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ATTITUDE ESTIMATION OF EARTH ORBITING SATELLITES BY DECOMPOSED LINEAR RECURSIVE FILTERS

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16. ABSTRACT

Attitude estimation of earth orbiting satellites (including Large Space Telescope) subjected to environmental disturbances and noises has been investigated. Modern control and estimation theory is used as a tool to design an efficient estimator for attitude estimation. This new derived estimator is called Decomposed Linear Recursive Filter. It is a statistical filter which is optimal in the sense of minimal mean square error. Comparison between this filter and Kalman filter has been discussed. This filter can overcome the computational difficulty of implementing Kalman filter digitally (integration error, roundoff error, etc.). It is shown that simplicity, accuracy and speed are the advantages of these filters. Decomposed linear recursive filters for both continuous-time systems and discrete-time systems are derived. By using this accurate estimation of the attitude of spacecrafts, state variable feedback controller may be designed to achieve (or satisfy) high requirements of system performance.

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DEFINITION OF SYMBOLS

Symbol	Definition
A	transition matrix of a linear system
A^T	transpose of a matrix A
$\underline{A}$	transition matrix of a composed system
A_k	transition matrix of a linear discrete time system, the elements of this matrix are evaluated at time t_k
$\underline{A}_k$	transition matrix of a composed discrete-time system
B	controlled input matrix
$\underline{B}$	controlled input matrix for composed system
B_k	controlled input matrix for discrete-time system
C	output matrix
$\underline{C}$	composed output matrix
C_1	output matrix corresponding to state variable x
C_2	output matrix corresponding to state variable z
C_k	output matrix for discrete-time system
$\underline{C}_k$	output matrix for composed discrete-time system
D	matrix related to the dynamics of disturbance z
D_k	the D matrix in the discrete time case

Symbol	Definition
F	related to deterministic disturbances
G	related to stochastic disturbances
$\underline{G}$	the G matrix for composed system
H	transformation matrix between variables w and z
$K(t)$	optimal gain matrix of Kalman filter
K_k	optimal gain matrix of Kalman filter for discrete time case
K_k^x	upper part of K_k with n rows
K_k^z	lower part of K_k with r rows
$\overline{K}(k)$	optimal gain matrix for a particular initial condition
$\overline{K}_x(k)$	upper part of $\overline{K}(t)$
$K_x(k)$	$= K_k^x$
$K_z(k)$	$= K_k^z$
M	a transformation matrix of Friedland's formula (Eq. (22))
M_1	a modification of M matrix
$M(k)$	the M matrix for discrete-time case
$p(w)$	power spectral density function
$P(t)$	covariance matrix

Symbol	Definition
$P_x(t)$	covariance matrix with respect to variable x
$P_{xz}(t)$	covariance matrix of variable x and z
$P_z(t)$	covariance matrix with respect to variable z
$\overline{P}_x(t)$	a special $P_x(t)$ corresponding to a certain initial values
P_1	a special case of P(t)
P_2	a general form of P(t)
P_k	P(t) in discrete time case
$P_x(k)$	$P_x(t)$ in discrete time case
$P_{xz}(k)$	$P_{xz}(t)$ in discrete time case
$P_z(k)$	$P_z(t)$ in discrete time case
$\overline{P}_x(k)$	$\overline{P}_x(t)$ in discrete time case
$\overline{P}(k)$	P_1 in discrete time case
Q	value of covariance function of s(t)
Q_k	Q in discrete time case
R	value of covariance function of v(t)
R_k	R in discrete time case
s	aerodynamic torque and solar pressure torque

Symbol	Definition
s_k	s in the discrete time case
$T(k)$	a posteriori covariance matrix
$T_x(k)$	the part of T(k) related to x
$T_{xz}(k)$	a posteriori covariance matrix of x and z
$T_z(k)$	the part of T(k) related to z
$\overline{T}(k)$	a special case of T(k)
$\overline{T}_x(k)$	the part of $\overline{T}(k)$ related to x
u	control input
$U(k)$	a transformation matrix of Friedland's formula (Eq. (A-16))
$U_x(k)$	upper n rows of U(k)
$U_z(k)$	lower r rows of U(k)
υ_s	covariance function of s(t)
υ_v	covariance function of v(t)
v	sensor noise
V	one of the transformation matrix of the formula eq. (22)
V_x	the first two rows of V
V_z	the last three rows of V

Symbol	Definition
V_1	modified V form
V_2	another modified V form
v_k	sensor noise in the discrete time case
$V(k)$	transformation matrix for the a posteriori covariance matrix
$V_x(k)$	upper n rows of V
$V_z(k)$	lower r rows of V
w	gravity gradient torque and earth magnetic torque
x	state of a system
$\dot{x}$	time derivative of the state
$\hat{x}$	estimated state of the filter
$\bar{x}$	estimated state corresponding to $\bar{P}_x(t)$
x_k	state of a discrete time system
$\hat{x}_k$	estimated state of x_k
$\bar{x}_k$	estimated state of x_k corresponding to $\bar{P}_x(k)$
y	output of a system
y_k	y in the discrete time case
z	transformed deterministic disturbances

Symbol	Definition
$\hat{z}$	estimated value of z
z_k	z in the discrete time case
$\hat{z}_k$	estimated value of z_k
θ	pitch angle of Large Space Telescope
$\dot{\theta}$	time derivative of θ
ω	mechanical angular frequency
$\delta(t)$	Dirac delta function
δ_{kj}	Kronecker delta function

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INTRODUCTION

The problem of attitude determination (or estimation) of earth orbiting satellites is considered in this report. The exactly true attitude of these satellites is not available because of the environmental disturbance torques (e.g. gravity gradient torque, earth magnetic torque, aerodynamic torque, etc.) and some undesirable noises (e.g. rate gyro noise, CMG tachometer noise, etc.). Hence it becomes necessary to estimate the attitude of satellites in some optimal way in order to obtain an estimated (or approximate) attitude and to implement a controller by state variable feedback design. It is well-known that Kalman estimation theory [1] is a useful tool used to estimate the state (or attitude) of linear dynamic systems. But a Kalman filter is difficult to be implemented for higher order systems. Furthermore, if it is implemented digitally then the computational accuracy and speed will decrease rapidly as the order of the system increases. There are several papers working for the reduction of computation burden in the Kalman filter calculations, for example, the works have been done by Johnson [2], Simon and Stubberud [3], Samant and Sorenson [4], Friedland [5], etc. The first three of these papers are so-called reduced order Kalman filters in which the required computations are greatly reduced when only part of the states are interesting and being estimated. Friedland's work [5] discussed a complete state estimation problem. In his paper only the situation of constant bias (or disturbance) is considered. This paper will extend Friedland's work [5] to include the time-varying disturbance case. It will be shown that the environmental disturbances and noises on the earth orbiting satellites are really either time-varying or random. We will use the Large Space Telescope (LST), which is an unmanned astronomical observatory facility with 3-meter diameter of the primary mirror, as an example. It is developed by NASA under the direction of Marshall Space Flight Center at Huntsville, Alabama. The description of LST system is shown in section 2. Special attention is given to model stochastic environmental disturbances and sensor noise as Gaussian stationary white noise processes. For the reason of comparison, a Kalman filter for estimating states of the LST system is included in section 3. The main result of this paper which is called "Decomposed Linear Recursive Filter" (a useful extention of Friedland's work [5]) is derived in section 4. In this filter, the covariance matrix has been transformed into smaller matrices in order to reduce the computation burden. Discussions and conclusions are in section 5. A Decomposed Linear Recursive Filter for discrete-time systems has also been developed. Its derivation is slightly different from that of continuous-time case and is included in the Appendix. References are in section 7.

SYSTEM DESCRIPTION

In order to explain dynamics of earth orbiting satellites and some physical quantities in detail we use Large Space Telescope as an example. The mathematical model of LST system described in this section is basically according to Schiehlen [6]. The LST is modelled as a rigid body having reaction wheels as actuators and subjected to gravitational and magnetic disturbance torques. These torques are persistent, deterministic disturbances and can be effectively described by linear differential equations (see Johnson [7]). In addition, the aerodynamic torque, the solar pressure torque and the fine guidance sensor noise are also considered which can be modelled as white noise processes. Now for a single-axis analysis LST system is described, according to [6], as

$$\dot{x} = Ax + Bu + Fw + Gs \qquad (1)$$

where x: 2x1 state vector and $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ with θ and $\dot{\theta}$ represent the pitch angle and the time rate of pitch angle of LST.

u: a scalar control variable which is related to the driving motor torque.

w: denotes the gravity gradient torque and earth magnetic torque.

s: denotes the stochastic (or random) aerodynamic and solar pressure torques.

and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The sensor output y corresponding to the pitch motion θ can be represented by

$$y = \theta + v$$

or equivalently

$$y = Cx + v \qquad (2)$$

where $C = [1 \quad 0]$ and v is the sensor noise.

Furthermore, the dynamics of the disturbance torques w can be modelled by

$$w = Hz \tag{3}$$

$$\dot{z} = Dz \tag{4}$$

where z is a 3x1 vector and

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\cdot 10^{-3} \\ 0 & -2\cdot 10^{-3} & 0 \end{bmatrix} , \qquad H = [1 \quad 1 \quad 0]$$

And the white noise process s(t) of Eq. (1) is characterized with zero mean and spectral density $1\cdot 10^{-12}$ arc $\sec^2/\sec^3$. The white noise process v(t) of Eq. (2) is characterized by zero mean and spectral density $8.394\cdot 10^{-6}$ arc $\sec^2$-sec. (These data are obtained from Proise [3]).

For the following analysis we still need to specify the covariances of these white noise processes s(t) and v(t). These covariances can be derived from their spectral densities. The covariance function and the power spectral density function form the Fourier transform relationship for stationary random processes. (see Astrom [9]). It is assumed that white noise processes s(t) and v(t) are stationary. Denote the power spectral density function of s(t) by

$$p(\omega) = 1\cdot 10^{-12}$$

for all angular frequency ω; then its covariance function

$$\begin{aligned} \nu_s(z) &\triangleq \text{Cov}\,[s(t),\ s(t+z)] \\ &= \int_{-\infty}^{\infty} p(\omega)\ e^{i\omega z}\ d\omega \\ &= 2\pi\cdot 10^{-12}\ \ \delta(z) \\ &\triangleq Q\ \delta(z) \end{aligned} \tag{5}$$

where $\delta(\cdot)$ is the Dirac delta function. Similarly, for the white noise v(t), we have

$$\begin{aligned} \nu_v(z) &\triangleq \text{Cov}\,[v(t),\ v(t+z)] \\ &= 2\pi\cdot 8.394\cdot 10^{-6}\ \delta(z) \\ &\triangleq R\ \delta(z) \end{aligned} \tag{6}$$

Here we introduce notations Q and R for the use of future formulation. Let us rewrite (1) - (4) as a composed system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \underline{A} \begin{bmatrix} x \\ z \end{bmatrix} + \underline{B}\, u + \underline{G}\, s \tag{7}$$

and

$$y = \underline{C} \begin{bmatrix} x \\ z \end{bmatrix} + v \tag{8}$$

where

$$\underline{A} = \begin{bmatrix} A & FH \\ 0 & D \end{bmatrix} \quad \underline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \underline{G} = \begin{bmatrix} G \\ 0 \end{bmatrix} \tag{9}$$

$$\underline{C} = [C \quad 0] = [1 \quad 0 \quad 0 \quad 0 \quad 0]$$

Since it is without loss of generality to omit the control term $\underline{B}u$ in the consideration of estimation problems, let us consider the following simplified system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \underline{A} \begin{bmatrix} x \\ z \end{bmatrix} + \underline{G}\, s \tag{10}$$

and

$$y = \underline{C} \begin{bmatrix} x \\ z \end{bmatrix} + v \tag{11}$$

where $E\,[s(t)\; s^T(\tau)] = Q\, \delta\, (t-\tau)$,

$E\,[v(t)\; v^T(\tau)] = R\, \delta\, (t-\tau)$,

and $s(t)$, $v(\tau)$ are independent.

STATE ESTIMATION BY KALMAN FILTER

The Kalman filter is a well-known state estimator for linear stochastic systems. There are numerous references about the derivation and discussions of Kalman filter (e.g. Sorenson [10]). In this section, we will show the use of a Kalman filter for state estimation of LST system. When we apply Kalman filter to LST system described by Eqs. (10), (11), we have

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{bmatrix} = \underline{A} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} + K(t)\, [y(t) - \underline{C} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix}] \tag{12}$$

where $\hat{x}$ and $\hat{z}$ are the estimated states of x and z respectively and the optimal gain K(t) is defined by

$$K(t) = P(t)\, \underline{C}^T R^{-1} \tag{13}$$

and P(t) is a 5x5 covariance matrix which satisfies the following Riccati equation

$$\dot{P} = \underline{A}\,P + P\,\underline{A}^T + \underline{G}\,Q\,\underline{G}^T - P\,\underline{C}^T\,R^{-1}\,\underline{C}\,P \tag{14}$$

From Eq. (12) we see that for the 5th order system (LST system), Kalman filter is also a 5th order dynamic system. The dynamic behavior of the covariance matrix P(t) is governed by Eq. (14) which is nonlinear matrix differential equation. Furthermore P(t) is a symmetric matrix function.

This is an optimal filter (or estimator) in the sense of minimal mean square error between the true state and the estimated state. Now for the special structures of matrices $\underline{A}$, $\underline{C}$ and $\underline{G}$ of the LST system (see Eq. (9)), the covariance matrix P(t) and the matrix equation (14) can be partitioned as follows:

$$P(t) = \begin{bmatrix} P_x & P_{xz} \\ P_{xz}^T & P_z \end{bmatrix} \tag{15}$$

where Px is the 2x2 covariance matrix of estimation of x. Pz is the 3x3 covariance matrix of estimation of z. Pxz is the 2x3 cross-covariance matrix for estimation of x and z. Then the matrix covariance equation (14) takes the form of the following three equations

$$\dot{P}_x = A\,P_x + FH\,P_{xz}^T + P_x\,A^T + P_{xz}(FH)^T - P_x\,C^TR^{-1}C\,P_x + G\,Q\,G^T \tag{16}$$

$$\dot{P}_{xz} = A\,P_{xz} + FH\,P_z + P_{xz}\,D^T - P_x\,C^T\,R^{-1}\,C\,P_{xz} \tag{17}$$

$$\dot{P}_z = D\,P_z + P_z\,D^T - P_{xz}^T\,C^T\,R^{-1}\,C\,P_{xz} \tag{18}$$

Eqs. (16)-(18) are composed of fifteen nonlinear differential equations. These equations are coupled to each other and should be solved simultaneously. Furthermore, Eq. (12) is composed of 5 differential equations which are coupled to each other too. So to obtain the estimated state $\hat{x}$ and $\hat{z}$ we have to solve the simultaneous 15 differential equations (Eqs. (16)-(18)) and simultaneous 5 differential equations (12). This is quite computationally time-consuming and has severe numerical integration errors. To solve these problems, we derive the Decomposed Linear Recursive Filter in the next section.

DECOMPOSED LINEAR RECURSIVE FILTER

We would like to point out that the objective of Friedland's research [5] is to estimate the state of the following types of systems

$$\begin{cases} \dot{x} = Ax + Bz + Gs \\ \dot{z} = o \end{cases} \qquad y = Cx + v \tag{19}$$

The second part of this composed system (19), $\dot{z} = o$, represents an unknown constant quantity. In this section, the Decomposed Linear Recursive Filter is derived which is an extension of the above work to include the case $\dot{z} = Dz$ where D is not a zero matrix, i.e. z is a time-varying quantity. Hence this result is applicable to the earth orbiting satellites including the LST system described by Eq. (10). The basic idea of this filter is to use matrix transformation of the covariance matrix and the particular system structure of earth orbiting satellites to simplify the estimation procedure. It turns out that we need to solve only lower order matrix equations and estimated states $\hat{x}$ and $\hat{z}$ can be obtained separately. This will greatly increase the computational speed and accuracy of estimation of the attitude of satellites.

At first we investigate a particular solution for the matrix equations (16)-(18). Note that Eqs. (17) and (18) are homogeneous in P_{xz} and P_z. Hence, if $P_{xz}(o) = o$ and $P_z(o) = o$ then $P_{xz}(t) \equiv o$ for all $t > o$, then Eq. (16) becomes

$$\dot{\overline{P}}_x = A\,\overline{P}_x + \overline{P}_x\,A^T + G\,Q\,G^T - \overline{P}_x\,C^T\,R^{-1}\,C\,\overline{P}_x \tag{20}$$

the notation $\overline{P}_x$ is used to point out the forms of Eqs. (16) and (20) are different. So for this particular selection of initial values of covariance matrix, let us denote the solution of Riccati equation (14) by

$$P_1(t) = \begin{bmatrix} \overline{P}_x(t) & 0 \\ 0 & 0 \end{bmatrix} \tag{21}$$

The relation between solutions of Riccati equations of the form (14) corresponding to different initial conditions can be expressed according to a result of Friedland [11].

Theorem ([11]): If $P_1(t)$ is a solution to Riccati equation (14), any other solution $P_2(t)$ of Eq. (14) corresponding to a different initial condition can be expressed as follows:

$$P_2(t) = P_1(t) + VMV^T \tag{22}$$

where V is a 5x3 matrix satisfying $\dot{V} = (\underline{A} - P_1\,\underline{C}^T\,R^{-1}\,\underline{C})\,V$ (23)

and M is a 3x3 matrix satisfying $\dot{M} = -M\,V^T\,\underline{C}^T\,R^{-1}\,\underline{C}\,V\,M$ (24)

and initial condition of $P_2(t)$ is

$$P_2(o) = \begin{bmatrix} \overline{P}_x(o) & P_{xz}(o) \\ P_{xz}^T(o) & P_z(o) \end{bmatrix} \tag{25}$$

where $\overline{P}_x(o)$ is the nonzero part of the initial condition for $P_1(t)$.

In this theorem, we see that M is a symmetric matrix and V can be partitioned as follows

$$V = \begin{bmatrix} V_x \\ V_z \end{bmatrix} \tag{26}$$

where V_x is 2x3 matrix and V_z is 3x3 matrix. Then by using the definition of $\underline{A}$ and P_1, Eqs. (23), (24) can be expressed by

$$\dot{V}_x = (A - \overline{P}_x C^T R^{-1} C) V_x + FH V_z \tag{27}$$

$$\dot{V}_z = D V_z \tag{28}$$

$$\dot{M} = -M V_x^T C^T R^{-1} C V_x M \tag{29}$$

Now let $P_2(t)$ be partitioned according to (15), then

$$\begin{aligned} P_x(t) &= \overline{P}_x(t) + V_x M V_x^T \\ P_{xz}(t) &= V_x M V_z^T \\ P_z(t) &= V_z M V_z^T \end{aligned} \tag{30}$$

So the initial conditions $V_x(o)$, $V_z(o)$, $M(o)$ of Eqs. (27)-(29) must satisfy

$$P_x(o) = \overline{P}_x(o) + V_x(o) M V_x^T(o)$$

$$P_{xz}(o) = V_x(o) M(o) V_z^T(o)$$

$$P_z(o) = V_z(o) M(o) V_z^T(o)$$

If $P_{xz}(o)$ is also assumed to be zero, then we have

$$\begin{aligned} V_x(o) &= o \\ V_z(o) &= I \\ M(o) &= P_z(o) \end{aligned} \tag{31}$$

We want to show that the inverse of the square matrix V_z exists. From Eqs. (28) and (31) we have

$$V_z(t) = e^{Dt} V_z(o) = e^{Dt}$$

which is nonsingular (see Brockett [12]). Hence its inverse V_z^{-1} exists. Now for the purpose of simplification of equations used for estimation of states of LST system, we introduce new variables V_1 and V_2

$$V_1 \triangleq V V_z^{-1} = \begin{bmatrix} V_x \\ V_z \end{bmatrix} V_z^{-1} = \begin{bmatrix} V_x V_z^{-1} \\ I \end{bmatrix} \tag{32}$$

$$V_2 \triangleq V_x V_z^{-1} \tag{33}$$

Here we have V_1 is also a 5x3 matrix consisting of two parts-- the lower part is a constant identity matrix. Again we introduce new variable M_1

$$M_1 = V_z M V_z^{-1} \tag{34}$$

Now we try to express $P_z(t)$ in terms of these new matrices V_1, V_2 and M_1. From Eq. (22), we have

$$\begin{aligned} P_2(t) &= P_1 + V_1 V_z M (V_1 V_z)^T \\ &= P_1 + V_1 M_1 V_1^T \end{aligned} \tag{35}$$

And from Eq. (30) we have

$$\begin{aligned} P_x(t) &= \overline{P}_x(t) + V_x V_z^{-1} V_z M V_z^T (V_z^T)^{-1} V_x^T \\ &= \overline{P}_x(t) + V_x V_z^{-1} V_z M V_z^T (V_z^{-1})^T V_x^T \\ &= \overline{P}_x(t) + V_2 M_1 V_2^T \end{aligned} \tag{36}$$

$$\begin{aligned} P_{xz}(t) &= V_x V_z^{-1} V_z M V_z^T \\ &= V_2 M_1 \end{aligned} \tag{37}$$

$$P_z(t) = V_z M V_z^T = M_1 \tag{38}$$

So far we know that to obtain P_x, P_{xz} and P_z we need only to have V_2, M_1 and $\overline{P}_x$. $\overline{P}_x$ can be obtained from Eq. (20). Now we will show that V_2 and M_1 satisfies the following equations:

$$\dot{V}_2 = (A - \overline{P}_x C^T R^{-1} C) V_2 + FH - V_2 D \tag{39}$$

$$\dot{M}_1 = D\,M_1 - M_1\,V_2^T\,C^T\,R^{-1}\,C\,V_2\,M_1 + M_1\,D^T \tag{40}$$

Proof: Since $V_2 = V_x\,V_z^{-1}$,

$$\dot{V}_2 = \dot{V}_x\,V_z^{-1} + V_x(\dot{V}_z^{-1})$$

$$= \dot{V}_x\,V_z^{-1} - V_x\,V_z^{-1}\,\dot{V}_z\,V_z^{-1}$$

$$= [(A - \overline{P}_x\,C^T\,R^{-1}\,C)\;V_x + FH\,V_z]\;V_z^{-1} - V_x\,V_z^{-1}\,D\,V_z\,V_z^{-1}$$

$$= (A - \overline{P}_x\,C^T\,R^{-1}\,C)\,V_2 + FH - V_2\,D$$

Similarly $M_1 = V_z\,M\,V_z^T$, so

$$\dot{M}_1 = \dot{V}_z\,M\,V_z^T + V_z\,\dot{M}\,V_z^T + V_z\,M\,\dot{V}_z^T$$

$$= D\,V_z\,M\,V_z^T + V_z\,(-M\,V_x^T\,C^T\,R^{-1}\,C\,V_x\,M)\,V_z^T + V_z\,M\,(D\,V_z)^T$$

$$= D\,M_1 - M_1\,V_2^T\,C^T\,R^{-1}\,C\,V_2\,M_1 + M_1\,D^T$$

Remark: In the above analysis, we see that to find covariance matrix $P_2(t)$ (which is governed by a nonlinear coupled matrix differential equation (14) and should be solved simultaneously as shown in section 3), we need only find out $V_2(t)$, $M_1(t)$ and $\overline{P}_x(t)$. Now the important thing is that the equations ((20), (39) and (40)) of V_2, M_1 and $\overline{P}_x$ are not coupled in the following sense. We can solve $\overline{P}_x$ by using Eq. (20) without using any information about V_2 and M_1. Next we obtain V_2 from Eq. (39) and information $\overline{P}_x$. Finally using V_2, $\overline{P}_x$ and Eq. (40) we obtain M_1. As a matter of fact these matrices can be solved separately. It is obvious that by using the method of this analysis to find covariance matrix, we can save a lot of computation time and increase greatly the numerical integration accuracy.

Next if we substitute the results Eqs. (36)-(38) into Eqs. (12) and (13) we have

$$\dot{\hat{x}} = A\,\hat{x} + FH\,\hat{z} + (\overline{P}_x + V_2\,M_1\,V_2^T)\,C^T\,R^{-1}\,(y - C\,\hat{x}) \tag{41}$$

$$\dot{\hat{z}} = D\,\hat{z} + M_1\,V_2^T\,C^T\,R^{-1}\,(y - C\,\hat{x}) \tag{42}$$

Now we want to show that how $\hat{x}$ and $\hat{z}$ can be obtained separately

Theorem: The solution $\hat{x}$ of Eq. (41) can be expressed by

$$\hat{x} = \bar{x} + V_2 \, \hat{z} \tag{43}$$

where $\bar{x}$ is the solution of the following system

$$\dot{\bar{x}} = A \, \bar{x} + \bar{P}_x \, C^T \, R^{-1} \, (y - C \, \bar{x}) \tag{44}$$

and V_2, $\hat{z}$ satisfies Eqs. (39) and (42).

Remark: $\bar{x}(t)$ can be obtained from Eq. (44) whenever we have $\bar{P}_x$ and $y(t)$. It is in fact the estimate of state $x(t)$ when there is no deterministic external disturbance torques, i.e. $z = o$.

Proof: To prove this theorem, we have to check that the expression (43) satisfies Eq. (41) or equivalently the following formula should be equal to zero

$$\dot{\hat{x}} - A \, \hat{x} - FH \, \hat{z} - (\bar{P}_x + V_2 \, M_1 \, V_2^T) \, C^T \, R^{-1} \, (y - C \, \hat{x}) = 0 \tag{45}$$

In the following we will substitute all $\hat{x}$ by $\bar{x} + V_2 \, \hat{z}$ to the left hand side of Eq. (45). First we give an expression for $\dot{\hat{x}}$,

$$\begin{aligned}
\dot{\hat{x}} &= \dot{\bar{x}} + \frac{d}{dt} V_2 \, \hat{z} \\
&= \dot{\bar{x}} + \dot{V}_2 \, \hat{z} + V_2 \, \dot{\hat{z}} \\
&= A \, \bar{x} + \bar{P}_x \, C^T \, R^{-1} \, (y - C \, \bar{x}) + [(A - \bar{P}_x \, C^T \, R^{-1} \, C) \, V_2 + FH - V_2 \, D] \, \hat{z} \\
&\quad + V_2 \, [D \, \hat{z} + M_1 \, V_2^T \, C^T \, R^{-1} \, (y - C \, (\bar{x} + V_2 \, \hat{z}))]
\end{aligned}$$

Now the left hand side of Eq. (45) becomes

$$\begin{aligned}
& A \, \bar{x} + \bar{P}_x \, C^T \, R^{-1} \, (y - C \, \bar{x}) + [(A - \bar{P}_x \, C^T \, R^{-1} \, C) \, V_2 + FH - V_2 \, D] \, \hat{z} \\
& + V_2 \, [D \, \hat{z} + M_1 \, V_2^T \, C^T \, R^{-1} \, (y - C \, (\bar{x} + V_2 \, \hat{z}))] - A \, (\bar{x} + V_2 \, \hat{z}) - FH \, \hat{z} \\
& - (\bar{P}_x + V_2 \, M_1 \, V_2^T) \, C^T \, R^{-1} \, (y - C \, (\bar{x} + V_2 \, \hat{z})) \\
= & \, (- \bar{P}_x \, C^T \, R^{-1} \, C \, V_2 - V_2 D) \, \hat{z} + V_2 [D \, \hat{z} + M_1 \, V_2^T C^T R^{-1} \, (y - C(\bar{x} + V_2 \, \hat{z}))] \\
& - V_2 \, M_1 \, V_2^T \, C^T \, R^{-1} \, (y - C(\bar{x} + V_2 \, \hat{z})) + \bar{P}_x \, C^T \, R^{-1} \, C \, V_2 \, \hat{z} \\
= & \, 0
\end{aligned}$$

This completes the proof.

Remark: Based on this theorem and the above analysis, we will summarize the procedures in finding the estimated states $\hat{x}$ and $\hat{z}$ as follows:

(a) From Eq. (20) we have $\overline{P}_x$.

(b) Using $\overline{P}_x$ and Eqs. (39) and (40) we get V_2 and M_1.

(c) Again using $\overline{P}_x$ and the system's output $y(t)$ we get $\overline{x}(t)$ from Eq (44). Note that in this step we don't use any information about $\hat{x}$ or $\hat{z}$.

(d) Now we use the results of (a) - (C) to find $\hat{z}$ without using any information about $\hat{x}$. That is from Eqs. (42) and (43).

$$\dot{\hat{z}} = D\,\hat{z} + M_1\,V_2^T\,C^T\,R^{-1}\,[y - C\,(\overline{x} + V_2\,\hat{z})] \qquad (46)$$

(e) At last from Eq. (43) we have

$$\hat{x} = \overline{x} + V_2\,\hat{z}$$

In this procedure we see that the estimation of states of LST system has been decomposed into two parts; $\hat{z}$ and $\hat{x}$. Estimation of z can be obtained by using data $\overline{x}$ instead of $\hat{x}$. Furthermore $\overline{x}$ and $\hat{z}$ are obtained from Eqs. (44) and (46) which are 2nd-order and 3rd-order systems respectively. On the contrary, if the Kalman filter is used, we have to build up a fifth order dynamic system (filter) to estimate the states of the LST system. So by using the analysis and techniques of this section, filters can be implemented to obtain estimated states $\hat{x}$ and $\hat{z}$ by lower order systems. In addition, the simplicity in structure of this Decomposed Linear Recursive Filter gives us considerable amount of reduction in computation burden and storage requirements.

DISCUSSIONS AND CONCLUSION

(a) Discussions

1. High order estimators usually are too complex for practical mechanization and are very parameter sensitive. Now if we use Decomposed Linear Recursive Filter to estimate the attitude of LST system, we need only to build up 2nd-order and 3rd-order dynamic systems (instead of a 5th order Kalman filter) as an estimator.

2. If the estimators or filters are implemented digitally, the computations required by a high-order filtering algorithm may become excessive (in memory capacity, accuracy and computational speed).

3. The characteristic of this new filter is that the low order covariance matrix Eq. (20) is independent of matrices V and M of Eq. (22). Similarly, the 2nd-order system $\bar{x}$ can be obtained from Eq. (44) independent of the value of $\hat{z}$. This property is a kind of decomposition of a set of equations and greatly simplifies the estimator design.

(b) Conclusion

In this paper, a Decomposed Linear Recursive Filter has been derived for the estimation of attitude of earth orbiting satellites in which the disturbances and noises are either time-varying or random. Simplicity, accuracy and speed are the advantages of this filter. This filter can be easily implemented by the on-board digital computer and give an accurate estimation of attitude of satellites. Furthermore, this is the necessary information for state variable feedback controller design.

The analysis of section 4 can be generalized to linear time-varying systems such as

$$\begin{cases} \dot{x} = A(t)\, x + B(t)\, z + s \\ \dot{z} = D(t)\, z \end{cases}$$

$$y = C_1\, x + C_2\, z + v$$

where A, B, D, C_1 and C_2 can be matrices with time-varying elements, and y may also include the disturbance z explicitly. The proof of this further extention is exactly the same as that of section 4.

APPENDIX

Discrete-Time Version of Decomposed Linear Recursive Filter

Most of the practical implementations of filters are by digital means. Here we derive the analog counterpart of Decomposed Linear Recursive Filter for discrete-time systems.

Consider

$$x_k = A_{k-1} x_{k-1} + B_{k-1} z_{k-1} + s_{k-1} \tag{A-1}$$

$$z_k = D_{k-1} z_{k-1} \tag{A-2}$$

where

x_k is the value of n-dimension state at time t_k,

z_k is the value of r-dimension deterministic environmental disturbances at time t_k,

A_{k-1}, B_{k-1}, D_{k-1} are time-varying matrices with dimensions n x n, n x r, r x r respectively and assume that D_k is nonsingular for all time t_k.

s_{k-1} is n-dimension white noise sequences with

$$E\,[s_k\, s_j^T] = Q_k\, \delta_{kj}$$

where δ_{kj} is the Kronecker delta function.

and the measurement equation is expressed by

$$y_k = C_k x_k + v_k \tag{A-3}$$

where

y_k is the value of m-dimension output at time t_k,

C_k is a m x n time varying matrix,

v_k is m-dimension white noise sequences with

$$E\,[v_k\, v_j^T] = R_k\, \delta_{kj}$$

Combine (A-1) and (A-2), we have

$$\begin{bmatrix} x_k \\ z_k \end{bmatrix} = \underline{A}_{k-1} \begin{bmatrix} x_{k-1} \\ z_{k-1} \end{bmatrix} + G\, s_{k-1} \tag{A-4}$$

and

$$y_k = \underline{C}_k \begin{bmatrix} x_k \\ z_k \end{bmatrix} + v_k \tag{A-5}$$

where

$$\underline{A}_{k-1} = \begin{bmatrix} A_{k-1} & B_{k-1} \\ 0 & D_{k-1} \end{bmatrix}, \quad G = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \underline{C}_k = [C_k \quad 0]$$

Kalman filter formulation for system (A-4), (A-5) is

$$\begin{bmatrix} \hat{x}_k \\ \hat{z}_k \end{bmatrix} = \underline{A}_{k-1} \begin{bmatrix} \hat{x}_{k-1} \\ \hat{z}_{k-1} \end{bmatrix} + K_k \left[y_k - \underline{C}_k \underline{A}_{k-1} \begin{bmatrix} \hat{x}_{k-1} \\ \hat{z}_{k-1} \end{bmatrix}\right] \tag{A-6}$$

where $\hat{x}_k$ and $\hat{z}_k$ are the estimated values of x_k and z_k,

$$K_k = P_k \underline{C}_k^T [\underline{C}_k P_k \underline{C}_k^T + R_k]^{-1} \tag{A-7}$$

P_k is the a priori covariance matrix satisfying the following difference equation

$$P_{k+1} = \underline{A}_k [I - K_k \underline{C}_k] P_k \underline{A}_k^T + G\, Q_{k+1} G^T \tag{A-8}$$

Let

$$P_k = \begin{bmatrix} P_x(k) & P_{xz}(k) \\ P_{xz}^T(k) & P_z(k) \end{bmatrix} \tag{A-9}$$

where

$P_x(k)$ is n x n covariance matrix of estimate of x at time t_k,

$P_z(k)$ is r x r covariance matrix of estimate of z at time t_k,

$P_{xz}(k)$ is n x r covariance matrix of estimates of x and z.

According to the submatrices of Eq. (A-9) and using (A-1), (A-2) and (A-3), we have Eq. (A-8) being expressed by the following three equations

$$P_x(k+1) = [\, A_k\, (I - K_k^x\, C_k) - B_k\, K_k^z\, C_k]\; [\, P_x(k)\, A_k^T + P_{xz}(k)\, B_k^T]$$

$$+ B_k\, [\, P_{xz}^T(k)\, A_k^T + P_z(k)\, B_k^T\,] + Q_{k+1} \qquad \text{(A-10)}$$

$$P_{xz}(k+1) = [\, A_k\, (I - K_k^x\, C_k) - B_k\, K_k^z\, C_k\,]\, P_{xz}(k)\, D_k^T + B_k\, P_z(k)\, D_k^T \qquad \text{(A-11)}$$

$$P_z(k+1) = -D_k\, K_k^z\, C_k\, P_{xz}(k)\, D_k^T + D_k\, P_z(k)\, D_k^T \qquad \text{(A-12)}$$

where $K_k = \begin{bmatrix} K_k^x \\ K_k^z \end{bmatrix}$, K_k is a $(n + r) \times m$ matrix and K_k^x, K_k^z are $n \times m$ and $r \times m$ respectively.

If $P_z(o) = o$, $P_{xz}(o) = o$, then we have the solution of (A-8) denoted by

$$\overline{P}_k = \begin{bmatrix} \overline{P}_x(k) & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(A-13)}$$

for all $k > o$ and $\overline{P}_x(k)$ satisfies

$$\overline{P}_x(k+1) = [\, A_k - (A_k\, K_k^x + B_k\, K_k^z)\, C_k\,]\, \overline{P}_x(k)\, A_k^T + Q_{k+1} \qquad \text{(A-14)}$$

By Kalman filtering theory (see Sorenson [10]), a priori covariance matrix $\overline{P}_x$ can be related with a posteriori covariance matrix $\overline{T}_x$ by

$$\overline{P}_x(k+1) = A_k\, \overline{T}_x(k)\, A_k^T + Q_{k+1} \qquad \text{(A-15)}$$

Note: By definition, a priori covariance matrix $P_x(k)$ is defined by the covariance matrix of the estimate of $x(t_k)$ given the observations $\{ y_1, y_2, \ldots, y_{k-1} \}$ but not y_k. A posteriori covariance matrix $T_x(k)$ is the covariance matrix of estimating $x(t_k)$ given the observation $\{ y_1, y_2, \ldots, y_k \}$, i.e. including the current output y_k.

Theorem (Friedland [11]) If $\overline{P}(k)$ is a solution of equation (A-8), then any other solution $P(k)$ of Eq. (A-8) corresponding to different initial conditions can be expressed by

$$P(k) = \overline{P}(k) + U(k)\, M(k)\, U^T(k) \qquad \text{(A-16)}$$

where

$U(k)$ is a $(n + r) \times r$ matrix and $M(k)$ is a $r \times r$ matrix satisfying

$$U(k+1) = \underline{A}_k\, [\, I - \overline{P}(k)\, \underline{C}_k^T\, (\underline{C}_k\, \overline{P}(k)\, \underline{C}_k^T + R_k)^{-1}\, \underline{C}_k\,]\, U(k) \qquad \text{(A-17)}$$

$$M(k+1) = M(k) - M(k)\, U^T(k)\, \underline{C}_k^T [\underline{C}_k \overline{P}(k) \underline{C}_k^T + R_k + \underline{C}_k U(k) M(k) U^T(k) \underline{C}_k^T]^{-1}$$

$$\cdot\, \underline{C}_k\, U(k)\, M(k) \qquad \text{(A-18)}$$

In terms of this transformation, (A-16) can be rewritten as

$$P(k+1) = \overline{P}(k+1) + U(k+1)\, M(k+1)\, U^T(k+1) \qquad \text{(A-19)}$$

$$= \overline{P}(k+1) + \underline{A}_k [\, I - \overline{K}(k)\, C_k \,]\, U(k)\, M(k+1)\, U^T(k)\, [I - \overline{K}(k)\, C_k]^T \underline{A}_k^T \qquad \text{(A-20)}$$

where

$$\overline{K}(k) = \overline{P}(k)\, \underline{C}_k^T\, [\, \underline{C}_k\, \overline{P}(k)\, \underline{C}_k^T + R_k \,]^{-1} \qquad \text{(A-21)}$$

Similarly, we have

$$P(k+1) = \underline{A}_k\, T(k)\, \underline{A}_k^T + G\, Q_{k+1}\, G^T \qquad \text{(A-22)}$$

$$\overline{P}(k+1) = \underline{A}_k\, \overline{T}(k)\, \underline{A}_k^T + G\, Q_{k+1}\, G^T \qquad \text{(A-23)}$$

Hence (A-20) can be expressed as

$$T(k) = \overline{T}(k) + V(k)\, M(k+1)\, V^T(k) \qquad \text{(A-24)}$$

where

$$V(k) = [\, I - \overline{K}(k)\, \underline{C}_k \,]\, U(k) \qquad \text{(A-25)}$$

and by (A-17) we have

$$U(k+1) = \underline{A}_k\, V(k) \qquad \text{(A-26)}$$

Using (A-13), Equation (A-21) turns out to be

$$\overline{K}(k) = \begin{bmatrix} \overline{P}_x(k)\, C_k^T\, [\, C_k\, \overline{P}_x(k)\, C_k^T + R_k \,]^{-1} \\ 0 \end{bmatrix} \triangleq \begin{bmatrix} \overline{K}_x(k) \\ 0 \end{bmatrix} \qquad \text{(A-27)}$$

Let U(k) and V(k) be also partitioned as follows

$$U(k) = \begin{bmatrix} U_x(k) \\ U_z(k) \end{bmatrix}, \quad V(k) = \begin{bmatrix} V_x(k) \\ V_z(k) \end{bmatrix} \qquad \text{(A-28)}$$

where U_x is n x r, U_z is r x r, V_x is n x r and V_z is r x r.
Then, using (A-27), V(k) and U(k+1) of (A-25), (A-26) become

$$\begin{cases} V_x(k) = [\, I - \overline{K}_x(k)\, C_k \,]\, U_x(k) \\ V_z(k) = U_z(k) \end{cases} \qquad \text{(A-29)}$$

and

$$\begin{cases} U_x(k+1) = A_k\ V_x(k) + B_k\ V_z(k) \\ U_z(k+1) = D_k\ V_z(k) \end{cases} \tag{A-30}$$

Finally, Eq. (A-19) becomes

$$\begin{aligned} P_x(k) &= \overline{P}_x(k) + U_x(k)\ M(k)\ U_x'^T(k) \\ P_{xz}(k) &= U_x(k)\ M(k)\ U_z^T(k) \\ P_z(k) &= U_z(k)\ M(k)\ U_z^T(k) \end{aligned} \tag{A-31}$$

and Eq. (A-24) becomes

$$\begin{aligned} T_x(k) &= \overline{T}_x(k) + V_x(k)\ M(k+1)\ V_x^T(k) \\ T_{xz}(k) &= V_x(k)\ M(k+1)\ V_z^T(k) \\ T_z(k) &= V_z(k)\ M(k+1)\ V_z^T(k) \end{aligned} \tag{A-32}$$

It can be shown that " of (A-7) can be expressed by

$$K_k = T(k)\ \underline{C}_k^T\ R_k^{-1} \tag{A-33}$$

or

$$\begin{bmatrix} K_x(k) \\ K_z(k) \end{bmatrix} = \begin{bmatrix} T_x\ C_k^T\ R_k^{-1} \\ T_{xz}^T\ C_k^T\ R_k^{-1} \end{bmatrix} \tag{A-34}$$

Substituting T_x and T_{xz}^T according to (A-32), we have

$$K_x(k) = (\overline{T}_x(k) + V_x\ M(k+1)\ V_x^T)\ C_k^T\ R_k^{-1} \tag{A-35}$$

$$K_z(k) = V_z(k)\ M(k+1)\ V_x^T(k)\ C_k^T\ R_k^{-1} \tag{A-36}$$

Now from (A-6),

$$\hat{x}_{k+1} = A_k\ \hat{x}_k + B_k\ \hat{z}_k + K_x(k+1)\ [y_{k+1} - C_{k+1}\ A_k\ \hat{x}_k - C_{k+1}\ B_k\ \hat{z}_k] \tag{A-37}$$

and

$$\hat{z}_{k+1} = D_k\ \hat{z}_k + K_z(k+1)\ [\ y_{k+1} - C_{k+1}\ A_k\ \hat{x}_k - C_{k+1}\ B_k\ \hat{z}_k\ ] \tag{A-38}$$

Theorem: The solution $\hat{x}_k$ of Eq. (A-37) can be expressed as

$$\hat{x}_k = \overline{x}_k + V_x(k)\ V_z^{-1}(k)\ \hat{z}_k \tag{A-39}$$

where $\bar{x}_k$ is the solution of the following equation

$$\bar{x}_{k+1} = A_k\ \bar{x}_k + \bar{K}_x(k+1)\ [\ y_{k+1} - C_{k+1}\ A_k\ \bar{x}_k\ ] \tag{A-40}$$

and

$$\bar{K}_x(k) = \bar{P}_x(k)\ C_k^{\ T}\ [\ C_k\ \bar{P}_x(k)\ C_k^{\ T} + R_k\ ]^{-1}$$

and V_x, V_z and $\hat{z}_k$ satisfying (A-29) and (A-38) respectively.

Proof: We first show that V_z^{-1} exists. Since

$$V_z(k+1) = D_k\ V_z(k)$$

$$V_z(o) = I$$

so

$$V_z(k+1) = D_k\ D_{k-1}\ \cdot\cdot\cdot\cdot\ D_o$$

is not singular since D_j are assumed to be nonsingular as in (A-2). So V_z^{-1} exists.

Next we will prove that both sides of Eq. (A-37) are equal by substituting the expression (A-3[illegible]) into Eq. (A-37).
Left hand side of Eq. (A-37) becomes

$$\hat{x}_{k+1} = \bar{x}_{k+1} + V_x(k+1)\ V_z^{-1}(k+1)\ \hat{z}_{k+1}$$

$$= A_k\ \bar{x}_k + \bar{K}_x(k+1)\ [y_{k+1} - C_{k+1}\ A_k\ \bar{x}_k] + V_x(k+1)\ V_z^{-1}(k+1)$$

$$[D_k\ \hat{z}_k + K_z(k+1)\ (y_{k+1} - C_{k+1}\ A_k\ \hat{x}_k - C_{k+1}\ B_k\ \hat{z}_k)\ ]$$

$$= A_k\ \bar{x}_k + \bar{K}_x(k+1)\ [y_{k+1} - C_{k+1}\ A_k\ \bar{x}_k] + V_x(k+1)\ V_z^{-1}(k+1)$$

$$[D_k\ \hat{z}_k + K_z(k+1)\ (y_{k+1} - C_{k+1}\ A_k\ \bar{x}_k - C_{k+1}\ (A_k\ V_k\ V_z^{-1} +$$

$$B_k)\ \hat{z}_k\ ] \tag{A-41}$$

Right hand side of Eq. (A-37) becomes

$$A_k\ \hat{x}_k + B_k\ \hat{z}_k + K_x(k+1)\ [y_{k+1} - C_{k+1}\ A_k\ \hat{x}_k - C_{k+1}\ B_k\ \hat{z}_k\ ]$$

$$= A_k\ \bar{x}_k + A_k\ V_x(k)\ V_z^{-1}(k)\ \hat{z}_k + B_k\ \hat{z}_k + K_x(k+1)\ [y_{k+1}-C_{k+1}\ A_k\ \hat{x}_k-C_{k+1}B_k\hat{z}_k]$$

$$= A_k \bar{x}_k + A_k V_x V_z^{-1} \hat{z}_k + B_k \hat{z}_k + K_x(k+1) [y_{k+1} - C_{k+1} A_k \bar{x}_k - C_{k+1}(A_k V_x V_z^{-1} + B_k) \hat{z}_k] \quad \text{(A-42)}$$

By (A-33) and (A-32) we have

$$K_x(k+1) = [\bar{T}_x(k+1) + V_x(k+1) M(k+2) V_x^T(k+1)] C_{k+1}^T R_{k+1}^{-1}$$

$$= \bar{T}_x(k+1) C_{k+1}^T R_{k+1}^{-1} + V_x(k+1) M(k+2) V_x^T(k+1) C_{k+1}^T R_{k+1}^{-1}$$

$$= \bar{K}_x(k+1) + V_x(k+1) V_z^{-1}(k+1) K_z(k+1) \quad \text{(A-43)}$$

Comparison of expressions (A-41) and (A-42), and using the relation (A-43), it turns out we need only to prove that

$$A_k V_x(k) V_z^{-1}(k) + B_k - K_x(k+1) C_{k+1} (A_k V_x(k) V_z^{-1}(k) + B_k) = V_x(k+1) V_z^{-1}(k+1) D_k - V_x(k+1) V_z^{-1}(k+1) K_z(k+1) C_{k+1}(A_k V_x(k) V_z^{-1}(k) + B_k) \quad \text{(A-44)}$$

Proof: Left-hand side of (A-44)

$$= A_k V_x(k) V_z^{-1}(k) + B_k - \bar{K}_x(k+1) C_{k+1} (A_k V_x(k) V_z^{-1}(k) + B_k)$$

$$- V_x(k+1) V_z^{-1}(k+1) K_z(k+1) C_{k+1} (A_k V_x V_z^{-1} + B_k)$$

$$= (I - \bar{K}_x(k+1) C_{k+1}) (A_k V_x V_z^{-1} + B_k) - V_x(k+1) V_z^{-1}(k+1) K_z(k+1) C_{k+1} (A_k V_x V_z^{-1} + B_k)$$

$$= (I - \bar{K}_x(k+1) C_{k+1}) (A_k V_x + B_k V_z) V_z^{-1} - V_x(k+1) V_z^{-1}(k+1) K_z(k+1) C_{k+1} (A_k V_x V_z^{-1} + B_k)$$

$$= V_x(k+1) V_z^{-1}(k) - V_x V_z^{-1} K_z C_{k+1} (A_k V_x V_z^{-1} + B_k)$$

$$= V_x(k+1) V_z^{-1}(k+1) D_k - V_x(k+1) V_z^{-1} K_z C_{k+1} (A_k V_x(k) V_z^{-1}(k) + B_k)$$

= right hand side of Eq. (A-44).

This completes the proof.

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APPROVAL

ATTITUDE ESTIMATION OF EARTH ORBITING SATELLITES BY DECOMPOSED LINEAR RECURSIVE FILTERS

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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