}{35} \right\} \\
45^M &= \int_S * \left\{ \hat{\rho} \left[4^{P'} \cdot 5^{P'} \right] \right\} = M\ell \left\{ \frac{-1}{120} \frac{h^2}{\ell^2} + \frac{13}{420} \right\} \\
46^M &= \int_S * \left\{ \hat{\rho} \left[4^{P'} \cdot 6^{P'} \right] \right\} = M\ell \left\{ \frac{-1}{120} \frac{h^2}{\ell^2} - \frac{11}{210} \right\} \\
55^M &= \int_S * \left\{ \hat{\rho} \left[5^{P'} \cdot 5^{P'} \right] \right\} = M\ell^2 \left\{ \frac{1}{90} \frac{h^2}{\ell^2} + \frac{1}{105} \right\} \\
56^M &= \int_S * \left\{ \hat{\rho} \left[5^{P'} \cdot 6^{P'} \right] \right\} = M\ell^2 \left\{ \frac{-1}{360} \frac{h^2}{\ell^2} - \frac{1}{140} \right\} \\
66^M &= \int_S * \left\{ \hat{\rho} \left[6^{P'} \cdot 6^{P'} \right] \right\} = M\ell^2 \left\{ \frac{1}{90} \frac{h^2}{\ell^2} + \frac{1}{105} \right\} \quad (C.2-11)
\end{aligned}$$

where the remaining elements are determined by symmetry (i.e., $i_j^M = j_i^M$). It may be noted that if the beam is slender ($\frac{h}{\ell} \ll 1$), then the terms involving $(h/\ell)^2$ are negligible. The mass matrix is then identical to the consistent mass matrix for a slender beam as presented in the literature.

C.3 Substructure Modes

In this section modes for a very simple substructure are calculated using Hurty's method. The substructure selected for this example consists of a single beam finite element in a stationary reference frame (see Sections C.1 and C.2 of this appendix). It will be assumed that the beam is attached to other substructures at 1^q , 2^q , 3^q and 5^q . Out of plane deformations are not included and the preload is assumed to be zero.

For this special case the equations of motion (2.3-5) become

$$j\ell^M \ell \ddot{\hat{q}} + j\ell^K \ell \hat{q} = j^Q \quad (C.3-1)$$

where M and K are obtained from Equations (C.1-17) and (C.2-11), respectively. Rearranging the rows and columns to 4,6,1,2,3,5 to group internal (4,6) and boundary (1,2,3,5) nodes, the equations may be partitioned

$$\begin{bmatrix} M_{II} & M_{IB} \\ - & - \\ M_{IB}^T & M_{BB} \end{bmatrix} \begin{Bmatrix} \ddot{q}_I \\ - \\ \ddot{q}_B \end{Bmatrix} + \begin{bmatrix} K_{II} & K_{IB} \\ - & - \\ K_{IB}^T & K_{BB} \end{bmatrix} \begin{Bmatrix} q_I \\ - \\ q_B \end{Bmatrix} = \begin{Bmatrix} Q_I \\ - \\ Q_B \end{Bmatrix} \quad (C.3-2)$$

The inverse of K_{II} may be computed as

$$\frac{\ell^2}{EI} \begin{bmatrix} \ell/3 & 1/2 \\ 1/2 & 1/\ell \end{bmatrix} \quad (C.3-3)$$

The static shapes are

$$\begin{aligned} -K_{II}^{-1} K_{IB} &= \frac{-\ell^2}{EI} \begin{bmatrix} \ell/3 & 1/2 \\ - & - \\ 1/2 & 1/\ell \end{bmatrix} \frac{EI}{\ell^2} \begin{bmatrix} 0 & 0 & -12/\ell & -6 \\ - & - & - & - \\ 0 & 0 & 6 & 2\ell \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & \ell \\ - & - & - & - \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (C.3-4)$$

The modes of the internal degrees of freedom satisfy

$$\left[K_{II}^{-1} M_{II} - {}_i\lambda U \right] \{ {}_i\phi \} = 0 \quad (C.3-5)$$

where ${}_i\lambda = 1/{}_i\omega^2$, the inverse of the square of the frequency in radians per second, and ${}_i\phi$ is the associated mode shape. This yields the characteristic equation

$$\begin{aligned}
0 &= \text{DET} \left\{ \frac{\ell^2}{EI} \begin{bmatrix} \ell/3 & 1/2 \\ 1/2 & 1/\ell \end{bmatrix} M \begin{bmatrix} 13/35 & -11\ell/210 \\ -11\ell/210 & \ell^2/105 \end{bmatrix} - \begin{bmatrix} {}_1\lambda & 0 \\ 0 & {}_1\lambda \end{bmatrix} \right\} \\
&= \text{DET} \left\{ \frac{M\ell^2}{EI} \begin{bmatrix} 52\ell/420 - 11\ell/420 - {}_1\lambda'\ell & -11\ell^2/630 + 3\ell^2/630 \\ \text{-----} & \text{-----} \\ 39/210 - 11/210 & -11\ell/420 + 4\ell/420 - {}_1\lambda'\ell \end{bmatrix} \right\} \\
&= \text{DET} \begin{bmatrix} 41/420 - {}_1\lambda' & -4\ell/315 \\ \text{-----} & \text{-----} \\ 2/15\ell & -1/60 - {}_1\lambda' \end{bmatrix} \\
&= {}_1\lambda'^2 - 17/210 {}_1\lambda' - 41/25200 + 8/4725 \\
&= {}_1\lambda'^2 - 17/210 {}_1\lambda' + 1/15120 \tag{C.3-6}
\end{aligned}$$

where $\lambda' = \frac{EI}{M\ell^3} \lambda$. The roots are

$${}_1\lambda' = .08013, \text{ and } {}_2\lambda' = .0008254 \tag{C.3-7}$$

with associated frequencies (radians/second)

$${}_1\omega = 3.533 \sqrt{\frac{EI}{M\ell^3}} \text{ and } {}_2\omega = 34.81 \sqrt{\frac{EI}{M\ell^3}} \tag{C.3-8}$$

It may be noted the first frequency is quite accurate (the exact continuum mechanics solution is $3.52 \sqrt{\frac{EI}{M\ell^3}}$). The second is much

less accurate (exact is $22.4 \sqrt{\frac{EI}{M\ell^3}}$). The associated mode shapes are

$${}_1\phi = \begin{Bmatrix} .5875 \\ .8092/\ell \end{Bmatrix} \text{ and } {}_2\phi = \begin{Bmatrix} .1301 \\ .9915/\ell \end{Bmatrix} \tag{C.3-9}$$

Now assuming only the lower frequency is of interest, or merely noting that this single element model cannot accurately reflect behavior for higher frequencies, the higher mode is discarded. Thus the original coordinates are related to a new set of coordinates by

$$\begin{Bmatrix} q_4 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \\ q_5 \end{Bmatrix} = \begin{bmatrix} .5875 & 0 & 0 & 1 & \ell \\ .8092/\ell & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta \\ q_1 \\ q_2 \\ q_3 \\ q_5 \end{Bmatrix} \quad (\text{C.3-10})$$

Transforming the mass matrix to these new coordinates gives

$$M = \begin{bmatrix} .08462 & 0. & 0. & .2263 & .1652\ell \\ 0. & 1/3 & 1/6 & 0. & 0. \\ 0. & 1/6 & 1/3 & 0. & 0. \\ .2263 & 0. & 0. & 1. & \ell/2 \\ .1652\ell & 0. & 0. & \ell/2 & \ell^2/3 \end{bmatrix} \quad (\text{C.3-11})$$

Similarly, transforming the stiffness matrix provides

$$\frac{\overline{EI}}{\ell^2} \begin{bmatrix} 1.0561 & 0. & 0. & 0. & 0. \\ 0. & A\ell/I & -A\ell/I & 0. & 0. \\ 0. & -A\ell/I & A\ell/I & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. \end{bmatrix} \quad (\text{C.3-12})$$

The reduced dimension mass and stiffness in an equation of the form (C.3-1) provide a reduced dimension set of differential equations in the modal amplitude, η , and the four boundary coordinates which describe the low frequency behavior. The reduction is not particularly impressive in this example since there are more boundary than internal nodes. However, in practice, the number of internal nodes is usually much greater than the number of boundary nodes. Discarding half of the modes in such cases results in substantial savings.

APPENDIX D

QUADRATIC TERMS

This appendix demonstrates the adequacy of a linear expansion. The unimportance of quadratic terms is demonstrated both analytically and by example. This material has been removed to this appendix in an effort to preserve the continuity of the text.

D.1 Quadratic Terms in Lagrangian

The purpose of this section is to analytically demonstrate that in a Lagrangian formulation truncation of an assumed quadratic expansion of the displacement field to linear order results in equivalent (although not identical) linearized equations. Let the assumed displacement field be of the form

$$\underline{P} = \underline{0}^P + \underline{i}^P \underline{i}^q + \underline{ij}^P \underline{i}^q \underline{j}^q, \quad (\text{D.1-1})$$

where the repeated Latin subscripts imply summation over all the generalized coordinates. Similarly, body and surface forces may be expressed in terms of the generalized coordinates as

$$\begin{aligned} \underline{X} &= \underline{0}^X + \underline{i}^X \underline{i}^q, \text{ and} \\ \underline{S} &= \underline{0}^S + \underline{i}^S \underline{i}^q. \end{aligned} \quad (\text{D.1-2})$$

The generalized forces may be computed.

$$\begin{aligned} \underline{i}^Q \delta \underline{i}^q &= \int_{\underline{S}} * \{ \underline{X} \cdot \delta \underline{P} \} + \int_{\partial M(t)} * \{ \underline{S} \cdot \delta \underline{P} \} \\ &= \int_{\underline{S}} * \left\{ \underline{X} \cdot \frac{\partial \underline{P}}{\partial \underline{i}^q} \delta \underline{i}^q \right\} + \int_{\partial M(t)} * \left\{ \underline{S} \cdot \frac{\partial \underline{P}}{\partial \underline{i}^q} \delta \underline{i}^q \right\} \\ \Rightarrow \underline{i}^Q &= \int_{\underline{S}} * \left\{ \underline{X} \cdot \frac{\partial \underline{P}}{\partial \underline{i}^q} \right\} + \int_{\partial M(t)} * \left\{ \underline{S} \cdot \frac{\partial \underline{P}}{\partial \underline{i}^q} \right\}. \end{aligned} \quad (\text{D.1-3})$$

Selecting an inertial frame of reference (to avoid the detail of a quasi-coordinate formulation) the velocity and acceleration are

$$\begin{aligned} \dot{\underline{P}} &= \frac{\partial \underline{P}}{\partial \underline{i}^q} \underline{i}^{\dot{q}} + \frac{\partial \underline{P}}{\partial t}, \text{ and} \\ \ddot{\underline{P}} &= \frac{\partial \underline{P}}{\partial \underline{i}^q} \underline{i}^{\ddot{q}} + \frac{\partial^2 \underline{P}}{\partial \underline{i}^q \partial \underline{j}^q} \underline{i}^{\dot{q}} \underline{j}^{\dot{q}} + 2 \frac{\partial^2 \underline{P}}{\partial \underline{i}^q \partial t} \underline{i}^{\dot{q}} + \frac{\partial^2 \underline{P}}{\partial t^2}. \end{aligned} \quad (\text{D.1-4})$$

The kinetic energy may be calculated as

$$T = \frac{1}{2} \int_S^* \left\{ \rho \left[\frac{\partial P}{\partial_i q} \cdot \frac{\partial P}{\partial_j q} \dot{q}_i \dot{q}_j + 2 \frac{\partial P}{\partial_i q} \cdot \frac{\partial P}{\partial t} \dot{q}_i + \frac{\partial P}{\partial t} \cdot \frac{\partial P}{\partial t} \right] \right\} \cdot (D.1-5)$$

Lagrange's equation (including all potential energy terms in the generalized force),

$${}_i Q = \frac{d}{dt} \left(\frac{\partial T}{\partial_i \dot{q}} \right) - \frac{\partial T}{\partial_i q} \quad , \quad (D.1-6)$$

provides

$$\begin{aligned} {}_i Q &= \int_S^* \left\{ \rho \left[\frac{d}{dt} \left(\frac{\partial P}{\partial_i q} \cdot \frac{\partial P}{\partial_j q} \dot{q}_j + \frac{\partial P}{\partial_i q} \cdot \frac{\partial P}{\partial t} \right) - \frac{\partial^2 P}{\partial_i q \partial_j q} \cdot \frac{\partial P}{\partial_k q} \dot{q}_j \dot{q}_k \right. \right. \\ &\quad \left. \left. - \frac{\partial^2 P}{\partial_i q \partial t} \cdot \frac{\partial P}{\partial_j q} \dot{q}_j - \frac{\partial P}{\partial t} \cdot \frac{\partial^2 P}{\partial_i q \partial_j q} \dot{q}_j - \frac{\partial^2 P}{\partial_i q \partial t} \cdot \frac{\partial P}{\partial t} \right] \right\} \\ &= \int_S^* \left\{ \rho \left[\frac{\partial P}{\partial_i q} \cdot \frac{\partial P}{\partial_j q} j \ddot{q}_j + \frac{\partial P}{\partial_i q} \cdot \frac{\partial^2 P}{\partial_j q \partial_k q} \dot{q}_j \dot{q}_k + \frac{\partial^2 P}{\partial_i q \partial_k q} \cdot \frac{\partial P}{\partial_j q} \dot{q}_j \dot{q}_k \right. \right. \\ &\quad + \frac{\partial P}{\partial_i q} \cdot \frac{\partial^2 P}{\partial_j q \partial t} \dot{q}_j + \frac{\partial^2 P}{\partial_i q \partial t} \cdot \frac{\partial P}{\partial_j q} \dot{q}_j + \frac{\partial^2 P}{\partial_i q \partial_j q} \cdot \frac{\partial P}{\partial t} \dot{q}_j + \frac{\partial^2 P}{\partial_i q \partial t} \cdot \frac{\partial P}{\partial t} \\ &\quad + \frac{\partial P}{\partial_i q} \cdot \frac{\partial^2 P}{\partial_j q \partial t} \dot{q}_j + \frac{\partial P}{\partial_i q} \cdot \frac{\partial^2 P}{\partial t^2} - \frac{\partial^2 P}{\partial_i q \partial_j q} \cdot \frac{\partial P}{\partial_k q} \dot{q}_j \dot{q}_k \\ &\quad \left. \left. - \frac{\partial^2 P}{\partial_i q \partial t} \cdot \frac{\partial P}{\partial_j q} \dot{q}_j - \frac{\partial P}{\partial t} \cdot \frac{\partial^2 P}{\partial_i q \partial_j q} \dot{q}_j - \frac{\partial^2 P}{\partial_i q \partial t} \cdot \frac{\partial P}{\partial t} \right] \right\} \cdot (D.1-7) \end{aligned}$$

Noting the cancellation of the third and tenth, fifth and eleventh, sixth and twelfth, and seventh and thirteenth terms, this becomes

$$\begin{aligned}
i^Q &= \int_S^* \left\{ \hat{\rho} \left[\frac{\partial P}{\partial i^q} \cdot \frac{\partial P}{\partial j^q} j^{\ddot{q}} + \frac{\partial P}{\partial i^q} \cdot \frac{\partial^2 P}{\partial j^q \partial k^q} j^{\dot{q}} j^{\dot{k}} + 2 \frac{\partial P}{\partial i^q} \cdot \frac{\partial^2 P}{\partial j^q \partial t} j^{\dot{q}} + \frac{\partial P}{\partial i^q} \cdot \frac{\partial^2 P}{\partial t^2} \right] \right\} \\
&= \int_S^* \left\{ \hat{\rho} \frac{\partial P}{\partial i^q} \cdot \left[\frac{\partial P}{\partial j^q} j^{\ddot{q}} + \frac{\partial^2 P}{\partial j^q \partial k^q} j^{\dot{q}} j^{\dot{k}} + 2 \frac{\partial^2 P}{\partial j^q \partial t} j^{\dot{q}} + \frac{\partial^2 P}{\partial t^2} \right] \right\} \\
&= \int_S^* \left\{ \hat{\rho} \frac{\partial P}{\partial i^q} \cdot \ddot{P} \right\}. \tag{D.1-8}
\end{aligned}$$

Substituting the expansions in Equations (D.1 - 1 and - 2) into Equations (D.1-3 and -8) and combining these two provides to linear order

$$\begin{aligned}
&\int_S^* \left\{ \left[0^X + j^X j^q \right] \cdot \left[i^P + (ij^P + ji^P) j^q \right] \right\} \\
&\quad + \int_{\partial M(t)}^* \left\{ \left[0^S + j^S j^q \right] \cdot \left[i^P + (ij^P + ji^P) j^q \right] \right\} \\
&= \int_S^* \left\{ \hat{\rho} \left[i^P + (ij^P + ji^P) j^q \right] \cdot \left[j^P j^{\ddot{q}} + 2 \frac{\partial j^P}{\partial t} j^{\dot{q}} + \frac{\partial^2 0^P}{\partial t^2} + \frac{\partial^2 j^P}{\partial t^2} j^q \right] \right\} \\
&\tag{D.1-9}
\end{aligned}$$

or

$$\begin{aligned}
&\int_S^* \left\{ 0^X \cdot i^P + 0^X \cdot (ij^P + ji^P) j^q + j^X \cdot i^P j^q \right\} \\
&\quad + \int_{\partial M(t)}^* \left\{ 0^S \cdot i^P + 0^S \cdot (ij^P + ji^P) j^q + j^S \cdot i^P j^q \right\} \\
&= \int_S^* \left\{ \hat{\rho} \left[i^P \cdot j^P j^{\ddot{q}} + 2 i^P \cdot \frac{\partial j^P}{\partial t} j^{\dot{q}} + i^P \cdot \frac{\partial^2 0^P}{\partial t^2} + (ij^P + ji^P) \cdot \frac{\partial^2 0^P}{\partial t^2} j^q \right. \right. \\
&\quad \left. \left. + i^P \cdot \frac{\partial^2 j^P}{\partial t^2} j^q \right] \right\}
\end{aligned}$$

Equation (D.1-10) continued

$$= \int_{\bar{S}} * \left\{ \hat{\rho} \left[\begin{aligned} & \left[\underset{i}{i}^P \cdot \frac{\partial^2 \underset{0}{0}^P}{\partial t^2} + (\underset{i}{i}^P + \underset{j}{j}^P) \cdot \frac{\partial^2 \underset{0}{0}^P}{\partial t^2} j^q + \underset{i}{i}^P \cdot \underset{j}{j}^P j^{\ddot{q}} + 2 \underset{i}{i}^P \cdot \frac{\partial \underset{j}{j}^P}{\partial t} j^{\dot{q}} \right. \right. \\ & \left. \left. + \underset{i}{i}^P \cdot \frac{\partial^2 \underset{j}{j}^P}{\partial t^2} j^q \right] \right\} . \end{aligned} \right. \quad (D.1-10)$$

The zero order term is

$$\int_{\bar{S}} * \left\{ \underset{0}{0}^X \cdot \underset{i}{i}^P \right\} + \int_{\partial M(t)} * \left\{ \underset{0}{0}^S \cdot \underset{i}{i}^P \right\} = \int_{\bar{S}} * \left\{ \hat{\rho} \underset{i}{i}^P \cdot \frac{\partial^2 \underset{0}{0}^P}{\partial t^2} \right\} . \quad (D.1-11)$$

Now either by requiring the $\underset{i}{i}^P$ be a complete function space, noting the region is arbitrary, or merely recognizing the above as $\underset{i}{i}^P$ times Newton's law, it may be seen that

$$\begin{aligned} & \int_{\bar{S}} * \left\{ \underset{0}{0}^X \cdot (\underset{i}{i}^P + \underset{j}{j}^P) j^q \right\} + \int_{\partial M(t)} * \left\{ \underset{0}{0}^S \cdot (\underset{i}{i}^P + \underset{j}{j}^P) j^q \right\} \\ & = \int_{\bar{S}} * \left\{ \hat{\rho} (\underset{i}{i}^P + \underset{j}{j}^P) \cdot \frac{\partial^2 \underset{0}{0}^P}{\partial t^2} j^q \right\} \end{aligned} \quad (D.1-12)$$

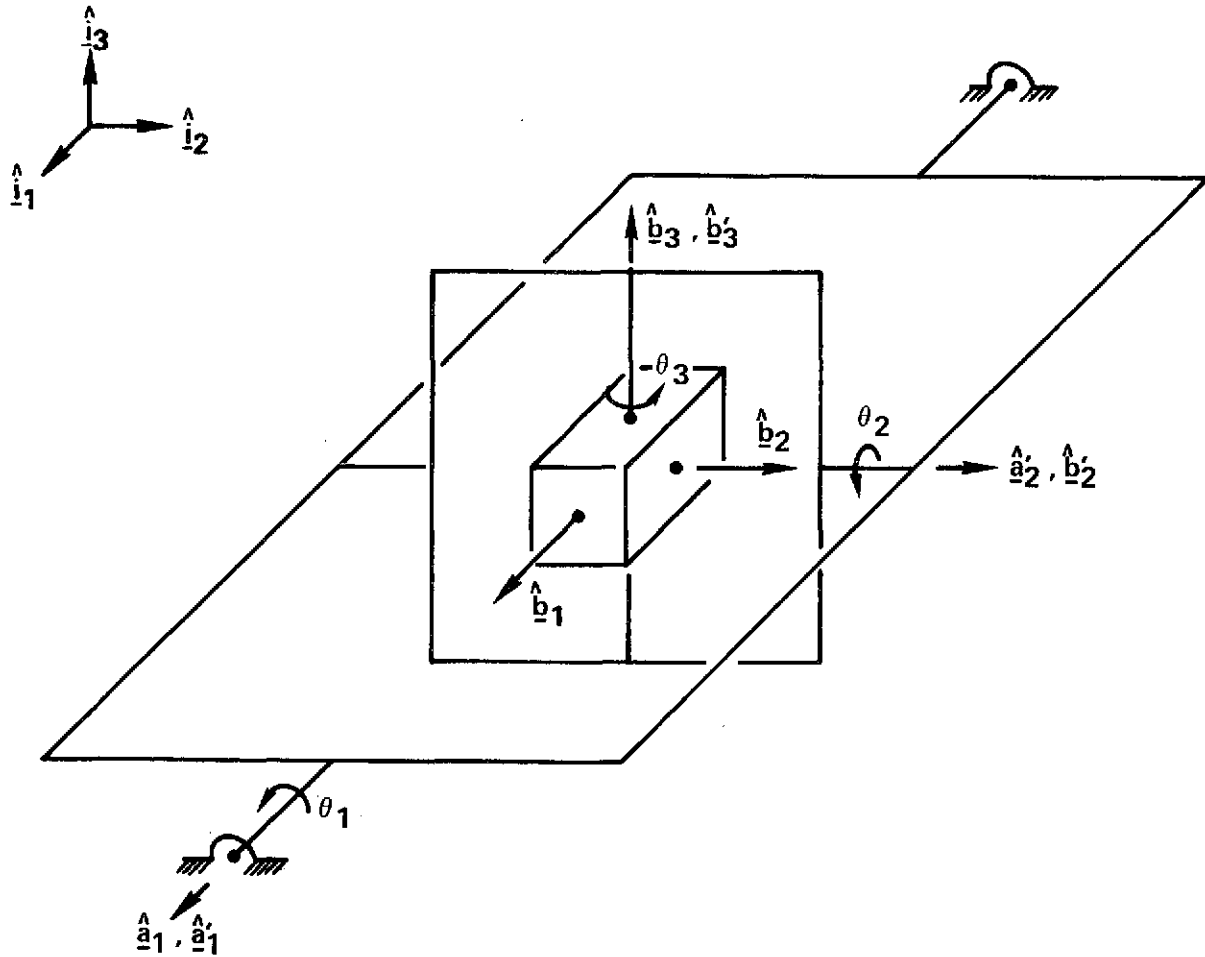
must also be satisfied. Subtracting this from the linear term in Equation (D.1-10) gives

$$\begin{aligned} & \int_{\bar{S}} * \left\{ \underset{j}{j}^X \cdot \underset{i}{i}^P j^q \right\} + \int_{\partial M(t)} * \left\{ \underset{j}{j}^S \cdot \underset{i}{i}^P j^q \right\} \\ & = \int_{\bar{S}} * \left\{ \hat{\rho} \underset{i}{i}^P \cdot \left[\underset{j}{j}^P j^{\ddot{q}} + 2 \frac{\partial \underset{j}{j}^P}{\partial t} j^{\dot{q}} + \frac{\partial^2 \underset{j}{j}^P}{\partial t^2} j^q \right] \right\} , \end{aligned} \quad (D.1-13)$$

which is the equation obtained by retaining only linear terms. Thus the retention of quadratic terms adds the terms in Equation (D.1-12) which are redundant with the zero order term (D.1-11) and thus may be eliminated.

D.2 Quadratic Terms in Gimbal System

The purpose of this section is to demonstrate, by example, that it is sufficient to retain only linear terms in an assumed quadratic expansion of the displacement field. To provide a reasonable test of the hypothesis, a system was selected for which rotations are the natural coordinates. The base of the illustrated gimbal system is experiencing a steady inertial spin, $\underline{\Omega}$. The equilibrium state is $\theta_\alpha = 0$, in which steady state torques T_α are carried through clocksprings.



The linear equations governing small motions about the steady state equilibrium will be obtained in three ways: first in the natural, Eulerian fashion, second retaining quadratic terms in a Lagrangian formulation, and finally retaining only linear terms in a Lagrangian formulation. It will then be shown that the equations are all equivalent.

This problem is not entirely academic. Structures are frequently represented by finite elements whose generalized displacements are three translations and three rotations at connecting nodes. Kinetic energy is sometimes associated with these models in the simplest possible way, by assuming all the mass is concentrated in infinitesimal rigid bodies at the nodes (which, however, also have rotary inertia). The gimbal system may be thought of as a mechanization of such a nodal body. In this case the rotations are a 1-2-3 set about the body fixed axes. An alternative mechanization that has been suggested by other researchers is a 1-2-3 set about axes fixed in the substructure reference frame. It can be shown that such a set is equivalent to a 3-2-1 set about the body fixed axes, so the example covers both types of mechanizations. (Of course, the question of mechanization is academic, since it is about to be demonstrated that linear terms are sufficient, and since it is only the quadratic terms that are affected by mechanization.)

Before proceeding with the development of equations, it will be convenient to express some of the relationships common to all of the derivations. It may be noted that the customary letter with a caret

(e.g., \hat{e}) notation has been used for unit vectors in place of the more cumbersome $\frac{\partial}{\partial x^i}$. The basis vectors are related by the direction cosine matrices

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \begin{bmatrix} c\theta_3 & s\theta_3 & 0 \\ -s\theta_3 & c\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{b}'_1 \\ \hat{b}'_2 \\ \hat{b}'_3 \end{Bmatrix},$$

$$\begin{Bmatrix} \hat{b}'_1 \\ \hat{b}'_2 \\ \hat{b}'_3 \end{Bmatrix} = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 \\ 0 & 1 & 0 \\ s\theta_2 & 0 & c\theta_2 \end{bmatrix} \begin{Bmatrix} \hat{a}'_1 \\ \hat{a}'_2 \\ \hat{a}'_3 \end{Bmatrix},$$

$$\begin{Bmatrix} \hat{a}'_1 \\ \hat{a}'_2 \\ \hat{a}'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_1 & s\theta_1 \\ 0 & -s\theta_1 & c\theta_1 \end{bmatrix} \begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix},$$

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \begin{bmatrix} c\theta_3 & c\theta_2 & s\theta_3 & -c\theta_3 & s\theta_2 \\ -c\theta_2 & s\theta_3 & c\theta_3 & s\theta_2 & s\theta_3 \\ s\theta_2 & 0 & 0 & c\theta_2 & 0 \end{bmatrix} \begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix},$$

$$\begin{Bmatrix} \hat{b}'_1 \\ \hat{b}'_2 \\ \hat{b}'_3 \end{Bmatrix} = \begin{bmatrix} c\theta_2 & s\theta_2 & s\theta_1 & -s\theta_2 & c\theta_1 \\ 0 & c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_2 & -c\theta_2 & s\theta_1 & c\theta_2 & c\theta_1 \end{bmatrix} \begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix}, \text{ and}$$

$$\begin{Bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{Bmatrix} = \begin{bmatrix} c\theta_3 c\theta_2 & s\theta_3 c\theta_1 + s\theta_1 c\theta_3 s\theta_2 & s\theta_3 s\theta_1 - c\theta_3 s\theta_2 c\theta_1 \\ -c\theta_2 s\theta_3 & c\theta_3 c\theta_1 - s\theta_1 s\theta_2 s\theta_3 & c\theta_3 s\theta_1 + s\theta_2 s\theta_3 c\theta_1 \\ s\theta_2 & -s\theta_1 c\theta_2 & c\theta_2 c\theta_1 \end{bmatrix} \begin{Bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{Bmatrix},$$

(D.2-1)

where the abbreviations $c\theta_\alpha$ and $s\theta_\alpha$ have been used for $\cos(\theta_\alpha)$ and $\sin(\theta_\alpha)$. The angular velocity of the body is

$$\underline{\omega} = \underline{\hat{a}}^T \Omega + \underline{\hat{a}}'_1 \dot{\theta}_1 + \underline{\hat{a}}'_2 \dot{\theta}_2 + \underline{\hat{b}}'_3 \dot{\theta}_3, \quad (\text{D.2-2})$$

where the absence of subscripts in the first term implies the complete array treated as a matrix, and the juxtaposition of the two terms implies the matrix product. Making use of the direction cosine relationships (D.2-1) this may be written

$$\underline{\omega} = \begin{Bmatrix} \Omega_1 + \dot{\theta}_1 + s\theta_2 \dot{\theta}_3 \\ \Omega_2 + c\theta_1 \dot{\theta}_2 - c\theta_2 s\theta_1 \dot{\theta}_3 \\ \Omega_3 + s\theta_1 \dot{\theta}_2 + c\theta_2 c\theta_1 \dot{\theta}_3 \end{Bmatrix}^T \begin{Bmatrix} \underline{\hat{a}}_1 \\ \underline{\hat{a}}_2 \\ \underline{\hat{a}}_3 \end{Bmatrix}. \quad (\text{D.2-3})$$

Linearizing this result gives

$$\underline{\omega} = \begin{Bmatrix} \Omega_1 + \dot{\theta}_1 \\ \Omega_2 + \dot{\theta}_2 \\ \Omega_3 + \dot{\theta}_3 \end{Bmatrix}^T \begin{Bmatrix} \underline{\hat{a}}_1 \\ \underline{\hat{a}}_2 \\ \underline{\hat{a}}_3 \end{Bmatrix} = (\Omega^T + \dot{\theta}^T) \underline{\hat{a}}. \quad (\text{D.2-4})$$

In the body fixed basis this is

$$\begin{aligned} \underline{\omega} &= \begin{Bmatrix} \Omega_1 + \dot{\theta}_1 + \Omega_2 \theta_3 - \Omega_3 \theta_2 \\ \Omega_2 + \dot{\theta}_2 - \Omega_1 \theta_3 + \Omega_3 \theta_1 \\ \Omega_3 + \dot{\theta}_3 + \Omega_1 \theta_2 - \Omega_2 \theta_1 \end{Bmatrix}^T \begin{Bmatrix} \underline{\hat{b}}_1 \\ \underline{\hat{b}}_2 \\ \underline{\hat{b}}_3 \end{Bmatrix} \\ &= (\Omega^T + \dot{\theta}^T - \theta^T \tilde{\Omega}) \underline{\hat{b}} \\ &= \underline{\hat{b}}^T (\Omega + \dot{\theta} + \tilde{\Omega} \theta). \end{aligned} \quad (\text{D.2-5})$$

The angular acceleration is

$$\dot{\underline{\omega}} = \underline{\hat{b}}^T (\ddot{\theta} + \tilde{\omega} \dot{\theta}) = \underline{\hat{a}}^T (\ddot{\theta} + \tilde{\omega} \dot{\theta}) . \quad (\text{D.2-6})$$

The components of the torque are

$$\begin{aligned} \underline{\tau} \cdot \underline{\hat{a}}_1 &= T_1 - K_1 \theta_1 , \\ \underline{\tau} \cdot \underline{\hat{a}}_2 &= T_2 - K_2 \theta_2 , \text{ and} \\ \underline{\tau} \cdot \underline{\hat{a}}_3 &= T_3 - K_3 \theta_3 . \end{aligned} \quad (\text{D.2-7})$$

Expressing $\underline{\tau}$ in the $\underline{\hat{a}}$ basis ,

$$\begin{aligned} \tau_1 &= T_1 - K_1 \theta_1 , \\ c\theta_1 \tau_2 + s\theta_1 \tau_3 &= T_2 - K_2 \theta_2 , \text{ and} \\ s\theta_2 \tau_1 + c\theta_2 s\theta_1 \tau_2 + c\theta_2 c\theta_1 \tau_3 &= T_3 - K_3 \theta_3 . \end{aligned} \quad (\text{D.2-8})$$

Multiplying the second equation by $c\theta_2 s\theta_1$ and adding $c\theta_1$ times the third equation gives

$$c\theta_2 s\theta_1 (T_2 - K_2 \theta_2) + c\theta_1 (T_3 - K_3 \theta_3) - c\theta_1 s\theta_2 (T_1 - K_1 \theta_1) = c\theta_2 \tau_3 . \quad (\text{D.2-9})$$

Linearizing this result provides

$$\begin{aligned} \tau_3 &= T_3 - K_3 \theta_3 + \theta_1 T_2 - \theta_2 T_1 , \text{ and} \\ \tau_2 &= T_2 - K_2 \theta_2 - \theta_1 T_3 . \end{aligned} \quad (\text{D.2-10})$$

So the torque may be expressed as

$$\underline{\tau} = \underline{\hat{a}}^T (T - K\theta + \begin{bmatrix} 0 & 0 & 0 \\ -T_3 & 0 & 0 \\ T_2 - T_1 & 0 & 0 \end{bmatrix} \theta) = \underline{\hat{b}}^T (T - K\theta + \begin{bmatrix} 0 & -T_3 & T_2 \\ 0 & 0 & -T_1 \\ 0 & 0 & 0 \end{bmatrix} \theta) . \quad (\text{D.2-11})$$

Before proceeding with the Eulerian derivation it will be demonstrated that a 1-2-3 rotation about axes fixed in the $\underline{\hat{a}}$ basis is the same as a 3-2-1 about body fixed axes. A set of three unit vectors, $\underline{\hat{u}}^{\alpha}$, initially parallel to the $\underline{\hat{a}}$ basis vectors, $\underline{\hat{u}} = U \underline{\hat{a}}$ (where U is an identity matrix), after a rotation θ_1 about the $\underline{\hat{a}}_1$ axis have an expression in the $\underline{\hat{a}}$ basis

$$\underline{\hat{u}} = U C(\theta_1) \underline{\hat{a}} . \quad (\text{D.2-12})$$

A subsequent rotation, θ_2 , about the $\underline{\hat{a}}_2$ axis yields unit vectors, $\underline{\hat{u}}_b$, with an expression in the $\underline{\hat{a}}$ basis

$$\underline{\hat{u}}_b = U C(\theta_1) C(\theta_2) \underline{\hat{a}} . \quad (\text{D.2-13})$$

A final rotation, θ_3 , about the $\underline{\hat{a}}_3$ axis yields unit vectors, $\underline{\hat{u}}_b$, with an expression in the $\underline{\hat{a}}$ basis

$$\underline{\hat{u}}_b = U C(\theta_1) C(\theta_2) C(\theta_3) \underline{\hat{a}} . \quad (\text{D.2-14})$$

It may now be recognized that the product direction cosine matrix is the same as that for a 3-2-1 rotation about body fixed axes.

The equations of motion will now be developed following the usual Eulerian formulation. The rotational equation is

$$\underline{\tau} = \int_S * \{ \overset{\wedge}{\rho} \underline{P} \times \ddot{\underline{P}} \} . \quad (\text{D.2-15})$$

For a rigid body

$$\begin{aligned} \underline{P} &= \underset{O}{P} \\ \ddot{\underline{P}} &= \dot{\underline{\omega}} \times \underset{O}{P} + \underline{\omega} \times (\underline{\omega} \times \underset{O}{P}) , \end{aligned} \quad (\text{D.2-16})$$

so that

$$\begin{aligned} \underline{\tau} &= \int_S * \left\{ \overset{\wedge}{\rho} \underset{O}{P} \times \left(\dot{\underline{\omega}} \times \underset{O}{P} + [\underline{\omega} \cdot \underset{O}{P}] \underline{\omega} - [\underline{\omega} \cdot \underline{\omega}] \underset{O}{P} \right) \right\} \\ &= \int_S * \left\{ \overset{\wedge}{\rho} \left[(\underset{O}{P} \cdot \underset{O}{P}) \underline{U} - \underset{O}{P} \underset{O}{P} \right] \cdot \dot{\underline{\omega}} + \underline{\omega} \times \overset{\wedge}{\rho} \left[(\underset{O}{P} \cdot \underset{O}{P}) \underline{U} - \underset{O}{P} \underset{O}{P} \right] \cdot \underline{\omega} \right. \\ &\quad \left. - \underline{\omega} \times (\underset{O}{P} \cdot \underset{O}{P}) \underline{U} \cdot \underline{\omega} \right\} \\ &= \underline{I} \cdot \dot{\underline{\omega}} + \underline{\omega} \times \underline{I} \cdot \underline{\omega} \end{aligned} \quad (\text{D.2-17})$$

where $\underline{I} = \underline{b}^T I \underline{b}$ and I is the array of moments and products of inertia. Writing this result in the \underline{b} basis and noting Equations (D.2-5, 6, and -11)

$$\tau - K\theta + \begin{bmatrix} 0 & -T_3 & T_2 \\ 0 & 0 & -T_1 \\ 0 & 0 & 0 \end{bmatrix} \theta = I(\ddot{\theta} + \tilde{\Omega}\dot{\theta}) + (\Omega + \dot{\theta} + \tilde{\Omega}\theta) \tilde{\Omega} I (\Omega + \dot{\theta} + \tilde{\Omega}\theta) . \quad (\text{D.2-18})$$

Retaining only terms to linear order

$$T - K\theta + \begin{bmatrix} 0 & -T_3 & T_2 \\ 0 & 0 & -T_1 \\ 0 & 0 & 0 \end{bmatrix} \theta = I\ddot{\theta} + I\tilde{\Omega}\dot{\theta} + \tilde{\Omega}I\Omega + \tilde{\Omega}I\dot{\theta} + \tilde{\Omega}I\tilde{\Omega}\theta + \tilde{\theta}I\Omega + (\tilde{\Omega}\theta)\tilde{\Omega}I\Omega. \quad (D.2-19)$$

The steady state and deviational equations are

$$T = \tilde{\Omega}I\Omega, \text{ and}$$

$$I\ddot{\theta} + [I\tilde{\Omega} + \tilde{\Omega}I - (I\Omega)\tilde{\Omega}]\dot{\theta} + [\tilde{\Omega}I\tilde{\Omega} + K - (I\Omega)\tilde{\Omega}\tilde{\Omega} - \begin{bmatrix} 0 & -T_3 & T_2 \\ 0 & 0 & -T_1 \\ 0 & 0 & 0 \end{bmatrix} \theta] = 0. \quad (D.2-20)$$

These equations may also be written in the $\hat{\underline{a}}$ basis. The inertia dyadic becomes

$$\underline{I} = \underline{\hat{a}}^T (E + \tilde{\theta}) I (E - \tilde{\theta}) \underline{\hat{a}} = \underline{\hat{a}}^T (I + \tilde{\theta}I - I\tilde{\theta}) \underline{\hat{a}}, \quad (D.2-21)$$

where the linear approximation to the direction cosine matrix $(E - \tilde{\theta})$ has been used. The governing equations are

$$T - K\theta + \begin{bmatrix} 0 & 0 & 0 \\ -T_3 & 0 & 0 \\ T_2 & -T_1 & 0 \end{bmatrix} \theta = (I + \tilde{\theta}I - I\tilde{\theta})(\ddot{\theta} + \tilde{\Omega}\dot{\theta}) + (\Omega + \dot{\theta})\tilde{\Omega}(I + \tilde{\theta}I - I\tilde{\theta})(\Omega + \dot{\theta}) = I\ddot{\theta} + I\tilde{\Omega}\dot{\theta} + \tilde{\Omega}I\Omega + \tilde{\Omega}\tilde{\theta}I\Omega - \tilde{\Omega}I\tilde{\theta}\Omega + \tilde{\theta}I\Omega + \tilde{\Omega}I\dot{\theta}. \quad (D.2-22)$$

The steady state and deviational equations are

$T = \tilde{\Omega} I \Omega$, and

$$I\ddot{\theta} + [I\tilde{\Omega} + \tilde{\Omega}I - (I\Omega)\tilde{\Omega}] \dot{\theta} + [\tilde{\Omega}I\tilde{\Omega} + K - \tilde{\Omega}(I\Omega)\tilde{\Omega}] \theta - \begin{bmatrix} 0 & 0 & 0 \\ -T_3 & 0 & 0 \\ T_2 & -T_1 & 0 \end{bmatrix} \theta = 0.$$

(D.2-23)

It may be noted that Equations (D.2-20 and -23) are equivalent although not identical since subtracting gives

$$[-(I\Omega)\tilde{\Omega} + \tilde{\Omega}(I\Omega)] \theta - \begin{bmatrix} 0 & -T_3 & T_2 \\ T_3 & 0 & -T_1 \\ -T_2 & T_1 & 0 \end{bmatrix} \theta = 0$$

(D.2-24)

or

$$\begin{aligned} \tilde{T} &= \tilde{\Omega}(I\Omega)\tilde{\Omega} - (I\Omega)\tilde{\Omega}\tilde{\Omega} \\ &= (\tilde{\Omega} I \Omega)\tilde{\Omega}, \end{aligned} \quad (D.2-25)$$

(where use has been made of the identity (A.13-12)) and this is merely a restatement of the steady state torque relation.

The equations will now be derived using Lagrange's equations retaining all quadratic terms. The notation is taken from Roberson and Likin's "The Quadratic Approximation in Rotational Dynamical Equations". With this notation the direction cosine matrix is

$$\underline{\hat{b}} = (U - \tilde{\theta} + \frac{1}{2} \tilde{\theta} \tilde{\theta} + \frac{1}{2} \theta U^{123} \tilde{\theta}) \underline{\hat{a}}, \quad (D.2-26)$$

where

$$\bar{\theta} = \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix}, \quad U^{123} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

The relative angular velocity, \underline{w} , is expressed in the $\underline{\hat{b}}$ basis as

$$\underline{\hat{b}}^T \underline{w} = \underline{\hat{b}}^T \left(U - \frac{1}{2} \tilde{\theta} + \frac{1}{2} \bar{U}^{123} \hat{\theta} \right) \dot{\theta}, \quad (\text{D.2-27})$$

where

$$\hat{\theta} = \begin{bmatrix} 0 & \theta_3 & \theta_2 \\ \theta_3 & 0 & \theta_1 \\ \theta_2 & \theta_1 & 0 \end{bmatrix}, \quad \bar{U}^{123} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The total angular velocity is then

$$\begin{aligned} \underline{\omega} &= \underline{\hat{a}}^T \Omega + \underline{\hat{b}}^T \underline{w} \\ &= \underline{\hat{b}}^T \left(U - \tilde{\theta} + \frac{1}{2} \tilde{\theta} \tilde{\theta} + \frac{1}{2} \bar{\theta} U^{123} \bar{\theta} \right) \Omega + \underline{\hat{b}}^T \underline{w} \\ &= \underline{\hat{b}}^T \left(\Omega - \tilde{\theta} \Omega + \frac{1}{2} \tilde{\theta} \tilde{\theta} \Omega + \frac{1}{2} \bar{\theta} U^{123} \bar{\theta} \Omega + \dot{\theta} - \frac{1}{2} \tilde{\theta} \dot{\theta} + \frac{1}{2} \bar{U}^{123} \hat{\theta} \dot{\theta} \right). \end{aligned} \quad (\text{D.2-28})$$

The generalized forces may be computed as follows. Noting that

$$\underline{0}^P = \underline{\hat{b}}^T \underline{0}^P = \underline{\hat{1}}^T C^T \underline{0}^P \quad (\text{D.2-29})$$

and that $\underline{\hat{a}}$ is a function only of time

$$\delta \underline{0}^P = \underline{\hat{1}}^T \frac{\partial C^T}{\partial q_\alpha} \delta q_\alpha \underline{0}^P, \quad (\text{D.2-30})$$

therefore

$$\begin{aligned}
Q_\alpha \delta q_\alpha &= \int_{\partial M(t)} * \{ \delta \underline{0}^P \cdot \underline{S} \} \\
&= \int_{\partial M(t)} * \left\{ \underline{S}^T \underline{C} \underline{\hat{a}} \cdot \underline{\hat{a}}^T \frac{\partial \underline{C}^T}{\partial \dot{q}_\alpha} \underline{0}^P \right\} \delta q_\alpha .
\end{aligned} \tag{D.2-31}$$

Now noting that \underline{C} is not an explicit function of time so that

$$\dot{\underline{C}}^T = \frac{\partial \underline{C}^T}{\partial \dot{q}_\alpha} \dot{q}_\alpha \tag{D.2-32}$$

and recalling that

$$\underline{C} \dot{\underline{C}}^T = \underline{\tilde{\omega}} , \tag{D.2-33}$$

the generalized forces become

$$\begin{aligned}
Q_\alpha &= \int_{\partial M(t)} * \left\{ \underline{S}^T \underline{C} \frac{\partial \dot{\underline{C}}^T}{\partial \dot{q}_\alpha} \underline{0}^P \right\} \\
&= \int_{\partial M(t)} * \left\{ \underline{S}^T \frac{\partial}{\partial \dot{q}_\alpha} \left(\underline{C} \dot{\underline{C}}^T \right) \underline{0}^P \right\} \\
&= \int_{\partial M(t)} * \left\{ \underline{S}^T \frac{\partial}{\partial \dot{q}_\alpha} \underline{\tilde{\omega}} \underline{0}^P \right\} \\
&= \int_{\partial M(t)} * \left\{ \frac{\partial}{\partial \dot{q}_\alpha} \underline{\omega}^T \underline{0}^P \underline{S} \right\} \\
&= \frac{\partial}{\partial \dot{q}_\alpha} \underline{\omega} \cdot \int_{\partial M(t)} * \left\{ \underline{0}^P \times \underline{S} \right\} \\
&= \frac{\partial}{\partial \dot{q}_\alpha} \underline{\omega} \cdot \underline{I} .
\end{aligned} \tag{D.2-34}$$

Now $\underline{\omega}$ may be written as

$$\underline{\omega} = \underline{\Omega} + \dot{\theta}_1 \underline{\hat{a}}_1 + \dot{\theta}_2 \underline{\hat{a}}'_2 + \dot{\theta}_3 \underline{\hat{b}}'_3 , \tag{D.2-35}$$

so

$$\begin{aligned}
\frac{\partial}{\partial \dot{\theta}_1} \underline{\omega} &= \underline{\hat{a}}_1 \\
\frac{\partial}{\partial \dot{\theta}_2} \underline{\omega} &= \underline{\hat{a}}'_2 \\
\frac{\partial}{\partial \dot{\theta}_3} \underline{\omega} &= \underline{\hat{b}}'_3 .
\end{aligned} \tag{D.2-36}$$

Thus the generalized forces are

$$\begin{aligned}
 Q_1 &= \underline{\hat{a}}_1 \cdot \underline{\tau} = T_1 - K_1 \theta_1 \\
 Q_2 &= \underline{\hat{a}}'_2 \cdot \underline{\tau} = T_2 - K_2 \theta_2 \\
 Q_3 &= \underline{\hat{b}}'_3 \cdot \underline{\tau} = T_3 - K_3 \theta_3
 \end{aligned} \tag{D.2-37}$$

and the potential energy is

$$V = -T^T \theta + \frac{1}{2} \theta^T K \theta, \tag{D.2-38}$$

which is just the sum of the strain energies in the three springs.

The kinetic energy is

$$\begin{aligned}
 T &= \frac{1}{2} \underline{\omega} \cdot \underline{I} \cdot \underline{\omega} \\
 &= \frac{1}{2} \omega^T I \omega \\
 &= \frac{1}{2} \left(\Omega^T + \dot{\theta}^T + \frac{1}{2} \dot{\theta}^T \tilde{\theta} + \frac{1}{2} \dot{\theta}^T \hat{U}^{-123} - \theta^T \tilde{\Omega} - \frac{1}{2} \theta^T \tilde{\Omega} \tilde{\theta} - \frac{1}{2} \theta^T \tilde{\Omega} U^{-123} \tilde{\theta} \right) \\
 &\quad I \left(\Omega + \dot{\theta} - \frac{1}{2} \tilde{\theta} \dot{\theta} + \frac{1}{2} U^{-123} \hat{\theta} \dot{\theta} + \tilde{\Omega} \theta - \frac{1}{2} \tilde{\theta} \tilde{\Omega} \theta + \frac{1}{2} \tilde{\theta} U^{-123} \tilde{\Omega} \theta \right)
 \end{aligned} \tag{D.2-39}$$

where it has been noted that $\bar{x} y = \bar{y} x$. So

$$\begin{aligned}
 T &= \frac{1}{2} \Omega^T I \Omega + \dot{\theta}^T I \Omega + \frac{1}{2} \dot{\theta}^T I \dot{\theta} - \frac{1}{2} \dot{\theta}^T (I \Omega) \tilde{\theta} + \frac{1}{2} \dot{\theta}^T \hat{U}^{-123} I \Omega \\
 &\quad + \dot{\theta}^T I \tilde{\Omega} \theta - \theta^T \tilde{\Omega} I \Omega - \frac{1}{2} \theta^T \tilde{\Omega} I \tilde{\theta} + \frac{1}{2} \theta^T \tilde{\Omega} (I \Omega) \tilde{\theta} - \frac{1}{2} \theta^T \tilde{\Omega} U^{-123} (I \Omega) \tilde{\theta}
 \end{aligned} \tag{D.2-40}$$

Now noting $\hat{x} y = \hat{y} x$, the elements of Lagrange's equations may be obtained

$$\begin{aligned}
\frac{\partial T}{\partial \dot{\theta}} &= I\Omega + I\dot{\theta} - \frac{1}{2} (I\Omega)^{\sim}\theta + I\tilde{\Omega}\theta + \frac{1}{2} (\bar{U}^{123}I\Omega)^{\wedge}\theta \\
\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) &= I\ddot{\theta} + \left[I\tilde{\Omega} - \frac{1}{2} (I\Omega)^{\sim} + \frac{1}{2} (\bar{U}^{123}I\Omega)^{\wedge} \right] \dot{\theta} \\
\frac{\partial T}{\partial \theta} &= \left[\frac{1}{2} (I\Omega)^{\sim} - \tilde{\Omega}I + \frac{1}{2} (\bar{U}^{123}I\Omega)^{\wedge} \right] \dot{\theta} - \tilde{\Omega}I\Omega + \left[-\tilde{\Omega}I\tilde{\Omega} \right. \\
&\quad \left. + \frac{1}{2} \tilde{\Omega} (I\Omega)^{\sim} + \frac{1}{2} (I\Omega)^{\sim} \tilde{\Omega} - \frac{1}{2} \bar{\Omega} U^{123} (I\Omega)^{-} + \frac{1}{2} (I\Omega)^{-} U^{123} \bar{\Omega} \right] \theta .
\end{aligned} \tag{D.2-41}$$

Lagrange's equations may now be written

$$\begin{aligned}
T - K\theta &= \tilde{\Omega}I\Omega + I\ddot{\theta} + [I\tilde{\Omega} + \tilde{\Omega}I - (I\Omega)^{\sim}] \dot{\theta} + [\tilde{\Omega}I\tilde{\Omega} \\
&\quad - \frac{1}{2} \{ \tilde{\Omega}(I\Omega)^{\sim} + (I\Omega)^{\sim} \tilde{\Omega} \} + \frac{1}{2} \{ \bar{\Omega} U^{123} (I\Omega)^{-} - (I\Omega)^{-} U^{123} \bar{\Omega} \}] \theta .
\end{aligned} \tag{D.2-42}$$

Thus the steady state and deviational equations are

$$\begin{aligned}
T &= \tilde{\Omega}I\Omega \\
I\ddot{\theta} + [I\tilde{\Omega} + \tilde{\Omega}I - (I\Omega)^{\sim}] \dot{\theta} + \left[\tilde{\Omega}I\tilde{\Omega} + K - \frac{1}{2} \{ \tilde{\Omega}(I\Omega)^{\sim} + (I\Omega)^{\sim} \tilde{\Omega} \} \right. \\
&\quad \left. + \frac{1}{2} \{ \bar{\Omega} U^{123} (I\Omega)^{-} - (I\Omega)^{-} U^{123} \bar{\Omega} \} \right] \theta = 0 . \tag{D.2-43}
\end{aligned}$$

Some of terms in the stiffness matrix can be identified by noting

$$\begin{aligned}
-\frac{1}{2} \{ \tilde{\Omega}(I\Omega)^{\sim} + (I\Omega)^{\sim} \tilde{\Omega} \} &= -\frac{1}{2} \{ 2\tilde{\Omega}(I\Omega)^{\sim} - [\tilde{\Omega}(I\Omega)^{\sim} - (I\Omega)^{\sim} \tilde{\Omega}] \} \\
&= -\tilde{\Omega}(I\Omega)^{\sim} + \frac{1}{2} (\tilde{\Omega}I\Omega)^{\sim} \\
&= -\tilde{\Omega}(I\Omega)^{\sim} + \frac{1}{2} \tilde{T} .
\end{aligned} \tag{D.2-44}$$

Subtracting the $\hat{\underline{a}}$ basis Eulerian equations (D.2-23) yields

$$\left\{ \frac{1}{2} \begin{bmatrix} 0 & -T_3 & T_2 \\ T_3 & 0 & -T_1 \\ -T_2 & T_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -T_3 & 0 & 0 \\ T_2 & -T_1 & 0 \end{bmatrix} + \frac{1}{2} [\bar{\Omega} U^{123} (I\Omega)^{-} - (I\Omega)^{-} U^{123} \bar{\Omega}] \right\} \theta = 0 \quad (D.2-45)$$

or

$$\begin{bmatrix} 0 & -T_3 & T_2 \\ -T_3 & 0 & -T_1 \\ T_2 & -T_1 & 0 \end{bmatrix} = (I\Omega)^{-} U^{123} \bar{\Omega} - \bar{\Omega} U^{123} (I\Omega)^{-} . \quad (D.2-46)$$

Thus if this equation is satisfied Euler's equations and Lagrange's equations retaining quadratic terms are equivalent. That this equation is true, may be verified by direct expansion. From the steady state equation

$$\begin{aligned} T_\alpha &= \epsilon_{\alpha\beta\gamma} \Omega_\beta I_{\gamma\delta} \Omega_\delta , \\ T_1 &= \Omega_2 I_{3\alpha} \Omega_\alpha - \Omega_3 I_{2\alpha} \Omega_\alpha , \\ T_2 &= \Omega_3 I_{1\alpha} \Omega_\alpha - \Omega_1 I_{3\alpha} \Omega_\alpha , \\ T_3 &= \Omega_1 I_{2\alpha} \Omega_\alpha - \Omega_2 I_{1\alpha} \Omega_\alpha . \end{aligned} \quad (D.2-47)$$

So

$$\begin{bmatrix} 0 & -T_3 & T_2 \\ -T_3 & 0 & -T_1 \\ T_2 & -T_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & | \Omega_2 I_{1\alpha} \Omega_\alpha - \Omega_1 I_{2\alpha} \Omega_\alpha | \Omega_3 I_{1\alpha} \Omega_\alpha - \Omega_1 I_{3\alpha} \Omega_\alpha \\ \Omega_2 I_{1\alpha} \Omega_\alpha - \Omega_1 I_{2\alpha} \Omega_\alpha & | 0 & | \Omega_3 I_{2\alpha} \Omega_\alpha - \Omega_2 I_{3\alpha} \Omega_\alpha \\ \Omega_3 I_{1\alpha} \Omega_\alpha - \Omega_1 I_{3\alpha} \Omega_\alpha & | \Omega_3 I_{2\alpha} \Omega_\alpha - \Omega_2 I_{3\alpha} \Omega_\alpha & | 0 \end{bmatrix} . \quad (D.2-48)$$

The elements of the other terms are

$$\{(I\Omega)^{-1} U^{123\bar{\Omega}}\}_{\alpha\beta} = I_{\alpha\gamma} \Omega_{\gamma} U_{\alpha\beta}^{123} \Omega_{\beta} \quad (\text{no sum } \alpha, \beta), \text{ and}$$

$$\{\bar{\Omega} U^{123} (I\Omega)^{-1}\}_{\alpha\beta} = \Omega_{\alpha} U_{\alpha\gamma}^{123} I_{\gamma\beta} \Omega_{\beta} \quad (\text{no sum } \alpha, \beta).$$

(D.2-49)

Thus they produce matrices

$$(I\Omega)^{-1} U^{123\bar{\Omega}} = \begin{bmatrix} 0 & | & u_{12} \Omega_2 I_{1\alpha} \Omega_{\alpha} & | & u_{13} \Omega_3 I_{1\alpha} \Omega_{\alpha} \\ u_{21} \Omega_1 I_{2\alpha} \Omega_{\alpha} & | & 0 & | & u_{23} \Omega_3 I_{2\alpha} \Omega_{\alpha} \\ u_{31} \Omega_1 I_{3\alpha} \Omega_{\alpha} & | & u_{32} \Omega_2 I_{3\alpha} \Omega_{\alpha} & | & 0 \end{bmatrix}, \text{ and}$$

$$\bar{\Omega} U^{123} (I\Omega)^{-1} = \begin{bmatrix} 0 & | & u_{12} \Omega_1 I_{2\alpha} \Omega_{\alpha} & | & u_{13} \Omega_1 I_{3\alpha} \Omega_{\alpha} \\ u_{21} \Omega_2 I_{1\alpha} \Omega_{\alpha} & | & 0 & | & u_{23} \Omega_2 I_{3\alpha} \Omega_{\alpha} \\ u_{31} \Omega_3 I_{1\alpha} \Omega_{\alpha} & | & u_{32} \Omega_3 I_{2\alpha} \Omega_{\alpha} & | & 0 \end{bmatrix}.$$

(D.2-50)

The difference between the two arrays, noting that $U_{\alpha\beta}^{123} = -U_{\beta\alpha}^{123}$ gives

$$\begin{bmatrix} 0 & | & u_{12} (\Omega_2 I_{1\alpha} \Omega_{\alpha} - \Omega_1 I_{2\alpha} \Omega_{\alpha}) & | & u_{13} (\Omega_3 I_{1\alpha} \Omega_{\alpha} - \Omega_1 I_{3\alpha} \Omega_{\alpha}) \\ u_{12} (\Omega_2 I_{1\alpha} \Omega_{\alpha} - \Omega_1 I_{2\alpha} \Omega_{\alpha}) & | & 0 & | & u_{23} (\Omega_3 I_{2\alpha} \Omega_{\alpha} - \Omega_2 I_{3\alpha} \Omega_{\alpha}) \\ u_{13} (\Omega_3 I_{1\alpha} \Omega_{\alpha} - \Omega_1 I_{3\alpha} \Omega_{\alpha}) & | & u_{23} (\Omega_3 I_{2\alpha} \Omega_{\alpha} - \Omega_2 I_{3\alpha} \Omega_{\alpha}) & | & 0 \end{bmatrix}.$$

(D.2-51)

Recalling that $u_{12} = u_{13} = u_{23} = 1$, it can be seen that the arrays in Equations (D.2-48 and -51) are identical, so Equation (D.2-46) is satisfied and the Eulerian and quadratic Lagrangian equations are equivalent. It may be noted, however, that the stiffness matrix in the Lagrangian formulation is symmetric and its elements are order of rotation dependent.

The equations of motion will now be obtained using Lagrange's equations, but retaining only linear terms in the displacement field.

So in this case the position is

$$\begin{aligned} \underline{0}^P &= \underline{\hat{a}}^T C^T \underline{0}^P \\ &= \underline{\hat{a}}^T (E + \tilde{\theta}) \underline{0}^P \end{aligned} \quad (D.2-52)$$

The generalized forces are computed by noting that

$$\delta \underline{0}^P = \underline{\hat{a}}^T \delta \tilde{\theta} \underline{0}^P \quad (D.2-53)$$

so

$$\begin{aligned} Q_\alpha \delta \theta_\alpha &= \int_{\partial M(t)} * \{ \delta \underline{0}^P \cdot \underline{s} \} \\ &= \delta \theta^T \int_{\partial M(t)} * \{ \underline{0}^{\tilde{P}} \underline{s} \} \end{aligned} \quad (D.2-54)$$

The velocity is

$$\begin{aligned} \underline{0}^{\dot{P}} &= \underline{\dot{\theta}} \times \underline{0}^P + \underline{\Omega} \times (\underline{0}^P + \underline{\theta} \times \underline{0}^P) \\ &= (\underline{\Omega} + \underline{\dot{\theta}}) \times \underline{0}^P + \underline{\Omega} \times (\underline{\theta} \times \underline{0}^P) \end{aligned} \quad (D.2-55)$$

so the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} \int_S * \{ \rho [(\underline{\Omega} + \underline{\dot{\theta}}) \times \underline{0}^P + \underline{\Omega} \times (\underline{\theta} \times \underline{0}^P)] \cdot [(\underline{\Omega} + \underline{\dot{\theta}}) \times \underline{0}^P + \underline{\Omega} \times (\underline{\theta} \times \underline{0}^P)] \} \\ &= \frac{1}{2} \int_S * \{ \rho [(\underline{\Omega} + \underline{\dot{\theta}}) \times \underline{0}^P \cdot (\underline{\Omega} + \underline{\dot{\theta}}) \times \underline{0}^P + 2(\underline{\Omega} + \underline{\dot{\theta}}) \times \underline{0}^P \cdot \underline{\Omega} \times (\underline{\theta} \times \underline{0}^P) \\ &\quad + \underline{\Omega} \times (\underline{\theta} \times \underline{0}^P) \cdot \underline{\Omega} \times (\underline{\theta} \times \underline{0}^P)] \} \\ &= \frac{1}{2} \int_S * \{ \rho [(\underline{\Omega} + \underline{\dot{\theta}}) \cdot \underline{0}^P \times [(\underline{\Omega} + \underline{\dot{\theta}}) \times \underline{0}^P] + 2(\underline{\Omega} + \underline{\dot{\theta}}) \cdot \underline{0}^P \times [\underline{\Omega} \times (\underline{\theta} \times \underline{0}^P)] \\ &\quad + \underline{\Omega} \cdot (\underline{\theta} \times \underline{0}^P) \times [\underline{\Omega} \times (\underline{\theta} \times \underline{0}^P)] \} \end{aligned}$$

Equation (D.2-56) continued

$$\begin{aligned}
&= \frac{1}{2} \underline{\Omega} \cdot \underline{I} \cdot \underline{\Omega} + \dot{\underline{\theta}} \cdot \underline{I} \cdot \underline{\Omega} + \frac{1}{2} \dot{\underline{\theta}} \cdot \underline{I} \cdot \dot{\underline{\theta}} + (\underline{\Omega} + \dot{\underline{\theta}}) \cdot \int_{\underline{S}} * \{ \hat{\rho} [\underline{0}^{\underline{P}} \times \underline{\theta} (\underline{\Omega} \cdot \underline{0}^{\underline{P}})] \} \\
&\quad + \frac{1}{2} \int_{\underline{S}} * \{ \hat{\rho} [\underline{\Omega} \cdot (\underline{\theta} \times \underline{0}^{\underline{P}}) \times (\underline{\Omega} \times (\underline{\theta} \times \underline{0}^{\underline{P}}))] \} \quad (D.2-56)
\end{aligned}$$

Now

$$\begin{aligned}
\int_{\underline{S}} * \{ \hat{\rho} [\underline{0}^{\underline{P}} \times \underline{\theta} (\underline{\Omega} \cdot \underline{0}^{\underline{P}})] \} &= \int_{\underline{S}} * \{ \hat{\rho} [\underline{\theta} \times (\underline{0}^{\underline{P}} \cdot \underline{0}^{\underline{P}} \underline{U} - \underline{0}^{\underline{P}} \underline{0}^{\underline{P}}) \cdot \underline{\Omega} - \frac{1}{2} 2 (\underline{0}^{\underline{P}} \cdot \underline{0}^{\underline{P}}) \underline{\theta} \times \underline{\Omega}] \} \\
&= \underline{\theta} \times \underline{I} \cdot \underline{\Omega} + \frac{1}{2} \text{TR}(\underline{I}) \underline{\Omega} \times \underline{\theta} \\
&= \underline{\theta} \times \underline{I} \cdot \underline{\Omega} + \frac{1}{2} (\underline{I} \cdot \underline{\Omega} \times \underline{\theta} + \underline{\Omega} \times \underline{I} \cdot \underline{\theta} + (\underline{I} \cdot \underline{\Omega}) \times \underline{\theta}) \quad (D.2-57)
\end{aligned}$$

where $\text{TR}(\underline{I})$ denotes the trace of the array. Inserting Equation (D.2-57) in (D.2-56) and expressing the result in the $\underline{\hat{a}}$ basis yields

$$\begin{aligned}
\underline{T} &= \frac{1}{2} \underline{\Omega}^{\underline{T}} \underline{I} \underline{\Omega} + \dot{\underline{\theta}}^{\underline{T}} \underline{I} \underline{\Omega} + \frac{1}{2} \dot{\underline{\theta}}^{\underline{T}} \underline{I} \dot{\underline{\theta}} + \frac{1}{2} (\underline{\Omega}^{\underline{T}} + \dot{\underline{\theta}}^{\underline{T}}) (\underline{I} \underline{\tilde{\theta}} \underline{\theta} + \underline{\tilde{\Omega}} \underline{I} \underline{\theta} + \underline{\tilde{\theta}} \underline{I} \underline{\Omega}) \\
&\quad + \frac{1}{2} \int_{\underline{S}} * \{ \hat{\rho} [\underline{\Omega}^{\underline{T}} (\underline{\tilde{\theta}} \cdot \underline{0}^{\underline{P}}) \underline{\tilde{\theta}} \underline{\tilde{\theta}} \cdot \underline{0}^{\underline{P}}] \} \quad (D.2-58)
\end{aligned}$$

Applying Lagrange's equations

$$\begin{aligned}
\int_{\partial M(\underline{t})} * \{ \underline{0}^{\underline{P}} \underline{S} \} &= \frac{d}{dt} \left[\underline{I} \underline{\Omega} + \underline{I} \dot{\underline{\theta}} + \frac{1}{2} \underline{I} \underline{\tilde{\theta}} \underline{\theta} + \frac{1}{2} \underline{\tilde{\theta}} \underline{I} \underline{\theta} - \frac{1}{2} (\underline{I} \underline{\Omega}) \underline{\tilde{\theta}} \underline{\theta} \right] \\
&\quad - \left[-\frac{1}{2} \underline{\tilde{\theta}} \underline{I} \dot{\underline{\theta}} - \frac{1}{2} \underline{I} \underline{\tilde{\theta}} \dot{\underline{\theta}} + \frac{1}{2} (\underline{I} \underline{\Omega}) \underline{\tilde{\theta}} \dot{\underline{\theta}} - \frac{1}{2} \underline{\tilde{\theta}} \underline{I} \underline{\Omega} - \frac{1}{2} \underline{I} \underline{\tilde{\theta}} \underline{\Omega} + \frac{1}{2} (\underline{I} \underline{\Omega}) \underline{\tilde{\theta}} \underline{\Omega} \right] \\
&\quad - \frac{\partial}{\partial \underline{\theta}} \frac{1}{2} \int_{\underline{S}} * \{ \hat{\rho} [\underline{\theta}^{\underline{T}} \underline{0}^{\underline{P}} \underline{\tilde{\theta}} \underline{\tilde{\theta}} \underline{0}^{\underline{P}} \underline{\theta}] \} \\
&= \underline{\tilde{\theta}} \underline{I} \underline{\Omega} + \underline{I} \ddot{\underline{\theta}} + [\underline{I} \underline{\tilde{\theta}} + \underline{\tilde{\theta}} \underline{I} - (\underline{I} \underline{\Omega}) \underline{\tilde{\theta}}] \dot{\underline{\theta}} - \int_{\underline{S}} * \{ \hat{\rho} [\underline{0}^{\underline{P}} \underline{\tilde{\theta}} \underline{\tilde{\theta}} \underline{0}^{\underline{P}} \underline{\theta}] \} \quad (D.2-59)
\end{aligned}$$

Now

$$\begin{aligned}
{}_0\tilde{P} \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P}\theta &= \tilde{\Omega} {}_0\tilde{P} \tilde{\Omega} {}_0\tilde{P} \theta - (\tilde{\Omega} {}_0\tilde{P}) \tilde{\Omega} {}_0\tilde{P} \theta \\
&= \tilde{\Omega} {}_0\tilde{P} \tilde{\Omega} {}_0\tilde{P} \theta + (\tilde{\Omega} \tilde{\Omega} {}_0\tilde{P}) \tilde{\Omega} {}_0\tilde{P} \theta - \tilde{\Omega} (\tilde{\Omega} {}_0\tilde{P}) \tilde{\Omega} {}_0\tilde{P} \theta \\
&= \tilde{\Omega} {}_0\tilde{P} \tilde{\Omega} {}_0\tilde{P} \theta + (\tilde{\Omega} \tilde{\Omega} {}_0\tilde{P}) \tilde{\Omega} {}_0\tilde{P} \theta - \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P} \tilde{\Omega} {}_0\tilde{P} \theta + \tilde{\Omega} {}_0\tilde{P} \tilde{\Omega} {}_0\tilde{P} \theta \\
&= - ({}_0\tilde{P}\theta) \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P} + \tilde{\Omega} ({}_0\tilde{P} \tilde{\Omega} {}_0\tilde{P} - \tilde{\Omega} {}_0\tilde{P} \tilde{\Omega} + {}_0\tilde{P} \tilde{\Omega} {}_0\tilde{P}) \theta \\
&= - ({}_0\tilde{P}\theta) \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P} + \tilde{\Omega} ({}_0\tilde{P} \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P} - \tilde{\Omega} {}_0\tilde{P} \tilde{\Omega} + {}_0\tilde{P} \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P}) \theta \\
&= - ({}_0\tilde{P}\theta) \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P} + \tilde{\Omega} (- [{}_0\tilde{P} \tilde{\Omega} \tilde{\Omega}] + {}_0\tilde{P} \tilde{\Omega} \tilde{\Omega}) \theta . \tag{D.2-60}
\end{aligned}$$

Inserting this in Equation (D.2-59) yields

$$\begin{aligned}
\int_{\partial M(t)} * \{ {}_0\tilde{P} \underline{S} \} &= \tilde{\Omega} \underline{I} \Omega + \underline{I} \ddot{\theta} + [\underline{I} \tilde{\Omega} + \tilde{\Omega} \underline{I} - (\underline{I} \Omega) \tilde{\Omega}] \dot{\theta} \\
&+ \int_{\underline{S}} * \{ \hat{\rho} [({}_0\tilde{P}\theta) \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P} + \tilde{\Omega} ({}_0\tilde{P} \tilde{\Omega} \tilde{\Omega} \theta - \tilde{\Omega} {}_0\tilde{P} \tilde{\Omega} \theta)] \} \tag{D.2-61}
\end{aligned}$$

or rearranging

$$\begin{aligned}
\int_{\partial M(t)} * \{ {}_0\tilde{P} \underline{S} \} - \int_{\underline{S}} * \{ \hat{\rho} [({}_0\tilde{P}\theta) \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P}] \} &= \tilde{\Omega} \underline{I} \Omega + \underline{I} \ddot{\theta} + [\underline{I} \tilde{\Omega} + \tilde{\Omega} \underline{I} - (\underline{I} \Omega) \tilde{\Omega}] \dot{\theta} \\
&- \tilde{\Omega} (\underline{I} \Omega) \tilde{\Omega} \theta + \tilde{\Omega} \underline{I} \tilde{\Omega} \theta \tag{D.2-62}
\end{aligned}$$

Now recognizing that to zero order $\underline{S} = \tilde{\Omega} \tilde{\Omega} {}_0\tilde{P}$ this can be written

$$\begin{aligned}
\int_{\partial M(t)} * \{ ({}_0\tilde{P} - {}_0\tilde{P}\theta) \tilde{\Omega} \underline{S} \} &= \tilde{\Omega} \underline{I} \Omega + \underline{I} \ddot{\theta} + [\underline{I} \tilde{\Omega} + \tilde{\Omega} \underline{I} - (\underline{I} \Omega) \tilde{\Omega}] \dot{\theta} \\
&+ [\tilde{\Omega} \underline{I} \tilde{\Omega} - \tilde{\Omega} (\underline{I} \Omega) \tilde{\Omega}] \theta \tag{D.2-63}
\end{aligned}$$

Recalling that

$$\underline{0}^P = \underline{\hat{a}}^T ({}_0\tilde{P} - {}_0\tilde{P}\theta) = \underline{\hat{a}}^T ({}_0\tilde{P} + {}_0\tilde{P}\theta) \tag{D.2-64}$$

noting that

$$\underline{I} = \int_{\partial M(t)} * \{ \underline{0}^P \times \underline{S} \}$$

and recalling Equation (D.2-11), the steady and deviational equations become

$$T = \tilde{\Omega} I \Omega$$

$$I\ddot{\theta} + [I \tilde{\Omega} + \tilde{\Omega} I - (I\tilde{\Omega})^{\sim}] \dot{\theta}$$

$$+ [\tilde{\Omega} I \tilde{\Omega} + K - \tilde{\Omega} (I\tilde{\Omega})^{\sim} - \begin{bmatrix} 0 & 0 & 0 \\ -T_3 & 0 & 0 \\ T_2 - T_1 & 0 & 0 \end{bmatrix}] \theta = 0$$

(D.2-65)

which is identical to the Eulerian formulation in the $\underline{\hat{a}}$ basis.