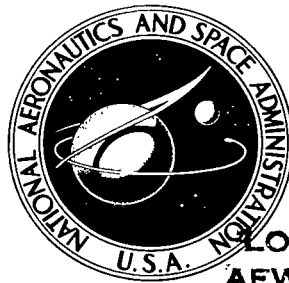


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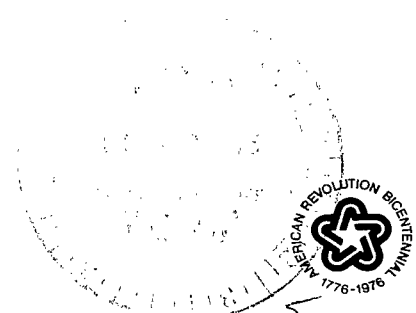
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A VECTOR-DYADIC DEVELOPMENT  
OF THE EQUATIONS OF MOTION  
FOR N-COUPLED FLEXIBLE BODIES  
AND POINT MASSES

*Harold P. Frisch*

*Goddard Space Flight Center*

*Greenbelt, Md. 20771*





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16. Abstract The equations of motion for a system of coupled flexible bodies, rigid bodies, point masses, and symmetric wheels are derived in vector-dyadic format. The equations are cast into a partitioned matrix form in which certain partitions become nontrivial when the effects of flexibility are treated. The equations contract to the coupled rigid body equations or expand to the coupled flexible body equations all within the same basic framework. Furthermore, the coefficient matrix always has the computationally desirable property of symmetry. Making use of the derived equations, a comparison is made between the equations which describe a flexible body model and those describing a rigid body model of the same elastic appendage attached to an arbitrary coupled body system. From the comparison, equivalence relations are developed which define how the two modeling approaches describe identical dynamic effects. These relations provide the analyst with a guide for determining which modeling technique is most efficient for a particular application and also for estimating the number of modes required for accurate flexible body simulation.					
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## CONTENTS

	<i>Page</i>
ABSTRACT . . . . .	i
SYMBOLS . . . . .	v
INTRODUCTION . . . . .	1
BASIC SYSTEM . . . . .	2
MOMENTUM EQUATIONS FOR A FLEXIBLE BODY . . . . .	8
EQUATIONS OF VIBRATION FOR A FLEXIBLE BODY . . . . .	9
NESTED BODY EQUATIONS OF MOTION . . . . .	20
PARTITIONED MATRIX FORM OF EQUATIONS OF MOTION . . . . .	30
ELIMINATION OF CONSTRAINT TORQUES . . . . .	34
PREDEFINED RELATIVE MOTION OF CONTIGUOUS BODIES . . . . .	37
FLEXIBLE VERSUS RIGID BODY MODELING . . . . .	39
INTERPRETATION OF RESULTS . . . . .	48
Problem Setup . . . . .	48
Computation Speed . . . . .	48
Interpretation of Numerical Solutions . . . . .	49
Choice of Modeling Technique . . . . .	49
REFERENCES . . . . .	52
APPENDIX—VECTOR-DYADIC IDENTITIES . . . . .	53



## SYMBOLS

$a_{n,\lambda}(t)$	Generalized displacement coordinate associated with the $n$ th mode of vibration of body $\lambda$
$\vec{A}_{n,\lambda}$	Mode-dependent parameter used to define body $\lambda$ center-of-mass (see equation 48)
$\vec{C}_\lambda$	Force associated with the mass of body $\lambda$ and its combined centripetal and Coriolis acceleration
$\vec{C}\vec{L}_m$	Control torque applied to symmetric wheel $m$
$E_{n,\lambda}$	Mode-dependent parameter used to define body $\lambda$ inertia tensor (see equation 52)
$F_{m,n,\lambda}$	Mode-dependent parameter (see equation 70)
$\vec{F}_{k-1}^c$	Resultant force of constraint acting on body $k$ through hinge point $k-1$
$\vec{F}_{k-1}^H$	Resultant force acting on body $k$ due to all mechanisms existing between bodies $J(k)$ and $k$ at hinge point $k-1$
$\vec{F}_k^{(e)}$	Resultant of all external forces applied to body $k$
$G_{k-1,i-1}^\lambda$	Pseudo-inertia tensor of body $\lambda$ with respect to the origin of nest $k-1$ and the hinge point $i-1$
$\vec{G}_{I,\lambda}$	Linear momentum of body $\lambda$ relative to the inertial origin
$\vec{G}_{I,i,\lambda}$	Linear momentum of element $i$ of body $\lambda$ relative to the inertial origin
$\vec{H}_m$	Inertial angular momentum of symmetric wheel $m$
$\vec{\approx}H_m$	Relative angular momentum of symmetric wheel $m$
$\vec{\circ}H_m$	Angular momentum of symmetric wheel $m$ relative to the undeformed state of body $MO(m)$
$H_{n,\lambda}$	Mode-dependent parameter (see equation 68)
$I_{W_m}$	Inertia tensor of symmetric wheel $m$ about its center-of-mass

$J(\lambda)$	Body label of the body to which body $\lambda$ is attached at hinge point $\lambda-1$
$\vec{K}_{m,n,\lambda}$	Mode-dependent parameter (see equation 72)
$\vec{L}_{I,\lambda}$	Inertial angular momentum of body $\lambda$ about the inertial origin
$\vec{L}_{i,i,\lambda}$	Inertial angular momentum of the element $i$ of body $\lambda$ about its own center-of-mass
$\vec{L}_{\lambda,\lambda}$	Inertial angular momentum of body $\lambda$ about its center-of-mass
$M$	Total number of symmetric wheels
$MO(m)$	Body label of the gyrostat in which symmetric wheel $m$ is embedded
$m_\lambda$	Total mass of body $\lambda$
$m_{i,\lambda}$	Mass of element $i$ of body $\lambda$
$N$	Total number of rigid bodies, flexible bodies, and point masses
$N_m$	Total number of flexible body modes (all flexible bodies)
$N_R$	Total number of rigid and flexible bodies
$\vec{q}_m$	Free coordinate vector $m$
$\vec{Q}_{n,\lambda}$	Mode-dependent parameter used to define relative angular momentum of body $\lambda$ due to deformation (see equation 56)
$\vec{r}_{i,\lambda}$	Position vector from hinge point $\lambda-1$ to the undeformed center-of-mass position of the element $i$ of body $\lambda$
$S_R$	Set of all rigid-body, body labels
$S_F$	Set of all flexible-body, body labels
$S_L$	Set of all point-mass body labels
$S_{k-1}$	Set of all body labels of those bodies outboard of hinge point $k-1$ relative to body 1; body labels of those bodies in nest $k-1$
$S_{k-1,\lambda-1}$	Set of all body labels associated with those bodies lying on the topological path from hinge point $k-1$ to the center-of-mass of body $\lambda$

$\mathcal{S}$	Tensor operator which maps vectors into skew symmetric tensors of rank 2, dyads
$\vec{\alpha}_\lambda$	Position vector from hinge point $\lambda-1$ of body $\lambda$ to the center-of-mass of body $\lambda$
$\vec{\alpha}_\lambda^0$	Position vector from hinge point $\lambda-1$ to the undeformed center-of-mass position of body $\lambda$
$\vec{\beta}_1$	Position vector from inertial origin to hinge point 0 of body 1; that is, to the center-of-mass of body 1
$\vec{\beta}_\lambda$	Position vector from hinge point $J(\lambda)-1$ to hinge point $\lambda-1$
$\vec{\beta}_\lambda^0$	Position vector from hinge point $J(\lambda)-1$ to the undeformed position of hinge point $\lambda-1$
$\vec{\gamma}_{1,\lambda}$	Position vector from inertial origin to center-of-mass of body $\lambda$
$\vec{\gamma}_{k,\lambda}$	Position vector from hinge point $k$ to the center-of-mass of body $\lambda$
$\Gamma_{k,\lambda}$	Skew symmetric tensor form of the vector $\vec{\gamma}_{k,\lambda}$
$\delta_{m,n}$	Kronecker delta function
$\vec{\delta}_{i,\lambda}$	Position vector from the deformed center-of-mass position of body $\lambda$ to the deformed center-of-mass position of the element $i$
$\vec{\Delta\alpha}_\lambda$	Position vector from the undeformed to the deformed center-of-mass position of body $\lambda$
$\vec{\Delta\beta}_\lambda$	Position vector from the undeformed to the deformed position of hinge point $\lambda-1$
$\Delta\Phi_\lambda$	Change in the inertia tensor of body $\lambda$ from the undeformed to the deformed state
$\overset{\circ}{\Delta\theta}_\lambda$	Angular rate of a reference frame fixed in body $J(\lambda)$ at hinge point $\lambda-1$ relative to a reference frame fixed at hinge point $J(\lambda)-1$ in body $J(\lambda)$
$\overset{\circ}{\Delta\mathbf{H}}_m$	Angular momentum of the symmetric wheel $m$ associated with the deformation-induced rotation of its spin axis relative to the undeformed state of body $MO(m)$



$\overset{\circ}{\Delta\vec{\theta}}_{W_m}$	Angular velocity of a reference frame fixed in the despun wheel $m$ relative to the undeformed state of body $MO(m)$
$\Delta\vec{L}_{\lambda,\lambda}$	Angular momentum of the flexible body $\lambda$ relative to its center-of-mass, due to the elastic deformation of all elements $i$ of body $\lambda$
$\vec{e}_{i,\lambda}$	Position vector from the undeformed to the deformed center-of-mass position of the element $i$ of body $\lambda$
$\zeta$	Damping ratio
$\vec{\theta}_{i,\lambda}$	Relative rotation vector of the element $i$ of body $\lambda$ from the undeformed to the deformed state
$\theta_m$	Displacement about or along free coordinate vector $\vec{q}_m$
$\Phi_\lambda$	Inertia tensor of body $\lambda$ about its center-of-mass
$\Phi_{i,i,\lambda}$	Inertia tensor of element $i$ of body $\lambda$ about its own center-of-mass
${}^\circ\Phi_{i,i,\lambda}$	Inertia tensor of element $i$ of body $\lambda$ about its own center-of-mass in the undeformed state of body $\lambda$
$\Phi_{\lambda,i,\lambda}$	Inertia tensor of element $i$ of body $\lambda$ about the deformed center-of-mass position of body $\lambda$
${}^\circ\Phi_\lambda$	Inertia tensor of body $\lambda$ in the undeformed state relative to the undeformed center-of-mass position
$\vec{\phi}_k^{(e)}$	Resultant torque acting on body $k$ due to external causes
$\vec{\phi}_{k-1}^c$	Resultant torques of constraint acting on body $k$ through hinge point $k-1$
$\vec{\phi}_{W_m}^c$	Resultant torque of constraint acting on symmetric wheel $m$
$\vec{\phi}_{k-1}^H$	Resultant torque acting on body $k$ due to all mechanisms existing between bodies $J(k)$ and $k$ at hinge point $k-1$
$\vec{\varphi}_{n,i,\lambda}^T$	The $n$ th normal mode displacement vector for the element $i$ of body $\lambda$
$\vec{\varphi}_{n,i,\lambda}^R$	The $n$ th normal mode rotation vector for the element $i$ of body $\lambda$

$\vec{\omega}_\lambda$	Angular velocity of the body $\lambda$ fixed coordinate frame relative to the inertially fixed coordinate frame
${}^\circ\vec{\omega}_\lambda$	Angular rate of a reference frame fixed in body $\lambda$ at hinge point $\lambda-1$ relative to a reference frame fixed in body $J(\lambda)$ at hinge point $\lambda-1$
$\vec{\omega}_\lambda$	Angular velocity of the body $\lambda$ fixed coordinate frame relative to the body $J(\lambda)$ fixed coordinate frame
$\vec{\omega}_{W_m}$	Inertial angular velocity of symmetric wheel $m$
${}^\circ\vec{\omega}_{W_m}$	Angular velocity of the symmetric wheel $m$ relative to the undeformed state of body $MO(m)$
$\vec{\omega}_{W_m}$	Relative angular velocity of symmetric wheel $m$
$\omega_{n,\lambda}$	Natural frequency of the $n$ th normal mode of vibration for body $\lambda$
$\{ \}$	Column matrix
$[ ]$	Square matrix
$\lfloor \rfloor$	Rectangular matrix
$\vec{R}$	Vector $\vec{R}$
$\{\vec{R}\}_\lambda$	$3 \times 1$ matrix of components of vector $\vec{R}$ relative to body $\lambda$ fixed coordinates
$\dot{\vec{R}}$	Time derivative of vector $\vec{R}$ relative to inertially fixed reference frame
$\overset{\circ}{\vec{R}}$	Time derivative of vector $\vec{R}$ relative to the local reference frame when no confusion exists as to which local reference frame the differentiation is with respect to
$1$	Unit dyad
$\sum_{i \in S_{k-1}}$	Summation over all indices $i$ contained in the set $S_{k-1}$

$$\sum_{i@k-1}$$

Summation over all indices  $i$  of vectors defined at hinge point  $k-1$

$$\sum_{i \in \lambda}$$

Sum over all elements of the body  $\lambda$

$$\sum_{\substack{m: \\ MO(m)=\lambda}}$$

Sum over all wheels imbedded in body  $\lambda$

$$\sum_{n,\lambda}$$

Sum over all modes to be used for the description of flexible body  $\lambda$  elastic deformation

$\approx$

Equivalence sign

# A VECTOR-DYADIC DEVELOPMENT OF THE EQUATIONS OF MOTION FOR N-COUPLED FLEXIBLE BODIES AND POINT MASSES

Harold P. Frisch  
*Goddard Space Flight Center*

## INTRODUCTION

The development of a general purpose simulation model to be used to study the attitude dynamics of a general class of multibody spacecraft having one or more flexible members presents a number of formidable problems. The model must be computationally efficient, yield an accurate representation of the spacecraft's attitude dynamics, and provide sufficient generality to be applicable to a broad class of spacecraft.

The burden of writing a computationally efficient program rests both with the analyst and the programmer. The analyst must write the equations in such a form that redundant computations and multiplications by zero can be readily avoided. The programmer must use computationally efficient programming techniques. If the programmer does not include the logic to avoid redundant and meaningless computation, and makes excessive unnecessary use of multidimensioned arrays, the speed of computation can easily be increased by more than an order of magnitude above what it should be.

The burden of defining a representative model of a complex spacecraft rests solely with the analyst. An overdefined model will yield accurate answers but will be useless from the standpoint of computational speed; an underdefined model will be highly efficient computationally but will yield erroneous results.

In reference 1, Velman's nested body approach has been used to derive the equations of motion of a topological tree of rigid bodies, point masses, and symmetric momentum wheels. These equations have been programmed and form the basis for the general purpose digital computer program, N-BOD. Too often, during the course of a particular study, the question has arisen as to the effects of body flexibility and its resultant perturbing effects on the response characteristics of the composite system. The quest for an answer to that question has motivated this study. It builds directly upon the nested body approach presented in reference 1. Excellent reviews of other methods of attacking the same general problem are provided by Likens in references 2 and 3.

The aim of this flexible body analysis is to derive a final set of coupled body equations which separate effects that are analogous to those found by the coupled rigid body analysis

from those effects which are unique to the coupled flexible body analysis. Achievement of this goal allows flexibility to be treated as simply an add-on effect.

The equations are first given as a set of simultaneous vector-dyadic differential equations and are then reduced, by a method which eliminates unknown forces and torques of constraint, to a set of simultaneous scalar differential equations. The equations are cast into a partitioned matrix format which has certain computationally desirable symmetry properties. The effects of body flexibility are introduced by adding a few simple terms onto the coupled rigid body equations of motion, along with a generalized displacement coordinate equation for each elastic mode of vibration. To describe a body's flexibility, clamped-free normal modes and frequencies of vibration are assumed obtainable for each flexible body. These may be obtained by closed-form continuum analysis, finite element analysis, or an educated guess. The mode shapes are not used per se within the final set of coupled body equations. They are used by a preprocessor to obtain resultant mode-dependent parameters for the stream of input data.

Frequently the analyst must decide whether it is better to model a flexible appendage as a spring-connected rigid body or as a flexible body having several significant natural modes of vibration. To make an intelligent decision, there must be some means of comparing the two approaches.

Because the equations derived are valid for either a rigid body or a flexible body model of any appendage, it is a routine process to step through the equations first with a rigid body model and then with a flexible body model of an arbitrary appendage. As both models simulate the same gyroscopic effects, equivalence relations can be established. These relations can then be reduced to a set of three fundamental equivalence relations, which provide the guide for determining the best modeling approach for any particular problem.

## BASIC SYSTEM

The basic system to be studied is identical to that defined in reference 1, with the exception that now one or more of the bodies may be flexible. The theoretical development and notation used here is a natural extension of the work presented in reference 1, and, therefore, the large volume of supportive material presented in that document will not be repeated.

Consider the pair of contiguous bodies shown in figure 1.

As in reference 1, the following definitions are made:

- Body 1 = principal body of the system of coupled bodies,
- Body  $J(\lambda)$  = body contiguous to and, relative to body 1, inboard of body  $\lambda$ ,
- Hinge point  $\lambda-1$  = point of connection between bodies  $J(\lambda)$  and  $\lambda$ ,
- $\vec{\alpha}_\lambda$  = position vector from hinge point  $\lambda-1$  to the center-of-mass of body  $\lambda$ ,

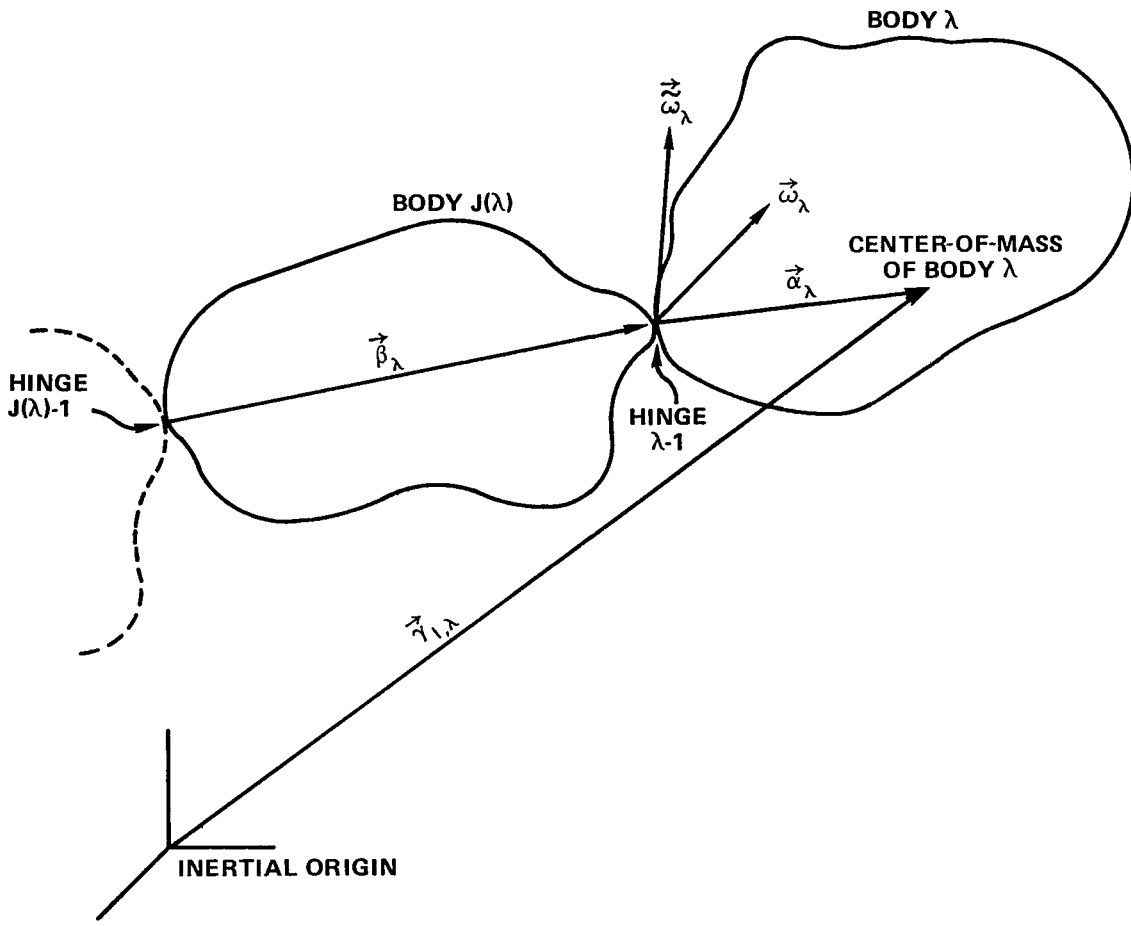


Figure 1. Notation for a pair of contiguous bodies.

- $\vec{\beta}_\lambda$  = position vector from hinge point  $J(\lambda)-1$  to hinge point  $\lambda-1$ ,
- $\Phi_\lambda$  = inertia tensor of body  $\lambda$  plus that of all imbedded despun symmetric wheels about the composite body center-of-mass,
- $m_\lambda$  = total mass of the body  $\lambda$ ,
- $\vec{\gamma}_{I,\lambda}$  = position vector from the inertial origin to the center-of-mass of body  $\lambda$ ,
- $\vec{\omega}_\lambda$  = angular velocity of a coordinate frame fixed at hinge point  $\lambda-1$  in body  $\lambda$  relative to a coordinate frame fixed at hinge point  $J(\lambda)-1$  in body  $J(\lambda)$ , and
- $\vec{\omega}_\lambda$  = angular velocity of a coordinate frame fixed at hinge point  $\lambda-1$  in body  $\lambda$  relative to a coordinate frame fixed inertially.

The present study permits any one or all of the coupled bodies to be flexible. To maintain the required generality, it is assumed that a finite element model of each flexible body of the system can be defined. The elements into which each flexible body is decomposed are assumed to have both mass and rotational inertia characteristics. Of course, as the number of elements used increases, the rotational inertia characteristics of the elements approach zero.

To establish the required notation, consider figure 2 for the arbitrary flexible body  $\lambda$ .

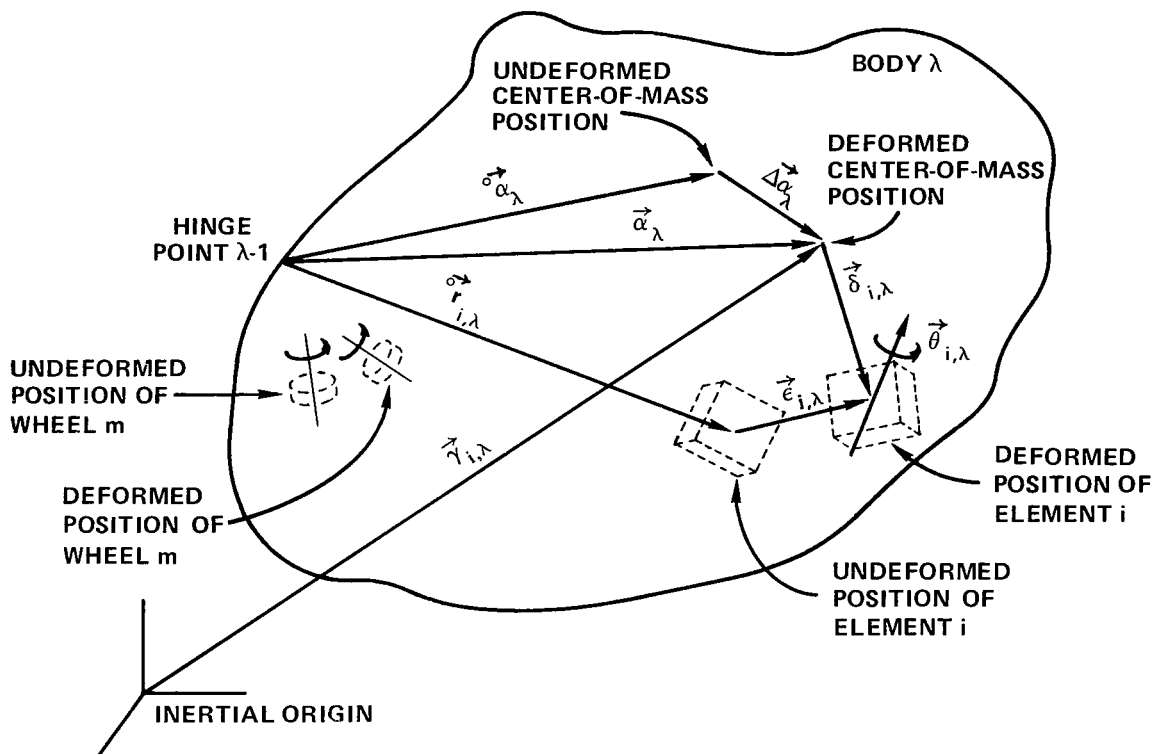


Figure 2. Notation for flexible body  $\lambda$ .

Let:

$\vec{\alpha}_\lambda$  = position vector from hinge point  $\lambda-1$  to the undeformed center-of-mass position of body  $\lambda$ ,

$\vec{\Delta\alpha}_\lambda$  = position vector from the undeformed to the deformed center-of-mass position of body  $\lambda$ ,

$\vec{r}_{i,\lambda}$  = position vector from hinge point  $\lambda-1$  to the undeformed center-of-mass position of element  $i$  of body  $\lambda$ ,

- $m_{i,\lambda}$  = mass of element  $i$  of body  $\lambda$ ,
- $\Phi_{i,i,\lambda}$  = inertia tensor of element  $i$  of body  $\lambda$  about its own center-of-mass,
- ${}^{\circ}\Phi_{i,i,\lambda}$  = inertia tensor of element  $i$  of body  $\lambda$  about its own center-of-mass in the undeformed state of body  $\lambda$ ,
- $\Phi_{\lambda,i,\lambda}$  = inertia tensor of the element  $i$  of body  $\lambda$  about the deformed center-of-mass position of body  $\lambda$ ,
- $\vec{e}_{i,\lambda}$  = position vector from the undeformed to the deformed center-of-mass position of the element  $i$  of body  $\lambda$ ,
- $\vec{\delta}_{i,\lambda}$  = position vector from the deformed center-of-mass position of body  $\lambda$  to the deformed center-of-mass position of element  $i$ ,
- $\vec{\theta}_{i,\lambda}$  = relative rotation vector of the element  $i$  of body  $\lambda$  from the undeformed to the deformed state,
- $MO(m)$  = body label of the body in which symmetric wheel  $m$  is imbedded,
- $\vec{H}_m$  = relative angular momentum of the symmetric wheel  $m$  imbedded in body  $MO(m)$ ,
- $I_{W_m}$  = inertia tensor of the symmetric wheel  $m$  about its own center-of-mass,
- $\vec{\beta}_{\lambda}$  = position vector from hinge point  $J(\lambda)-1$  to the undeformed position of hinge point  $\lambda-1$ ,
- $\Delta\vec{\beta}_{\lambda}$  = position vector from the undeformed to the deformed position of hinge point  $\lambda-1$ ,
- ${}^{\circ}\Phi_{\lambda}$  = inertia tensor of body  $\lambda$  in the undeformed state relative to the undeformed center-of-mass position,
- $\Delta\Phi_{\lambda}$  = change in the inertia tensor of body  $\lambda$  from the undeformed to the deformed state,
- ${}^{\circ}\vec{\omega}_{\lambda}$  = angular rate of a reference frame fixed in body  $\lambda$  at hinge point  $\lambda-1$  relative to a reference frame fixed in body  $J(\lambda)$  at hinge point  $\lambda-1$  (relative rate minus the effects of body  $J(\lambda)$  elastic deformation),
- $\Delta\vec{\theta}_{\lambda}$  = angular rate of a reference frame fixed in body  $J(\lambda)$  at hinge point  $\lambda-1$  relative to a reference frame fixed at hinge point  $J(\lambda)-1$  in body  $J(\lambda)$  (the effect of elastic deformation of body  $J(\lambda)$ ),



- $\overset{\circ}{\vec{H}}_m$  = angular momentum of the symmetric wheel m relative to the undeformed state of body MO(m),
- $\Delta\vec{H}_m$  = angular momentum of the symmetric wheel m associated with the deformation-induced rotation of its spin axis relative to the undeformed state of body MO(m),
- $\overset{\circ}{\vec{\omega}}_{W_m}$  = angular velocity of the symmetric wheel m relative to the undeformed state of body MO(m), and
- $\Delta\overset{\circ}{\vec{\theta}}_{W_m}$  = angular velocity of a reference frame fixed in the despun wheel m relative to the undeformed state of body MO(m) (the effect of elastic deformation of body MO(m)).

Making use of these definitions, summing over all elements i, and deleting terms which are quadratic in deformation, the following equations are obtained:

$$\vec{\alpha}_\lambda = \overset{\circ}{\vec{\alpha}}_\lambda + \Delta\vec{\alpha}_\lambda \quad (1)$$

$$\vec{\beta}_\lambda = \overset{\circ}{\vec{\beta}}_\lambda + \Delta\vec{\beta}_\lambda \quad (2)$$

$$\Phi_\lambda = \overset{\circ}{\Phi}_\lambda + \Delta\Phi_\lambda \quad (3)$$

$$\vec{\omega}_\lambda = \overset{\circ}{\vec{\omega}}_\lambda + \Delta\overset{\circ}{\vec{\theta}}_\lambda \quad (4)$$

$$\vec{H}_m = \overset{\circ}{\vec{H}}_m + \Delta\vec{H}_m \quad (5)$$

$$\vec{\omega}_{W_m} = \overset{\circ}{\vec{\omega}}_{W_m} + \Delta\overset{\circ}{\vec{\theta}}_{W_m} \quad (6)$$

$$m_\lambda = \sum_{i \in \lambda} m_{i,\lambda} \quad (7)$$

$$\overset{\circ}{\vec{\alpha}}_\lambda = \frac{1}{m_\lambda} \sum_{i \in \lambda} m_{i,\lambda} \overset{\circ}{\vec{r}}_{i,\lambda} \quad (8)$$

$$\Delta\vec{\alpha}_\lambda = \frac{1}{m_\lambda} \sum_{i \in \lambda} m_{i,\lambda} \vec{e}_{i,\lambda} \quad (9)$$

$$\vec{\delta}_{i,\lambda} = \vec{\sigma}_{\Gamma_{i,\lambda}} - \vec{\sigma}_{\alpha_\lambda} + \vec{\epsilon}_{i,\lambda} - \Delta\vec{\alpha}_\lambda \quad (10)$$

$$0 = \sum_{i \in \lambda} m_{i,\lambda} \vec{\delta}_{i,\lambda} \quad (11)$$

$$\begin{aligned} \Phi_{i,i,\lambda} &= {}^\circ\Phi_{i,i,\lambda} + \vec{\theta}_{i,\lambda} \times \left( {}^\circ\Phi_{i,i,\lambda} \times \vec{\theta}_{i,\lambda} \right) \\ &= {}^\circ\Phi_{i,i,\lambda} + 0(\epsilon^2) \end{aligned} \quad (12)$$

$$\begin{aligned} \Phi_{\lambda,i,\lambda} &= \Phi_{i,i,\lambda} + m_{i,\lambda} \left[ \vec{\delta}_{i,\lambda} \cdot \vec{\delta}_{i,\lambda} \mathbf{1} - \vec{\delta}_{i,\lambda} \vec{\delta}_{i,\lambda} \right] \\ &= {}^\circ\Phi_{\lambda,i,\lambda} + \Delta\Phi_{\lambda,i,\lambda} + 0(\epsilon^2) \end{aligned} \quad (13)$$

$$\Phi_\lambda = \sum_{i \in \lambda} \Phi_{\lambda,i,\lambda} \quad (14)$$

$${}^\circ\Phi_{\lambda,i,\lambda} = {}^\circ\Phi_{i,i,\lambda} + m_{i,\lambda} \left[ \left( \vec{\sigma}_{\Gamma_{i,\lambda}} - \vec{\sigma}_{\alpha_\lambda} \right) \cdot \left( \vec{\sigma}_{\Gamma_{i,\lambda}} - \vec{\sigma}_{\alpha_\lambda} \right) \mathbf{1} - \left( \vec{\sigma}_{\Gamma_{i,\lambda}} - \vec{\sigma}_{\alpha_\lambda} \right) \left( \vec{\sigma}_{\Gamma_{i,\lambda}} - \vec{\sigma}_{\alpha_\lambda} \right) \right] \quad (15)$$

$$\begin{aligned} \Delta\Phi_{\lambda,i,\lambda} &= m_{i,\lambda} \left[ \left( \vec{\sigma}_{\Gamma_{i,\lambda}} - \vec{\sigma}_{\alpha_\lambda} \right) \cdot \vec{\epsilon}_{i,\lambda} \mathbf{1} - \left( \vec{\sigma}_{\Gamma_{i,\lambda}} - \vec{\sigma}_{\alpha_\lambda} \right) \vec{\epsilon}_{i,\lambda} \right] \\ &\quad + m_{i,\lambda} \left[ \vec{\epsilon}_{i,\lambda} \cdot \left( \vec{\sigma}_{\Gamma_{i,\lambda}} - \vec{\sigma}_{\alpha_\lambda} \right) \mathbf{1} - \vec{\epsilon}_{i,\lambda} \left( \vec{\sigma}_{\Gamma_{i,\lambda}} - \vec{\sigma}_{\alpha_\lambda} \right) \right] \end{aligned} \quad (16)$$

$${}^\circ\Phi_\lambda = \sum_{i \in \lambda} {}^\circ\Phi_{\lambda,i,\lambda} \quad (17)$$

$$\Delta\Phi_\lambda = \sum_{i \in \lambda} \Delta\Phi_{\lambda,i,\lambda} \quad (18)$$

$$\vec{\Delta}\vec{H}_m = \mathbf{I}_{W_m} \cdot \vec{\Delta}\vec{\theta}_{W_m} \quad (19)$$

where

$$\sum_{i \in \lambda} = \text{sum over all elements } i \text{ of body } \lambda,$$

$$\mathbf{1} = \text{unit dyad},$$

and

$$0(\epsilon^2) = \text{terms quadratic in deformation.}$$

## MOMENTUM EQUATIONS FOR A FLEXIBLE BODY

The basic definitions for linear and angular momentum found in any textbook on rigid body dynamics may be applied to obtain the following relations:

- $\vec{G}_{I,\lambda}$  = linear momentum of flexible body  $\lambda$  relative to the inertial origin

$$\vec{G}_{I,\lambda} = \sum_{i \in \lambda} m_{i,\lambda} \left( \dot{\vec{\gamma}}_{I,\lambda} + \dot{\vec{\delta}}_{i,\lambda} \right) = m_\lambda \dot{\vec{\gamma}}_{I,\lambda}. \quad (20)$$

- $\vec{L}_{I,\lambda}$  = angular momentum of flexible body  $\lambda$  relative to the inertial origin

$$\vec{L}_{I,\lambda} = \sum_{i \in \lambda} \left[ \Phi_{i,i,\lambda} \cdot \dot{\vec{\theta}}_{i,\lambda} + m_{i,\lambda} \left( \vec{\gamma}_{I,\lambda} + \vec{\delta}_{i,\lambda} \right) \times \left( \dot{\vec{\gamma}}_{I,\lambda} + \dot{\vec{\delta}}_{i,\lambda} \right) \right] + \sum_{\substack{m: \\ \text{MO}(m)=\lambda}} \vec{H}_m \quad (21)$$

where

$\sum_{\substack{m: \\ \text{MO}(m)=\lambda}}$  = sum over all wheels imbedded in body  $\lambda$ ,

$$\dot{\vec{\theta}}_{i,\lambda} = \vec{\omega}_\lambda + \overset{\circ}{\vec{\theta}}_{i,\lambda}, \quad (22)$$

$$\dot{\vec{\delta}}_{i,\lambda} = \vec{\omega}_\lambda \times \vec{\delta}_{i,\lambda} + \overset{\circ}{\vec{\delta}}_{i,\lambda}, \quad (23)$$

and the closed dot implies an inertial derivative while the open dot implies differentiation relative to the local body  $\lambda$  fixed reference frame.

Direct substitution and application of previously made definitions yields:

$$\vec{L}_{I,\lambda} = \Phi_\lambda \cdot \vec{\omega}_\lambda + m_\lambda \vec{\gamma}_{I,\lambda} \times \dot{\vec{\gamma}}_{I,\lambda} + \Delta \vec{L}_{\lambda,\lambda} + \sum_{\substack{m: \\ \text{MO}(m)=\lambda}} \vec{H}_m \quad (24)$$

where

$\Delta \vec{L}_{\lambda,\lambda}$  = angular momentum of the flexible body  $\lambda$  relative to its deformed center-of-mass position due to the elastic deformation of all elements  $i$  of body  $\lambda$

$$\Delta \vec{L}_{\lambda,\lambda} = \sum_{i \in \lambda} \left[ \Phi_{i,i,\lambda} \cdot \overset{\circ}{\vec{\theta}}_{i,\lambda} + m_{i,\lambda} \vec{\delta}_{i,\lambda} \times \overset{\circ}{\vec{\delta}}_{i,\lambda} \right]. \quad (25)$$

- $\vec{H}_m$  = angular momentum of symmetric wheel m, relative to the inertial origin

$$\vec{H}_m = \mathbf{I}_{W_m} \cdot \vec{\omega}_\lambda + \vec{H}_m \quad (26)$$

Straightforward differentiation yields the rates of change of momentum relative to the inertial reference:

$$\dot{\vec{G}}_{I,\lambda} = m_\lambda \ddot{\vec{\gamma}}_{I,\lambda}, \quad (27)$$

$$\begin{aligned} \dot{\vec{L}}_{I,\lambda} &= \Phi_\lambda \cdot \dot{\vec{\omega}}_\lambda + m_\lambda \vec{\gamma}_{I,\lambda} \times \ddot{\vec{\gamma}}_{I,\lambda} \\ &\quad + \Delta \vec{L}_{\lambda,\lambda} + \sum_{\substack{m: \\ \text{MO}(m)=\lambda}} \left[ \overset{\circ}{\vec{H}}_m + \vec{\omega}_\lambda \times \overset{\circ}{\vec{H}}_m \right] \\ &\quad + \overset{\circ}{\Phi}_\lambda \cdot \vec{\omega}_\lambda + \vec{\omega}_\lambda \times \left( \Phi_\lambda \cdot \vec{\omega}_\lambda + \Delta \vec{L}_{\lambda,\lambda} \right) \end{aligned} \quad (28)$$

and

$$\dot{\vec{H}}_m = \mathbf{I}_{W_m} \cdot \dot{\vec{\omega}}_\lambda + \overset{\circ}{\vec{H}}_m + \vec{\omega}_\lambda \times \left( \mathbf{I}_{W_m} \cdot \vec{\omega}_\lambda + \overset{\circ}{\vec{H}}_m \right). \quad (29)$$

## EQUATIONS OF VIBRATION FOR A FLEXIBLE BODY

A set of equations which define the vibrational motion of an arbitrary flexible body relative to its undeformed state is required for the complete definition of all terms contained in the momentum equations (27, 28, and 29). It is not essential, but it is most convenient, to work toward a set of equations which define deformation in terms of the flexible body's normal modes of vibration.

At the hinge point  $\lambda-1$  of the flexible body  $\lambda$ , a body-fixed reference frame is defined. Body  $\lambda$  is assumed to be clamped in both translation and rotation at the origin of this body-fixed reference frame; all elastic deformation is measured relative to it. While the position and orientation of this body  $\lambda$  fixed reference is independent of deformation, it may have as many as three degrees of rotational freedom relative to its contiguous body, body  $J(\lambda)$ .

Each element  $i$  of the finite element model of the flexible body  $\lambda$  has, at most, six degrees of freedom: three translational and three rotational. In addition, the elements are each subject to viscoelastic loads and conditions of constraint.

Let

$\vec{G}_{I,i,\lambda}$  = linear momentum of element  $i$  of body  $\lambda$  relative to an inertial reference

$$\vec{G}_{I,i,\lambda} = m_{i,\lambda} \left( \dot{\vec{\gamma}}_{I,\lambda} + \dot{\vec{\delta}}_{i,\lambda} \right) \quad (30)$$

and

$\vec{L}_{i,i,\lambda}$  = angular momentum of the element  $i$  of body  $\lambda$  about its own center-of-mass relative to an inertially fixed reference frame (small angle deformation assumed)

$$\vec{L}_{i,i,\lambda} = {}^{\circ}\Phi_{i,i,\lambda} \cdot \left( \vec{\omega}_{\lambda} + \vec{\theta}_{i,\lambda}^{\circ} \right) . \quad (31)$$

The equations of motion for the elements of body  $\lambda$  are obtained by defining the holonomic conditions of constraint which limit the number of allowable degrees of freedom for each element and by equating the time rate of change of momentum for each element to its viscoelastic loading.

In order to separate relative-translational and relative-rotational deformation, partitioned matrix rotation is used to write

$$\begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} = \begin{Bmatrix} \vec{\epsilon}_{1,\lambda} \\ \vec{\epsilon}_{2,\lambda} \\ \cdot \\ \cdot \\ \cdot \\ \vec{\theta}_{1,\lambda} \\ \vec{\theta}_{2,\lambda} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \quad (32)$$

The holonomic conditions of constraint can, in general, be written in the form

$$\mathbf{[g]} \begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} = \{0\} \quad (33)$$

where the number of rows of the rectangular matrix  $\mathbf{[g]}$  is equal to the number of constraint conditions and hence the number of degrees of freedom to be suppressed.

The inertial time rate of change of momentum can, in general, be written in the form

$$\begin{Bmatrix} \dot{\vec{G}} \\ \dot{\vec{L}} \end{Bmatrix} = \frac{d^I}{dt} \begin{bmatrix} m\mathbf{1} & \\ & \Phi \end{bmatrix} \begin{Bmatrix} \dot{\vec{\gamma}} + \dot{\vec{\delta}} \\ \dot{\vec{\omega}} + \dot{\vec{\theta}} \end{Bmatrix} \quad (34)$$

where row  $i$  of the upper partition is  $\dot{\vec{G}}_{I,i,\lambda}$ , row  $i$  of the lower partition is  $\dot{\vec{L}}_{I,i,\lambda}$ , and  $d^I/dt$  is the time derivative relative to the inertial reference.

The viscoelastic conditions existing between contiguous elements can be expressed in terms of resultant viscoelastic loads. In general, this will take the form

$$- \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} \dot{\vec{\epsilon}} \\ \dot{\vec{\theta}} \end{Bmatrix} \quad (35)$$

where row  $i$  of the upper partition defines the resultant viscoelastic force acting on the element  $i$ , and row  $i$  of the lower partition defines the resultant viscoelastic torque acting on the element  $i$ .

By a direct application of the preceding, the general form for the equation of relative elastic deformation is found to be as follows:

$$\frac{d^I}{dt} \begin{bmatrix} m1 & 0 \\ 0 & \Phi \end{bmatrix} \begin{Bmatrix} \dot{\vec{\gamma}} + \dot{\vec{\delta}} \\ \vec{\omega} + \dot{\vec{\theta}} \end{Bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} \overset{\circ}{\vec{\epsilon}} \\ \overset{\circ}{\vec{\theta}} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (36)$$

subject to the constraint equation

$$[g] \begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \quad (33)$$

In order to put the deformation equation into a form which is amenable to an eigenvalue solution, it may be rewritten as

$$\begin{bmatrix} m1 & 0 \\ 0 & \Phi \end{bmatrix} \begin{Bmatrix} \overset{\circ\circ}{\vec{\epsilon}} \\ \overset{\circ\circ}{\vec{\theta}} \end{Bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} \overset{\circ}{\vec{\epsilon}} \\ \overset{\circ}{\vec{\theta}} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} = \begin{Bmatrix} \vec{F}^T \\ \vec{F}^R \end{Bmatrix} \quad (37)$$

where, on the right-hand side of equation 37,

$$\begin{Bmatrix} \vec{F}^T \\ \vec{F}^R \end{Bmatrix} = \begin{bmatrix} m1 & 0 \\ 0 & \Phi \end{bmatrix} \begin{Bmatrix} \overset{\circ\circ}{\vec{\epsilon}} \\ \overset{\circ\circ}{\vec{\theta}} \end{Bmatrix} - \frac{d^I}{dt} \begin{bmatrix} m1 & 0 \\ 0 & \Phi \end{bmatrix} \begin{Bmatrix} \dot{\vec{\gamma}} + \dot{\vec{\delta}} \\ \vec{\omega} + \dot{\vec{\theta}} \end{Bmatrix}, \quad (38)$$

and the constraint equation (33)

$$[g] \begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix} \quad (33)$$

remains unchanged.

An eigenvalue solution is obtained by assuming a solution of the form

$$\begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} = e^{i\omega t} \begin{Bmatrix} \vec{\varphi}^T \\ \vec{\varphi}^R \end{Bmatrix} \quad (39)$$

to the load-free equation

$$\begin{bmatrix} m1 & 0 \\ 0 & \Phi \end{bmatrix} \begin{Bmatrix} \overset{\circ\circ}{\vec{\epsilon}} \\ \overset{\circ\circ}{\vec{\theta}} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (40)$$

subject to the conditions of constraint

$$[g] \begin{Bmatrix} \vec{\epsilon} \\ \vec{\theta} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} . \quad (33)$$

Let

$\omega_{n,\lambda}$  = natural frequency of the  $n$ th normal mode of vibration of body  $\lambda$ ,

$$\begin{Bmatrix} \vec{\varphi}_{n,\lambda}^T \\ \vec{\varphi}_{n,\lambda}^R \end{Bmatrix} = n\text{th orthogonal mode of vibration of body } \lambda,$$

and impose the following normalization conditions:

$$\sum_{i \in \lambda} \left[ m_{i,\lambda} \vec{\varphi}_{m,i,\lambda}^T \cdot \vec{\varphi}_{n,i,\lambda}^T + \vec{\varphi}_{m,i,\lambda}^R \cdot ({}^\circ\Phi_{i,i,\lambda} \cdot \vec{\varphi}_{n,i,\lambda}^R) \right] = m_\lambda \delta_{m,n} \quad (41)$$

where

$\vec{\varphi}_{n,i,\lambda}^T$  =  $n$ th normal mode displacement vector for the element  $i$  of body  $\lambda$ ,

$\vec{\varphi}_{n,i,\lambda}^R$  =  $n$ th normal mode rotation vector for the element  $i$  of body  $\lambda$ ,

and

$\delta_{m,n}$  = Kronecker delta function.

It should be noted that, in the normalization equation, as the number of elements used increases and their respective mass properties decrease in magnitude, the contributions due to their respective rotational inertia effects approach zero. Hence, for all  $m$  and  $n$ ,

$$\lim_{N_{i,\lambda} \rightarrow \infty} \sum_{i \in \lambda} \vec{\varphi}_{m,i,\lambda}^R \cdot ({}^\circ\Phi_{i,i,\lambda} \cdot \vec{\varphi}_{n,i,\lambda}^R) = 0 \quad (42)$$

where

$N_{i,\lambda}$  = total number of elements  $i$  of body  $\lambda$ .

To obtain a modal solution to the deformation equation, a solution of the form

$$\vec{\epsilon}_{i,\lambda} = \sum_{n,\lambda} a_{n,\lambda}(t) \vec{\varphi}_{n,i,\lambda}^T \quad (43)$$



$$\vec{\theta}_{i,\lambda} = \sum_{n,\lambda} a_{n,\lambda}(t) \vec{\varphi}_{n,i,\lambda}^R \quad (44)$$

is assumed for the motion of the element  $i$  of body  $\lambda$ , where

$a_{n,\lambda}(t)$  = generalized displacement coordinate associated with the  $n$ th mode of vibration of body  $\lambda$

and

$\sum_{n,\lambda} \equiv$  sum over all modes to be used for the description of flexible body  $\lambda$  elastic deformation.

Substitution of the assumed solution into the deformation equation (37), along with the standard application of orthogonality conditions, readily yields the uncoupled equations of motion for the generalized displacement coordinate  $a_{n,\lambda}(t)$ . That is,

$$m_\lambda \left[ \ddot{a}_{n,\lambda}(t) + 2\zeta\omega_{n,\lambda} \dot{a}_{n,\lambda}(t) + \omega_{n,\lambda}^2 a_{n,\lambda}(t) \right] = \sum_{i \in \lambda} \left[ \vec{\varphi}_{n,i,\lambda}^T \cdot \vec{F}_{i,\lambda}^T + \vec{\varphi}_{n,i,\lambda}^R \cdot \vec{F}_{i,\lambda}^R \right] \quad (45)$$

where proportional damping has been assumed and  $\zeta$  is the damping ratio.

Two points should be especially noted. First, each eigenvector satisfies the constraint equation (33); that is,

$$\lfloor g \rfloor \begin{Bmatrix} \vec{\varphi}_{n,\lambda}^T \\ \vec{\varphi}_{n,\lambda}^R \end{Bmatrix} = \{0\} \quad \text{for } n = 1, 2, \dots \quad (46)$$

Secondly, it is not necessary to actually perform the finite element analysis outlined for every problem. The specific method used to obtain mode shapes is immaterial. The boundary conditions and normalization procedures, however, must be adhered to as follows:

- **Boundary conditions**

The flexible body must be clamped in both translation and rotation at one and at only one point; this point is the origin of the body-fixed reference frame at the hinge point. Reference frames fixed at other points are assumed free to both translate and rotate relative to it.

- **Normalization of modes**

A somewhat unconventional normalization procedure was found useful in uncoupling the generalized displacement coordinate equations. Six parameter

mode shapes are assumed available: three translation components of  $\vec{\varphi}_{n,i,\lambda}^T$  for the element  $i$  and three rotation components of  $\vec{\varphi}_{n,i,\lambda}^R$  for the minicoordinate frame fixed in the element  $i$ . They are normalized as follows:

$$\sum_{i \in \lambda} \left[ m_{i,\lambda} \vec{\varphi}_{m,i,\lambda}^T \cdot \vec{\varphi}_{n,i,\lambda}^T + \vec{\varphi}_{m,i,\lambda}^R \cdot \left( {}^\circ\Phi_{i,i,\lambda} \cdot \vec{\varphi}_{n,i,\lambda}^R \right) \right] = m_\lambda \delta_{m,n} \quad (41)$$

The problem is essentially solved at this point. A tedious exercise in vector-tensor operations must now be performed to put the equations together into a computationally efficient form. The reader is referred to reference 1 for supportive material and to references 4 and 5 for proofs of the vector-tensor identities used and listed in the Appendix.

There is no point in carrying out summations over all elements  $i$  at every integration step. All summations of this type may be replaced by resultant mode-dependent parameters as follows:

- Center-of-mass vector

$$\vec{\alpha}_\lambda = \overset{\circ}{\alpha}_\lambda + \frac{1}{m_\lambda} \sum_{i \in \lambda} m_{i,\lambda} \vec{\epsilon}_{i,\lambda} = \overset{\circ}{\alpha}_\lambda + \sum_{n,\lambda} \vec{A}_{n,\lambda} a_{n,\lambda}(t) \quad (47)$$

where

$$\vec{A}_{n,\lambda} = \frac{1}{m_\lambda} \sum_{i \in \lambda} m_{i,\lambda} \vec{\varphi}_{n,i,\lambda}^T \quad (48)$$

- Inertia tensor

$$\Phi_\lambda = \sum_{i \in \lambda} \left[ {}^\circ\Phi_{\lambda,i,\lambda} + \Delta\Phi_{\lambda,i,\lambda} \right] = {}^\circ\Phi_\lambda + \sum_{n,\lambda} \mathbf{E}_{n,\lambda} a_{n,\lambda}(t) \quad (49)$$

where

$$\mathbf{D}_{n,i,\lambda} = m_{i,\lambda} \left[ \left( \overset{\circ}{\Gamma}_{i,\lambda} - \overset{\circ}{\alpha}_\lambda \right) \cdot \vec{\varphi}_{n,i,\lambda}^T \mathbf{1} - \left( \overset{\circ}{\Gamma}_{i,\lambda} - \overset{\circ}{\alpha}_\lambda \right) \vec{\varphi}_{n,i,\lambda}^T \right], \quad (50)$$

$$\mathbf{D}_{n,\lambda} = \sum_{i \in \lambda} \mathbf{D}_{n,i,\lambda}, \quad (51)$$

and using the post superscript T to imply “transpose”

$$\mathbf{E}_{n,\lambda} = \mathbf{D}_{n,\lambda} + \mathbf{D}_{n,\lambda}^T \quad (52)$$

- Angular momentum due to deformation

$$\vec{\Delta L}_{\lambda,\lambda} = \sum_{i \in \lambda} \left[ \Phi_{i,i,\lambda} \cdot \overset{\circ}{\vec{\theta}}_{i,\lambda} + m_{i,\lambda} \vec{\delta}_{i,\lambda} \times \overset{\circ}{\vec{\delta}}_{i,\lambda} \right] = \sum_{n,\lambda} \vec{Q}_{n,\lambda} \dot{a}_{n,\lambda}(t) \quad (53)$$

where

$$\vec{B}_{n,\lambda} = \frac{1}{m_\lambda} \sum_{i \in \lambda} m_{i,\lambda} \overset{\sigma}{\vec{r}}_{i,\lambda} \times \vec{\varphi}_{n,i,\lambda}^T, \quad (54)$$

$$\vec{C}_{n,\lambda} = \sum_{i \in \lambda} \overset{\circ}{\Phi}_{i,i,\lambda} \cdot \vec{\varphi}_{n,i,\lambda}^R, \quad (55)$$

$$\vec{Q}_{n,\lambda} = \vec{C}_{n,\lambda} + m_\lambda \left( \vec{B}_{n,\lambda} - \overset{\sigma}{\vec{\alpha}}_\lambda \times \vec{A}_{n,\lambda} \right), \quad (56)$$

and terms quadratic in deformation have been deleted.

- Generalized force

$$\sum_{i \in \lambda} \left[ \vec{\varphi}_{n,i,\lambda}^T \cdot \vec{F}_{i,\lambda}^T + \vec{\varphi}_{n,i,\lambda}^R \cdot \vec{F}_{i,\lambda}^R \right] \quad (57)$$

From the partition matrix equation (38),

$$\vec{F}_{i,\lambda}^T = m_{i,\lambda} \left[ \overset{\circ\circ}{\vec{\epsilon}}_{i,\lambda} - \left( \ddot{\vec{\gamma}}_{I,\lambda} + \ddot{\vec{\delta}}_{i,\lambda} \right) \right] \quad (58)$$

and

$$\vec{F}_{i,\lambda}^R = \overset{\circ}{\Phi}_{i,i,\lambda} \cdot \overset{\circ\circ}{\vec{\theta}}_{i,\lambda} - \frac{d^I}{dt} \left[ \overset{\circ}{\Phi}_{i,i,\lambda} \cdot \left( \vec{\omega}_\lambda + \overset{\circ}{\vec{\theta}}_{i,\lambda} \right) \right]. \quad (59)$$

It is convenient, in the process of removing all summations over the elements  $i$ , to separate and collect terms which can be associated with the various components of acceleration.

Accordingly, write

$$\vec{\gamma}_{I,\lambda} + \vec{\delta}_{i,\lambda} = \vec{\gamma}_{I,\lambda} - \vec{\alpha}_\lambda + \overset{\sigma}{\vec{\alpha}}_\lambda + \overset{\sigma}{\vec{r}}_{i,\lambda} - \overset{\sigma}{\vec{\alpha}}_\lambda + \vec{\epsilon}_{i,\lambda} \quad (60)$$

and carry out the prescribed differentiations in equations 58 and 59 to obtain

$$\begin{aligned} \vec{F}_{i,\lambda}^T = -m_{i,\lambda} \left[ \ddot{\vec{\gamma}}_{I,\lambda} - \ddot{\vec{\alpha}}_\lambda + \overset{\sigma}{\vec{\alpha}}_\lambda + \vec{\omega}_\lambda \times \left( \overset{\sigma}{\vec{r}}_{i,\lambda} - \overset{\sigma}{\vec{\alpha}}_\lambda + \vec{\epsilon}_{i,\lambda} \right) \right. \\ \left. + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \left( \overset{\sigma}{\vec{r}}_{i,\lambda} - \overset{\sigma}{\vec{\alpha}}_\lambda + \vec{\epsilon}_{i,\lambda} \right) \right) + 2\vec{\omega}_\lambda \times \overset{\circ}{\vec{\epsilon}}_{i,\lambda} \right] \quad (61) \end{aligned}$$

and

$$\vec{F}_{i,\lambda}^R = - \left[ \overset{\circ}{\Phi}_{i,i,\lambda} \cdot \dot{\vec{\omega}}_{\lambda} + \vec{\omega}_{\lambda} \times \overset{\circ}{\Phi}_{i,i,\lambda} \cdot \left( \vec{\omega}_{\lambda} + \overset{\circ}{\theta}_{i,\lambda} \right) \right]. \quad (62)$$

Consistent application of the initial assumption that terms quadratic in deformation are negligible yields the following components of the generalized force. They are obtained from a direct substitution of equations 61 and 62 into equation 57.

- Generalized force component which is due to the linear acceleration of the undeformed center-of-mass position of the body  $\lambda$

$$- \sum_{i \in \lambda} m_{i,\lambda} \left( \ddot{\vec{\gamma}}_{I,\lambda} - \ddot{\vec{\alpha}}_{\lambda} + \overset{\circ}{\ddot{\alpha}}_{\lambda} \right) \cdot \vec{\varphi}_{n,i,\lambda}^T = -m_{\lambda} \left( \ddot{\vec{\gamma}}_{I,\lambda} - \ddot{\vec{\alpha}}_{\lambda} + \overset{\circ}{\ddot{\alpha}}_{\lambda} \right) \cdot \vec{A}_{n,\lambda} \quad (63)$$

- Generalized force component which is due to the inertial angular acceleration of the reference frame fixed at the origin (hinge point  $\lambda-1$  of body  $\lambda$ )

$$\begin{aligned} & - \sum_{i \in \lambda} \left\{ \left( \overset{\circ}{\Phi}_{i,i,\lambda} \cdot \dot{\vec{\omega}}_{\lambda} \right) \cdot \vec{\varphi}_{n,i,\lambda}^R + m_{i,\lambda} \left[ \dot{\vec{\omega}}_{\lambda} \times \left( \overset{\sigma}{\vec{\gamma}}_{I,\lambda} - \overset{\sigma}{\vec{\alpha}}_{\lambda} + \vec{\epsilon}_{i,\lambda} \right) \right] \cdot \vec{\varphi}_{n,i,\lambda}^T \right\} \\ & = -\dot{\vec{\omega}}_{\lambda} \cdot \sum_{i \in \lambda} \left\{ \overset{\circ}{\Phi}_{i,i,\lambda} \cdot \vec{\varphi}_{n,i,\lambda}^R + m_{i,\lambda} \left( \overset{\sigma}{\vec{\gamma}}_{I,\lambda} - \overset{\sigma}{\vec{\alpha}}_{\lambda} \right) \times \vec{\varphi}_{n,i,\lambda}^T \right\} = -\vec{Q}_{n,\lambda} \cdot \dot{\vec{\omega}}_{\lambda} \quad (64) \end{aligned}$$

where  $\vec{Q}_{n,\lambda}$  is defined by equation 56.

- Generalized force component which is due to the inertial angular velocity of the reference frame fixed at the origin (hinge point  $\lambda-1$  of body  $\lambda$ )

$$\begin{aligned} & - \sum_{i \in \lambda} \left\{ m_{i,\lambda} \left[ \vec{\omega}_{\lambda} \times \left( \vec{\omega}_{\lambda} \times \left( \overset{\sigma}{\vec{\gamma}}_{I,\lambda} - \overset{\sigma}{\vec{\alpha}}_{\lambda} + \vec{\epsilon}_{i,\lambda} \right) \right) + 2\vec{\omega}_{\lambda} \times \overset{\circ}{\vec{\epsilon}}_{i,\lambda} \right] \cdot \vec{\varphi}_{n,i,\lambda}^T \right. \\ & \quad \left. + \left[ \vec{\omega}_{\lambda} \times \overset{\circ}{\Phi}_{i,i,\lambda} \cdot \left( \vec{\omega}_{\lambda} + \overset{\circ}{\theta}_{i,\lambda} \right) \right] \cdot \vec{\varphi}_{n,i,\lambda}^R \right\} \\ & = \vec{\omega}_{\lambda} \cdot \left( \mathbf{H}_{n,\lambda} + \sum_{m,\lambda} \mathbf{F}_{m,n,\lambda} a_{m,\lambda}(t) \right) \cdot \vec{\omega}_{\lambda} - 2\vec{\omega}_{\lambda} \cdot \sum_{m,\lambda} \dot{a}_{m,\lambda}(t) \vec{K}_{m,n,\lambda} \quad (65) \end{aligned}$$

where, by making use of equations 43 and 44

$$\vec{\epsilon}_{i,\lambda} = \sum_{n,\lambda} a_{n,\lambda}(t) \vec{\varphi}_{n,i,\lambda}^T \quad (43)$$

$$\vec{\theta}_{i,\lambda} = \sum_{n,\lambda} a_{n,\lambda}(t) \vec{\varphi}_{n,i,\lambda}^R \quad (44)$$

and a manipulation of vector-tensor identities, the following relationships are established:

$$\begin{aligned} \text{a.} - \sum_{i \in \lambda} \left\{ m_{i,\lambda} \left[ \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \left( \vec{\sigma}_{i,\lambda}^T - \vec{\alpha}_\lambda \right) \right) \right] \cdot \vec{\varphi}_{n,i,\lambda}^T + \left( \vec{\omega}_\lambda \times {}^\circ\Phi_{i,i,\lambda} \cdot \vec{\omega}_\lambda \right) \cdot \vec{\varphi}_{n,i,\lambda}^R \right\} \\ = \vec{\omega}_\lambda \cdot \mathbf{H}_{n,\lambda} \cdot \vec{\omega}_\lambda \end{aligned} \quad (66)$$

where

$$\mathbf{J}_{n,\lambda} = \sum_{i \in \lambda} \vec{\varphi}_{n,i,\lambda}^R \times {}^\circ\Phi_{i,i,\lambda} \quad (67)$$

and

$$\mathbf{H}_{n,\lambda} = \mathbf{D}_{n,\lambda} + \mathbf{J}_{n,\lambda} \quad (68)$$

$$\text{b.} - \sum_{i \in \lambda} m_{i,\lambda} \left[ \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \vec{e}_{i,\lambda} \right) \right] \cdot \vec{\varphi}_{n,i,\lambda}^T = \vec{\omega}_\lambda \cdot \sum_{m,\lambda} \mathbf{F}_{m,n,\lambda} a_{m,\lambda}(t) \cdot \vec{\omega}_\lambda \quad (69)$$

where

$$\mathbf{F}_{m,n,\lambda} = \sum_{i \in \lambda} m_{i,\lambda} \left[ \vec{\varphi}_{m,i,\lambda}^T \cdot \vec{\varphi}_{n,i,\lambda}^T \mathbf{1} - \vec{\varphi}_{m,i,\lambda}^T \vec{\varphi}_{n,i,\lambda}^T \right] \cdot \quad (70)$$

$$\begin{aligned} \text{c.} - \sum_{i \in \lambda} \left\{ 2m_{i,\lambda} \left( \vec{\omega}_\lambda \times \overset{\circ}{\vec{e}}_{i,\lambda} \right) \cdot \vec{\varphi}_{n,i,\lambda}^T + \left[ \vec{\omega}_\lambda \times {}^\circ\Phi_{i,i,\lambda} \cdot \overset{\circ}{\vec{\theta}}_{i,\lambda} \right] \cdot \vec{\varphi}_{n,i,\lambda}^R \right\} \\ = -2\vec{\omega}_\lambda \cdot \sum_{m,\lambda} \dot{a}_{m,\lambda}(t) \vec{K}_{m,n,\lambda} \end{aligned} \quad (71)$$

where

$$\vec{K}_{m,n,\lambda} = \sum_{i \in \lambda} \left\{ m_{i,\lambda} \vec{\varphi}_{m,i,\lambda}^T \times \vec{\varphi}_{n,i,\lambda}^T + \frac{1}{2} \left( {}^\circ\Phi_{i,i,\lambda} \cdot \vec{\varphi}_{m,i,\lambda}^R \right) \times \vec{\varphi}_{n,i,\lambda}^R \right\} \cdot \quad (72)$$

A direct summation of the above components yields the desired form of the generalized force for the  $n$ th mode. That is,

$$\begin{aligned} \sum_{i \in \lambda} \left[ \vec{\varphi}_{n,i,\lambda}^T \cdot \vec{F}_{i,\lambda}^T + \vec{\varphi}_{n,i,\lambda}^R \cdot \vec{F}_{i,\lambda}^R \right] = -m_\lambda \left( \ddot{\vec{\gamma}}_{I,\lambda} - \ddot{\vec{\alpha}}_\lambda + \overset{\circ}{\ddot{\vec{\alpha}}}_\lambda \right) \cdot \vec{A}_{n,\lambda} - \vec{Q}_{n,\lambda} \cdot \dot{\vec{\omega}}_\lambda \\ + \vec{\omega}_\lambda \cdot \left[ \mathbf{H}_{n,\lambda} + \sum_{m,\lambda} \mathbf{F}_{m,n,\lambda} a_{m,\lambda}(t) \right] \cdot \vec{\omega}_\lambda - 2\vec{\omega}_\lambda \cdot \sum_{m,\lambda} \dot{a}_{m,\lambda}(t) \vec{K}_{m,n,\lambda} . \end{aligned} \quad (73)$$

These definitions, established by equations 47 through 73, are used to rewrite the rate of change-of-momentum equations (27, 28, and 29) as

$$\dot{\vec{G}}_{I,\lambda} = m_\lambda \ddot{\vec{\gamma}}_{I,\lambda} , \quad (74)$$

$$\begin{aligned} \dot{\vec{L}}_{I,\lambda} = \Phi_\lambda \cdot \dot{\vec{\omega}}_\lambda + m_\lambda \vec{\gamma}_{I,\lambda} \times \ddot{\vec{\gamma}}_{I,\lambda} + \sum_{n,\lambda} \vec{Q}_{n,\lambda} \ddot{a}_{n,\lambda}(t) \\ + \sum_{\substack{m: \\ \text{MO}(m)=\lambda}} \left( \overset{\circ}{\vec{H}}_m + \vec{\omega}_\lambda \times \overset{\circ}{\vec{H}}_m \right) + \dot{\Phi}_\lambda \cdot \vec{\omega}_\lambda + \vec{\omega}_\lambda \times \left( \Phi_\lambda \cdot \vec{\omega}_\lambda + \Delta \vec{L}_{\lambda,\lambda} \right) , \end{aligned} \quad (75)$$

$$\dot{\vec{H}}_m = \mathbf{I}_{W_m} \cdot \dot{\vec{\omega}}_\lambda + \overset{\circ}{\vec{H}}_m + \vec{\omega}_\lambda \times \left( \mathbf{I}_{W_m} \cdot \vec{\omega}_\lambda + \overset{\circ}{\vec{H}}_m \right) , \quad (76)$$

and the generalized displacement coordinate equation (44) as

$$\begin{aligned} m_\lambda \left[ \ddot{a}_{n,\lambda}(t) + 2\xi\omega_{n,\lambda} \dot{a}_{n,\lambda}(t) + \omega_{n,\lambda}^2 a_{n,\lambda}(t) \right] \\ = -m_\lambda \left( \ddot{\vec{\gamma}}_{I,\lambda} - \ddot{\vec{\alpha}}_\lambda + \overset{\circ}{\ddot{\vec{\alpha}}}_\lambda \right) \cdot \vec{A}_{n,\lambda} - \vec{Q}_{n,\lambda} \cdot \dot{\vec{\omega}}_\lambda \\ + \vec{\omega}_\lambda \cdot \left( \mathbf{H}_{n,\lambda} + \sum_{m,\lambda} \mathbf{F}_{m,n,\lambda} a_{m,\lambda}(t) \right) \cdot \vec{\omega}_\lambda \\ - 2\vec{\omega}_\lambda \cdot \sum_{m,\lambda} \dot{a}_{m,\lambda}(t) \vec{K}_{m,n,\lambda} . \end{aligned} \quad (77)$$

In summary, the resultant mode-dependent parameters used in the analysis are:

$$\vec{A}_{n,\lambda} = \frac{1}{m_\lambda} \sum_{i \in \lambda} m_{i,\lambda} \vec{\varphi}_{n,i,\lambda}^T \quad (48)$$

$$\mathbf{D}_{n,i,\lambda} = m_{i,\lambda} \left[ \left( \vec{\sigma}_{i,\lambda}^T - \vec{\sigma}_{\lambda}^T \right) \cdot \vec{\varphi}_{n,i,\lambda}^T \mathbf{1} - \left( \vec{\sigma}_{i,\lambda}^T - \vec{\sigma}_{\lambda}^T \right) \vec{\varphi}_{n,i,\lambda}^T \right] \quad (50)$$

$$\mathbf{D}_{n,\lambda} = \sum_{i \in \lambda} \mathbf{D}_{n,i,\lambda} \quad (51)$$

$$\mathbf{E}_{n,\lambda} = \mathbf{D}_{n,\lambda} + \mathbf{D}_{n,\lambda}^T \quad (52)$$

$$\vec{\mathbf{B}}_{n,\lambda} = \frac{1}{m_{\lambda}} \sum_{i \in \lambda} m_{i,\lambda} \vec{\sigma}_{i,\lambda}^T \times \vec{\varphi}_{n,i,\lambda}^T \quad (54)$$

$$\vec{\mathbf{C}}_{n,\lambda} = \sum_{i \in \lambda} {}^{\circ}\Phi_{i,i,\lambda} \cdot \vec{\varphi}_{n,i,\lambda}^R \quad (55)$$

$$\vec{\mathbf{Q}}_{n,\lambda} = \vec{\mathbf{C}}_{n,\lambda} + m_{\lambda} \left( \vec{\mathbf{B}}_{n,\lambda} - \vec{\sigma}_{\lambda}^T \times \vec{\mathbf{A}}_{n,\lambda} \right) \quad (56)$$

$$\mathbf{J}_{n,\lambda} = \sum_{i \in \lambda} \vec{\varphi}_{n,i,\lambda}^R \times {}^{\circ}\Phi_{i,i,\lambda} \quad (67)$$

$$\mathbf{H}_{n,\lambda} = \mathbf{D}_{n,\lambda} + \mathbf{J}_{n,\lambda} \quad (68)$$

$$\mathbf{F}_{m,n,\lambda} = \sum_{i \in \lambda} m_{i,\lambda} \left[ \vec{\varphi}_{m,i,\lambda}^T \cdot \vec{\varphi}_{n,i,\lambda}^T \mathbf{1} - \vec{\varphi}_{m,i,\lambda}^T \vec{\varphi}_{n,i,\lambda}^T \right] \quad (70)$$

$$\vec{\mathbf{K}}_{m,n,\lambda} = \sum_{i \in \lambda} \left\{ m_{i,\lambda} \vec{\varphi}_{m,i,\lambda}^T \times \vec{\varphi}_{n,i,\lambda}^T + \frac{1}{2} \left( {}^{\circ}\Phi_{i,i,\lambda} \cdot \vec{\varphi}_{m,i,\lambda}^R \right) \times \vec{\varphi}_{n,i,\lambda}^R \right\} \quad (72)$$

## NESTED BODY EQUATIONS OF MOTION

In reference 1, the nested body approach was used to write the coupled rigid body equations of motion. This same approach can be used if the bodies are flexible. The extension of the equations to account for body flexibility simply involves carrying along a few more terms. The basic form of the equations remains unchanged.

As in reference 1, several sets of body labels must be defined. The bodies are each given distinct integer labels such that, along the topological path from body 1 to any limb end, the body labels are of increasing numerical magnitude. Let

$S_R$  = the set of all body labels associated with rigid and flexible bodies,

$S_L$  = the set of all body labels associated with point masses,

- $S_F$  = the set of all body labels associated with flexible bodies,  
 $S_{k-1}$  = the set of all body labels outboard, relative to body 1, of hinge point k-1 of body k (these body labels define the bodies contained in the nest k-1), and  
 $S_{k-1,\lambda-1}$  = the set of all body labels associated with those bodies on the topological path outboard from hinge point k-1 to the center-of-mass of body  $\lambda$ .

Furthermore, as in reference 1, let

$\vec{\beta}_1$  = position vector from the inertial origin to the hinge point of body 1, which, by definition, is its center-of-mass, and

$\vec{\gamma}_{k-1,\lambda}$  = position vector from hinge point k-1 to the center-of-mass of body  $\lambda$ ,

where

$$\vec{\gamma}_{I,\lambda} = \vec{\beta}_1 + \vec{\gamma}_{0,\lambda}, \quad (78)$$

$$\vec{\gamma}_{k-1,\lambda} = \sum_{\substack{i \in S_{k-1,\lambda-1} \\ i \neq k}} \vec{\beta}_i + \vec{\alpha}_\lambda, \quad (79)$$

$\vec{F}_{k-1}^c$  = resultant force of constraint acting on body k through hinge point k-1,

$\vec{\phi}_{k-1}^c$  = resultant torque of constraint acting on body k through hinge point k-1,

$\vec{\phi}_{W_m}^c$  = resultant torque of constraint acting on the symmetric wheel m which prevents motion about any axis normal to its spin axis,

$\vec{F}_{k-1}^H$  = resultant force acting on body k due to all mechanisms existing between bodies J(k) and k at hinge point k-1,

$\vec{\phi}_{k-1}^H$  = resultant torque acting on body k due to all mechanisms existing between bodies J(k) and k at hinge point k-1,

$\vec{C}\vec{L}_m$  = resultant torque acting on the symmetric wheel m due to all mechanisms existing between it and the body MO(m) in which it is embedded,

$\vec{F}_k^{(e)}$  = resultant external force acting on body k, and

$\vec{\phi}_k^{(e)}$  = resultant external torque acting on body k.



Making use of the above definitions and Newton's fundamental laws of motion, the equations of motion for the nest k-1 are:

$$\sum_{\lambda \in S_{k-1}} \dot{\vec{G}}_{I,\lambda} = \vec{F}_{k-1}^H + \vec{F}_{k-1}^c + \sum_{\lambda \in S_{k-1}} \vec{F}_{\lambda}^{(e)}, \quad (80)$$

$$\begin{aligned} \sum_{\lambda \in S_{k-1}} \dot{\vec{L}}_{I,\lambda} &= \vec{\phi}_{k-1}^H + \vec{\phi}_{k-1}^c + \left( \vec{\gamma}_{I,k} - \vec{\alpha}_k \right) \times \left( \vec{F}_{k-1}^c + \vec{F}_{k-1}^H \right) \\ &+ \sum_{\lambda \in S_{k-1}} \left( \vec{\gamma}_{I,\lambda} \times \vec{F}_{\lambda}^{(e)} + \vec{\phi}_{\lambda}^{(e)} \right), \end{aligned} \quad (81)$$

and

$$\dot{\vec{H}}_m = \vec{\phi}_{W_m}^c + \vec{C}\vec{L}_m. \quad (82)$$

By direct elimination of the forces of constraint from the angular momentum equations, the following relations can be shown:

- Rotation equation for nest k-1,  $k \in S_R$

$$\begin{aligned} \sum_{\lambda \in S_{k-1}} \left[ \dot{\vec{L}}_{I,\lambda} - \left( \vec{\gamma}_{I,k} - \vec{\alpha}_k \right) \times \dot{\vec{G}}_{I,\lambda} \right] \\ = \vec{\phi}_{k-1}^H + \vec{\phi}_{k-1}^c + \sum_{\lambda \in S_{k-1}} \left[ \vec{\gamma}_{k-1,\lambda} \times \vec{F}_{\lambda}^{(e)} + \vec{\phi}_{\lambda}^{(e)} \right] \end{aligned} \quad (83)$$

- Translation equation for nest k-1,  $k \in S_R$  and  $k \in S_L$

$$\sum_{\lambda \in S_{k-1}} \dot{\vec{G}}_{I,\lambda} = \vec{F}_{k-1}^H + \vec{F}_{k-1}^c + \sum_{\lambda \in S_{k-1}} \vec{F}_{\lambda}^{(e)} \quad (84)$$

- Symmetric wheel equation for wheel m

$$\dot{\vec{H}}_m = \vec{\phi}_{W_m}^c + \vec{C}\vec{L}_m \quad (82)$$

- Generalized displacement coordinate equation

$$\begin{aligned}
& m_\lambda \left[ \ddot{a}_{n,\lambda}(t) + 2\xi\omega_{n,\lambda} \dot{a}_{n,\lambda}(t) + \omega_{n,\lambda}^2 a_{n,\lambda}(t) \right] \\
& = -m_\lambda \left( \ddot{\vec{\gamma}}_{I,\lambda} - \ddot{\vec{\alpha}}_\lambda + \overset{\circ}{\ddot{\alpha}}_\lambda \right) \cdot \vec{A}_{n,\lambda} - \vec{Q}_{n,\lambda} \cdot \dot{\vec{\omega}}_\lambda \\
& \quad + \dot{\vec{\omega}}_\lambda \cdot \left( \mathbf{H}_{n,\lambda} + \sum_{m,\lambda} \mathbf{F}_{m,n,\lambda} a_{m,\lambda}(t) \right) \cdot \dot{\vec{\omega}}_\lambda \\
& \quad - 2\dot{\vec{\omega}}_\lambda \cdot \sum_{m,\lambda} \dot{a}_{m,\lambda}(t) \vec{K}_{m,n,\lambda}
\end{aligned} \tag{85}$$

where, from equations 74, 75, and 76,

$$\begin{aligned}
\dot{\vec{G}}_{I,\lambda} &= m_\lambda \ddot{\vec{\gamma}}_{I,\lambda}, \\
\dot{\vec{L}}_{I,\lambda} - \left( \vec{\gamma}_{I,k} - \vec{\alpha}_k \right) \times \dot{\vec{G}}_{I,\lambda} &= \Phi_\lambda \cdot \dot{\vec{\omega}}_\lambda + m_\lambda \vec{\gamma}_{k-1,\lambda} \times \ddot{\vec{\gamma}}_{I,\lambda} \\
& \quad + \sum_{n,\lambda} \vec{Q}_{n,\lambda} \ddot{a}_{n,\lambda} + \sum_{m:} \left( \overset{\circ}{\vec{H}}_m + \dot{\vec{\omega}}_\lambda \times \vec{\tilde{H}}_m \right) \\
& \quad \text{MO}(m)=\lambda \\
& \quad + \overset{\circ}{\Phi}_\lambda \cdot \dot{\vec{\omega}}_\lambda + \dot{\vec{\omega}}_\lambda \times \left( \Phi_\lambda \cdot \dot{\vec{\omega}}_\lambda + \Delta \vec{L}_{\lambda,\lambda} \right),
\end{aligned} \tag{87}$$

and

$$\dot{\vec{H}}_m = \mathbf{I}_{W_m} \cdot \dot{\vec{\omega}}_\lambda + \vec{\tilde{H}}_m + \dot{\vec{\omega}}_\lambda \times \left( \mathbf{I}_{W_m} \cdot \dot{\vec{\omega}}_\lambda + \vec{\tilde{H}}_m \right). \tag{88}$$

Special consideration must be directed toward the vector representation of the translational acceleration of the center-of-mass of the body  $\lambda$ . Since body  $\lambda$  may be at the end of a chain of flexible bodies, several terms which were zero in reference 1 must now be included. That is, from equations 78 and 79,

$$\begin{aligned}
\ddot{\vec{\gamma}}_{I,\lambda} &= \ddot{\vec{\beta}}_1 + \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left[ \dot{\vec{\omega}}_{J(i)} \times \vec{\beta}_i + \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) + 2\vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i + \overset{\circ\circ}{\vec{\beta}}_i \right] \\
& \quad + \dot{\vec{\omega}}_\lambda \times \vec{\alpha}_\lambda + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \vec{\alpha}_\lambda \right) + 2\vec{\omega}_\lambda \times \overset{\circ}{\vec{\alpha}}_\lambda + \overset{\circ\circ}{\vec{\alpha}}_\lambda.
\end{aligned} \tag{89}$$

From the standpoint of optimizing computational efficiency, it is desirable to establish symmetry relations. Accordingly, as in reference 1, all inertial angular acceleration vectors are replaced by summations of relative angular acceleration vectors. The identity

$$\dot{\vec{\omega}}_\lambda = \sum_{i \in S_{0,\lambda-1}} \dot{\vec{\omega}}_i \quad (90)$$

is therefore introduced and equation 89 rewritten as

$$\begin{aligned} \ddot{\vec{\gamma}}_{I,\lambda} = & \ddot{\vec{\beta}}_1 + \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \overset{\circ}{\vec{\beta}}_i + \overset{\circ}{\vec{\alpha}}_\lambda + \sum_{i \in S_{0,\lambda-1}} \dot{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda} \\ & + \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left[ \vec{\omega}_{J(i)} \times (\vec{\omega}_{J(i)} \times \vec{\beta}_i) + 2\vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \right] + \vec{\omega}_\lambda \times (\vec{\omega}_\lambda \times \vec{\alpha}_\lambda) + 2\vec{\omega}_\lambda \times \overset{\circ}{\vec{\alpha}}_\lambda. \end{aligned} \quad (91)$$

Substituting the various vector identities provided and recombining terms yields:

- Rotation equation for the nest k-1,  $k \in S_R$

$$\begin{aligned} & \sum_{\lambda \in S_{k-1}} \left[ \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left( \overset{\circ}{\vec{\beta}}_1 + \overset{\circ}{\vec{\alpha}}_\lambda \right) + \sum_{\substack{m \\ MO(m)=\lambda}} \overset{\circ}{\vec{H}}_m + \sum_{n,\lambda} \vec{Q}_{n,\lambda} \ddot{a}_{n,\lambda}(t) \right] \\ & + \sum_{\substack{i \in S_{0,k-1} \\ i \neq k}} \left[ \sum_{\substack{\lambda \in S_{k-1} \\ \lambda \notin S_L}} \Phi_\lambda \cdot \dot{\vec{\omega}}_i + \sum_{\lambda \in S_{k-1}} \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left( \dot{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda} \right) + \sum_{\substack{\lambda \in S_{k-1} \\ i \neq 1}} \vec{\gamma}_{k-1,\lambda} \times m_\lambda \overset{\circ}{\vec{\beta}}_i \right] \\ & + \sum_{i \in S_{k-1}} \left[ \sum_{\substack{\lambda \in S_{i-1} \\ \lambda \notin S_L}} \Phi_\lambda \cdot \dot{\vec{\omega}}_i + \sum_{\lambda \in S_{i-1}} \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left( \dot{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda} \right) + \sum_{\lambda \in S_{i-1}} \vec{\gamma}_{k-1,\lambda} \times m_\lambda \overset{\circ}{\vec{\beta}}_i \right] \\ = & - \sum_{\lambda \in S_{k-1}} \left[ \overset{\circ}{\Phi}_\lambda \cdot \vec{\omega}_\lambda + \vec{\omega}_\lambda \times (\Phi_\lambda \cdot \vec{\omega}_\lambda + \Delta \vec{L}_{\lambda,\lambda}) + \sum_{\substack{m \\ MO(m)=\lambda}} \vec{\omega}_\lambda \times \overset{\circ}{\vec{H}}_m \right] \end{aligned}$$

$$\begin{aligned}
& - \sum_{\lambda \in S_{k-1}} \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left[ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left( \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) + 2 \vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \right) \right. \\
& \qquad \qquad \qquad \left. + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \vec{\alpha}_\lambda \right) + 2 \vec{\omega}_\lambda \times \overset{\circ}{\vec{\alpha}}_\lambda \right] \\
& + \vec{\phi}_{k-1}^H + \vec{\phi}_{k-1}^c + \sum_{\lambda \in S_{k-1}} \left[ \vec{\gamma}_{k-1,\lambda} \times \vec{F}_\lambda^{(e)} + \vec{\phi}_\lambda^{(e)} \right] \tag{92}
\end{aligned}$$

- Translation equation for the nest k-1,  $k \in S_R$  and  $k \in S_L$

$$\begin{aligned}
& \sum_{\lambda \in S_{k-1}} m_\lambda \left( \overset{\circ}{\vec{\beta}}_1 + \overset{\circ}{\vec{\alpha}}_\lambda \right) + \sum_{\substack{i \in S_{0,k-1} \\ i \neq k}} \left[ \sum_{\lambda \in S_{k-1}} m_\lambda \overset{\cdot}{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda} + \sum_{\lambda \in S_{k-1}} m_\lambda \overset{\circ}{\vec{\beta}}_i \right] \\
& + \sum_{i \in S_{k-1}} \left[ \sum_{\lambda \in S_{i-1}} m_\lambda \overset{\cdot}{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda} + \sum_{\lambda \in S_{i-1}} m_\lambda \overset{\circ}{\vec{\beta}}_i \right] \\
& = - \sum_{\lambda \in S_{k-1}} m_\lambda \left[ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left( \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) + 2 \vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \right) \right. \\
& \left. + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \vec{\alpha}_\lambda \right) + 2 \vec{\omega}_\lambda \times \overset{\circ}{\vec{\alpha}}_\lambda \right] + \vec{F}_{k-1}^H + \vec{F}_{k-1}^c + \sum_{\lambda \in S_{k-1}} \vec{F}_\lambda^{(e)} \tag{93}
\end{aligned}$$

- Symmetric wheel equation for wheel m,  $\lambda = MO(m)$

$$\overset{\circ}{\vec{H}}_m + \sum_{i \in S_{0,\lambda-1}} I_{W_m} \cdot \overset{\cdot}{\vec{\omega}}_i = -\vec{\omega}_\lambda \times \left( I_{W_m} \cdot \vec{\omega}_\lambda + \vec{H}_m \right) + C\vec{L}_m + \vec{\phi}_{W_m}^c \tag{94}$$

- Generalized displacement coordinate equation

$$\begin{aligned}
& m_\lambda \left[ \ddot{a}_{n,\lambda}(t) + \overset{\circ}{\vec{\beta}}_1 \cdot \vec{A}_{n,\lambda} + \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \overset{\circ}{\vec{\beta}}_i \cdot \vec{A}_{n,\lambda} \right] \\
& + \sum_{i \in S_{0,\lambda-1}} \left[ m_\lambda \left( \overset{\cdot}{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda} \right) \cdot \vec{A}_{n,\lambda} + \vec{Q}_{n,\lambda} \cdot \overset{\cdot}{\vec{\omega}}_i \right]
\end{aligned}$$

$$\begin{aligned}
&= -m_\lambda \left[ 2\xi\omega_{n,\lambda} \dot{a}_{n,\lambda}(t) + \omega_{n,\lambda}^2 a_{n,\lambda}(t) \right] \\
&\quad -m_\lambda \left[ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left( \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) + 2\vec{\omega}_{J(i)} \times \dot{\vec{\beta}}_i \right) + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \overset{\circ}{\alpha}_\lambda \right) \right] \cdot \vec{A}_{n,\lambda} \\
&\quad + \vec{\omega}_\lambda \cdot \left[ \mathbf{H}_{n,\lambda} + \sum_{m,\lambda} \mathbf{F}_{m,n,\lambda} a_{m,\lambda}(t) \right] \cdot \vec{\omega}_\lambda \\
&\quad - 2\vec{\omega}_\lambda \cdot \sum_{m,\lambda} \dot{a}_{m,\lambda}(t) \vec{K}_{m,n,\lambda} \tag{95}
\end{aligned}$$

where the second order effect assumption has been exercised to write

$$\sum_{i \in S_{0,\lambda-1}} \dot{\vec{\omega}}_i \times \left( \vec{\gamma}_{i-1,\lambda} - \vec{\alpha}_\lambda + \overset{\circ}{\alpha}_\lambda \right) = \sum_{i \in S_{0,\lambda-1}} \dot{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda} \tag{96}$$

in equation 95.

These equations may now be put into their final vector tensor form by the introduction of several more definitions. As in reference 1, let:

$\mathcal{P}$  = skew operator such that for vector  $\vec{v}$  having components  $(v_1, v_2, v_3)$ ,

$$\mathcal{P}(\vec{v}) = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} \tag{97}$$

$$\mathbf{G}_{k-1,i-1}^\lambda = m_\lambda \left[ \left( \vec{\gamma}_{i-1,\lambda} \cdot \vec{\gamma}_{k-1,\lambda} \right) \mathbf{1} - \vec{\gamma}_{i-1,\lambda} \vec{\gamma}_{k-1,\lambda} \right], \tag{98}$$

and

$$\Gamma_{k-1,\lambda} = \mathcal{P}(\vec{\gamma}_{k-1,\lambda}). \tag{99}$$

Furthermore, define

$$\vec{C}_\lambda = \text{force associated with the centripetal plus Coriolis acceleration of the center-of-mass of body } \lambda \text{ relative to the inertial reference}$$

$$\vec{c}_\lambda = m_\lambda \left[ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left( \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) + 2\vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \right) + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \vec{\alpha}_\lambda \right) + 2\vec{\omega}_\lambda \times \overset{\circ}{\vec{\alpha}}_\lambda \right] , \quad (100)$$

$\vec{L}_{\lambda,\lambda}$  = angular momentum of body  $\lambda$  about its own center-of-mass relative to the inertial reference

$$\vec{L}_{\lambda,\lambda} = \Phi_\lambda \cdot \vec{\omega}_\lambda + \Delta \vec{L}_{\lambda,\lambda} + \sum_{\substack{m: \\ \text{MO}(m)=\lambda}} \vec{\tilde{H}}_m , \quad (101)$$

$\vec{H}_m$  = angular momentum of wheel  $m$  about its own center-of-mass relative to the inertial reference

$$\vec{H}_m = \mathbf{I}_{W_m} \cdot \vec{\omega}_\lambda + \vec{\tilde{H}}_m , \quad (102)$$

and make use of the vector-tensor identities provided in the Appendix to recognize that

$$\vec{\gamma}_{k-1,\lambda} \times \ddot{\vec{\beta}}_1 = -\Gamma_{k-1,\lambda} \cdot \ddot{\vec{\beta}}_1 , \quad (103)$$

$$\vec{\gamma}_{k-1,\lambda} \times m_\lambda \left( \dot{\vec{\omega}}_i \times \vec{\gamma}_{i-1,\lambda} \right) = \mathbf{G}_{k-1,i-1}^\lambda \cdot \dot{\vec{\omega}}_i , \quad (104)$$

$$\overset{\circ}{\vec{\tilde{H}}}_m = \mathbf{I}_{W_m} \cdot \overset{\circ}{\vec{\omega}}_{W_m} , \quad (105)$$

and

$$\vec{\gamma}_{i-1,\lambda} \times \vec{A}_{n,\lambda} = -\Gamma_{i-1,\lambda} \cdot \vec{A}_{n,\lambda} . \quad (106)$$

Substitution and rearrangement of terms yields the following relations:

- Rotation equation for nest  $k-1$ ,  $k \in S_R$

$$\begin{aligned}
& \sum_{\substack{i \in S_{0,k-1} \\ i \neq k}} \left[ \sum_{\substack{\lambda \in S_{k-1} \\ \lambda \notin S_L}} \Phi_\lambda + \sum_{\lambda \in S_{k-1}} G_{k-1,i-1}^\lambda \right] \cdot \overset{\circ}{\vec{\omega}}_i \\
& + \sum_{\substack{i \in S_{k-1} \\ i \in S_R}} \left[ \sum_{\substack{\lambda \in S_{i-1} \\ \lambda \notin S_L}} \Phi_\lambda + \sum_{\lambda \in S_{i-1}} G_{k-1,i-1}^\lambda \right] \cdot \overset{\circ}{\vec{\omega}}_i \\
& + \sum_{\substack{\lambda \in S_{k-1} \\ \lambda \in S_L}} \left[ -m_\lambda \Gamma_{k-1,\lambda} \cdot \overset{\circ}{\vec{\alpha}}_\lambda \right] + \sum_{\lambda \in S_{k-1}} \left[ -m_\lambda \Gamma_{k-1,\lambda} \cdot \overset{\circ}{\vec{\beta}}_1 \right] \\
& + \sum_{\substack{i \in S_{0,k-1} \\ i \neq k \\ i \neq 1}} \left[ \sum_{\lambda \in S_{k-1}} -m_\lambda \Gamma_{k-1,\lambda} \cdot \overset{\circ}{\vec{\beta}}_i \right] + \sum_{i \in S_{k-1}} \left[ \sum_{\lambda \in S_{i-1}} -m_\lambda \Gamma_{k-1,\lambda} \cdot \overset{\circ}{\vec{\beta}}_i \right] \\
& + \sum_{\lambda \in S_{k-1}} \left[ \sum_{\substack{m: \\ MO(m)=\lambda}} I_{W_m} \cdot \overset{\circ}{\vec{\omega}}_{W_m} + \sum_{\substack{n,\lambda \\ \lambda \in S_F}} \left( -m_\lambda \Gamma_{k-1,\lambda} \cdot \vec{A}_{n,\lambda} + \vec{Q}_{n,\lambda} \right) \ddot{a}_{n,\lambda}(t) \right] \\
& = - \sum_{\lambda \in S_{k-1}} \left[ \overset{\circ}{\Phi}_\lambda \cdot \vec{\omega}_\lambda + \vec{\omega}_\lambda \times \vec{L}_{\lambda,\lambda} + \vec{\gamma}_{k-1,\lambda} \times \vec{C}_\lambda \right] \\
& + \vec{\phi}_{k-1}^H + \vec{\phi}_{k-1}^c + \sum_{\lambda \in S_{k-1}} \left[ \vec{\gamma}_{k-1,\lambda} \times \vec{F}_\lambda^{(e)} + \vec{\phi}_\lambda^{(e)} \right]
\end{aligned} \tag{107}$$

- Translation equation for nest  $k-1$ ,  $k \in S_R$  and  $k \in S_L$

$$\begin{aligned}
& \sum_{\lambda \in S_{k-1}} m_\lambda \mathbf{1} \cdot \ddot{\vec{\beta}}_1 + \sum_{\substack{\lambda \in S_{k-1} \\ \lambda \in S_L}} m_\lambda \mathbf{1} \cdot \overset{\circ}{\vec{\alpha}}_\lambda + \sum_{\substack{\lambda \in S_{k-1} \\ \lambda \in S_F}} \sum_{n,\lambda} m_\lambda \mathbf{1} \cdot \vec{A}_{n,\lambda} \ddot{a}_{n,\lambda}(t) \\
& + \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq k}} \left[ \sum_{\lambda \in S_{k-1}} m_\lambda \Gamma_{i-1,\lambda} \cdot \dot{\vec{\omega}}_i + \sum_{\substack{\lambda \in S_{k-1} \\ i \neq 1}} m_\lambda \mathbf{1} \cdot \overset{\circ}{\vec{\beta}}_i \right] \\
& + \sum_{i \in S_{k-1}} \left[ \sum_{\lambda \in S_{i-1}} m_\lambda \Gamma_{i-1,\lambda} \cdot \dot{\vec{\omega}}_i + \sum_{\lambda \in S_{i-1}} m_\lambda \mathbf{1} \cdot \overset{\circ}{\vec{\beta}}_i \right] \\
& = - \sum_{\lambda \in S_{k-1}} \vec{C}_\lambda + \vec{F}_{k-1}^H + \vec{F}_{k-1}^c + \sum_{\lambda \in S_{k-1}} \vec{F}_\lambda^{(e)} \tag{108}
\end{aligned}$$

- Symmetric wheel equation for wheel  $m$ ,  $\lambda = MO(m)$

$$\sum_{i \in S_{0,\lambda-1}} I_{W_m} \cdot \dot{\vec{\omega}}_i + I_{W_m} \cdot \overset{\circ}{\vec{\omega}}_{W_m} = -\vec{\omega}_\lambda \times \vec{H}_m + \vec{C}\vec{L}_m + \vec{\phi}_{W_m}^c \tag{109}$$

- Generalized displacement coordinate equation for mode  $n$  of body  $\lambda$ ,  $\lambda \in S_F$

$$\begin{aligned}
& \sum_{i \in S_{0,\lambda-1}} \left( -m_\lambda \Gamma_{i-1,\lambda} \cdot \vec{A}_{n,\lambda} + \vec{Q}_{n,\lambda} \right) \cdot \dot{\vec{\omega}}_i + m_\lambda \vec{A}_{n,\lambda} \cdot \ddot{\vec{\beta}}_i \\
& + m_\lambda \ddot{a}_{n,\lambda}(t) + m_\lambda \vec{A}_{n,\lambda} \cdot \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \overset{\circ}{\vec{\beta}}_i \\
& = -m_\lambda \left[ 2\zeta \omega_{n,\lambda} \dot{a}_{n,\lambda}(t) + \omega_{n,\lambda}^2 a_{n,\lambda}(t) \right] \\
& - m_\lambda \left[ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left( \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) + 2\vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \right) + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \overset{\circ}{\vec{\alpha}}_\lambda \right) \right] \cdot \vec{A}_{n,\lambda} \\
& + \vec{\omega}_\lambda \cdot \left[ \vec{H}_{n,\lambda} + \sum_{m,\lambda} \vec{F}_{m,n,\lambda} a_{m,\lambda}(t) \right] \cdot \vec{\omega}_\lambda - 2\vec{\omega}_\lambda \cdot \sum_{m,\lambda} \dot{a}_{m,\lambda}(t) \vec{K}_{m,n,\lambda} \tag{110}
\end{aligned}$$



## PARTITIONED MATRIX FORM OF EQUATIONS OF MOTION

The preceding equations may now be put into a compact partitioned matrix format. To do this, the number of bodies, wheels, and flexible modes must be known. Let

- N = total number of rigid bodies, flexible bodies, and point masses,
- M = total number of symmetric wheels,
- $N_m$  = total number of flexible body modes, and
- $N_R$  = total number of rigid and flexible bodies.

By inspection,\* it is readily seen that the equations of motion, equations 107 through 110, may all be expressed by the single partitioned matrix equation

$$\begin{bmatrix} \mathbf{X} & \mathbf{I}^c & \mathbf{F} & \mathbf{\Gamma} \\ \mathbf{I}^{cT} & \mathbf{I}^s & \mathbf{0} & \mathbf{0} \\ \mathbf{F}^T & \mathbf{0} & \mathbf{M}^1 & \mathbf{A} \\ \mathbf{\Gamma}^T & \mathbf{0} & \mathbf{A}^T & \mathbf{M}^2 \end{bmatrix} \cdot \begin{bmatrix} \ddot{\tilde{\omega}} \\ \ddot{\tilde{\omega}}_w \\ \ddot{\mathbf{a}} \\ \ddot{\beta} \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} + \begin{bmatrix} \phi^c \\ \phi_w^c \\ \mathbf{0} \\ \mathbf{F}^c \end{bmatrix} + \begin{bmatrix} \phi^{(e)} \\ \mathbf{CL} \\ \mathbf{0} \\ \mathbf{F}^{(e)} \end{bmatrix} \quad (111)$$

where the coefficient matrix on the left-hand side of the equation is symmetric (post superscript T implies “transpose”) and the following definitions are applied:

- $$\mathbf{X} \cdot \ddot{\tilde{\omega}} + \mathbf{I}^c \cdot \ddot{\tilde{\omega}}_w + \mathbf{F}\ddot{\mathbf{a}} + \mathbf{\Gamma} \cdot \ddot{\beta} = \eta_1 + \phi^c + \phi^{(e)} \quad (112)$$

This is an  $(N+1) \times 1$  column matrix of vector equations defining the equation of rotation for each nest containing one or more rigid or flexible bodies, the equation of translation for each nest containing a single point mass, and the equation of translation for the composite system.

- $$\mathbf{I}^{cT} \cdot \ddot{\tilde{\omega}} + \mathbf{I}^s \cdot \ddot{\tilde{\omega}}_w = \eta_2 + \phi_w^c + \mathbf{CL} \quad (113)$$

This is an  $M \times 1$  column matrix of vector equations defining the equation of motion for each symmetric wheel.

- $$\mathbf{F}^T \cdot \ddot{\tilde{\omega}} + \mathbf{M}^1 \ddot{\mathbf{a}} + \mathbf{A} \cdot \ddot{\beta} = \eta_3 \quad (114)$$

This is an  $N_m \times 1$  column matrix of scalar equations defining the generalized displacement coordinate equation for each mode of vibration of each flexible body.

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\*The reader is referred to the section “Equations of Motion (Matrix Form)” in reference 1 for supportive material.

- $$\Gamma^T \cdot \dot{\tilde{\omega}} + A^T \ddot{a} + M^2 \cdot \ddot{\beta} = \eta_4 + F^c + F^{(e)} \quad (115)$$

This is an  $(N_R - 1) \times 1$  column matrix of vector equations defining the translation equation for each nest containing one or more rigid or flexible bodies but not the translation equation for the composite system. Since relative translation is, by definition, not permitted between contiguous rigid or flexible bodies, these equations do not introduce any independent state variables. Their inclusion, however, provides certain computationally useful symmetry relations and the equations which define the unknown forces of constraint acting between contiguous rigid or flexible bodies.

The partitioned matrix equation defined above makes use of an expanded set of state variables which are not all independent. In order to arrive at a set of simultaneous equations which will not lead to singular coefficient matrices, a set of linearly independent state variables and a rectangular linear transformation matrix must be defined.

Throughout the development, an attempt has been made to separate those effects which are obtained by a straightforward coupled rigid body analysis from those which must be added on to account for the effects of body flexibility. Accordingly, it is convenient to choose as the set of independent state variables those rate vectors which define the relative motion of contiguous bodies at hinge points and the generalized displacement coordinate rates which define the effects of flexible body vibration. It should be noted that a somewhat loose definition of independence is being employed at this point as the relative rate vectors used coalesce as many as three degrees of relative freedom into a single independent state variable.

Making use of previously made definitions, the various elements of the expanded set of state variables may be expressed as follows:

- Relative angular rate between the reference frame fixed in body  $\lambda$  and the reference frame fixed in flexible body  $J(\lambda)$

$$\tilde{\omega}_\lambda = {}^\circ\tilde{\omega}_\lambda + \Delta\tilde{\theta}_\lambda \quad \lambda \in S_R \quad (116)$$

and

$$\tilde{\omega}_\lambda = \Delta\tilde{\theta}_\lambda \quad \lambda \in S_L \quad (117)$$

where

$$\Delta\tilde{\theta}_\lambda = \sum_{n, J(\lambda)} \dot{a}_{n, J(\lambda)}(t) \vec{\varphi}_{n, H(\lambda-1), J(\lambda)}^R \quad (118)$$

$\vec{\varphi}_{n,H(\lambda-1),J(\lambda)}^R = nth$  normal mode rotation vector at hinge point  $\lambda-1$  on the flexible body  $J(\lambda)$ ,

and

$$\dot{\vec{\omega}}_{\lambda} = {}^{\circ}\dot{\vec{\omega}}_{\lambda} + \Delta\dot{\vec{\theta}}_{\lambda} + \vec{\omega}_{\lambda} \times \Delta\dot{\vec{\theta}}_{\lambda} . \quad (119)$$

- Relative translation rates

For all point masses contiguous to either rigid or flexible bodies, the relative translation rate

$$\dot{\vec{\alpha}}_{\lambda} \quad \lambda \in S_L$$

is an independent state variable. Similarly, the translation rate of hinge point 0 relative to the inertial reference

$$\dot{\vec{\beta}}_1$$

is also an independent state variable.

- Relative angular rate between symmetric wheel  $m$  and flexible body  $MO(m)$  fixed reference frames

$$\dot{\vec{\omega}}_{W_m} = {}^{\circ}\dot{\vec{\omega}}_{W_m} + \Delta\dot{\vec{\theta}}_{W_m}, \quad (120)$$

where

$$\Delta\dot{\vec{\theta}}_{W_m} = \sum_{n, MO(m)} \dot{a}_{n,MO(m)}(t) \vec{\varphi}_{n,W(m),MO(m)}^R, \quad (121)$$

$\vec{\varphi}_{n,W(m),MO(m)}^R = nth$  normal mode rotation vector at the attachment point of the wheel  $m$  in the flexible body  $MO(m)$ ,

and

$$\dot{\vec{\omega}}_{W_m} = {}^{\circ}\dot{\vec{\omega}}_{W_m} + \Delta\dot{\vec{\theta}}_{W_m}. \quad (122)$$

- Translational rate of hinge point  $\lambda-1$  relative to its undeformed position in flexible body  $J(\lambda)$

$$\dot{\vec{\beta}}_{\lambda} = \Delta\dot{\vec{\beta}}_{\lambda} \quad (123)$$

where

$$\Delta \vec{\beta}_\lambda^{\circ} = \sum_{n, J(\lambda)} \dot{a}_{n, J(\lambda)}(t) \vec{\varphi}_{n, H(\lambda-1), J(\lambda)}^T, \quad (124)$$

$\vec{\varphi}_{n, H(\lambda-1), J(\lambda)}^T$  =  $n$ th normal mode displacement vector at hinge point  $\lambda-1$  in the flexible body  $J(\lambda)$ , and

$$\vec{\beta}_\lambda^{\circ} = \Delta \vec{\beta}_\lambda^{\circ}. \quad (125)$$

Combining the above definitions, it is possible to write the following rectangular linear transformation matrix equation that relates the expanded set of state variables to the chosen set of linearly independent state variables. That is, in partitioned matrix form,

$$\begin{Bmatrix} \dot{\lambda} \\ \dot{\lambda}_W \\ \dot{a} \\ \beta^{\circ} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & \varphi_H^R \\ 0 & 1 & \varphi_W^R \\ 0 & 0 & 1 \\ 0 & 0 & \varphi_H^T \end{bmatrix} \cdot \begin{Bmatrix} \dot{\lambda} \\ \dot{\lambda}_W \\ \dot{a} \end{Bmatrix} \quad (126)$$

and

$$\begin{Bmatrix} \dot{\lambda} \\ \dot{\lambda}_W \\ \dot{a} \\ \beta^{\circ} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & \varphi_H^R \\ 0 & 1 & \varphi_W^R \\ 0 & 0 & 1 \\ 0 & 0 & \varphi_H^T \end{bmatrix} \cdot \begin{Bmatrix} \dot{\lambda} \\ \dot{\lambda}_W \\ \dot{a} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & \omega \times \varphi_H^R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} \dot{\lambda} \\ \dot{\lambda}_W \\ \dot{a} \end{Bmatrix}. \quad (127)$$

Direct substitution of equation 127 into 111 yields the final form of the simultaneous vector-dyadic equations of motion for the total system and the equations which define the internal forces of constraint. That is,

$$\begin{bmatrix} \mathbf{X} & \mathbf{I}^c & \mathbf{F} & \mathbf{\Gamma} \\ \mathbf{I}^{cT} & \mathbf{I}^s & \mathbf{0} & \mathbf{0} \\ \mathbf{F}^T & \mathbf{0} & \mathbf{M}^1 & \mathbf{A} \\ \mathbf{\Gamma}^T & \mathbf{0} & \mathbf{A}^T & \mathbf{M}^2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{1} & \mathbf{0} & \varphi_H^R \\ \mathbf{0} & \mathbf{1} & \varphi_W^R \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \varphi_H^T \end{bmatrix} \cdot \left\{ \begin{array}{c} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \\ \mathbf{a} \end{array} \right\} = \left\{ \begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{array} \right\}$$

$$\begin{bmatrix} \mathbf{X} & \mathbf{I}^c & \mathbf{F} & \mathbf{\Gamma} \\ \mathbf{I}^{cT} & \mathbf{I}^s & \mathbf{0} & \mathbf{0} \\ \mathbf{F}^T & \mathbf{0} & \mathbf{M}^1 & \mathbf{A} \\ \mathbf{\Gamma}^T & \mathbf{0} & \mathbf{A}^T & \mathbf{M}^2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{0} & \mathbf{0} & \omega \times \varphi_H^R \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \cdot \left\{ \begin{array}{c} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \\ \dot{\omega}_4 \\ \mathbf{a} \end{array} \right\} + \left\{ \begin{array}{c} \phi^c \\ \phi_W^c \\ \mathbf{0} \\ \mathbf{F}^c \end{array} \right\} + \left\{ \begin{array}{c} \phi^{(e)} \\ \mathbf{CL} \\ \mathbf{0} \\ \mathbf{F}^{(e)} \end{array} \right\} \quad (128)$$

By leaving the equations in this form it is obvious that the effects of body flexibility may be taken into account by simply adding a few extra terms to the coupled rigid body equations. In practical applications, a simplified coupled rigid body model is usually the starting point. At this level there are no flexible modes of vibration and the equations reduce to those derived in reference 1. In the next step, some of the bodies are taken to be flexible, but hinge points and wheel attachment points are assumed to be either at node points of flexible bodies or in rigid bodies. At this level the elements of the partitions  $\varphi_H^R$ ,  $\varphi_W^R$ , and  $\varphi_H^T$  are all zero. In the final step of analysis, both hinge points and wheel attachment points are subject to the effects of flexibility. At this level all partitions have nontrivial entries and must be included. Computationally, the above form is advantageous since simple logic flags can be created from an interrogation of the input data. These can then be used to determine whether various computation loops should or should not be executed.

## ELIMINATION OF CONSTRAINT TORQUES

To solve the coupled vector-dyadic equations of motion, the unknown constraint torques must be deleted and the equations reduced to a set of simultaneous scalar equations. The procedure is identical to that used in reference 1.

At every hinge point, free coordinate vectors are defined. These define the vector directions about or along which relative motion of contiguous bodies is allowed (free). These may be sequentially indexed and the relative angular velocity vectors expressed as

$$\dot{\varepsilon}_k^{\rightarrow} = \sum_{m @ k-1} \dot{\theta}_m \vec{q}_m \quad (129)$$

where

$\vec{q}_m$  = free coordinate vector defined at hinge point k-1,

$\dot{\theta}_m$  = scalar rate of body k relative to body J(k) about or along the free vector  $\vec{q}_m$ , and

$\sum_{m@k-1}$  = sum over all free coordinate indices m, defined at hinge point k-1.

Differentiation of equation 129 yields

$${}^{\circ}\dot{\tilde{\omega}}_k = \sum_{m@k-1} \left[ \ddot{\theta}_m \vec{q}_m + \dot{\theta}_m \dot{\vec{q}}_m \right] . \quad (130)$$

Similarly, for symmetric wheels we may write

$${}^{\circ}\tilde{\omega}_{w_m} = \ddot{\theta}_j \vec{q}_j , \quad (131)$$

and for point masses,

$${}^{\circ\circ}\ddot{\vec{a}}_k = \sum_{m@k-1} \ddot{\theta}_m \vec{q}_m . \quad (132)$$

A partitioned rectangular matrix can be defined such that the system state vector can be rewritten as

$$\begin{Bmatrix} {}^{\circ}\tilde{\omega} \\ {}^{\circ}\tilde{\omega}_w \\ \ddot{a} \end{Bmatrix} = \begin{bmatrix} q & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{\theta}_w \\ \dot{a} \end{Bmatrix} \quad (133)$$

and

$$\begin{Bmatrix} {}^{\circ}\tilde{\omega} \\ {}^{\circ\circ}\tilde{\omega}_w \\ \ddot{a} \end{Bmatrix} = \begin{bmatrix} q & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\theta}_w \\ \ddot{a} \end{Bmatrix} + \begin{bmatrix} \dot{q} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{\theta}_w \\ \dot{a} \end{Bmatrix} . \quad (134)$$

Furthermore, at each hinge point, the free coordinate vectors are orthogonal to the constraint torque, hence

$$\begin{bmatrix} \mathbf{q}^T & 0 & 0 & 0 \\ 0 & \mathbf{h}^T & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} \phi^c \\ \phi_W^c \\ 0 \\ \mathbf{F}^c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{F}^c \end{Bmatrix} \quad (135)$$

In a manner identical to that used in reference 1, the following set of simultaneous scalar equations is obtained by direct substitution of equation 134 into 128 and application of the orthogonality relation defined above in equation 135:

$$\begin{aligned} & \begin{bmatrix} \mathbf{q}^T & 0 & 0 & 0 \\ 0 & \mathbf{h}^T & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X} & \mathbf{I}^c & \mathbf{F} & \Gamma \\ \mathbf{I}^{cT} & \mathbf{I}^s & 0 & 0 \\ \mathbf{F}^T & 0 & \mathbf{M}^1 & \mathbf{A} \\ \Gamma^T & 0 & \mathbf{A}^T & \mathbf{M}^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \phi_H^R \\ 0 & 1 & \phi_W^R \\ 0 & 0 & 1 \\ 0 & 0 & \phi_H^T \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q} & 0 & 0 \\ 0 & \mathbf{h} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\theta}_W \\ \ddot{\mathbf{a}} \end{Bmatrix} \\ & = \begin{bmatrix} \mathbf{q}^T & 0 & 0 & 0 \\ 0 & \mathbf{h}^T & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \left\{ - \begin{bmatrix} \mathbf{X} & \mathbf{I}^c & \mathbf{F} & \Gamma \\ \mathbf{I}^{cT} & \mathbf{I}^s & 0 & 0 \\ \mathbf{F}^T & 0 & \mathbf{M}^1 & \mathbf{A} \\ \Gamma^T & 0 & \mathbf{A}^T & \mathbf{M}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \phi_H^R \\ 0 & 1 & \phi_W^R \\ 0 & 0 & 1 \\ 0 & 0 & \phi_H^T \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{\theta}_W \\ \dot{\mathbf{a}} \end{Bmatrix} \right. \\ & \left. - \begin{bmatrix} \mathbf{X} & \mathbf{I}^c & \mathbf{F} & \Gamma \\ \mathbf{I}^{cT} & \mathbf{I}^s & 0 & 0 \\ \mathbf{F}^T & 0 & \mathbf{M}^1 & \mathbf{A} \\ \Gamma^T & 0 & \mathbf{A}^T & \mathbf{M}^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & \omega \times \phi_H^R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \omega \\ \omega_W \\ \dot{\mathbf{a}} \end{Bmatrix} \right. \\ & \left. + \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{Bmatrix} + \begin{Bmatrix} \phi^{(e)} \\ \text{CL} \\ 0 \\ \mathbf{F}^{(e)} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{F}^c \end{Bmatrix} \right\}, \quad (136) \end{aligned}$$

where the last  $N_R - 1$  vector equations of the partitioned matrix equation (136) may be evaluated if the forces of constraint between contiguous rigid or flexible bodies must be known.

Equation 136 implies that a horrendous amount of matrix multiplication must be carried out. From a computational standpoint, however, the matrices are sparse, and relatively simple logic statements can be created which will eliminate all unnecessary numerical multiplications by zero.

## PREDEFINED RELATIVE MOTION OF CONTIGUOUS BODIES

In practical application, it is frequently desirable to model an appendage deployment or the relative motion of a scanning antenna without introducing the details of a high frequency onboard control system. The ability to do this often results in an enormous saving of computer simulation time, since the high frequency loops of the control system need not be modeled. The solution method is rather straightforward and can readily be outlined in general terms.

The simultaneous vector-dyadic equations which define the motion of the system of coupled bodies take on the general form

$$[Z] \{ \tilde{\omega} \} = \{ \mu \} + \{ \phi^c \} \quad (137)$$

where, from equation 128,

$$[Z] = \begin{bmatrix} X & I^c & F + X \cdot \varphi_H^R + I^c \cdot \varphi_W^R + \Gamma \cdot \varphi_H^T \\ I^{cT} & I^s & I^{cT} \cdot \varphi_H^R + I^s \cdot \varphi_W^R \\ F^T & 0 & M^1 + F^T \cdot \varphi_H^R + A \cdot \varphi_H^T \end{bmatrix}, \quad (138)$$

$$\{ \tilde{\omega} \} = \begin{Bmatrix} \overset{\circ}{\tilde{\omega}} \\ \overset{\circ}{\tilde{\omega}}_W \\ \ddot{a} \end{Bmatrix}, \quad (139)$$

$$\{ \phi^c \} = \begin{Bmatrix} \phi^c \\ \phi_W^c \\ 0 \end{Bmatrix}, \quad (140)$$



and

$$\left\{ \mu \right\} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} + \begin{Bmatrix} \phi^{(e)} \\ \text{CL} \\ 0 \end{Bmatrix} - \begin{Bmatrix} \mathbf{X} \cdot (\omega \times \phi_H^R) \dot{\mathbf{a}} \\ \mathbf{I}^{cT} \cdot (\omega \times \phi_H^R) \dot{\mathbf{a}} \\ \mathbf{F}^T \cdot (\omega \times \phi_H^R) \dot{\mathbf{a}} \end{Bmatrix} \quad (141)$$

If one or more of the bodies of the system has its relative motion defined, the number of independent state variables is reduced, additional unknown constraint torques are introduced, and special solution techniques must be used. Let

$\dot{\tilde{\omega}}_c$  = column matrix of all relative acceleration vectors which are given by known functions of time and state,

$\dot{\tilde{\omega}}_F$  = column matrix of all independent relative acceleration vectors, and

[R] = reordering matrix such that

$$\left\{ \dot{\tilde{\omega}} \right\} = [\mathbf{R}] \begin{Bmatrix} \dot{\tilde{\omega}}_F \\ \dot{\tilde{\omega}}_c \end{Bmatrix} \quad (142)$$

Direct substitution of the identity into the basic form and premultiplication by the transpose of the reordering matrix yields

$$\begin{bmatrix} \mathbf{Z}_{11}^* & \mathbf{Z}_{12}^* \\ \mathbf{Z}_{21}^* & \mathbf{Z}_{22}^* \end{bmatrix} \begin{Bmatrix} \dot{\tilde{\omega}}_F \\ \dot{\tilde{\omega}}_c \end{Bmatrix} = \begin{Bmatrix} \mu_1^* \\ \mu_2^* \end{Bmatrix} + \begin{Bmatrix} \phi_F^* \\ \phi_c^* \end{Bmatrix} \quad (143)$$

where

$\left\{ \phi_F^* \right\}$  = column matrix of constraint torques acting between rigid or flexible bodies and constraint forces acting between point masses and their contiguous bodies at hinge points at which relative motion is undefined, and

$\left\{ \phi_c^* \right\}$  = column matrix of constraint torques acting between rigid or flexible bodies and constraint forces acting between point masses and their contiguous bodies at hinge points at which relative motion is defined.

Since all elements of the column matrix

$$\left\{ \dot{\tilde{\omega}}_c \right\}$$

are given functions of time and state, the equations of motion for the coupled body system are defined by

$$\left[ \mathbf{Z}_{11}^* \right] \cdot \left\{ \dot{\tilde{\omega}}_F \right\} = \left\{ \mu_1^* \right\} + \left\{ \phi_F^* \right\} - \left[ \mathbf{Z}_{12}^* \right] \cdot \left\{ \dot{\tilde{\omega}}_c \right\}, \quad (144)$$

and, if desired, the unknown constraint torques can be obtained from the equation

$$\left\{ \phi_c^* \right\} = \left[ \mathbf{Z}_{21}^* \right] \cdot \left\{ \dot{\tilde{\omega}}_F \right\} + \left[ \mathbf{Z}_{22}^* \right] \cdot \left\{ \dot{\tilde{\omega}}_c \right\} - \left\{ \mu_2^* \right\}. \quad (145)$$

## FLEXIBLE VERSUS RIGID BODY MODELING

In establishing a representative model of a spacecraft with flexible appendages, the analyst must decide whether to model each appendage as a spring-connected rigid body or as a flexible body having a finite number of significant modes of vibration.

Neither modeling technique is perfect. Both have advantages and limitations which, if not taken into account, can severely compromise the conclusions drawn from the simulation model. It is possible, by a detailed study of the analysis presented in this report, to assess the analyst's choice of modeling technique. This is done by deriving a series of equivalence relations which define how the two modeling approaches describe the fundamental dynamic characteristics of the elastic appendage and their dynamic coupling into the system equations of motion.

Whether the analyst chooses to model a flexible appendage as a spring-connected rigid body or as a flexible body, certain fundamental dynamic effects exist and must be properly accounted for in the system equations of motion. These fundamental dynamic effects are independent of the choice of modeling technique; hence, the terms which simulate them, as derived by the two modeling approaches, are equivalent.

If the topological tree of connected bodies has a flexible body which is not at a limb end, terms appear in the equations of motion which cannot be simulated by a simple spring-connected rigid body model of that flexible body. These terms are related to the fact that coordinate frames at the hinge points on the flexible body are not fixed relative to each other.

The most common problem the spacecraft analyst is faced with, however, is the modeling of a flexible appendage, such as a boom, antenna, or solar array, which is at a limb end in the topological tree model. By making use of equations 107 through 110 and equation 136, the arbitrary body, body  $\lambda$ , may first be modeled as a spring-connected rigid body and then as a flexible body having several normal modes of vibration. In a step-by-step manner, terms which model identical dynamic effects may be equivalenced to each other, and the desired equivalence relations, which provide the measure of comparison between techniques, may be obtained. In the following development, body  $\lambda$  is an arbitrary body at a limb end.

If body  $\lambda$  is modeled as a rigid body, and if its contiguous body, body  $J(\lambda)$ , is a rigid body, then, from equations 116 and 129, its relative angular velocity is defined by the vector equation

$$\vec{\omega}_\lambda = \sum_{m@k-1} \dot{\theta}_m \vec{q}_m \quad (146)$$

The angular velocity of body  $\lambda$ , relative to the inertial frame, is given by

$$\vec{\omega}_\lambda = \vec{\omega}_{J(\lambda)} + \sum_{m@k-1} \dot{\theta}_m \vec{q}_m \quad (147)$$

If body  $\lambda$  is modeled as a flexible body which is clamped to rigid body  $J(\lambda)$ , the angular velocity of its body-fixed reference frame, relative to the inertial frame, is

$$\vec{\omega}_\lambda = \vec{\omega}_{J(\lambda)} \quad (148)$$

It will be shown that there are three fundamental dynamic characteristics of the body  $\lambda$  which must be modeled. The modeling technique which best simulates these three characteristics will provide the most accurate simulation model. The fundamental dynamic characteristics are as follows:

- Relative linear velocity of the center-of-mass of body  $\lambda$

The linear velocity of the center-of-mass of body  $\lambda$  relative to a reference frame fixed in the contiguous body, body  $J(\lambda)$ , must be modeled by both techniques. The equivalence relation is given by

$$\vec{\omega}_\lambda \times \vec{a}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda} \quad (149)$$

where the left-hand side pertains to a rigid body simulation of body  $\lambda$  and the right-hand side to a flexible body simulation of the same body  $\lambda$ .

- Relative derivative of the inertia tensor of body  $\lambda$

The time derivative of the inertia tensor of body  $\lambda$  relative to the reference frame fixed in body  $J(\lambda)$  must be modeled by both techniques. The equivalence relation is given by

$$\vec{\omega}_\lambda \times \Phi_\lambda - \Phi_\lambda \times \vec{\omega}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \mathbf{E}_{n,\lambda} \quad (150)$$

- Relative angular momentum of body  $\lambda$  about its center-of-mass

The angular momentum of body  $\lambda$  about its own center-of-mass relative to the reference frame fixed in body  $J(\lambda)$  must be modeled by both techniques. The equivalence relation is given by

$$\Phi_\lambda \cdot \vec{\omega}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{Q}_{n,\lambda} \cdot \quad (151)$$

To arrive at relations 149 through 151, the system equations of motion, equations 107 through 110 and 136, are examined in detail.

The rotational motion of all nests containing at least one rigid or flexible body is given by equation 107. If body  $\lambda$  is at a limb end in the nest  $k-1$ , the form of the cross-coupling term on the left-hand side of the equation is dependent upon how body  $\lambda$  is modeled.

On the left-hand side of equation 107, the cross-coupling term which is associated with a rigid body model of body  $\lambda$  is

$$\left[ \Phi_\lambda + \mathbf{G}_{k-1,\lambda}^\lambda \right] \cdot \dot{\vec{\omega}}_\lambda \quad (152)$$

where

$$\dot{\vec{\omega}}_\lambda = \dot{\vec{\omega}}_\lambda^\circ + \left( \vec{\omega}_{J(\lambda)} + \vec{\omega}_\lambda \right) \times \vec{\omega}_\lambda = \dot{\vec{\omega}}_\lambda^\circ + \vec{\omega}_{J(\lambda)} \times \vec{\omega}_\lambda \cdot \quad (153)$$

If body  $\lambda$  is treated as a flexible body clamped to body  $J(\lambda)$ , the cross-coupling term on the left-hand side of equation 107 is

$$\sum_{n,\lambda} \left[ -m_\lambda \Gamma_{k-1,\lambda} \cdot \vec{A}_{n,\lambda} + \vec{Q}_{n,\lambda} \right] \ddot{a}_{n,\lambda}(t) \cdot \quad (154)$$

The equivalence relation for the cross-coupling term on the left-hand side of equation 136 is

$$\left[ \Phi_\lambda + \mathbf{G}_{k-1,\lambda}^\lambda \right] \cdot \dot{\vec{\omega}}_\lambda \approx \sum_{n,\lambda} \left[ -m_\lambda \Gamma_{k-1,\lambda} \cdot \vec{A}_{n,\lambda} + \vec{Q}_{n,\lambda} \right] \ddot{a}_{n,\lambda}(t) \quad (155)$$

where the term proportional to relative angular velocity is carried over to the right-hand side of equation 136.

For small angle relative motion,

$$\overset{\circ}{\omega}_\lambda = \sum_{m@k-1} \ddot{\theta}_m \vec{q}_m \quad (156)$$

and the relative angular velocity term

$$\vec{\omega}_{J(\lambda)} \times \vec{\omega}_\lambda = \sum_{m@k-1} \dot{\theta}_m \dot{\vec{q}}_m . \quad (157)$$

Making use of equations 79, 99, 104, and 106, a routine substitution into equation 155 yields

$$\Phi_\lambda \cdot \overset{\circ}{\omega}_\lambda + \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left( \overset{\circ}{\omega}_\lambda \times \vec{\alpha}_\lambda \right) \approx \sum_{n,\lambda} \vec{Q}_{n,\lambda} \ddot{a}_{n,\lambda}(t) + \vec{\gamma}_{k-1,\lambda} \times m_\lambda \sum_{n,\lambda} \ddot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda} , \quad (158)$$

which readily implies the following relations:

$$\Phi_\lambda \cdot \overset{\circ}{\omega}_\lambda \approx \sum_{n,\lambda} \vec{Q}_{n,\lambda} \ddot{a}_{n,\lambda}(t) \quad (159)$$

and

$$\vec{\omega}_\lambda \times \vec{\alpha}_\lambda \approx \sum_{n,\lambda} \vec{A}_{n,\lambda} \ddot{a}_{n,\lambda}(t) . \quad (160)$$

The terms on the right-hand side of equation 107, which are dependent upon body deformation and the inertial angular rates of the bodies contained within the nest  $k-1$ , are given by the expression

$$- \sum_{\lambda \in S_{k-1}} \left[ \overset{\circ}{\Phi}_\lambda \cdot \vec{\omega}_\lambda + \vec{\omega}_\lambda \times \vec{L}_{\lambda,\lambda} + \vec{\gamma}_{k-1,\lambda} \times \vec{C}_\lambda \right] . \quad (161)$$

If body  $\lambda$  is modeled as a rigid body, then the expression,

$$+ \sum_{\lambda \in S_{k-1}} \left[ \Phi_\lambda + \mathbf{G}_{k-1,\lambda}^\lambda \right] \cdot \left( \vec{\omega}_{J(\lambda)} \times \vec{\omega}_\lambda \right) , \quad (162)$$

appearing on the left-hand side of equation 107, is carried to the right-hand side of equation 136 and must be used for a determination of the equivalence relations.

Consider only the contribution of body  $\lambda$  contained in the nest  $k-1$ . If body  $\lambda$  is modeled as a rigid body, then it follows from equations 100, 101, 104, 146, and 147, along with the fact that body  $\lambda$  deformation is zero, that

$$\vec{\omega}_\lambda = \vec{\omega}_{J(\lambda)} + \vec{\tilde{\omega}}_\lambda, \quad (163)$$

$$\overset{\circ}{\vec{\alpha}}_\lambda = \dot{0}, \quad (164)$$

$$\overset{\circ}{\vec{\Phi}}_\lambda = 0, \quad (165)$$

$$\begin{aligned} \vec{C}_\lambda = m_\lambda \left\{ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left[ \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) + 2 \vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \right] \right. \\ \left. + \left( \vec{\omega}_{J(\lambda)} + \vec{\tilde{\omega}}_\lambda \right) \times \left( \left( \vec{\omega}_{J(\lambda)} + \vec{\tilde{\omega}}_\lambda \right) \times \vec{\alpha}_\lambda \right) \right\}, \end{aligned} \quad (166)$$

$$\Delta \vec{L}_{\lambda,\lambda} = 0, \quad (167)$$

$$\vec{L}_{\lambda,\lambda} = \Phi_\lambda \cdot \left( \vec{\omega}_{J(\lambda)} + \vec{\tilde{\omega}}_\lambda \right), \quad (168)$$

and

$$\begin{aligned} \left[ \Phi_\lambda + \mathbf{G}_{k-1,\lambda-1}^\lambda \right] \cdot \left( \vec{\omega}_{J(\lambda)} \times \vec{\tilde{\omega}}_\lambda \right) = \Phi_\lambda \cdot \left( \vec{\omega}_{J(\lambda)} \times \vec{\tilde{\omega}}_\lambda \right) \\ + \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left( \left( \vec{\omega}_{J(\lambda)} \times \vec{\tilde{\omega}}_\lambda \right) \times \vec{\alpha}_\lambda \right), \end{aligned} \quad (169)$$

where it is assumed (for simplicity) that there are no momentum wheels in body  $\lambda$ .

If body  $\lambda$  is modeled as a flexible body clamped to rigid body  $J(\lambda)$ , then it follows from equations 47, 49, 53, 100, 101, and 148 that

$$\vec{\omega}_\lambda = \vec{\omega}_{J(\lambda)}, \quad (170)$$

$$\overset{\circ}{\vec{\alpha}}_\lambda = \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda}, \quad (171)$$

$$\overset{\circ}{\vec{\Phi}}_\lambda = \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \mathbf{E}_{n,\lambda}, \quad (172)$$

$$\Delta \vec{L}_{\lambda,\lambda} = \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{Q}_{n,\lambda}, \quad (173)$$

$$\vec{C}_\lambda = m_\lambda \left\{ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left[ \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) + 2 \vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \right] \right. \\ \left. + \vec{\omega}_{J(\lambda)} \times \left( \vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda \right) + 2 \vec{\omega}_{J(\lambda)} \times \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda} \right\}, \quad (174)$$

and

$$\vec{L}_{\lambda,\lambda} = \Phi_\lambda \cdot \vec{\omega}_{J(\lambda)} + \sum_{n,\lambda} \dot{a}_{n,\lambda} \vec{Q}_{n,\lambda}. \quad (175)$$

The desired equivalence relation is obtained by a substitution of equations 163 through 169 into equation 161 minus 162 and equivalencing the result to that obtained by a substitution of equations 170 through 175 into equation 161 alone. The above substitutions, along with a slight rearrangement of terms and deletion of terms quadratic in deformation, yield:

$$\vec{\omega}_{J(\lambda)} \times \Phi_\lambda \cdot \vec{\omega}_{J(\lambda)} + \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left\{ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left[ \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) \right. \right. \\ \left. \left. + 2 \vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \right] + \vec{\omega}_{J(\lambda)} \times \left( \vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda \right) \right\} \\ + \vec{\omega}_{J(\lambda)} \times \left( \Phi_\lambda \cdot \vec{\omega}_\lambda \right) + \vec{\omega}_\lambda \times \left( \Phi_\lambda \cdot \vec{\omega}_{J(\lambda)} \right) + \Phi_\lambda \cdot \left( \vec{\omega}_{J(\lambda)} \times \vec{\omega}_\lambda \right) \\ + \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left[ \vec{\omega}_{J(\lambda)} \times \left( \vec{\omega}_\lambda \times \vec{\alpha}_\lambda \right) + \vec{\omega}_\lambda \times \left( \vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda \right) + \left( \vec{\omega}_{J(\lambda)} \times \vec{\omega}_\lambda \right) \times \vec{\alpha}_\lambda \right] \\ \approx \vec{\omega}_{J(\lambda)} \times \Phi_\lambda \cdot \vec{\omega}_{J(\lambda)} + \vec{\gamma}_{k-1,\lambda} \times m_\lambda \left\{ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left[ \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) \right. \right. \\ \left. \left. + 2 \vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \right] + \vec{\omega}_{J(\lambda)} \times \left( \vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda \right) \right\}$$

$$\begin{aligned}
& + 2\vec{\omega}_{J(i)} \times \overset{\circ}{\vec{\beta}}_i \Big] + \vec{\omega}_{J(\lambda)} \times \left( \vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda \right) \Big\} \\
& + \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \mathbf{E}_{n,\lambda} \cdot \vec{\omega}_{J(\lambda)} + \vec{\omega}_{J(\lambda)} \times \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{Q}_{n,\lambda} \\
& + \vec{\gamma}_{k-1,\lambda} \times \left( 2 m_\lambda \vec{\omega}_{J(\lambda)} \times \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda} \right) \tag{176}
\end{aligned}$$

By equivalencing equivalent dynamic effects in equation 176 on a term-by-term basis and applying the vector-tensor identities of the Appendix, the following three equivalence equations are obtained:

$$\vec{\omega}_{J(\lambda)} \times \left( \Phi_\lambda \cdot \vec{\omega}_\lambda \right) \approx \vec{\omega}_{J(\lambda)} \times \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{Q}_{n,\lambda}, \tag{177}$$

$$\begin{aligned}
& \vec{\omega}_\lambda \times \left( \Phi_\lambda \cdot \vec{\omega}_{J(\lambda)} \right) + \Phi_\lambda \cdot \left( \vec{\omega}_{J(\lambda)} \times \vec{\omega}_\lambda \right) \\
& = \left( \vec{\omega}_\lambda \times \Phi_\lambda - \Phi_\lambda \times \vec{\omega}_\lambda \right) \cdot \vec{\omega}_{J(\lambda)} \\
& \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \mathbf{E}_{n,\lambda} \cdot \vec{\omega}_{J(\lambda)}, \tag{178}
\end{aligned}$$

and

$$\begin{aligned}
& \vec{\omega}_{J(\lambda)} \times \left( \vec{\omega}_\lambda \times \vec{\alpha}_\lambda \right) + \vec{\omega}_\lambda \times \left( \vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda \right) + \left( \vec{\omega}_{J(\lambda)} \times \vec{\omega}_\lambda \right) \times \vec{\alpha}_\lambda \\
& = 2 \vec{\omega}_{J(\lambda)} \times \left( \vec{\omega}_\lambda \times \vec{\alpha}_\lambda \right) \\
& \approx 2 \vec{\omega}_{J(\lambda)} \times \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda}. \tag{179}
\end{aligned}$$

Direct elimination of common factors from both sides of equations 177, 178, and 179 yields the three fundamental equivalence relations given by equations 149, 150, and 151. That is,

$$\vec{\omega}_\lambda \times \vec{\alpha}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda}, \tag{149}$$



$$\vec{\omega}_\lambda \times \Phi_\lambda - \Phi_\lambda \times \vec{\omega}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \mathbf{E}_{n,\lambda}, \quad (150)$$

and

$$\Phi_\lambda \cdot \vec{\omega}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{Q}_{n,\lambda}. \quad (151)$$

In an analogous manner, equivalence relations can be derived from the translation equation (108) for the nest  $k-1$ . The equivalence relation for terms which appear on the left-hand side of equation 136 is

$$m_\lambda \Gamma_{\lambda-1,\lambda} \cdot \overset{\circ}{\vec{\omega}}_\lambda \approx m_\lambda \mathbf{1} \cdot \sum_{n,\lambda} \ddot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda}, \quad (180)$$

and the equivalence relation for the terms on the right-hand side of the equation is

$$\begin{aligned} & \left[ \vec{\omega}_{J(\lambda)} \times \left( \vec{\omega}_\lambda \times \vec{\alpha}_\lambda \right) + \vec{\omega}_\lambda \times \left( \vec{\omega}_{J(\lambda)} \times \vec{\alpha}_\lambda \right) + \left( \vec{\omega}_{J(\lambda)} \times \vec{\omega}_\lambda \right) \times \vec{\alpha}_\lambda \right] \\ &= 2 \vec{\omega}_{J(\lambda)} \times \left( \vec{\omega}_\lambda \times \vec{\alpha}_\lambda \right) \\ &\approx 2 \vec{\omega}_{J(\lambda)} \times \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda}. \end{aligned} \quad (181)$$

Direct cancellation of common factors in equations 180 and 181 leads to equations 160 and 149. That is,

$$\overset{\circ}{\vec{\omega}}_\lambda \times \vec{\alpha}_\lambda \approx \sum_{n,\lambda} \ddot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda} \quad (160)$$

and

$$\vec{\omega}_\lambda \times \vec{\alpha}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda} \quad (149)$$

where use of the fact that

$$\vec{\gamma}_{\lambda-1,\lambda} = \vec{\alpha}_\lambda \quad (182)$$

has been made.

If body  $\lambda$  is modeled as a rigid body, its equation of motion is the rotation equation for the nest  $\lambda-1$ , equation 107. This equation defines the rate of change of angular momentum of the body  $\lambda$  relative to the inertial origin.

If body  $\lambda$  is modeled as a flexible body clamped to rigid body  $J(\lambda)$ , its equation of motion is the set of simultaneous generalized displacement coordinate equations given by equation 110. These equations have the units of translation; however, the generalized displacement coordinates are used to define both the translational and rotational motion of all elements of the body  $\lambda$ .

Since the units of equation 107 and 110 are different, it is not possible to define equivalence relations. It should be noted, however, that the coefficient matrix on the left-hand side of equation 136 is symmetric for either modeling approach. Hence, the coupling coefficients appearing in the body  $\lambda$  equation also appear in the equation for each nest which contains body  $\lambda$ .

In the right-hand side of the deformation equation (110), a back substitution of original definitions provided by equations 48 and 65 yields

$$\begin{aligned}
& -m_\lambda \left[ \sum_{\substack{i \in S_{0,\lambda-1} \\ i \neq 1}} \left( \vec{\omega}_{J(i)} \times \left( \vec{\omega}_{J(i)} \times \vec{\beta}_i \right) + 2 \vec{\omega}_{J(i)} \times \overset{\circ}{\beta}_i \right) + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \overset{\circ}{\alpha}_\lambda \right) \right] \cdot \vec{A}_{n,\lambda} \\
& + \vec{\omega}_\lambda \cdot \left[ \mathbf{H}_{n,\lambda} + \sum_{m,\lambda} \mathbf{F}_{m,n,\lambda} a_{m,\lambda}(t) \right] \cdot \vec{\omega}_\lambda - 2\vec{\omega}_\lambda \cdot \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{K}_{m,n,\lambda} \\
= & - \sum_{i \in \lambda} \left\{ m_{i,\lambda} \left[ \sum_{\substack{j \in S_{0,\lambda-1} \\ j \neq 1}} \left( \vec{\omega}_{J(j)} \times \left( \vec{\omega}_{J(j)} \times \vec{\beta}_j \right) + 2 \vec{\omega}_{J(j)} \times \overset{\circ}{\beta}_j \right) \right. \right. \\
& \left. \left. + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \left( \overset{\circ}{r}_{i,\lambda} + \vec{e}_{i,\lambda} \right) \right) + 2 \vec{\omega}_\lambda \times \overset{\circ}{e}_{i,\lambda} \right] \cdot \vec{\varphi}_{n,i,\lambda}^T + \left[ \vec{\omega}_\lambda \times \left( \overset{\circ}{\Phi}_{i,i,\lambda} \cdot \overset{\circ}{\theta}_{i,\lambda} \right) \right] \cdot \vec{\varphi}_{n,i,\lambda}^R \right\}, \quad (183)
\end{aligned}$$

where

$$\begin{aligned}
& m_{i,\lambda} \left[ \sum_{\substack{j \in S_{0,\lambda-1} \\ j \neq 1}} \left( \vec{\omega}_{J(j)} \times \left( \vec{\omega}_{J(j)} \times \vec{\beta}_j \right) + 2 \vec{\omega}_{J(j)} \times \overset{\circ}{\beta}_j \right) \right. \\
& \left. + \vec{\omega}_\lambda \times \left( \vec{\omega}_\lambda \times \left( \overset{\circ}{r}_{i,\lambda} + \vec{e}_{i,\lambda} \right) \right) + 2 \vec{\omega}_\lambda \times \overset{\circ}{e}_{i,\lambda} \right] \quad (184)
\end{aligned}$$

is the force associated with the element  $i$  of body  $\lambda$  due to its centripetal and Coriolis acceleration, relative to an inertial reference, and

$$\vec{\omega}_\lambda \times \left( {}^o\Phi_{i,i,\lambda} \cdot \vec{\theta}_{i,\lambda} \right) \quad (185)$$

is the torque associated with the rate of change of relative angular momentum of the element  $i$  of body  $\lambda$  about its own center-of-mass due to the angular velocity of the body  $\lambda$  fixed reference frame.

The other term on the right-hand side of equation 110, namely

$$-m_\lambda \left[ 2\zeta\omega_{n,\lambda} \dot{a}_{n,\lambda}(t) + \omega_{n,\lambda}^2 a_{n,\lambda}(t) \right] \quad (186)$$

is simply the standard spring and viscous damping effects which are normally expected in the equations of vibration.

## INTERPRETATION OF RESULTS

The work presented in this document and in reference 1 provides a complete derivation of the equations of motion for a topological tree of coupled rigid bodies, flexible bodies, point masses, and symmetric momentum wheels. Furthermore, for the modeling of an elastic appendage, equivalence relations have been derived which define exactly how rigid body modeling and flexible body modeling procedures purport to describe identical dynamic effects.

From an analysis of the derived equations and a knowledge of the digital computation which will be required to obtain numerical solutions, several points should be discussed. These points are intended to provide the analyst with a set of benchmarks which may be used in the development of a simulation model for a particular problem.

### Problem Setup

The setup of the input data for a coupled rigid body model is extremely simple and can usually be done in a matter of minutes; furthermore, the input data consists of physically realizable quantities which can be readily checked. The setup of the input data for a coupled flexible body model is slightly more complex. Several mode-dependent parameters must be computed which do not have an obvious physical analogy. Depending upon the particular problem, these can be time consuming to compute and check.

### Computation Speed

The internal computation which must be carried out by the computer should increase roughly as the square of the number of bodies used. If only first-mode bending vibration is to be modeled, the internal computation which will be performed if the body is treated as a flexible body is roughly equivalent to that which will be done if it is treated as a spring-connected rigid body.

If higher mode vibrational response is to be modeled, the input of modal data for the flexible body model short circuits much of the internal machinery which must be gone through every integration step for a coupled rigid body model of the same appendage. In essence, a coupled rigid body model of a flexible body is forced to set up and solve an eigenvalue problem every integration step, while the flexible body model solves the problem once and uses the results to form the required input data. For this situation, flexible body modeling is usually mandatory from the standpoint of computation speed. One problem which should be recognized is that, for large angle elastic deformation, mode shapes and frequencies based upon linear elasticity can be an error source. A coupled rigid body model for this case may be necessary in spite of the impact upon computation speed.

### **Interpretation of Numerical Solutions**

The interpretation of the results of a coupled body simulation can at times be extremely difficult, especially if superfluous degrees of freedom are permitted. The reader should be cautioned about using high-degree-of-freedom simulation models. Not only is computation speed compromised, but a veil of noise can be created which will mask important effects such as slowly growing fundamental instabilities. These remarks apply equally to both rigid and flexible body simulation models.

### **Choice of Modeling Technique**

The analyst must inevitably decide if the particular problem at hand can be adequately simulated as a system of coupled rigid bodies or if some of the bodies must be treated as flexible bodies having several natural modes of vibration. Neither modeling approach is perfect.

The rigid body modeling approach permits only fundamental vibrational motion to be modeled. This is done by the introduction of a spring constraint which will yield the desired natural frequency of vibration for the spring-connected rigid body model. If this is acceptable, the equations as derived introduce no further assumptions; they are the exact nonlinear equations of motion of the defined coupled rigid body system.

The flexible body modeling approach permits any number of vibration modes to be used to describe the dynamic characteristics of the body. If this approach is used, all terms quadratic in appendage deformation are assumed of second order importance and are dropped from the system equations of motion. It is emphasized that this assumption perpetuates itself throughout the entire set of system equations. It does not, definitely not, localize itself in the generalized coordinate equations alone.

It is up to the analyst to decide whether it is more important for a particular application to include higher modes of vibration or to include terms which are nonlinear in appendage deformation. The simulation model may be provided with the capability to have one effect or the other but not both.

The detailed study of the equations of motion which result from both the rigid body and the flexible body modeling approach has provided three fundamental equivalence relations, namely:

$$\vec{\omega}_\lambda \times \vec{\alpha}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda}, \quad (149)$$

$$\vec{\omega}_\lambda \times \Phi_\lambda - \Phi_\lambda \times \vec{\omega}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \mathbf{E}_{n,\lambda}, \quad (150)$$

and

$$\Phi_\lambda \cdot \vec{\omega}_\lambda \approx \sum_{n,\lambda} \dot{a}_{n,\lambda}(t) \vec{Q}_{n,\lambda}. \quad (151)$$

In addition to these relations, the related supplementary relations

$$\vec{\alpha}_\lambda \approx \overset{\circ}{\alpha}_\lambda + \sum_{n,\lambda} a_{n,\lambda}(t) \vec{A}_{n,\lambda}, \quad (47)$$

$$\Phi_\lambda \approx \overset{\circ}{\Phi}_\lambda + \sum_{n,\lambda} a_{n,\lambda}(t) \mathbf{E}_{n,\lambda}, \quad (49)$$

$$\overset{\circ}{\omega}_\lambda \times \vec{\alpha}_\lambda \approx \sum_{n,\lambda} \ddot{a}_{n,\lambda}(t) \vec{A}_{n,\lambda}, \quad (160)$$

and

$$\Phi_\lambda \cdot \overset{\circ}{\omega}_\lambda \approx \sum_{n,\lambda} \ddot{a}_{n,\lambda}(t) \vec{Q}_{n,\lambda} \quad (159)$$

should also be considered. It is essential to realize that the equivalence relations must be viewed relative to a common frame of reference.

To determine which modeling approach is best for a particular application, the analyst need only substitute the expected parameters of the problem into equations 149, 150, and 151. The side of the equation which yields the most representative description of the respective dynamic effects dictates the modeling approach to be used. Furthermore, the

right-hand side of the equations provides a guide to the number of mode shapes required for representative modeling.

Goddard Space Flight Center  
National Aeronautics and Space Administration  
Greenbelt, Maryland      November 26, 1974

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## APPENDIX

### VECTOR-DYADIC IDENTITIES

The following vector-tensor identities have been used extensively throughout this document. They are scattered throughout various texts on the subject and collected here for the convenience of the reader.

Let

$\vec{P}$  = vector

$\vec{Q}$  = vector

$\vec{R}$  = vector

$\mathbf{T}$  = dyad\*

$\mathbf{1}$  = unit dyad

$\mathcal{S}$  = skew operator such that for the vector  $\vec{P}$  having components  $(p_1, p_2, p_3)$ , relative to a particular frame of reference,

$$\mathcal{S}(\vec{P}) = \begin{bmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{bmatrix}$$

Making use of the above definitions, the following vector-dyadic identities may be proven:

$$\begin{aligned} \vec{P} \times \vec{Q} &= -\mathcal{S}(\vec{P}) \cdot \vec{Q} \\ \vec{P} \cdot (\vec{Q} \times \vec{R}) &= (\vec{P} \times \vec{Q}) \cdot \vec{R} \\ \vec{P} \cdot (\vec{Q} \times \vec{R}) &= \vec{Q} \cdot (\vec{R} \times \vec{P}) \\ \vec{P} \times (\vec{Q} \times \vec{R}) &= (\vec{R} \cdot \vec{P}) \vec{Q} - \vec{R} (\vec{P} \cdot \vec{Q}) \\ \vec{P} \times (\vec{Q} \times \vec{R}) &= \left[ (\vec{R} \cdot \vec{P}) \mathbf{1} - \vec{R} \vec{P} \right] \cdot \vec{Q} \end{aligned}$$

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\*Also referred to as dyadic or tensor of rank 2.



$$\begin{aligned}
(\vec{P} \cdot \tau) \cdot \vec{R} &= \vec{P} \cdot (\tau \cdot \vec{R}) \\
(\tau \times \vec{P}) \cdot \vec{Q} &= \tau \cdot (\vec{P} \times \vec{Q}) \\
\vec{P} \cdot (\vec{Q} \times \tau) &= (\vec{P} \times \vec{Q}) \cdot \tau \\
\vec{P} \cdot (\tau \times \vec{Q}) &= (\vec{P} \cdot \tau) \times \vec{Q} \\
(\vec{P} \times \tau) \cdot \vec{Q} &= \vec{P} \times (\tau \cdot \vec{Q}) \\
(\vec{P} \times \tau) \times \vec{Q} &= \vec{P} \times (\tau \times \vec{Q}) \\
(\tau \times \vec{P}) \times \vec{Q} &= -\tau (\vec{P} \cdot \vec{Q}) + (\tau \cdot \vec{Q}) \vec{P} \\
\vec{P} \times (\vec{Q} \times \tau) &= -(\vec{P} \cdot \vec{Q}) \tau + \vec{Q} (\vec{P} \cdot \tau) \\
\tau \times (\vec{P} \times \vec{Q}) &= -\tau \cdot (\vec{P} \vec{Q} - \vec{Q} \vec{P}) \\
(\vec{P} \times \vec{Q}) \times \tau &= -(\vec{P} \vec{Q} - \vec{Q} \vec{P}) \cdot \tau
\end{aligned}$$



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