Difference Equation for Superradiance*

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We study the evolution of a completely excited system of N two-level atoms, distributed over a large region, interacting with all modes of radiation field. We pay special attention to the distinction between r-conserving (RC) and r-nonconserving (RNC) processes. Considering the number of photons emitted as the discrete independent variable, the evolution is described by a partial difference equation. Numerical solution of this equation shows the transition from RNC dominance at the beginning to RC dominance later. This is also a transition from incoherent to coherent emission of radiation.

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In 1954 R. H. Dicke[1] discussed the spontaneous emission of radiation from an excited system of N identical two-level atoms. By considering the entire collection of atoms as a single quantum mechanical system, he found that under certain conditions the individual atoms cooperate to emit radiation at a rate proportional to N^2 which is much greater than their incoherent emission rate. This phenomenon is called superradiance. The quantum number that plays the central role in the description of such system is the so-called cooperation number r which can be any integer between 0 and N/2, assuming N to be an even number.

In the vast literature on superradiance[2], most of the investigation have been limited to cases in which the emiting atoms either are confined to a region smaller than the wavelength of emitted radiation or are able to couple with only one radiation mode. In these situations r is conserved; which permits important mathematical simplifications. Current attention in this field is being focused on the more difficult, but more realistic, problem of N two-level atoms, distributed over a space of dimension greater than the radiation wavelength, interacting with all the modes of radiation field.[3] In this situation r is no longer conserved.[4] However, the effects of r-nonconserving processes have not received sufficient attention in the literature.

Recently, we[5] have developed a simple and systematic diagrammatic technique for calculating the matrix elements of collective operators; which gives the transition probabilities of both r-conserving (RC) and r-nonconserving (RNC) processes as follows:

$$T^{RC}_{\vec{k}}(r,m) \propto (r+m)(r-m+1)\ \Gamma(\vec{k}-\vec{k}_1) \quad , \tag{1}$$

$$T^{RNC}_{\vec{k}}(r,m) \propto \frac{(r+m)(r+m-1)}{2r}\ [1 - \Gamma(\vec{k}-\vec{k}_1)] \quad , \tag{2}$$

where $\vec{k}$ is the wavevector of the photon emitted and $\vec{k}_1$ is a specific mode which may be the wavevector of the exciting pulse or the end-fire mode; m is 1/2 of the difference of the numbers of atoms in the excited state and the ground state. $\Gamma(\vec{k}-\vec{k}_1)$ is independent of r as long as r is very close to N/2; but it depends on the geometry of the sample. For a circular cylinder of length ℓ and radius a, it is known to be

$$\Gamma(\vec{k}-\vec{k}_1) = \frac{\sin^2[(\vec{k}-\vec{k}_1)_z \ell/2]}{[(\vec{k}-\vec{k}_1)_z \ell/2]^2} \cdot \frac{4J_1^2[(\vec{k}-\vec{k}_1)_\rho a]}{[(\vec{k}-\vec{k}_1)_\rho a]^2} \quad , \tag{3}$$

where $(\vec{k}-\vec{k}_1)_{z,\rho}$ are the transversal and axial components of $(\vec{k}-\vec{k}_1)$ and J_1 is the Bessel function of order 1. Rehler and Eberly[6] have carried out the numerical calculation of $\Gamma(\vec{k}-\vec{k}_1)$ which shows a very sharp peak along the direction of $\vec{k}_1$. This implies that a photon emitted through RC process will be predominantly along the direction of $\vec{k}_1$; while RNC process will give radiation along all other random directions.

When (1) and (2) are integrated over all possible directions, we obtain

$$T^{RC}(r,m) \propto (r+m)(r-m+1)\mu \quad , \tag{4}$$

$$T^{RNC}(r,m) \propto [(r+m)(r+m-1)/2r](1-\mu) \quad , \tag{5}$$

where μ has the same meaning as given by Ref. 6 and can have values much smaller than 1.

Suppose we start with a completely excited system with $r = m = N/2$. Then at the beginning photons will be emitted predominantly through RNC processes and, therefore, along random directions. For each photon emitted through RNC process, both r and m decrease by 1; which does not improve the chance of RC processes. On the other hand, for each photon emitted through RC process, r remains the same while m decreases by 1; which will improve the chance of RC processes. Hence, the more photons emitted along $\vec{k}_1$ direction, the higher the probability of additional photons emitted along this direction. The RC and RNC processes will be about equally important when $(r-m)\mu$ is of the order of 1.

In this article, we are most interested in studying the transition from RNC dominance to RC dominance. To study the evolution of a system of N two-level atoms from a completely excited state, we will replace the time as the continuos independent variable by the number of photons emitted as the discrete independent variable. Then our "dynamic equation" will be a difference equation.

Let $P(r,m)$ be the probability that our system has cooperation number r after $N/2-m$ photons have been emitted. Then (4) and (5) gives the following partial difference equation for $P(r,m)$

$$P(r,m) = \frac{2r(r-m)\sigma}{2r(r-m)\sigma + (r+m)} P(r,m+1) + \frac{(r+m+1)}{(2r+2)(r-m+1)\sigma + (r+m+1)} P(r+1,m+1) \tag{6}$$

with the initial condition $P(N/2,N/2) = 1$ and the obvious restrictions: $0 \le r \le N/2$ and $-r \le m \le r$. We have defined $\sigma \equiv \mu/(1-\mu)$ which is essentially the same as μ when μ is much less than 1. In obtaining (6), we have ignored the possibility that a photon may be absorbed by the atomic system and the possibility that r may increase. The latter is justified in Ref. 5.

It is convenient to define two new variables

$$s \equiv N/2 - r \quad , \qquad t \equiv r - m \tag{7}$$

and let $Q(s,t) \equiv P(r,m)$. Assuming that both s and t are much less than N,

(6) becomes

$$Q(s,t) = [t\sigma/(1+t\sigma)]Q(s,t-1) + [1 + (t+1)\sigma]^{-1}Q(s-1,t) \tag{8}$$

with initial condition $Q(0,0) = 1$.

By direct observation, the solution to (8) can be seen to be

$$Q(s,t) = \left[\sum_{\beta} \prod_{i=1}^{t+1} (1+i\sigma)^{-\beta_i}\right]\left[\prod_{i=1}^{t} (1+i\sigma)^{-1}\right] t!\, \sigma^t \quad , \tag{9}$$

where $\beta_i = 0, 1, 2, \cdots$; and the summation is over all possible choices of β_i such that

$$\sum_{i=1}^{t+1} \beta_i = s \ . \tag{10}$$

The result of this summation can be expressed as a contour integral as follows:

$$Z \equiv \sum_{\beta} \prod_{i=1}^{t+1} (1 + i\sigma)^{-\beta_i}$$

$$= \frac{1}{2\pi i} \oint z^{-s-1} \prod_{i=1}^{t+1} [1 + (1+i\sigma)^{-1}z + (1+i\sigma)^{-2}z^2 + \cdots]\, dz \ . \tag{11}$$

A good approximation for Z can be obtained by integrating through a saddle point on the real axis, the so-called steepest descent method[7], as follows:

$$Z \simeq (2\pi)^{-\frac{1}{2}} A^{-s} \left[\prod_{i=1}^{t+1} \frac{(1 + i\sigma)}{(1 - A + i\sigma)}\right]\left[\sum_{i=1}^{t+1} \frac{(1 + i\sigma)A}{(1 - A + i\sigma)^2}\right]^{-\frac{1}{2}} , \tag{12}$$

where $z = A$ is the location of the saddle point to be determined by the equation

$$A \sum_{i=1}^{t+1} (1 - A + i\sigma)^{-1} = s+1 \ . \tag{13}$$

Substituting (12) into (9) and using Stirling's formula for t!, we obtain

$$Q(s,t) = A^{-s}[1+(t+1)\sigma]\left[\prod_{i=1}^{t+1}(1-A+i\sigma)^{-1}\right]\left[\sum_{i=1}^{t+1}\frac{(1+i\sigma)A}{(1-A+i\sigma)^2}\right]^{-1/2}$$

$$\times\ e^{-t}\ t^{t+1/2}\ \sigma^t \quad . \tag{14}$$

Replacing s and t by the original variables r and m, we have a good approximation for P(r,m). We want to know what cooperation number has the highest probability after a certain number of photons have been emitted; i.e., we want to locate the maximum of P(r,m) with respect to r for fixed m. We will defined

$$x \equiv (N/2 - m)\sigma \quad , \qquad y \equiv (N/2 - r)\sigma \quad , \tag{15}$$

consider x and y as continuos variables, and replace all summations encountered by integrations. Then, from (13) and (14), we have

$$A\ \ln[1 + (x-y)/(1-A)] = y \quad , \tag{16}$$

$$\ln P(y,x) = \ln(1+x-y) - (y/\sigma+1/2)\ln A + (1/2)\ln[(x-y)/\sigma]$$

$$+ [(1-A+x-y)\ln(1-A+x-y) + (1-A)\ln(1-A) + (x-y)\ln(x-y)]/\sigma$$

$$- \frac{1}{2}\ln\left\{\frac{1}{\sigma}\left[\frac{y}{A} + \frac{A}{1-A} - \frac{A}{1-A+x-y}\right] + \frac{1}{A}\right\} \quad . \tag{17}$$

Letting $\partial \ln P(y,x)/\partial y = 0$ and using (16), we obtain

$$1/(1+x-y) + 1/[2(x-y)] + (1/\sigma)\ln[A(x-y)/(1-A+x-y)]$$

$$+ (1+x-y)[A^2(x-y) - (1-A)^2(1+x-y)y]/2[A^2(x-y) + (1-A)(1-A+x-y)]^2$$

$$= 0 \ . \tag{18}$$

Simultaneous numerical solution of (16) and (18) gives y as a function of x which, in turn, gives the most probable cooperation number as a function of the number of photons emitted. The results for five different values of σ are presented in Fig. 1. A slop of 1 of these curves indicates pure RNC processes and a slop of 0 indicates pure RC processes.

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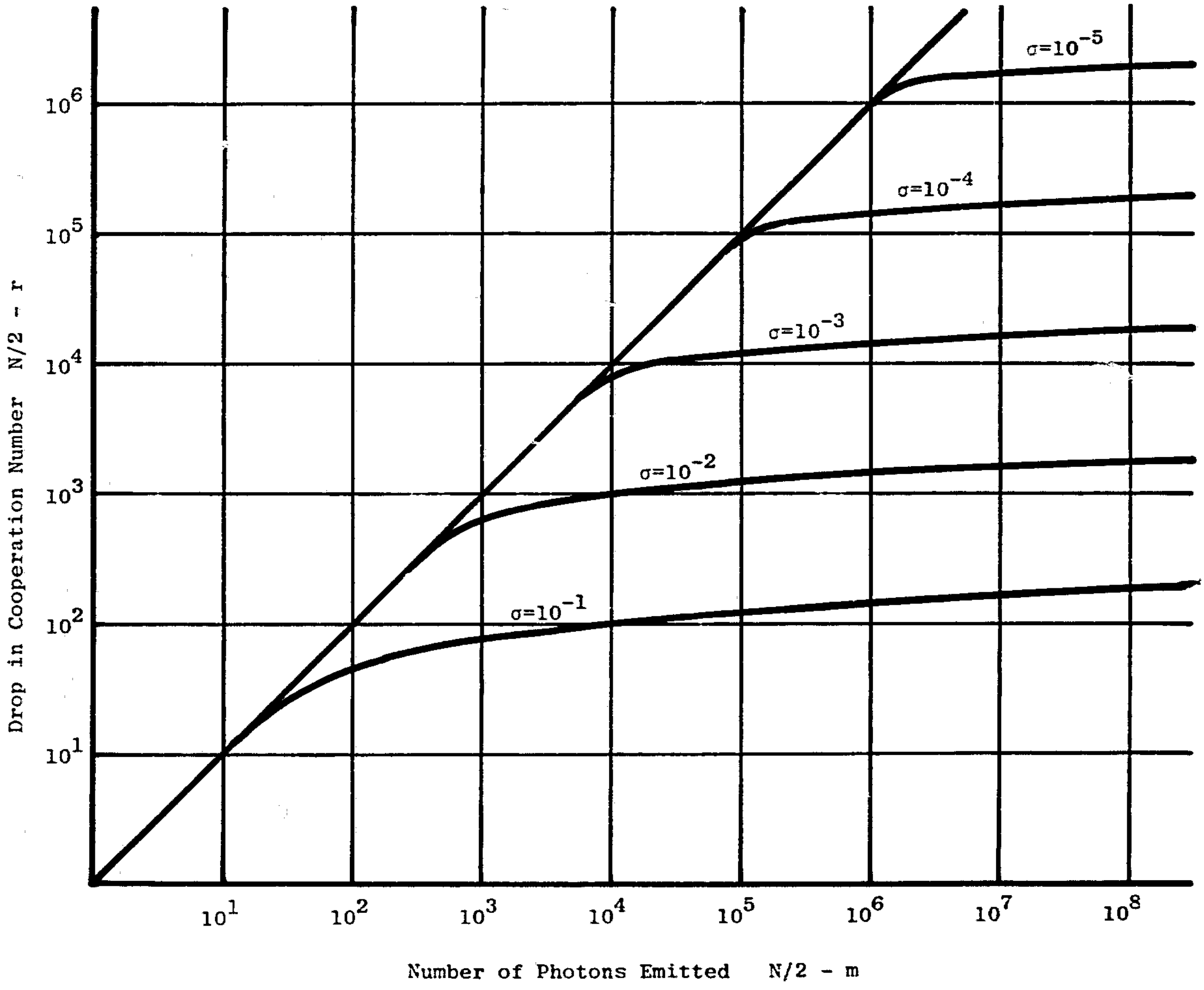
FIGURE CAPTIONS

Fig. 1. Drop in cooperation number (N/2-r) as a function of the number of photons emitted (N/2-m) with σ as a parameter.


σ=10^-5
σ=10^-4
σ=10^-3
σ=10^-2
σ=10^-1
10^6
10^5
10^4
10^3
10^2
10^1
10^1
10^2
10^3
10^4
10^5
10^6
10^7
10^8
Drop in Cooperation Number N/2 - r
Number of Photons Emitted N/2 - m