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THE EFFECT OF THIN TURBULENT SHEAR LAYERS ON THE OPTICAL QUALITY OF IMAGING SYSTEMS

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ABSTRACT

The presence of H_2O and CO_2 in the earth's atmosphere has for centuries frustrated astronomers in their attempt to "see" objects radiating outside the visible window. Recently, NASA has outfitted a modified C141 transport with a 91.5 cm reflector telescope designed to view in the infrared range from 1μ to 1000μ. The telescope is situated in a cavity which (because of the lack of infrared-passing windows) is operated open port. Spoilers have been designed which reduce turbulence-induced excitation of the cavity.

Since the aircraft is designed to operate at altitudes up to 15 km, the effect of the H_2O and CO_2 is reduced significantly. Furthermore, the optically degrading influence of the large scale atmospheric turbulence on land-based telescopes is thus replaced by the, as yet little understood, effect of the turbulent shear layer resulting from the spoiler upstream of the cavity. The purpose of this report is to establish a mathematical model appropriate to describe the effect of turbulent shear layers on imaging systems as well as to examine the parameters of interest relevant to potential wind-tunnel experimentation.

Because of the relative thinness of the turbulent shear layer, it is argued that the zeroth order geometrical optics plane wave approximation adequately describes the radiation field in the vicinity of the telescope. With rather mild assumptions, one then finds the following expressions for the average and mean square of the optical transfer function:

$$\langle\tau(\frac{kx}{f}, \frac{ky}{f})\rangle = (1/A) \int_{-\infty}^{\infty} G_0(\vec{x}_1+\vec{x})G_0(\vec{x}_1)\exp\{-k^2K^2/2 \int_0^L\int_0^L \mathcal{D}(t,u;v,w)dz_1dz_2\}d\vec{x}_1$$

and

$$<|\tau(\frac{kx}{f}, \frac{ky}{f})|^2> = (1/A)^2 \iint_{-\infty}^{\infty} G_0(\vec{x}_1+\vec{x})G_0(\vec{x}_1)G_0(\vec{u}_1+\vec{x})G_0(\vec{u}_1)$$

$$X\exp\{-k^2K^2/2\int_0^L\int_0^L[\mathcal{D}(t,u;v,w)+\mathcal{D}(t,s;v,q)+\mathcal{D}(r,u;p,w)+\mathcal{D}(r,s;p,q)]dz_1dz_2\}d\vec{x}_1d\vec{u}_1$$

where $\mathcal{D}$ is defined in terms of the density correlation function, R_ρ, by

$$\mathcal{D}(t,u,;v,w) = R_\rho(v;w) - R_\rho(t;w) - R_\rho(u;v) + R_\rho(t;u)$$

and $p=(\vec{u}_1+\vec{x},z_1)$, $q=(\vec{u}_1+\vec{x},z_2)$, $r=(\vec{u}_1,z_1)$, $s=(\vec{u}_1,z_2)$, $t=(\vec{x}_1+\vec{x},z_1)$, $u=(\vec{x}_1+\vec{x},z_2)$, $v=(\vec{x}_1,z_1)$, and $w=(\vec{x}_1,z_2)$. In the special case that R_ρ is Gaussian, $<\tau>$ can be found explicitly in closed form, and a bound for $<|\tau|^2>$ obtained.

I. INTRODUCTION

The adverse effect of the earth's atmosphere on man's study of celestial bodies has been of concern since the first telescopes were built in the seventeenth century. Electromagnetic waves traverse the universe relatively unhindered in their travels only to be absorbed or dispersed in the last few kilometers of their journey to man's eye. For example, the presence of H_2O and CO_2 in the atmosphere accounts for the absorption of essentially all infrared radiation (1μ to 1000μ) except for several narrow windows in the spectrum up to 20 microns.

An observatory at an altitude of 15 km would be above 95% of the H_2O and 75% of the CO_2 in the atmosphere and objects emitting in the infrared would be detectable. With this in mind, NASA has installed a 91.5 cm reflector telescope in a C-141 aircraft (ref. 1). Though not matching the height advantage of satellites, the aircraft has some advantages in cost and flexibility.

Since no known material is transparent to radiation at all wavelengths in the infrared range, the airborne telescope is designed to operate open port. This circumstance has created the potential for acoustic resonance of the telescope cavity. To avert the concomitant telescope vibrations, spoilers were placed just upstream of the cavity (ref. 2), and, in this configuration, the telescope is able to "see" in the infrared portion of the spectrum. However, the quality of seeing varies randomly in time due to the refractive index fluctuations introduced by the spoilers. Thus, the large scale turbulent fluctuations of the atmospheric boundary layer, which are responsible for such phenomena as the twinkling of stars, have been replaced by the relatively small scale fluctuations in the spoiler-induced free shear layer.

The phenomenon just described is similar to optical degradation experienced by photo reconnaissance or earth resources missions as well as in the use of laser beams for communication systems. The occurrence of optical degradation due to "thin" turbulent boundary layers and shear layers common to so many varied endeavors has stimulated much work, both experimental and theoretical. It is the purpose of this paper to summarize and develop the requisite mathematical models which will provide a basis for an experimental investigation of the fundamental nature of the influence of turbulent shear layers on electromagnetic wave propagation, especially in the infrared range.

What follows is a short history of relevant work performed during the last 20 years. The most thorough experimental investigation of the effect of turbulent boundary layers on light propagation has been that of Stine and Winovich (ref. 3) in 1956. Their results indicate that the loss in resolution due to turbulent boundary layers can be several times that experienced by land-based telescopes resulting from atmospheric turbulence. Hufnagel and Stanley (ref. 4) showed theoretically that the optical degradation due to turbulence can be related to the distribution of the mutual intensity function across the aperture of the telescope. Fried (ref. 5) further distinguishes between "long-exposure" (corresponding to Hufnagel and Stanley's results) and "short-exposure" average optical transfer functions. In the present paper, the author will describe those aspects of the theoretical models devised thus far which are particularly relevant to propagation of infrared radiation through turbulent shear layers.

II. THE ELECTROMAGNETIC FIELD

The electromagnetic field is a vector field satisfying Maxwell's equations together with appropriate constitutive relations. (See Born and Wolf (ref. 6, pp. 1-3)). By assuming that the magnetic permeability is constant and that the dielectric constant does not vary appreciably over times of order λ/c or distances of order λ, the electric and magnetic vectors are found to satisfy identical wave equations. (λ represents the wavelength and c the free space propagation speed.) Hence, each component of the field vectors satisfies the scalar wave equation:

$$\nabla^2 V - (n/c)^2 \ddot{V} = 0, \qquad (2.1)$$

where $V = V(\vec{x},t)$ represents any of the components of the electromagnetic field vectors, $\vec{x}$ is the position vector in the field, t represents time, and $n = n(\vec{x},t)$ is the refractive index. If there is no preferential polarization direction, then the study of the scalar equation (2.1) suffices to yield the measurable quantities of interest, such as intensity or mutual coherence. (See Born and Wolf (ref. 6, pp. 387-392).)

Assuming V is polychromatic, and assuming further that V is square integrable, we have the following Fourier representation:

$$V(\vec{x},t) = \int_{-\infty}^{\infty} u(\vec{x},\omega)e^{-i\omega t}d\omega. \qquad (2.2)$$

In the form (2.2), V is in general complex, which proves to be convenient. However, in the physical problem we will, of course, be interested in the real part of V.

Substituting (2.2) into (2.1) we have

$$\nabla^2 \int_{-\infty}^{\infty} u(\vec{x},\omega)e^{-i\omega t}d\omega - \left(\frac{n}{c}\right)^2 \frac{d^2}{dt^2} \int_{-\infty}^{\infty} u(\vec{x},\omega)e^{-i\omega t}d\omega = 0. \qquad (2.3)$$

Assuming the required uniform convergence of the improper integrals involved, the order of integration and differentiation in (2.3) may be interchanged to give

$$\int_{-\infty}^{\infty} [\nabla^2 u+(n/c)^2\omega^2 u]e^{-i\omega t}d\omega = 0. \qquad (2.4a)$$

If we define $k = \omega/c$ to be the free space wave number (rad/m), then

$$\int_{-\infty}^{\infty} (\nabla^2 u+k^2n^2u)e^{-i\omega t}d\omega = 0. \qquad (2.4b)$$

Finally, we have

$$\nabla^2 u+k^2n^2u = 0; \qquad (2.5)$$

that is, each Fourier component of the complex signal, V, satisfies the Helmholtz time-independent wave equation. If the Fourier components $u(\vec{x},\omega)$ are known, then the signal V is given by (2.2). Furthermore, if polarization is neglected, then intensity, $I(\vec{x})$, at a point satisfies the relation

$$I(\vec{x}) \propto \int_{-\infty}^{\infty} |u(\vec{x},\omega)|^2 d\omega \quad ([6, p.392]).$$

We will henceforth concentrate our attention on $u(\vec{x},\omega)$ rather than $V(\vec{x},t)$ since one determines the other.

We wish now to specialize the problem to that of a plane wave propagating through a turbulent shear layer of thickness, L. The turbulence will be considered to occupy the infinite layer bounded by the planes z=0 and z=L. The undistorted plane wave travels in the positive z direction, entering the shear layer at the plane z=0 and exiting at z=L. It will furthermore be assumed that the wave radiates to ∞ in the positive z direction, unaffected except for the turbulent region just described; in particular, it is assumed that the body which borders the shear layer does not reflect any of the incident radiation. (This is sometimes referred to as the Sommerfeld radiation condition.) With these boundary conditions understood, each Fourier component of the radiation satisfies the Helmholtz equation (2.5).

It should be pointed out that since the refractive index n varies randomly in space, equation (2.5) is a random partial differential equation with variable coefficient, and hence it would be fruitless to seek an exact solution. However, in most cases of practical interest, one can write

$$n(\vec{x}) = 1+n_1(\vec{x}), \qquad (2.6)$$

where $|n_1|<<1$; i.e., the refractive index is perturbed only slightly from its free space value. Because of this fortuitous circumstance, perturbation solutions of (2.5) abound. Most of these methods are described in references 7 and 8. Because of the thinness of the turbulent layer under consideration, however, it is the author's opinion that diffraction effects can be ignored, and a geometrical optics description suffices. Evidence to support this contention follows in section III.

Tatarski [8, p. 174] seeks a solution of (2.5) of the form:

$$u(\vec{x}) = [u_0(\vec{x}) + \frac{1}{k} u_1(\vec{x}) + \frac{1}{k^2} u_2(\vec{x}) + \ldots]e^{ik\theta(\vec{x})}.$$

(The dependence of u on ω will henceforth be implied rather than explicitly stated.) He finds the following expressions for u_0, u_1, and θ:

$$u_0(x,y,z) = A_0 \exp\{-\tfrac{1}{2}\int_0^z (z-\zeta)\nabla_\perp^2 n_1(x,y,\zeta)d\zeta\}$$

$$u_1(x,y,z) = -\frac{iA_0}{8}\int_0^z (z-\zeta)^2\ \nabla_\perp^2[\nabla_\perp^2 n_1(x,y,\zeta)]d\zeta$$

$$\theta(x,y,z) = \int_0^z n(x,y,\zeta)d\zeta = z + \int_0^z n_1(x,y,\zeta)d\zeta$$

where $\nabla_\perp^2$ denotes the lateral (perpendicular to the direction of propagation) Laplacian operator

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad ,$$

and A_0 denotes the initial undistorted wave amplitude. Tatarski argues that terms higher than u_0 can be neglected if (i) $1/k \ll l_i$ and (ii) $\sqrt{\lambda L} \ll l_i$, where l_i denotes the Kolmogoroff inner scale of turbulence. Condition (i) places an upper limit on the wavelength

$$\lambda \ll 2\pi\ l_i \ ,$$

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whereas (ii) essentially restricts the thickness of the shear layer as follows:

$$L \ll l_i^2/\lambda .$$

(Restated, condition (ii) requires that the radius of the first Fresnel zone, $\sqrt{\lambda L}$, be much less than the inner scale of turbulence.) In the analysis which follows, then, we will take:

$$\begin{aligned} u(x,y,z) &= u_0(x,y,z)e^{ik\theta(x,y,z)} \\ &= A_0 e^{ikz} e^{X(x,y,z)+ikY(x,y,z)} \end{aligned} \tag{2.7}$$

where

$$X(x,y,z) = -\tfrac{1}{2}\int_0^z (z-\zeta)\nabla_\perp^2 n_1(x,y,\zeta)d\zeta$$

and

$$Y(x,y,z) = \int_0^z n_1(x,y,\zeta)d\zeta \ .$$

III. THE OPTICAL TRANSFER FUNCTION

The question of how to describe the effect of turbulence on the quality of an imaging system is, to some extent, a matter of taste. Several parameters have been used by designers to describe optical quality depending on the design aspect of interest. A description of several of these quality factors is given by O'Neill [9, Ch. 7]. With few exceptions, all can be related to the so-called optical transfer function (OTF), τ, which is defined to be the normalized two-dimensional spatial Fourier transform of the image plane intensity distribution due to a monochromatic point source; viz:

$$\tau(f,g) = \frac{\iint_{-\infty}^{\infty} s(x,y)\, e^{-i(fx+gy)}\, dx\, dy}{\iint_{-\infty}^{\infty} s(x,y)\, dx\, dy} , \qquad (3.1)$$

where f and g are spatial frequencies with units, radians per meter. In (3.1), the function s(x,y), which denotes the intensity at the point (x,y) in the image plane due to a point source at (0,0) in the object plane, is sometimes referred to as the point spread function. If one views an extended object emitting incoherent light, then

$$I(f,g) = \tau(f,g)\ O(f,g) , \qquad (3.2)$$

where I and O are the two-dimensional spatial Fourier transforms of the intensity distributions in the image and object planes respectively. If one were viewing an object with randomly varying intensity distribution, then instead of (3.2), one has:

$$\Phi_{ii}(f,g) = |\tau(f,g)|^2\ \Phi_{00}(f,g) , \qquad (3.3)$$

where Φ_{ii} and Φ_{00}, which denote the Fourier transforms of the intensity autocorrelation functions in the image and object planes respectively, are the image and object Wiener spectra respectively. ($\Phi(f,g)$ corresponds to the power spectral density for time varying random functions.) To summarize, then, for incoherently illuminated objects, the optical transfer function relates the Fourier transforms of image and object intensities in a linear manner.

Let us define the pupil (or aperture) function, G(x,y), to be the complex disturbance at the point (x,y) in the aperture plane due to a point source; G(x,y) will be defined to be zero if (x,y) lies outside the aperture. (Note that we restrict our source to lie within an isoplanatic patch of the working field.) It can be shown (see, for example, Born and Wolf [6, p.485] or O'Neill [9, p. 77]) that the optical transfer function is related to G as follows:

$$\tau\left(\frac{kx}{R}, \frac{ky}{R}\right) = \frac{\iint_{-\infty}^{\infty} G(x'+x,y'+y)G^*(x',y')\ dx'dy'}{\iint_{-\infty}^{\infty} G(x',y')G^*(x'y')\ dx'dy'}, \qquad (3.4)$$

where R denotes the focal length of the system. (Note that the integrations in (3.4) are only formally infinite since G vanishes identically outside the aperture.) The representation (3.4) will be particularly convenient as we now proceed to describe the effect of the turbulent layer on optical quality.

If we assume that a point source is located at a distance far removed from the optical system under consideration, then the radiation entering the turbulent layer is, to good approximation, a plane wave. But then we have:

$$\tau\left(\frac{kx}{R}, \frac{ky}{R}\right) = \frac{\iint_{-\infty}^{\infty} G_0(x'+x,y'+y)G_0(x',y')u(x'+x,y'+y,L)u^*(x',y',L)dx'dy'}{\iint_{-\infty}^{\infty} G_0^2(x',y')u(x',y',L)u^*(x',y',L)dx'dy'} \qquad (3.5)$$

where

$$G_0(x,y) = \begin{cases} 1, & (x,y) \text{ in the aperture} \\ 0, & (x,y) \text{ not in the aperture} \end{cases}$$

and u(x,y,z) is given by (2.7).

Before proceeding further, a word is in order here regarding the efficacy of τ as a measure of optical degradation due to turbulence. By definition, τ is the transfer function relating the intensity distribution in an extended object to that of the image. Its value as a transfer function is thus limited to incoherently illuminated objects. On the other hand, if one observes a star with a telescope, then the intensity distribution in the focal plane is exactly the point spread function, whose normalized Fourier transform is given by (3.1). Hence, even though the use of τ as a transfer function must be limited to the ideal, and rarely realized, case of incoherent illumination, its usefulness as a measure of optical degradation is broader. (The derivation of transfer functions appropriate to coherent or partially coherent extended objects is given in Born and Wolf [6, Sections 9.5.1 and 10.5.3 respectively].)

Since, by (2.7), u depends on n_1, which varies randomly with position in the turbulent field, then (3.5) implies that for each fixed spatial frequency, τ is a complex random variable. Hence, a complete description of τ requires determination of its probability density, or, equivalently, its moments. In practice, one is fortunate to determine the first two moments, namely the mean, or expectation, and the mean square. If we denote mean by $< >$, then from (3.5),

$$<\tau(\frac{kx}{R},\frac{ky}{R})> \approx \frac{\iint_{-\infty}^{\infty} G_0^{(2)}(x',y';x,y) M_2(x',y';x,y)\ dx'dy'}{\iint_{-\infty}^{\infty} G_0^{(2)}(x',y';0,0) M_2(x',y';0,0)\ dx'dy'}, \quad (3.6)$$

where

$$G_0^{(2)}(x',y';x,y) = G_0(x'+x,y'+y)G_0(x',y')$$

and

$$M_2(x',y';x,y) = <u(x'+x,y'+y,L)\ u^*(x',y',L)>.$$

An exact equality is, unfortunately, not realized since, in general, the mean of a quotient is not equal to the quotient of the means. It is reasonable, however, to accept (3.6) as a first approximation. (See [10, p. 151] for the derivation of a series expansion for $<\tau>$ of which (3.6) gives the first term.) Similarly, the mean square of τ is given by:

$$<|\tau(\frac{kx}{R},\frac{ky}{R})|^2> \approx \frac{\iiiint_{-\infty}^{\infty} G_0^{(4)}(x',y';u',v';x,y)M_4(x',y';u',v';x,y)dx'dy'du'dv'}{\iiiint_{-\infty}^{\infty} G_0^{(4)}(x',y';u',v';0,0)M_4(x',y';u',v';0,0)dx'dy'du'dv'} \quad (3.7)$$

where

$$G_0^{(4)}(x',y';u',v';x,y)=G_0(x'+x,y'+y)G_0(x'y')G_0(u'+x,v'+y)G_0(u',v')$$

$$M_4(x',y';u',v';x,y)=<u(x'+x,y'+y,L)u^*(x',y',L)u(u',v',L)u^*(u'+x,v'+y,L)>.$$

Note that the influence of the turbulence on $<\tau>$ is felt through the second moments of the disturbance in the aperture plane, namely the mutual intensity (in the numerator) and intensity (in the denominator), whereas the stochastic contribution to $<|\tau|^2>$ arises from the fourth order moments of the disturbance in the aperture plane. Although the expectations or means which occur here are ensemble averages, they may be calculated in practice by taking time averages if the light disturbance can be assumed stationary, an assumption almost surely warranted.

Equations (3.6) and (3.7) are quite general, and do not depend on the particular model of the disturbance. If we utilize the geometrical optics approximation, described in section II, then the mutual intensity is given by:

$$M_2(x',y';x,y) = |A_0|^2<e^{A+iB}>, \quad (3.8)$$

where

$$A = X(x'+x,y'+y,L) + X(x',y',L)$$
$$= -\tfrac{1}{2}\int_0^L (L-\zeta)[\nabla_\perp^2 n_1(x'+x,y'+y,\zeta) + \nabla_\perp^2 n_1(x',y',\zeta)]d\zeta$$

and

$$B = k[Y(x'+x,y'+y,L) - Y(x',y',L)]$$
$$= k\int_0^L [n_1(x'+x,y'+y,\zeta) - n_1(x',y',\zeta)]\, d\zeta.$$

Similarly, the fourth order correlation function occurring in (3.7) is given by:

$$M_4(x',y';u',v';x,y) = |A_0|^4 \langle e^{A'+iB'}\rangle, \tag{3.9}$$

where

$$A' = -\tfrac{1}{2}\int_0^L (L-\zeta)[\nabla_\perp^2 n_1(x'+x,y'+y,\zeta)+\nabla_\perp^2 n_1(x',y',\zeta)+\nabla_\perp^2 n_1(u',v',\zeta)+\nabla_\perp^2 n_1(u'+x,v'+y,\zeta)]d\zeta$$

and

$$B'=k\int_0^L [n_1(x'+x,y'+y,\zeta)-n_1(x',y',\zeta)+n_1(u',v',\zeta)-n_1(u'+x,v'+y,\zeta)]d\zeta.$$

Unless we know the probability density for n_1, we have reached an impasse. However, if we are willing to assume that the variables A, B, A', and B' are Gaussian, then further progress can be made. In particular, we will assume that A and B are jointly Gaussian and similarly for A' and B'. (From the definitions of A, B, A', and B', an appeal to the central limit theorem seems to lend credence to such an assumption.) Then, by carrying out the integration inferred by < > with respect to the Gaussian density, one finds

$$\langle e^{A+iB}\rangle = e^{\langle A\rangle+i\langle B\rangle} e^{\frac{1}{2}(\sigma_A^2 - \sigma_B^2)} e^{i\,\mathrm{cov}(A,B)} \tag{3.10a}$$

and, similarly,

$$\langle e^{A'+iB'}\rangle = e^{\langle A'\rangle + i\langle B'\rangle}\, e^{\frac{1}{2}(\sigma^2_{A'} - \sigma^2_{B'})}\, e^{i\,\mathrm{cov}(A',B')} , \tag{3.10b}$$

where σ^2 denotes variance and cov denotes covariance. In order to carry out the calculations indicated by (3.10), we proceed to determine the joint moments for A and B. (Exactly similar results will follow for A' and B'.)

For convenience in notation, we make the definitions:

$$R(x_1,y_1,z_1;x_2,y_2,z_2) = \langle n_1(x_1,y_1,z_1)n_1(x_2,y_2,z_2)\rangle \tag{3.11a}$$

$$P(x_1,y_1,z_1;x_2,y_2,z_2) = \langle \nabla^2_\perp n_1(x_1,y_1,z_1)\nabla^2_\perp n_1(x_2,y_2,z_2)\rangle \tag{3.11b}$$

$$Q(x_1,y_1,z_1;x_2,y_2,z_2) = \langle [\nabla^2_\perp n_1(x_1,y_1,z_1)]n_1(x_2,y_2,z_2)\rangle . \tag{3.11c}$$

Because the operation of ensemble averaging is commutative with both integration and differentiation, then from (3.8)

$$\begin{aligned} \langle A\rangle &= -\tfrac{1}{2}\int_0^L (L-\zeta)[\nabla^2_\perp\langle n_1(x'+x,y'+y,\zeta)\rangle + \nabla^2_\perp\langle n_1(x',y',\zeta)\rangle]d\zeta \\ \langle B\rangle &= k\int_0^L [\langle n_1(x'+x,y'+y,\zeta)\rangle - \langle n_1(x',y',\zeta)\rangle]d\zeta . \end{aligned} \tag{3.12}$$

Note that if one assumes $\langle n_1\rangle$ = constant throughout the turbulent field (a not unreasonable assumption), then $\langle A\rangle = \langle B\rangle = 0$. Furthermore, from (3.8) and (3.11) one finds:

$$\begin{aligned} \langle A^2\rangle = \tfrac{1}{4}\int_0^L\int_0^L (L-\zeta_1)(L-\zeta_2)[&P(x',y',\zeta_1;x',y',\zeta_2) \\ &+P(x'+x,y'+y,\zeta_1;x',y',\zeta_2)+P(x',y',\zeta_1;x'+x,y'+y,\zeta_2) \\ &+P(x'+x,y'+y,\zeta_1;x'+x,y'+y,\zeta_2)]\, d\zeta_1 d\zeta_2 \end{aligned} \tag{3.13a}$$

$$\langle B^2\rangle = k^2 \int_0^L\int_0^L [R(x',y',\zeta_1;x',y',\zeta_2)-R(x'+x,y'+y,\zeta_1;x',y',\zeta_2)$$
$$-R(x',y',\zeta_1;x'+x,y'+y,\zeta_2) \tag{3.13b}$$
$$+R(x'+x,y'+y,\zeta_1;x'+x,y'+y,\zeta_2)]\ d\zeta_1 d\zeta_2$$

and

$$\langle AB\rangle = k/2 \int_0^L\int_0^L (L-\zeta_1)[Q(x',y',\zeta_1;x',y',\zeta_2)$$
$$+Q(x'+x,y'+y,\zeta_1;x',y',\zeta_2)-Q(x',y',\zeta_1;x'+x,y'+y,\zeta_2)$$
$$-Q(x'+x,y'+y,\zeta_1;x'+x,y'+y,\zeta_2)]\ d\zeta_1 d\zeta_2. \tag{3.13c}$$

Similar expressions hold for A' and B'. Clearly, without further simplifying assumptions the problem remains, in a practical sense, intractable. To elaborate, observe that from (3.11)

$$P(x_1,y_1,z_1;x_2,\ y_2,\ z_2) = \frac{\partial^4 R}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 R}{\partial y_1^2 \partial x_2^2} + \frac{\partial^4 R}{\partial x_1^2 \partial y_2^2} + \frac{\partial^4 R}{\partial y_1^2 \partial y_2^2}$$
$$Q(x_1,y_1,z_1;x_2,y_2,z_2) = \frac{\partial^2 R}{\partial x_1^2} + \frac{\partial^2 R}{\partial y_1^2}\ . \tag{3.14}$$

Hence, even if one were able to experimentally determine the refractive index (density) correlation function, evaluation of $\langle A^2\rangle$ and $\langle AB\rangle$ would require a numerical differentiation of fourth order and second order respectively. Although this is feasible in principle, it is very difficult to do with any accuracy in practice, without a very fine mesh measurement scheme.

To conclude our general discussion, it should be noted that in practice (see, for example, [11]), it is the modulus of the OTF, called the modulation transfer function (MTF), which is measured, rather than the OTF itself. The statistics of the MTF are related to those of the OTF as follows:

$$0 \leq |\langle\tau\rangle| \leq \langle|\tau|\rangle \leq 1. \tag{3.15}$$

Hence, if the expressions for $\langle\tau\rangle$ obtained in this section are used to estimate the average MTF, $\langle|\tau|\rangle$, the estimate can be expected to be conservative. Furthermore, the variance of the MTF is bounded as follows:

$$0 \leq \sigma^2_{|\tau|} = \langle|\tau|^2\rangle - \langle|\tau|\rangle^2 \leq \langle|\tau|^2\rangle - |\langle\tau\rangle|^2 = \sigma^2_\tau. \tag{3.16}$$

Knowledge of the variance, together with application of Chebyshev's inequality (see, for example, [12, p. 20]), yields information about the variability of "seeing" conditions. In particular,

$$\Pr(|\tau - \langle\tau\rangle| > \alpha\sigma_\tau) \leq 1/\alpha^2$$

and

$$\Pr(\big||\tau| - \langle|\tau|\rangle\big| > \alpha\sigma_{|\tau|}) \leq 1/\alpha^2 ,$$

for the OTF and MTF respectively.

In concluding this discussion, it is interesting to note that from (3.15), the MTF suffers less degradation on the average than the OTF. Furthermore, from (3.16), the variance of the MTF is, in general, smaller than that of the OTF.

Homogeneous Turbulence. If one is willing to assume that the refractive index spatial variations are statistically homogeneous, then certain simplifications can be made. In particular, $\langle n_1\rangle$ = constant and, hence, from (3.9) and (3.12),

$$\langle A\rangle = \langle B\rangle = \langle A'\rangle = \langle B'\rangle = 0.$$

Furthermore, the correlation function R is now a function only of the difference in the coordinates rather than the actual coordinates of the two points under consideration; i.e.,

$$R(x_1,y_1,z_1;x_2,y_2,z_2) = R(x_2-x_1,y_2-y_1,z_2-z_1). \tag{3.17}$$

Then, the relations (3.13) become:

$$\langle A^2\rangle = \tfrac{1}{4} \iint_0^L (L-\zeta_1)(L-\zeta_2)[2P(0,0,\zeta_2-\zeta_1)+P(x,y,\zeta_2-\zeta_1) +P(-x,-y,\zeta_2-\zeta_1)]\, d\zeta_1 d\zeta_2 \tag{3.18a}$$

$$\langle B^2\rangle = k^2 \iint_0^L [2R(0,0,\zeta_2-\zeta_1)-R(x,y,\zeta_2-\zeta_1)-R(-x,-y,\zeta_2-\zeta_1]d\zeta_1 d\zeta_2 \tag{3.18b}$$

$$\langle AB\rangle = (k/2) \iint_0^L (L-\zeta_1)[Q(-x,-y,\zeta_2-\zeta_1)-Q(x,y,\zeta_2-\zeta_1)]\, d\zeta_1 d\zeta_2, \tag{3.18c}$$

where, if $R=R(\xi,\eta,\zeta)$, then

$$P(\xi,\eta,\zeta) = \frac{\partial^2 R}{\partial \xi^4} + 2\frac{\partial^4 R}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 R}{\partial \eta^4}$$

$$Q(\xi,\eta,\zeta) = \frac{\partial^2 R}{\partial \xi^2} + \frac{\partial^2 R}{\partial \eta^2} \tag{3.19}$$

Hufnagel and Stanley [4, Theorem II, p. 61] have shown that

$$Q(-x,-y,\zeta_2-\zeta_1) = Q(x,y,\zeta_2-\zeta_1),$$

which implies that $\langle AB\rangle = 0$ and, hence, that A and B are uncorrelated. Some simplification in the expressions for $\langle A^2\rangle$ and $\langle B^2\rangle$ can be achieved by making a change of variable (see Appendix). In particular, the double integrals are reduced to single integrals as follows:

$$\langle A^2\rangle = (1/12)\int_0^L (2L+z)(L-z)^2[2P(0,0,z)+P(x,y,z)+P(x,y,-z)]dz$$

$$\langle B^2\rangle = 2k^2\int_0^L (L-z)[2R(0,0,z)-R(x,y,z)-R(x,y,-z)]\,dz. \tag{3.20}$$

If we should assume further that isotropic conditions prevail, then $R(x,y,z)=R(x,y,-z)$ and $P(x,y,z)=P(x,y,-z)$ and (3.20) can be simplified even further. (In fact, one need only assume that the correlation function is an even function of z to obtain the same result.)

Similarly, from (3.9), one finds:

$$\begin{aligned}\langle A'^2\rangle = 2\langle A^2\rangle + (1/12)\int_0^L (2L+z)(L-z)^2\{&2[P(u'-x',v'-y',z)\\ &+P(u'-x',v'-y',-z)] + P(u'-x'-x,v'-y'-y,z)\\ &+P(u'-x'-x,v'-y'-y,-z) + P(u'-x'+x,v'-y'+y,z)\\ &+P(u'-x'+x,v'-y'+y,-z)\}\,dz\end{aligned} \tag{3.21a}$$

and

$$\begin{aligned}\langle B'^2\rangle = 2\langle B^2\rangle - 2k^2\int_0^L (L-z)\{&2[R(u'-x',v'-y',z)+R(u'-x',v'-y',-z)]\\ &-R(u'-x'-x,v'-y'-y,z)-R(u'-x'-x,v'-y'-y,-z)\\ &-R(u'-x'+x,v'-y'+y,z)-R(u'-x'+x,v'-y'+y,-z)\}\,dz\end{aligned} \tag{3.21b}$$

Finally, combining (3.6), (3.7), (3.8), (3.9), (3.10), (3.20), and (3.21) we have:

$$\langle\tau\rangle = \tau_0\ \tau_R\quad, \tag{3.22}$$

where

$$\tau_R=\exp\left\{-\tfrac{1}{2}\int_0^L\left[\frac{(2L+z)(L-z)^2}{12}F_1(x,y,z)+2k^2(L-z)G_1(x,y,z)\right]dz\right\}$$

$$F_1(x,y,z) = 2P(0,0,z) - P(x,y,z) - P(x,y,-z)$$

$$G_1(x,y,z) = 2R(0,0,z) - R(x,y,z) - R(x,y,-z)$$

and τ_0 denotes the diffraction limited transfer function of the imaging system. The function τ_R, representing the random influence of the turbulence, is independent of properties of the imaging system. This decomposition of $<\tau>$ into the product of the diffraction limited OTF and a turbulent degradation is, in general, possible only if an assumption like homogeneity (or local homogeneity) is invoked. Furthermore,

$$<|\tau|^2>= \tau_R^2 A_p^2 \frac{\iiiint\limits_{-\infty}^{\infty} G_0^{(4)}(x',y';u',v';x,y)e^{U(x',y',u',v';x,y,L)}dx'dy'du'dv'}{\iiiint\limits_{-\infty}^{\infty} G_0^{(4)}(x',y';u',v';0,0)e^{U(x',y',u',v';0,0,L)}dx'dy'du'dv'}, \quad (3.23a)$$

where

$$U(x',y',u',v';x,y,L)=\tfrac{1}{2}\ \{[<A'^2>-<B'^2>]-\ 2[<A^2>-<B^2>]\},$$

and A_p denotes the aperture area. If we invoke the second mean value theorem for integrals, then (3.23a) can be rewritten as follows:

$$<|\tau|^2>=\tau_0^2\tau_R^2\exp\left\{\tfrac{1}{2}\int_0^L\{\frac{(2L+z)(L-z)^2}{12}\ [F_2(s_1,t_1;x,y,z)-F_2(s_2,t_2;0,0,z)] + 2k^2(L-z)G_2(s_1,t_1;x,y,z)\}dz\right\}, \quad (3.23b)$$

where

$$F_2(s,t;x,y,z) = P(s-x,t-y,z) + P(s-x,t-y,-z) + P(s+x,t+y,z) + P(s+x,t+y,-z) -2[P(s,t,z) + P(s,t,-z)]$$

and

$$G_2(s,t;x,y,z) = 2[R(s,t,z) + R(s,t,-z)] -R(s-x,t-y,z) -R(s-x,t-y,-z) -R(s+x,t+y,z) -R(s+x,t+y,-z).$$

Since the mean value theorem is nonconstructive, we can guarantee the existence of s_i and t_i $(i=1,2)$ which validate (3.23b), but no algorithm exists for determining their values. For circular and rectangular apertures, one can argue, however, that

$$0 \le s_i^2 + t_i^2 \le (D-r)(D+r)$$

( D)

and

$$0 \le s_i^2 + t_i^2 \le (D_x-x)^2+(D_y-y)^2$$

( D_y) ,

D_x

respectively, where $r^2=x^2+y^2$. Clearly, equation (3.23b) is suggestive, but not computationally useful. On the other hand, in the following example, (3.23b) will be helpful in constructing an interesting upper bound for $<|\tau|^2>$.

<u>Homogeneous Turbulence with Gaussian Correlation</u>. Suppose the refractive index correlation function is Gaussian; i.e.,

$$R(x,y,z)= \sigma_{n_1}^2 \, e^{-[(\frac{x}{a})^2+(\frac{y}{b})^2+(\frac{z}{c})^2]} + <n_1>^2 .$$

(A brief discussion of this example can be found in [13, p.46].) Then from the definition (3.19) of P, one finds

$$P(x,y,z)= \sigma_{n_1}^2 \{(\frac{2}{a^2})^2[4(\frac{x}{a})^4-12(\frac{x}{a})^2+3] + 2(\frac{2}{a^2})(\frac{2}{b^2})[1-2(\frac{x}{a})^2][1-2(\frac{y}{b})^2]$$
$$+ (\frac{2}{b^2})^2[4(\frac{y}{b})^4-12(\frac{y}{b})^2+3]\} \exp\{-[(\frac{x}{a})^2+(\frac{y}{b})^2+(\frac{z}{c})^2]\}.$$

It might be noted that unless a=b=c, this model is homogeneous, but not isotropic. Even so, the correlation function is even in its arguments and so R(x,y,z)=R(x,y,-z) and P(x,y,z)=P(x,y,-z). Then, calculation of the constituents of τ_R as given by (3.22) gives:

$$\int_0^L \frac{(2L+z)(L-z)^2}{12} F_1(x,y,z)dz = \frac{\sigma_{n_1}^2}{6} \Big\{3(\tfrac{2}{a^2})^2+2(\tfrac{2}{a^2})(\tfrac{2}{b^2})+3(\tfrac{2}{b^2})^2$$

$$+\{(\tfrac{2}{a^2})^2[4(\tfrac{x}{a})^4-12(\tfrac{x}{a})^2+3] + 2(\tfrac{2}{a^2})(\tfrac{2}{b^2})[1-2(\tfrac{x}{a})^2][1-2(\tfrac{y}{b})^2] \qquad (3.24)$$

$$+(\tfrac{2}{b^2})^2[4(\tfrac{y}{b})^4-12(\tfrac{y}{b})^2+3]\} \exp\{-[(\tfrac{x}{a})^2+ (\tfrac{y}{b})^2]\}\Big\} I_z ,$$

where

$$I_z = \int_0^L (2L+z)(L-z)^2 \, e^{-(z/c)^2} dz$$

$$=3c^4\{(\sqrt{\pi}/3)(L/c)^3\mathrm{erf}(L/c)+(1/3)[(L/c)^2-\tfrac{1}{2}]e^{-(L/c)^2}+(1/6)-\tfrac{1}{2}(L/c)^2\}.$$

(<u>NOTE</u>: erf denotes the error function, which is defined in the usual way as:

$$\mathrm{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-u^2} du.)$$

Similarly,

$$\int_0^L 2k^2(L-z)G_1(x,y,z)dz = 4k^2\sigma_{n_1}^2 [1-\exp\{-[(\tfrac{x}{a})^2+ (\tfrac{y}{b})^2]\}] \; J_z , \qquad (3.25)$$

where

$$J_z = \int_0^L (L-z)e^{-(z/c)^2} dz$$

$$= (c^2/2)\{\sqrt{\pi}(L/c)\mathrm{erf}(L/c) - 1 + e^{-(L/c)^2}\} .$$

In the special case a=b, then, from (3.22), (3.24), and (3.25), we find

$$\tau_R = \exp\left\{ -\sigma_{n_1}^2 \{4(\frac{c}{a})^4 [2(e^{-(r/a)^2} - 1) + (\frac{r}{a})^2 [(\frac{r}{a})^2 - 4]^{\; -r/a)^2}] I_z / 3c^4 + 2k^2c^2(1 - e^{-(r/a)^2}) J_z / c^2\} \right\} . \tag{3.26}$$

Several calculations of <τ> for this case were carried out on a CDC 3300, and the results are illustrated in figures 1 through 3. Some tentative conclusions can be drawn. In particular, high degradation can be attributed to:

(i) large diameter optics,
(ii) low altitudes,
(iii) thick shear layers,

and

(iv) short wavelengths.

As might have been anticipated, the contribution of the F_1 term appearing in the integral in (3.22) (or the I_z term in (3.26)) is negligible. The term containing the wave number, k, which is very large in the geometrical optics approximation, essentially "swamps" the result. It is to be expected that, similarly, the F_2 term occurring in (3.23b) will prove negligible when compared with the k^2 term. If one ignores the contribution of the F_2 term in (3.23b), it is easily seen that

$$<|\tau|^2> \; < \; \tau_0^2 \tau_R^2 \exp\{4k^2c^2\sigma_{n_1}^2 [1 - \exp\{-[(\frac{x}{a})^2 + (\frac{y}{b})^2]\} J_z/c^2 , \tag{3.27a}$$

and, hence, in the special case a=b,

$$<|\tau|^2> \; < \; \tau_0^2 . \tag{3.27b}$$

Finally, for the case a=b, from (3.16), (3.22), and (3.27b), we have

$$0 \le \sigma^2_{|\tau|} \le \sigma^2_\tau < \tau_0^2 - \tau_0^2\tau_R^2 = \tau_0^2(1-\tau_R^2) < \tau_0^2. \tag{3.28}$$

It follows that

$$3\,\sigma_{|\tau|} < 3\,\tau_o\,\sqrt{1-\tau_R^2}. \tag{3.29}$$

It is the bound (3.29) which gives the 3σ boundaries of figure 4. Equation (3.29) leads to the interesting conclusion that the smaller the influence of the turbulence ($\tau_R \approx 1$) on the average MTF, the smaller the variance of the MTF. Indeed, in addition to the data presented in figures 1 to 4, the author performed calculations for propagation at $\lambda = 100\mu$ for all permutations of the parameters D= .01 and .915 meters, a=b=2c = .007 and .2 meters, L = .1 and .3 meters, and $\langle\rho\rangle$ = .905 kg/m^3 (3km) and .187 kg/m^3 (15km). The influence of the turbulence in these configurations was found to be negligible, and so the optical system is diffraction limited within small fluctuations. Finally, note that figures 1 through 4 are conservative in the sense that the correct MTF would, in general, be larger and the true deviations smaller than illustrated.

The influence of scale of turbulence is somewhat more complicated than the effects described above. In particular, an asymptotic analysis of the expression containing k^2 on the right hand side of (3.26) leads to the following representations of τ_R:

$$\tau_R \sim \exp\{-\beta\} ; \qquad L/c<<1,\ D/a>>1,$$

$$\tau_R \sim \exp\{-\beta(r/D)^2(D/a)^2\}; \qquad L/c,\ D/a<<1,$$

$$\tau_R \sim \exp\{-\beta(L/c)^{-1}[\sqrt{\pi} - (L/c)^{-1}]\}; \qquad L/c,\ D/a>>1,$$

$$\tau_R \sim \exp\{-\beta(r/D)^2(D/a)^2(L/c)^{-1}[\sqrt{\pi} - (L/c)^{-1}]\}; \ L/c>>1,\ D/a<<1,$$

where

$$\beta = [2\pi\ \sigma_{n_1}(L/\lambda)]^2.$$

If $\beta>>1$, then clearly the first case cited experiences the worst degradation. That is, if the upstream spoiler induces a shear layer which exhibits relatively low frequency fluctuations in the propagation direction and, at the same time, relatively high frequency fluctuations in the plane perpendicular to propagation, then degradation can be quite large. On the other hand, the converse situation (i.e., $L/c>>1$ and $D/a<<1$) tends to mitigate against large optical degradation.

Locally Homogeneous Turbulence. It is, of course, rare that turbulence satisfies exactly the homogeneous hypothesis. Perhaps the simplest variety of nonhomogeneous turbulence is that labeled locally homogeneous. In this case, it is assumed that while the refractive index field is not necessarily homogeneous, the spatial increments of n_1 are. The relevant correlation function is then

$$\mathcal{D}(\vec{x}_1,\vec{x}_2;\vec{x}_3,\vec{x}_4) = <[n_1(\vec{x}_3)-n_1(\vec{x}_1)][n_1(\vec{x}_4)-n_1(\vec{x}_2)]>, \tag{3.30}$$

which can be rewritten as follows:

$$\mathcal{D}(\vec{x}_1,\vec{x}_2;\vec{x}_3,\vec{x}_4) = \tfrac{1}{2}\{D(\vec{x}_1,\vec{x}_4)+D(\vec{x}_2,\vec{x}_3)-D(\vec{x}_1,\vec{x}_2)-D(\vec{x}_3,\vec{x}_4)\}, \tag{3.31}$$

where

$$D(\vec{x}_i,\vec{x}_j) = \langle[n_1(\vec{x}_j)-n_1(\vec{x}_i)]^2\rangle$$

is called the structure function of the random field, n_1. If the field n_1 is locally homogeneous, then

$$D(\vec{x}_i,\vec{x}_j) = D(x_j-x_i,y_j-y_i,z_j-z_i). \tag{3.32}$$

Our purpose here is to calculate the statistics for the OTF in terms of the structure function.

Based on the results for homogeneous turbulence discussed above, it seems reasonable to assume $\langle n_1\rangle$ = constant, and, hence, that

$$\langle A\rangle = \langle B\rangle = \langle A'\rangle = \langle B'\rangle = 0.$$

Furthermore, we will ignore the (presumably small) contributions of $\langle AB\rangle$, $\langle A'B'\rangle$, $\langle A^2\rangle$, and $\langle A'^2\rangle$ to the statistics of τ. Hence, we need only derive expressions for $\langle B^2\rangle$ and $\langle B'^2\rangle$. But, from (3.8),

$$\begin{aligned}\langle B^2\rangle &= k^2 \iint_0^L \mathcal{D}(x',y',\zeta_1;x',y',\zeta_2;x'+x,y'+y,\zeta_1;x'+x,y'+y,\zeta_2)d\zeta_1 d\zeta_2 \\ &= \tfrac{1}{2}k^2 \iint_0^L [D(x,y,\zeta_2-\zeta_1)+D(-x,-y,\zeta_2-\zeta_1)-2D(0,0,\zeta_2-\zeta_1)]d\zeta_1 d\zeta_2.\end{aligned} \tag{3.33}$$

By a change of variables (see Appendix), we can write:

$$\langle B^2\rangle = k^2 \int_0 (L-z)[D(x,y,z)+D(x,y,-z)-2D(0,0,z)]dz. \tag{3.34}$$

Similarly, one finds:

$$\langle B'^2\rangle = 2\langle B^2\rangle + k^2 \int_0^L (L-z)\Big\{2[D(u'-x',v'-y',z)+D(u'-x',v'-y',-z)]$$
$$-D(u'-x'-x,v'-y'-y,z)-D(u'-x'-x,v'-y'-y,-z) \tag{3.35}$$
$$-D(u'-x'+x,v'-y'+y,z)-D(u'-x'+x,v'-y'+y,-z)\Big\}dz.$$

Finally, combining equations (3.6)-(3.10), (3.34), and (3.35), we have:

$$\langle\tau\rangle = \tau_0\tau_R , \tag{3.36}$$

where

$$\tau_R = \exp\{-k^2/2 \int_0^L (L-z)[D(x,y,z)+D(x,y,-z)-2D(0,0,z)]dz\};$$

also,

$$\langle|\tau|^2\rangle = (\tau_R^2/A_p^2) \iiiint_{-\infty}^{\infty} G_0^{(4)}(x',y';u',v';x,y)\ e^{V(x',y',u',v';x,y,L)}dx'dy'du'dv', \tag{3.37a}$$

where

$$V(x',y',u',v';x,y,L) = -\tfrac{1}{2}[\langle B'^2\rangle - 2\langle B^2\rangle].$$

If we again invoke the second mean value theorem for integrals, then

$$\langle|\tau|^2\rangle = \tau_0^2\tau_R^2 \exp\{-k^2/2 \int_0^L (L-z)\ H(s,t;x,y,z)\ dz\} , \tag{3.37b}$$

where

$$H(s,t;x,y,z)=2[D(s,t,z) + D(s,t,-z)] - D(s-x,t-y,z)$$
$$-D(s-x,t-y,-z) -D(s+x,t+y,z) -D(s+x,t+y,-z).$$

The remarks following equation (3.23b) concerning s and t are applicable here.

If the turbulence is assumed to be locally isotropic as well as locally homogeneous, then (3.34) and (3.35) become

$$\langle B^2\rangle = 2k^2 \int_0^L (L-z)[D(x,y,z) - D(0,0,z)]\, dz$$

and

$$\langle B'^2\rangle = 2\{\langle B^2\rangle + k^2 \int_0^L (L-z)[2D(u'-x',v'-y',z)$$
$$-D(u'-x'-x,v'-y'-y,z) - D(u'-x'+x,v'-y'+y,z)]dz\},$$

respectively. Hence, the above results for $\langle\tau\rangle$ and $\langle|\tau|^2\rangle$ simplify to:

$$\langle\tau\rangle = \tau_0\tau_R = \tau_0 \exp\{-k^2 \int_0^L (L-z)[D(x,y,z) - D(0,0,z)]\, dz\}$$

and

$$\langle|\tau|^2\rangle = (\tau_R^2/A_p^2) \iiiint_{-\infty}^{\infty} G_0^{(4)}(x',y';u',v';x,y)$$
$$\times \exp\{-k^2 \int_0^L (L-z)[2D(u'-x',v'-y',z) - D(u'-x'-x,v'-y'-y,z)$$
$$-D(u'-x'+x,v'-y'+y,z)]\, dx'dy'du'dv'\}$$
$$= \tau_0^2\tau_R^2 \exp\{-k^2 \int_0^L (L-z)[2D(s,t,z) - D(s-x,t-y,z) - D(s+x,t+y,z)]dz\}.$$

A few remarks are in order here concerning the relation between correlation functions and structure functions. If the correlation function is known, then the structure function is given by:

$$D(\vec{x}_1,\vec{x}_2) = R(\vec{x}_1,\vec{x}_1) + R(\vec{x}_2,\vec{x}_2) - 2R(\vec{x}_1,\vec{x}_2). \qquad (3.38)$$

Unfortunately, there is no analogue for recovering the correlation function from a measured structure function (unless the turbulence is homogeneous, in which case the two statistics are equivalent). Hence, in general, it is recommended that the correlation function be measured, rather than the structure function. The structure function can be calculated using (3.38) if it should prove of interest. If the sole object is to verify the simplified model presented here, then either correlation or structure functions are acceptable, since the structure function occurs naturally in $\langle B^2 \rangle$ and $\langle B'^2 \rangle$ (see equations (3.34) and (3.35)). A thorough discussion of structure functions and their relation to correlation functions can be found in [14, p. 86] and [8, pp. 13-38].

IV. CONCLUDING REMARKS

Several conclusions and recommendations can be drawn, and observations made, based on the above analysis:

1. If one assumes from the start that

(i) $\langle n_1 \rangle$ = constant,

and

(ii) $\langle A^2 \rangle$, $\langle A'^2 \rangle$, $\langle AB \rangle$, and $\langle A'B' \rangle \ll 1$,

then the plane wave solution of the geometrical optics approximation yields:

$$\langle \tau(\frac{kx}{R},\frac{ky}{R}) \rangle = (1/A_p) \iint_{\infty}^{\infty} G_0(x'+x,y'+y)G_0(x',y')\exp\{-k^2/2 \iint_0^L \mathcal{D}(t,u;v,w)d\zeta_1 d\zeta_2\}\, dx'dy', \tag{4.1}$$

where $t=(x'+x,y'+y,\zeta_1)$, $u=(x'+x,y'+y,\zeta_2)$, $v=(x',y',\zeta_1)$, and $w=(x',y',\zeta_2)$. From the definition, (3.30), $\mathcal{D}$ is clearly related to the correlation function of n_1 as follows:

$$\mathcal{D}(t,u;v,w) = R(v;w) - R(t;w) - R(u;v) + R(t;u).$$

Similarly, the mean square $<|\tau|^2>$ is given by

$$<|\tau(\frac{kx}{R},\frac{ky}{R})|^2> = (1/A_p^2) \iiiint_{-\infty}^{\infty} G_0^{(4)}(x',y';u',v';x,y)$$

$$\times \exp\{-k^2/2 \iint_0^L [\mathcal{D}(t,u;v,w) + \mathcal{D}(t,s;v,q) \tag{4.2}$$

$$+ \mathcal{D}(r,u;p,w) + \mathcal{D}(r,s;p,q)]d\zeta_1 d\zeta_2 \; dx'dy'du'dv',$$

where t, u, v, and w are defined as above, and $p=(u'+x,v'+y,\zeta_1)$, $q=(u'+x,v'+y,\zeta_2)$, $r=(u',v',\zeta_1)$, and $s=(u',v',\zeta_2)$. Hence, by measuring correlation functions for n_1 (or, equivalently, density) on an appropriate mesh, one can evaluate $<\tau>$ and $<|\tau|^2>$ by the quadratures (4.1) and (4.2). If the density fluctuations are homogeneous or locally homogeneous, then, of course, the simpler relations (3.22)-(3.23) or (3.36)-(3.37), respectively, hold. It might be noted that the more realistic spherical wave approximations could be expected to imply less serious degradation than indicated by the plane wave theory discussed here. (See [4, figure 8, p. 60].)

2. Since n_1 is related to the density of the shear layer by

$$n_1 = K\rho ,$$

where $K=.000223 \; m^3/kg$ is the Gladstone-Dale constant, then

$$R(\vec{x}_i;\vec{x}_j) = K^2 \; R_\rho(\vec{x}_i;\vec{x}_j).$$

Hence, all of the results discussed above can be written in terms of density correlations instead of correlations of n_1. The measurement of the statistics of ρ is a nontrivial endeavor, and much activity has been devoted toward that end. Hot wire anemometers [15], laser velocimeters [16], crossed beam techniques ([17], [18], [19]) and sphere probes [20] all offer (either separately or jointly) some hope for carrying out such measurements.

3. As noted earlier (see (3.6) and (3.7)), determination of the mutual intensity, M_2, and fourth order moments, M_4, of the light disturbance in the aperture plane yields the pertinent statistics of τ. Hence, the moments of the disturbance provide an even more fundamental description of optical degradation than does the OTF. Some effort has been expended towards the measurement of such statistics. Typical results are described in [21], [22], [23], [24], and [25]. It might be noted that Kelsall's shearing interferometer [11] is capable of measuring the mutual intensity, M_2, under certain conditions.

4. It is interesting to conjecture that if the influence of the turbulence on the optical performance were known, then the resulting image might be "corrected" and the turbulence influence eliminated. In fact, research has been carried out to just this end ([26], [27], and [28]). Korff [27] makes the argument that $<|\tau|^2>$ may provide more useful information than does $<\tau>$ in that signal/noise is reduced for high spatial frequencies.

5. As remarked earlier, the choice of the OTF, τ, as the indicator of optical quality is somewhat arbitrary. In fact, if one were concerned with propagating, rather than imaging, systems, the Strehl intensity, S, is perhaps more appropriate. It is defined to be the ratio of the maximum intensity (in a particular plane of observation) to the maximum intensity which would have

been obtained had no disturbance been present. (See, for example, Born and Wolf [6, p. 462] and O'Neill [9, p. 106]). In terms of OTF, we have

$$S = \frac{\iint_{-\infty}^{\infty} \tau(f,g)\, df\, dg}{\iint_{-\infty}^{\infty} \tau_0\, (f,g)\, df\, dg} .$$

(The definition of S is thus the analog of system bandwidth for time-varying systems.) Hence, if one can calculate τ and its moments, similar results can be obtained for S. In particular,

$$<S> = \frac{\iint_{-\infty}^{\infty} <\tau(f,g)>\, df\, dg}{\iint_{-\infty}^{\infty} \tau_0(f,g)\, df\, dg}$$

and

$$<|S|^2> = \frac{\iiiint_{-\infty}^{\infty} <\tau(f,g)\tau^*(f',g')>\, df\, dg\, df'\, dg'}{\iiiint_{-\infty}^{\infty} \tau_0(f,g)\tau_0(f',g')\, df\, dg\, df'\, dg'} .$$

Note that $<|S|^2>$ is not directly related to $<|\tau|^2>$.

6. Finally, the evaluation of experimental results and comparison with full scale results requires knowledge of scaling laws. A derivation of such similarity laws is given in [13, p. 100]. In addition to the usual aerodynamic parameters such as Mach number and Reynolds number, it is clear from the analysis of part III that the ratio of model characteristic length to the diameter of the receiving optics should be maintained.

V. APPENDIX

We wish to derive the following formulae:

$$\int_0^L\int_0^L F(y-x)\ dx\ dy = \int_0^L (L-\xi)\ [F(\xi) + F(-\xi)]\ d\xi \tag{A.1}$$

$$\int_0^L\int_0^L (L-x)(L-y)G(y-x)dx\ dy = (1/6)\int_0^L (2L+\xi)(L-\xi)^2[G(\xi)+G(-\xi)]d\xi. \tag{A.2}$$

To this end, we make the change of variables

$$\xi = y-x, \quad \eta = L-(x+y)/2. \tag{A.3}$$

Then, in the ξ-η plane, the double integrals (A.1) and (A.2) are performed over the area indicated in figure A1.

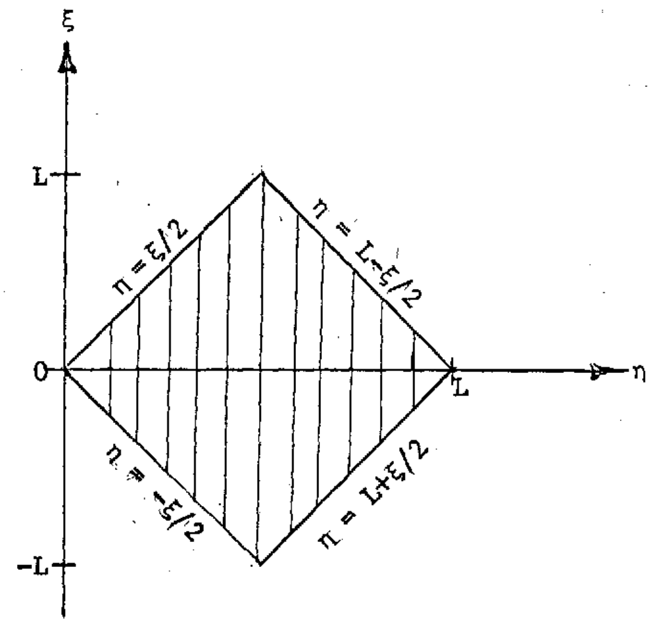

FIGURE A1

From (A.3), we have

$$x = L-\eta - \xi/2, \qquad y = L-\eta + \xi/2 \quad ;$$

Since the Jacobian $J = \partial(x,y)/\partial(\xi,\eta) = 1$, then the double integrals of (A.1) and (A.2) can be written:

$$\int_0^L\int_0^L F(y-x)\,dx\,dy = \int_{-L}^{0} F(\xi)\left[\int_{-\xi/2}^{L+\xi/2} d\eta\right] d\xi + \int_0^L F(\xi)\left[\int_{\xi/2}^{L-\xi/2} d\eta\right] d\xi \qquad (A.4)$$

$$= \int_0^L [F(\xi)+F(-\xi)]\left[\int_{\xi/2}^{L-\xi/2} d\eta\right] d\xi$$

$$= \int_0^L (L-\xi)[F(\xi)+F(-\xi)]\,d\xi$$

and

$$\int_0^L\int_0^L (L-x)(L-y)G(y-x)\,dx\,dy = \int_{-L}^{0} G(\xi)\left[\int_{-\xi/2}^{L+\xi/2}(\eta+\xi/2)(\eta-\xi/2)\,d\eta\right] d\xi \qquad (A.5)$$

$$+\int_0^L G(\xi)\left[\int_{-\xi/2}^{L-\xi/2}(\eta+\xi/2)(\eta-\xi/2)\,d\eta\right] d\xi$$

$$= \int_0^L [G(\xi)+G(-\xi)]\left[\int_{\xi/2}^{L-\xi/2}(\eta+\xi/2)(\eta-\xi/2)\,d\eta\right] d\xi \;.$$

But $\int_{\xi/2}^{L-\xi/2}(\eta+\xi/2)(\eta-\xi/2)\,d\eta = (1/6)(2L+\xi)(L-\xi)^2$, and, from (A.4) and (A.5), formulae (A.1) and (A.2) follow immediately.

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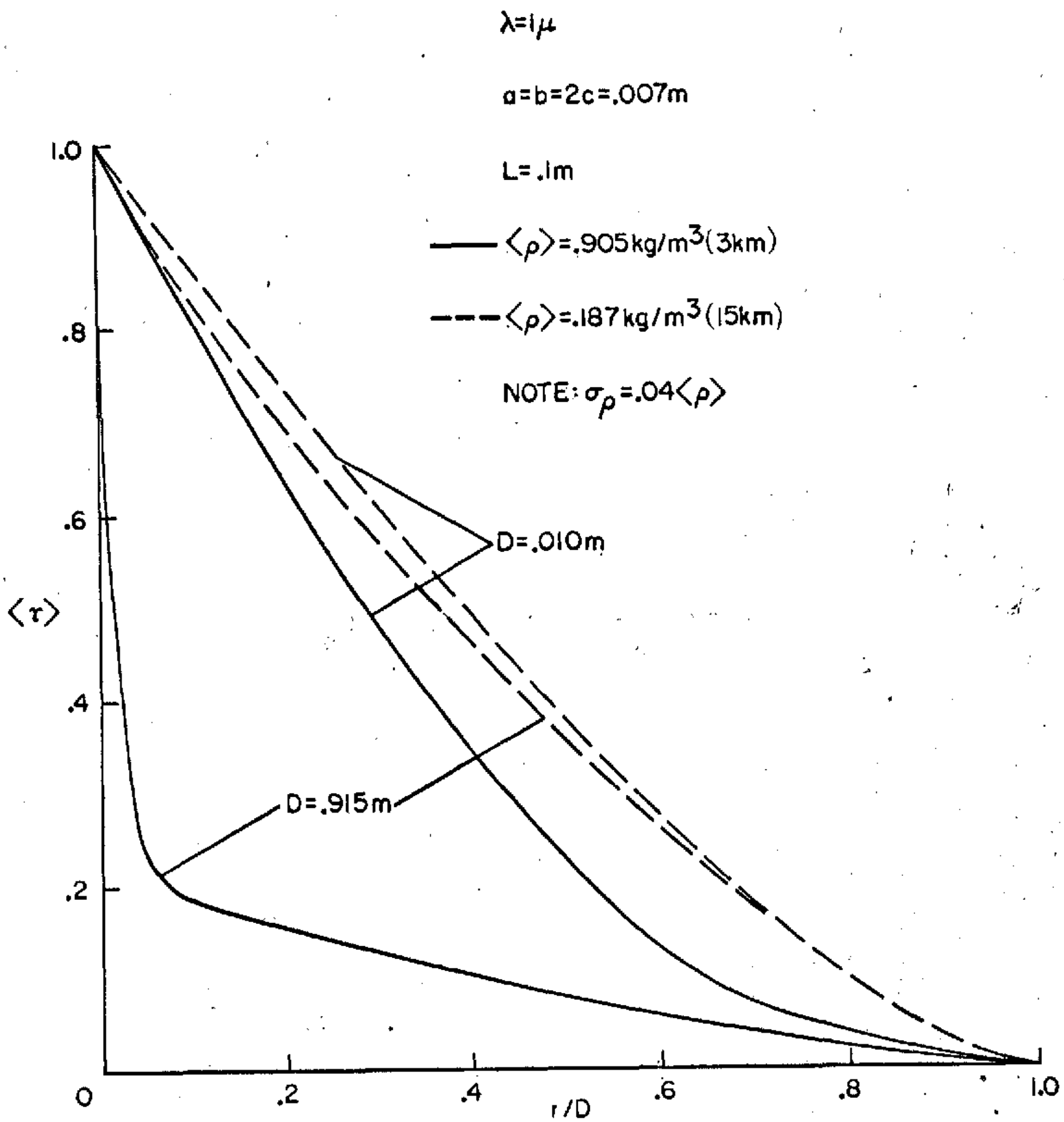

Figure 1.- The effect of telescope diameter and altitude on the average OTF

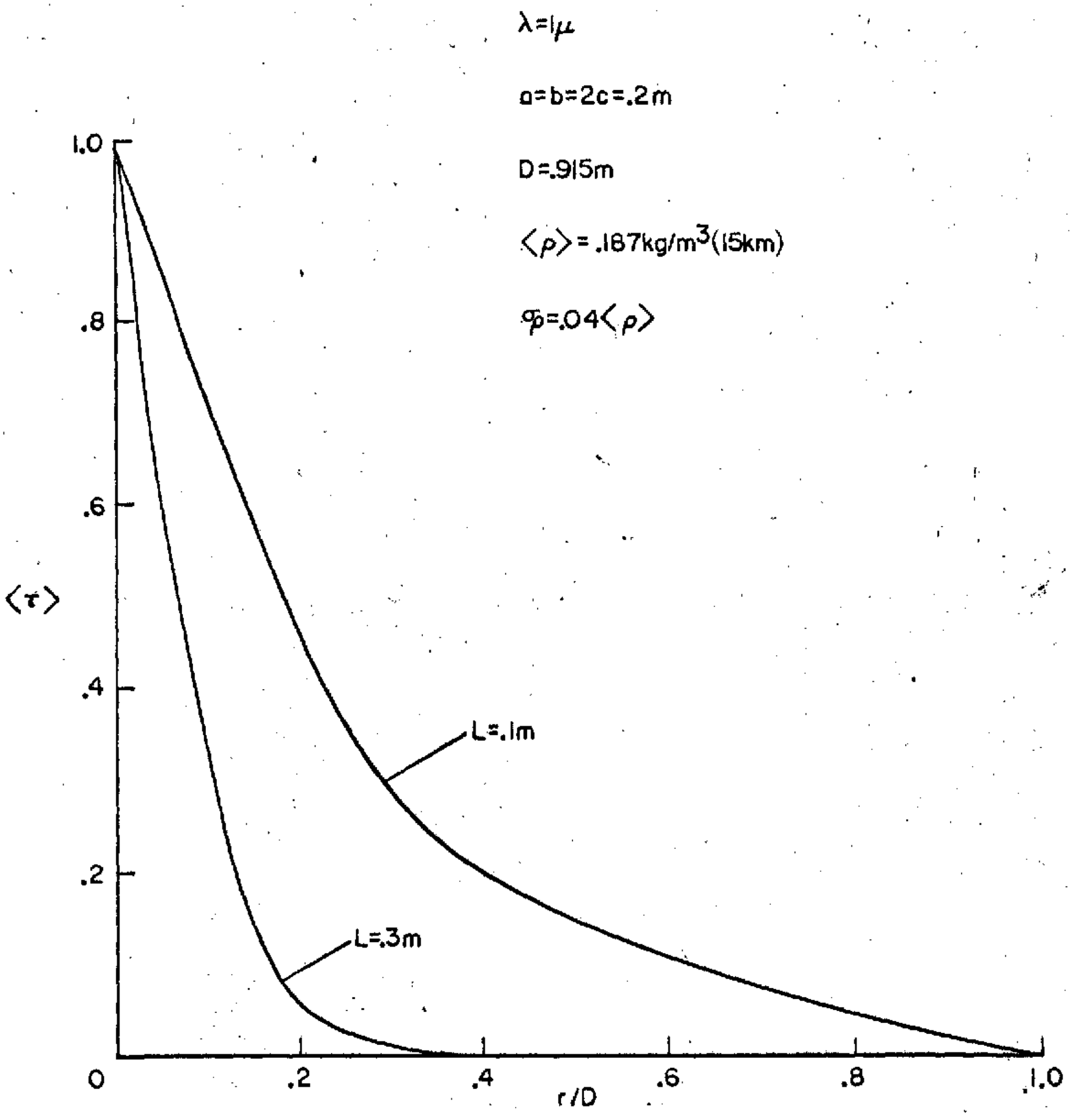

Figure 2.- The effect of shear layer thickness on average OTF

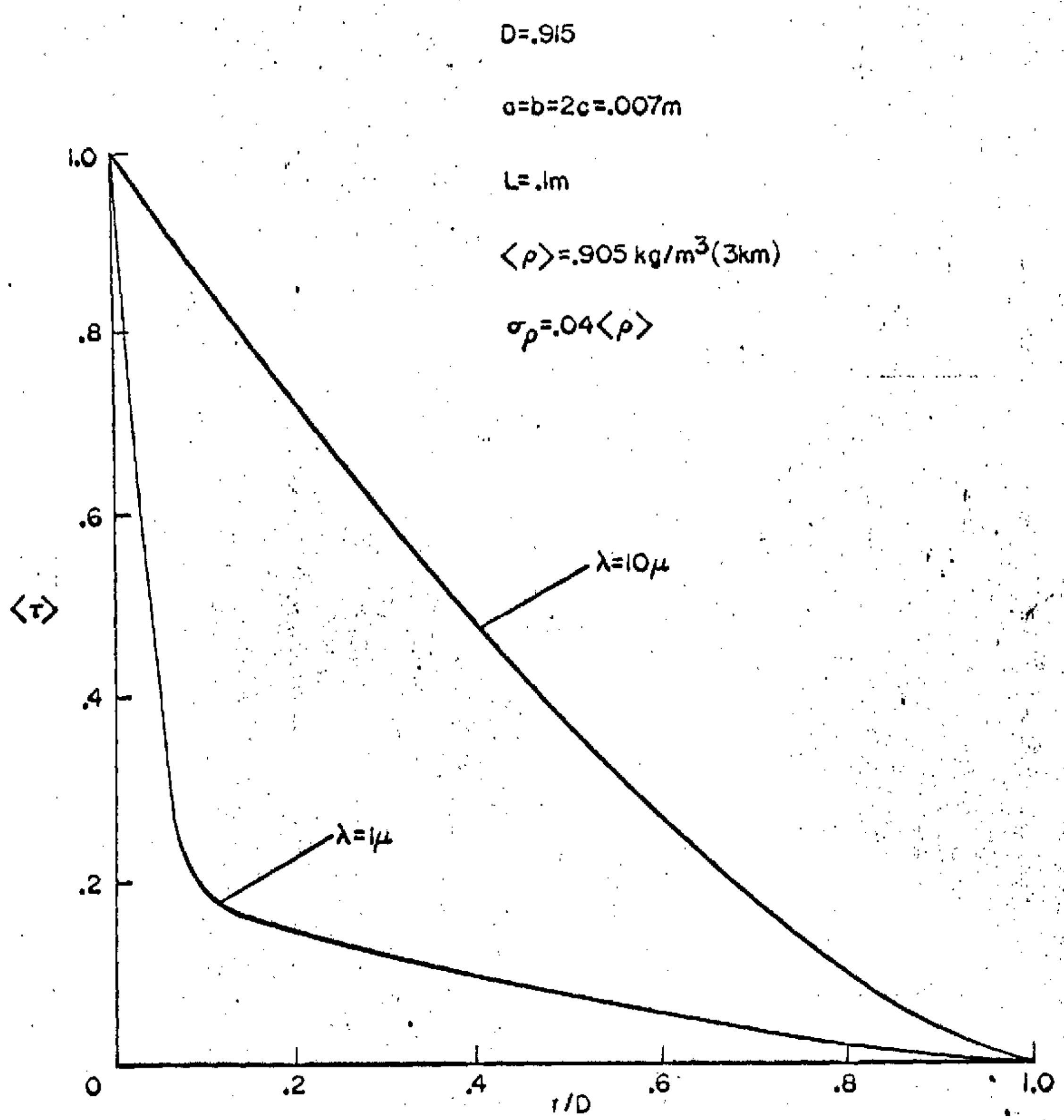

Figure 3.- The effect of wavelength on average OTF

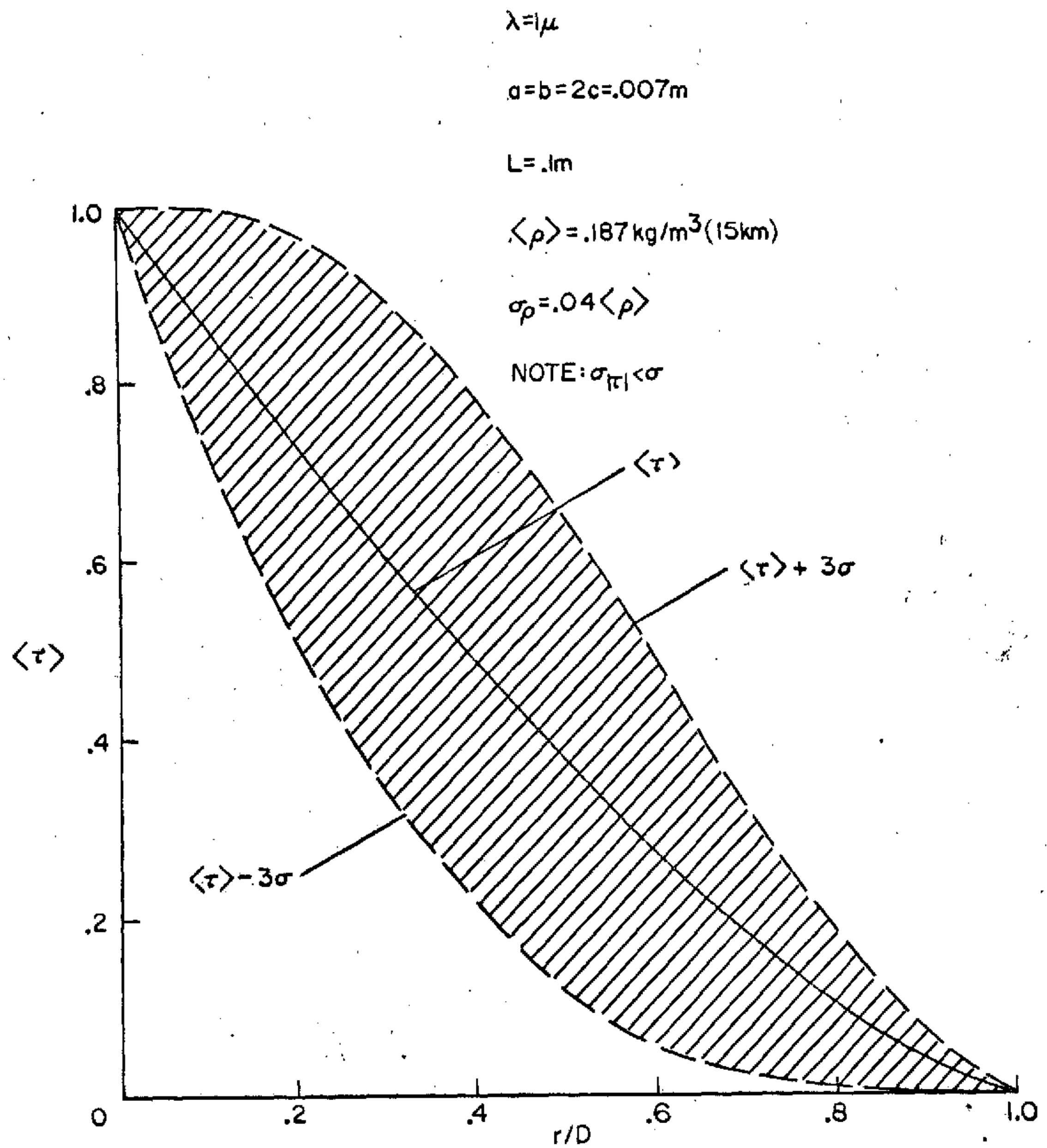

Figure 4.- A typical average OTF with 3-σ band