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Reports of the Department of Geodetic Science

Report No. 221

DOWNWARD CONTINUATION OF GRAVITY INFORMATION FROM SATELLITE TO SATELLITE TRACKING OR SATELLITE GRADIOMETRY IN LOCAL AREAS

by

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Prepared for

National Aeronautics and Space Administration

Goddard Space Flight Center

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Grant No. NGR 36-008-161

OSURF Project No. 3210

(NASA-CR-143236) DOWNWARD CONTINUATION OF GRAVITY INFORMATION FROM SATELLITE TO SATELLITE TRACKING OR SATELLITE GRADIOMETRY IN LOCAL AREAS (Ohio State Univ. Research Foundation) 54 p HC \$4.25 N75-28594 Unclas 29865 CSCI 08N G3/46



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Research Foundation
Columbus Ohio 43212

April, 1975

Foreword

This report was prepared by Dr. R. Rummel, The Ohio State University, under NASA Grant NGR 36-008-161, The Ohio State University Research Foundation Project No. 3210 which is under the direction of Professor Richard H. Rapp. The grant supporting this research is administered through the Goddard Space Flight Center, Greenbelt, Maryland, Mr. James Marsh, Technical Officer.

Acknowledgements

The writer wishes to thank Dr. Rapp for many helpful comments and suggestions and Dr. D. P. Hajela and Dr. K. -P. Schwarz for valuable discussions.

Abstract

The derivation of gravity anomalies at the surface of the earth from satellite to satellite tracking or satellite gradiometry observations is a downward continuation problem. In developing the spectral interrelations in terms of spherical harmonics between different gravity quantities such as the disturbing potential T , gravity anomalies Δg , and $\frac{\partial^2 T}{\partial r^2}$ at the surface and in satellite altitude a discussion of the features of these two methods for the estimation of surface gravity information is possible.

For a local improvement of our knowledge about the gravity field integral formulas in the parameter domain have to be applied instead of a representation by spherical harmonics. The neglected regions will cause a truncation error. The application of the discrete form of the integral equations connecting the satellite observations with surface gravity anomalies is discussed in comparison with the least squares prediction method.

One critical point of downward continuation is the proper choice of the boundary surface. Practical feasibilities are in conflict with theoretical considerations. The properties of different approaches for this question are analyzed.

As a result the considerations indicate the possibility of deriving mean gravity anomalies at the surface of the earth. By taking into account theoretical restrictions these anomalies are comparable with terrestrial gravity anomalies.

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1. Introduction

By the "Earth and Ocean Physics Applications Program"(1972) from NASA the development of some new satellite techniques is supported that will enable the determination of the earth's gravity field in a different way. Specifically, the methods of altimetry, satellite to satellite tracking, and satellite gradiometry are expected to lead to a considerable improvement in our current knowledge of the gravity field. In several theoretical studies the capability of these techniques is analyzed and compared, for example by Kaula (1969), Glaser and Sherry (1971), and Forward (1971). Since the successful results of Muller and Sjogren (1968) for the recovery of the gravitational field of the front face of the moon by a direct mapping, satellite methods received growing attention not only for a global but also for a local analysis of the gravity field. Local solutions would considerably reduce the number of unknowns and can be concentrated in unsurveyed areas. Simulation studies by Reed (1973) for satellite gradiometry and by C.R. Schwarz, (1970) and Hajela (1974) for satellite to satellite tracking brought promising results for this purpose.

Chovitz (1973) turned new attention to the satellite analysis methods as a downward continuation problem. Especially for the local recovery serious trouble can be expected and the theoretical considerations for example by Moritz (1970), Bjerhammar (1973), and Krarup (1969) have great practical importance.

This report has the intention to present a unified smoothing scheme by spectral analysis according to the idea of Meissl (1971). It shall allow us to compare different satellite techniques—here satellite to satellite tracking and satellite gradiometry—by their sensitivity with respect to a certain unknown. The analysis of the downward continuation operator involved in the solution equations will illuminate the scale of difficulties connected with the evaluation of surface gravity anomalies due to this problem. Thereby the specific properties of the prediction and the deterministic approach will be compared.

2. The spectral properties of the operators related to the earth's potential.

For an optimal design of geodetic satellite experiments it is of basic importance to analyze the resolution and accuracy of the desired quantity based on the expected observation precision. The final aim will be to derive the equipotential surface at mean sea level, the geoid. This surface may be computed from gradients, gravity disturbances, deflections of the vertical, gravity anomalies, or from the disturbing potential itself. The derivation of these quantities is possible by terrestrial measurements but also from different types of satellite data or from a combination of two or more of these techniques. In order to find out the specific character of these different methods with respect to the desired quantity a unified model for their comparison would be of great use.

The intention of an unified model will not be to present a precise analysis for a special experiment but it shall serve a feeling of possibilities and limitations for the derivation of a gravity quantity from different types of observations. Therefore to a certain degree the conclusions will always suffer from simplifications introduced to obtain an easily understandable closed model.

Because many properties of a physical quantity and its relations to others may be well described by spectral analysis we develop the desired unified model by expanding gravity quantities into spherical harmonics. An analysis based on this idea was initiated by Meissl (1971).

In a first step we concentrate our interest on the interrelations between the disturbing potential T , the gravity anomalies Δg , and the first and second derivative of the disturbing potential, $\frac{\partial T}{\partial r}$ and $\frac{\partial^2 T}{\partial r^2}$ later on denoted as gravity quantities all of them on a sphere σ with mean radius R of the earth. This collection is perhaps not representative but it has the advantage that these quantities are harmonic, their interrelations are simple, and their spectral properties reach from a high amplitude low frequency spectrum for T to a low amplitude high frequency spectrum for $\frac{\partial^2 T}{\partial r^2}$. Therefore the isosurface for T will be very smooth whereas the isosurface for $\frac{\partial^2 T}{\partial r^2}$ will have an unsmooth topography.

The general form of a spectral representation by expansion into spherical harmonics for a gravity anomaly on σ is (Heiskanen-Moritz, p. 255)

$$(2.1) \quad \Delta g(P') = \sum_{n=0}^{\infty} \sum_{m=0}^n (c(\Delta g)_{nm} \bar{R}_{nm}(P') + s(\Delta g)_{nm} \bar{S}_{nm}(P'))$$

$\Delta g(P')$... gravity anomaly at a surface point $P' \in \sigma$

$c(\Delta g)_{nm}$ } ... coefficients of the expansion of Δg into spherical harmonics of degree n and order m [in units of gravity]

$\bar{R}_{nm}(P')$ } ... fully normalized spherical harmonics in P' .

The relation between the disturbing potential T and the gravity anomalies Δg expressed by Stokes formula is

$$(2.2) \quad T(Q') = \frac{R}{4\pi} \int_{\sigma} \Delta g(P') \text{St}(\psi) d\sigma$$

Expansion of the Stokes operator into spherical harmonics (Heiskanen-Moritz, p. 97 (2-169) and p. 33 (1-82')) leads to

$$(2.3) \quad \text{St}(P', Q') = \sum_{n=2}^{\infty} \frac{1}{n-1} \sum_{m=0}^n (\bar{R}_{nm}(P') \bar{R}_{nm}(Q') + \bar{S}_{nm}(P') \bar{S}_{nm}(Q'))$$

Insertion of equation (2.1) and (2.3) into (2.2) and using the orthogonality relationships will give

$$(2.4) \quad T(Q') = \sum_{n=2}^{\infty} \frac{R}{n-1} \sum_{m=0}^n (c(\Delta g)_{nm} \bar{R}_{nm}(Q') + s(\Delta g)_{nm} \bar{S}_{nm}(Q')),$$

a spherical harmonic expression for $T(Q')$.

We may also develop $T(Q')$ directly into spherical harmonics and get, similar as for $\Delta g(P')$ in equ. (2.1)

$$(2.5) \quad T(Q') = \sum_{n=2}^{\infty} \sum_{m=0}^n (c(T)_{nm} \bar{R}_{nm}(Q') + s(T)_{nm} \bar{S}_{nm}(Q')),$$

with $\left. \begin{array}{l} c(T)_{nm} \\ s(T)_{nm} \end{array} \right\} \dots$ coefficients for the spherical harmonic representation of T [in units of potential].

A comparison of the coefficients in equ. (2.4) and (2.5) shows

$$(2.6) \quad \left\{ \begin{array}{l} c(T)_{nm} \\ s(T)_{nm} \end{array} \right\} = \frac{R}{n-1} \left\{ \begin{array}{l} c(\Delta g)_{nm} \\ s(\Delta g)_{nm} \end{array} \right\}$$

The same relationship holds true for the coefficients of an expansion into Legendre polynomials (Heiskanen-Moritz, p. 97)

$$(2.7) \quad T_n = \frac{R}{n-1} \Delta g_n$$

$T_n, \Delta g_n \dots$ coefficients of the Legendre polynomial for T and Δg [units of potential and gravity].

Similar expressions can be derived for the other harmonic gravity quantities. From the spectral representation of their interrelating operators we find the least upper bounds (LUB) and greatest lower bounds (GLB) of the operators. For the quotient $\frac{R}{n-1}$ in equ. (2.7) we derive for example

$$\text{GLB} \left(\frac{R}{n-1} \right) = 0 \quad \text{for } \lim_{n \rightarrow \infty} \text{ and}$$

$$\text{LUB} \left(\frac{R}{n-1} \right) = R \quad \text{for } n=2.$$

The bounds are the result of the frequency properties of the operator, which in this example will damp high frequencies and reach its maximal amplitude in the low frequencies (maximum R for $n=2$). The theoretical background of these considerations for different types of gravity quantities is described in Meissl (1971).

The spectral interrelations of the quantities in mind and the GLB and LUB for the operators are collected in Table 1.

	T_n	Δg_n	$\left(\frac{\partial T}{\partial r}\right)_n$	$\left(\frac{\partial^2 T}{\partial r^2}\right)_n$
T_n	/	$R/(n-1)$ 0; R	$-R/(n+1)$ 0; $R/3$	$R^2/(n+1)(n+2)$ 0; $R^2/12$
Δg_n	$(n-1)/R$ $1/R; \infty$	/	$-(n-1)/(n+1)$ $1/3; 1$	$R(n-1)(n+1)(n+2)$ 0; $R/12$
$\left(\frac{\partial T}{\partial r}\right)_n$	$-(n+1)/R$ $3/R; \infty$	$-(n+1)/(n-1)$ $1; 3$	/	$-R/(n+2)$ 0; $R/4$
$\left(\frac{\partial^2 T}{\partial r^2}\right)_n$	$(n+1)(n+2)/R^2$ $12/R^2; \infty$	$(n+1)(n+2)/R(n-1)$ $12/R; \infty$	$-(n+2)/R$ $4/R; \infty$	/

Table 1: Spectral formulas for the operators connecting T , Δg , $\frac{\partial T}{\partial r}$, and $\frac{\partial^2 T}{\partial r^2}$, and their least upper bound (LUB) and greatest lower bound (GLB).

In Table 1 the quantities in the first column are expressed in terms of those of the first row, for example,

$$\Delta g_n = \frac{R(n-1)}{(n+1)(n+2)} \left(\frac{\partial^2 T}{\partial r^2}\right)_n.$$

The relations to the right of the diagonal show a damping of higher frequencies that indicates a smoother isosurface for the derived gravity quantity than for the original one. This principle is also expressed by the two bounds which have a finite range between zero and a certain maximum. The reverse rule is valid for the formulas to the left of the diagonal in the diagram. Here the higher frequencies are amplified and we speak of an unsmoothing procedure. Because the GLB are infinite the convergency properties of these quantities have to be carefully analyzed. A small amplitude in the spectrum for a frequency $n \rightarrow \infty$ may increase to ∞ by multiplying with an unsmoothing operator. Small errors in the original quantity may raise to a high amount in the derived data.

Because the square of the spectral formulas of Table 1 shows the relations of the degree variances $v_n(T)$, $v_n(\Delta g)$, $v_n\left(\frac{\partial T}{\partial r}\right)$, and $v_n\left(\frac{\partial^2 T}{\partial r^2}\right)$ for the considered

gravity quantities, and for the observation errors too, the diagram is of additional use for the analysis of covariance formulas and error propagation.

Assume, for example a gravity anomaly Δg is estimated with a certain amount of white noise; the higher frequencies of the noise will be damped in calculating the disturbing potential from the anomaly, the high frequencies of the noise will be amplified in a $\frac{\partial^2 T}{\partial r^2}$ derivation. This is one of the well known reasons why the computation of $\frac{\partial^2 T}{\partial r^2}$ from gravity anomalies is connected with more problems than the geoid computation from Δg .

For the discussion of satellite experiments the scheme of table one-valid for surface quantities-has to be expanded in radial direction. The object will be the analysis of the height dependence of gravity quantities. We introduce three altitude levels, R near the earth's surface, r_1 and r_2 for low and high satellite altitude. The operator necessary for the generalization is the upward continuation operator, discussed later in detail. For the potential the spectral formula in spherical harmonics for this operator is (Heiskanen - Moritz, p. 35; 33 (1-82'))

$$(2.8) \quad \frac{R(r^2 - R^2)}{\ell(P-P')^3} = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \sum_{m=0}^n (\bar{R}_{nm}(P') \cdot \bar{R}_{nm}(P) + \bar{S}_{nm}(P') \bar{S}_{nm}(P))$$

- R ... radius of an inner sphere (near the earth's surface) with surface σ .
 r ... radius of a concentric sphere with surface τ $P \in \tau$, $P' \in \sigma$
 ℓ ... spatial distance between P and P' .

With the upward continuation operator of equ. (2.8) derivation of the potential in altitude r from the potential given on σ is possible. A more detailed discussion of the continuation of gravity quantities is given in Chapter three. For the spherical harmonic coefficients of T the upward continuation is carried out with equ. (2.8) by:

$$(2.9) \quad \begin{Bmatrix} c(T)_{nm}^{r_1} \\ s(T)_{nm}^{r_1} \end{Bmatrix} = \left(\frac{R}{r_1}\right)^{n+1} \begin{Bmatrix} c(T)_{nm}^R \\ s(T)_{nm}^R \end{Bmatrix}$$

where the indices R and r_1 of the coefficients indicate their relation to σ and τ . Similarly we have for the coefficients of the Legendre polynomial

$$(2.10) \quad T_n^{r_1} = \left(\frac{R}{r_1}\right)^{n+1} T_n^R$$

Thus, it is easy to generalize the diagram of Table 1 in radial direction. In order to avoid confusion we omit the normal derivative $\frac{\partial T}{\partial r}$, which has about the same spectral features as Δg .

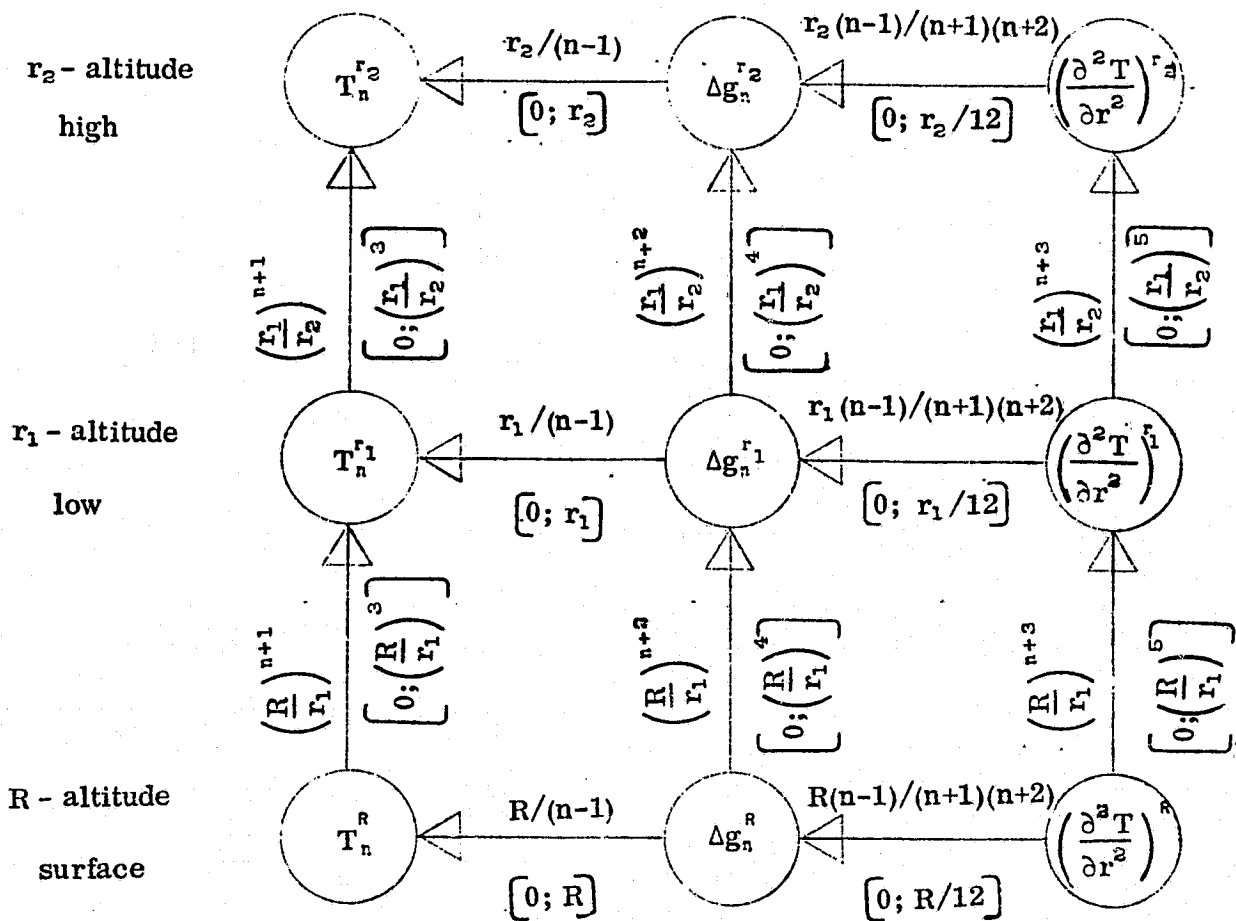


Table 2: Spectral formulas for the operators connecting T, Δg, and $\frac{\partial^2 T}{\partial r^2}$ and their [LUB;GLB] in three altitude levels R, r₁, and r₂. Only the operators in smoothing direction (damping of high frequencies) are introduced; the direction is indicated by \leftarrow . The spectral formulas in reverse direction where high frequencies are amplified, are reciprocal to these expressions; the LUB are inverse of the GLB in smoothing direction and the GLB are inverse of the LUB.

To get an idea about the spectra of the quantities of Table 2, their frequency distribution from n=2 to n=70 for two different models for altitude h = 250 km and altitude h = 0 km are shown in Figure 1, 2, and 3, where h = r - R (R ... mean radius of the earth). The two models are:

$$\text{Tscherning-Rapp (1974)} \quad \text{var}_n(\Delta g) = s^{n+2} \frac{425.28 (n-1)}{(n-2)(n+24)} \text{ mgal}^2, \quad s = \frac{R_{Bje}^2}{r^2}$$

$$R_{Bje} = 6369.8 \text{ km}$$

and Kaula (1969)
$$\text{var}_n(\bar{C}^*, \bar{S}^*) = \sum_{n=0}^n (\bar{C}_{nm}^{*2} + \bar{S}_{nm}^{*2}) = \frac{160 \cdot 10^{-12}}{n^3},$$

(*) dimensionless

which becomes

$$\text{var}_n(N) = s^{n+1} \left[\frac{\text{km}}{\text{RG}} \right]^2 \frac{160 \cdot 10^{-12}}{n^3} \text{ m}^2, \quad s = \frac{R^2}{r^2}$$

$$R = 6371 \text{ km}$$

We assumed

$$\Delta g_n = | \text{r. m. s. } (\Delta g)_n | = | \text{var}_n(\Delta g_n)^{\frac{1}{2}} |$$

and

$$N_n = | \text{r. m. s. } (N)_n | = | \text{var}_n(N)^{\frac{1}{2}} |.$$

By the formulas inserted in Table 2 we computed for both models (T-R, and K) N_n , Δg_n , and $\left(\frac{\partial^2 T}{\partial r^2} \right)_n$. The result may give an impression about the spectra of these gravity quantities. The different results for the model Tscherning-Rapp (based on currently available gravity material) and the model of Kaula (based on the information available in (1966) especially for the $\frac{\partial^2 T}{\partial r^2}$ component

indicate the problems that may be related in deriving an "a priori" analysis about instrumentation resolutions, error influence, and so on.

Example: Derivation of T_n^R from $\Delta g_n^{r_1}$.

A: Let us first look at the smoothing relation:

$$T_n^{r_1} = + \frac{r_1}{n-1} \Delta g_n^{r_1} \quad (\text{bounds } 0, r_1) \quad (a)$$

then opposite to the smoothing direction

$$T_n^R = \left[\frac{r_1}{R} \right]^{n+1} T_n^{r_1}; \left(\left[\frac{r_1}{R} \right]^3; \infty \right) \quad (b)$$

with (a) and (b) follows

$$T_n^R = \left[\frac{r_1}{R} \right]^{n+1} \frac{r_1}{n-1} \Delta g_n^{r_1}; (0; \infty) \quad (c)$$

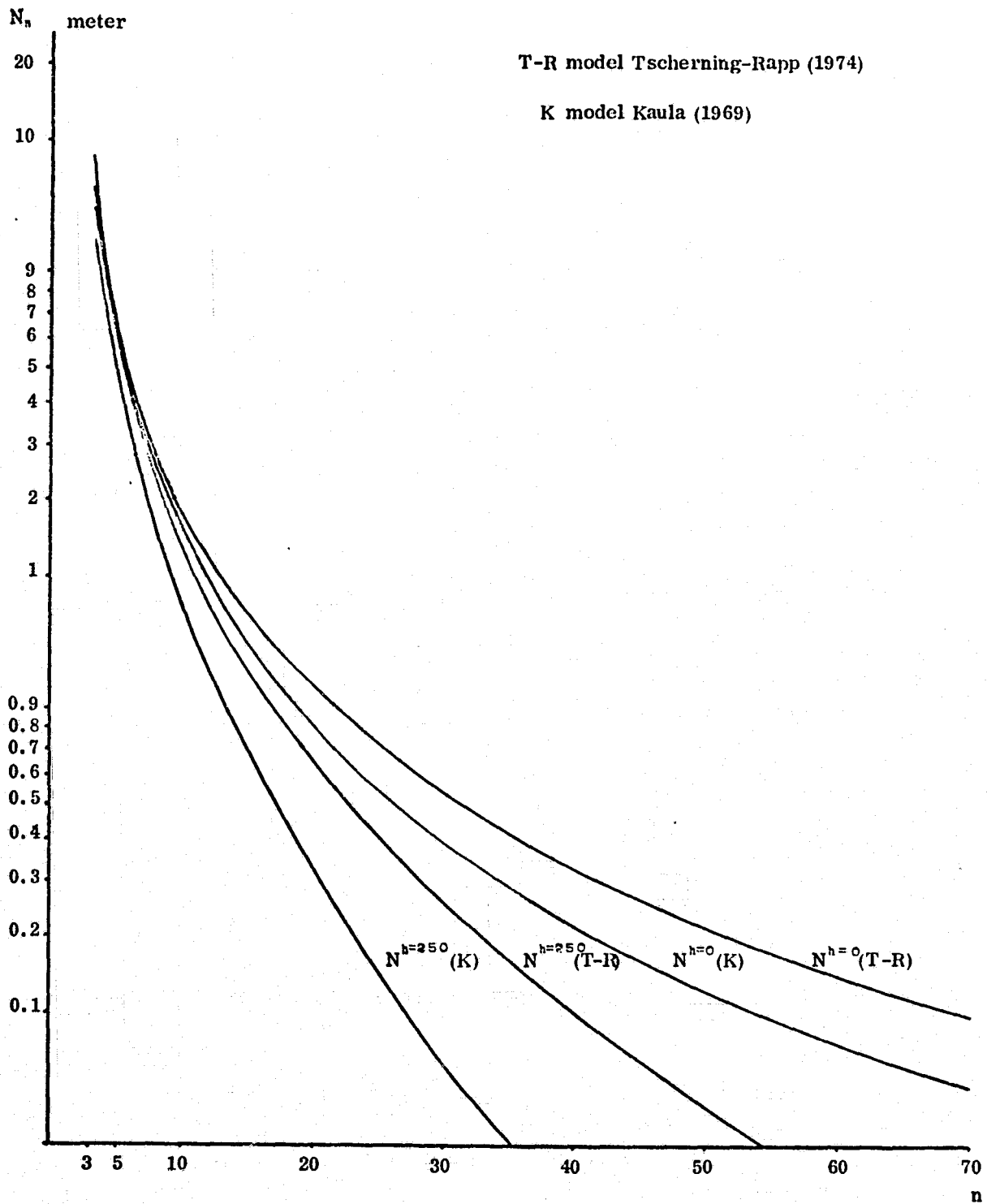


Figure 1: Amplitude spectrum of the Legendre polynomial coefficients N_n of the undulation for the two models K and (T-R) and for altitude 0 and 250 km.

Δg_n
mgal

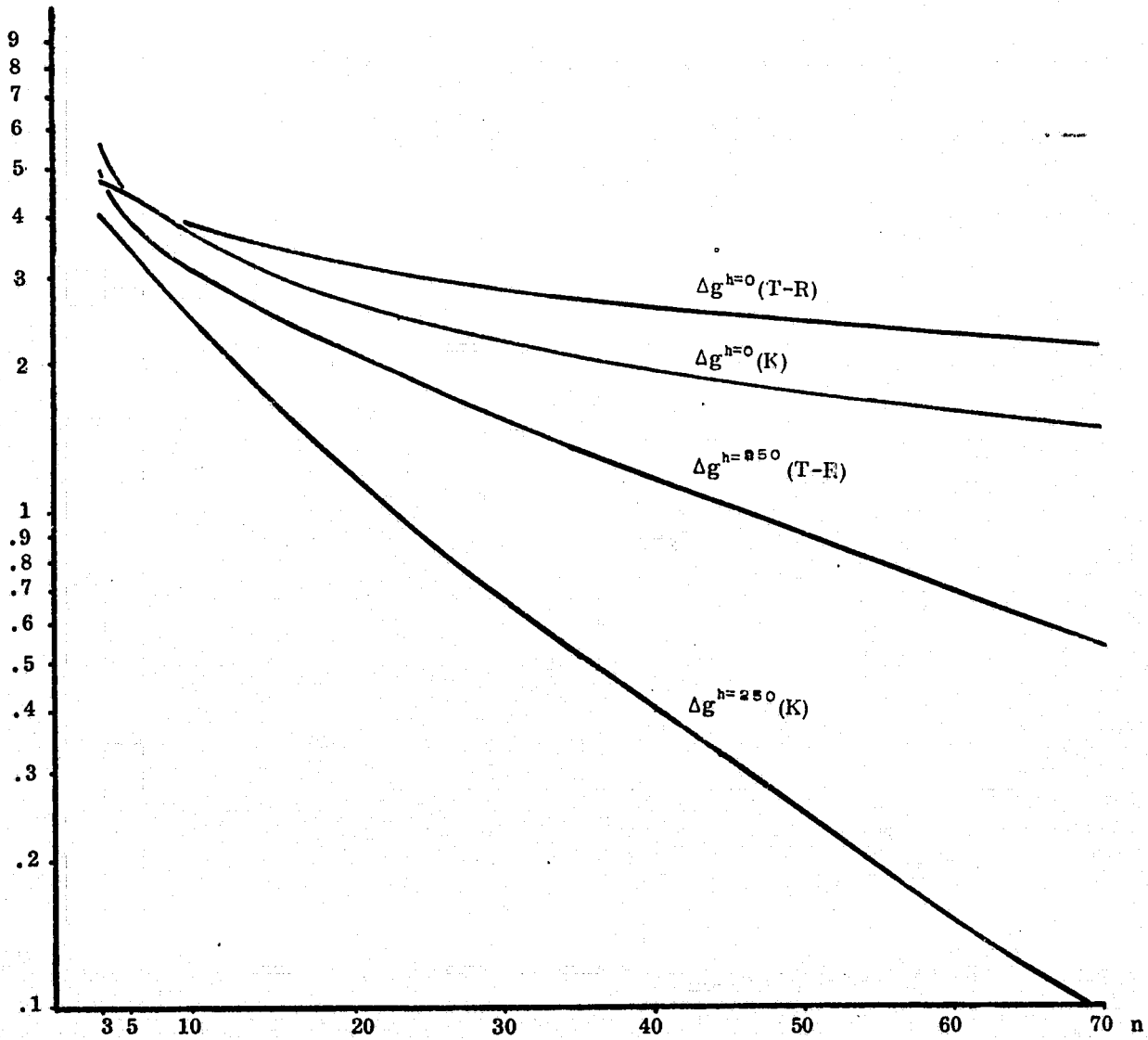


Figure 2: Amplitude spectrum of the Legendre polynomial coefficients Δg_n of gravity anomalies.

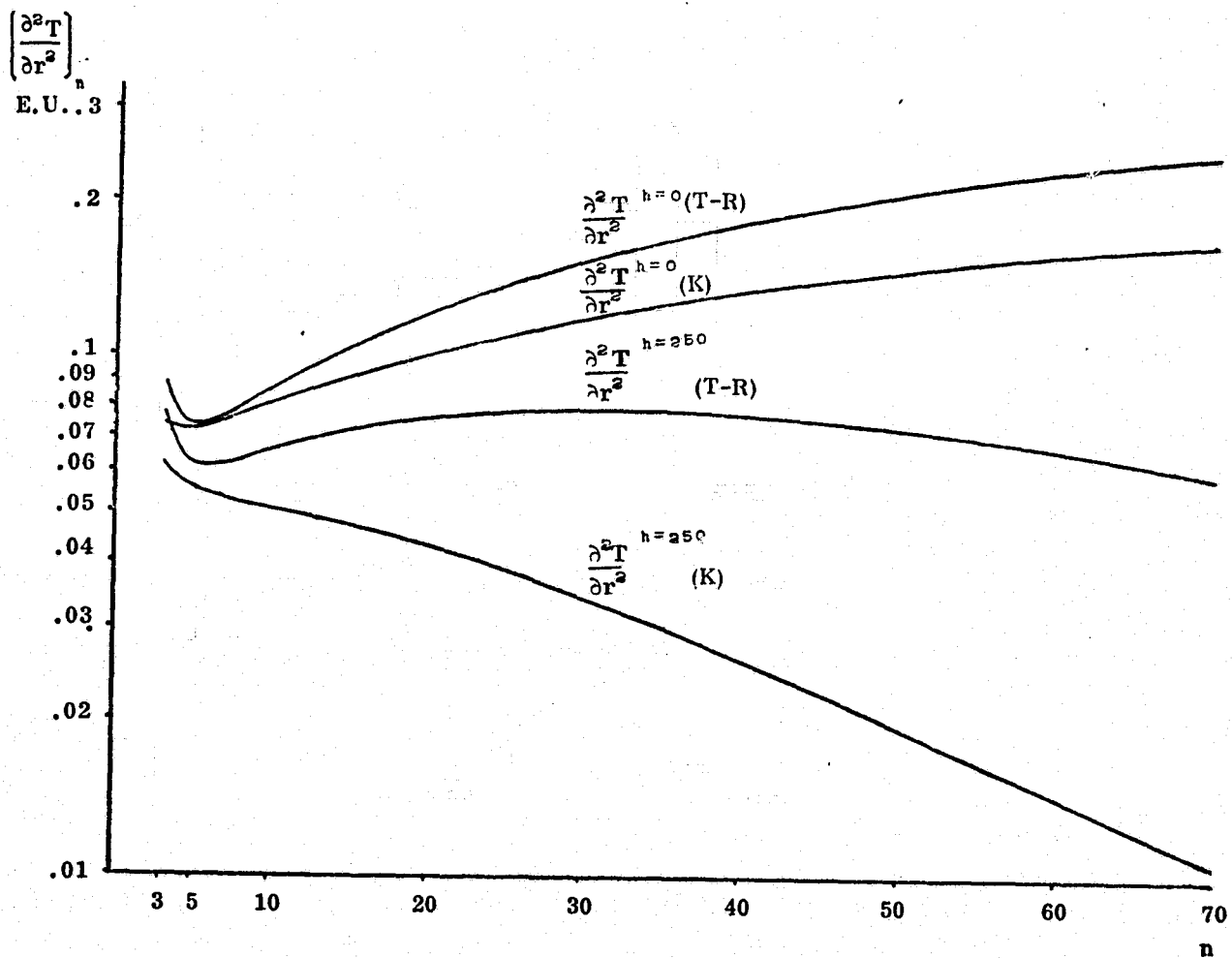


Figure 3: Amplitude spectrum of the Legendre polynomial coefficients $\left(\frac{\partial^2 T}{\partial r^2}\right)_n$ for the second radial derivative of the disturbing potential T.

B: We first go in an unsmoothing direction:

$$\Delta g_n^R = \left(\frac{r_1}{R}\right)^{n+2} \Delta g_n^{r_1}; \left(\frac{r_1}{R}\right)^4; \infty \quad (a)'$$

then

$$T_n^R = \frac{R}{n-1} \Delta g_n^R; (0; R) \quad (b)'$$

The result from (a) and (b) will be the same

$$\begin{aligned} T_n^R &= \left(\frac{r_1}{R}\right)^{n+2} \frac{R}{n-1} \Delta g_n^{r_1} \\ &= \left(\frac{r_1}{R}\right)^{n+1} \frac{r_1}{n-1} \Delta g_n^{r_1}; (0, \infty) \end{aligned} \quad (c)' = (c)$$

As Table 2 indicates the same problem may be solved by using several different ways, A and B are only the shortest of them. Without considering observation errors the result will always be the same. In addition, we see that at least one time an unsmoothing step has to be done in order to reduce the gravity quantity from altitude r_1 to R . As already mentioned this step has to be analyzed very carefully by considering the convergency properties of the involved data.

The columns of Table 2 indicate the attenuation of the frequencies of surface gravity quantities by the factor $\left(\frac{R}{r}\right)^{n+1}$, $\left(\frac{R}{r}\right)^{n+2}$, or $\left(\frac{R}{r}\right)^{n+3}$, $r > R$, for upward continuation to altitude r . The higher frequencies are, therefore strongly damped with altitude. For that reason it is easier to get information about high frequencies from observations to a low orbit satellite than from a high altitude satellite.

It becomes clear that under the assumption of comparable observation accuracy for the quantities involved in Table 2, the disturbing potential T^R , which defines the geoid, may be optimally derived from $\left(\frac{\partial^2 T}{\partial r^2}\right)^R$. For this second derivative no step in an unsmoothing direction has to be carried out by deriving T and disturbances in high frequencies are damped by the order of about $\frac{1}{n^2}$ (exactly by $\frac{R}{(n+1)(n+2)}$, compare Table 1). On the other hand the high frequency features in $\left(\frac{\partial^2 T}{\partial r^2}\right)^R$ will be very significant or what is the same it will be strongly affected by local disturbances. An illustrative example for the difficulties caused by these local disturbances is the trouble necessary for an accurate calculation of the topographic correction for torsion balance measurements. The derivation of a global data field with this instrumentation would be absolutely impossible.

In order to get gravity information in short time for the entire world, satellite techniques are necessary. Their loss of information by attenuation

may only be compensated by a higher sensitivity in the instrumentation.

The disturbing potential T and its gradient ∇T are not directly measured by satellite techniques but have to be derived from observations of the satellites position \underline{s} or velocity $\dot{\underline{s}}$. Therefore it is convenient to also introduce these two quantities into the spectral scheme of Table 2. The time dependent position vector \underline{s} and velocity vector $\dot{\underline{s}}$ are connected with the space dependent potential by the well known relation

$$(2.11) \quad \delta \ddot{\underline{s}} = \nabla T$$

with $\delta \ddot{\underline{s}}$... difference between the actual acceleration $\ddot{\underline{s}}$ and a "normal" acceleration $\ddot{\underline{s}}_c$ computed from a low degree field.

∇T ... gradient of the disturbing potential.

The tracking elements to the satellite are direction, range, or range rate, i.e. components of the vectors \underline{s} and $\dot{\underline{s}}$ or linear combinations of single components of these vectors.

Spectral analysis for the gravity quantities should therefore include the relations connecting \underline{s} , $\dot{\underline{s}}$, and $\ddot{\underline{s}}$. Kaula (1969) derived an interesting proportionality between the potential V , i.e. the space dependent integral of acceleration and the velocity $\dot{\underline{s}}$, the time dependent integral of acceleration. He gives:

$$(2.12) \quad v_n(\dot{\underline{s}}) = \frac{GM}{a_e} \left(\frac{a_e}{a} \right)^{n+1} v_n(V^*)$$

where

$v_n(\dot{\underline{s}})$... degree variance of the satellite velocity.

$\frac{GM}{a_e} \frac{a_e}{a}$... proportionality factor, G ... gravitational constant,

M ... earth's mass, a_e ... equatorial radius, a ... semi-major axis of the satellite orbit.

$v_n(V^*)$... degree variance of the non-dimensionalized (*) surface gravity potential.

The same relation holds true between the disturbing potential T^* and the residual velocity $\delta \dot{\underline{s}}$. The amplification of high frequencies in the spectrum of $\delta \ddot{\underline{s}}$ in deriving $\delta \dot{\underline{s}}$ from $\delta \ddot{\underline{s}}$, will therefore be proportional to that for the derivation of ∇T from T .

The spectral relations between \underline{s} , $\dot{\underline{s}}$ and $\ddot{\underline{s}}$ are in practice also dependent on the applied method of numerical or analytical integration and on the sample rate of the observations. It would be of great use to find closed expressions for

the amplification of the high frequencies in deriving $\ddot{\underline{s}}$ from $\dot{\underline{s}}$ and \underline{s} , for it would enable us to estimate the error propagation from \underline{s} or $\dot{\underline{s}}$ to T_n^R , Δg_n^R , or $\frac{\partial^2 T}{\partial r^2}^R$ by inserting this propagation law into the spectral scheme of Table 2.

Having equation (2.11) and (2.12) in mind the spectral formulas between T_n , Δg_n and $\frac{\partial^2 T}{\partial r^2}$ in elevation r_1 or r_2 may be expanded by \underline{s} , $\dot{\underline{s}}$ and $\ddot{\underline{s}}$ for which we don't have an explicit spectral expression for the connecting operators. This is shown in Table 3.

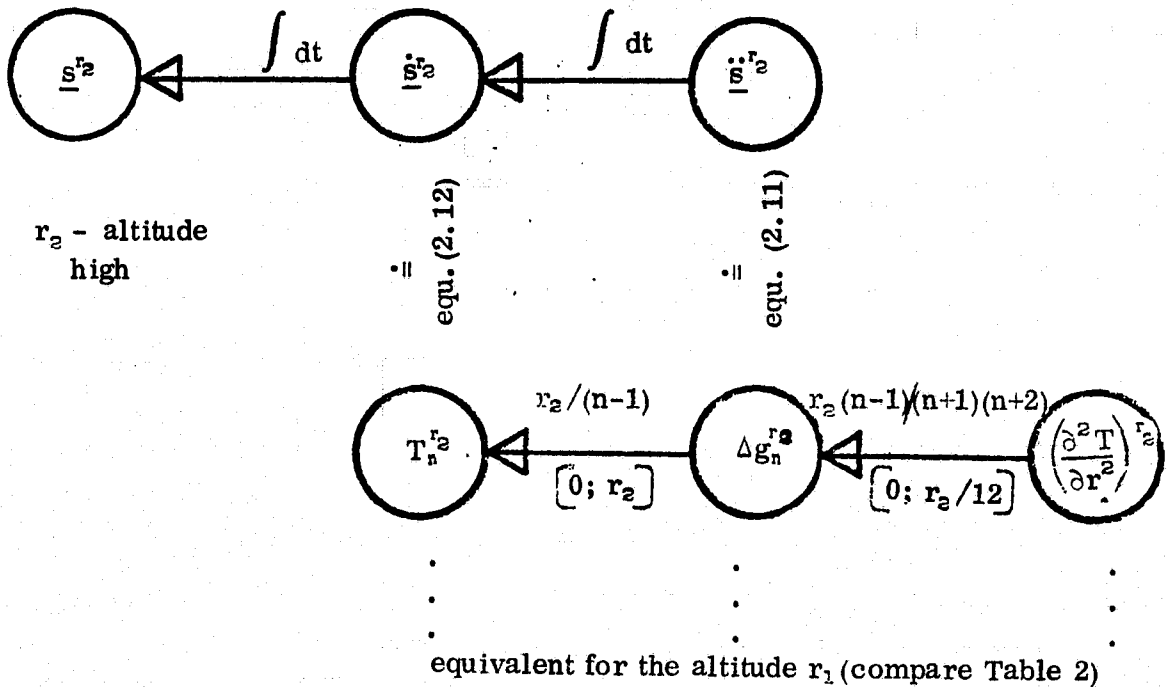


Table 3: Interrelations between \underline{s} , $\dot{\underline{s}}$, and $\ddot{\underline{s}}$ and their connection with the spectral formulas for the gravity quantities of Table 2 in order to get a more general impression of the smoothing and amplifying properties in the frequency domain.

The position vector \underline{s} of a satellite is a very smooth function, its spectral content is concentrated at low frequencies. As we see from Table 3, $\dot{\underline{s}}$ will have about the same amplitude distribution in the spectrum as the disturbing potential, and the spectrum of the satellite acceleration $\ddot{\underline{s}}$ is identical to that of the gradient ∇T by equ. (2.11), and will therefore in the radial component be about the same as the frequency spectrum of Δg . It is therefore more difficult to derive the spherical harmonic coefficients for higher frequencies for Δg and also for T from position observations than from range rate measurements. From this point of view direct observation of $\ddot{\underline{s}}$ would be optimal.

Finally we try to reflect the character of satellite observations, which are carried out from one specific height level to another. Assuming the observations have limited resolution but are not affected by noise, they may be seen as connections from the frequency level in elevation one to the frequency level in

elevation two. In reality this means the measurements transfer the estimation accuracy from one altitude to the other. As we know, we need fewer coefficients to determine a satellite position with a certain precision for high than for low altitude. Thus, when a high satellite observes a low flying satellite by distance measurements with the postulated attributes, the position accuracy of the low satellite becomes the same as that of the high satellite. Because of this a certain amount of additional information about the higher frequency part is derived for the low altitude orbit. This principle is used in high-low satellite experiments and in altimetry.

Summarizing the topic of this chapter we have derived a spatial spectral scheme (Table 1 and 2). Adding the smoothing properties of satellite position, velocity and acceleration (Table 3) allows us to analyze the essential attributes of terrestrial and satellite measurement techniques. The discussion is supported by also taking into account the characteristics of satellite measurements between different altitude levels.

Because these considerations are only valid for a global representation of gravity quantities in spherical harmonics the conclusions cannot be applied without modification to methods of local improvement of the gravity knowledge. For this purpose a detailed analysis of the special technique in mind is necessary.

3. Boundary value problem.

In recent time, concepts for local improvement of our knowledge about the earth's gravity by satellite methods have become more important. If the benefit of the higher accuracy of new techniques is utilized any global solution is coupled with a rapidly increasing number of unknowns in a least square adjustment. For a representation of the gravity field by spherical harmonics where the coefficients are the unknown parameters, the number of unknowns for a representation up to degree and order 36 ($\doteq 5^\circ$ equal area solution) is already 1363. For this reason a local adjustment where only mean values for a limited number of surface blocks are derived would be preferable. This kind of solution seems to be in contrast to some theoretical considerations which have their origin in the downward continuation problem for gravity quantities.

For a better understanding, a short review of the boundary value problem will be given:

Problem statement: Given any function w on a surface S , derive a function f defined outside S and harmonic which approaches on S the given boundary function w .

This first boundary value problem or Dirichlet's problem has an easy solution for a boundary sphere σ . ($R \dots$ radius of the sphere, $\Sigma \dots$ space outside σ) in form of the Poisson integral,

$$(3.1) \quad f(P) = \frac{R(r^2 - R^2)}{4\pi} \int_{\sigma} \frac{f(P')}{\ell(P, P')^3} d\sigma$$

(All terms except f are already explained for equ. (2.8), where the Poisson operator was introduced into the spectral diagram).

The Poisson integral solves the Dirichlet problem for the sphere σ ; that is, given any continuous function $f(P')$, $P' \in \sigma$, the Poisson integral of f defines a function exterior to the sphere, and harmonic in Σ which approaches the given boundary function.

Given the boundary function $f(P')$ on σ the integral formula (3.1) allows the determination of $f(P)$ for all $P \in \Sigma$. In this formulation equation (3.1) defines the upward continuation problem.

But far more complicated is the inverse problem where our aim is the derivation of the boundary function $f(P')$ from $f(P)$. Equ. (3.1) becomes an integral equation of the first kind and the problem is named improperly posed. For a general but given boundary surface the problem was analytically handled for example by Weck (1972) and Krarup (1969). Discrete and statistical solutions of this so called downward continuation problem are proposed by Bjerhammar (1966), (1969), Moritz (1970) or Schwarz (1971). The solution of the integral equation is essentially facilitated by choosing a spherical boundary surface σ as we already assumed for the Poisson integral of equ. (3.1). Equation (3.1) may be more abstractly written as:

$$(3.2) \quad f(P) = \frac{1}{4\pi} \int_{\sigma} a(|P-P'|) f(P') d\sigma$$

with

$a(|P-P'|) \dots$ Poisson operator, where $|P-P'|$ indicates that $a(x)$ is a function of only the distance $|P-P'|$.

Expression (3.2) formulates a convolution integral. It has the advantageous property that it becomes a simple product after transformation into the frequency domain. Functions defined on the real line are transformed into the frequency domain by Fourier expansion. The analogic procedure for a function defined on a sphere is the expansion into surface spherical harmonics.

Our convolution integral will therefore become by inserting the spherical harmonic expressions for $f(P)$, $f(P')$ and $a(|P-P'|)$

$$(3.3) \quad f_{nm} = a_{nm} f_{nm}^R$$

Recalling the spherical harmonic expression for the Poisson operator (2.8)

$$(2.8) \quad a(|P-P'|) = \frac{R(r^2 - R^2)}{\ell(P-P')^3} = \sum_{n=0}^{\infty} \left(\frac{R}{r} \right)^{n+1} \sum_{m=0}^n (\bar{R}_{nm}(P') \bar{R}_{nm}(P) + \bar{S}_{nm}(P') \bar{S}_{nm}(P))$$

and expanding the potential V into spherical harmonics

$$(3.4) \quad V(P') = \sum_{n=0}^{\infty} \sum_{m=0}^n (c(V)_{nm}^R \bar{R}_{nm}(P') + s(V)_{nm}^R \bar{S}_{nm}(P'))$$

we obtain by inserting equ. (2.8) and (3.4) into equ. (3.1) with $V(P)$ for $f(P)$ and $V(P')$ for $f(P')$

$$(3.5) \quad \begin{Bmatrix} c(V)_{nm} \\ s(V)_{nm} \end{Bmatrix} = \left(\frac{R}{r} \right)^{n+1} \begin{Bmatrix} c(V)_{nm}^R \\ s(V)_{nm}^R \end{Bmatrix}$$

which is in accordance with equ. (3.3) and the expression for the disturbing potential T , equ. (2.9). Also the inverse formula - the downward continuation - holds

$$(3.6) \quad \begin{Bmatrix} c(V)_{nm}^R \\ s(V)_{nm}^R \end{Bmatrix} = \left(\frac{r}{R} \right)^{n+1} \begin{Bmatrix} c(V)_{nm} \\ s(V)_{nm} \end{Bmatrix}$$

In the spectral representation an easy solution of the downward continuation problem in the form of equ. (3.6) seems to exist. But in order to be able to expand a function $f(P')$ into spherical harmonics it has to be given, independent of the frequency (n, m) in which we are interested, as a continuous function covering a sphere, expressed by

$$(3.7) \quad \begin{aligned} c(f)_{nm} &= \frac{1}{4\pi} \int_{\sigma} f(P') \bar{R}_{nm}(P') d\sigma \\ s(f)_{nm} &= \frac{1}{4\pi} \int_{\sigma} f(P') \bar{S}_{nm}(P') d\sigma \end{aligned}$$

Because the coefficients derived by equ. (3.7) are independent, one from another, it is possible first to estimate the coefficients up to a low degree and order and then to complete them step by step. Thus, when we have given a smoothed

form of $f(P')$ that contains only frequencies up to degree n_N we may derive the coefficients $c(f)$ and $s(f)$ up to degree n_N by (3.7). For such a procedure a solution of upward and downward continuation of gravity information in form of spherical harmonic coefficients is possible. These theoretical considerations form the background for actual global satellite solutions.

But there is a gap between theory and practice: The data from which the coefficients are derived does not cover a global sphere with a certain radius but vary in altitude because of the elliptic shape of the orbit; will not cover all sub-regions with the same density; and is given in discrete but not in continuous form. Thus the integral formulas (3.7) have to be approximated by a system of discrete linear equations applicable for least square adjustment. The coefficients for the spherical harmonic expansion will therefore lose their independence and will be affected by systematic errors. In addition, not incorporated higher frequencies in the data will falsify low frequencies by aliasing which also raises the correlation between different coefficients.

More difficult is the problem of local improvement of surface gravity information where we are directly faced with integral equation (3.1). Frequency representation can only be used in a modified form.

4. Satellite to satellite tracking and satellite gradiometry.

The large number of unknowns connected with a refinement of our knowledge of the gravity field by global analysis and the varying density of satellite observations depending on the orbit elements, the technical equipment, and the distribution of ground tracking stations leads our attention to local methods. This idea is supported by the fact that terrestrial measurements are given with high accuracy and density in some areas with no data in other areas. Our main interest lies in gravity anomalies from which by combination with terrestrial anomalies a geoid computation can be carried out. The satellite techniques in mind for this purpose are satellite to satellite tracking (SST) and satellite gradiometry.

We do not include satellite altimetry into these considerations, though this technique is very promising. But the altimeter observation connects directly the satellite orbit in altitude r with the topography of the earth, i. e. with respect to the discussion of chapter two, downward continuation from altitude r to the surface is carried out by the observation itself whereas we want to analyze the problem connected with the analytical downward continuation from satellite altitude.

Directional and range measurements from ground tracking stations are not accurate enough to provide boundary gravity anomalies for smaller block size with sufficient precision. The high frequencies in the spectrum of the disturbing potential T in satellite altitude r are damped by the factor $\left(\frac{R}{r}\right)^{n+1}$

as shown in Table 2. By deriving the satellite position \underline{s} from the velocity $\dot{\underline{s}}$ which is proportional to T , equ. (2.12), the spectrum is again damped. The instrumentation for direction and range measurements which are components of the satellite position, are therefore usually only sensitive enough to resolve the low frequencies of the disturbing potential.

Studies by Forward (1971) and by Glaser and Sherry (1972) assume the same frequency properties for SST and the disturbing potential and also for gradiometry and $\frac{\partial^2 T}{\partial r^2}$. Therefore we may expect a low frequency improvement from SST and better knowledge of higher frequencies by gradiometry. These conclusions are based on the essential simplification that SST has the same frequency features as $\dot{\underline{s}}$ itself and the only important component for the disturbing potential tensor is the radial component $\frac{\partial^2 T}{\partial r^2}$.

When we try to avoid any misleading simplification we are faced with the complex formula systems connecting the observation data with the final gravity anomaly. In spite of that we will try to obtain an insight into the problem by dividing it into several steps and analyzing their individual features.

4.1 Satellite to satellite tracking

The problem with which we are mainly concerned here is to derive gravity anomalies $\Delta g(P')$ from a special type of satellite observation $O(P)$, $P'\epsilon\sigma$ and $P\epsilon\tau$. The most common way of establishing a linearized system of observation equations is, Kaula (1966),

$$(4.1) \quad \frac{O}{n} + \frac{dO}{n} = \frac{C}{n} + \frac{W}{n \times m} \quad \frac{d\beta}{m}$$

\underline{O} ... n-dimensional vector of observation
 \underline{dO} ... " " " residuals
 \underline{C} ... " " " the calculated observation elements
 \underline{W} ... n x m sensitivity matrix (coefficient matrix)
 $\underline{d\beta}$... m-dimensional vector of the unknown parameters

For range rate measurements \dot{R} equ. (4.1) will become

$$(4.2) \quad \underline{\dot{R}}_{0-c} + \underline{d\dot{R}} = \underline{W} \underline{\Delta g}, \text{ with } \underline{\dot{R}}_{0-c} = \underline{\dot{R}}_0 - \underline{\dot{R}}_c \text{ and}$$

$\underline{\Delta g}$... vector of unknown gravity anomalies.

The sensitivity matrix \underline{W} consists of the coefficients $\frac{\partial \dot{R}_i}{\partial \Delta g_j}$ and shows the

interrelation between the range rate observation at a certain moment and a distinct gravity anomaly. As we know, equ. (4.2) can be divided into different parts by expanding

$$(4.3a) \quad \dot{R}_{o-c} + \underline{dR} = \underline{A} \underline{d\alpha}$$

where \underline{A} is the sensitivity matrix with respect to the intermedium parameters $\underline{d\alpha}$ and

$$(4.3b) \quad \underline{d\alpha} = \underline{B} \Delta g$$

where the coefficient matrix contains the relations between $d\alpha$ and Δg . By comparing (4.2) and (4.3 a,b) we see

$$(4.3c) \quad \underline{W} = \underline{AB}.$$

Applied to our problem we may first analyze the evaluation of the acceleration $\ddot{\underline{s}}(P)$ or what is the same $\nabla T(P)$, (compare equ. (2.11)), from range rate data $\dot{R}(P)$. Then we try to clarify the derivation of $\Delta g(P')$ from $\nabla T(P)$. For a term w_{ik} with indices (i,k) of the sensitivity matrix W for range and range rate observation in SST we get, compare Martin (1972) and Hajela (1974)

$$(4.4) \quad w_{ik}^r \doteq \frac{\partial R_i^{(1d)}}{\partial T_x^{r_1 j}} \frac{\partial T_x^{r_1 j}}{\partial \Delta g_k} + \frac{\partial R_i^{(2u)}}{\partial T_x^{r_1 j}} \frac{\partial T_x^{r_1 j}}{\partial \Delta g_k} - \frac{\partial R_i^{(2u)}}{\partial T_x^{r_2 j}} \frac{\partial T_x^{r_2 j}}{\partial \Delta g_k} \quad \text{and}$$

$$(4.5) \quad w_{ik}^r \doteq \frac{\partial \dot{R}_i^{(1d)}}{\partial T_x^{r_1 j}} \frac{\partial T_x^{r_1 j}}{\partial \Delta g_k} + \frac{\partial \dot{R}_i^{(2u)}}{\partial T_x^{r_1 j}} \frac{\partial T_x^{r_1 j}}{\partial \Delta g_k} - \frac{\partial \dot{R}_i^{(2u)}}{\partial T_x^{r_2 j}} \frac{\partial T_x^{r_2 j}}{\partial \Delta g_k}$$

with the index conventions

i ... range or range rate observation number i with respect to

k ... gravity anomaly number k

j ... sub and superindex j means summation for $j = 1, 2, 3$ of the vector components T_{x_1} , T_{x_2} , and T_{x_3} of ∇T

r_1, r_2 ... as already introduced, height level r_1 (low) and r_2 (high).

1d ... observation from relay satellite down to the close satellite

2u ... observation from the close satellite upward to the relay satellite, compare again Hajela (1974).

Each term $\frac{\partial R}{\partial T_x}$ is formed by

$$(4.6) \quad \frac{\partial \dot{R}}{\partial T_x} = \frac{\partial \dot{R}}{\partial X_\ell} \frac{\partial X_\ell}{\partial T_x} + \frac{\partial \dot{R}}{\partial \dot{X}_\ell} \frac{\partial \dot{X}_\ell}{\partial T_x}$$

and each term $\frac{\partial R}{\partial T_x}$ by

$$(4.7) \quad \frac{\partial R}{\partial T_x} = \frac{\partial R}{\partial X_\ell} \frac{\partial X_\ell}{\partial T_x}$$

where $\sum_{\ell} \dots$ is summation for $\ell = 1, 2, 3$.

According to equations (4.3 a-c) we partition the sensitivity matrix W into a product of three matrices A, B, and C. Matrix A contains the elements $\frac{\partial \dot{R}}{\partial X_\ell}$ and $\frac{\partial \dot{R}}{\partial \dot{X}_\ell}$ for range rate measurements, equ. (4.6), and the elements $\frac{\partial R}{\partial X_\ell}$ for range observations, equ. (4.7). Analogously the coefficients of B are $\frac{\partial X_\ell}{\partial T_x}$ and $\frac{\partial \dot{X}_\ell}{\partial T_x}$ for the range rate and $\frac{\partial X_\ell}{\partial T_x}$ for the range method. The terms of matrix C are $\frac{\partial T_x^{-1}}{\partial \Delta g_k}$ and $\frac{\partial T_x^{-2}}{\partial \Delta g_k}$. We divide the sensitivity analysis for Δg estimation from range or range rate observations into the analysis of the matrices A, B, and C.

Matrix A: The coefficients $\frac{\partial R}{\partial X_\ell}$, $\frac{\partial \dot{R}}{\partial X_\ell}$, and $\frac{\partial \dot{R}}{\partial \dot{X}_\ell}$ are the projections of the range or range rate vector onto the components of the position and velocity vector of the satellite, expressed in the inertial coordinate system (X_1, X_2, X_3). Their magnitude and therefore their influence on the final result is dependent on the geometrical configuration of the two satellites. The sensitivity coefficients in A may be transformed either into an earth fixed (φ, λ, r)-system into radial, longitudinal, or latitudinal direction or into a radial, along track and cross track component. In the latter system Koch and Argentiero (1974) carried out sensitivity considerations for the GRAVSAT/GEOPAUSE mission.

In a "high low" SST experiment the projection onto the radial coordinate axis of the range or range rate vector between two satellites will be predominant. Therefore the radial variations in range and range rate will be essential for the gravity anomaly determination.

The geometrical configuration in a "low-low" SST experiment will change continuously, no predominant coefficient occurs. The three components with respect to the axis of a certain coordinate system will contribute equally to the final result regardless of their importance with respect to Δg .

Equations (4.6) and (4.7) show the presence of derivatives with respect to the satellite position X and the satellite velocity X for range rate experiments whereas for range measurements, as expected, only derivatives with respect to

\underline{X} are involved. This is the essential difference in the solution equations for both types of observations. As discussed in detail in Chapter 2, the satellite position is only in a very smoothed form dependent on the gravity field. For example, Chovitz (1973) derived that the effect of high harmonics in the potential ($n \geq 40$) on a satellite position in altitude 900 km is absolutely negligible. Sufficient sensitivity for the gravity anomaly determination may be only expected from satellite velocity dependent terms. Therefore in the following our considerations are limited to range rate experiments.

Matrix B: The coefficients $\frac{\partial X}{\partial T_{x_j}}$ and $\frac{\partial \dot{X}}{\partial T_{x_j}}$ in \underline{B} connect the satellite

position and velocity vector with the components of the gradient $\underline{\nabla T}$ of the disturbing potential. The gradient $\underline{\nabla T}$ is equal to the acceleration vector $\underline{\ddot{X}}$. The magnitude of these terms is dependent on the method of integration from \ddot{X}_t to \dot{X}_t and X_t at every instant t . For high precision in a numerical integration many terms of type X_{t+i} , $i = -k(0)l$, have to be introduced into a linear combination. In order to avoid high error correlation between these neighboring data and a considerable loss of resolution caused by the method of numerical integration a high sample rate is desired which has its limit in the technical equipment.

Matrix C: The elements of C are $\frac{\partial T_x^1}{\partial \Delta g_k}$ and $\frac{\partial T_x^2}{\partial \Delta g_k}$. The transformation of the components T_{x_j} , $j = 1, 2, 3$ of $\underline{\nabla T}$ in the inertial system to the components $T_r = \frac{\partial T}{\partial r}$, $T_\varphi = \frac{\partial T}{\partial \varphi}$, and $T_\lambda = \frac{\partial T}{\partial \lambda}$ in the earth fixed (r, φ, λ) system is expressed, Rapp (1974, p. 5), by

$$\begin{bmatrix} T_{x_1} \\ T_{x_2} \\ T_{x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial X_1} & \frac{\partial \varphi}{\partial X_1} & \frac{\partial \lambda}{\partial X_1} \\ \frac{\partial r}{\partial X_2} & \frac{\partial \varphi}{\partial X_2} & \frac{\partial \lambda}{\partial X_2} \\ \frac{\partial r}{\partial X_3} & \frac{\partial \varphi}{\partial X_3} & \frac{\partial \lambda}{\partial X_3} \end{bmatrix} \begin{bmatrix} T_r \\ T_\varphi \\ T_\lambda \end{bmatrix}$$

The gradient in spherical coordinates $\underline{\nabla T}_{\text{sph}}$ has the form

$$\underline{\nabla T}_{\text{sph}} = \begin{bmatrix} T_r \\ \frac{1}{r} T_\varphi \\ \frac{1}{r \cos \varphi} T_\lambda \end{bmatrix}$$

and is related to the gravity anomalies by the expression, (Heiskanen-Moritz, p. 234) :

$$\nabla T_{sph} = \frac{R}{4\pi} \left[\begin{array}{l} \iint_{\sigma} \Delta g \frac{\partial St(r, \psi)}{\partial r} d\sigma \\ - \frac{1}{r} \iint_{\sigma} \Delta g \frac{\partial St(r, \psi)}{\partial \psi} \cos \alpha d\sigma \\ - \frac{1}{r} \iint_{\sigma} \Delta g \frac{\partial St(r, \psi)}{\partial \psi} \sin \alpha d\sigma \end{array} \right] \quad (4.8)$$

with $St(r, \psi)$... Pizzetti's extension of Stokes' function, (Heiskanen-Moritz, p. 92, Hotine, 1969, p. 310)

It is easy to find from (4.8) the derivative of T_x with respect to a certain mean anomaly Δg_k by approximating the integral by a finite sum, (Hajela 1974). The influence of each of the three terms of equ. (4.8) in their linear combination expressed by C is decided by the magnitude of the derivatives of Pizzetti's function with respect to ψ and r and the area of the finite surface elements $\Delta\sigma$ for the block mean Δg_k . The term $\frac{\partial St(r, \psi)}{\partial \psi}$ in the T_ϕ and T_λ component is rotation invariant and projected by $\cos \alpha$ and $\sin \alpha$ onto a desired azimuth. Therefore a comparison of $-\frac{1}{r} \frac{\partial St(r, \psi)}{\partial \psi}$ and $\frac{\partial St(r, \psi)}{\partial r}$ is sufficient.

By considering the spectral properties of these two functions we get an impression of their behavior in the frequency domain. Both functions are connected with Δg by the convolution integrals in (4.8) and therefore act like filters which decide about the frequency content of the derived gravity anomaly.

Pizzetti's function $St(r, \psi)$ is expressed in an expansion into Legendre functions, by (Hotine, 1969, p. 311)

$$(4.9) \quad St(k, \psi) = \sum_{n=2}^{\infty} \frac{1}{n-1} k^{n+1} (2n+1) P_n(\cos \psi)$$

which becomes with $k = \frac{R}{r}$ and $k < 1$

$$(4.10) \quad St(r, \psi) = R \sum_{n=2}^{\infty} \frac{1}{n-1} \left[\frac{R}{r} \right]^{n+1} (2n+1) P_n(\cos \psi)$$

The radial derivative is

$$(4.11) \quad \frac{\partial St(r, \psi)}{\partial r} = - \sum_{n=2}^{\infty} \frac{n+1}{n-1} \left(\frac{R}{r} \right)^{n+2} (2n+1) P_n(\cos \psi)$$

and the derivative with respect to ψ

$$(4.12) \quad - \frac{1}{r} \frac{\partial St(r, \psi)}{\partial \psi} = - \sum_{n=2}^{\infty} \frac{1}{n-1} \left(\frac{R}{r} \right)^{n+2} (2n+1) \frac{\partial P_n(\cos \psi)}{\partial \psi}$$

In the radial component of ∇T_{sph} the spectral content is connected with the spectrum of a certain gravity anomaly Δg_k by the factor $\frac{n+1}{n-1} \left(\frac{R}{r} \right)^{n+2}$. Thus, beside the attenuation effect $\left(\frac{R}{r} \right)^{n+2}$ the two spectra are in a one to one correspondence, because $\lim_{n \rightarrow \infty} \frac{n+1}{n-1} = 1$.

For both other components of ∇T , which have only poor high frequency information, the derived amplitudes in the Δg spectrum are amplified by $(n-1)$, because of the factor $\frac{1}{n-1}$ in equ. (4.12). It will be difficult to obtain from these two terms higher frequency information in Δg which is not seriously affected by errors.

Since the high frequencies at high altitudes, where the relay satellite is located are very strongly damped, the $\frac{\partial T_x^{i2}}{\partial \Delta g_k}$ coefficients will hardly contribute to the derivation of gravity anomalies of small block size. The influence of each of the components of equation (4.9) is dependent on its magnitude in the linear combination which is expressed by the matrix product

$$\underline{W} \Delta g = \underline{A} \underline{B} \underline{C} \Delta g$$

and is by \underline{A} mainly a function of the geometrical configuration of the two satellites.

4.2 Satellite gradiometry

In satellite gradiometry the equations are more transparent. The gravity quantity is measured directly by a satellite mounted instrumentation. Whereas in SST the gravity gradient was derived from range rate perturbations on a predetermined satellite orbit.

Two different gradiometer types are in discussion, the hardmounted and the rotating gradiometer, (Forward, 1971, Reed, 1973). For the amplitude of

the output of both types an extended sensitivity analysis was done by Glaser and Sherry (1972) and in more detail by Reed (1973).

We are interested in the resolution of the measurement elements with respect to the spectral content of gravity anomalies. The components of the gravity tensor are measured in a moving frame. By coordinate transformation they are transformed to the earth fixed (φ, λ, r) -system. Thus, in a least squares solution the sensitivity matrix \underline{W} will again consist of several terms with more or less desired properties.

The interrelations of the tensor components of T in the local rectangular coordinate system (η, ξ, ζ) -- ζ ... radial direction, η ... tangential latitudinal direction, ξ ... tangential longitudinal direction to the components in the earth fixed (φ, λ, r) -system are derived in Reed (1973),

$$\begin{aligned}
 T_{\eta\eta} &= \frac{1}{\cos^2\varphi} \left(\frac{1}{r^2} T_{\lambda\lambda} \right) + \frac{\tan\varphi}{r} \left(\frac{1}{r} T_{\varphi} \right) + \frac{1}{r} T_r \\
 T_{\xi\xi} &= \left(\frac{1}{r^2} T_{\varphi\varphi} \right) + \frac{1}{r} T_r \\
 T_{\zeta\zeta} &= T_{rr} \\
 (4.13) \quad T_{\xi\eta} &= T_{\eta\xi} = \frac{1}{\cos\varphi} \left(\frac{1}{r^2} T_{\lambda\varphi} \right) + \frac{\sin\varphi}{r \cos^2\varphi} \left(\frac{1}{r} T_{\lambda} \right) \\
 T_{\zeta\eta} &= T_{\eta\zeta} = \frac{1}{\cos\varphi} \left(\frac{1}{r} T_{\lambda r} \right) - \frac{1}{r \cos\varphi} \left(\frac{1}{r} T_{\lambda} \right) \\
 T_{\zeta\xi} &= T_{\xi\zeta} = \left(\frac{1}{r} T_{\varphi r} \right) - \frac{1}{r} \left(\frac{1}{r} T_{\varphi} \right)
 \end{aligned}$$

The connection of the second derivatives of the disturbing potential in the (φ, λ, r) -system to the desired gravity anomalies are again expressed by derivations of the extended Stokes' formula with respect to r, φ and λ , for example

$$T_{rr} = \frac{R}{4\pi} \iint_{\sigma} \Delta g \frac{\partial^2 St(r, \psi)}{\partial r^2} d\sigma$$

Therefore the discussion of the C-matrix for the range rate experiments can be repeated analogously for gradiometry. To get an insight into the frequency

behavior of the kernel functions $\frac{\partial^2 St(r, \psi)}{\partial r^2}$, $\frac{\partial^2 St(r, \psi)}{\partial \psi \partial r}$ (representing the kernel functions for $T_{r\phi}$ and $T_{r\lambda}$) and $\frac{\partial^2 St(r, \psi)}{\partial \psi^2}$ (representing the kernel functions for $T_{\phi\phi}$, $T_{\lambda\lambda}$, and $T_{\lambda\phi}$) we derive their spectral formulas:

$$\frac{\partial^2 St(r, \psi)}{\partial r^2} = \frac{1}{r} \sum_{n=2}^{\infty} \frac{(n+1)(n+2)}{(n-1)} \left[\frac{R}{r} \right]^{n+2} (2n+1) P_n(\cos \psi)$$

$$(4.14) \quad \frac{1}{r} \frac{\partial^2 St(r, \psi)}{\partial r \partial \psi} = - \frac{1}{r} \sum_{n=2}^{\infty} \frac{n+1}{n-1} \left[\frac{R}{r} \right]^{n+2} (2n+1) \frac{\partial P_n(\cos \psi)}{\partial \psi}$$

$$\frac{1}{r^2} \frac{\partial^2 St(r, \psi)}{\partial \psi^2} = \frac{1}{r} \sum_{n=2}^{\infty} \frac{1}{n-1} \left[\frac{R}{r} \right]^{n+2} (2n+1) \frac{\partial^2 P_n(\cos \psi)}{\partial \psi^2}$$

As already discussed in Chapter 2, the second radial derivative has very good high frequency information. Errors in the radial tensor component T_{rr} are damped in deriving Δg by the factor $\frac{n-1}{(n+1)(n+2)}$, equ. (4.14a), and high resolution can be expected. For the purpose of Δg - computation in small block sizes (which have high frequency content), the T_{rr} component seems to be well suited.

All derivatives of T in the tensor with respect to r and an additional component (ϕ or λ) are in a one to one correspondence with the gravity anomalies, compare (4.14b).

For second derivatives not containing a radial component, the kernel (in equ. (1.14c)) function will amplify the frequencies of the tensor components in $T_{\phi\phi}$, $T_{\lambda\lambda}$ and $T_{\phi\lambda}$ by $(n-1)$. These tensor components have only good low frequency information. Therefore we may expect an error influence on the higher frequencies in the mean block anomalies for these components.

The other members in (4.13), T_r , $\frac{1}{r} T_{\phi}$ and $\frac{1}{r} T_{\lambda}$ are all damped by $\frac{1}{r}$.

Their influence on the resulting gravity anomalies will be small.

The signal amplitude for the rotating gradiometer is, Reed (1973),

$$\text{amp} = ((V_{33} - V_{11})^2 + 4V_{13}^2)^{\frac{1}{2}}$$

subindices $1,2,3 \dots$ moving frame, 1-axis directed outward.

The component V_{11} would have the advantageous properties of T_{rr} but cannot be separately evaluated. By the combination with V_{33} and V_{13} the result is dependent on the worse properties of these terms. The deduced gravity anomalies cannot be resolved with the same accuracy than from T_{rr} alone.

Only for the hard-mounted system it seems to be possible to separate the radial derivative T_{rr} if an accurate space orientation is available. In a closed adjustment for five independent tensor components the special features of T_{rr} would be disturbed. But as far as we can see only the rotating system has a chance to be realized in the near future with sufficient accuracy.

These considerations show that the advantageous properties of $\frac{\partial^2 T(P)}{\partial r^2}$ and $\frac{\partial T(P)}{\partial r}$ for a high resolution in gravity anomalies, indicated by the discussion of Chapter 2, are with the up to now available techniques not realizable. Sensitivity studies based only on the radial components are therefore to a certain amount misleading.

The aim of the investigations in this chapter was to demonstrate the advantage of an "a priori" - sensitivity analysis of an experiment. The coefficients of the sensitivity matrix \underline{W} can be evaluated by the mathematical interrelations between measurement and unknown quantity. No simulations or practical data are assumed. For SST it seems to be possible to deduce objective criteria about the optimal geometrical configuration of the relay and the tracked satellite, a possible need for more than one relay satellite to get a better geometrical configuration and the requirements in the numerical integration. An accurate comparison of the sensitivity matrices in SST and gradiometry should give an impression of the individual advantages of both techniques. Detailed conclusions may be drawn, as indicated by a similar "a posteriori" sensitivity study for traditional satellite techniques by Gaposchkin (1970).

5. Downward Continuation from Satellite Altitudes.

As already mentioned in Chapter 3 we are mainly interested in deriving surface gravity anomalies from satellite measurements. In order to limit the number of unknowns a local improvement seems to be more promising. The derivation shall be done from the gradient of the disturbing potential ∇T by satellite to satellite tracking techniques or from the components of the anomalous potential tensor in satellite gradiometry.

Up to now this topic received little attention as a downward continuation problem. Theoretical and numerical studies treated the continuation from aerial altitudes or the reduction from the earth's surface to the geoid, for example Dean (1958), Madkour (1966), Koch, K-R. (1968), Moritz (1970), Schwarz, K-P. (1971), Bjerhammar (1966) and (1973).

An impression of the downward continuation effect on the gravity anomaly variance for ranges of n which are essential for $5^\circ \times 5^\circ$, $2.5^\circ \times 2.5^\circ$, and $1^\circ \times 1^\circ$ anomalies is given in Table 4.

The formula used for Table 4 and Figure 4 is:

$$\text{var}(\Delta g)_{n_1, n_2} = \sum_{n=n_1}^{n_2} \left[\frac{R}{r} \right]^{2n+4} \frac{A}{n+B} \quad (\text{A})$$

which is a simplified version of the expression

$$\text{var}(\Delta g)_{n_1, n_2} = \sum_{n=n_1}^{n_2} s^{n+2} \frac{A(n-1)}{(n-2)(n+B)}, \quad s = \frac{R_{Bje}^2}{r^2} \quad (\text{B})$$

derived by Tscherning and Rapp (1974). It may be evaluated either computationally in a summation routine or analytically. Then we get

$$\begin{aligned} \text{var}(\Delta g)_{n_1, n_2} &= \sum_{n=n_1}^{n_2} s^{n+2} \frac{A}{n+B} \doteq A s^2 \int_{n_1}^{n_2} \frac{s^n}{n+B} dn \\ &= A s^2 [e^{-B \ln s} \overline{\text{Ei}(\ln s (n+B))}]_{n_1}^{n_2}, \end{aligned}$$

because $\int \frac{e^{kx}}{a+bx} = \frac{1}{b} e^{-ka/b} \overline{\text{Ei}}\left(\frac{k}{b}(a+bx)\right)$, (Meyer Zur Capellen W., *Integraltafeln*, Berlin, 1950).

The exponential integral $\text{Ei}(x)$ is listed in standard tables like (Akademija Nauk SSR, *Tafelnder Exponentialintegrale*, Moskau, 1954).

Two facts are of particular interest:

1. The changing ratio of the high to the low frequency part for different altitudes, for example

$$\frac{\text{var}(\Delta g)_{3,20}}{\text{var}(\Delta g)_{73,180} \text{ h} = 0 \text{ km}} \doteq \frac{0.8}{1} \quad \text{whereas} \quad \frac{\text{var}(\Delta g)_{3,20}}{\text{var}(\Delta g)_{73,180} \text{ h} = 200 \text{ km}} \doteq \frac{196}{1}$$

2. The immense attenuation effect for higher frequencies as can be seen in Fig. 4. For example the amount $\text{var}(\Delta g)_{73,180}$ i.e. the additional information content distinguishing a $1^\circ \times 1^\circ$ anomaly from a $2.5^\circ \times 2.5^\circ$ anomaly (assuming the simplified rule that the highest frequency degree n in a $\theta \times \theta$ anomaly is $n = \frac{180^\circ}{\theta}$) is amplified from 0.6 mgal^2 at 200 km elevation by 510%

Table 4

Sum of Degree Variances for Gravity Anomalies $\text{var}(\Delta g)_{n_1, n_2}$ Between Degree n_1 and n_2 for different altitudes.

[km] altitude	[mgal ²] n =				
	3	20	36	72	180
0	235.7	129.2	194.5	305.2	
100	163.9	51.1	36.8	10.3	
200	117.8	20.9	7.8	0.6	
300	87.1	8.8	1.8	0.0 ₄	
400	66.1	3.8	0.5	0.0 ₀₀₀₄	

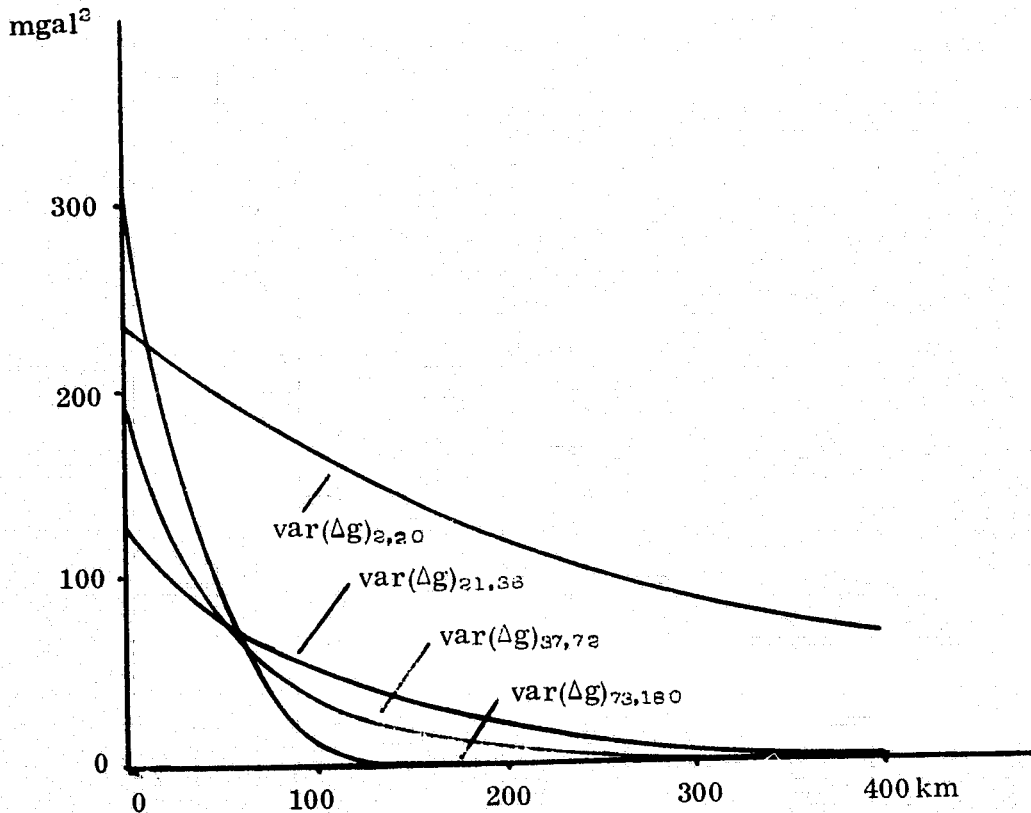


Figure 4: Attenuation of the variance in Δg between degree n_1 and degree n_2 from altitude 0km to 400 km.

to 305.2 mgal² at the earth's surface. An observation error of only ±0.5 mgal in this frequency range (at 200 km) will rise to 128 mgal² at the surface and cause an uncertainty of about 15% in the 1° x 1° anomaly with respect to surface variance of 865 mgal². These few numbers illustrate what extreme accuracy should be reached by the observation equipment for a 1° x 1° anomaly resolution. It also explains why it is necessary to discuss the topic as a downward continuation problem.

The center of the following considerations will be

$$(5.1) \quad T(P) = \frac{R}{4\pi} \int_{\sigma} St(r, \psi) \Delta g(P') d\sigma,$$

with $St(r, \psi)$... Pizzetti's extension of Stokes' function
 $P' \in \sigma$... as in Chapter 3.

Equation (5.1) connects the anomalous potential T in a space point $P \in \Sigma$ with the boundary function $\Delta g(P')$ on the sphere σ . In what way the continuation operator of equ. (2.9) is involved becomes obvious by expanding (5.1) into spherical harmonics. From (4.10) we get for $St(r, \psi)$,

$$(5.2) \quad St(r, \psi) = \sum_{n=2}^{\infty} \frac{1}{n-1} \left[\frac{R}{r} \right]^{n+1} \sum_{m=0}^n (\bar{R}_{nm}(P) \bar{R}_{nm}(P') + \bar{S}_{nm}(P) \bar{S}_{nm}(P'))$$

and we end up in analogy to equ. (2.6) with

$$(5.3) \quad \begin{Bmatrix} c(T)_{nm} \\ s(T)_{nm} \end{Bmatrix} = R \frac{1}{n-1} \left[\frac{R}{r} \right]^{n+1} \begin{Bmatrix} c'(\Delta g)_{nm} \\ s'(\Delta g)_{nm} \end{Bmatrix}$$

The same result can immediately be derived from Table 2 by connecting Δg_{nm}^R with T_{nm}^R . The inverse formula is

$$(5.4) \quad \begin{Bmatrix} c'(\Delta g)_{nm} \\ s'(\Delta g)_{nm} \end{Bmatrix} = \frac{1}{R} (n-1) \left[\frac{r}{R} \right]^{n+1} \begin{Bmatrix} c(T)_{nm} \\ s(T)_{nm} \end{Bmatrix}$$

The problem is to solve the integral equation of the first kind (5.1) for a finite number of gravity anomalies given discrete measurements of the gradient or the tensor components of T . Integral equation (5.1) formulates an "improperly posed problem", as the inverse operator of $St(r, \psi)$ is not bounded as may be

seen from (5.4). For an analytic study methods of an approximate solution are indicated by Weck (1972) and Bjerhammar (1973) or deduced in more detail for the planar approximation by Schwarz (1971). On the other hand there is also proposed the method of collocation in the statistical or analytical sense (reproducing kernel Hilbert spaces) by Moritz (1970) and Krarup (1969).

As we will see there is a lot of complications connected with the deterministic approach to find a solution of the downward continuation problem by means of equ. (5.1) whereas the solution by prediction methods seems to be very easy and extremely elegant. Logically one may ask why the prediction shall be so much easier. Where have the difficulties for the deterministic solution gone in the prediction approach?

Krarup (1969) defined a Hilbert space H of all given potentials φ regular in Σ with square integrable boundary values on σ . (This last restriction is essential and enables a solution of the downward-continuation for φ to be found). In addition, with the assumption of rotational invariance an expression for the covariance function of the disturbing potential is:

$$(5.5) \quad C_{TT}(P, Q) = \sum_{n=0}^{\infty} v_n(T) \left[\frac{R^2}{r_p r_q} \right]^{n+1} P_n(\cos \psi_{PQ})$$

where P and Q may be both $\in \Sigma$ or $\in \sigma$,

$v_n(T)$... degree variances of T (units : potential)

The kernels in the integral formulas of physical geodesy are usually known from the solution of the boundary value problems. In a first approximation these boundary value problems are solved for a spherical boundary surface. The spherical reference surface generates rotational invariant integral kernels with a reproducing property in H . The reproducing property of the operators allows us to derive auto- and cross-covariances for all other quantities related to the earth's potential. The Poisson kernel, eq. (2.8), to be applicable for both P and $Q \in \Sigma$ is generalized to

$$(5.6) \quad K_{\rho\sigma}(P, Q) = \sum_{n=0}^{\infty} \left[\frac{R^2}{r_p r_q} \right]^{n+1} (2n+1) P_n(\cos \psi_{PQ})$$

$$= \sum_{n=0}^{\infty} \left[\frac{R^2}{r_p r_q} \right]^{n+1} \sum_{m=0}^n (\bar{R}_{nm}(P) \bar{R}_{nm}(P') + \bar{S}_{nm}(P) \bar{S}_{nm}(P'))$$

A linear prediction approach for the derivation of gravity anomalies Δg from disturbing potential values T may be formulated neglecting measurement errors by

$$(5.7) \quad \underline{\Delta g}_{P'} = \underline{B} \underline{T}_P$$

$\underline{\Delta g}_{P'}$... vector of unknown quantities

\underline{B} ... unknown system matrix

\underline{T}_P ... given data set

Given T continuous on a sphere τ with radius r in satellite altitude and concentric with σ equation (5.7) becomes

$$(5.8) \quad \Delta g(P') = \int_{\tau} B(P'-P) T(P) d\tau$$

First we try to find an expression for the system function $B(P'-P)$. In a statistical approach we take the expectation $E(\cdot)$ on both sides of (5.8) with respect to the given function $T(Q)$, $Q \in \tau$.

$$(5.9) \quad E(\Delta g(P')T(Q)) = \int_{\tau} B(P'-P)E(T(P)T(Q)) d\tau$$

Assuming rotational invariance for the "stochastic" process $T(Q)$ we insert $C_{\tau\tau}$ from equ. (5.5) for $E(T(P)T(Q))$. For $E(\Delta g(P')T(Q))$ we introduce

$$(5.9a) \quad C_{\Delta g_{\tau}}(P', Q) = \sum_{n=0}^{\infty} \frac{1}{R} (n-1) v_n(T) \left[\frac{R}{r_Q} \right]^{n+1} P_n(\cos \psi'_{P'Q})$$

The unknown system function related to a sphere can be developed into Legendre polynomials

$$(5.9b) \quad B(P'-P) = \sum_{n=0}^{\infty} x_n P_n(\cos \psi'_{P'P})$$

Inserting (5.5), (5.9a), and (5.9b) into equation (5.9) gives using the orthogonality relationships

$$(5.10) \quad \frac{1}{R} (n-1) v_n(T) \left[\frac{R}{r_Q} \right]^{n+1} = x_n v_n(T) \left[\frac{R^2}{r_P r_Q} \right]^{n+1} \text{ and}$$

$$x_n = \frac{1}{R} (n-1) \left[\frac{r_P}{R} \right]^{n+1}$$

The x_i terms are exactly the coefficients of the spectral representation of the "downward continuation" operator we derived in equ. (5.4). We see that for the global case the solution of the linear prediction -- assuming no data errors -- has the same result as the analytic approach.

The basic equation is the "Wiener-Hopf" equation (5.9) which may be rewritten for this example as

$$(5.11) \quad C_{\Delta g_{\tau}}(P', Q) = \int_{\tau} B(P' - P) C_{\tau\tau}(P, Q) d\tau.$$

By (5.11) the previously mentioned reproducing property of the integral kernels in the spherical approximation of the basic equations in physical geodesy becomes obvious. Equ. (5.11) is the expression for the so called "covariance law of propagation" which allows us to derive the auto- and cross-covariances of all harmonic quantities related to the earth's potential from $C_{\tau\tau}$ of equ. (5.5).

Because of the orthogonality relationships, equations (5.8) to (5.11) hold true also when they involve functions containing only frequencies up to a finite degree N , as it is applied for global satellite solutions, compare with equ. (3.7). Therefore, the prediction method will not be connected with an essential advantage for global applications compared with the deterministic method. But already for combination solutions with terrestrial data prediction may be more convenient. For a finite amount of data not covering the total sphere the integral formulas become matrix equations which approximate the continuous expressions only to a certain degree.

The "Wiener-Hopf" integral equation degenerates to the matrix equation

$$(5.12) \quad \underline{C}_{\Delta g_p, \tau_Q} = \underline{B} \underline{C}_{\tau_P, \tau_Q} \quad \text{and}$$

$$(5.13) \quad \underline{B} = \underline{C}_{\Delta g_p, \tau_Q} \underline{C}_{\tau_P, \tau_Q}^{-1}$$

When we insert the finite expressions of $C_{\tau\tau}$ and $C_{\Delta g_{\tau}}$ into equ. (5.12) the orthogonality relations are lost and \underline{B} will not be the finite, discretized form of the "downward continuation operator". Logically the deterministic approach, derived from the discretisation of equ. (5.1) and the prediction approach, deduced by inserting equ. (5.13) into equ. (5.7) will lead to different results. In a smoothing process such as upward continuation or calculation of undulations from gravity anomalies the continuous formulas are integrals, not integral equations. Here the discretisation cannot be very misleading. The discretisation error will be about the same magnitude as the prediction error. Generally, prediction will result in smoother quantities--in this case a desired feature--and can be easily carried out for non-homogeneous data distribution.

The solution of the downward continuation problem is an unsmoothing procedure in the manner of the computation of gravity anomalies from undulations. For such problems the approximate solution of an integral equation is desired.

Equ. (5.1) becomes:

$$(5.14) \quad \underline{T}_p + \underline{n} = \underline{A} \underline{\Delta g}_p'$$

\underline{n} ... observation noise, with $E(\underline{n}) = 0$ and $E(\underline{n} \underline{n}') = C_{nn}$ and the problem is to find a proper inverse of \underline{A} . A conceptual analogy between the question of finding an optimal inverse of \underline{A} by linear estimation in terms of generalized inverses and the collocation (generalized prediction and adjustment) concept was noted by Moritz (1973). For problems of this type the prediction method seems to be more elegant because it apparently avoids the trouble connected with finding the proper inverse of \underline{A} .

5.1 Least Squares Prediction Solution

The prediction formula for the discrete downward continuation will be (with equ. (5.7) and equ. (5.13)):

$$(5.15) \quad \underline{\Delta g}_p' = \underline{C} \underline{\Delta g}_p' \tau_q \underline{C}_{\tau_p \tau_q}^{-1} \underline{T}_p$$

By taking into account observation noise \underline{n} as in equ. (5.14) with $E(\underline{T}_p \underline{n}') = 0$ equ. (5.15) has to be modified to

$$(5.16) \quad \underline{\Delta g}_p' = \underline{C} \underline{\Delta g}_p' \tau_q \underline{\bar{C}}^{-1} \underline{X}_p \quad \text{and} \quad \underline{X}_p = \underline{T}_p + \underline{n}_p$$

$$\underline{\bar{C}} = \underline{C}_{\tau_p \tau_q} + \underline{C}_{n_p n_q}$$

For all practical applications \underline{T}_p will consist of a very limited number of observations. Now $\underline{C}_{\tau_p \tau_q}$ expresses the real physical relation between the observations and $\underline{C}_{\Delta g_p' \tau_q}$ the real physical relation between the observations and the desired unknowns. An autocovariance matrix $\underline{C}_{\tau_p \tau_q}$ estimated from the observed values \underline{T}_p may not have a high confidence because of the limited number of samples and $\underline{C}_{\Delta g_p' \tau_q}$ cannot be calculated on the basis of the given data for the Δg_p are unknown. Therefore the sample covariances are supplied under the assumption of isotropy and stationarity by the global covariances. In addition these general covariances make the results of the limited number of T_{p_i} consistent with later observations in other areas and a global solution.

Statistically speaking a continuous correlated process on the sphere is approximated by a discrete n-order Markov model. For this model the optimal solution would be the conditional expectation

$$\hat{\Delta g}_p' = E(\Delta g_p' | T_p + n_p)$$

with the conditions

$$E(\hat{\Delta g}_p') = E(B(T_p + n_p)) = E(\Delta g_p') \quad [\text{unbiased}]$$

and

$$E \left| \Delta g_p' - \hat{\Delta g}_p' \right|_c^2 \leq E \left| \Delta g_p' - B(T_p + n_p) \right|_c^2$$

with basis \underline{C} ... nonnegative definite and symmetrical.

In our case the first condition is not fulfilled because Δg is "absolute" defined on the basis of $T(P)$. Therefore it is not desirable to solve the problem on the basis of sample covariances which would lead to a biased result. The drawback of introducing the global covariances into equ. (5.15) is that their system of eigenfunctions and eigenvalues is not the system of eigenvectors and eigenvalues of the discrete finite solution. The results will be very smooth and optimally consistent. Smoothness is not at all desirable in downward continuation where we try to find the unsmoothed quantity from the damped. But the control of high frequencies by prediction will bound the errors and therefore make a solution of the "improperly posed problem" feasible. The following conceptual disadvantages are essential for the prediction application.

--The finite approach predicts Δg_p only with the least possible prediction error $\epsilon_p = (\Delta g_p - \hat{\Delta g}_p)$ dependent on the introduced covariances

$$(5.17) \quad \text{var}(\epsilon_p) = \underline{C}_{\Delta g_p' \Delta g_p'} - \underline{C}_{\Delta g_p' \tau_p} \overline{\underline{C}_{\tau_p \tau_p}}^{-1} \underline{C}_{\Delta g_p' \tau_0}$$

The variance of the prediction error in equ. (5.17) tends only for $\lim_{T_p \rightarrow T(P)}$ to zero.

--The local deviation of the actual gravity field from the assumed isotropic and stationary global field causes an error ΔC in the covariances. The influence of ΔC on the prediction result will be of low order but considerable for the prediction error ϵ_p (Rummel (1975)).

It should be mentioned that the solution of Bjerhammar (1973) is not nonstationary. For nonstationary quantities related to a sphere it is impossible to find a covariance function in a closed form expanded into Legendre polynomials.

In the application of linear prediction methods we are faced with two problems:

1. In order to avoid filtering of desired frequencies the covariances have to be at least of the same degree as the output quantities. This means that for prediction of $1^\circ \times 1^\circ$ anomalies the degree of the covariances has to be higher than $n=180$. This causes serious trouble because gravity material necessary for a precise computation of covariances of such high degree is not available with sufficient density. The available terrestrial anomalies will lead to a not representative covariance function. The best possible covariance function based on the current gravity material is derived by Tscherning and Rapp (1974).

2. For a high number of observations the autocovariance matrix \overline{C} of equation (5.16) grows to a large dimension. The inversion of this large \overline{C} -matrix is a serious problem comparable with the difficulties connected with the necessary inversion of the matrix A in the deterministic approach. By applying stepwise methods the trouble can be reduced.

5.2 Deterministic approach

The problems related to the nonstatistical solution for the downward continuation are worked out in some detail by Schwarz (1971) for the planar approximation. The results derived there are valid with some modification for the spherical approach too and need not be repeated in detail. We have to solve the discrete version (5.14) of an integral equation of the first kind (5.1) for $\Delta g_p'$. Restrictions of two types are usually not avoidable:

The sample rate of the observations and their noise level allows only a limited resolution up to degree N . This restriction is not inconvenient because the frequency limitation is also necessary to stabilize the solution of the downward continuation problem. Because of the orthogonality relation between different frequencies, the finite solution is consistent with the infinite one. The results up to degree N may be treated as reference for the evaluation from $N+1$ to a higher frequency $N+k$, made possible by a more sensitive technical equipment.

In addition, data will be usually gathered only in a limited area. When we assume the satellite moves in a near circular orbit then with a certain approximation the observations are given on an outer sphere with surface τ . The surface of the sphere may be divided into a part τ_1 where observations are given and a part τ_0 with no information, $\tau = \tau_1 + \tau_0$. On the other hand it is impossible to derive from the limited distribution of observations a global solution.

A $1^\circ \times 1^\circ$ block global solution would require 64,800 unknowns (\doteq 44,000 equal area blocks). Thus also the surface σ of the boundary sphere is divided

into an inner region σ_1 , where gravity anomalies will be derived and a truncated outer region σ_0 , $\sigma_1 + \sigma_0 = \sigma$. The part σ_1 will usually be somewhat larger than the subregion of τ_1 (the projection of τ_1 onto σ).

We rewrite equation (5.14) as:

$$(5.18) \quad T_{P_1} = \underline{A}_1 \Delta g_{P_1} + \underline{A}_0 \Delta g_{P_0},$$

where, for example \underline{A}_0 transforms gravity anomalies from the domain σ_0 to disturbing potential values in the domain τ_1 . The problem is always underdetermined because of the limited number of given T_{P_1} -values. Only by smoothing, i.e. choosing the block size of the unknown gravity anomalies large the number of unknowns decreases to a finite number m . If the number of T_{P_1} -values is higher than m we get from this smoothing procedure an "overdetermined" problem.

For the finite solution with m unknown Δg -values in σ_1 , T_{P_1} has to be corrected by $\underline{A}_0 \Delta g_{P_0}$, equation (5.18) becomes

$$(5.19) \quad \underline{T}_{P_1} - \underline{A}_0 \Delta g_{P_0} = \underline{A}_1 \Delta g_{P_1}$$

When no gravity information on σ_0 is available we have to neglect the correction $\underline{A}_0 \Delta g_{P_0}$. This will lead to a systematic error in the results. The magnitude of this error depends on the size covered by σ_0 , for $\lim \sigma_0 \rightarrow 0$ the solution converges to a global solution.

If σ_1 is exactly the projection of τ_1 onto σ with respect to the common origin of both spheres the influence of $\underline{A}_0 \Delta g_{P_0}$ on the derived Δg_{P_1} will grow from the center of τ_1 to its border and result in a strong bias in $\underline{\Delta g}_{P_0}$.

But it is also possible to build up every element of T_{P_1} by a linear combination with Δg_{P_1} elements covering a certain cap, compare with Schwarz, (1971, p.40). Then σ_1 will be larger than the projection of τ_1 onto σ and the error influence of $\underline{A}_0 \Delta g_{P_0}$ will be about the same for every T_{P_1} . Usually this type of solution is preferred.

But we may also try to avoid or at least reduce the influence of $\underline{A}_0 \Delta g_{P_0}$ by computing it from available information in σ_0 . Most precisely the outer part is taken into account by computing a correction $\underline{A}_0 \Delta g_{P_0}$ in σ_0 from all available terrestrial gravity anomalies and interpolated values in unsurveyed regions for all elements of \underline{T}_{P_1} in satellite elevation. This procedure will lead to an optimal combination with gravity information especially when σ_1 consists mainly of ocean regions. The large drawback is that the computation would be very time consuming.

An approximation for the r. m. s. influence of $\underline{A}_0 \Delta g_{P_0}$ may be derived assuming a special mode for the degree variances $v_n(\Delta g)$ such as expressions shown by Rapp (1972) or by Tscherning and Rapp (1974).

Modifying the well known truncation formula of Molodenskii (Heiskanen and Moritz, p. 260) we get

$$\underline{A}_0' \underline{\Delta g}_{p_0} \doteq \frac{R}{4\pi} \int_{\psi=\psi_0}^{180^\circ} \int_{\alpha} St(r, \psi) \Delta g(P') d\sigma = \delta T(P)$$

$$\text{r. m. s. } \delta T(P) = \pm \frac{R}{2} \sum_{n=2}^N Q_n \left[\frac{R}{r} \right]^{n+1} v_n (\Delta g)^{\frac{1}{2}}$$

In any case the choice of a high reference field will cut down the influence of $\underline{A}_0' \underline{\Delta g}_{p_0}$, for the remaining frequencies are more of local nature. For the solution of the discrete problem (5.14) as well in the nonsingular as in the singular case, Bjerhammar (1973) gave some equations depending on the desired optimality criterion. The solutions are identical and equal to the prediction solution, equ. (5.7) for Δg and T assumed continuous on the spheres τ and σ .

Bjerhammar's method for $\underline{\Delta g}_{p_1}' \underline{\Delta g}_{p_1} = \min.$ and \underline{A}_0' with dimensions $n \times m$, $n < m$ (underdetermined) leads to:

$$\begin{aligned} (5.20) \quad \underline{\Delta g}_{p_1} &= (\underline{A}_1')^{-1}_{10} \underline{T}_{p_1} - (\underline{A}_1')^{-1}_{10} \underline{A}_0' \underline{\Delta g}_{p_0} \\ &= (\underline{A}_1')' [(\underline{A}_1') (\underline{A}_1')']^{-1} \underline{T}_{p_1} - (\underline{A}_1')' [(\underline{A}_1') (\underline{A}_1')']^{-1} \underline{A}_0' \underline{\Delta g}_{p_0} \end{aligned}$$

where usually the second term $(\underline{A}_1')' [(\underline{A}_1') (\underline{A}_1')']^{-1} \underline{A}_0' \underline{\Delta g}_{p_0}$ is neglected. The inverse $(\underline{A}_1')^{-1}_{10}$ is the result of an iteration or approximation process, compare again Schwarz, (1971).

Summarizing the problems for this type of solution we are faced with

- (1) A frequency limitation expressed by a certain block size which is caused by the limited resolution in the observation equipment and by the spacing between successive observations.

- (2) The error influence of the neglected outer zone σ_0 on the boundary sphere in the least squares solution.

- (3) The optimal choice of the inverse matrix $(\underline{A}_1')^{-1}_{10}$.

Now we try another solution for the downward continuation based on the inverse formula (5.4). We assume that there exists a downward continuation operator B which allows a continuation of the type

$$\Delta g(P') = \int_{\tau} B(P', P) T(P) d\tau \text{ with } P \in \tau$$

In the global continuous limit such an operator exists in form of the inverse upward continuation operator developed into spherical harmonics, equ. (5.4). The discrete form of this operator shall build up the matrix \underline{B} for the inverse problem

$$(5.21) \quad \Delta g_{p'} = \underline{B}_{p'p} \underline{T}_p$$

with

$$b_{p'p} = \sum_{n=2}^N \frac{n-1}{R} \left(\frac{r}{R} \right)^{n+1} (2n+1) P_n(\cos \psi_{p'p})$$

for a certain element of \underline{B} .

Equation (5.21) looks similar to the prediction solution, but in contrast to the prediction we try to approximate the operator directly from the continuous formula and not from the properties of the data expressed in covariances. This solution seems surprisingly simple, i.e. no matrix inversion seems necessary which with the same assumptions as before should hardly be true. Data T_p are only given in τ_1 , and we get

$$(5.22) \quad \Delta g_{p'} = \underline{B}_1^1 \underline{T}_1 + \underline{B}_0^1 \underline{T}_0$$

where now \underline{B}_0^1 maps from the domain τ_0 into the domain σ_1 . In the solution we try to take into account the second term of equ. (5.22) $\underline{B}_0^1 \underline{T}_0$. The elements of \underline{T}_0 for the zone τ_0 may be expressed in terms of gravity anomalies on σ by

$$(5.23) \quad \underline{T}_0 = \underline{A}_1^0 \Delta g_{p'} + \underline{A}_0^0 \Delta g_p$$

Inserted into equ. (5.22) it will lead to

$$(5.24) \quad \Delta g_{p'} (\underline{I} - \underline{B}_0^1 \underline{A}_1^0) = \underline{B}_1^1 \underline{T}_1 + \underline{B}_0^1 \underline{A}_0^0 \Delta g_p$$

and we have to solve again a difficult problem as before for $\Delta g_{p'}$, as long as \underline{B}_0^1 and \underline{A}_1^0 are not orthogonal - which they never will become due to integral equation (5.1), and as long as \underline{B}_0^1 or \underline{A}_1^0 are not zero matrices - which is only true for the continuous and global limit where equ. (5.24) converges to equ. (5.14).

By comparing these two methods expressed by equ. (5.20) and equ. (5.24) it becomes obvious that in both cases only the limitation of σ and not that of τ has to be kept in mind. The limitation of T_p to τ_1 causes a high smoothing of the solutions for the condition $\underline{\Delta g}_{p'} \underline{\Delta g}_{p'} = \min$.

In order to make the problem in the prediction and in the least squares adjustment analysis more local a N_{low} degree reference field can be introduced. For the deterministic solution it reduces the influence of $\Delta T_p = A_0^1 \Delta g_p$. In the least square prediction the residual covariances will decrease more rapidly to the zero line and then oscillate with $(N_{low} + 1)$ zeroes, compare Meissl (1971), which diminishes the importance of the not available data of the outer zone. Because of the nonglobal distribution of all, up to now, derived satellite data and terrestrial measurements, the sets of spherical harmonic coefficients estimated from the limited information are disturbed by aliasing and dependence of the theoretically independent coefficients. Noise to signal ratios are rapidly increasing to 1:1 almost for low degree potential coefficients. The small confidence for the low degree field and the very systematic effect of aliasing caused by frequencies higher the Nyquist frequency, will result in a bias in the desired gravity anomalies based on this reference field. It is therefore necessary to accompany the local experiments by a global analysis to achieve a considerable improvement of the low degree and order harmonic coefficients. The simulation for the GRAVSAT/GEOPAUSE mission done by Koch-Argentiero (1974) indicates a realistic possibility for the low degree improvement. For the same experiment a local derivation of gravity anomalies may also be carried out by satellite to satellite tracking.

The downward continuation problem was considered in this chapter in the form of equ. (5.1) i.e., as a computation of gravity anomalies on the surface σ from the disturbing potential in satellite altitude, though for the discussed satellite techniques- satellite to satellite tracking and gradiometry- the gravity anomalies shall be derived from the gradient ∇T of the disturbing potential or its tensor components. But this fact is not essential for the topic as a downward continuation problem, considering the circumstance that the case formulated with equ. (5.1) is the worst possible. As shown in Table 2 and Chapter 4, the high frequency content of the quantities in consideration i.e. ∇T and gravity tensor, is more significant than in the disturbing potential itself. The problem will therefore be more local. The operators connecting ∇T and the gravity tensor with Δg will not amplify uncertainties in higher frequencies to the same amount as the operator connecting T and Δg . The specific formulas necessary for the least square adjustment given the components of ∇T are explained in detail by Hajela (1974, p. 25); those for the Δg computation from the tensor components of the disturbing potential by Reed (1973 p. 68).

The application of the statistical prediction method for the analysis of satellite to satellite tracking should be divided into two steps. First, one may deduce the components of ∇T from satellite observations by least squares adjustment. Then from the gradient the gravity anomalies are computed by prediction. The partitioned analysis avoids the difficulties connected with the computation of auto- and cross-covariances for the sampled observations necessary in a direct one step prediction solution.

In addition, it is possible to estimate in the first step the ∇T - values only for a grid. This would reduce the matrices involved in the prediction step to a convenient size but on the other hand reduce the resolution.

To predict gravity anomalies from ∇T means estimation of a scalar quantity from a vector process. Thereby we have to keep carefully in mind that orthogonal vector components for two distinct points P and Q are correlated even for a stationary and isotropic process. The covariance functions of a vector process becomes a tensor.

$$C_{ij}(\nabla T(P), \nabla T(Q)) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

It may be expressed in the Kármán-Taylor decomposition, (Grafarend (1972)), as a function of the longitudinal and transversal covariances. The derivation of the tensor components for a vector process is explained by Moritz (1973, p. 60). The prediction formula (5.15) is modified for the case of satellite to satellite tracking to

$$(5.25) \quad \Delta g_P' = C_{\Delta g_P' \nabla T_Q} C_{\nabla T_Q}^{-1} \nabla T_P$$

For the gradiometer experiment the prediction equations for the tensor components of the disturbing potential are derived in Moritz, (1970 p. 34) and need no further explanation.

It should be mentioned that the prediction formulas are well suited for combination solutions with terrestrial and satellite observations of different type. For example the combination of satellite to satellite tracking data with the gradiometer data to take advantage of the specific wavelength properties of both methods does not have fundamental difficulties.

6. Proper Choice of the Inner Radius R

In all considerations up to this point, reduction of the gravity data to a sphere σ with radius R was assumed without specifying the semi-diameter in detail. Because the potential is regular and harmonic only outside the attracting masses also the continuation integral (3.1) is valid only in the outer space.

For the spherical formulation of the deterministic downward continuation problem a minimum semi-diameter for the boundary sphere fulfilling this condition would be $R = 6384.403$ km due to Snowden (1968), (Rapp (1969)). The sphere would enclose all masses and touch the earth at a latitude of $\varphi = -1^\circ 28'$ in an elevation of 6272m. The advantage of reducing all gravity

quantities, i. e. satellite and terrestrial data to such an outer sphere- called geosphere by Bjerhammar (1967)- is mentioned by Eeg and Krarup (1973). They also explain the essential drawback: In a second step the derived potential has to be reduced from the geosphere to the actual surface of the earth where we finally need the information. Therefore we are again faced with all theoretical disadvantages of downward continuation as an "improperly posed problem". In addition, the reduction has to be carried out to a complicated surface ω " a closed smooth surface in Ω surrounding and arbitrarily near to the boundary of Ω " (Krarup, 1969) ($\Omega \dots$ outer space of the earth).

It complicates the solution essentially. More generalized one may interpret the problem as a free boundary value problem with additional physical and geometrical restrictions necessary for its solution, Grafarend (1972). But for its practical realization there is by far no ray of hope.

The deviation δR for a point P_e on the surface of the earth with geocentric radius vector $\underline{\rho}$ to the geosphere with radius R - compare Figure 5 - reaches from zero to about 28 km.

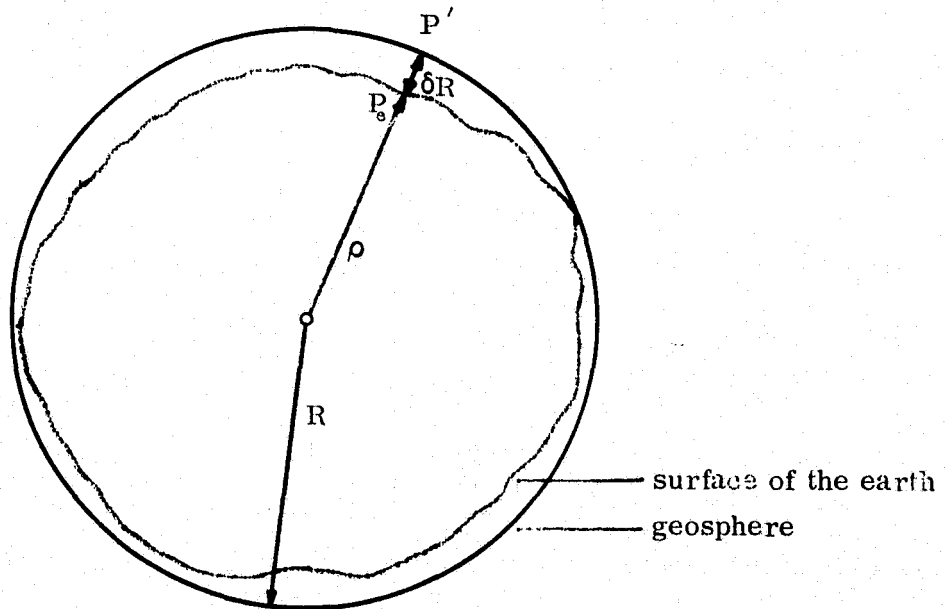


Figure 5: Geosphere and Surface of the Earth

In satellite geodesy up to now only global solutions for a low degree field are computed. The introduced radius R is usually the

equatorial radius $a_e = 6378.16 \text{ km}$

-compare for example Kaula (1969, equ. (5) and equ. (7)). Calculating with the derived coefficients gravity quantities by using formulas like equ. (2.1) or equ. (2.5) is nothing but a downward continuation of the gravity information to a sphere with radius a_e . The drawbacks of this common practice are:

1. A sphere with semidiameter a_e does not enclose all masses. Therefore, for large areas the deduced quantities lose their physical meaning.

2. The deviation δR of the earth's surface point P_e to the point P' on the geosphere should be considered at least theoretically. The amount of δR for $R = a_e$ varies from -6 km to $+22 \text{ km}$. For low harmonics the downward continuation effect due to δR is not very high, as may be seen from Table 5a and 5b. The deviation in r. m. s. $(\Delta g)_{n_1 n_2}$ caused by neglecting δR is for the gravity field up to $n_2 = 20$ about 1.5% and reaches 4% for some extreme areas. For the current accuracy level this influence may be neglected. For small block sizes the same deviation grows up to 10% and should no longer be neglected because of its strong systematic dependence on latitude and elevation.

altitude \ mgal ²	$\text{var}(\Delta g)_{3,20}$	$\text{var}(\Delta g)_{3,36}$ $\doteq 5^\circ \times 5^\circ$	$\text{var}(\Delta g)_{3,72}$ $\doteq 2.5^\circ \times 2.5^\circ$	$\text{var}(\Delta g)_{3,180}$ $\doteq 1^\circ \times 1^\circ$
$R - R_{eje}$ 1 km	235.7	364.9	559.4	864.6
$R - R_{eje}$ 10 km	227.8	346.5	513.1	730.5
$R - R_{eje}$ 22 km	217.8	323.8	459.5	599.5

Table 5a: Gravity Anomaly Variance Between Degree n_1 and n_2 For Low Altitudes

$$\text{var}(\Delta g)_{n_1 n_2} = \sum_{n=n_1}^{n_2} \left[\frac{R_{eje}}{R} \right]^{2n+4} \frac{A}{n+B} \text{ mgal}^2;$$

$$A = 425.28 \text{ mgal}^2, \quad B = 24$$

n	3-20	3-36	3-72	3-180
1 km to 10 km %	1.4	2.5	4.2	6.8
1 km to 22 km %	3.8	5.7	9.3	10.6

Table 5b: Change of $|\text{r.m.s } (\Delta g)_{n_1, n_2}|$ in percent from altitude 1 km to altitude 10 or 22 km.

This error source may be considerably reduced by introducing a spheroidal boundary surface instead of the geosphere as proposed by Hotine (1967). The computations for ellipsoidal harmonics are by far not as convenient but an ellipsoidal boundary surface will become necessary in the near future for several applications as for example high accurate geoid computation, Lelgemann (1970). The transformation formulas from ellipsoidal to spherical harmonics are shown in Hotine, (1967, equ. (12) and (13)).

A second and more promising approach is the computation of "gravity anomalies" at a boundary sphere completely embedded in the earth's surface with its origin in the earth's center—the so called Bjerhammar sphere. A possible choice for the Bjerhammar sphere would be

$$R_{Bje} = 6356.7 \text{ km. (which is somewhat smaller than the semiminor axis of the earth)}$$

The application of this method is today generally recommended, Hotine (1969), Krarup (1969), Moritz (1970). Although the physical meaning of the reduced values is lost it is in best agreement with the practical restriction of an always finite number of measurements. In a second step the deduced 'gravity anomalies' have to be upward continued and can be compared with the terrestrial gravity information.

By applying the least squares prediction approach the question of a proper choice of R seems not to be very troublesome. The geocentric radius ρ to a surface point P₀ may be inserted into the necessary covariance $C_{\Delta g_p r_0}$ expressed by equ. (5.9a) to lead with equ. (5.7) immediately to the desired boundary value. It is a very convenient procedure, but it does not ensure convergence as long as the convergence of the applied covariance function on the surface of the earth is not proved. When in the covariance expressions the empirical degree variances are replaced by practical applicable analytic expression a fixed radius has to be found. A detailed analysis for a proper formula was done by Tscherning and Rapp (1974). The introduced expression is

$$(6.1) \quad \text{var}(\Delta g_{P'}) = \sum_{n=3}^{\infty} s^{n+2} \frac{A(n-1)}{(n-2)(n+B)}$$

with the unknown parameters s , A and B where

$$s = \frac{R^2}{R_{P'}^2}$$

R ... radius of an inner reference sphere (Bjerhammar sphere)

$R_{P'}$... central distance for a surface point P' , for the analysis to be replaced by

R_e ... mean radius of the earth

The result of the least square fit is (Ibid, p. 22, Table 7)

$s = 0.999617$, $A = 425.28 \text{ mgal}^2$, $B = 24$ and gives $R = 6369.8 \text{ km}$.

A Bjerhammar sphere with this semi-diameter would not be embedded in the earth and makes another choice for R necessary for downward continuation applications. For our problem the fitting procedure shall lead to a regular and harmonic expression for the covariance function that converges on the Bjerhammar sphere with minimum variance and least squares deviation to the empirical data at the earth's surface. The theoretical formulation of such an adjustment for the potential was done by Krarup (1969, p. 54) based on the Runge's theorem. The analysis for equ. (6.1) based on the semidiameter $R_{Bje} = 6356.7 \text{ km}$ gives the results, using the same procedure and data as in Tscherning and Rapp (1974).

$s = 0.995516$, $A = 705.094 \text{ mgal}^2$, $B = 43$.

The covariance expression for downward continuation based on these values is

$$(6.2) \quad C_{\Delta g \Delta g}(P', Q) = \sum_{n=3}^{\infty} \left[\frac{R_{Bje}^2}{r_{P'} r_Q} \right]^{n+2} \frac{705.094(n-1)}{(n-2)(n+43)} P_n(\cos \psi_{P'Q}) \text{ mgal}^2$$

The drawback of the result lies in the fact that the variance of the point anomalies is too low ($\text{var}(\Delta g)_{3,500} = 1081.7 \text{ mgal}^2$) and will therefore lead to an additional smoothing in the least squares prediction solution. Especially for high elevations the covariance function decreases very rapidly with increasing degree. In addition for all expressions of the form of equ. (6.1) convergency on the Bjerhammar sphere cannot be ensured. Unrealistic amplifying of the high frequency content in the predicted quantity near or on the Bjerhammar sphere may be the result.

The quotient Q of the $(n+1)^{\text{th}}$ term a_{n+1} and of the n^{th} term a_n of $\text{var}(\Delta g')$, equ. (6.1), with $s=1$ leads to

$$Q = \frac{a_{n+1}}{a_n} = \frac{(n^2 - 2n)(n+B)}{(n^2 + 2n+1)(n+B+1)}$$

and has the limit

$$\lim_{n \rightarrow \infty} Q = 1,$$

which should be smaller one for convergency (d'Alembert criterion). Therefore expanded analysis with more complicated analytical expressions for $\text{var}(\Delta g)$ fulfilling the demanded properties would be very useful.

7. Conclusions

Any derivation of surface gravity information from satellite observations is a downward continuation procedure. If the downward continuation is carried out in a global solution by means of a spherical harmonic expansion of the desired gravity quantity to a sphere containing the masses of the earth (geosphere) no essential difficulty arise but the values related to the geosphere are because of their distance to the earth's surface or the geoid, of no use for practical applications.

Therefore, in practice the spherical harmonic coefficients are usually derived in a satellite method with respect to a sphere with radius a_e (a_e ... semi-major axis of the earth). A sphere with radius a_e intersects the earth's surface in some areas, and in most areas lies outside of it. Thus, the disturbing potential T derived from this set of spherical harmonic coefficients by

$$T(P') = \sum_{n=2}^N \sum_{m=0}^n (c_{nm}(T)\bar{R}_{nm}(P') + s_{nm}(T)\bar{S}_{nm}(P')), \quad P' \in \sigma$$

i. e. with factors $\left(\frac{a_e}{r}\right)^{n+1} = 1$, or $r = a_e$

has in a strict sense no physical meaning because the solution of the boundary value problem is valid only for functions harmonic and regular outside the boundary surface. The resulting problem of divergency for the spherical harmonic expansion at and inside the surface of the earth becomes not obvious in a low degree solution, but the physical meaning of gravity quantities calculated from this set of spherical harmonic coefficients is doubtful. These considerations

get practical relevancy in a combination of satellite derived gravity values with terrestrial gravity data, Rapp (1969).

Because of a strictly analytical continuation of gravity quantities to the actual surface and its interior is not possible, (Moritz (1961), Bjerhammar (1969)), the purpose of physical geodesy may be reformulated to the "determination of a gravity field that is compatible with the given discrete observations", (Heiskanen and Moritz, p. 321). This formulation is in agreement with the real situation where always only a finite number of observations is given.

With this more realistic statement we may look at the local derivation of gravity anomalies from satellite to satellite tracking or satellite gradiometry rather optimistically. For this approach we are theoretically faced not only with the question of a proper choice of the boundary surface but also with the restriction of the boundary surface to a finite region where we try to derive gravity information. Results of simulation studies of Reed (1973) for satellite gradiometry and Hajela (1974) for SST seem to be promising. But as any simulation they suffer from some simplifications which shall be kept in mind for planned practical experiments:

- a) For simulation studies the input gravity anomalies are thought to be referred to a sphere with earth's mean radius 6371 km. The gravity anomalies resulting from the simulation are again related to the same sphere and can easily be compared with the input data. In a practical experiment the derived gravity quantities related to the boundary sphere have to be transformed to be comparable with terrestrial gravity material.
- b) The simulation data are generated by mean gravity anomalies of a certain block size. White noise is added to simulate observation errors. In reality the satellite is affected in its motion by the whole gravity spectrum. The observations are done with a certain amount of normal distributed white noise and with some systematic errors. The error level defines the limit of the resolution in the frequency range. But this limit is very flexible. Therefore in a certain frequency range the noise will be somewhat biased and cannot be filtered out by least squares methods. In practical experiments, this bias will affect the resolved spectral content of the derived mean gravity anomalies.
- c) The downward continuation problem expressed by integral equation (5.1) is in any way underdetermined, because we try to derive a continuous function on the boundary surface from a finite number of observations.

For practical applications we approximate the integral equation by a system of linear equations with m unknowns for n observations. The three different cases $m < n$, $m = n$, and $m > n$ express thereby the degree of approximation of the integral equation by the set of linear equations. For a practical situation a solution with a certain m and n , and $m < n$ may be optimal in terms of optimal mean anomaly recovery. From the theoretical point of view, with equ. (5.1) in mind, it does not express an overdetermined least squares adjustment problem.

d) The local solution is in contrast to integral equation (5.1) or its finite approximation by linear equations which assume global recovery of anomalies. In the simulation studies the satellite "observations" are generated from gravity anomalies of a finite region at the surface of the earth whereas in reality a satellite is influenced in its motion by the global gravity field of the earth. Gravity anomalies are then derived in a somewhat smaller region from the simulated observations. Therefore a numerical analysis of the influence of the outer zone on the derived gravity anomalies would be of great use.

Based on these considerations we propose the following solution for the local recovery of gravity anomalies which is in best possible agreement with theoretical demands and practical restrictions. We may choose between two different approaches, one is the least squares adjustment or inversion procedure based on the approximation of the integral equations for the ∇T - components and the disturbing potential tensor-components; the other is the least squares prediction method. For both types two steps are necessary in their solution.

In least squares adjustment we first derive "gravity anomalies" with no real physical meaning at a Bjerhammar sphere totally embedded in the surface of the earth by downward continuation by such procedures as described by Reed (1973) or Hajela (1974). Then by upward continuation these quantities are transformed to the surface of the earth, where the resulting anomalies will be compared with terrestrial gravity anomalies. The second step, the upward continuation procedure is not connected with principal problems.

In the least squares prediction solution we derive first from SST-observations or gradiometer observations gravity quantities in satellite altitude - the components of ∇T^r in SST and the independent tensor components $T_{x_{ij}}^r$, $i = 1, 2, 3$ and $j = 1, 2, 3$ in gradiometry. This intermediate step is necessary because we have no information about the covariance functions for the observations and about the crosscovariances connecting the observations with the gravity quantities. In a second step we derive with downward continuation by least squares prediction immediately point- or mean anomalies at the surface of the earth, (Moritz (1970), (1973)), that may be compared with terrestrial anomalies. Practical problems related with both procedures are described in Chapter 5.

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