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INSTITUTE OF FRACTURE AND SOLID MECHANICS

LEHIGH UNIVERSITY

THE EFFECT OF BENDING ON THE STRESSES IN ADHESIVE JOINTS

BY

U. YUCEOGLU
D. P. UPDIKE

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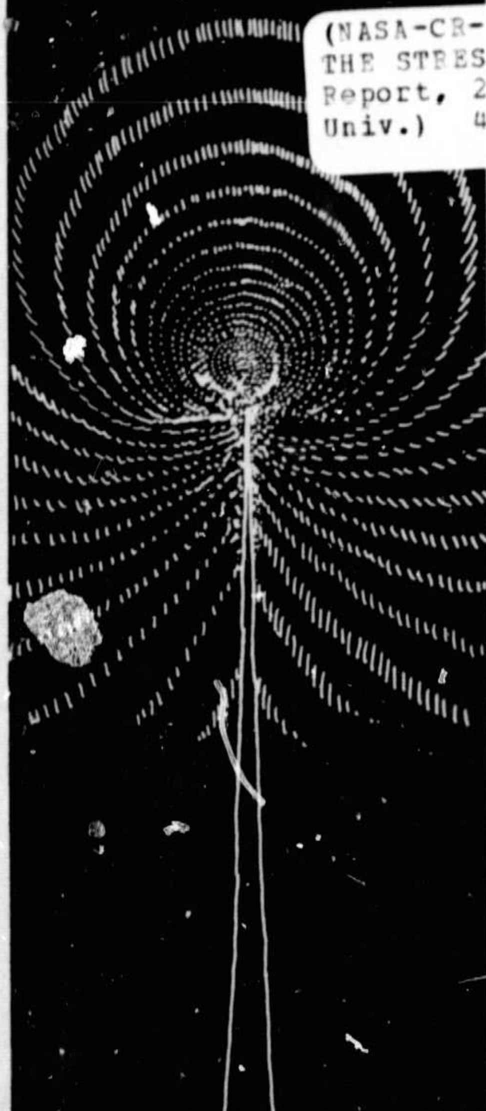

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THE EFFECT OF BENDING
ON THE STRESSES IN ADHESIVE JOINTS

by

U. Yuceoglu and D. P. Updike
Lehigh University, Bethlehem, Pa.

ABSTRACT

The problem of stress distribution in adhesive joints where two orthotropic plates are bonded through a flexible adhesive layer is analyzed. It is shown that the effect of bending of the adherends on the stresses in the adhesive layer is very significant. However, the transverse shear deformations of the adherends have in general very little influence on the adhesive layer stresses and therefore these shear strains of adherends can be neglected in many practical cases. It is shown that the maxiumum transverse normal stress in the adhesive is, in general, larger than the maximum longitudinal shear stress.

The method of solution is applied to several examples of specific joint geometries and material combinations. It is also shown that the formulation and the solution of the problem of adhesive joints as presented in this case is general enough to be applicable to other related problems such as "scarf joints", "stiffener plates", etc. in a similar fashion.

1. INTRODUCTION

The joining and extension of structural components in the form of "adhesive (or bonded) joints" has been a very common feature in all kinds of lightweight structures. In recent years, the developments in very strong epoxy based adhesives and advanced composites as well as new fabrication methods of joints have made feasible the extensive use of adhesive joints in flight vehicle structures in which lightweight and high fatigue strength are prime requirements. Consequently, considerable amount of analytical and experimental research has been carried out on the stress distribution in adhesive joints. In this connection, one may mention some early analytical work by Goland and Reissner [1], later Mylonas [2], Cornell [3], Lubkin and Demarkles [5] and more recently Erdogan and Ratwani [6], Sainsbury-Carter [7] and Adams and Peppiatt [8]. A good survey of the papers on adhesive joints up to 1964 can be found in Kutscha [9]. The practical aspects of the design of adhesive joints and adhesives are given in a recent book by Bikerman [10]. For scarf joints, Lubkin [11], Erdogan and Ratwani [6] can be mentioned.

Adhesive joints are also being used increasingly in prestressed post-tensioned concrete structures. For instance, Papault [12], Levy [13,14] and particularly Abeles [15] among others investigated the practical applications of adhesive joints to prestressed concrete structural elements.

It is of interest to observe here that in all the references given above, excluding Coland and Reissner [1] the effect of bending of adherends on the stress distribution in adhesive layer, has been ignored. Also, in [6], [7] the transverse normal stress is neglected. In the case of Goland and Reissner [1], the cylindrical bending of identical adherends of isotropic material is considered in combination with the transverse normal stress and longitudinal shear stress in the adhesive layer. However, they assumed adherends with equal thickness and identical isotropic material in order to obtain a closed form solution to the problem. This severely limits the applicability and the practical range of the closed form solutions given in [1].

Therefore, the purpose of this paper is to develop an analytical model for adhesive joints in order to find the transverse normal stress as well as longitudinal shear stress distribution in the adhesive layer due to the bending deformations in unidentical orthotropic adherends. Furthermore, the thickness shear deformations in the adherends will also be taken into account.

The results indicate that the bending of adherends drastically change both the normal stress and shear stress concentrations in the adhesive layer particularly in a joint composed of two adherends with different elastic constants. It will be shown that the formulation and method of solution of the problem of adhesive joints as presented here is general enough to handle other related problems such as "scarf joints", "double joints", etc. with relative ease.

2. FORMULATION OF THE PROBLEM

For all practical purposes, the adhesive joints in terms of geometry may be divided into three basic types: 1) lap joint, 2) stepped joint, and 3) scarf (or tapered) joint as shown in Fig. 1. In general, almost all other joints may be obtained through some combination or repetition of these basic types. In a similar fashion, from the standpoint of mechanics, the "stepped joint" may be considered same as the "lap joint"[*] and the "scarf joint" is a limiting case of infinite number of "stepped joints" put together between two end points of the scarf joint. Therefore in this work only the "lap joint" will be investigated in detail. Other types of joints and related problems will be presented in a forthcoming report.

A typical "lap joint" of length L shown in Fig. 2, consists of upper and lower adherends (or plates) of different orthotropic materials with thicknesses h_1 and h_2 respectively and a thin isotropic adhesive layer of thickness t. The principal directions of orthotropy in both adherends are assumed to coincide with the coordinate axes as shown in Fig. 2a and 2b. Both upper and lower adherends are treated as orthotropic plates subjected to in-plane stretching, bending and thickness (or transverse) shear deformations. The thin adhesive layer can be considered as composed of longitudinal shear and transverse tension-compression springs connecting the two adherends. In other words, it is assumed that, in the adhesive layer, the dominant stresses are the longitudinal shear and transverse (or thickness) normal stress and, furthermore, these stresses do not change across the thickness of the adhesive (see Fig. 2 and 3).[**] The sign convention for u^i, v^i, w^i (i=1,2) displacements, the strain quantities, the stresses and stress-resultants for both adherends and adhesive layer are those of the theory of elasticity. (See also the Appendix)

A self-contained treatment of the field equations of the shear theory of othotropic plates which takes into account in the average the transverse (or thickness) shear deformations are given in the Appendix (see

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[*] Provided the vertical slits of the "stepped joint" are not filled with adhesive or their effect is ignored. This is not an unrealistic assumption specially if one considers the fact that the thickness of the adherends are in general very small.

[**] This last assumption implies that $\sigma^1(x,y) = \sigma^2(x,y) = \sigma$, $\tau^1_x(x,y) = \tau^2_x(x,y) = \tau_x$ and $\tau^1_y(x,y) = \tau^2_y(x,y) = \tau_y$ providing that adhesive thickness $t << (h_1, h_2)$.

also [16]). Thus, after some algebric manipulations, these equations can be reduced, in terms of the so-called "fundamental variables" N_x^i, N_{xy}^i, Q_x^i, M_x^i, M_{xy}^i, u^i, v^i, w^i, β_x^i, β_y^i ($i = 1,2$), into the following system of partial differential equations:

$$
\begin{aligned}
N^i_{x,x} &= - p^i_x - N_{xy,y} \\
N^i_{xy,x} &= - p^i_y - (C^i_{12}\, u,^i_{xy} + C^i_{22}\, v,^i_{yy}) \\
Q^i_{x,x} &= - p^i_z - L^i_2\, (w,^i_{yy} + \beta^i_{y,y}) \qquad (i = 1,2) \\
M^i_{x,x} &= - m^i_x + Q^i_x - M^i_{xy,y} \\
M^i_{xy,x} &= - m^i_y + L^i_2\, (w,^i_y + \beta^i_y) - (D^i_{12}\, \beta^i_{x,xy} + D^i_{22}\, \beta^i_{y,yy})
\end{aligned}
\tag{1a-e}
$$

and,

$$
\begin{aligned}
u,^i_x &= (N^i_x - C^i_{12}\, v,^i_y)/C^i_{11} \\
v,^i_x &= (N^i_{xy} - F^i_{22}\, u,^i_y)/F^i_{12} \\
\beta^i_{x,x} &= (M^i_x - D^i_{12}\, \beta^i_{y,y})/D^i_{11} \qquad (i = 1,2) \\
\beta^i_{y,x} &= (M^i_{xy} - K^i_{12}\, \beta^i_{x,y})/K^i_{11} \\
w,^i_x &= Q^i_x/L^i_1 - \beta^i_x
\end{aligned}
\tag{1f-j}
$$

where $i = 1$ and $i = 2$ correspond to the upper and lower adherends respectively. The sum of the distributed surface loads p_x^i, p_y^i, p_z^i ($i = 1,2$) and distributed surface moments m_x^i, m_y^i ($i = 1,2$) acting on the reference planes are given as,

$$p^1_x = q^1_x - \tau_x \quad , \qquad p^2_x = - q^2_x + \tau_x$$

$$p^1_y = q^1_y - \tau_y \quad , \qquad p^2_y = - q^2_y + \tau_y$$

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$$p_z^1 = q_z^1 - \sigma \quad , \qquad p_z^2 = - q_z^2 + \sigma \tag{2a-e}$$

$$m_x^1 = q_x^1 \frac{h_1}{2} + \tau_x \frac{h_1+t}{2} \quad , \quad m_x^2 = q_x^2 \frac{h_2}{2} + \tau_x \frac{h_2+t}{2}$$

$$m_y^1 = q_y^1 \frac{h_1}{2} + \tau_y \frac{h_1+t}{2} \quad , \quad m_y^2 = q_y^2 \frac{h_2}{2} + \tau_y \frac{h_2+t}{2}$$

and the stresses in the adhesive layer are

$$\sigma(x,y) = \frac{B}{t} (w^1 - w^2)$$

$$\tau_x(x,y) = \frac{G}{t} (u^1 - \beta_x^1 \frac{h_1}{2} - u^2 - \beta_x^2 \frac{h_2}{2}) \tag{3a-c}$$

$$\tau_y(x,y) = \frac{G}{t} (v^1 - \beta_y^1 \frac{h_1}{2} - v^2 - \beta_y^2 \frac{h_2}{2})$$

where B is an elastic constant related to the Young's modulus and Poisson's ratio ν and the shear modulus G of the adhesive, $\sigma(x,y)$ is the transverse normal stress, $\tau_x(x,y)$, $\tau_y(x,y)$ are longitudinal shear stresses of the adhesive layer in x and y directions respectively.

The equations (3a-c) define the mechanical behavior of the adhesive layer and they also correspond to the compatibility equations of the problem i.e. peeling off, cracking or separation are not permitted on the interfaces between the adhesive layer and adherends.

The elastic constant B in (3.2) can be found from the following elastic stress-strain relation for the adhesive layer.

$$\sigma(x,y) \cong \sigma_z = \lambda(e_x + e_y + e_z) + 2Ge_z \tag{4}$$

where σ is the transverse normal stress, e_x, e_y, e_z are the strain components and,

$$\lambda = \nu E/[(1+\nu)(1-2\nu)] \tag{5}$$

Because of the compatibility of strains on the inter faces between adherends and adhesive layer, e_x and e_y in the adhesive layer must be equal in magnitude to the adherend strains e_x^i, e_y^i $(i = 1,2)$ on the interfaces, whereas adhesive layer strain e_z which is given by,

$$e_z = (w^1 - w^2)/t \qquad (6)$$

can be much larger or $|e_z| >> (e_x, e_y)$. If e_x, e_y are neglected in comparison with e_z in equation (5), then the elastic constant B of the adhesive is,

$$B \cong \lambda + 2G = (1-\nu)E/(1-2\nu) \qquad (7)$$

making it somewhat larger than Young's modulus E.

The surface load terms p_x^i, p_y^i, p_z^i and m_x^i, m_y^i $(i = 1,2)$ in (1a-e) can be easily eliminated by simply substituting (3a-c) into (2a-e). Finally, the system (1a-e) and (1f-j) reduces to the following matrix form of a system of twenty partial differential equations with the appropriate boundary conditions in the region $(a_1 \leq x \leq b_1)$ and $(a_2 \leq y \leq b_2)$,

$$\begin{aligned}
&\frac{\partial}{\partial x} Y_j(x,y) = F_j(x,y, \frac{\partial Y_k}{\partial y}, \frac{\partial^2 Y_k}{\partial y \partial x}, \ldots) \qquad (k = j = 1,2,..,20) \\
&T_{mr}^{a_1}(y)\, Y_r(a_1,y) = U_m^{a_1}(y) \qquad (m = n = 1,2,..,10) \\
&T_{ns}^{b_1}(y)\, Y_s(b_1,y) = U_n^{b_1}(y) \qquad (r = s = 1,2,..,20) \qquad (8a\text{-}e) \\
&T_{mn}^{a_2}(x)\, Y_r(x,a_2) = U_m^{a_2}(x) \\
&T_{ns}^{b_2}(x)\, Y_s(x,b_2) = U_n^{b_2}(x)
\end{aligned}$$

where $Y_j(x,y)$ is a column matrix of order 20 which includes all the "fundamental variables" and $U_m^{a_1}(y)$ $U_n^{b_1}(y)$ are the specified boundary conditions at $x = a_1$ and $x = b_1$ respectively. Similarly $U_m^{a_2}(x)$, $U_n^{b_2}(x)$ are boundary conditions specified at $y = a_2$ and $y = b_2$ respectively. The matrices $T_{mr}^{a_1}$, $T_{ns}^{b_1}$, $T_{mr}^{a_2}$, $T_{ns}^{b_2}$ are coefficient matrices depending on the support conditions along the boundaries. In general, they are unit matrices.

In the case of a joint with finite width in y direction the system (8a-c) has to be solved. However, if it is assumed that the dimension in y direction is large and that cylindrical bending occurs along the joint in x direction, then the equations (1a-j) reduce to the twelfth order system of ordinary differential equations given by,

$$\frac{dN_x^i}{dx} = - p_x^i$$

$$\frac{dQ_x^i}{dx} = - p_z^i \qquad (i = 1,2) \qquad \text{(9a-c)}$$

$$\frac{dM_x^i}{dx} = - m_x^i + Q_x^i$$

and,

$$\frac{du^i}{dx} = N_x^i/C_{11}^i$$

$$\frac{d\beta_x^i}{dx} = M_x^i/D_{11}^i \qquad (i = 1,2) \qquad \text{(9d-f)}$$

$$\frac{dw^i}{dx} = Q_x^i/L_1^i - \beta_x^i$$

where p_x^i, p_z^i and m_x^i ($i = 1,2$) are given in (2a,c,d) however $\sigma(x,y)$, $\tau_x(x,y)$ and $\tau_y(x,y)$ become,

$$\sigma(x,y) = \sigma(x) \quad , \qquad \tau_x(x,y) = \tau(x) \quad , \qquad \tau_y(x,y) \equiv 0 \qquad (10)$$

with the righthand sides of the equations (3a,b) still being valid (see Fig. 3).

Consequently, (8a-c) reduces to a simpler matrix form in terms of a system of twelve ordinary differential equations with the boundary conditions along the joint in the region with $a_1 = -\ell$, $b_1 = +\ell$ or $(-\ell \leq x \leq +\ell)$ and $(-\infty \leq y \leq +\infty)$,

$$\frac{d}{dx} Y_j(x) = A_{jk}(x) Y_k(x) + P_j(x) \qquad (j = k = 1,2,..,12)$$

$$T^a_{mr}\ Y_r(-\ell) = U^a_m \qquad (m = n = 1,2,..,6) \qquad (11a\text{-}c)$$

$$T^b_{ns}\ Y_s(+\ell) = U^b_n \qquad (r = s = 1,2,..,12)$$

where $A_{jk}(x)$ is a coefficient matrix of order (12,12) which includes the elastic constants and geometric dimensions such as thickness, etc. of the adherends and the adhesive layer. $P_j(x)$ is a column matrix of order 12, corresponding to the distributed loads q^i_x, q^i_y, q^i_z. The coefficient matrix A_{jk} is not in general a function of x unless the thickness or the material constants of the adherends (or the adhesive layer) or both varies along the length of the joint (i.e. scarf joint). The matrix $Y_j(x)$ is again a column matrix of order 12, including all the "fundamental variables" as its elements. The twelve fundamental variables N^i_x, Q^i_x, M^i_x, u^i, β^i_x, w^i $(i = 1,2)$ are the unknown functions of the independent variable x. In the boundary conditions (11b,c), the matrices T^a_{mr} and T^b_{ns} are constant matrices with the order of (6,12) and (6,12) respectively. The quantities U^a_m and U^b_n are column matrices corresponding to the stress-resultants and displacements prescribed at the ends of the adhesive joint $x = -\ell$ and $x = +\ell$ respectively.

The boundary conditions in (11b) and (11c) are obtained from the known stress resultants and displacements of the adherends at the ends of the joint. In the lap joint in Fig. 2, the six boundary conditions to be prescribed at each end may be found using the free body diagrams in Fig. 3 and Fig. 4. For instance, N^1_*, Q^1_*, M^1_* and N^2_*, Q^2_*, M^2_* where subscript $*$ designates prescribed quantities at $x = \mp \ell$, are calculated from the statics in terms of the distributed external force P (or in terms of basic loads N_o, Q_o, M_o in Fig. 4) and the geometry. Then, the boundary conditions are:

at $x = -\ell$, the column matrix U^a_m in (11b),

$$u^1 = 0 \ , \ w^1 = 0 \ , \ \beta^1_x = 0 \ , \ N^2_x = 0 \ , \ Q^2_x = 0 \ , \ M^2_x = 0 \qquad (12a)$$

at $x = +\ell$, the column matrix U^b_n in (11c),

$$N^1_x = 0 \ , \ Q^1_x = 0 \ , \ M^1_x = 0 \ , \ N^2_x = N^2_* \ , \ Q^2_x = Q^2_* \ , \ M^2_x = M^2_* \qquad (12b)$$

Note here that in (12a) displacement boundary conditions rather than the force conditions prescribed for the upper adherend at the left end of the joint. The reason for this is twofold. If the three displacements (i.e. $u^1 = 0$, $w^1 = 0$, $\beta^1 = 0$) are not prescribed, then the displacements throughout the joint cannot be found from (9a-f) or from (11a-c). Because any arbitrary rigid body displacements can be added to these system of equations without violating the mathematical conditions of the problem; hence, the solution to these equations would not be unique. Also, the prescribed stress resultants N_*^1, Q_*^1, M_*^1 must necessarily be in equilibrium with N_*^2, Q_*^2, M_*^2. This equilibrium, however, is already expressed through (9a-c). Therefore, in (12a), the inclusion of equilibrium values of the external forces as boundary conditions, instead of displacements $u_*^1 = 0$, $w_*^1 = 0$, $\beta_*^1 = 0$, would be redundant. (With the assumed fully fixed condition applied to the one section of the upper adherend, the displacements u^i, v^i, w^i ($i = 1$) now represent displacements relative to this section. The choice of the joint end section assumed fixed in no way influences the calculated stresses).

Thus, at $x = -\ell$, the displacement conditions for the upper adherend in combination with the force conditions for the lower adherend represent the "appropriate" boundary conditions for U_m^a ($m = 1,2,..,6$). In (11b,c) the matrices T_{mr}^a, T_{ns}^b are unit matrices, however, in special cases with spring and other type of support conditions at $x = \mp \ell$ they may have other nonzero components.

Thus, the equations (11a) with the appropriate boundary conditions (11b,c) represent a system of twelfth order linear ordinary differential equations. The entire system (11a-c) constitutes a so-called "two-point boundary value problem" of all the three basic types of the adhesive joint.

3. METHOD OF SOLUTION OF DIFFERENTIAL EQUATIONS

The system of equations similar to (1a-j) or (4a-c) and also (9a-f) or (11a-c) has been investigated among others by Kalnins [16,17]. In general, they can be solved by making use of numerical methods such as the "multi-segment method of integration" or "finite difference methods" or both. However, in the case of adhesive joints under special conditions, the equations (9a-f) (or 11a-c) have a closed form solution. This "special case" and the more "general cases" are considered next.

The problem in both special and general cases will be solved for external tension N_o, external shear Q_o and external bending moment M_o which are defined as the three "basic loading cases" in Fig. 4. Any other loading case can be treated as a superposition of these basic loadings (provided there are no distributed surface loads on adherends).

a) Special Case (Adherends with idential thickness and material):

In order to gain some idea about the effect of the thickness shear deformations of adherends on the stresses $\tau(x)$ and $\sigma(x)$, consider a special case in which adherends have the same thickness and material. (or $h_1=h_2=h_a$, $E_1^1=E_1^2=E_a$, $G_{13}^1=G_{13}^2=G_a$, $\nu_{12}^1=\nu_{12}^2=\nu_a$, $C_{11}^1=C_{11}^2=C_a$, $D_{11}^1=D_{11}^2=D_a$, $L_1^1=L_1^2=L_a$ where subscript "a" denotes x direction and "b" for y direction in identical and orthotropic adherends).

In such a case the system of ordinary differential equations (9a-f) and (11a-c) can easily be reduced to two coupled ordinary differential equations in terms of the two unknown adhesive layer stresses $\sigma(x)$ and $\tau(x)$ so that,

$$d^2\tau/dx^2 - (c)^2\tau = - Gh_a(Q_*^1+Q_*^2)/(2D_a t) \qquad (13\ a,b)$$

$$d^4\sigma/dx^4 - 2(\alpha)^2\, d^2\sigma/dx^2 + (\beta)^4\,\sigma = 0$$

$$(c)^2 = 2G/(c_a t) + Gh_a(h_a+t)/(2D_a t)$$

$$(\alpha)^2 = B/(L_a t) \qquad (14a\text{-}c)$$

$$(\beta)^4 = 2B/(D_a t)$$

The quantity $(Q_*^1+Q_*^2)$ in the equation (13a) can be considered as the total shear resultant transmitted through the joint. It is of interest to observe here that if the transverse shear strain in the adhereads is neglected, then (13a) remains the same, but the parameter α in (13b) becomes zero. In such a case, the equations reduce to these of Goland and Reissner [1]. Thus, it seems that transverse shear strains in adherends effect mainly the transverse normal stress $\sigma(x)$ in the adhesive layer.

The general solution of (13a) is,

$$\tau(x) = A_1 \sinh cx + A_2 \cosh cx + Gh_a(Q_*^1 + Q_*^2)/(2D_a tc^2) \quad (15)$$

However, the solution of (13b) is dependent on the relative values of α and β. Thus, if $\alpha<\beta$, the general solution of (13b) is,

$$\sigma(x) = A_3 \sinh ax \cos bx + A_4 \sinh ax \sin bx + A_5 \cosh ax \sin bx + A_6 \cosh ax \cos bx \quad (16)$$

where

$$a = [(\beta^2 + \alpha^2)/2]^{1/2}$$
$$b = [(\beta^2 - \alpha^2)/2]^{1/2} \quad (17a,b)$$

If $\alpha>\beta$, the general solution of (13b) becomes,

$$\sigma(x) = A_3 \sinh ax + A_4 \cosh ax + A_5 \sinh bx + A_6 \cosh bx \quad (18)$$

where

$$a = \left[\alpha^2 + \sqrt{\alpha^4 - \beta^4}\right]^{1/2}$$
$$b = \left[\alpha^2 - \sqrt{\alpha^4 - \beta^4}\right]^{1/2} \quad (19a,b)$$

The maximum values of $\tau(x)$ and $\sigma(x)$ corresponding to each basic loading case can easily be calculated from (15), (16) and (18) and they occur in all loading cases at the ends of the joint. For example, in the case of external normal tension load N_o in Fig. 4, $\tau(x)$ and $\sigma(x)$ are even functions of x so that the arbitrary constants A_1, A_3, A_5 of the solutions in (15) and (16), (18) and the remaining three constants may be determined from the boundary conditions (or the external equilibrium conditions) of the joint. Thus, for external tension load N_o,

$$2\ell\tau_{max}/N_o = c\ell \coth c\ell \tag{20a,b}$$

$$tD_a\beta^2 G_{max}/BM_\star = k_2/k_1$$

where

$$M_\star = N_o(h_a + t)/2 \tag{21}$$

and for $\alpha<\beta$,

$$k_1 = b\ell \sinh 2a\ell + a\ell \sin 2b\ell$$

$$k_2 = b\ell \sinh 2a\ell - a\ell \sin 2b\ell \tag{22a,b}$$

whereas, for $\alpha>\beta$,

$$k_1 = b\ell \sinh b\ell \cosh a\ell - a\ell \sinh a1 \cosh b\ell$$

$$k_2 = b\ell \cosh b\ell \sinh a\ell - a\ell \cosh a\ell \sinh b\ell \tag{23a,b}$$

Similarly, in the case of external shear force Q_o shown in Fig. 4, the stresses $\tau(x)$ and $\sigma(x)$ are even functions of the coordinate x. The maximum values of $\tau(x)$ and $\sigma(x)$, which occur at the joint edges, are,

$$2tD_a c\tau_{max}/Gh_a Q_o\ell = \coth c\ell - 1/c\ell \tag{24a,b}$$

$$tD_a\beta^2 G_{max}/B\ Q_o\ell = (k_2 + k_3)/k_1$$

where, for $\alpha<\beta$,

$$k_3 = 2ab(\cosh 2a\ell + \cos 2b\ell)/\beta^2 \tag{25a}$$

and, for $\alpha > \beta$,

$$k_3 = -(a^2 - b^2)(\cosh a\ell \cosh b\ell)/\beta^2 \qquad (25,b)$$

and k_1 and k_2 are given in (23a,b).

In the case of bending moment loading M_o of Fig. 4 the expressions for $\tau(x)$ and $\sigma(x)$ are odd functions of x and maximum values occuring at the edges are,

$$2tD_a c\tau_{max}/Gh_a M_o = \tanh c\ell$$
$$tD_a \beta^2 \sigma_{max}/BM_o = k_1/k_2 \qquad (26a,b)$$

where k_i and k_2 have been defined previously.

In order to calculate these maximum stresses, values of the ratios k_2/k_1 and k_3/k_1 as functions of $\beta\ell$ and α/β have been plotted in Figures 5 and 6, respectively. In these plots, the parameter describing the influence of the transverse shearing strain of the adherends is the ratio α/β given by,

$$\alpha/\beta = \left[0.5\ BD_a/\{(L_a)^2 t\}\right]^{1/4} \qquad (27)$$

Substituting for D_a and L_a the ratio α/β becomes for orthotropic adherends of equal thickness and the same material,

$$\alpha/\beta = \left[3BE_a h_a/\{50(1-\nu_a^2)(G_a)^2 t\}\right]^{1/4} \qquad (28,a)$$

and for isotropic adherends of equal thickness and same material,

$$\alpha/\beta = \left[12(1+\nu_a)Bh_a/\{50E_a(1-\nu_a)t\}\right]^{1/4} \qquad (28b)$$

By calculating the ratio α/β from (28a) or (28b), one can determine for any given case whether or not the thickness shear strains in adherends significantly influence the maximum value of the normal stress in the adhesive. According to Fig. 5 and Fig. 6, the curves for the values

$\alpha/\beta \leq 1$ do not differ significantly from those for $\alpha/\beta=0$, thus, the influence of the thickness shear strains of adherends on maximum normal stress σ_{max} can be ignored provided $\alpha/\beta \leq 1.0$.

(This is more or less valid in practical cases. For example, a typical joint of two aluminum plates, each of $h_a=0.1$ inch thickness with $E_a=10 \times 10^6$psi, $\nu_a=0.33$ bonded by a layer of epoxy with $E=4.5 \times 10^5$ psi, $\nu=0.35$ and $t=0.01$ inch thickness, has a ratio $\alpha/\beta=0.77$ which is smaller than unity). On the other hand, in the case of adherends with large thicknesses and small longitudinal shear modulus G_{13}, the effect of the adherend thickness shear deformations on the adhesive layer stresses may not be ignored.

b) General Case (Dissimilar Adherends):

In the case of adherends with unequal thicknesses and different elastic properties, a closed form solution (such as given in the preceeding section) of the two-point boundary value problem of (11a-c) seems not to be possible. Therefore, numerical methods have to be employed in order to solve this system. The "multi-segment method of integration" as given in [17] is most suitable for this purpose. In fact, this method of numerical integration is used here in solving the general case in (9a-f) or in (11a-c). This way, the two-point boundary value problem of the adhesive joint reduces into a series of initial value problems and then integrated numerically between the boundary points.

For the sake of simplicity, one can drop the quantities corresponding to the distributed surface loads such as q_x^i, q_z^i $(i = 1,2)$ in (9a-f) and consequently P_j matrix in (11a-c) associated with the external surface loads. Then, in (11a-c),

$$q_x^i = q_z^i = 0 \text{ or } P_j(x) = 0 \; (i = 1,2;\; j = 1,2,..,12) \qquad (29)$$

It should be emphasized here that this last additional assumption in no way affects either the general form of the system in (11a-c) or the applicability of the method of solution employed here to the lap joint as well as other type of joints.

Thus, a computer program based on multi-segment method of integration has been developed to solve the equations in (9a-f) or (11a-c) and has been applied to several joints under various edge loads. The results for a typical lap joint subjected to basic external load cases of Fig. 4 and also for a stepped joint are presented in the next section.

As it is explained at the beginning, the analysis of a "stepped joint" is not different from that of a lap joint. Therefore, it will not be treated as a separate case. However, in order to point out the influence of bending on the stresses in a stepped joint, even when it is under uniaxial tension, the results of a numerical example given in Fig. 10 will be discussed briefly in the next section.

In passing, it may be of interest to note here that in the case of a scarf joint (see Fig. 1) the formulation of the problem and the method of solution as presented in this paper can easily be employed. In such a case the equations (9a-f) and (11a-c) are exactly the same except that the terms of the coefficient matrix A_{jk} become functions of x due to variation of adherend thicknesses rather than elastic constants. The computer program which was developed for the general case can also easily handle the scarf joint problem.

4. DISCUSSION OF RESULTS

The non-dimensional expressions (20a), (24a), (26a) for τ_{max} and (20b), (24b), (26b) for σ_{max} corresponding to the basic loading cases in a joint of identical thickness and material, are functions of the parameters $c\ell$, $\beta\ell$ and the ratio α/β. As it is noted earlier in the solution of the "special case", the transverse shear strains in adherends may be neglected. Thus, for practical applications, the ratio α/β in the "special case" becomes

$$\alpha/\beta \rightarrow 0 \tag{30}$$

and similarly in the "general case", in (9f)

$$(Q_x^i/L_1^i) \rightarrow 0 \quad (i = 1,2) \tag{31}$$

which corresponds to the vanishing of adherend transverse shear strains or consequently to the classical bending theory of thin plates.

For practical purposes, in a typical joint α/β ratio may be assumed as

$$\alpha/\beta \lessapprox 1 \tag{32}$$

and the adhesive thickness t is in general considered to be very small i.e. $t \ll h_a (h_1 = h_2 = h_a)$. Then, the parameters $c\ell$ and $\beta\ell$ become large and for the values of,

$$c\ell \gtrapprox 3, \quad \beta\ell \gtrapprox 5 \tag{33}$$

the τ_{max} and σ_{max} equations (20a,b), (24a,b), (26a,b) for the basic external loading cases can be expressed in terms of asymptotic expansions for large ℓ.

Thus, for the external tension load N_o case of Fig. 4a, the asymptotic values of τ_{max} and σ_{max} are

$$\tau_{max} \cong N_o c/2$$
$$\sigma_{max} \cong N_o (h_a + t) B / (2tD_a \beta^2) \tag{34a,b}$$

by introducing (14a) and (14c) for c and β and also for $D_a/C_a = (h_a)^2/12$, τ_{max} and σ_{max} can be further reduced to,

$$\tau_{max} \cong \frac{N_o}{\sqrt{t}} (2G/C_a)^{1/2}$$
$$\sigma_{max} \cong \frac{N_o}{\sqrt{t}} (3B/C_a)^{1/2} \tag{35a,b}$$

In the case of a joint subjected to external bending moment M_o only (Fig. 4c), the asymptotic expressions for τ_{max} and σ_{max}, derived in the same manner, are

$$\tau_{max} \cong \frac{M_o}{\sqrt{t}} (0.375\ G/D_a)^{1/2}$$

$$\sigma_{max} \cong \frac{M_o}{\sqrt{t}} (0.5\ B/D_a)^{1/2} \qquad (36a,b)$$

Similarly, if the joint is under external shear force Q_o in Fig. 4b, the asymptotic expressions for τ_{max} and σ_{max} are,

$$\tau_{max} \cong \frac{Q_o(\ell_o-\ell)}{\sqrt{t}} (0.375\ G/D_a)^{1/2}$$

$$\sigma_{max} \cong \frac{Q_o(\ell_o-\ell)}{\sqrt{t}} (0.5\ B/D_a)^{1/2} \qquad (37a,b)$$

The equations (34a,b), (36a,b) and (37a,b) all indicate that for joints composed of identical adherends in thickness and material, maximum stresses occur at the edges of the adhesive joint. Furthermore, both σ_{max} and τ_{max} are of $0(1/\sqrt{t})$ as $t \to 0$, showing the singular behavior of these stresses around the edges. Similar results are obtained in the more general case with unidentical adherends. For example, in Fig. 7, 8 and 9, a lap joint of aluminum-epoxy-steel subjected to basic external loads demonstrate behavior similar to that of a joint with identical adherends. Both stresses $\sigma(x)$ and $\tau(x)$ shoot up within the boundary layer region in the neighborhood of the joint edges. This is specially so in the pure bending case (see Fig. 9) which clearly illustrates the effect of bending of adherends.

Another interesting result can be obtained by estimating the ratio σ_{max}/τ_{max} for basic external load cases of Fig. 4. Thus, from (35a,b), for the basic tension load N_o,

$$\sigma_{max}/\tau_{max} \cong (1.5\ B/G)^{1/2} \qquad (38)$$

and for the basic external moment loading M_o,

$$\sigma_{max}/\tau_{max} \cong \frac{2}{\sqrt{3}} (B/G)^{1/2} \text{ as } t \to 0 \qquad (39)$$

the ratio for the basic shear loading Q_o is equal to that of (39). These stress ratios, in the case of identical adherends, depend only on the Poisson's ratio ν of the adhesive and for practical values of ν, one can conclude that,

$$\sigma_{max}/\tau_{max} \geq 1.0 \text{ as } t \to 0 \tag{40}$$

and gets larger as the bending of the joint increases (depending on the particular loading this ratio can easily reach the value of 2 or more). These results can also be verified for the general case with unidentical adherends by simply comparing σ_{max} and τ_{max} values in Fig. 7, 8 and 9. These plots again indicate that higher stress concentrations occur at the edge corresponding to less stiffer adherend and the normal stress, σ_{max}, because of bending deformation, is in general the dominant stress. As a result, the tearing apart of the adhesive layer and adherends along the joint is likely to start and grow due to these transverse normal stress concentrations. Therefore, one may rightly call the maximum transverse normal stress σ_{max} as the "tearing stress" of the adhesive joint. On this basis, it may be said that theories which do not take into account the bending of the adherends cannot correctly predict the maximum stress in the adhesive layer. (For instance, in Erdogan and Ratwani [6], Sainsbury-Carter [7], Adams and Peppiatt [8], Lubkin [11], the effect of bending is completely ignored).

The significance of the bending effect even on the relatively smaller longitudinal shear stress of the adhesive can best be demonstrated by simply comparing the results presented here with those which neglect both the bending deformation and transverse shear deformations in the adherends. The equations (15) and (16) reduce to that of [6] with $\sigma(x)\equiv 0$ and $\tau(x)\neq 0$ if the bending stiffnesses D^1_{11}, D^2_{11} and the transverse shear stiffnesses L^1_1, L^2_1 are assumed to be infinite. Thus, "~" defining the quantities with bending neglected, (15) and (16) become,

$$\frac{d^2\tilde{\tau}(x)}{dx^2} - (\tilde{c})^2\tilde{\tau}(x) = 0$$

$$\tilde{\sigma}(x) \equiv 0 \tag{41a,b}$$

$$(\tilde{c})^2 = 2G/(C_a t) \tag{42}$$

then the corresponding maximum shear stress $\tilde{\tau}_{max}$ for the basic external tension load N_o of Fig. 4 is given by,

$$2\ell\tilde{\tau}_{max}/N_o = \tilde{c}\ell \coth \tilde{c}\ell \tag{43}$$

and the asymptotic value is,

$$\tilde{\tau}_{max} \cong N_o\tilde{c}/2 \tag{44}$$

by using (20a) for τ_{max} which includes the effect bending and transverse shear deformation and (43) for $\tilde{\tau}_{max}$,

$$\tau_{max}/\tilde{\tau}_{max} = c \coth c\ell/(\tilde{c} \coth \tilde{c}\ell) \tag{45}$$

For the small values of adhesive thickness, i.e. $t \ll h_a$, the ratio is almost $c/\tilde{c} = 2$. Then,

$$\tau_{max}\tilde{\tau}_{max} \geq 1 \tag{46}$$

However, from the asymptotic values in (34a) and (44)

$$\tau_{max}/\tilde{\tau}_{max} \cong 2 \qquad (\text{for large } \tilde{c}\ell \text{ and } t \to 0) \tag{47}$$

Thus, for practical joints, this $\tau_{max}/\tilde{\tau}_{max}$ ratio will have a value somewhat close to 2. Consequently, the theories [6,7,8,11] which neglect the bending effect, even in the case of uniaxial external tension only, might underestimate the maximum shear stress by nearly 50%.

Similar results are obtained for the more "general case" with unidentical adherends. For instance, the comparison of the longitudinal shear stress plot of the same "stepped joint" of Aluminum-Epoxy-Steel in Erdogan and Ratwani [6] with that of the shear stress plot obtained by the present authors and which includes the bending effect is given in Fig. 10. It is obvious that, in spite of the uniaxial external tension load, the actual maximum shear stress is almost twice as large as the shear stress obtained by [6]. Also, it is important to observe here that, in Fig. 10, the maximum transverse normal stress, because of the bending deformations along the joint, is again very much larger than the maximum transverse shear stress.

The influence of protruding lengths ℓ_1 and ℓ_2 outside of the joint on the normal and shear stresses in the adhesive is also considered. As pointed out previously, even under uniaxial tension the adhesive joint, is in the state of bending regardless of the size of the protruding lengths. As an example, the stresses in Aluminum-Epoxy-Boron Epoxy joint under uniaxial tension P=1.0 are computed for protruding length values $\ell_1=\ell_2=\ell_*(\ell_*=0,L,5L)$ as shown in Fig. 11. It is seen that, after $\ell_*=5L$ is reached both the maximum normal stress and shear stress in the adhesive layer level off and remain more or less same in magnitude. Again, as expected, the maximum normal stress is larger than the maximum shear stress.

5. CONCLUSIONS

A general method of stress analysis of adhesive joints of relatively rigid adherends bonded through a flexible adhesive layer has been developed and applied to several types of joints. Based on the numerical examples and the discussion in preceeding section, one can conclude the following:

1. Bending of one or both adherends is a dominant factor on the stress distribution in adhesive joints and it occurs even in a stepped joint under external uniaxial tension load.

2. Due to the influence of bending of adherends, the distribution of the transverse normal stress $\sigma(x)$ as well as the shear stress $\tau(x)$ in adhesive layer is drastically changed and σ_{max} is in general larger than τ_{max} and in some cases at least twice as large. (This point should be taken into account in the design of adhesive joints.)

3. Stress concentrations for $\sigma(x)$ and $\tau(x)$ occur at both ends of the joint within the so-called boundary layer region, with higher stress concentrations taking place at less stiff adherend side. Otherwise, with equal thickness and identical

adherends both $\sigma(x)$ and $\tau(x)$ are symmetric or skew-symmetric as the case may be.

4. As the adhesive thickness t decreases the magnitudes of stress concentrations at the ends of the joint increase sharply and finally as $t \to 0$ the stresses become singular i.e. $\tau_{max} \to \infty$, $\sigma_{max} \to \infty$. (For identical adherends these limiting expressions have the form $\tau_{max} = 0(1/\sqrt{t})$ and $\sigma_{max} = 0(1/\sqrt{t})$ as $t \to 0$ and for unidentical adherends, similar forms can be expected).

5. The thickness shear deformations in adherends do not significantly influences $\tau(x)$ and $\sigma(x)$ distribution in the adhesive layer. For practical purposes thickness shear deformations in adherends can be neglected unless the adherends are extermely thick and deformable in shear.

6. The formulation of the problem and the method of solution as presented in this work is very general and can easily be applied to other types of joints such as "scarf joints", "double joints", "cover plates", "joints with layered adherends", etc. without any difficulty. (In fact, these problems have already been solved by the present authors and will be presented in a forthcoming report as the continuation of this work.

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APPENDIX

For the sake of completeness and easy reference the field equations of orthotropic plates which include the transverse shear deformations will be reproduced in this section. Referring to Kalnins [16] (similar equations were also obtained by other well known authors), one can write the equilibrium equations of the orthotropic plates in terms of the coordinate system given in Fig. 2 in the following form,

$$
\begin{aligned}
&N^i_{x,x} + N^i_{xy,y} + p^i_x = 0 \\
&N^i_{xy,x} + N^i_{y,y} + p^i_y = 0 \\
&Q^i_{x,x} + Q^i_{y,y} + p^i_z = 0 \qquad (i = 1,2) \qquad \text{(A.1a-f)} \\
&M^i_{x,x} + M^i_{xy,y} - Q^i_x + m^i_x = 0 \\
&M^i_{xy,x} + M^i_{y,y} - Q^i_y + m^i_y = 0 \\
&\quad N^i_{xy} = N^i_{yx}
\end{aligned}
$$

where i=1 indicates upper plate (or adherend) and i=2 corresponds to the lower plate (or adherend) and p^i_x, p^i_y, p^i_z are distributed loads and m^i_x and m^i_y are distributed moments both acting on the middle plane (or the reference plane) of the upper and lower plates. The Hooke's Law or the stress-strain relations for an orthotropic material,

$$
\begin{aligned}
&\sigma^i_x = B^i_{11}\, e^i_x + B^i_{12}\, e^i_y \\
&\sigma^i_y = B^i_{12}\, e^i_x + B^i_{22}\, e^i_y \\
&\sigma^i_z \simeq 0 \\
&\sigma^i_{xy} = 2\, e^i_{xy}\, G^i_{12} \qquad (i = 1,2) \qquad \text{(A.2a-f)} \\
&\sigma^i_{yz} = 2\, e^i_{yz}\, G^i_{23}
\end{aligned}
$$

$$\sigma^i_{xz} = 2\, e^i_{xz}\, G^i_{13}$$

where

$$B^i_{11} = E^i_1/(1 - \nu^i_{12}\nu^i_{21})$$

$$B^i_{12} = \nu^i_{12}E^i_1/(1 - \nu^i_{12}\nu^i_{21}) = \nu^i_{21}E^i_2/(1 - \nu^i_{12}\nu^i_{21}) \quad (i = 1,2) \qquad \text{(A.3a-c)}$$

$$B^i_{22} = E^i_2/(1 - \nu^i_{12}\nu^i_{21})$$

and where E^i_1, E^i_2 are the Young's Moduli of the material in the direction of x and y axes respectively. Similarly G^i_{12}, G^i_{23}, G^i_{13} are the shear moduli with the subscripts corresponding to coordinate axes $x \to 1$, $y \to 2$, $z^i \to 3$. Also, ν^i_{12}, ν^i_{21} are the Poisson's ratios of the material. The expressions for displacement $\bar{u}^i$, $\bar{v}^i$, $\bar{w}^i$ are,

$$\bar{u}^i = u^i + z^i\, \beta^i_x$$

$$\bar{v}^i = v^i + z^i\, \beta^i_y \qquad (i = 1,2) \qquad \text{(A.4a-c)}$$

$$\bar{w}^i = w^i$$

where u^i, v^i, w^i are middle surface displacements along x, y, z^i coordinate lines and β^i_x and β^i_y are angles of rotation of the normal to the middle surface. The strains e_{ij}, in terms of middle surface strains ε^i_j, γ^i_{xz}, γ^i_{yz}, γ^i_1, γ^i_2, δ^i_1, δ^i_2 ,

$$e^i_x = \varepsilon^i_x + z^i\, k_x$$

$$e^i_y = \varepsilon^i_y + z^i\, k_y \qquad \text{(A.5a-f)}$$

$$e^i_z = 0$$

$$2e^i_{xy} = (\delta^i_1 + z^i\, \delta^i_1) + (\delta^i_2 + z^i\, \delta^i_2) \quad (i = 1,2)$$

$$2e^i_{xz} = \gamma^i_{xz}$$

$$2e^i_{yz} = \gamma^i_{yz}$$

Where middle surface strains ε^i_j and other strain quantities γ^i_{xz}, γ^i_{yz}, k^i_x, k^i_y, γ^i_1, γ^i_2, δ^i_1, δ^i_2 are defined in terms of displacement quantities u^i, v^i, w^i and rotations β^i_x, β^i_y in the following way,

$$\varepsilon^i_x = u,^i_x \quad , \qquad \varepsilon^i_y = u,^i_y$$

$$\gamma^i_1 = v,^i_x \quad , \qquad \gamma^i_2 = v,^i_y$$

$$k^i_x = \beta^i_{x,x} \quad , \qquad k^i_y = \beta^i_{y,y} \qquad (i = 1,2) \qquad \text{(A.6a-d)}$$

$$\delta^i_1 = \beta^i_{y,x} \quad , \qquad \delta^i_2 = \beta^i_{x,y}$$

$$\gamma^i_{xz} = w,^i_x + \beta^i_x$$

$$\gamma^i_{yz} = w,^i_y + \beta^i_y \qquad (i = 1,2) \qquad \text{(A.6e-g)}$$

$$\gamma^i_{xy} = \gamma^i_1 + \gamma^i_2$$

The stress-resultant and strain relations can be expressed as,

$$N^i_x = C^i_{11}\ \varepsilon^i + C^i_{12}\ \varepsilon^i_y \ , \qquad N^i_y = C^i_{12}\ \varepsilon^i_x + C^i_{22}\ \varepsilon^i_y$$

$$M^i_x = D^i_{11}\ k^i_x + D^i_{12}\ k^i_y \ , \qquad M^i_y = D^i_{12}\ k^i_x + D^i_{22}\ k^i_y$$

$$(i = 1,2) \qquad \text{(A7a-d)}$$

$$N^i_{xy} = F^i_{11}\ \gamma^i_1 + F^i_{12}\ \gamma^i_2 = N^i_{yx} \ , \qquad M^i_{xy} = K^i_{11}\ \delta^i_1 + K^i_{12}\ \delta^i_2 = M^i_{yx}$$

$$Q^i_x = L^i_1\ \gamma^i_{xz} \ , \qquad Q^i_y = L^i_2\ \gamma^i_{yz}$$

Where extentional, bending, twisting and shearing rigidities are given as,

$$C_{11}^{i} = \int B_{11}^{i}\, dz^{i}, \quad C_{12}^{i} = \int B_{12}^{i}\, dz^{i}, \quad C_{22}^{i} = \int B_{22}^{i}\, dz^{i}$$

$$D_{11}^{i} = \int B_{11}^{i}\, (z^{i})^{2}\, dz^{i}, \; D_{12} = \int B_{12}^{i}\, (z^{i})^{2}\, dz^{i}, \; D_{22}^{i} = \int B_{22}^{i}\, (z^{i})^{2}\, dz^{i}$$

$$F_{11}^{i} = F_{12}^{i} = \int G_{12}^{i}\, dz^{i} \qquad (i = 1,2) \;\; (A.8a\text{-}e)$$

$$K_{11}^{i} = K_{12}^{i} = \int G_{12}^{i}\, (z^{i})^{2}\, dz^{i}$$

$$L_{1}^{i} = \int G_{13}^{i}\, dz^{i}, \quad L_{2}^{i} = \int G_{23}^{i}\, dz^{i}$$

The integrals above are to be taken across the thicknesses of the plates.

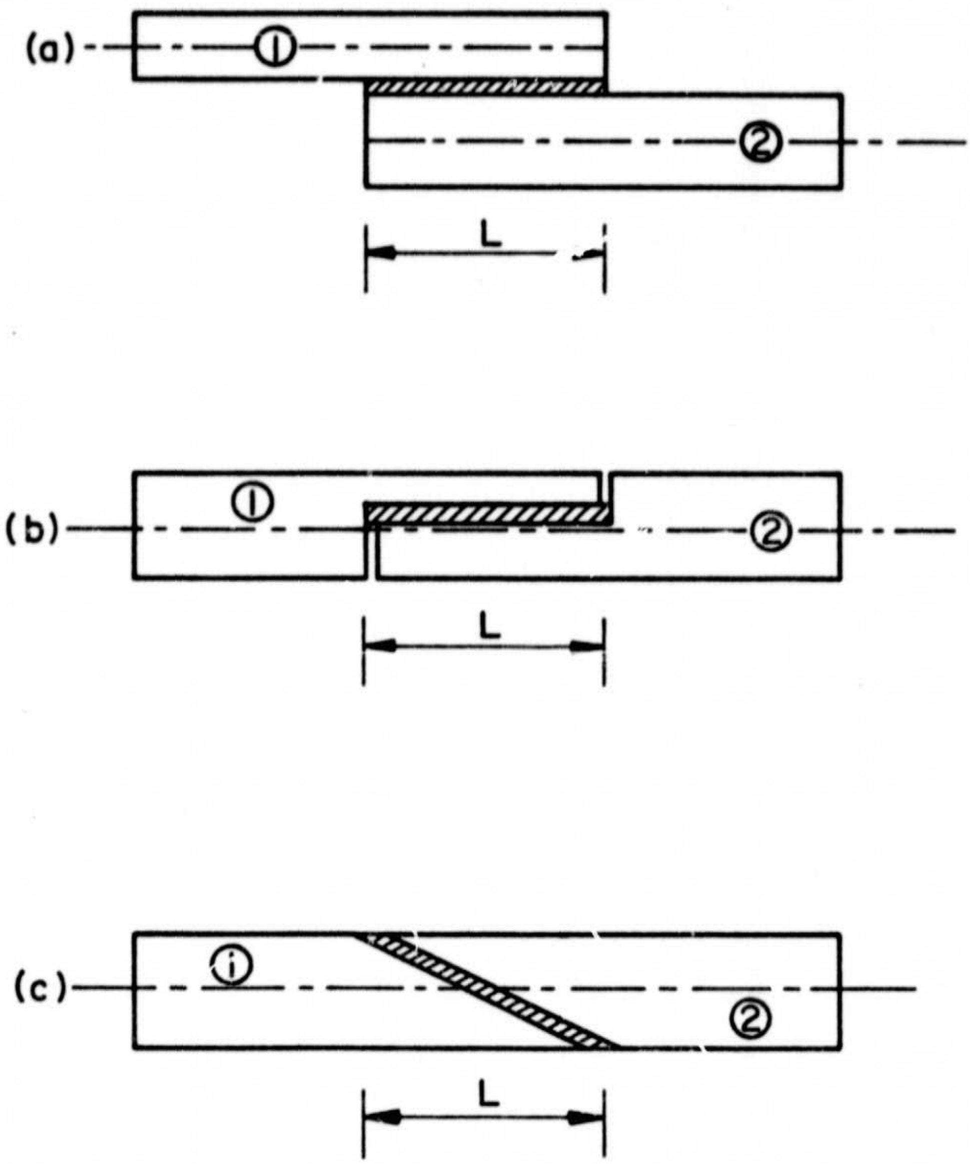

Fig. 1 Basic Types of Adhesive Joints (a) Lap Joint (b) Stepped Joint (c) Scarf Joint

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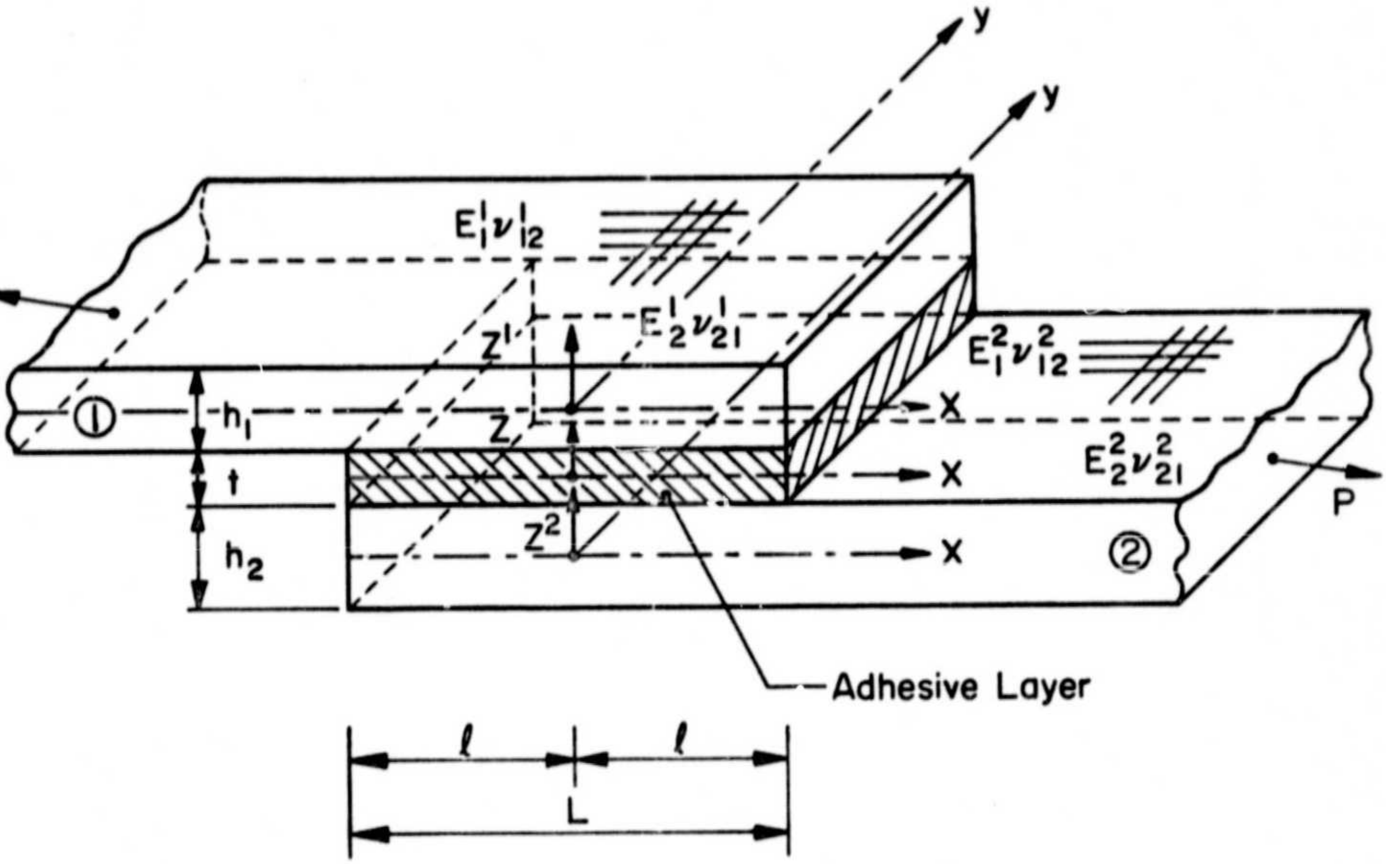

Fig. 2a Perspective View of a Lap Joint

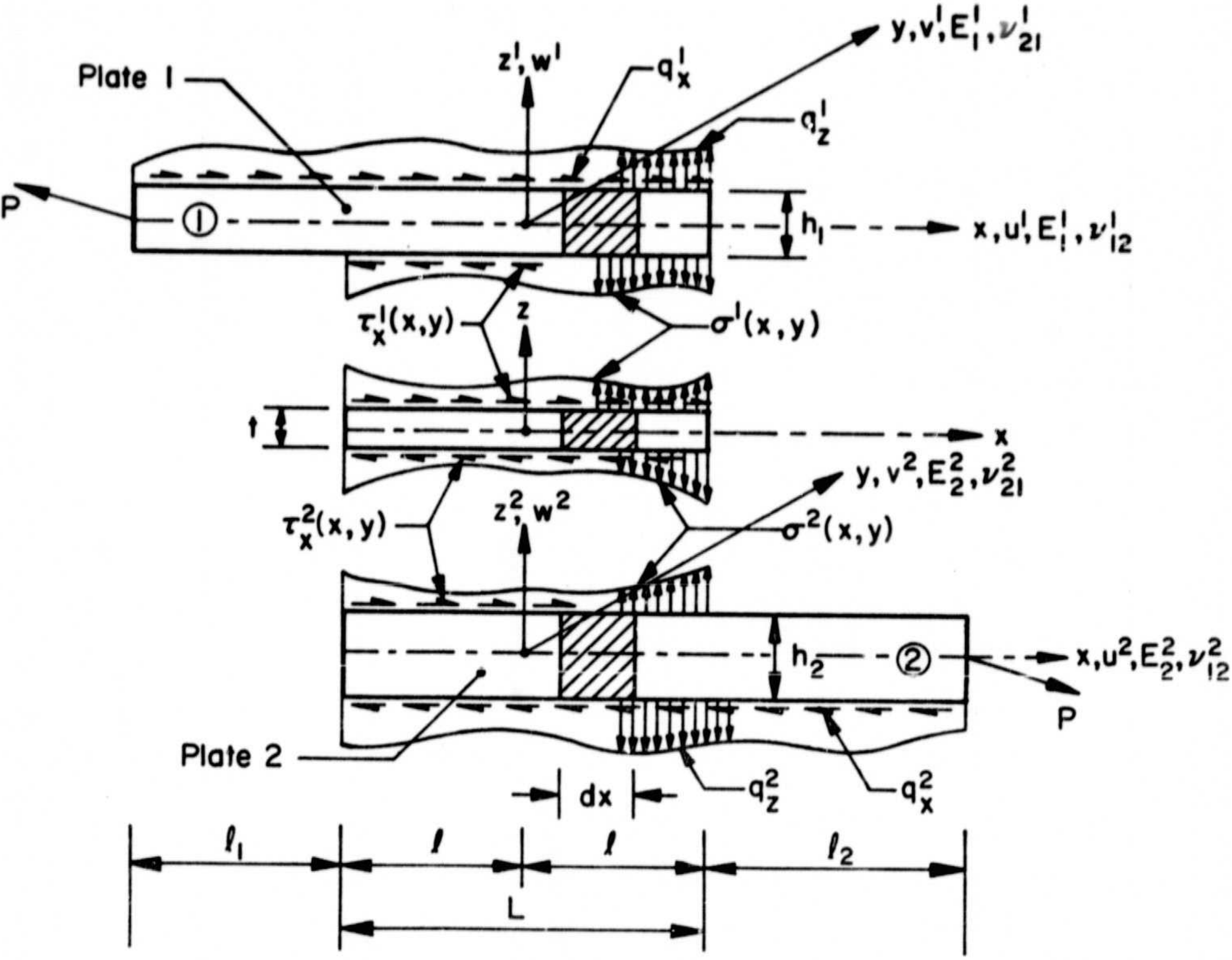

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Fig. 2b Stress Distribution and Sign Convention in a Lap Joint

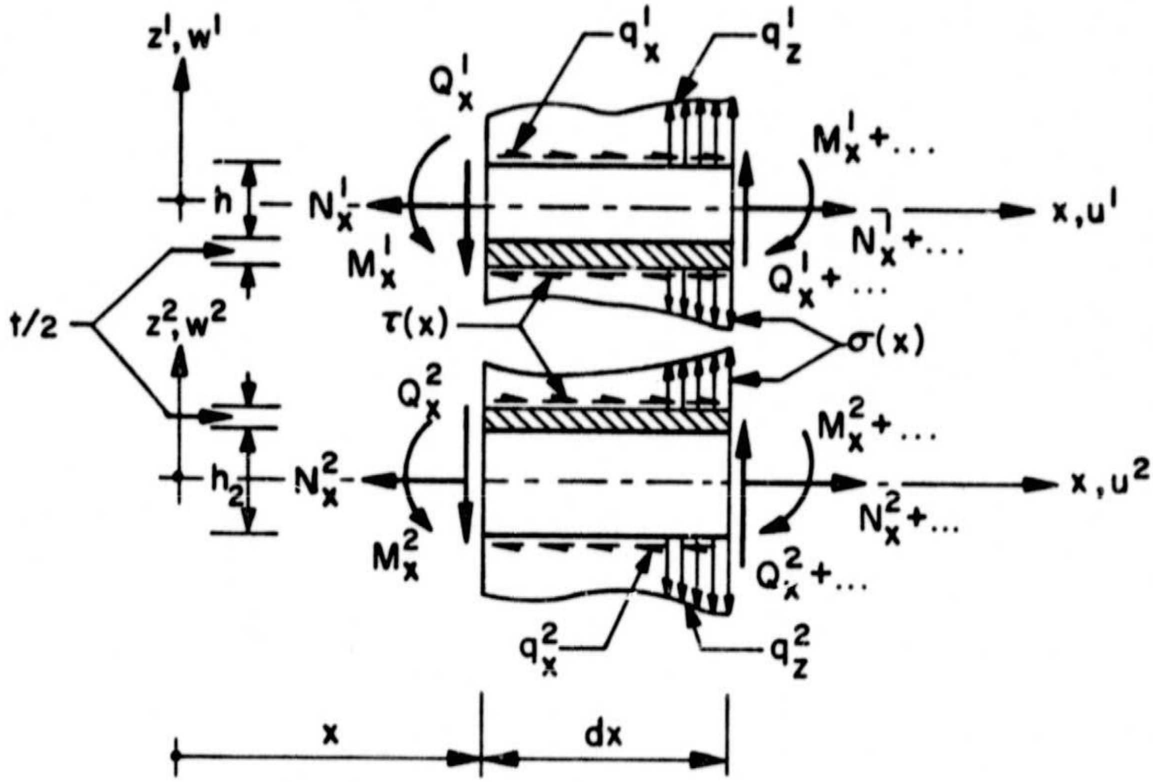

Fig. 3 Equilibrium Element in a Lap Joint (Cylindrical Bending)

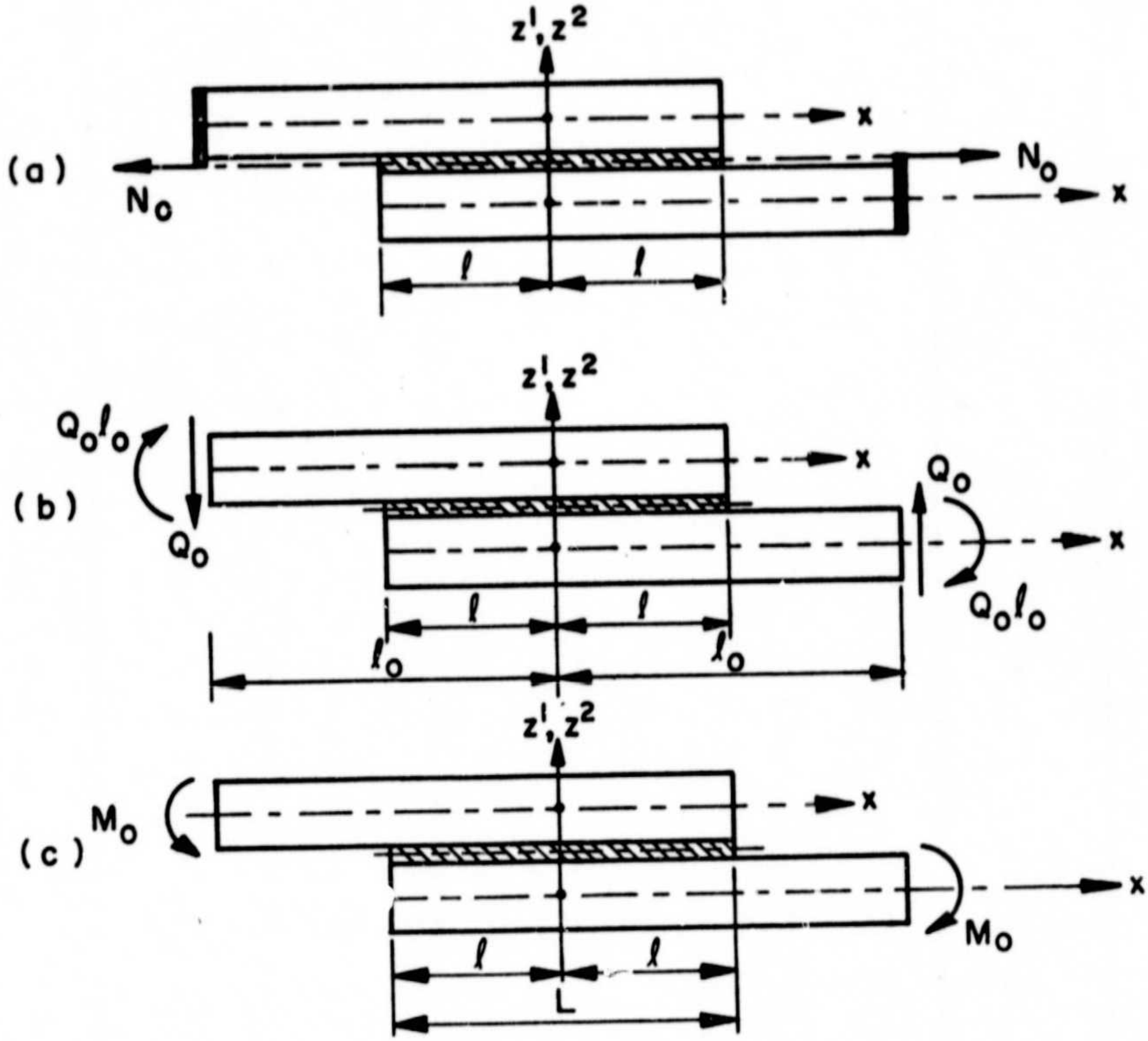

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Fig. 4 Basic Loading Cases for a Lap Joint (a) External Tension N_o, (b) External Shear Q_o (c) External Bending Moment M_o

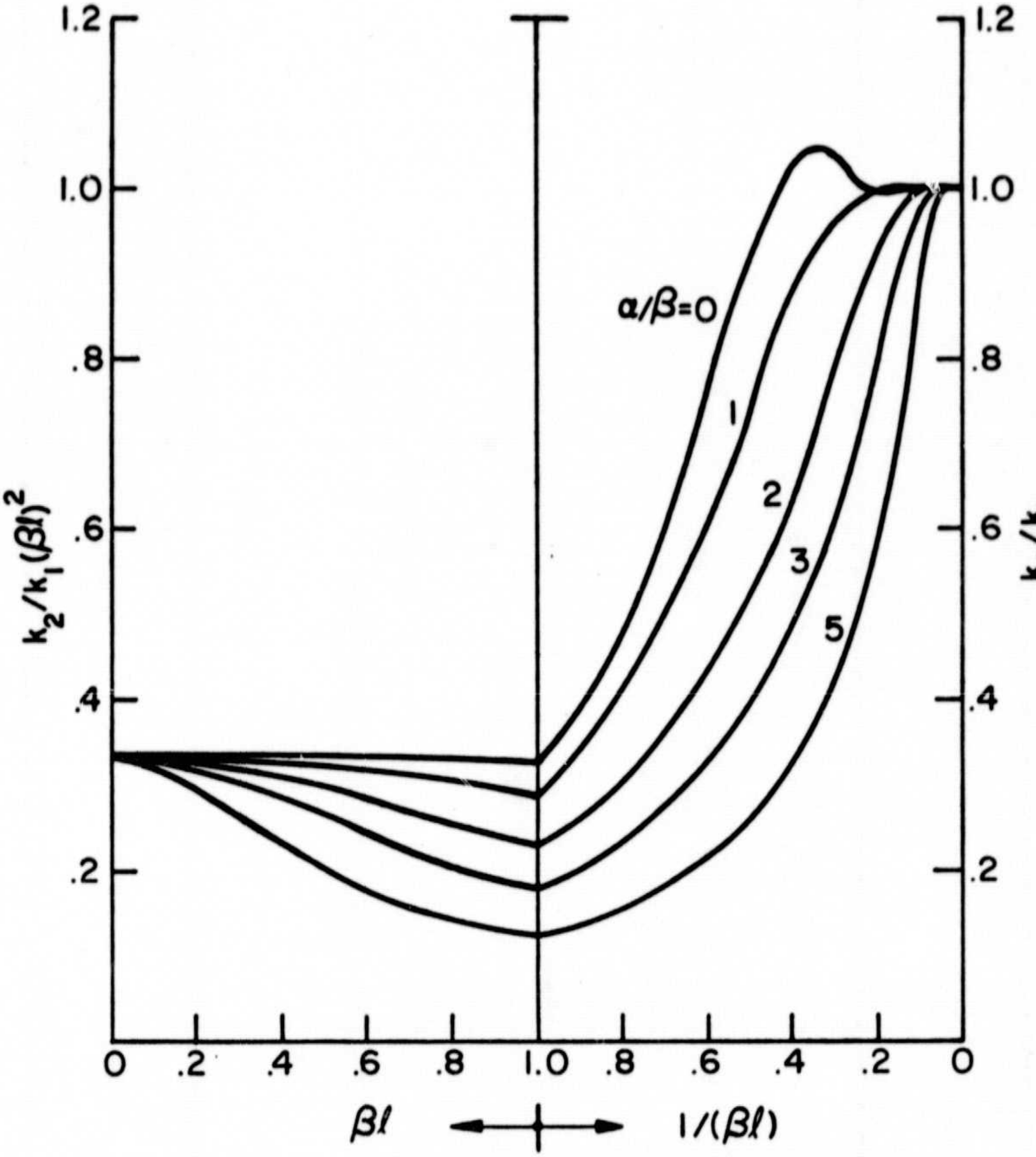

Fig. 5 Values of k_2/k_1

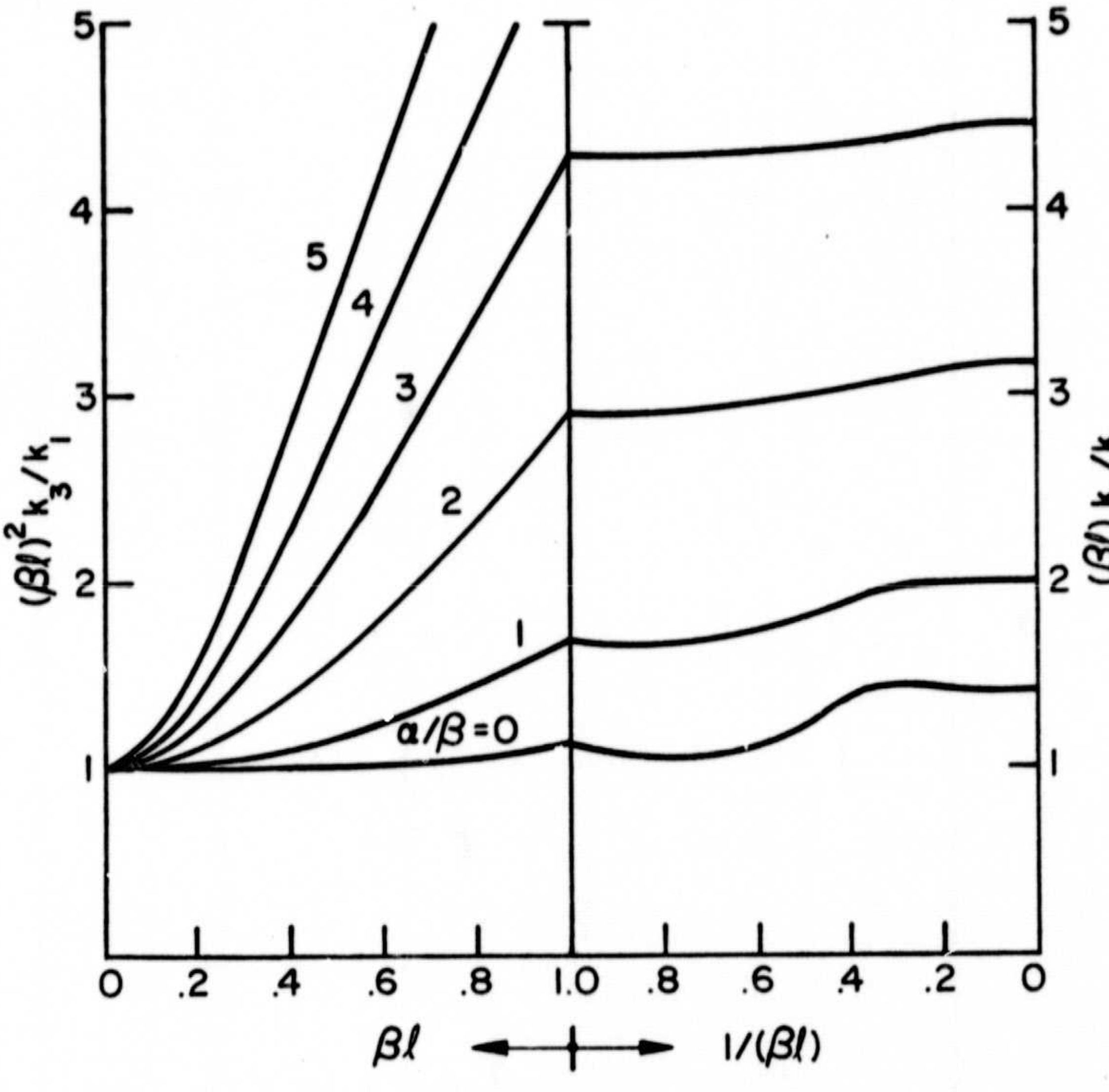

Fig. 6 Values of k_3/k_1

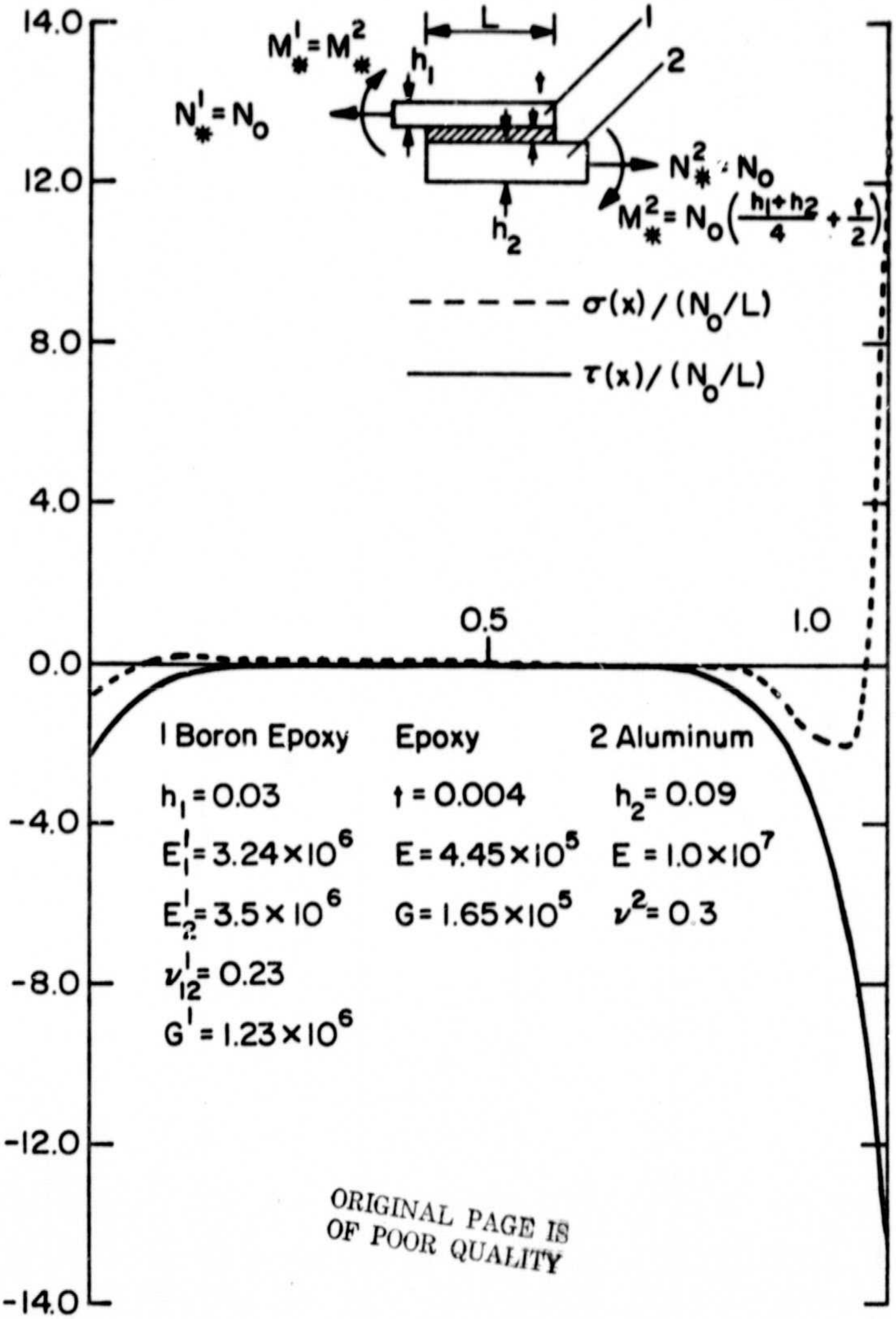

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Fig. 7 Joint with Dissimilar Adherends- External Tension (N_o=1.0, L=1.0)

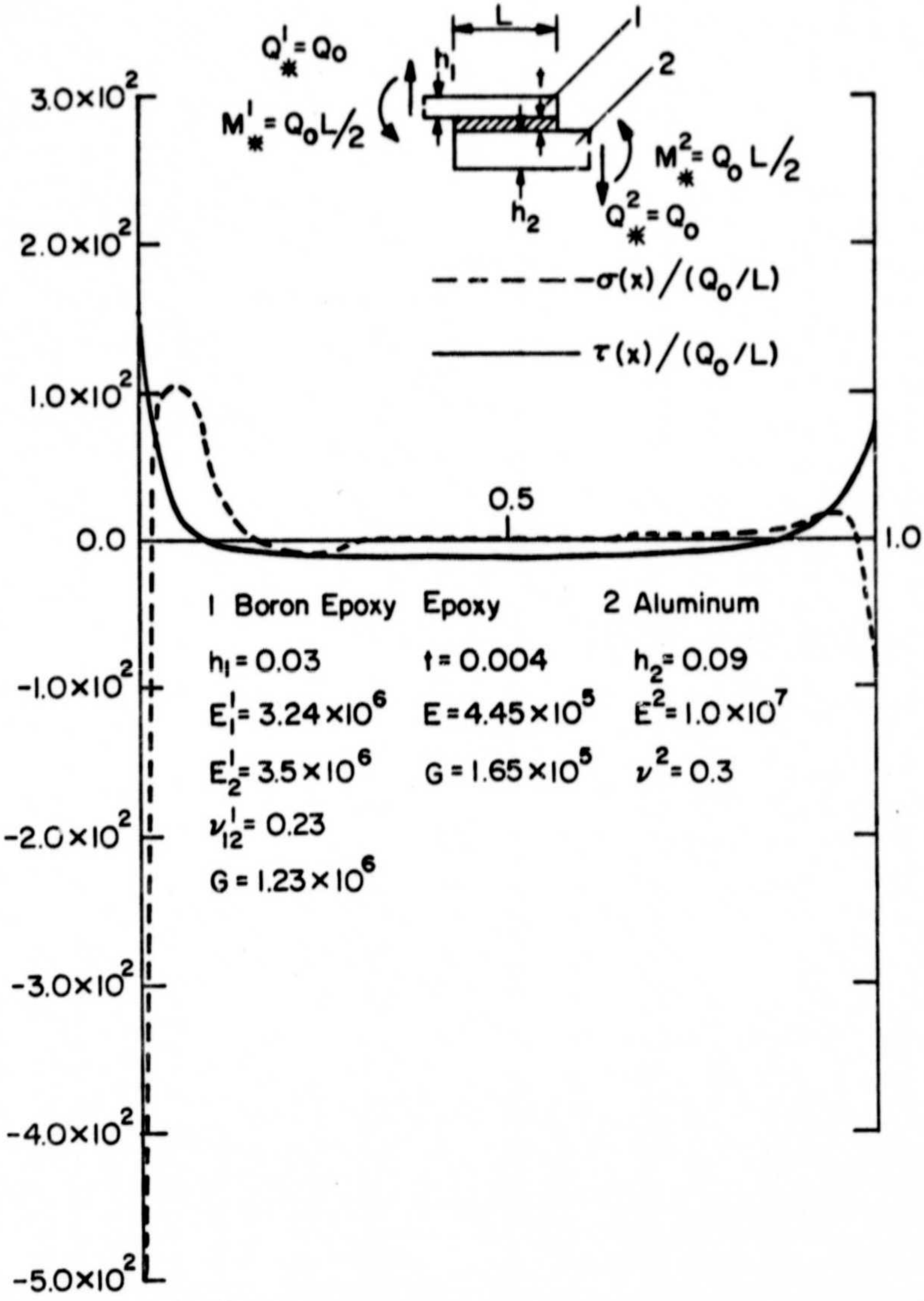

Fig. 8 Joint with Dissimilar Adherends- External Shear (Q_o=1.0, L=1.0)

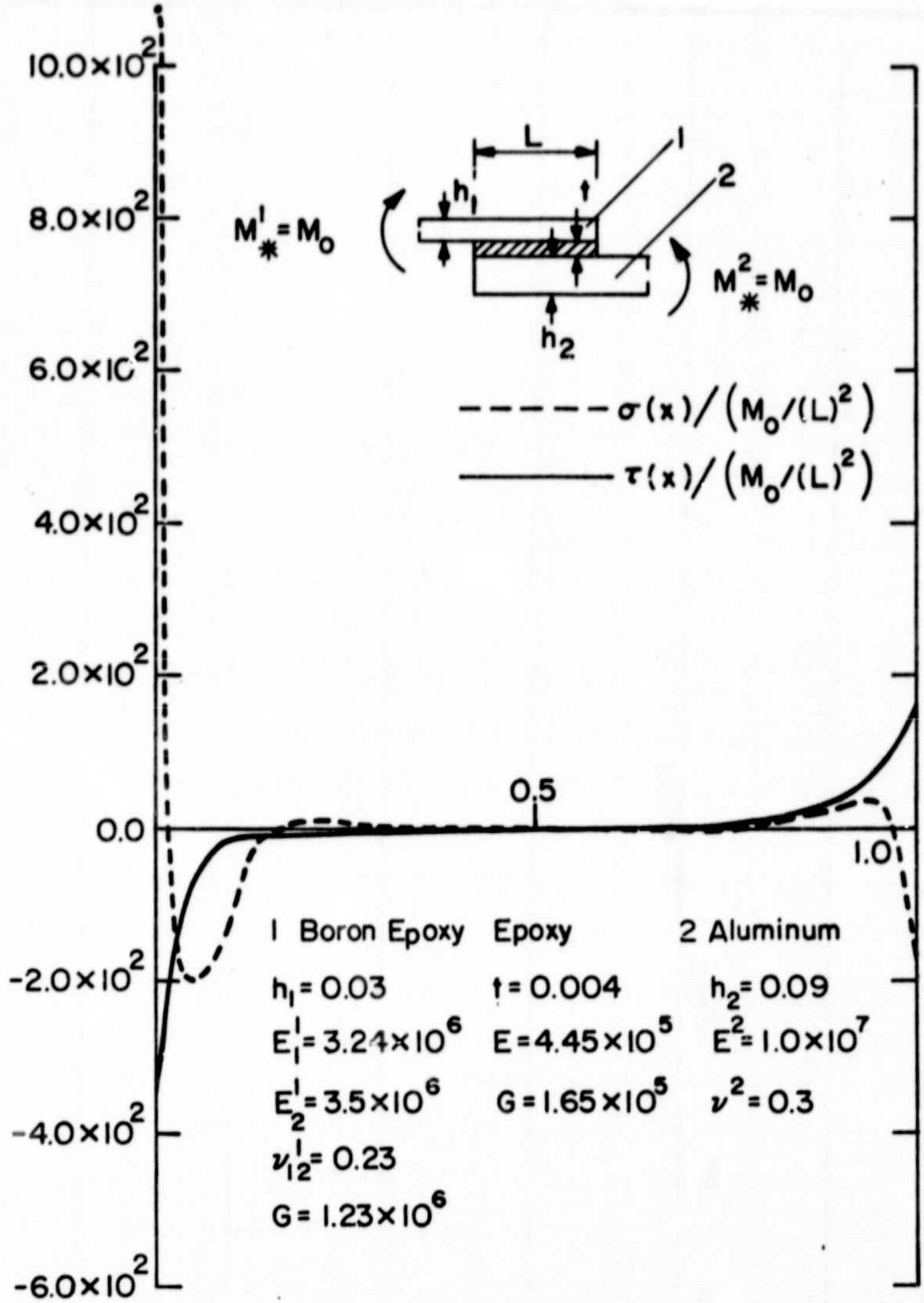

Fig. 9 Joint with Dissimilar Adherends- External Bending Moment (M_o=1.0, L=1.0)

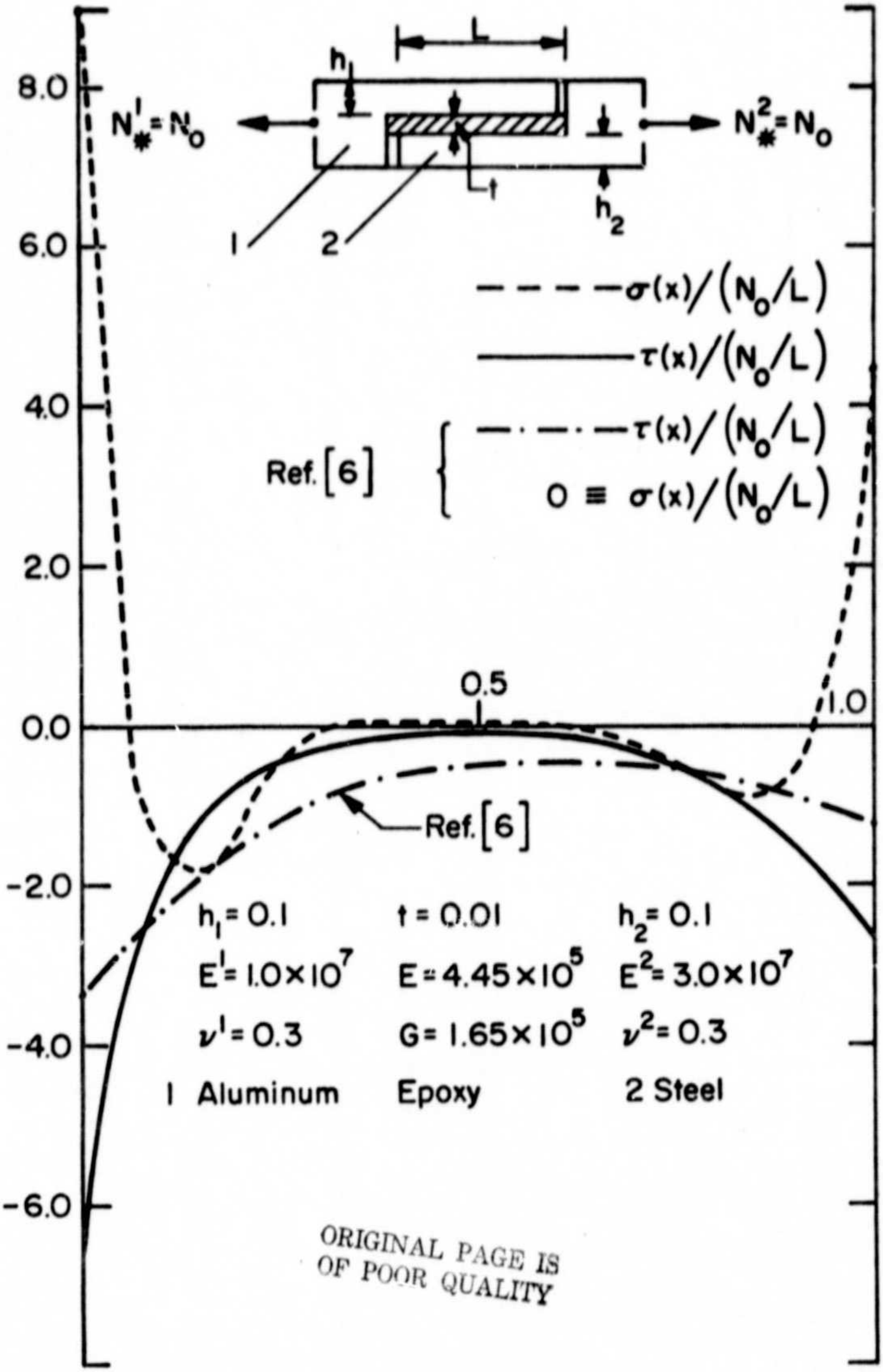

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Fig. 10 The Bending Effect on the Adhesive Shear Stresses $\tau(x)$ in a Stepped Joint with Dissimilar Adherends. (N_o=1.0, L=1.0)

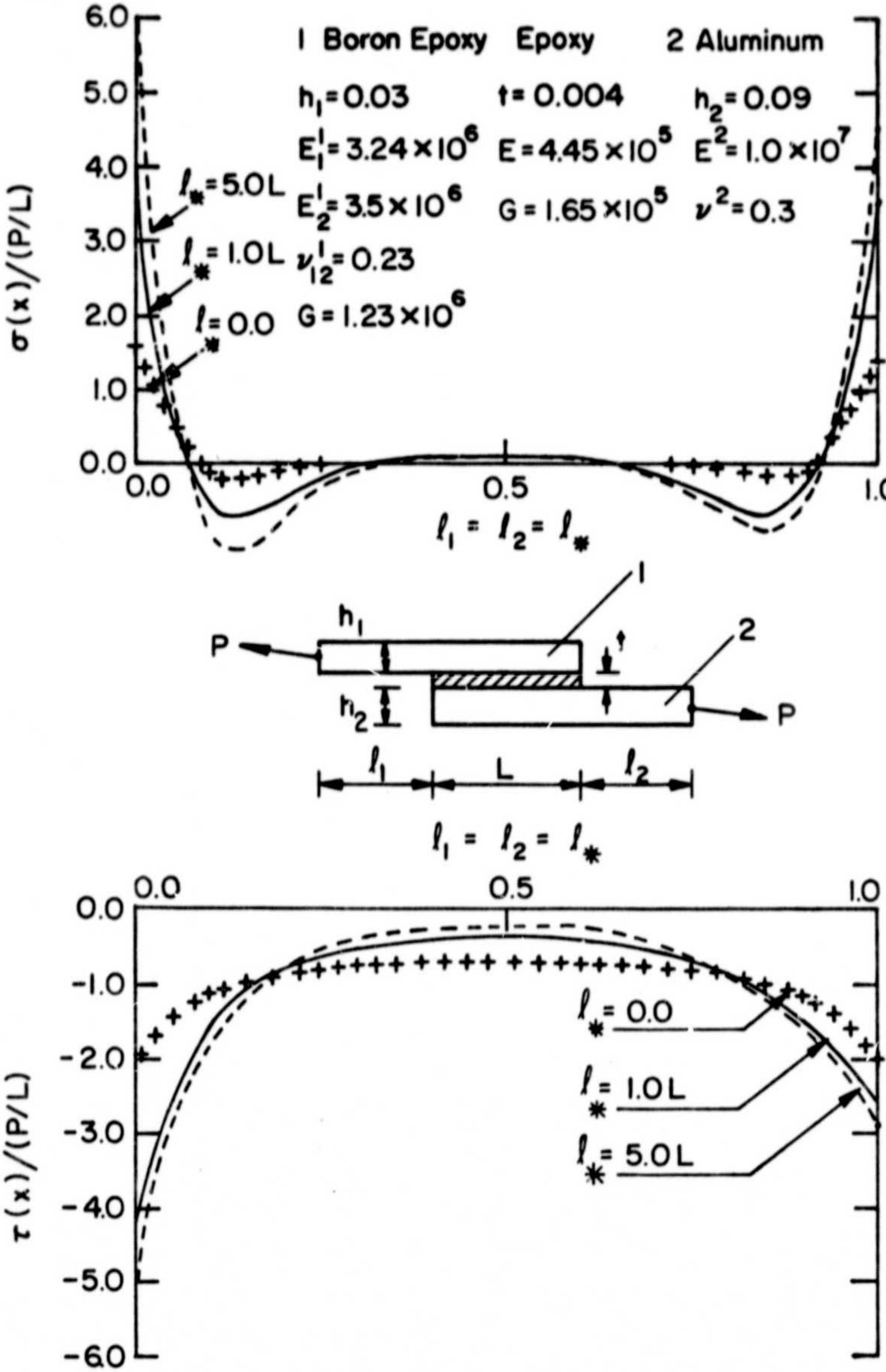

Fig. 11 Effect of Protruding Lengths on the Stresses in a Lap Joint with Dissimilar Adherends. (P=1.0, L=1.0)