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SOME SELF-STARTING INTEGRATORS FOR $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$

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ABSTRACT

This report discusses the integration of the vector differential equation $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ from time t_i to t_{i+1}, where only the values of $\underline{x}_i$ are available for the integration. No previous values of $\underline{x}$ or $\dot{\underline{x}}$ are used. Using an orbit integration problem, comparisons are made between Taylor series integrators and various types and orders of Runge-Kutta integrators. A very outstanding fourth order Runge-Kutta type integrator for orbital work is presented. Approximate (there may be no exact) fifth order Runge-Kutta integrators are discussed. Also discussed and compared is a self starting integrator using $\partial \underline{f}/\partial \underline{x}$. A numerical method for controlling the accuracy of integration is given. And, the special equations for accurately integrating accelerometer data are shown.

1. INTRODUCTION

Within recent years, the increasing use of sequential data processors (e.g. Kalman filters) has increased the need for high quality, self-starting integrators. A self-starting integrator is one which takes a current estimate of the state vector, $\underline{x}_i$, and, using the equation $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$, propagates this estimate ahead ΔT seconds to obtain $\underline{x}_{i+1}$. Non self-starting integrators (e.g. Adams-Moulten integrators) need the previous values of $\dot{\underline{x}}_i$, $\dot{\underline{x}}_{i-1}$, $\dot{\underline{x}}_{i-2}$, etc. in order to obtain an estimate of $\underline{x}_{i+1}$.

This report will discuss several self-starting integrators. An empirical evaluation of these integrators will then be made with regard to their efficacy in an orbit determination problem. A numerical method of determining the accuracy of these integrators over each integration step is shown. This determination is quite useful in such areas as calculating interplanetary trajectories, where equal accuracy integration step sizes may vary from a few seconds (close to a planet) to as long as several hours.

A section has been included which shows how to accurately integrate accelerometer data. This data generally comes into the integrator as the integral of sensed acceleration, not sensed acceleration itself.

To my knowledge, no one has ever developed a fifth order Runge-Kutta integrator for the vector differential equation $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$. We will show that 6 evaluations of $\underline{f}(\underline{x}, t)$ are needed, not 5 as one might expect. Further, even with 6 evaluations there may be no exact set of integrator constants. Two approximate sets of fifth order integrator constants are shown and evaluated.

2. INTEGRATION BY TAYLOR SERIES

One of the oldest and best known methods of numerically integrating is the Taylor series expansion. It is the standard of comparison against which all other methods are evaluated. Despite the fact that the method is old, doesn't mean that it is obsolete. Indeed it is frequently and efficiently used in digital computer programs. Integration of the vector differential equation, $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$, is accomplished by

$$\underline{x}_{i+1} = \underline{x}_i + \dot{\underline{x}}_i \, \Delta T + \ddot{\underline{x}}_i \frac{\Delta T^2}{2!} + \dddot{\underline{x}}_i \frac{\Delta T^3}{3!} + \ddddot{\underline{x}}_i \frac{\Delta T^4}{4!} + \cdots \qquad (1)$$

where

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t)$$

$$\ddot{\underline{x}} = \frac{\partial \dot{\underline{x}}}{\partial \underline{x}} \dot{\underline{x}} + \frac{\partial \dot{\underline{x}}}{\partial t} = \frac{\partial \underline{f}}{\partial \underline{x}} \underline{f} + \frac{\partial \underline{f}}{\partial t}$$

$$\dddot{\underline{x}} = \frac{\partial \ddot{\underline{x}}}{\partial \underline{x}} \dot{\underline{x}} + \frac{\partial \ddot{\underline{x}}}{\partial t}$$

$$\ddddot{\underline{x}} = \frac{\partial \dddot{\underline{x}}}{\partial \underline{x}} \dot{\underline{x}} + \frac{\partial \dddot{\underline{x}}}{\partial t} \qquad \text{etc.}$$

The partial derivative of a vector, $\underline{g}$, with respect to a vector, $\underline{z}$, is defined to be the matrix

$$\frac{\partial \underline{g}}{\partial \underline{z}} = \begin{bmatrix} \frac{\partial g_1}{\partial z_1} & \frac{\partial g_1}{\partial z_2} & \frac{\partial g_1}{\partial z_3} & \cdots & \frac{\partial g_1}{\partial z_n} \\ \frac{\partial g_2}{\partial z_1} & \frac{\partial g_2}{\partial z_2} & \frac{\partial g_2}{\partial z_3} & \cdots & \frac{\partial g_2}{\partial z_n} \\ \frac{\partial g_3}{\partial z_1} & \frac{\partial g_3}{\partial z_2} & \frac{\partial g_3}{\partial z_3} & \cdots & \frac{\partial g_3}{\partial z_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial g_m}{\partial z_1} & \frac{\partial g_m}{\partial z_2} & \frac{\partial g_m}{\partial z_3} & \cdots & \frac{\partial g_m}{\partial z_n} \end{bmatrix}$$

where the subscripts indicate the particular element of the vector in question.

3. SECOND ORDER RUNGE-KUTTA TYPE INTEGRATOR

A second order integrator is one that gives a solution which agrees with a Taylor series expansion up through the ΔT^2 terms. Such an integrator is

$$\underline{k}_1 = \Delta T \underline{f}(\underline{x}_i, t_i + \delta_1 \Delta T) \tag{2}$$

$$\underline{k}_2 = \Delta T \underline{f}(\underline{x}_i + a_1 \underline{k}_1, t_i + \delta_2 \Delta T) \tag{3}$$

$$\underline{x}_{i+1} = \underline{x}_i + b_1 \underline{k}_1 + b_2 \underline{k}_2 + O(\Delta T^3) \tag{4}$$

The integration constants in the above equation are not arbitrary. In order to achieve an error of order ΔT^3, the following constraint equations must be satisfied

$$b_2 a_1 = 1/2 \tag{5}$$

$$b_1 + b_2 = 1 \tag{6}$$

$$b_1 \delta_1 + b_2 \delta_2 = 1/2 \tag{7}$$

Note that there are only three equations in five unknowns. The constraint equations could be rewritten as

$b_1 = \dfrac{1 - 2\delta_2}{2(\delta_1 - \delta_2)}$	$b_2 = \dfrac{-1 + 2\delta_1}{2(\delta_1 - \delta_2)}$	$a_1 = \dfrac{\delta_1 - \delta_2}{-1 + 2\delta_1}$

TABLE 1: SECOND ORDER CONSTRAINT EQUATIONS

δ_1 and δ_2 can be experimentally chosen to reduce the $O(\Delta T^3)$ truncation error. A special case arises when $\delta_1 = \delta_2$. The constraint equations for this case are shown in Table 2. Here, b_2 is chosen to reduce the truncation error.

$\delta_1 = \frac{1}{2}$	$\delta_2 = \frac{1}{2}$	$a_1 = \frac{1}{2b_2}$	$b_1 = 1 - b_2$

TABLE 2: 2ND ORDER CONSTRAINTS WHEN $\delta_1 = \delta_2$

It is interesting to note that fourth order integration of $\dot{\underline{x}} = \underline{f}(t)$ can be achieved by adding the two additional constraint equations

$$b_1\delta_1^2 + b_2\delta_2^2 = 1/3 \qquad (8)$$

$$b_1\delta_1^3 + b_2\delta_2^3 = 1/4 \qquad (9)$$

There are two solutions of the five constraint equations (equations (5) through (9)). These solutions are shown in Tables 3 and 4.

$\delta_1 = (3 - \sqrt{3})/6 = .2113\ 2486\ 5405\ 1888$		
$\delta_2 = (3 + \sqrt{3})/6 = .7886\ 7513\ 4594\ 8112$		
$a_1 = 1$	$b_1 = 1/2$	$b_2 = 1/2$

TABLE 3: 2/4 ORDER INTEGRATOR CONSTANTS, SET #1

$\delta_1 = (3 + \sqrt{3})/6 = .7886\ 7513\ 4594\ 8112$		
$\delta_2 = (3 - \sqrt{3})/6 = .2113\ 2486\ 5405\ 1888$		
$a_1 = 1$	$b_1 = 1/2$	$b_2 = 1/2$

TABLE 4: 2/4 ORDER INTEGRATOR CONSTANTS, SET #2

When $\dot{\underline{x}} = \underline{f}(t)$, the constants in Tables 3 and 4 will cause the solution to be

$$\hat{\underline{x}}_{i+1} = \underline{x}_i + \dot{\underline{x}}_i \Delta T + \ddot{\underline{x}}_i \frac{\Delta T^2}{2!} + \dddot{\underline{x}}_i \frac{\Delta T^3}{3!} + \ddddot{\underline{x}}_i \frac{\Delta T^4}{4!} + \frac{35}{36} \frac{d^5 \underline{x}_i}{dt^5} \frac{\Delta T^5}{5!} + \cdots$$

$\hat{\underline{x}}_{i+1}$ means approximate value of $\underline{x}_{i+1}$. A similar equation also applies to any element of $\underline{f}(\underline{x}, t)$ which is a function of time alone. Note that the general solution of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ is still only second order, hence the terminology 2/4 order.

If, instead of using equation (9) as an additional constraint equation, we use the equation

$$\delta_1 = 0 \tag{10}$$

then we obtain a third order solution when $\dot{\underline{x}} = \underline{f}(t)$. The 2/3 order integrator constants are shown in Table 5.

$\delta_1 = 0$	$\delta_2 = 2/3$	$a_1 = 2/3$	$b_1 = 1/4$	$b_2 = 3/4$

TABLE 5: 2/3 ORDER INTEGRATOR CONSTANTS

4. THIRD ORDER RUNGE-KUTTA TYPE INTEGRATOR

Third order integration of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ can be achieved by

$$\underline{k}_1 = \Delta T \, \underline{f}(\underline{x}_i, t_i + \delta_1 \Delta T) \quad (11)$$

$$\underline{k}_2 = \Delta T \, \underline{f}(\underline{x}_i + a_1 \underline{k}_1, t_i + \delta_2 \Delta T) \quad (12)$$

$$\underline{k}_3 = \Delta T \, \underline{f}(\underline{x}_i + b_1 \underline{k}_1 + b_2 \underline{k}_2, t_i + \delta_3 \Delta T) \quad (13)$$

$$\underline{x}_{i+1} = \underline{x}_i + c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3 + 0(\Delta T^4) \quad (14)$$

The constraint equations for the integration constants are

$$\delta_1 = 0 \quad (15)$$

$$a_1 = \delta_2 \quad (16)$$

$$b_1 + b_2 = \delta_3 \quad (17)$$

$$b_2 c_3 \delta_2 = 1/6 \quad (18)$$

$$c_1 + c_2 + c_3 = 1 \quad (19)$$

$$c_2 \delta_2 + c_3 \delta_3 = 1/2 \quad (20)$$

$$c_2 \delta_2^2 + c_3 \delta_3^2 = 1/3 \quad (21)$$

Here again there are two more unknowns than there are equations. Thus all the integrator constants can be expressed as functions of δ_2 and δ_3, as shown in Table 6.* δ_2 and δ_3 can be experimentally adjusted to reduce the $O(\Delta T^4)$ truncation error in equation (14).

$\delta_1 = 0$	$a_1 = \delta_2$	$b_1 = \delta_3 - b_2$	$b_2 = \frac{\delta_3}{\delta_2}\frac{\delta_3 - \delta_2}{2 - 3\delta_2}$
$c_1 = 1 - c_2 - c_3$	$c_2 = \frac{1}{6\delta_2}\frac{3\delta_3 - 2}{\delta_3 - \delta_2}$		$c_3 = \frac{1}{6\delta_3}\frac{2 - 3\delta_2}{\delta_3 - \delta_2}$

TABLE 6: THIRD ORDER CONSTRAINT EQUATIONS

Two special cases arise when $\delta_2 = \delta_3$ and when $\delta_3 = 0$. The constraint equations for these cases are shown in Tables 7 and 8. In both cases c_3 is emperically determined to reduce truncation error.

$\delta_1 = 0$	$\delta_2 = \delta_3 = \frac{2}{3}$	$a_1 = \frac{2}{3}$	$b_1 = \frac{2}{3} - b_2$
$b_2 = \frac{1}{4c_3}$	$c_1 = \frac{1}{4}$	$c_2 = \frac{3}{4} - c_3$	

TABLE 7: THIRD ORDER CONSTRAINTS WHEN $\delta_2 = \delta_3$

* Note from equation (18) that b_2, c_3, $\delta_2 = 0$ are not allowable solutions.

$\delta_1 = \delta_3 = 0$	$\delta_2 = \frac{2}{3}$	$a_1 = \frac{2}{3}$	$b_1 = -\frac{1}{4c_3}$
$b_2 = \frac{1}{4c_3}$	$c_1 = \frac{1}{4} - c_3$		$c_2 = \frac{3}{4}$

TABLE 8: THIRD ORDER CONSTRAINTS WHEN $\delta_3 = 0$

Fifth order integration of $\dot{\underline{x}} = \underline{f}(t)$ can be achieved by adding the two additional constraint equations

$$c_2\delta_2^3 + c_3\delta_3^3 = 1/4 \tag{22}$$

$$c_2\delta_2^4 + c_3\delta_3^4 = 1/5 \tag{23}$$

There are two solutions of the nine constraint equations (equations (15) through (23)). These solutions are shown in Tables 9 and 10.

$\delta_1 = 0$
$\delta_2 = (6 - \sqrt{6})/10 = .3550\ 5102\ 5721\ 6822$
$\delta_3 = (6 + \sqrt{6})/10 = .8449\ 4897\ 4278\ 3178$
$a_1 = (6 - \sqrt{6})/10 = .3550\ 5102\ 5721\ 6822$
$b_1 = -(54 + 19\sqrt{6})/250 = -.4021\ 6122\ 0451\ 5215$
$b_2 = 2(51 + 11\sqrt{6})/125 = 1.247\ 1101\ 9472\ 98393$
$c_1 = 1/9 = .1111\ 1111\ 1111\ 1111$
$c_2 = (16 + \sqrt{6})/36 = .5124\ 8582\ 6188\ 4216$
$c_3 = (16 - \sqrt{6})/36 = .3764\ 0306\ 2700\ 4673$

TABLE 9: 3/5 ORDER INTEGRATOR CONSTANTS, SET NO. 1

$\delta_1 = 0$

$\delta_2 = (6 + \sqrt{6})/10 = .8449\ 4897\ 4278\ 3178$

$\delta_3 = (6 - \sqrt{6})/10 = .3550\ 5102\ 5721\ 6822$

$a_1 = (6 + \sqrt{6})/10 = .8449\ 4897\ 4278\ 3178$

$b_1 = -(54 - 19\sqrt{6})/250 = -.02983\ 8779\ 5484\ 7846$

$b_2 = 2(51 - 11\sqrt{6})/125 = .3848\ 8980\ 5270\ 1607$

$c_1 = 1/9 = .1111\ 1111\ 1111\ 1111$

$c_2 = (16 - \sqrt{6})/36 = .3764\ 0306\ 2700\ 4673$

$c_3 = (16 + \sqrt{6})/36 = .5124\ 8582\ 6188\ 4216$

TABLE 10: 3/5 ORDER INTEGRATOR CONSTANTS, SET NO. 2

When $\dot{\underline{x}} = \underline{f}(t)$, the constants in the two tables above give a solution of

$$\hat{\underline{x}}_{i+1} = \underline{x}_i + \dot{\underline{x}}_i \Delta T + \ddot{\underline{x}}_i \frac{\Delta T^2}{2!} + \cdots + \frac{d^5 \underline{x}_i}{dt^5} \frac{\Delta T^5}{5!} + .99 \frac{d^6 \underline{x}_i}{dt^6} \frac{\Delta T^6}{6!} + \cdots$$

5. FOURTH ORDER RUNGE-KUTTA TYPE INTEGRATOR

Fourth order integration of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ can be achieved by

$$\underline{k}_1 = \Delta T \ \underline{f}(\underline{x}_i, \ t_i + \delta_1 \Delta T) \tag{24}$$

$$\underline{k}_2 = \Delta T \ \underline{f}(\underline{x}_i + a_1 \underline{k}_1, \ t_i + \delta_2 \Delta T) \tag{25}$$

$$\underline{k}_3 = \Delta T \ \underline{f}(\underline{x}_i + b_1 \underline{k}_1 + b_2 \underline{k}_2, \ t_i + \delta_3 \Delta T) \tag{26}$$

$$\underline{k}_4 = \Delta T \ \underline{f}(\underline{x}_i + c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3, \ t_i + \delta_4 \Delta T) \tag{27}$$

$$\underline{x}_{i+1} = \underline{x}_i + d_1 \underline{k}_1 + d_2 \underline{k}_2 + d_3 \underline{k}_3 + d_4 \underline{k}_4 + 0(\Delta T^5) \tag{28}$$

The constraint equations for the integration constants are

$$\delta_1 = 0 \tag{29}$$

$$a_1 = \delta_2 \tag{30}$$

$$b_1 + b_2 = \delta_3 \tag{31}$$

$$c_1 + c_2 + c_3 = \delta_4 \tag{32}$$

$$b_2 d_3 \delta_2 + (c_2 \delta_2 + c_3 \delta_3) d_4 = 1/6 \tag{33}$$

$$b_2 d_3 \delta_2 \delta_3 + (c_2 \delta_2 + c_3 \delta_3) d_4 \delta_4 = 1/8 \tag{34}$$

$$b_2 d_3 \delta_2^2 + (c_2 \delta_2^2 + c_3 \delta_3^2) d_4 = 1/12 \tag{35}$$

$$b_2 c_3 d_4 \delta_2 = 1/24 \tag{36}$$

$$d_1 + d_2 + d_3 + d_4 = 1 \tag{37}$$

$$d_2\delta_2 + d_3\delta_3 + d_4\delta_4 = 1/2 \tag{38}$$

$$d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2 = 1/3 \tag{39}$$

$$d_2\delta_2^3 + d_3\delta_3^3 + d_4\delta_4^3 = 1/4 \tag{40}$$

It is interesting to note that equations (34) and (35) cannot be derived from consideration of the scalar equation $\dot{x} = f(x)$, as can most of the other constraint equations. Consideration of only the scalar equation leads to a constraint equation which is a linear combination of equations (34) and (35).

It can be shown that the only value of δ_4 which satisfies the constraint equations is $\delta_4 = 1$. As before, the constraint equations can be rewritten as shown in Table 11. δ_2 and δ_3 can be emperically determined to reduce the $O(\Delta T^5)$ truncation error in equation (28).

$$\delta_1 = 0 \qquad \delta_4 = 1 \qquad a_1 = \delta_2 \qquad b_1 = \delta_3 - b_2 \qquad b_2 = \frac{\delta_3}{2\delta_2}\,\frac{\delta_3 - \delta_2}{1 - 2\delta_2}$$

$$c_1 = 1 - c_2 - c_3 \qquad c_2 = \frac{1 - \delta_2}{2\delta_2(\delta_2 - \delta_3)}\,\frac{4\delta_3^2 - 5\delta_3 - \delta_2 + 2}{-4\delta_2 - 4\delta_3 + 6\delta_2\delta_3 + 3}$$

$$c_3 = \frac{(1-\delta_2)\ (1-\delta_3)\ (2\delta_2 - 1)}{\delta_3(\delta_2 - \delta_3)(-4\delta_2 - 4\delta_3 + 6\delta_2\delta_3 + 3)} \qquad d_1 = 1 - d_2 - d_3 - d_4$$

$$d_2 = \frac{2\delta_3 - 1}{12\delta_2(\delta_2 - \delta_3)(\delta_2 - 1)} \qquad d_3 = \frac{2\delta_2 - 1}{12\delta_3(\delta_3 - \delta_2)(\delta_3 - 1)}$$

$$d_4 = \frac{-4\delta_2 - 4\delta_3 + 6\delta_2\delta_3 + 3}{12(1 - \delta_2)(1 - \delta_3)}$$

TABLE 11: FOURTH ORDER CONSTRAINT EQUATIONS

There is an interesting array of nonacceptable solutions of the constraint equations. Some of these are

$$\delta_2 \neq 0$$

$$\delta_3 \neq 1$$

$$-4\delta_2 - 4\delta_3 + 6\delta_2\delta_3 + 3 \neq 0 \text{ unless } \delta_2 = 1$$

$$2\delta_2 - 1 \neq 0 \text{ unless } \delta_2 = \delta_3 \text{ or unless } \delta_3 = 0$$

$$b_2 \neq 0$$

$$c_3 \neq 0$$

$$d_3 \neq 0$$

$$d_4 \neq 0$$

These unacceptable solutions give rise to the three special forms of the constraint equations shown in Tables 12, 13, and 14. If d_3 is set to 1/3 in Table 12, it will be seen that the standard, fourth-order, Runge-Kutta integration constants will be obtained. If d_3 is set equal to $(1 + 1/\sqrt{2})/3$ the Runge-Kutta-Gill integration constants will be obtained.

$\delta_1 = 0$	$\delta_2 = 1/2$	$\delta_3 = 1/2$	$\delta_4 = 1$	$a_1 = 1/2$
$b_1 = \frac{1}{2} - b_2$	$b_2 = \frac{1}{6d_3}$	$c_1 = 0$	$c_2 = 1 - c_3$	
$c_3 = 3d_3$	$d_1 = \frac{1}{6}$	$d_2 = \frac{2}{3} - d_3$	$d_4 = \frac{1}{6}$	

TABLE 12: 4th ORDER CONSTRAINT EQUATIONS FOR $\delta_2 = \delta_3$

$\delta_1 = 0$	$\delta_2 = 1$	$\delta_3 = \frac{1}{2}$	$\delta_4 = 1$	$a_1 = 1$
$b_1 = 3/8$	$b_2 = 1/8$	$c_1 = 1-c_2-c_3$	$c_2 = -\frac{1}{12d_4}$	
$c_3 = \frac{1}{3d_4}$	$d_1 = \frac{1}{6}$	$d_2 = \frac{1}{6} - d_4$		$d_3 = \frac{2}{3}$

TABLE 13: 4th ORDER CONSTRAINT EQUATIONS FOR $\delta_2 = 1$

$\delta_1 = 0$	$\delta_2 = \frac{1}{2}$	$\delta_3 = 0$	$\delta_4 = 1$	$a_1 = 1/2$
$b_1 = -\frac{1}{12d_3}$	$b_2 = \frac{1}{12d_3}$	$c_1 = 1-c_2-c_3$		$c_2 = 3/2$
$c_3 = 6d_3$	$d_1 = \frac{1}{6} - d_3$	$d_2 = \frac{2}{3}$		$d_4 = \frac{1}{6}$

TABLE 14: 4th ORDER CONSTRAINT EQUATIONS FOR $\delta_3 = 0$

Sixth order integration of $\dot{\underline{x}} = \underline{f}(t)$ can be accomplished by adding the two additional constraint equations

$$d_2\delta_2^4 + d_3\delta_3^4 + d_4 = 1/5 \tag{41}$$

$$d_2\delta_2^5 + d_3\delta_3^5 + d_4 = 1/6 \tag{42}$$

There are two solutions of the 14 constraint equations (equations (29) through (42)). These solutions are shown in Tables 15 and 16.

$$\delta_1 = 0$$

$$\delta_2 = (5 - \sqrt{5})/10 = .2763 \; 9320 \; 2250 \; 0210$$

$$\delta_3 = (5 + \sqrt{5})/10 = .7236 \; 0679 \; 7749 \; 9790$$

$$\delta_4 = 1$$

$$a_1 = (5 - \sqrt{5})/10 = .2763 \; 9320 \; 2250 \; 0210$$

$$b_1 = -(5 + 3\sqrt{5})/20 = -.5854 \; 1019 \; 6624 \; 9685$$

$$b_2 = (3 + \sqrt{5})/4 = 1.309 \; 0169 \; 9437 \; 49475$$

$$c_1 = -(1 - 5\sqrt{5})/4 = 2.545 \; 0849 \; 7187 \; 4737$$

$$c_2 = -(5 + 3\sqrt{5})/4 = -2.927 \; 0509 \; 8312 \; 4842$$

$$c_3 = (5 - \sqrt{5})/2 = 1.381 \; 9660 \; 1125 \; 0105$$

$$d_1 = 1/12 = .08333 \; 3333 \; 3333 \; 3333$$

$$d_2 = 5/12 = .4166 \; 6666 \; 6666 \; 66667$$

$$d_3 = 5/12 = .4166 \; 6666 \; 6666 \; 66667$$

$$d_4 = 1/12 = .08333 \; 3333 \; 3333 \; 3333$$

TABLE 15: 4/6 ORDER INTEGRATOR CONSTANTS, SET NO. 1

$\delta_1 = 0$

$\delta_2 = (5 + \sqrt{5})/10 = .7236\ 0679\ 7749\ 9790$

$\delta_3 = (5 - \sqrt{5})/10 = .2763\ 9320\ 2250\ 0210$

$\delta_4 = 1$

$a_1 = (5 + \sqrt{5})/10 = .7236\ 0679\ 7749\ 9790$

$b_1 = -(5 - 3\sqrt{5})/20 = .08541\ 0196\ 6249\ 6845$

$b_2 = (3 - \sqrt{5})/4 = .1909\ 8300\ 5625\ 05256$

$c_1 = -(1 + 5\sqrt{5})/4 = -3.045\ 0849\ 7187\ 47373$

$c_2 = -(5 - 3\sqrt{5})/4 = .4270\ 5098\ 3124\ 8423$

$c_3 = (5 + \sqrt{5})/2 = 3.618\ 0339\ 8874\ 98950$

$d_1 = 1/12 = .08333\ 3333\ 3333\ 3333$

$d_2 = 5/12 = .4166\ 6666\ 6666\ 66667$

$d_3 = 5/12 = .4166\ 6666\ 6666\ 66667$

$d_4 = 1/12 = .08333\ 3333\ 3333\ 3333$

TABLE 16: 4/6 ORDER INTEGRATOR CONSTANTS, SET NO. 2

When $\dot{\underline{x}} = \underline{f}(t)$, the constants in the two above tables give a solution of

$$\hat{\underline{x}}_{i+1} = \underline{x}_i + \dot{\underline{x}}_i \Delta T + \ddot{\underline{x}}_i \frac{\Delta T^2}{2!} + \cdots + \frac{d^6 \underline{x}_i}{dt^6} \frac{\Delta T^6}{6!} + \frac{301}{300} \frac{d^7 \underline{x}_i}{dt^7} \frac{\Delta T^7}{7!} + \cdots$$

A 4/5 order integrator can be obtained from Table 11 if we set

$$\delta_3 = \frac{1}{5} (3 - 5\delta_2)/(1 - 2\delta_2)$$

δ_2 is then adjusted to give optimum performance. There are, however, several nonallowable values of δ_2. They are

$$\delta_2 \neq 0,\ .4,\ .5,\ .6,\ 1,\ \frac{1}{10} (6 \pm \sqrt{6})$$

It is sometimes of value to have an estimate of $\underline{x}$ at $t_i + \delta\Delta T$. From the equations shown in Appendix A, it is easily seen that

$$\underline{x}_{i+\delta} = \underline{x}_i + \beta_1 \underline{k}_1 + \beta_2 \underline{k}_2 + \beta_3 \underline{k}_3 + \beta_4 \underline{k}_4 + O(\Delta T^4) \tag{43}$$

where

$$\beta_3 = \frac{\delta^2}{6} \frac{(c_2\delta_2 + c_3\delta_3)(2\delta - 3\delta_2) - (1 - \delta_2)\delta}{\delta_3(\delta_3 - \delta_2)(c_2\delta_2 + c_3\delta_3) - b_2\delta_2(1 - \delta_2)} \tag{44}$$

$$\beta_4 = \frac{\delta^2}{6} \frac{\delta_3(\delta_3 - \delta_2)\ \delta - b_2\delta_2\ (2\delta - 3\delta_2)}{\delta_3(\delta_3 - \delta_2)(c_2\delta_2 + c_3\delta_3) - b_2\delta_2(1 - \delta_2)} \tag{45}$$

$$\beta_2 = \frac{1}{\delta_2} \left(\frac{1}{2} \delta^2 - \beta_3\delta_3 - \beta_4\right) \tag{46}$$

$$\beta_1 = \delta - \beta_2 - \beta_3 - \beta_4 \tag{47}$$

The $\underline{k}$'s are the same as those used to obtain $\underline{x}_{i+1}$. Thus no new derivative evaluations are necessary.

6. FIFTH ORDER RUNGE-KUTTA TYPE INTEGRATOR WITH FIVE DERIVATIVE EVALUATIONS

Fifth order integration of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ can be attempted by

$$\underline{k}_1 = \Delta T \underline{f}(\underline{x}_i, t_i + \delta_1 \Delta T) \quad (48)$$

$$\underline{k}_2 = \Delta T \underline{f}(\underline{x}_i + a_1 \underline{k}_1, t_i + \delta_2 \Delta T) \quad (49)$$

$$\underline{k}_3 = \Delta T \underline{f}(\underline{x}_i + b_1 \underline{k}_1 + b_2 \underline{k}_2, t_i + \delta_3 \Delta T) \quad (50)$$

$$\underline{k}_4 = \Delta T \underline{f}(\underline{x}_i + c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3, t_i + \delta_4 \Delta T) \quad (51)$$

$$\underline{k}_5 = \Delta T \underline{f}(\underline{x}_i + d_1 \underline{k}_1 + d_2 \underline{k}_2 + d_3 \underline{k}_3 + d_4 \underline{k}_4, t_i + \delta_5 \Delta T) \quad (52)$$

$$\underline{x}_{i+1} = \underline{x}_i + e_1 \underline{k}_1 + e_2 \underline{k}_2 + e_3 \underline{k}_3 + e_4 \underline{k}_4 + e_5 \underline{k}_5 + O(\Delta T^?) \quad (53)$$

The constraint equations for the integration constants are

$$\delta_1 = 0 \quad (54)$$

$$a_1 = \delta_2 \quad (55)$$

$$b_1 + b_2 = \delta_3 \quad (56)$$

$$c_1 + c_2 + c_3 = \delta_4 \quad (57)$$

$$d_1 + d_2 + d_3 + d_4 = \delta_5 \quad (58)$$

$$e_1 + e_2 + e_3 + e_4 + e_5 = 1 \quad (59)$$

$$e_2 \delta_2 + e_3 \delta_3 + e_4 \delta_4 + e_5 \delta_5 = 1/2 \quad (60)$$

$$e_2\delta_2^2 + e_3\delta_3^2 + e_4\delta_4^2 + e_5\delta_5^2 = 1/3 \tag{61}$$

$$e_2\delta_2^3 + e_3\delta_3^3 + e_4\delta_4^3 + e_5\delta_5^3 = 1/4 \tag{62}$$

$$e_2\delta_2^4 + e_3\delta_3^4 + e_4\delta_4^4 + e_5\delta_5^4 = 1/5 \tag{63}$$

$$b_2\delta_2 e_3 + (c_2\delta_2 + c_3\delta_3)e_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5 = 1/6 \tag{64}$$

$$b_2\delta_2 e_3\delta_3 + (c_2\delta_2 + c_3\delta_3)e_4\delta_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5\delta_5 = 1/8 \tag{65}$$

$$b_2\delta_2 e_3\delta_3^2 + (c_2\delta_2 + c_3\delta_3)e_4\delta_4^2 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5\delta_5^2 = 1/10 \tag{66}$$

$$(b_2\delta_2)^2 e_3 + (c_2\delta_2 + c_3\delta_3)^2 e_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)^2 e_5 = 1/20 \tag{67}$$

$$b_2\delta_2^2 e_3 + (c_2\delta_2^2 + c_3\delta_3^2)e_4 + (d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)e_5 = 1/12 \tag{68}$$

$$b_2\delta_2^2 e_3\delta_3 + (c_2\delta_2^2 + c_3\delta_3^2)e_4\delta_4 + (d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)e_5\delta_5 = 1/15 \tag{69}$$

$$b_2\delta_2^3 e_3 + (c_2\delta_2^3 + c_3\delta_3^3)e_4 + (d_2\delta_2^3 + d_3\delta_3^3 + d_4\delta_4^3)e_5 = 1/20 \tag{70}$$

$$b_2\delta_2 c_3 e_4\delta_4 + [b_2\delta_2 d_3 + (c_2\delta_2 + c_3\delta_3)d_4]e_5\delta_5 = 1/30 \tag{71}$$

$$b_2\delta_2 c_3 e_4 + [b_2\delta_2 d_3 + (c_2\delta_2 + c_3\delta_3)d_4]e_5 = 1/24 \tag{72}$$

$$b_2\delta_2 c_3 e_4\delta_3 + [b_2\delta_2 d_3\delta_3 + (c_2\delta_2 + c_3\delta_3)d_4\delta_4]e_5 = 1/40 \tag{73}$$

$$b_2\delta_2^2 c_3 e_4 + [b_2\delta_2^2 d_3 + (c_2\delta_2^2 + c_3\delta_3^2)d_4]e_5 = 1/60 \tag{74}$$

$$b_2\delta_2 c_3 d_4 e_5 = 1/120 \tag{75}$$

Since the above equations are so difficult to derive, we mention the fact that they have been triple checked for accuracy and may be used with confidence by the reader. There are 22 equations in 20 unknowns, leading one to suspect that there may be no exact solution of these equations. However, the equations are nonlinear and there may be a possibility of a solution. In Appendix B we explore this possibility and arrive at the conclusion that there is no exact solution of these equations. Thus it appears that a fifth order Runge-Kutta integrator will require at least 6 derivative evaluations, not 5 as one would expect.

We have made though an approximate solution of the above equations. Equations (54) through (59) are solved exactly. Equations (60) through (75) were solved to an average 1σ error of $\pm 1.6 \cdot 10^{-6}$. The constants are shown in Table 17.

δ_1 = 0	δ_2 = .0013290 82957	δ_3 = .37046 68626
δ_4 = .74985 56689	δ_5 = 1.0000 07268	
a_1 = .0013290 82957		
b_1 = -51.257 65184	b_2 = 51.628 11870	
c_1 = 165.34 07125	c_2 = -165.94 52689	c_3 = 1.3544 12081
d_1 = -598.90 49520	d_2 = 601.87 06163	d_3 = -3.0940 32895
d_4 = 1.1283 75862		
e_1 = -7.8256 11091	e_2 = 8.0068 96701	e_3 = .38071 98964
e_4 = .35851 70449	e_5 = .079477 4487	

TABLE 17: FIFTH ORDER INTEGRATION CONSTANTS FOR 5 DERIVATIVE EVALUATIONS (APPROXIMATE SOLUTION)

7. FIFTH ORDER RUNGE-KUTTA TYPE INTEGRATOR WITH SIX DERIVATIVE EVALUATIONS

As we saw in the previous section, we need more than five derivative evaluations to obtain a fifth order Runge-Kutta type integrator. In this section we attempt to obtain the fifth order integrator using six derivative evaluations, as shown below.

$$\underline{k}_1 = \Delta T \underline{f}(\underline{x}_i,\ t_i + \delta_1 \Delta T) \tag{76}$$

$$\underline{k}_2 = \Delta T \underline{f}(\underline{x}_i + a_1 \underline{k}_1,\ t_i + \delta_2 \Delta T) \tag{77}$$

$$\underline{k}_3 = \Delta T \underline{f}(\underline{x}_i + b_1 \underline{k}_1 + b_2 \underline{k}_2,\ t_i + \delta_3 \Delta T) \tag{78}$$

$$\underline{k}_4 = \Delta T \underline{f}(\underline{x}_i + c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3,\ t_i + \delta_4 \Delta T) \tag{79}$$

$$\underline{k}_5 = \Delta T \underline{f}(\underline{x}_i + d_1 \underline{k}_1 + d_2 \underline{k}_2 + d_3 \underline{k}_3 + d_4 \underline{k}_4,\ t_i + \delta_5 \Delta T) \tag{80}$$

$$\underline{k}_6 = \Delta T \underline{f}(\underline{x}_i + e_1 \underline{k}_1 + e_2 \underline{k}_2 + e_3 \underline{k}_3 + e_4 \underline{k}_4 + e_5 \underline{k}_5,\ t_i + \delta_6 \Delta T) \tag{81}$$

$$\underline{x}_{i+1} = \underline{x}_i + f_1 \underline{k}_1 + f_2 \underline{k}_2 + f_3 \underline{k}_3 + f_4 \underline{k}_4 + f_5 \underline{k}_5 + f_6 \underline{k}_6 + 0(\Delta T^{?}) \tag{82}$$

The constraint equations that the integration constants must satisfy are

$$\delta_1 = 0 \tag{83}$$

$$a_1 = \delta_2 \tag{84}$$

$$b_1 + b_2 = \delta_3 \tag{85}$$

$$c_1 + c_2 + c_3 = \delta_4 \tag{86}$$

$$d_1 + d_2 + d_3 + d_4 = \delta_5 \qquad (87)$$

$$e_1 + e_2 + e_3 + e_4 + e_5 = \delta_6 \qquad (88)$$

$$f_1 + f_2 + f_3 + f_4 + f_5 + f_6 = 1 \qquad (89)$$

$$f_2\delta_2 + f_3\delta_3 + f_4\delta_4 + f_5\delta_5 + f_6\delta_6 = \frac{1}{2} \qquad (90)$$

$$f_2\delta_2^2 + f_3\delta_3^2 + f_4\delta_4^2 + f_5\delta_5^2 + f_5\delta_6^2 = \frac{1}{3} \qquad (91)$$

$$f_2\delta_2^3 + f_3\delta_3^3 + f_4\delta_4^3 + f_5\delta_5^3 + f_6\delta_6^3 = \frac{1}{4} \qquad (92)$$

$$f_2\delta_2^4 + f_3\delta_3^4 + f_4\delta_4^4 + f_5\delta_5^4 + f_6\delta_6^4 = \frac{1}{5} \qquad (93)$$

$$b_2\delta_2 f_3 + (c_2\delta_2 + c_3\delta_3)f_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)f_5$$
$$+ (e_2\delta_2 + e_3\delta_3 + e_4\delta_4 + e_5\delta_5)f_6 = \frac{1}{6} \qquad (94)$$

$$b_2\delta_2 f_3\delta_3 + (c_2\delta_2 + c_3\delta_3)f_4\delta_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)f_4\delta_5$$
$$+ (e_2\delta_2 + e_3\delta_3 + e_4\delta_4 + e_5\delta_5)f_6\delta_6 = \frac{1}{8} \qquad (95)$$

$$b_2\delta_2 f_3\delta_3^2 + (c_2\delta_2 + c_3\delta_3)f_4\delta_4^2 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)f_5\delta_5^2$$
$$+ (e_2\delta_2 + e_3\delta_3 + e_4\delta_4 + e_5\delta_5)f_6\delta_6^2 = \frac{1}{10} \qquad (96)$$

$$(b_2\delta_2)^2 f_3 + (c_2\delta_2 + c_3\delta_3)^2 f_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)^2 f_5$$
$$+ (e_2\delta_2 + e_3\delta_3 + e_4\delta_4 + e_5\delta_5)^2 f_6 = \frac{1}{20} \qquad (97)$$

$$b_2\delta_2^2 f_3 + (c_2\delta_2^2 + c_3\delta_3^2)f_4 + (d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)f_5$$
$$+ (e_2\delta_2^2 + e_3\delta_3^2 + e_4\delta_4^2 + e_5\delta_5^2)f_6 = \frac{1}{12} \qquad (98)$$

$$b_2\delta_2^2 f_3\delta_3 + (c_2\delta_2^2 + c_3\delta_3^2)f_4\delta_4 + (d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)f_5\delta_5$$
$$+ (e_2\delta_2^2 + e_3\delta_3^2 + e_4\delta_4^2 + e_5\delta_5^2)f_6\delta_6 = \frac{1}{15} \qquad (99)$$

$$b_2\delta_2^3 f_3 + (c_2\delta_2^3 + c_3\delta_3^3)f_4 + (d_2\delta_2^3 + d_3\delta_3^3 + d_4\delta_4^3)f_5$$
$$+ (e_2\delta_2^3 + e_3\delta_3^3 + e_4\delta_4^3 + e_5\delta_5^3)f_6 = \frac{1}{20} \qquad (100)$$

$$b_2\delta_2(c_3f_4\delta_4 + d_3f_5\delta_5 + e_3f_6\delta_6) + (c_2\delta_2 + c_3\delta_3)(d_4f_5\delta_5 + e_4f_6\delta_6)$$
$$+ (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5f_6\delta_6 = \frac{1}{30} \qquad (101)$$

$$b_2\delta_2(c_3f_4 + d_3f_5 + e_3f_6) + (c_2\delta_2 + c_3\delta_3)(d_4f_5 + e_4f_6)$$
$$+ (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5f_6 = \frac{1}{24} \qquad (102)$$

$$b_2\delta_2(c_3f_4 + d_3f_5 + e_3f_6)\delta_3 + (c_2\delta_2 + c_3\delta_3)(d_4f_5 + e_4f_6)\delta_4$$
$$+ (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5f_6\delta_5 = \frac{1}{40} \qquad (103)$$

$$b_2\delta_2^2(c_3f_4 + d_3f_5 + e_3f_6) + (c_2\delta_2^2 + c_3\delta_3^2)(d_4f_5 + e_4f_6)$$
$$+ (d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)e_5f_6 = \frac{1}{60} \qquad (104)$$

$$b_2\delta_2c_3(d_4f_5 + e_4f_6) + b_2\delta_2d_3e_5f_6 + (c_2\delta_2 + c_3\delta_3)d_4e_5f_6 = \frac{1}{120} \qquad (105)$$

There are 23 constraint equations in 27 unknowns. This apparently allows us to supply 4 more constraint equations of our own choice. I picked the following 4 additional constraint equations.

$$f_2\delta_2^5 + f_3\delta_3^5 + f_4\delta_4^5 + f_5\delta_5^5 + f_6\delta_6^5 = \frac{1}{6} \quad (106)$$

$$f_2\delta_2^6 + f_3\delta_3^6 + f_4\delta_4^6 + f_5\delta_5^6 + f_6\delta_6^6 = \frac{1}{7} \quad (107)$$

$$f_2\delta_2^7 + f_3\delta_3^7 + f_4\delta_4^7 + f_5\delta_5^7 + f_6\delta_6^7 = \frac{1}{8} \quad (108)$$

$$f_2\delta_2^8 + f_3\delta_3^8 + f_4\delta_4^8 + f_5\delta_5^8 + f_6\delta_6^8 = \frac{1}{9} \quad (109)$$

These additional constraint equations will cause $\dot{\underline{x}} = \underline{f}(t)$ to be integrated to ninth order accuracy, giving us a 5/9 order integrator.

We now have 27 equations in 27 unknowns. We tried diligently to find a solution to these equations. We used three different computer programs, each with a different method, but were only able to obtain an approximate solution of the 27 equations. Equations (83) through (89) were solved exactly. Equations (90) through (109) were solved to an average 1σ accuracy of $\pm 1.2 \cdot 10^{-6}$. The integration constants that we obtained are shown in Table 18. We then made a brief attempt to find any solution of the original 23 equations in 27 unknowns. We were unsuccessful in that attempt, but the effort that we expended was not as great as it might have been. Thus we are reasonably sure that the 27 equations in 27 unknowns has no exact solution. However, we can only say that we suspect that the 23 equations in 27 unknowns has no solution.

One final comment concerning a sixth order Runge-Kutta type integrator. We have ascertained that the sixth order integration of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ requires 9 derivative evaluations, giving 38 constraint equations in 45 unknowns, a truly formidable problem which again may have no exact solution.

δ_1 = 0	δ_2 = .0015763 74117	δ_3 = .30514 37420
δ_4 = .71385 17583	δ_5 = .61937 61366	δ_6 = .93389 85460
a_1 = .0015763 74117		
b_1 = -29.265 36037	b_2 = 29.570 50411	
c_1 = 19.658 98673	c_2 = -19.883 49164	c_3 = .93835 66728
d_1 = -18.777 16241	d_2 = 19.272 14754	d_3 = -.17709 86412
d_4 = .30148 96548		
e_1 = 13.387 20508	e_2 = -13.325 28099	e_3 = .30208 02382
e_4 = .12617 48124	e_5 = .44371 94067	
f_1 = -3.5372 73077	f_2 = 3.6754 69584	f_3 = .34296 07771
f_4 = .17209 54083	f_5 = .18162 04139	f_6 = .16512 68939

TABLE 18: 5/9 ORDER INTEGRATION CONSTANTS FOR 6 DERIVATIVE EVALUATIONS (APPROXIMATE SOLUTION)

8. A SELF-STARTING INTEGRATOR USING $\partial \underline{f}/\partial \underline{x}$

In conjunction with integrating $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$, the equations defining the matrix, $\partial \underline{f}/\partial \underline{x}$, are frequently available. A self-starting integrator making use of this matrix is shown below.

Given	$\dot{\underline{x}} = \underline{f}(\underline{x}, t) \qquad \ddot{\underline{x}} = \underline{g}(\underline{x}, t) = \frac{\partial \underline{f}}{\partial \underline{x}} \underline{f} + \frac{\partial \underline{f}}{\partial t}$
Let	$\hat{\underline{x}}_{i+.5} = \underline{x}_i + \frac{\Delta T}{2} \dot{\underline{x}}_i + \frac{1}{2}\left(\frac{\Delta T}{2}\right)^2 \ddot{\underline{x}}_i$
Then	$\hat{\underline{x}}_{i+1} = \underline{x}_i + \Delta T \, \dot{\underline{x}}_i + \frac{1}{6} \Delta T^2 \, \ddot{\underline{x}}_i + \frac{1}{3} \Delta T^2 \underline{g}(\hat{\underline{x}}_{i+5}, t_i + .5 \, \Delta T)$
Where	$\hat{\underline{x}}_{i+1} = \underline{x}_{i+1} + 0(\Delta T^5)$

the proof is

$$\hat{\underline{x}}_{i+.5} = \underline{x}_{i+.5} + \underline{\varepsilon}\Delta T^3$$

Therefore

$$\underline{g}(\hat{\underline{x}}_{i+.5}, t_i + .5\Delta T) = \underline{g}(\underline{x}_{i+.5}, t_i + .5\Delta T) + \frac{\partial \underline{g}}{\partial \underline{x}} \underline{\varepsilon}\Delta T^3 = \ddot{\underline{x}}_{i+.5} + 0(\Delta T^3)$$

So

$$\hat{\underline{x}}_{i+1} = \underline{x}_i + \Delta T \dot{\underline{x}}_i + \frac{1}{6} \Delta T^2 \, \ddot{\underline{x}}_i + \frac{1}{3} \Delta T^2 \, \ddot{\underline{x}}_{i+.5} + 0(\Delta T^5)$$

But

$$\ddot{\underline{x}}_{i+.5} = \ddot{\underline{x}}_i + \frac{\Delta T}{2}\dddot{\underline{x}}_i + \frac{1}{2}\left(\frac{\Delta T}{2}\right)^2 \ddddot{\underline{x}}_i + O(\Delta T^3)$$

so

$$\hat{\underline{x}}_{i+1} = \underline{x}_i + \Delta T\, \dot{\underline{x}}_i + \frac{\Delta T^2}{2}\ddot{\underline{x}}_i + \frac{\Delta T^3}{6}\dddot{\underline{x}}_i + \frac{\Delta T^4}{24}\ddddot{\underline{x}}_i + O(\Delta T^5)$$

The integration coefficients in a self-starting integrator utilizing $\partial \underline{f}/\partial \underline{x}$, are unique. Thus there is no opportunity to adjust integrator constants to reduce truncation error, as there is with the Runge-Kutta type of integrators.

9. ESTIMATION OF TRUNCATION ERROR

It is frequently desirable to be able to estimate integrator accuracy over the integration interval. This information might then be used to control the integration step size. A logical approach to this problem would to be make one extra derivative evaluation (generate one more $\underline{k}$), and then use this extra value of $\underline{k}$, combined with the other $\underline{k}$'s, to estimate the truncation error. This would be equivalent to taking a, say, third order integrator, making one more derivative evaluation (evaluate $\underline{k}_4$), and obtaining a fourth order integration. This would then be compared with the third order answer to determine the truncation error. Unfortunately it is not possible to do this. From Table 6 it is seen that a third order integrator must have

$$b_2 = \frac{\delta_3}{\delta_2}\frac{\delta_3 - \delta_2}{2 - 3\delta_2}$$

From Table 11 it is seen that a fourth order integrator must have

$$b_2 = \frac{\delta_3}{2\delta_2}\frac{\delta_3 - \delta_2}{1 - 2\delta_2}$$

It is clearly seen that the two values of b_2 can never equal one another. Thus a third order integrator can not be made into a fourth order integrator. In other words, you can't make a silk purse out of a sow's ear.

In the following estimation of truncation error, we will assume that three integrations are made when integrating from t_i to t_{i+1}: one integration from t_i to $t_{i+.5}$; the result being integrated from $t_{i+.5}$ to t_{i+1}; and one integration from t_i to t_{i+1} directly.

Integration from t_i to $t_{i+.5}$, for nth order integrator, may be represented by

$$\underline{\hat{x}}_{i+.5} = \underline{h}(\underline{x}_i, t_i) + \underline{e}\left(\frac{\Delta T}{2}\right)^{n+1} = \underline{x}_{i+.5} + \underline{e}\left(\frac{\Delta T}{2}\right)^{n+1}$$

where $\Delta T = t_{i+1} - t_i$, and where $\underline{e}(\Delta T/2)^{n+1}$ is the truncation error associated with the particular integration scheme being used.

Assuming $\underline{e}$ remains relatively constant, integration from $t_{i+.5}$ to t_{i+1} is given by

$$\underline{\hat{\hat{x}}}_{i+1} = \underline{h}\left[\underline{x}_{i+.5} + \underline{e}\left(\frac{\Delta T}{2}\right)^{n+1}, t_i + .5\,\Delta T\right] + \underline{e}\left(\frac{\Delta T}{2}\right)^{n+1}$$

or

$$\underline{\hat{\hat{x}}}_{i+1} = \underline{x}_{i+1} + \frac{\partial \underline{h}}{\partial \underline{x}}\,\underline{e}\left(\frac{\Delta T}{2}\right)^{n+1} + \underline{e}\left(\frac{\Delta T}{2}\right)^{n+1} \tag{110}$$

A single integration from t_i to t_{i+1} would yield

$$\underline{\hat{x}}_{i+1} = \underline{x}_{i+1} + \underline{e}\,\Delta T^{n+1} \tag{111}$$

Subtracting equation (110) from (111) gives

$$\left(\frac{\Delta T}{2}\right)^{n+1}\left[(2^{n+1}-1)\,I - \frac{\partial \underline{h}}{\partial \underline{x}}\right]\underline{e} = \underline{\hat{x}}_{i+1} - \underline{\hat{\hat{x}}}_{i+1}$$

Thus

$$\underline{e} = \left(\frac{2}{\Delta T}\right)^{n+1}\left[(2^{n+1}-1)\,I - \frac{\partial \underline{h}}{\partial \underline{x}}\right]^{-1}(\underline{\hat{x}}_{i+1} - \underline{\hat{\hat{x}}}_{i+1}) \tag{112}$$

As a first approximation. $\partial \underline{h}/\partial \underline{x} = I$, so

$$\underline{e} \approx \frac{1}{2^{n+1}-2}\left(\frac{2}{\Delta T}\right)^{n+1} (\hat{\underline{x}}_{i+1} - \hat{\hat{\underline{x}}}_{i+1}) \tag{113}$$

For $\partial \underline{h}/\partial \underline{x} \approx I$

In situations where $\underline{x} = \begin{bmatrix} \underline{x}_i \\ \dot{\underline{x}}_i \end{bmatrix}$, a better approximation is

$$\frac{\partial \underline{h}}{\partial \underline{x}} = \begin{bmatrix} I & I\Delta T/2 \\ 0 & I \end{bmatrix}$$

In this case

$$\underline{e} \approx \frac{1}{2^{n+1}-2}\left(\frac{2}{\Delta T}\right)^{n+1} \begin{bmatrix} I & I\,\dfrac{\Delta T}{2(2^{n+1}-2)} \\ 0 & I \end{bmatrix} (\hat{\underline{x}}_{i+1} - \hat{\hat{\underline{x}}}_{i+1}) \tag{114}$$

For $\underline{x} = \begin{bmatrix} \underline{x}_i \\ \dot{\underline{x}}_i \end{bmatrix}$ only.

Knowing $\underline{e}$, and assuming $\underline{e}$ remains relatively constant from one cycle to the next, a value of ΔT for the next cycle may be chosen to give any desired integration accuracy. For example, suppose that $\underline{x} = [\underline{x}_1^T \quad \dot{\underline{x}}_1^T]^T$ and the error in $\hat{\underline{x}}_1$ (the position error) is to be controlled. From equation (110), and assuming $\partial \underline{h}/\partial \underline{x} = I$, the position error in the next cycle is seen to be

$$2\underline{e}_1\left(\frac{\Delta T_{i+1}}{2}\right)^{n+1}$$

Let the magnitude of the allowable position error vector be

$$\delta_p = \frac{\text{Units of position (feet, radians, etc.)}}{\text{Units of integration time (secs, days, etc.)}}$$

Then

$$\delta_p \Delta T_{i+1} = 2|\underline{e}_1|\left(\frac{\Delta T_{i+1}}{2}\right)^{n+1}$$

Substituting equation (113) into the above, and solving for ΔT_{i+1} gives

$$\Delta T_{i+1} = \left[\frac{(2^n-1)\ \delta_p \Delta T_i}{|\hat{\underline{x}}_1 - \hat{\underline{x}}_1|}\right]^{\frac{1}{n}} \Delta T_i \tag{115}$$

The above equation has been used very successfully with a fourth order integration of an earth to Mars trajectory. Here the position vector consisted of the positions of the earth, moon, sun, and Mars with respect to the spacecraft.

It was interesting to note that the motion of the earth-moon system was the dominant factor in controlling ΔT during most of the voyage. This is something that many people might overlook, yet it was readily detected using equation (115).

When operating with very small values of δ_p (and small integration step sizes), there may be a very large loss of significant figures in the quantity $\hat{\underline{x}} - \hat{\hat{\underline{x}}}$. Generally, integrator equations can be written in the form

$$\underline{x}_{i+1} = \underline{x}_i + \Delta \underline{x}$$

Thus the loss of significant figures is dimished if we let $\hat{\underline{x}} - \hat{\hat{\underline{x}}} = \Delta\hat{\underline{x}} - \Delta\hat{\hat{\underline{x}}}$. The reader may object that, in these cases, computer roundoff error is the dominant error source - not truncation error. He is correct. In these situations δ_p will not have absolute control over the integrator accuracy. However, the ΔT generated by the preceding equations will be an indicator of the "dynamic activity" of the system.

We are now led to the interesting question of what is the best ΔT to use to obtain maximum integration accuracy. Theoretically, integration accuracy improves as $\Delta T \to 0$. However, as a practical matter, computer roundoff error will, at some point, cause the integration error to start increasing again as $\Delta T \to 0$. Suppose that the computer stores position accurate to ε_p (feet, radians, km, etc.). Again we will assume $\hat{\underline{x}}$ is being used as the output of the integration. As before, the truncation error in position for one integration, for $\hat{\underline{x}}$, is

$$\underline{e}_1 \left(\frac{\Delta T_{i+1}}{2}\right)^{n+1}$$

Now let the truncation error be the same size as the computer roundoff error.

Thus

$$|\underline{e}_1|\left(\frac{\Delta T_{i+1}}{2}\right)^{n+1} = \varepsilon_p$$

Using equation (113) and solving ΔT_{i+1} gives

$$\Delta T_{i+1} = \left[\frac{(2^{n+1}-2)\varepsilon_p}{|\hat{\underline{x}}_1 - \hat{\hat{\underline{x}}}_1|}\right]^{\frac{1}{n+1}} \Delta T_i \tag{116}$$

Had we been saving $\hat{\underline{x}}$ instead of $\hat{\hat{\underline{x}}}$ as the output of the integrator, then we would have obtained the equation

$$\Delta T_{i+1} = \frac{1}{2}\left[\frac{(2^{n+1}-2)\varepsilon_p}{|\hat{\underline{x}}_1 - \hat{\hat{\underline{x}}}_1|}\right]^{\frac{1}{n+1}} \Delta T_i \tag{117}$$

Note that $\Delta\hat{\underline{x}}_1 - \Delta\hat{\hat{\underline{x}}}_1$ should be used in place of $\hat{\underline{x}}_1 - \hat{\hat{\underline{x}}}_1$ in the above equations.

10. COMPUTER ALGORITHM FOR RUNGE-KUTTA TYPE INTEGRATORS

The computing algorithm for Runge-Kutta type integrators is particularly simple. For convenience of notation, let

$$a_1 = A_{11}$$

$$b_1, b_2 = A_{21}, A_{22}$$

$$c_1, c_2, c_3 = A_{31}, A_{32}, A_{33}$$

etc.

Then, in engineering notation, Nth order integration of $\dot{\underline{x}} = \underline{f}(\underline{x})$ is accomplished by

$$\underline{x}_{i+1} = \underline{x}_i$$

$$\underline{k}_1 = \Delta T \underline{f}(\underline{x}_{i+1})$$

$$\underline{x}_{i+1} = \underline{x}_i + A_{11}\underline{k}_1$$

$$\underline{k}_2 = \Delta T \underline{f}(\underline{x}_{i+1})$$

$$\underline{x}_{i+1} = \underline{x}_i + A_{21}\underline{k}_1 + A_{22}\underline{k}_2$$

.

.

.

$$\underline{k}_N = \Delta T \underline{f}(\underline{x}_{i+1})$$

$$\underline{x}_{i+1} = \underline{x}_i + A_{N1}\underline{k}_1 + A_{N2}\underline{k}_2 + \dots + A_{NN}\underline{k}_N$$

In order to express the preceding equations in FORTRAN notation, let (for I = 1,2,..,M and J = 1,2,...,N)

X(I) = $\underline{\ddot{x}}_{i+1}$ XX(I) = $\underline{x}_i$ F(I) = $\underline{f}$ DT = ΔT

A(I,J) = A_{ij} K(I,1) = $\underline{k}_1$ K(I,2) = $\underline{k}_2$ K(I,N) = $\underline{k}_N$

T = t   TI = T_i D(1) = δ_1 D(2) = δ_2 ...D(N) = δ_N

Then the FORTRAN equations for integrating $\underline{\dot{x}} = \underline{f}(\underline{x})$ are

```
              DO 1   I = 1,M
1             XX(I) = X(I)
              DO 2   J = 1,N
comment       EVALUATE F(I) = FUNCTION OF X(I)
              DO 2   I = 1,M
              K(I,J) = DT*F(I)
              X(I) = XX(I)
              DO 2   L = 1,J
2             X(I) = X(I) + A(J,L)*K(I,L)
              T = T + DT
```

N^{th} Order Integrator for $\underline{\dot{x}} = \underline{f}(\underline{x})$

The FORTRAN equations for integrating $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ are

```
          TI = T
          DO 1  I = 1,M
1         XX(I) = X(I)
          DO 2  J = 1,N
          T = TI + D(J)*DT
comment   EVALUATE F(I) = FUNCTION OF X(I),T
          DO 2  I = 1,M
          K(I,J) = DT*F(I)
          X(I) = XX(I)
          DO 2  L = 1,J
2         X(I) = X(I) + A(J,L)*K(I,L)
          TI = TI + DT
```

N^{th} Order Integrator for $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$

Though the above algorithms have the advantage of simplicity, the reader should be aware that if maximum speed of computing is desired, then the algorithms should be rewritten so that the doubly subscripted variables, A(I,J) and K(I,J), are replaced by singly subscripted variables. Singly subscripted variables require less time to locate in core than do doubly subscripted variables.

11. SOME EMPERICAL EVALUATIONS

The truncation error terms, in the previously discussed integrators, are extremely complicated. The only practical way to investigate integrator accuracy is by emperical methods. One way to do this is to pick an $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ equation whose solution is known and whose form is similar to the actual equation of interest. We have done this here in this section for an orbit determination problem. Another way is to use Eq. (116) (or similar equation) to obtain the most accurate solution of $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$ that the computer can give. Other solutions would then be compared with this one.

In order to provide an illustrative example of some of the integrators discussed in this report, a satellite orbit determination problem was run for 10 orbits. The dynamical equation of motion, for a spherical gravitational field, is given by the nonlinear, vector differential equation

$$\ddot{\underline{R}} = -\frac{\mu}{|\underline{R}|^3}\underline{R}$$

$\underline{R}$ is the vector from the center of mass of the attracting body to the satellite. μ is the gravitational parameter. The elements of the state vector were taken to be x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$. The state vector equation of motion, $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$, is then

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ -\mu x/[x^2 + y^2 + z^2]^{1.5} \\ -\mu y/[x^2 + y^2 + z^2]^{1.5} \\ -\mu z/[x^2 + y^2 + z^2]^{1.5} \end{bmatrix}$$

The initial conditions were precisely set to give a circular orbit, inclined at 45°, and with a period of exactly 6144 seconds. The accuracies of the various integrators with the above equation are shown in Tables 20 and 21. The term "2.5 order Taylor series" needs explaining. If x and $\dot{x}$ are elements of the state vector, then a second order Taylor series solution of the vector differential equation is

$$x_{i+1} = x_i + \dot{x}_i \Delta T + \ddot{x}_i \Delta T^2/2$$

$$\dot{x}_{i+1} = \dot{x}_i + \ddot{x}_i \Delta T + \dddot{x}_i \Delta T^2/2$$

However, one would rarely use the equations in a computer program in this form. As long as $\dddot{x}_i$ is being computed, one would also use it to "improve" the estimate of x_{i+1}. Thus we define a 2.5 order Taylor series solution to be

$$x_{i+1} = x_i + \dot{x}_i \Delta T + \ddot{x}_i \Delta T^2/2 + \dddot{x}_i \Delta T^3/6$$

$$\dot{x}_{i+1} = \dot{x}_i + \ddot{x}_i \Delta T + \dddot{x}_i \Delta T^2/2$$

Similarly a 3.5 order Taylor series solution goes out to fourth order in position and third order in velocity. Likewise for 4.5 and 5.5 solutions.

The term "4/6 RK Set #1" refers to the Runge-Kutta integration constants in Table 15. They provide fourth order integration of $\underline{\dot{x}} = \underline{f}(\underline{x})$ and sixth order for $\underline{\dot{x}} = \underline{f}(t)$. The set of integration constants "4th RK: $\delta_2 = \delta_3 = .5$, $d_3 = .5$" are fourth order Runge-Kutta integration constants which came from Table 12, where d_3 was optimized to give the best average position error over 10 orbits of integration.

The set of integration constants "4th RK: $\delta_2 = .15$, $\delta_3 = .192$" deserves special mention. This set of fourth order Runge-Kutta constants was obtained from Table 11 by optimizing the two free parameters, δ_2 and δ_3. The optimization was not done with the orbit determination problem studied here. They were optimized with an entirely different orbit and with a different gravity field (equations of motion).* Yet they worked extremely well with our current problem, which indicates that they probably would work very well with any general orbit determination problem. Their accuracy was even better than the 5th or 5.5 order Taylor series solutions. This outstanding set of integration constants is shown in Table 19 below.

$\delta_1 = 0$	$\delta_2 = .15$	$\delta_3 = .192$	$\delta_4 = 1$
$a_1 = .15$			
$b_1 = .1536$	$b_2 = .0384$		
$c_1 = 6.745\ 2657\ 1119\ 01$		$c_2 = -38.77\ 8319\ 5429\ 47$	
$c_3 = 33.03\ 3053\ 8317\ 57$			
$d_1 = 1.414\ 3518\ 5185\ 20$		$d_2 = -9.586\ 0566\ 4488\ 03$	
$d_3 = 8.952\ 7181\ 8848\ 55$		$d_4 = .2189\ 3660\ 4542\ 81$	

TABLE 19: <u>OPTIMUM FOURTH ORDER RUNGE-KUTTA INTEGRATION CONSTANTS FOR ORBIT DETERMINATION</u>

* Wm. M. Lear, "Direct Integration of Orbital Equations Using a Fourth Order Runge-Kutta Integrator", TRW Technical Report 20029-6013-T0-00, 31 August 1972.

Table 20 shows the position error at the end of 10 orbits. Table 21 shows the average position error during the 10 orbits. In general the results of the two tables are in agreement as far as ranking the integrators for accuracy. However, there are a few cases of disagreement. For an integration step size of 256 seconds, the Runge-Kutta-Gill integration constants had a final position error of 1274 meters, lower than with $\Delta T = 128$ seconds which was 2193 meters. However, the average position error in Table 21 gives a truer picture of the overall accuracy. We see that for $\Delta T = 256$ seconds, the Runge-Kutta-Gill constants had a rather large average error and thus indicated that the absolute position error was passing through an abnormally low minimum at the end of 10 orbits. In other words, the average position error is a better indication of overall accuracy. We have not shown tables of velocity accuracy since, without exception, all the velocity errors are approximately 1/1000 of the position errors.

Figures 1 and 2 show how the integrator error "grows" with time for several of the integrators. We see that the errors are not necessarily always increasing with time, but may exhibit periods where the error may decrease with time. Note particularly the excellent performance of the Table 19 integrator constants shown in Figure 2.

Figure 3 shows the accuracy of various orders of Taylor series integrators versus integration step size. The important thing shown in this figure is the desirability of the fourth order integrator. Notice that for a constant integrator accuracy of 100 meters, the fourth order integration step size can be 13 times larger than the third order integration step. Yet when we go from fourth order to fifth order, we can increase our integration step by only a factor of 1.3, hardly worth the effort of evaluating the extra derivative term. Also if one examines Tables 20 and 21, we also reach the conclusion that, at least for orbit integration, the fourth order integrators are perhaps the most desirable all purpose integrators.

TABLE 20: POSITION ERROR (METERS) AT THE END OF 10 ORBITS

INTEGRATOR	ΔT = .25 sec	.5	1	2	4	8	16	32	64	128	256	512
5.5 Order Taylor Series						.157	.00420	2.20	71.6	2284.	72 131.	$2\cdot10^6$
5th Order Taylor Series						.129	.0730	4.44	143.	4548.	142 365.	$4\cdot10^6$
4.5 Order Taylor Series						.0960	.204	.333	147.	6929.	257 150.	$9\cdot10^6$
4th Order Taylor Series						.112	.481	18.0	646.	21 727.	712 854.	
3.5 Order Taylor Series		1.76	2.67	15.7	123.	980.	7839.	62 728.	502 785.			
3rd Order Taylor Series		1.76	4.47	30.9	245.	1960.	15 674.	125 416.	$1\cdot10^6$			
2.5 Order Taylor Series	12.0	40.3	156.	605.	2296.	8204.	24 970.	37 181.				
2nd Order Taylor Series	6.50	19.3	68.4	226.	536.	804.	26 666.	293 217.				
5th RK, 5 Deriv. Eval.						2.84	6.08	13.2	67.3	4120.	146 653.	$5\cdot10^6$
5/9 RK, 6 Deriv. Eval.						15.5	31.9	70.6	212.	1421.	24 999.	790 364.
Standard 4th Order RK						.175	1.50	34.4	908.	26 032.	795 167.	
4th Runge-Kutta-Gill						.0498	.841	13.5	191.	2193.	1274.	$2\cdot10^6$
4/6 RK Set #1						.510	8.49	194.	4967.	137 233.	$4\cdot10^6$	
4/6 RK Set #2						.349	16.8	372.	9283.	256 099.	$8\cdot10^6$	
4th RK: $\delta_2 = \delta_3 = .5$, $d_3 = .5$						.0741	.384	4.17	23.4	3304.	152 748.	
4th RK: $\delta_2 = .15$, $\delta_3 = .192$						.109	.0988	.460	2.07	322.	13 363	338 548.
3/5 RK Set #1		1.53	2.57	15.8	124.	989.	7 908.	63 279.	507 825.			
3/5 RK Set #2		.580	14.3	119.	956.	7649.	61 152.	487 479.	$4\cdot10^6$			
2/3 RK	23.2	93.9	400.	1618.	6597.	27 363.	117 205.	529 695.				
2/4 RK	38.2	159.	646.	2633.	10 898.	46 522.	209 397.	$1\cdot10^6$				
4th Order, Section 8						.0610	.320	7.13	159.	3970.	108 450.	

TABLE 21: AVERAGE POSITION ERROR (METERS) OVER 10 ORBITS

INTEGRATOR	ΔT = .25 sec	.5	1	2	4	8	16	32	64	128	256	512
5.5 Order Taylor Series						.0522	.00143	.733	23.9	763.	24 172.	730 051.
5th Order Taylor Series						.0431	.0243	1.48	47.6	1518.	47 646.	$1\cdot10^6$
4.5 Order Taylor Series						.0271	.159	1.38	37.0	1987.	80 328.	$3\cdot10^6$
4th Order Taylor Series						.0347	.127	5.35	204.	7067.	235 146.	
3.5 Order Taylor Series		.587	.890	5.22	40.8	326.	2610.	20 892.	167 510.			
3rd Order Taylor Series		.587	1.49	10.3	81.6	653.	5221.	41 778.	335 101.			
2.5 Order Taylor Series	5.66	20.1	78.4	308.	1190.	4433.	15 120.	39 582.				
2nd Order Taylor Series	2.99	9.73	36.0	128.	391.	695.	6 805.	85 642.				
5th RK, 5 Deriv. Eval.						.959	2.08	4.59	21.5	1365.	48 748.	$2\cdot10^6$
5/9 RK, 6 Deriv. Eval.						5.11	10.5	23.1	68.9	466.	8330.	266 863.
Standard 4th Order RK						.0663	.625	13.5	335.	9201.	273 888.	
4th Runge-Kutta-Gill						.0126	.438	7.02	104.	1369.	9982.	530 777.
4/6 RK Set #1						.223	3.62	77.3	1856.	48 899.	$1\cdot10^6$	
4/6 RK Set #2						.423	7.30	151.	3529.	92 296.	$3\cdot10^6$	
4th RK: $\delta_2 = \delta_3 = .5$, $d_3 = .5$						.0198	.230	3.02	21.4	845.	45 255.	
4th RK: $\delta_2 = .15$, $\delta_3 = .192$						.0369	.0438	.334	2.36	82.3	3841.	101 553.
3/5 RK Set #1		.509	.856	5.24	41.1	329.	2629.	21 043.	168 836.			
3/5 RK Set #2		.193	4.76	39.7	318.	2547.	20 367.	162 498.	$1\cdot10^6$			
2/3 RK	11.9	49.5	200.	805.	3260.	13 366.	56 040.	244 368.				
2/4 RK	19.4	79.7	322.	1302.	5330.	22 294.	96 920.	448 223.				
4th Order, Section 8						.0178	.152	3.10	64.7	1511.	39 276.	

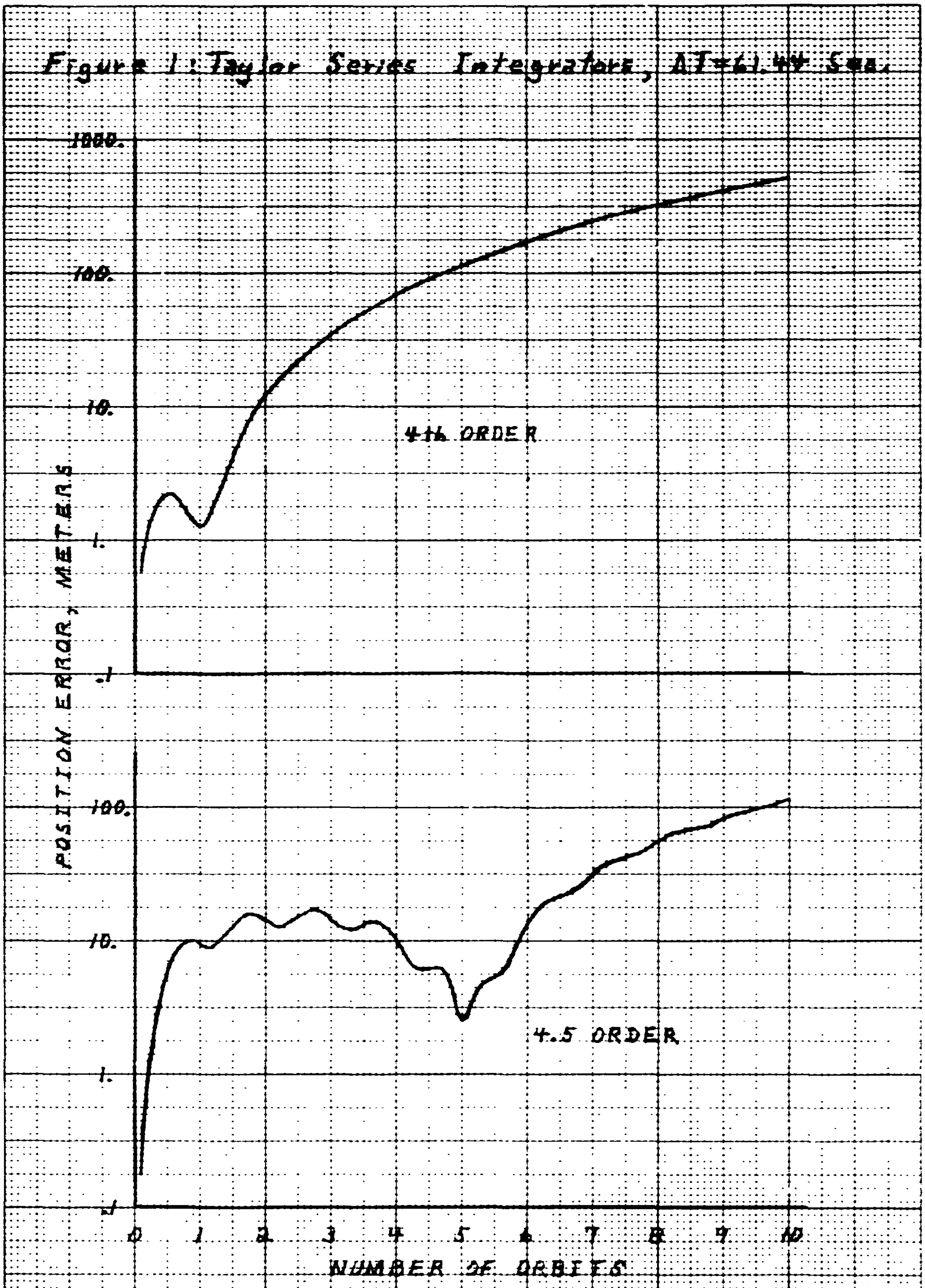


Figure 1: Taylor Series Integrators, ΔT=61.44 Sec.
1000.
100.
10.
1.
.1
4th ORDER
POSITION ERROR, METERS
100.
10.
1.
.1
4.5 ORDER
0
1
2
3
4
5
6
7
8
9
10
NUMBER OF ORBITS

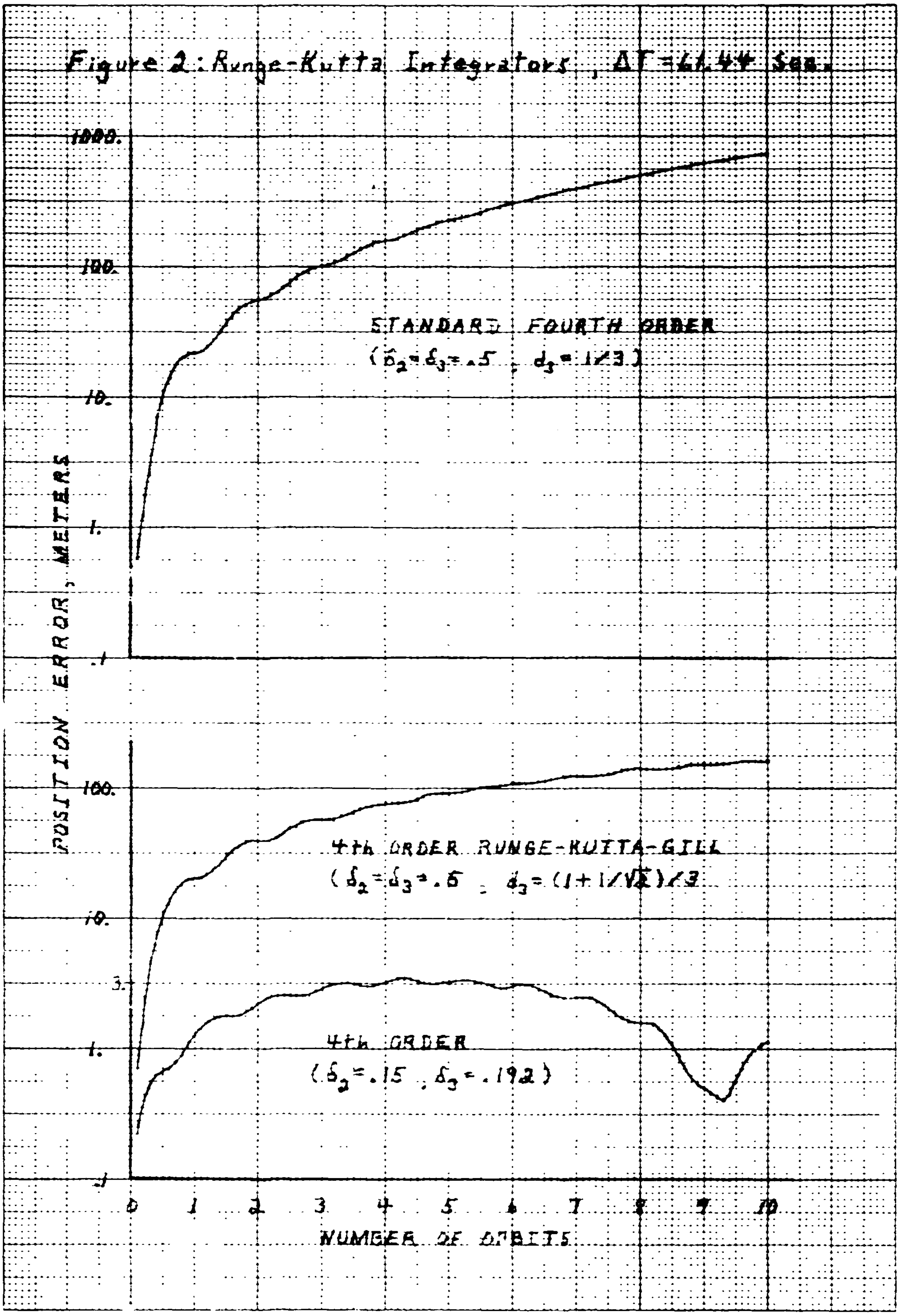


Figure 2: Runge-Kutta Integrators, ΔT = 61.44 Sec.
STANDARD FOURTH ORDER
(δ2 = δ3 = .5 ; d3 = 1/3)
4th ORDER RUNGE-KUTTA-GILL
(δ2 = δ3 = .5 ; d3 = (1+1/√2)/3
4th ORDER
(δ2 = .15 ; δ3 = .192)
POSITION ERROR, METERS
NUMBER OF ORBITS

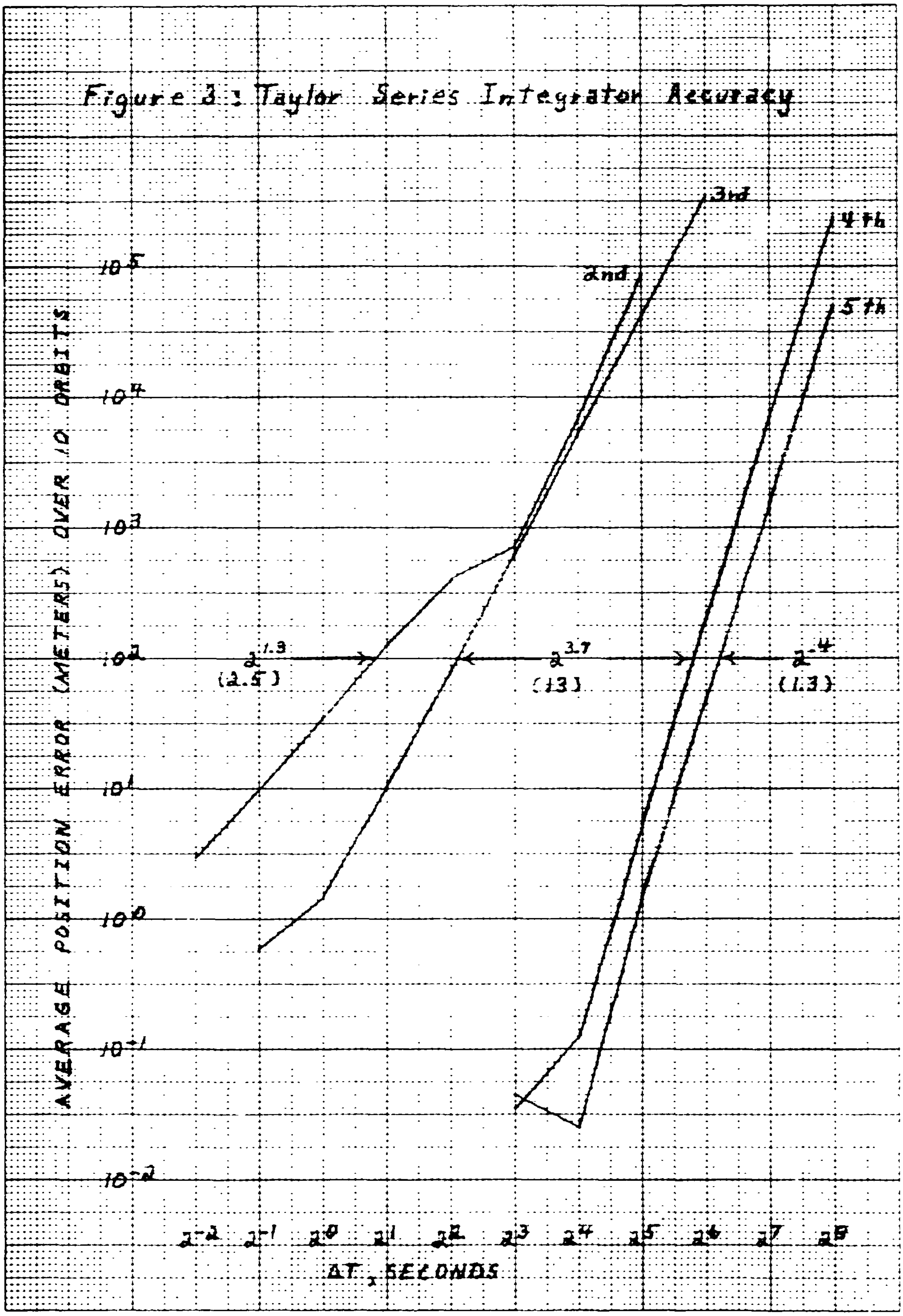


Figure 3: Taylor Series Integrator Accuracy
AVERAGE POSITION ERROR (METERS) OVER 10 ORBITS
ΔT, SECONDS
2nd
3rd
4th
5th
(2.5)
(13)
(1.3)

12. INTEGRATION USING ACCELEROMETER DATA

12.1 INTRODUCTION

In aerospace work we frequently have the equation of motion

$$\ddot{\underline{R}} = \underline{a}_g + \underline{a}_s \tag{118}$$

where

$$\underline{R} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \underline{a}_g = \underline{a}_g(\underline{R}, t)$$

$\underline{a}_g$ is acceleration due to gravity and is an analytic function of position and time (if higher order gravitational harmonics are used). The function $\underline{a}_s$ is sensed acceleration, sensed by accelerometers, and is due to external forces other than gravity. However, the output of the accelerometers is not $\underline{a}_s$ itself, but the integral of $\underline{a}_s$. Thus what we have available for integration of the equations of motion is

$$\underline{v}(t) = \int_{t_0}^{t} \underline{a}_s(\tau)d\tau \tag{119}$$

For "nondestruct" accelerometer data, t_0 is the time at which the system was turned on. For "destruct" accelerometer data the integral is only over small time intervals, say from t_i to t_{i+1}. We will assume nondestruct data here, the resulting equations can be easily modified for destruct data.

12.2 ONE COMMON SOLUTION OF EQ. (118)

Solve (integrate) the free-flight equation

$$\ddot{\underline{R}} = \underline{a}_g(\underline{R}, t)$$

to obtain

$$\underline{R}_{i+1} = \underline{f}(\underline{R}_i, \dot{\underline{R}}_i, t_i, \Delta T) \qquad \dot{\underline{R}}_{i+1} = \underline{g}(\underline{R}_i, \dot{\underline{R}}_i, t_i, \Delta T) \tag{120}$$

Let

$$\underline{P} = \frac{\partial \underline{a}_g}{\partial \underline{R}} \tag{121}$$

Note that P is a 3 by 3 matrix which is a function of $\underline{R}$ and t.

Then the solution of Eq. (118),

$$\ddot{\underline{R}} = \underline{a}_g + \underline{a}_s(t)$$

is given by

$$\underline{R}_{i+1} = \underline{f}(\underline{R}_i, \dot{\underline{R}}_i, t_i, \Delta T) + \int_{t_i}^{t_{i+1}} \underline{a}_s(\tau)(t_{i+1} - \tau)d\tau$$
$$+ \frac{1}{4!} P_i\underline{a}_{s,i}\Delta T^4 + \frac{1}{5!} [3\dot{P}_i\underline{a}_{s,i} + \underline{P}_i\dot{\underline{a}}_{s,i}]\Delta T^5 + \cdots$$

$$\dot{\underline{R}}_{i+1} = \underline{g}(\underline{R}_i, \dot{\underline{R}}_i, t_i, \Delta T) + \int_{t_i}^{t_{i+1}} \underline{a}_s(\tau)d\tau$$
$$+ \frac{1}{3!} P_i\underline{a}_{s,i}\Delta T^3 + \frac{1}{4!} [3\dot{P}_i\underline{a}_{s,i} + P_i\dot{\underline{a}}_{s,i}]\Delta T^4 + \cdots$$

Let

$$\underline{I} = \int_{t_i}^{t_{i+1}} \underline{a}_s(\tau)(t_{i+1}-\tau)d\tau \tag{122}$$

and note that

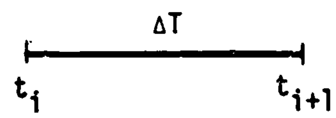
$$\int_{t_i}^{t_{i+1}} \underline{a}_s(\tau)d\tau = \underline{v}_{i+1} - \underline{v}_i \text{ exactly} \tag{123}$$

Thus our solution of the equations of motion is given by

$$\underline{R}_{i+1} = \underline{f}(\underline{R}_i, \dot{\underline{R}}_i, t_i, \Delta T) + \underline{I} + \frac{1}{4!} P_i \underline{a}_{s,i} \Delta T^4 + \frac{1}{5!} [3\dot{P}_i \underline{a}_{s,i} + P_i \dot{\underline{a}}_{s,i}]\Delta T^5 + \cdots \tag{124}$$

$$\dot{\underline{R}}_{i+1} = \underline{g}(\underline{R}_i, \dot{\underline{R}}_i, t_i, \Delta T) + \underline{v}_{i+1} - \underline{v}_i + \frac{1}{3!} P_i \underline{a}_{s,i} \Delta T^3 + \frac{1}{4!} [3\dot{P}_i \underline{a}_{s,i} + P_i \dot{\underline{a}}_{s,i}]\Delta T^4 + \cdots \tag{125}$$

Generally the last two terms in the above equations are ignored. Note however, it is not particularly difficult to at least include the $P_i \underline{a}_{s,i} \approx P_i (\underline{v}_{i+1} - \underline{v}_i)/\Delta T$ term. Approximations for the integral, $\underline{I}$, are shown below.

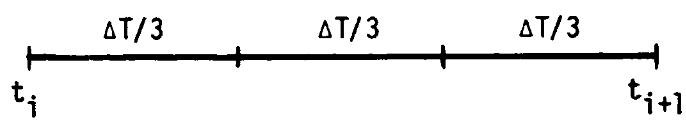


$$\hat{\underline{I}} = \frac{\Delta T}{2}\left(-\underline{v}_i + \underline{v}_{i+1}\right)$$

$$\hat{\underline{I}} - \underline{I} = \dot{\underline{a}}_{s,i}\,\frac{\Delta T^3}{12}$$

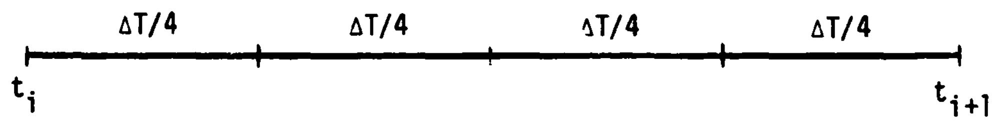
$$\hat{\underline{I}} = \frac{\Delta T}{6}\left[-5\underline{v}_i + 4\underline{v}_{i+.5} + \underline{v}_{i+1}\right]$$

$$\hat{\underline{I}} - \underline{I} = \dddot{\underline{a}}_{s,i}\,\frac{\Delta T^5}{2880}$$ (Eq. is correct.)



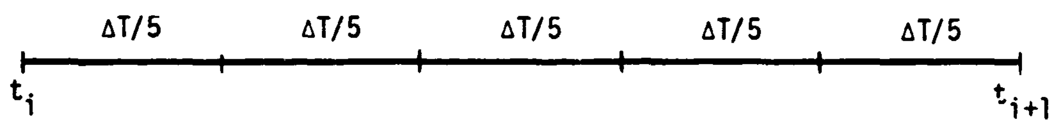
$$\hat{\underline{I}} = \frac{\Delta T}{8}\left[-7\underline{v}_i + 3\underline{v}_{i+1/3} + 3\underline{v}_{i+2/3} + \underline{v}_{i+1}\right]$$

$$\hat{\underline{I}} - \underline{I} = \dddot{\underline{a}}_{s,i}\,\frac{\Delta T^5}{6480}$$



$$\underline{\hat{I}} = \frac{\Delta T}{90}\left[-83\underline{v}_i + 32\underline{v}_{i+1/4} + 12\underline{v}_{i+2/4} + 32\underline{v}_{i+3/4} + 7\underline{v}_{i+1}\right]$$

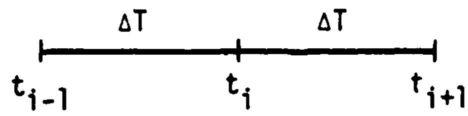
$$\underline{\hat{I}} - \underline{I} = \frac{d^5\underline{a}_{s,i}}{dt^5}\,\frac{\Delta T^7}{1935360}$$



$$\underline{\hat{I}} = \frac{\Delta T}{288}\left[-269\underline{v}_i + 75\underline{v}_{i+.2} + 50\underline{v}_{i+.4} + 50\underline{v}_{i+.6} + 75\underline{v}_{i+.8} + 19\underline{v}_{i+1}\right]$$

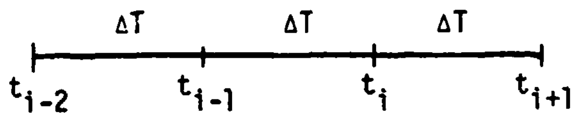
$$\underline{\hat{I}} - \underline{I} = \frac{d^5\underline{a}_{s,i}}{dt^5}\,\frac{11\,\Delta T^7}{37\,800\,000}$$

We have shown the higher order integrators above, not because of their accuracy, but because the engineer doing the integration frequently has no control over when the accelerometers are read out. The approximations of $\underline{I}$ shown above are much preferred to those shown below because of such things as engine cutoff in the middle of an integration step.



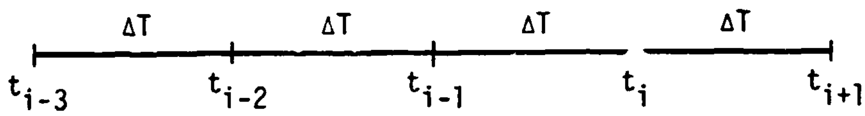
$$\hat{\underline{I}} = \frac{\Delta T}{12}\left[-\underline{v}_{i-1} - 4\underline{v}_i + 5\underline{v}_{i+1}\right]$$

$$\hat{\underline{I}} - \underline{I} = \ddot{\underline{a}}_{s,i}\,\frac{\Delta T^4}{24}$$



$$\hat{\underline{I}} = \frac{\Delta T}{24}\left[\underline{v}_{i-2} - 5\underline{v}_{i-1} - 5\underline{v}_i + 9\underline{v}_{i-1}\right]$$

$$\hat{\underline{I}} - \underline{I} = \dddot{\underline{a}}_{s,i}\,\frac{19\Delta T^5}{720}$$

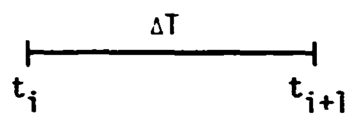


$$\hat{\underline{I}} = \frac{\Delta T}{720}\left[-19\underline{v}_{i-3} + 106\underline{v}_{i-2} - 264\underline{v}_{i-1} - 74\underline{v}_i + 251\underline{v}_{i+1}\right]$$

$$\hat{\underline{I}} - \underline{I} = \frac{d^4\underline{a}_{s,i}}{dt^4}\,\frac{3\Delta T^6}{160}$$

12.3 RUNGE-KUTTA INTEGRATION OF ACCELEROMETER DATA

Runge-Kutta integrators require the value of $\underline{a}_s(t)$ at various times within the integration interval from t_i to t_{i+1}. The equations below supply this information. Note that Runge-Kutta integration of accelerometer data is generally preferable to the previous method since the Runge-Kutta method easily accounts for the higher order terms, generally left out in the previous method (Eqs. (124) and (125)).



$$\hat{\underline{a}}_s(t_i + \delta\Delta T) = -\frac{1}{\Delta T}\underline{v}_i + \frac{1}{\Delta T}\underline{v}_{i+1}$$

$$\hat{\underline{a}}_s - \underline{a}_s \approx \dot{\underline{a}}_{s,i}\Delta T\left(\frac{1}{2} - \delta\right)$$

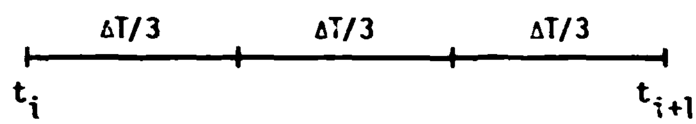
$\hat{\underline{a}}_s$ For 1st Order R-K Integration

$$\hat{\underline{a}}_s(t_i + \delta\Delta T) = A_0\underline{v}_i + A_1\underline{v}_{i+.5} + A_2\underline{v}_{i+1}$$

$$A_1 = \frac{-8\delta+4}{\Delta T} \qquad A_2 = \frac{4\delta-1}{\Delta T} \qquad A_0 = -A_1 - A_2$$

$$\hat{\underline{a}}_s - \underline{a}_s \approx \ddot{\underline{a}}_{s,i}\frac{\Delta T^2}{12}(-6\delta^2 + 6\delta - 1)$$

$\hat{\underline{a}}_s$ For 2nd Order R-K Integration



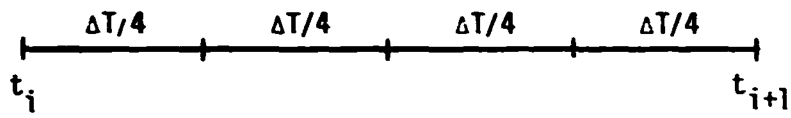
$$\underline{\hat{a}}_s(t_i + \delta\Delta T) = A_o\underline{v}_i + A_1\underline{v}_{i+1/3} + A_2\underline{v}_{i+2/3} + A_3\underline{v}_{i+1}$$

$$A_1 = \frac{81\delta^2 - 90\delta + 18}{2\Delta T} \qquad A_2 = \frac{-81\delta^2 + 72\delta - 9}{2\Delta T}$$

$$A_3 = \frac{27\delta^2 - 18\delta + 2}{2\Delta T} \qquad A_o = -A_1 - A_2 - A_3$$

$$\underline{\hat{a}}_s - \underline{a}_s \approx \underline{\dddot{a}}_{s,i} \frac{\Delta T^3}{108}(-18\delta^3 + 27\delta^2 - 11\delta + 1)$$

$\underline{\hat{a}}_3$ For 3rd Order R-K Integration



$$\hat{\underline{a}}_s(t_i + \delta\Delta T) = A_o\underline{v}_i + A_1\underline{v}_{i+.25} + A_2\underline{v}_{i+.5} + A_3\underline{v}_{i+.75} + A_4\underline{v}_{i+1}$$

$$A_1 = \frac{-512\delta^3 + 864\delta^2 - 416\delta + 48}{3\Delta T}$$

$$A_2 = \frac{768\delta^3 - 1152\delta^2 + 456\delta - 36}{3\Delta T}$$

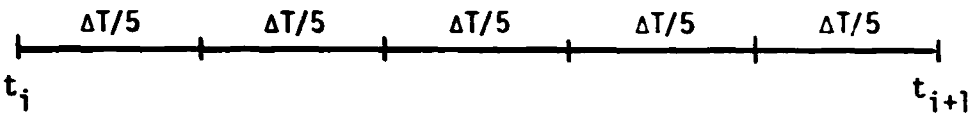
$$A_3 = \frac{-512\delta^3 + 672\delta^2 - 224\delta + 16}{3\Delta T}$$

$$A_4 = \frac{128\delta^3 - 144\delta^2 + 44\delta - 3}{3\Delta T}$$

$$A_o = -A_1 - A_2 - A_3 - A_4$$

$$\hat{\underline{a}}_s - \underline{a}_s \approx \overset{.....}{\underline{a}}_{s,i} \frac{\Delta T^4}{3840} (-160\delta^4 + 320\delta^3 - 210\delta^2 + 50\delta - 3)$$

$\hat{\underline{a}}_s$ For 4th Order R-K Integration



$$\hat{\underline{a}}_s(t_i + \delta\Delta T) = A_o\underline{v}_i + A_1\underline{v}_{i+.2} + A_2\underline{v}_{i+.4} + A_3\underline{v}_{i+.6} + A_4\underline{v}_{i+.8} + A_5\underline{v}_{i+1}$$

$$A_1 = \frac{15\,625\delta^4 - 35\,000\delta^3 + 26\,625\delta^2 - 7700\delta + 600}{24\Delta T}$$

$$A_2 = \frac{-31\,250\delta^4 + 65\,000\delta^3 - 44\,250\delta^2 + 10\,700\delta - 600}{24\Delta T}$$

$$A_3 = \frac{31\,250\delta^4 - 60\,000\delta^3 + 36\,750\delta^2 - 7800\delta + 400}{24\Delta T}$$

$$A_4 = \frac{-15\,625\delta^4 + 27\,500\delta^3 - 15\,375\delta^2 + 3050\delta - 150}{24\Delta T}$$

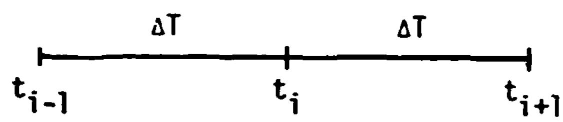
$$A_5 = \frac{3125\delta^4 - 5000\delta^3 + 2625\delta^2 - 500\delta + 24}{24\Delta T}$$

$$A_o = -A_1 - A_2 - A_3 - A_4 - A_5$$

$$\hat{\underline{a}}_s - \underline{a}_s \approx \frac{d^5\underline{a}_{s,i}}{dt^5}\,\frac{\Delta T^5}{450\,000}\,(-3750\delta^5 + 9375\delta^4 - 8500\delta^3 + 3375\delta^2 - 548\delta + 24)$$

$\hat{\underline{a}}_s$ For 5th Order R-K Integration

Note that any of the previous approximations of $\underline{a}_s$ may be used with any lower or higher order integrator. The frequency with which the accelerometers are read out may determine the approximation that is used. The approximations of $\underline{a}_s$ shown above are much preferred to those shown below because of such things as engine cutoff in the middle of an integration step.



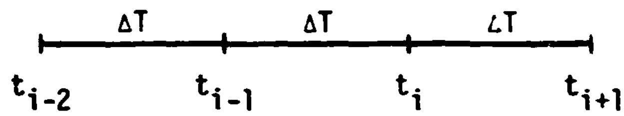
$$\hat{\underline{a}}_s(t_i + \delta\Delta T) = A_{-1}\underline{v}_{i-1} + A_0\underline{v}_i + A_1\underline{v}_{i+1}$$

$$A_{-1} = \frac{2\delta - 1}{2\Delta T} \qquad A_1 = \frac{2\delta + 1}{2\Delta T}$$

$$A_0 = -A_{-1} - A_1$$

$$\hat{\underline{a}}_s - \underline{a}_s \approx \ddot{\underline{a}}_{s,i}\,\frac{\Delta T^2}{6}\,(-3\delta^2 + 1)$$

$\hat{\underline{a}}_s$ for 2nd Order R-K Integration
(Accelerometer Sample Interval = ΔT)



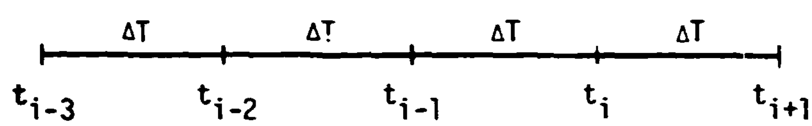
$$\hat{\underline{a}}_s(t_i + \delta\Delta T) = A_{-2}\underline{v}_{i-2} + A_{-1}\underline{v}_{i-1} + A_o\underline{v}_i + A_1\underline{v}_{i+1}$$

$$A_{-2} = \frac{-3\delta^2 + 1}{6\Delta T} \qquad A_{-1} = \frac{12\delta^2 + 6\delta - 9}{6\Delta T}$$

$$A_1 = \frac{6\delta^2 + 6\delta - 1}{6\Delta T} \qquad A_o = -A_{-2} - A_{-1} - A_1$$

$$\hat{\underline{a}}_s - \underline{a}_s \approx \dddot{\underline{a}}_{s,i}\, \frac{\Delta T^3}{24} (-4\delta^3 - 5\delta^2 + 2\delta + 1)$$

$\hat{\underline{a}}_s$ for 3rd Order R-K Integration
(Accelerometer Sample Interval = ΔT)



$$\hat{\underline{a}}_s(t_i + \delta\Delta T) = A_{-3}\underline{v}_{i-3} + A_{-2}\underline{v}_{i-2} + A_{-1}\underline{v}_{i-1} + A_o\underline{v}_i + A_1\underline{v}_{i+1}$$

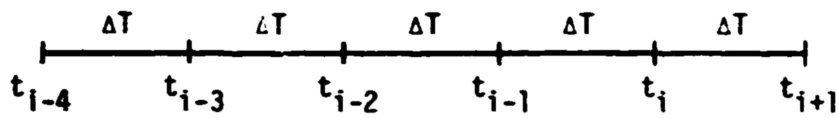
$$A_{-3} = \frac{2\delta^3 + 3\delta^2 - \delta - 1}{12\Delta T} \qquad A_{-2} = \frac{-8\delta^3 - 18\delta^2 + 4\delta + 6}{12\Delta T}$$

$$A_{-1} = \frac{12\delta^3 + 36\delta^2 + 6\delta - 18}{12\Delta T} \qquad A_1 = \frac{2\delta^3 + 9\delta^2 + 11\delta + 3}{12\Delta T}$$

$$A_o = -A_{-3} - A_{-2} - A_{-1} - A_1$$

$$\hat{\underline{a}}_s - \underline{a}_s \approx \ddddot{\underline{a}}_{s,i}\, \frac{\Delta T^4}{360} (-15\delta^4 - 80\delta^3 - 45\delta^2 + 30\delta + 18)$$

$\hat{\underline{a}}_s$ For 4th Order R-K Integration
(Accelerometer Sample Interval = ΔT)



$$\underline{\hat{a}}_s(t_i + \delta\Delta T) = A_{-4}\underline{v}_{i-4} + A_{-3}\underline{v}_{i-3} + A_{-2}\underline{v}_{i-2} + A_{-1}\underline{v}_{i-1} + A_0\underline{v}_i + A_1\underline{v}_{i+1}$$

$$A_{-4} = \frac{-5\delta^4 - 20\delta^3 - 15\delta^2 + 10\delta + 6}{120\Delta T}$$

$$A_{-3} = \frac{25\delta^4 + 120\delta^3 + 105\delta^2 - 60\delta - 40}{120\Delta T}$$

$$A_{-2} = \frac{-50\delta^4 - 280\delta^3 - 330\delta^2 + 140\delta + 120}{120\Delta T}$$

$$A_{-1} = \frac{50\delta^4 + 320\delta^3 + 510\delta^2 - 40\delta - 240}{120\Delta T}$$

$$A_1 = \frac{5\delta^4 + 40\delta^3 + 105\delta^2 + 100\delta + 24}{120\Delta T}$$

$$A_0 = -A_{-4} - A_{-3} - A_{-2} - A_{-1} - A_1$$

$$\underline{\hat{a}}_s - \underline{a}_s \approx \frac{d^5\underline{a}_{s,i}}{dt^5}\frac{\Delta T^5}{720}(-6\delta^5 - 45\delta^4 - 100\delta^3 - 45\delta^2 + 52\delta + 24)$$

$\underline{\hat{a}}_s$ For 5th Order R-K Integration
(Accelerometer Sample Interval = ΔT)

APPENDIX A

THE $\underline{k}$ EXPANSIONS

Consider the vector differential equation, $\dot{\underline{x}} = \underline{f}(\underline{x})$. Let f^i be the i^{th} element of $\underline{f}$. Let f^i_j be the partial derivative of f^i with respect to the j^{th} element of $\underline{x}$. Let

$$A^i = f^i \qquad B^i = f^i_j f^j \qquad C^i = f^i_{jk} f^j f^k \qquad D^i = f^i_j f^j_k f^k$$

$$E^i = f^i_{jk\ell} f^j f^k f^\ell \qquad F^i = f^i_{jk} f^j_\ell f^k f^\ell \qquad G^i = f^i_j f^j_{k\ell} f^k f^\ell$$

$$H^i = f^i_j f^j_k f^k_\ell f^\ell \qquad I^i = f^i_{jk\ell m} f^j f^k f^\ell f^m$$

$$J^i = f^i_{jk\ell} f^j_m f^k f^\ell f^m \qquad K^i = f^i_{jk} f^j_{\ell m} f^k f^\ell f^m$$

$$L^i = f^i_{jk} f^j_\ell f^k_m f^\ell f^m \qquad M^i = f^i_{jk} f^j_\ell f^\ell_m f^k f^m$$

$$N^i = f^i_j f^j_{k\ell m} f^k f^\ell f^m \qquad P^i = f^i_j f^j_{k\ell} f^k_m f^\ell f^m$$

$$Q^i = f^i_j f^j_k f^k_{\ell m} f^\ell f^m \qquad R^i = f^i_j f^j_k f^k_\ell f^\ell_m f^m$$

where the repeated indices indicate summation.

Let the $\underline{A}$ vector be composed of the elements A^i; let the $\underline{B}$ vector be composed of the elements B^i, etc. Then it can be shown that

$$\dot{\underline{x}} = \underline{A}$$

$$\ddot{\underline{x}} = \underline{B}$$

$$\dddot{\underline{x}} = \underline{C} + \underline{D}$$

$$\ddddot{\underline{x}} = \underline{E} + 3\underline{F} + \underline{G} + \underline{H}$$

$$\overset{\cdot\cdot\cdot\cdot\cdot}{\underline{x}} = \underline{I} + 6\underline{J} + 4\underline{K} + 3\underline{L} + 4\underline{M} + \underline{N} + 3\underline{P} + \underline{Q} + \underline{R}$$

We also note that

$$f^i(\underline{x} + \underline{\varepsilon}) = f^i + f^i_j \varepsilon^j + \frac{1}{2!} f^i_{jk} \varepsilon^j \varepsilon^k + \frac{1}{3!} f^i_{jk\ell} \varepsilon^j \varepsilon^k \varepsilon^\ell + \cdots$$

Let*

$$\delta_2 = a_1$$

$$\delta_3 = b_1 + b_2$$

$$\delta_4 = c_1 + c_2 + c_3$$

$$\delta_5 = d_1 + d_2 + d_3 + d_4$$

$$\delta_6 = e_1 + e_2 + e_3 + e_4 + e_5$$

* If t is an element of $\underline{x}$, then t will get "bumped" by $a_1 \Delta T$, $(b_1+b_2)\Delta T$, $(c_1+c_2+c_3)\Delta T$, etc.

The fifth-order expansions of the $\underline{k}$'s are now shown below.

$$\underline{k}_1 = \Delta T \underline{f}(\underline{x}) = \Delta T \underline{A}$$

$$\underline{k}_2 = \Delta T \underline{f}(\underline{x} + a_1 \underline{k}_1) = \Delta T \underline{A} + \Delta T^2 \delta_2 \underline{B} + \Delta T^3 \frac{1}{2} \delta_2^2 \underline{C} + \Delta T^4 \frac{1}{6} \delta_2^3 \underline{E} + \Delta T^5 \frac{1}{24} \delta_2^4 \underline{I}$$

$$\underline{k}_3 = \Delta T \underline{f}(\underline{x} + b_1 \underline{k}_1 + b_2 \underline{k}_2) = \Delta T \underline{A} + \Delta T^2 \delta_3 \underline{B} + \Delta T^3 \left[\frac{1}{2} \delta_3^2 \underline{C} + b_2 \delta_2 \underline{D} \right] + \Delta T^4 \left[\frac{1}{6} \delta_3^3 \underline{E} + b_2 \delta_2 \delta_3 \underline{F} + \frac{1}{2} b_2 \delta_2^2 \underline{G} \right] + \Delta T^5 \left[\frac{1}{24} \delta_3^4 \underline{I} + \frac{1}{2} b_2 \delta_2 \delta_3^2 \underline{J} + \frac{1}{2} b_2 \delta_2^2 \delta_3 \underline{K} + \frac{1}{2} b_2^2 \delta_2^2 \underline{L} + \frac{1}{6} b_2 \delta_2^3 \underline{N} \right]$$

$$\underline{k}_4 = \Delta T \underline{f}(\underline{x} + c_1 \underline{k}_1 + c_2 \underline{k}_2 + c_3 \underline{k}_3) = \Delta T \underline{A} + \Delta T^2 \delta_4 \underline{B} + \Delta T^3 \left[\frac{1}{2} \delta_4^2 \underline{C} + (c_2 \delta_2 + c_3 \delta_3) \underline{D} \right] + \Delta T^4 \left[\frac{1}{6} \delta_4^3 \underline{E} + (c_2 \delta_2 + c_3 \delta_3) \delta_4 \underline{F} + \frac{1}{2} (c_2 \delta_2^2 + c_3 \delta_3^2) \underline{G} + b_2 \delta_2 c_3 \underline{H} \right] + \Delta T^5 \left[\frac{1}{24} \delta_4^4 \underline{I} + \frac{1}{2} (c_2 \delta_2 + c_3 \delta_3) \delta_4^2 \underline{J} + \frac{1}{2} (c_2 \delta_2^2 + c_3 \delta_3^2) \delta_4 \underline{K} + \frac{1}{2} (c_2 \delta_2 + c_3 \delta_3)^2 \underline{L} + b_2 \delta_2 c_3 \delta_4 \underline{M} + \frac{1}{6} (c_2 \delta_2^3 + c_3 \delta_3^3) \underline{N} + b_2 \delta_2 c_3 \delta_3 \underline{P} + \frac{1}{2} b_2 \delta_2^2 c_3 \underline{Q} \right]$$

$$
\begin{aligned}
\underline{k}_5 &= \Delta T \underline{f}(\underline{x} + d_1\underline{k}_1 + d_2\underline{k}_2 + d_3\underline{k}_3 + d_4\underline{k}_4) = \Delta T \underline{A} + \Delta T^2 \delta_5 \underline{B} + \Delta T^3 \Big[\tfrac{1}{2}\, \delta_5^2 \underline{C} \\
&\quad + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)\underline{D} \Big] + \Delta T^4 \Big[\tfrac{1}{6}\, \delta_5^3 \underline{E} + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)\delta_5 \underline{F} \\
&\quad + \tfrac{1}{2}(d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)\underline{G} + \{b_2\delta_2 d_3 + (c_2\delta_2 + c_3\delta_3)d_4\}\underline{H} \Big] \\
&\quad + \Delta T^5 \Big[\tfrac{1}{24}\, \delta_5^4 \underline{I} + \tfrac{1}{2}(d_2\delta_2 + d_3\delta_3 + d_4\delta_4)\delta_5^2 \underline{J} + \tfrac{1}{2}(d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)\delta_5 \underline{K} \\
&\quad + \tfrac{1}{2}(d_2\delta_2 + d_3\delta_3 + d_4\delta_4)^2 \underline{L} + \{b_2\delta_2 d_3 + (c_2\delta_2 + c_3\delta_3)d_4\}\delta_5 \underline{M} \\
&\quad + \tfrac{1}{6}(d_2\delta_2^3 + d_3\delta_3^3 + d_4\delta_4^3)\underline{N} + \{b_2\delta_2 d_3\delta_3 + (c_2\delta_2 + c_3\delta_3)d_4\delta_4\}\underline{P} \\
&\quad + \tfrac{1}{2}\, \{b_2\delta_2^2 d_3 + (c_2\delta_2^2 + c_3\delta_3^2)d_4\}\underline{Q} + b_2\delta_2 c_3 d_4 \underline{R} \Big]
\end{aligned}
$$

$$
\begin{aligned}
\underline{k}_6 &= \Delta T \underline{f}(\underline{x} + e_1\underline{k}_1 + e_2\underline{k}_2 + e_3\underline{k}_3 + e_4\underline{k}_4 + e_5\underline{k}_5) = \Delta T\underline{A} + \Delta T^2\delta_6\underline{B} \\
&+ \Delta T^3\left[\frac{1}{2}\delta_6^2\underline{C} + (e_2\delta_2 + e_3\delta_3 + e_4\delta_4 + e_5\delta_5)\underline{D}\right] + \Delta T^4\left[\frac{1}{6}\delta_6^3\underline{E}\right. \\
&+ (e_2\delta_2 + e_3\delta_3 + e_4\delta_4 + e_5\delta_5)\delta_6\underline{F} + \frac{1}{2}(e_2\delta_2^2 + e_3\delta_3^2 + e_4\delta_4^2 + e_5\delta_5^2)\underline{G} \\
&\left.+ \{b_2\delta_2e_3 + (c_2\delta_2 + c_3\delta_3)e_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5\}\underline{H}\right] \\
&+ \Delta T^5\left[\frac{1}{24}\delta_6^4\underline{I} + \frac{1}{2}(e_2\delta_2 + e_3\delta_3 + e_4\delta_4 + e_5\delta_5)\delta_6^2\underline{J}\right. \\
&+ \frac{1}{2}(e_2\delta_2^2 + e_3\delta_3^2 + e_4\delta_4^2 + e_5\delta_5^2)\delta_6\underline{K} + \frac{1}{2}(e_2\delta_2 + e_3\delta_3 + e_4\delta_4 + e_5\delta_5)^2\underline{L} \\
&+ \{b_2\delta_2e_3 + (c_2\delta_2 + c_3\delta_3)e_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5\}\delta_6\underline{M} \\
&+ \frac{1}{6}(e_2\delta_2^3 + e_3\delta_3^3 + e_4\delta_4^3 + e_5\delta_5^3)\underline{N} + \{b_2\delta_2e_3\delta_3 + (c_2\delta_2 + c_3\delta_3)e_4\delta_4 \\
&+ (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5\delta_5\}\underline{P} + \frac{1}{2}\{b_2\delta_2^2e_3 + (c_2\delta_2^2 + c_3\delta_3^2)e_4 \\
&\left.+ (d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)e_5\}\underline{Q} + \{b_2\delta_2c_3e_4 + b_2\delta_2d_3e_5 + (c_2\delta_2 + c_3\delta_3)d_4e_5\}\underline{R}\right]
\end{aligned}
$$

APPENDIX B

A FIFTH ORDER R-K INTEGRATOR WITH FIVE DERIVATIVE EVALUATIONS?

Since one infrequently hears about al˘ ned fifth order Runge-Kutta integrators using five derivative evaluations, ı appeared worthwhile to try and show that no such solution exists for the vector differential equation $\dot{\underline{x}} = \underline{f}(\underline{x})$ or $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$.

The integrator constant constraint equations are as shown below.

$$\delta_1 = 0 \tag{B-1}$$

$$a_1 = \delta_2 \tag{B-2}$$

$$b_1 + b_2 = \delta_3 \tag{B-3}$$

$$c_1 + c_2 + c_3 = \delta_4 \tag{B-4}$$

$$d_1 + d_2 + d_3 + d_4 = \delta_5 \tag{B-5}$$

$$e_1 + e_2 + e_3 + e_4 + e_5 = 1 \tag{B-6}$$

$$e_2\delta_2 + e_3\delta_3 + e_4\delta_4 + e_5\delta_5 = 1/2 \tag{B-7}$$

$$e_2\delta_2^2 + e_3\delta_3^2 + e_4\delta_4^2 + e_5\delta_5^2 = 1/3 \tag{B-8}$$

$$e_2\delta_2^3 + e_3\delta_3^3 + e_4\delta_4^3 + e_5\delta_5^3 = 1/4 \tag{B-9}$$

$$e_2\delta_2^4 + e_3\delta_3^4 + e_4\delta_4^4 + e_5\delta_5^4 = 1/5 \tag{B-10}$$

$$b_2\delta_2 e_3 + (c_2\delta_2 + c_3\delta_3)e_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5 = 1/6 \qquad \text{(B-11)}$$

$$b_2\delta_2 e_3\delta_3 + (c_2\delta_2 + c_3\delta_3)e_4\delta_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5\delta_5 = 1/8 \qquad \text{(B-12)}$$

$$b_2\delta_2 e_3\delta_3^2 + (c_2\delta_2 + c_3\delta_3)e_4\delta_4^2 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)e_5\delta_5^2 = 1/10 \qquad \text{(B-13)}$$

$$(b_2\delta_2)^2 e_3 + (c_2\delta_2 + c_3\delta_3)^2 e_4 + (d_2\delta_2 + d_3\delta_3 + d_4\delta_4)^2 e_5 = 1/20 \qquad \text{(B-14)}$$

$$b_2\delta_2^2 e_3 + (c_2\delta_2^2 + c_3\delta_3^2)e_4 + (d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)e_5 = 1/12 \qquad \text{(B-15)}$$

$$b_2\delta_2^2 e_3\delta_3 + (c_2\delta_2^2 + c_3\delta_3^2)e_4\delta_4 + (d_2\delta_2^2 + d_3\delta_3^2 + d_4\delta_4^2)e_5\delta_5 \text{ [illegible]} \qquad \text{(B-16)}$$

$$b_2\delta_2^3 e_3 + (c_2\delta_2^3 + c_3\delta_3^3)e_4 + (d_2\delta_2^3 + d_3\delta_3^3 + d_4\delta_4^3)e_5 = 1/\text{[illegible]} \qquad \text{(B-17)}$$

$$b_2\delta_2 c_3 e_4\delta_4 + [b_2\delta_2 d_3 + (c_2\delta_2 + c_3\delta_3)d_4]e_5\delta_5 = 1/30 \qquad \text{(B-18)}$$

$$b_2\delta_2 c_3 e_4 + [b_2\delta_2 d_3 + (c_2\delta_2 + c_3\delta_3)d_4]e_5 = 1/24 \qquad \text{(B-19)}$$

$$b_2\delta_2 c_3 e_4\delta_3 + [b_2\delta_2 d_3\delta_3 + (c_2\delta_2 + c_3\delta_3)d_4\delta_4]e_5 = 1/40 \qquad \text{(B-20)}$$

$$b_2\delta_2^2 c_3 e_4 + [b_2\delta_2^2 d_3 + (c_2\delta_2^2 + c_3\delta_3^2)d_4]e_5 = 1/60 \qquad \text{(B-21)}$$

$$b_2\delta_2 c_3 d_4 e_5 = 1/120 \qquad \text{(B-22)}$$

22 equations in 20 unknowns. Solving Eqs. (B-7) through (B-10) gives

$$e_2 = \frac{(\delta_4 + \delta_5)(20\delta_3 - 15) + \delta_4\delta_5(20 - 30\delta_3) + 12 - 15\delta_3}{60\delta_2(\delta_2 - \delta_3)(\delta_2 - \delta_4)(\delta_2 - \delta_5)} \quad \text{(B-23)}$$

$$e_3 = \frac{(\delta_4 + \delta_5)(20\delta_2 - 15) + \delta_4\delta_5(20 - 30\delta_2) + 12 - 15\delta_2}{60\delta_3(\delta_3 - \delta_2)(\delta_3 - \delta_4)(\delta_3 - \delta_5)} \quad \text{(B-24)}$$

$$e_4 = \frac{(\delta_2 + \delta_3)(20\delta_5 - 15) + \delta_2\delta_3(20 - 30\delta_5) + 12 - 15\delta_5}{60\delta_4(\delta_4 - \delta_5)(\delta_4 - \delta_3)(\delta_4 - \delta_2)} \quad \text{(B-25)}$$

$$e_5 = \frac{(\delta_2 + \delta_3)(20\delta_4 - 15) + \delta_2\delta_3(20 - 30\delta_4) + 12 - 15\delta_4}{60\delta_5(\delta_5 - \delta_4)(\delta_5 - \delta_3)(\delta_5 - \delta_2)} \quad \text{(B-26)}$$

Equations (B-11) and (B-12) give

$$b_2\delta_2e_3(\delta_3 - \delta_5) + (c_2\delta_2 + c_3\delta_3)e_4(\delta_4 - \delta_5) = \frac{3-4\delta_5}{24} \quad \text{(B-27)}$$

Equations (B-12) and (B-13) give

$$b_2\delta_2e_3\delta_3(\delta_3 - \delta_5) + (c_2\delta_2 + c_3\delta_3)e_4\delta_4(\delta_4 - \delta_5) = \frac{4-5\delta_5}{40} \quad \text{(B-28)}$$

The above two equations yield

$$b_2\delta_2 = \frac{-15(\delta_4 + \delta_5) + 20\,\delta_4\delta_5 + 12}{120e_3(\delta_3 - \delta_4)(\delta_3 - \delta_5)} \qquad \text{(B-29)}$$

Substituting Eq. (B-24), for e_3, into Eq. (B-29) gives

$$b_2\delta_2 = \frac{\delta_3(\delta_3 - \delta_2)[-15(\delta_4 + \delta_5) + 20\,\delta_4\delta_5 + 12]}{(40\delta_2 - 30)(\delta_4 + \delta_5) + (40-60\delta_2)\,\delta_4\delta_5 + 24 - 30\delta_2} \qquad \text{(B-30)}$$

Now Eqs. (B-19) and (B-21) yield

$$c_3d_4e_5 = \frac{1}{120\delta_3}\,\frac{2 - 5\delta_2}{\delta_3 - \delta_2} \qquad \text{(B-31)}$$

But from Eq. (B-22)

$$c_3d_4e_5 = \frac{1}{120}\,\frac{1}{b_2\delta_2} \qquad \text{(B-32)}$$

Setting Eq. (B-31) equal to Eq. (B-32) gives

$$b_2\delta_2 = \frac{\delta_3(\delta_3 - \delta_2)}{2 - 5\delta_2} \qquad \text{(B-33)}$$

Setting Eq. (B-30) equal to Eq. (B-33) yields

$$\delta_2[(7 - 8\delta_4)\delta_5 + 7\delta_4 - 6] = 0$$

But from Eq. (B-22), $\delta_2 \neq 0$, therefore

$$\delta_5 = \frac{7\delta_4 - 6}{8\delta_4 - 7} \tag{B-34}$$

Now Eqs. (B-19) and (B-20) give

$$c_3e_4 + d_3e_5 = \frac{1}{120\ b_2\delta_2}\ \frac{3 - 5\delta_4}{\delta_3 - \delta_4} \tag{B-35}$$

Substituting Eq. (B-33) into the above yields

$$c_3e_4 + d_3e_5 = \frac{(2 - 5\delta_2)(3 - 5\delta_4)}{120\ \delta_3(\delta_3 - \delta_2)(\delta_3 - \delta_4)} \tag{B-36}$$

Equations (B-17) and (B-15) give

$$c_3\delta_3^2(\delta_3 - \delta_2)e_4 + d_3\delta_3^2(\delta_3 - \delta_2)e_5 + d_4\delta_4^2(\delta_4 - \delta_2)e_5 = \frac{3 - 5\delta_2}{60} \tag{B-37}$$

Equations (B-17) and (B-11) give

$$c_3\delta_3(\delta_3^2 - \delta_2^2)e_4 + d_3\delta_3(\delta_3^2 - \delta_2^2)e_5 + d_4\delta_4(\delta_4^2 - \delta_2^2)e_5 = \frac{3 - 10\delta_2^2}{60} \tag{B-38}$$

Combining the above two equations yields

$$c_3e_4 + d_3e_5 = \frac{-10\delta_4 - 10\delta_2 + 20\delta_2\delta_4 + 6}{120\delta_3(\delta_3 - \delta_2)(\delta_3 - \delta_4)} \qquad \text{(B-39)}$$

Setting this equation equal to Eq. (B-36) results in

$$\delta_2(\delta_4 - 1) = 0$$

But from Eq. (B-22) we see that $\delta_2 \neq 0$. Therefore

$$\delta_4 = 1 \qquad \text{(B-40)}$$

Substituting this into Eq. (B-34) yields

$$\delta_5 = 1 \qquad \text{(B-41)}$$

Now comparing Eq. (B-18) with (B-19), we see that $\delta_4 = \delta_5 = 1$ can not be a solution. Thus we have a contradiction, and there appears to be no valid solution. Also in going back to investigate the singular points of the various equations, I still find contradictions. And subjecting the above equations to two different least-squares, computer solutions, I find no exact solutions. Thus it appears that one more derivative evaluation is needed for a fifth order Runge-Kutta type of integrator for the vector differential equation, $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$.