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GRAVITATIONAL HARMONICS FROM SHALLOW RESONANT ORBITS

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ABSTRACT

GRAVITATIONAL HARMONICS FROM SHALLOW RESONANT ORBITS

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Until very recently, there has been no identification of the significant gravitational constraints on the many common orbits in shallow resonance. Without them it is difficult to compare results derived for different sets of harmonics from different orbits. With them it is possible to extend these results to any degree without reintegration of the orbits. Five such (strong) constraints have been derived for the GEOS II orbit (order 13, to 30th degree) whose principal resonant period is 6 days. The constraints explain the sinusoidal variation with argument of perigee of a lumped harmonic found from 41 6-day arcs of optical and laser data in 1968-69. For example, the constant terms derived are:

$$10^9(38.1,-55.9) = -.872(C,S)_{13,13}$$

$$+ (C,S)_{15,13} + .462(C,S)_{17,13} + \dots,$$

in terms of fully normalized spherical harmonics.

The condition equations, derived from elementary perturbation theory are shown to account for almost all (>98%) of the resonant information in the tracking data. They agree well with recent gravitational models which include substantial amounts of GEOS II tracking data.

INTRODUCTION

We can calculate the amplitudes and phases of the geopotential perturbations from the formulas given by Kaula, 1966. He has expressed the potential entirely in terms of Kepler elements as

$$V = \sum_{\ell} \sum_{m} \sum_{p} \sum_{q} V_{\ell mpq}, \text{ where}$$

$$V_{\ell mpq} = \mu \frac{a_e^{\ell}}{a^{\ell+1}} J_{\ell m} F_{\ell mp}(i) \; G_{\ell pq}(e) \begin{bmatrix} \cos \\ \sin \end{bmatrix} \begin{matrix} (\ell-m) \text{ even} \\ \cos\psi_{\ell mpq} \\ (\ell-m) \text{ odd} \end{matrix} , \quad (1)$$

and

$$\psi_{\ell mpq} = (\ell-2p)\omega+(\ell-2p+q)M+m(\Omega-\theta-\lambda_{\ell m}).$$

In Equation (1):

a, e, i, ω Ω, and M are the mean Kepler elements: semi-major axis, eccentricity, inclination, argument of perigee, right ascension of the ascending node and mean anomaly respectively;

a_e is the semi-major axis of the earth;

μ is the gravitational constant of the earth;

$\dot{\theta}$ is the rotation rate of the earth; and

$m\lambda_{\ell m}$ is $\tan^{-1} \dfrac{S_{\ell m}}{C_{\ell m}}$.

The indexes ℓ,m are the degree and order of a fully normalized spherical harmonic term; the p,q quantities identify a particular component of that term. The function $F_{\ell mp}(i)$ (fully normalized) and $G_{\ell pq}(e)$ arise when the potential is converted from position coordinates to Kepler elements. The $G_{\ell pq}(e)$ functions are identical to Hansen's coefficients. The $F_{\ell mp}(i)$ functions are sinusoidal with wave length about $2\pi/(\ell-m+1)$.

In terms of Kepler elements, the equations of satellite motion are [Kaula, 1966, p. 29]

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial V}{\partial M} ,$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2e}\frac{\partial V}{\partial M} - \frac{(1-e^2)^{1/2}}{na^2e} \cdot \frac{\partial V}{\partial \omega} ,$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial V}{\partial i} + \frac{(1-e)^{1/2}}{na^2e} \cdot \frac{\partial V}{\partial e} ,$$

$$\frac{di}{dt} = -\frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial V}{\partial \omega} - \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial V}{\partial \Omega} ,$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2(1-e)^{1/2}\ \sin i}\frac{\partial V}{\partial i} ,$$

$$\frac{dM}{dt} = -\frac{1-e^2}{na^2e}\frac{\partial V}{\partial e} - \frac{2}{na}\frac{\partial V}{\partial a} + n , \qquad (2)$$

where n is the mean motion $(\mu^{1/2}\ a^{-3/2})$.

The equations of motion can be approximately integrated under certain assumptions. Because of the smallness of the effects of tesseral harmonics, they can be treated (to first order) as linear perturbations about the orbit produced by only the central force and the secular second zonal harmonic term (ℓmpq = 2010) in the potential. Under these assumptions, Kaula (1966, pp. 40, 49) gives the solutions for the amplitudes of the perturbations of the Kepler elements due to <u>each</u> harmonic component $V_{\ell mpq}$ as

$$\Delta a_{\ell mpq} = \mu a_e^{\ell} \frac{2F_{\ell mp}G_{\ell pq}(\ell-2p+q) \quad J_{\ell m}}{na^{\ell+2}[(\ell-2p)\dot{\omega}+(\ell-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]} \quad ,$$

$$\Delta e_{\ell mpq} = \mu a_e^{\ell} \frac{F_{\ell mp}G_{\ell pq}(1-e^2)^{1/2}[(1-e^2)^{1/2}(\ell-2p+q)-(\ell-2p)] \; J_{\ell m}}{na^{\ell+3}e[(\ell-2p)\dot{\omega}+(\ell-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]} \quad ,$$

$$\Delta\omega_{\ell mpq} = \mu a_e^{\ell} \frac{[(1-e^2)^{1/2}e^{-1}F_{\ell mp}(\partial G_{\ell pq}/\partial e) - \cot i(1-e^2)^{-1/2}(\partial F_{\ell mp}/\partial i)G_{\ell pq}] \, J_{\ell m}}{na^{\ell+3}[(\ell-2p)\dot{\omega}+(\ell-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]} \quad ,$$

$$\Delta i_{\ell mpq} = \mu a_e^{\ell} \frac{F_{\ell mp}G_{\ell pq}[(\ell-2p)\cos i \; -m] \quad J_{\ell m}}{na^{\ell+3}(1-e^2)^{1/2}\sin i[(\ell-2p)\dot{\omega}+(\ell-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]} \quad , \qquad (3)$$

$$\Delta\Omega_{\ell mpq}=\mu a_e^{\ell}\frac{(\partial F_{\ell mp}/\partial i)G_{\ell pq}\ J_{\ell m}}{na^{\ell+3}(1-e^2)^{1/2}\sin i[(\ell-2p)\dot{\omega}+(\ell-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]},$$

$$\Delta M_{\ell mpq}=\mu a_e^{\ell}\frac{[-(1-e^2)e^{-1}(\partial G_{\ell pq}/\partial e)+2(\ell+1)G_{\ell pq}]F_{\ell mp}\ J_{\ell m}}{na^{\ell+3}[(\ell-2p)\dot{\omega}+(\ell-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]}$$

$$-\frac{3\mu a_e^{\ell}\ F_{\ell mp}G_{\ell pq}(\ell-2p+q)\ J_{\ell m}}{a^{\ell+3}[(\ell-2p)\dot{\omega}+(\ell-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]^2}\ .$$

When examining these expressions, one can see that under certain conditions the frequencies $(\dot{\psi})$ in their denominators can go to zero:

$$\dot{\psi}_{\ell mpq} = (\ell-2p)\dot{\omega}+(\ell-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta}) = 0 \qquad (4)$$

This is known as the resonance condition. When this happens, one has <u>exact</u> commensurability between satellite motion and the earth's rotation yielding a perturbation from the longitudinal dependent terms of the geopotential analytically approaching infinity. This is known as <u>deep resonance</u>. Of course, other forces are acting on the satellite, so that the orbit usually simply passes through this condition of perfect commensurability. Typically, atmospheric drag is the dominant other force.

However, for a very large number of satellites, a situation exists where the resonance condition is only approximated, yielding substantial perturbations on the satellite nevertheless (Wagner and Douglas, 1969). This happenstance is called shallow resonance. These effects start becoming a problem for orbital operations and precise orbit determination when the resonant period, which is

$$\frac{1}{\dot{\psi}} = \frac{1}{(\ell-2p)\dot{\omega} + (\ell-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})} \tag{5}$$

starts to approach a few days duration. Unfortunately, in shallow resonance, all the frequencies of a given order satisfying Equation (4) are almost the same. It will be difficult therefore to distinguish these effects over a short time. The problem is to separate the information in order to determine specific gravitational harmonics.

Let $\psi_{m,q} = \psi_{m,0} = \omega + M + m(\Omega - \theta)$, ($m \simeq n$ in revolutions/day) be the dominant resonant longitude, and E be any of the Kepler elements. The effect associated with this longitude will be of order one compared to the fringe resonances, with longitudes $\psi_{m,\pm 1} = \psi_{m,0} \pm \omega$, which are of order 10e (Allan, 1973, p. 224). Note that in one period of $\psi_{m,0}$, ω will be essentially constant if $\dot{\psi}_{m,0} >> \dot{\omega}$ as it usually is for shallow resonant orbits.

As long as the resonant period is less than about 100 days, Kaula's (1966) linear perturbations, (equations (3)) are valid (Gedeon, 1969):

$$E = E_0 + \sum_{\substack{\text{Relevant} \\ \ell mpq}} \left\{ \Delta E_C \cos \psi_{\ell mpq} + \Delta E_S \sin \psi_{\ell mpq} \right\} \qquad (6)$$

where

$$\psi_{\ell mpq} = (\ell - 2p)\omega + (\ell - 2 + q)M + m(\Omega - \theta) \ ,$$

ignoring the phase $\lambda_{\ell m}$ as in equation (1) ,

$$\Delta E_{C,S} = \Delta E_{\ell mpq} \begin{cases} \begin{bmatrix} -S_{\ell m}, & C_{\ell m} \\ -C_{\ell m}, & -S_{\ell m} \end{bmatrix}_{\ell-m \text{ odd}}^{\ell-m \text{ even}} & \text{for } E = \omega, \Omega, M \\ \begin{bmatrix} C_{\ell m}, & S_{\ell m} \\ -S_{\ell m}, & C_{\ell m} \end{bmatrix}_{\ell-m \text{ odd}}^{\ell-m \text{ even}} & \text{for } E = a, e, I \end{cases} \tag{7}$$

Note that $(C_{\ell m}, S_{\ell m}) = J_{\ell m}$ $(\cos m\,\lambda_{\ell m}, \sin m\,\lambda_{\ell m})$ and $\Delta E_{\ell mpq}$ (given by a right-hand side of (3) without $J_{\ell m}$) is inversely proportional to $\dot{\psi}_{\ell mpq}$ (or $\dot{\psi}^2$ for ΔM) which is small in resonance.

For the five principal resonant frequencies $\dot{\psi}_{m,0}$, $\dot{\psi}_{m,-1}$, $\dot{\psi}_{m,+1}$, $\dot{\psi}_{m,-2}$, $\dot{\psi}_{m,2}$, Equation (6) becomes simply:

$$E = E_0 + \sum_{\substack{\ell \text{ RESONANT}, \\ q=0}} [\Delta E_C \cos \psi_{m,0} + \Delta E_S \sin \psi_{m,0}] \quad +$$

$$\sum_{\substack{\ell \text{ RESONANT} \\ q=\mp 1}} [\Delta E_C \cos (\psi_{m,0} \pm \omega) + \Delta E_S \sin (\psi_{m,0} \pm \omega)] \quad +$$

$$\sum_{\substack{\ell \text{ RESONANT} \\ q=\mp 2}} [\Delta E_C \cos (\psi_{m,0} \pm 2\omega) + \Delta E_S \sin (\psi_{m,0} \pm 2\omega)] \ .$$

In Equation (8) the resonant ℓ's for the $q=\pm1$ terms are always even and the resonant ℓ's for the $q=0$ (dominant) and $q=\pm2$ terms are always odd.

Expanding the cos and sin terms in (8) and collecting terms in $\cos\psi_{m,0}$, $\sin\psi_{m,0}$, $\cos\omega$ and $\sin\omega$, $\cos 2\omega$ and $\sin 2\omega$:

$$
\begin{aligned}
E = E_0 &+ \cos\psi_{m,0}\Bigg[\sum_{\substack{\ell\ RES\\ q=0}} \Delta E_C + \cos\omega\Bigg(\sum_{\substack{\ell\ RES\\ q=1}} \Delta E_C + \sum_{\substack{\ell\ RES\\ q=-1}} \Delta E_C\Bigg)\\
&+ \sin\omega\Bigg(-\sum_{\substack{\ell\ RES\\ q=1}} \Delta E_S + \sum_{\substack{\ell\ RES\\ q=-1}} \Delta E_S\Bigg)\\
&+ \cos 2\omega\Bigg(\sum_{\substack{\ell\ RES\\ q=2}} \Delta E_C + \sum_{\substack{\ell\ RES\\ q=-2}} \Delta E_C\Bigg) + \sin 2\omega\Bigg(-\sum_{\substack{\ell RES\\ q=2}} \Delta E_S + \sum_{\substack{\ell\ RES\\ q=-2}} \Delta E_S\Bigg)\Bigg]\\
&+ \sin\psi_{m,0}\Bigg[\sum_{\substack{\ell\ RES\\ q=0}} \Delta E_S + \cos\omega\Bigg(\sum_{\substack{\ell\ RES\\ q=1}} \Delta E_S + \sum_{\substack{\ell\ RES\\ q=-1}} \Delta E_S\Bigg)\\
&+ \sin\omega\Bigg(\sum_{\substack{\ell\ RES\\ q=1}} \Delta E_C - \sum_{\substack{\ell\ RES\\ q=-1}} \Delta E_C\Bigg)\\
&+ \cos 2\omega\Bigg(\sum_{\substack{\ell\ RES\\ q=2}} \Delta E_S + \sum_{\substack{\ell\ RES\\ q=-2}} \Delta E_S\Bigg) + \sin 2\omega\Bigg(\sum_{\substack{\ell\ RES\\ q=2}} \Delta E_C - \sum_{\substack{\ell\ RES\\ q=-2}} \Delta E_C\Bigg)\Bigg]
\end{aligned}
\tag{9}
$$

Equation (9) shows that the (lumped) coefficients (for each element) of the $\cos\psi_{m,0}$ and $\sin\psi_{m,0}$ terms (determinable in one period of $\psi_{m,0}$) are themselves sinusoidal functions of a slowly varying argument of perigee:

$$
\begin{aligned}
C^* &= C_0 + C_c \cos\omega + C_s \sin\omega + C_{2c} \cos 2\omega + C_{2s} \sin 2\omega + \ldots \\
S^* &= S_0 + S_c \cos\omega + S_s \sin\omega + S_{2c} \cos 2\omega + S_{2s} \sin 2\omega + \ldots
\end{aligned}
\qquad (10)
$$

The components of the lumped coefficients (C^*, S^*) depend on the resonant geopotential coefficients, as well as the a, e and i of the orbit.

The relation between the lumped coefficients and the actual tracking information is straightforward, but tedious to write out in detail. Essentially it is the same as the relation between the tracking information and the orbital elements. If '0' is a tracking observation, it is clearly a function of the orbital elements E. These in turn are given (to first order) by:

$$E = E_0 + E^* \begin{matrix} \cos\psi_{m,0} \\ \sin\psi_{m,0} \end{matrix}$$

over as many lumped coefficients E^* as are necessary to describe the variation. This suggests a simple scheme for determining all the lumped coefficients for a single tracking arc. Tracking residuals in '0' can be resolved by differential correction to E_0 and E^* through the condition equations:

$$\Delta 0 = \frac{\partial 0}{\partial E} \left[\Delta E_0 + \Delta E^* \begin{matrix} \cos\psi_{m,0} \\ \sin\psi_{m,0} \end{matrix} \right]$$

The observation 'partials' $\partial O/\partial E$ are laborious to calculate but are readily available in existing differential programs [Lerch et. al. 1974]. This is analogous to the method chosen by Riegber (1973) to determine shallow resonant constraints. Riegber however corrects a boundary (rather than an initial) value solution to the orbit. He also appears to make no use of the known frequencies of the problem, preferring a general Fourier analysis of the resonant elements. However, the principal periods in Reigber's solutions are the full arc lengths, chosen as $1/\dot{\psi}_{m,0}$. His GEOS II solution for these are compatible with our results, as will be shown later.

One of the goals of our analysis, however, is to identify the dominant information in current fields. Where possible, we want to use the good geopotential solutions, already developed at great expense. Our central hypothesis is that the resonance information is almost entirely in the along track ($\Delta\omega+\Delta M+\Delta\Omega\cos i$) variation. To give a concrete example, consider the orbit of GEOS II (a = 1.209 e.r., e = .033, i = 105.8°). The root sum of squares of all resonant terms contributing to this orbital component is indeed 95% of the total perturbation. The information in the semi-major axis variation is almost the same as this since its integral controls the along track change in resonance. Table 1 give the resonant 'a' perturbations on GEOS II from $J_{\ell m} = \sqrt{2} \times 10^{-5}/\ell^2$ to (30,13).

Clearly, the q=0 terms dominate but the q=$\pm$1 terms are also significant. To illustrate the development of the lumped coefficients, the quantities $\Delta(a)$ are the components ΔE_C or ΔE_S (for E=a) for coefficients $C_{\ell m}$ or $S_{\ell m}$ of $\sqrt{2} \times 10^{-5}/\ell^2$ in equation (9), where $\Delta E_{C,S}(q=\pm 1) = \Delta E[-S_{\ell m}, C_{\ell m}]$ and $\Delta E_{C,S}(q=0) = \Delta E[C_{\ell m}, S_{\ell m}]$. Therefore, to find the partial contribution (or sensitivity) due to each (unknown) coefficient (ΔE) the perturbations in the table must be divided by $\sqrt{2} \times 10^{-5}/\ell^2$.

In fact they can also be adjusted by a constant (as a set) without changing the relative information content of the terms in the lumped coefficients C* and S*. Multiplying each term of Table 1 by $\ell^2/400$ gives the partial contributions presented in Table 2.

To obtain non-dimensional sensitivities, these partials are divided by the maximum contribution (from $\ell=15$, $q=0$) yielding the sensitivities presented in Table 3.

Thus, from this final sensitivity table the lumped coefficients determinable from observation of just the 'a' variation are (to $q=\pm 2$ terms)

$$
\begin{aligned}
(C_0, S_0) = {} & -.886(C,S)_{13,13} + 1.000(C,S)_{15,13} \\
& +.456(C,S)_{17,13} - .020(C,S)_{19,13} - .181(C,S)_{21,13} \\
& -.156(C,S)_{23,13} - .076(C,S)_{25,13} - 0.010(C,S)_{27,13} \\
& +.022(C,S)_{29,13} + \ldots
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
(C_C, S_C) = {} & -.514(-S,C)_{14,13} + .034(-S,C)_{16,13} \\
& +.160(-S,C)_{18,13} + .110(-S,C)_{20,13} \\
& +.035(-S,C)_{22,13} - .011(-S,C)_{24,13} \\
& -.025(-S,C)_{26,13} - .021(-S,C)_{28,13} \\
& -.011(-S,C)_{30,13} \ldots
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
C_S,S_S) = \; & .158(C,S)_{14,13} + .188(C,S)_{16,13} \\
& +.041(C,S)_{18,13} - .068(C,S)_{20,13} - .092(C,S)_{22,13} \\
& -.064(C,S)_{24,13} - .022(C,S)_{26,13} + .009(C,S)_{28,13} \\
& +.021(C,S)_{30,13} + \ldots
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
(C_{C2},S_{C2}) = \; & -.039(C,S)_{13,13} + .021(C,S)_{15,13} \\
& +.012(C,S)_{17,13} + .007(C,S)_{19,13} \\
& +.006(C,S)_{21,13} + .005(C,S)_{23,13} \\
& +.002(C,S)_{25,13} - .001(C,S)_{27,13} \\
& -.003(C,S)_{29,13} + \ldots
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
(C_{S2},S_{S2}) = \; & +.031(S,-C)_{13,13} + .010(S,-C)_{15,13} \\
& -.023(S,-C)_{17,13} - .027(S,-C)_{19,13} \\
& -.015(S,-C)_{21,13} - .001(S,-C)_{23,13} \\
& +.007(S,-C)_{25,13} + .008(S,-C)_{27,13} \\
& +.006(S,-C)_{29,13} + \ldots
\end{aligned}
\tag{15}
$$

One can also repeat the above computations using the sum of the perturbations $\Delta\omega$, ΔM and $\cos i \cdot \Delta\Omega$ to derive influence coefficients for observation of an effective along track variation:

$$(C_0,S_0) = -.872(-S,C)_{13,13} + 1.00(-S,C)_{15,13} + .462(-S,C)_{17,13}$$
$$-.020(-S,C)_{19,13} - .189(-S,C)_{21,13} - .165(-S,C)_{23,13}$$
$$-.082(-S,C)_{25,13} - .011(-S,C)_{27,13} + .025(-S,C)_{29,13}\cdots \quad (16)$$

$$(C_C,S_C) = -.516(-C,-S)_{14,13} + .029(-C,-S)_{16,13} + .162(-C,-S)_{18,13}$$
$$+.116(-C,-S)_{20,13} + .039(-C,-S)_{22,13} - .010(-C,-S)_{24,13}$$
$$-.027(-C,-S)_{26,13} - .024(-C,-S)_{28,13} - .013(-C,-S)_{30,13}\cdots \quad (17)$$

$$(C_S,S_S) = .171(-S,C)_{14,13} + .189(-S,C)_{16,13} + .037(-S,C)_{18,13}$$
$$-.073(-S,C)_{20,13} - .098(-S,C)_{22,13} - .068(-S,C)_{24,13}$$
$$-.023(-S,C)_{26,13} + .010(-S,C)_{28,13} + .024(-S,C)_{30,13}\cdots \quad (18)$$

$$(C_{C2},S_{C2}) = -.040(-S,C)_{13,13} + .020(-S,C)_{15,13}$$
$$+.014(-S,C)_{17,13} + .008(-S,C)_{19,13}$$
$$+.007(-S,C)_{21,13} + .006(-S,C)_{23,13}$$
$$+.002(-S,C)_{25,13} - .001(-S,C)_{27,13}$$
$$-.004(-S,C)_{29,13}\cdots \quad (19)$$

$$(C_{S2},S_{S2}) = +.033(C,S)_{13,13} + .009(C,S)_{15,13}$$
$$-.024(C,S)_{17,13} - .029(C,S)_{19,13}$$
$$-.016(C,S)_{21,13} - .001(C,S)_{23,13}$$
$$+.007(C,S)_{25,13} + .009(C,S)_{27,13}$$
$$+.007(C,S)_{29,13}\cdots \quad (20)$$

TABLE 1
PERTURBATION AMPLITUDE IN a Δ(a) meters
(m = 13)

ℓ =	q = −2	q = −1	q = 0	q = +1	q = +2
13	−.011		−2.589		−.103
14		−.448		−.847	
15	.034		2.195		.011
16		.214		−.149	
17	−.009		.779		.030
18		.153		.091	
19	−.014		−.026		.023
20		.026		.109	
21	.005		−.203		.012
22		−.029		.065	
23	−.002		−.146		.003
24		−.032		.023	
25	.004		−.060		−.002
26		−.017		−.001	
27	.003		−.007		−.003
28		−.004		−.009	
29	.001		.013		−.003
30		.003		−.009	
RSS	.04	0.51	3.2	0.86	.11

TABLE 2
PARTIAL CONTRIBUTION Δ(a) m
m = 13

ℓ =	q = −2	q = −1	q = 0	q = +1	q = +2
13	−.005		−1.034		−.043
14		−.220		−.415	
15	.019		1.235		.006
16		.137		−.096	
17	−.007		.563		.022
18		.123		.073	
19	−.013		−.024		.021
20		.026		.109	
21	−.005		−.224		.013
22		−.035		.079	
23	.002		−.193		.004
24		−.046		.033	
25	.006		−.094		−.003
26		−.029		−.002	
27	.005		−.013		−.007
28		−.008		−.019	
29	.002		.028		−.006
30		−.006		−.020	

TABLE 3
SENSITIVITY COEFFICIENTS
FOR GEOS–II RESONANCE
ΔE(a)

ℓ =	q = −2	q = −1	q = 0	q = +1	q = +2
13	−.004		−.886		−.035
14		−.178		−.336	
15	.016		1.000		.005
16		.111		−.077	
17	−.005		.456		.018
18		.100		.059	
19	−.010		−.020		.017
20		.021		.089	
21	−.004		−.181		.011
22		−.029		.064	
23	.002		−.156		.003
24		−.037		.026	
25	.005		−.076		−.002
26		−.024		−.002	
27	.004		−.010		−.005
28		−.006		−.015	
29	.001		.022		−.005
30		+.005		−.016	

As predicted, these sensitivities only vary slightly from the values derived for Δa.

Similarly, the information content from the element variations in 'e' and 'i' is almost the same or predictable from the 'a' variation since:

$$\Delta i/\Delta a = \frac{[(1-q)\cos i - m]}{2a(1-e^2)^{1/2}\sin i}, \text{ and} \tag{21}$$

$$\frac{\Delta e}{\Delta a} = \frac{(1-e^2)^{1/2}[(1-e^2)^{1/2} - (1-q)]}{2ae}$$

Note that these ratios depend on the frequency q as well as the resonant order m, but they are independent of degree. Therefore, the sensitivity tables for e and i information are the same as those for 'a' with the following adjustments (Table 4):

TABLE 4
FACTORS OF THE Δa TABLE FOR
GEOS II OBSERVATIONS Δe AND ΔI

	q = −1	q = 0	q = + 1
ΔI	−13.54	−13.27	−13.00
Δe	−1.	-5.45×10^{-4}	+ 1.
$\Delta I' = \Delta I / -13.54$	1.	.980	.960

Table 4 shows that the sensitivities for Δi observations are virtually unchanged from those for Δa while the Δe information appears to be significantly altered. But in fact, the Δe information is almost entirely in the $q=\pm 1$ terms which are predictable from the 'a' variation. In fact:

$$
\begin{aligned}
C_c(\Delta e) &= S_s(\Delta a) \\
S_s(\Delta e) &= C_c(\Delta a) \\
C_s(\Delta e) &= -S_c(\Delta a) \\
S_c(\Delta e) &= -C_s(\Delta a)
\end{aligned}
\qquad (22)
$$

so that no new information is added by observations of eccentricity variation.

In summary, we have hypothesized that nearly all the information in shallow resonance is contained in observations of either the semi-major axis or along track variation. As a consequence, analysis of tracking data within each short period $(1/\dot{\psi}_{m,0})$ may be made in terms of a simple "lumped" coefficient set. Such a set will vary sinusoidally with the long period of the argument of perigee. A similar analysis can be made for the higher order resonances of 2m, 3m, ... , on the same orbit.

We have chosen the GEOS-II (1968 2A) satellite to test our method of analysis. GEOS-II has a principal resonance period which is approximately 6.5 days. This satellite was selected for three major reasons:

1) It has been heavily tracked with very accurate instrumentation,

2) This satellite afforded us the opportunity to calibrate our new technique for identifying resonant constraints since it was used in almost all recent global geopotential solutions,

3) The satellite is largely unaffected by atmospheric drag.

The total resonance perturbation on GEOS-II along track is approximately 600m, and is an enormous effect when compared to the 1m laser ranging and ~1$''$.50 camera instrumentational accuracy which was employed to track this satellite.

ANALYSIS OF GEOS-II DATA FOR RESONANCE DETERMINATION

Following the hypothesis developed in the previous section, the entire laser and camera data set available on GEOS-II (in 1968-69) was divided into forty-one 6.5 day segments. The GEM1 gravity model (Smith, Lerch and Wagner, 1973) was then employed and the orbital state was estimated using this tracking data. We, however, removed all m=13 terms from the GEM1 model and recovered a lumped value of C(13,13) and S(13,13). These recovered values are plotted (fully normalized) against ω in Figure 1. The results for the various GEODYN solutions are summarized in Table 6.

The GEODYN orbit determination system (T. Martin, 1972) was used for the resonance determination. GEODYN is a Bayesian least-squares, multiarc, multiple satellite orbit and geodetic parameter estimation system based upon Cowell type numerical integration techniques. Modeled parameters normally include luni-solar gravitational perturbations, solar radiation pressure, atmospheric drag, BIH polar motion and UT1 data and the GEM1 geopotential model. Initially drag was modeled but not adjusted in the single arc solutions. Considerable correlation was found between the drag coefficient and the recovered resonant coefficients. Later, small but significant adjustments of drag were made in a multi-arc solution, having low correlation with the lumped coefficients.

A weighted least-squares solution was then performed to find the values for the sinusoidal terms in equation (10) from the lumped values for (13,13) recovered from the GEODYN orbital analysis over each 6.5 day arc. This solution found the following values describing the lumped coefficients:

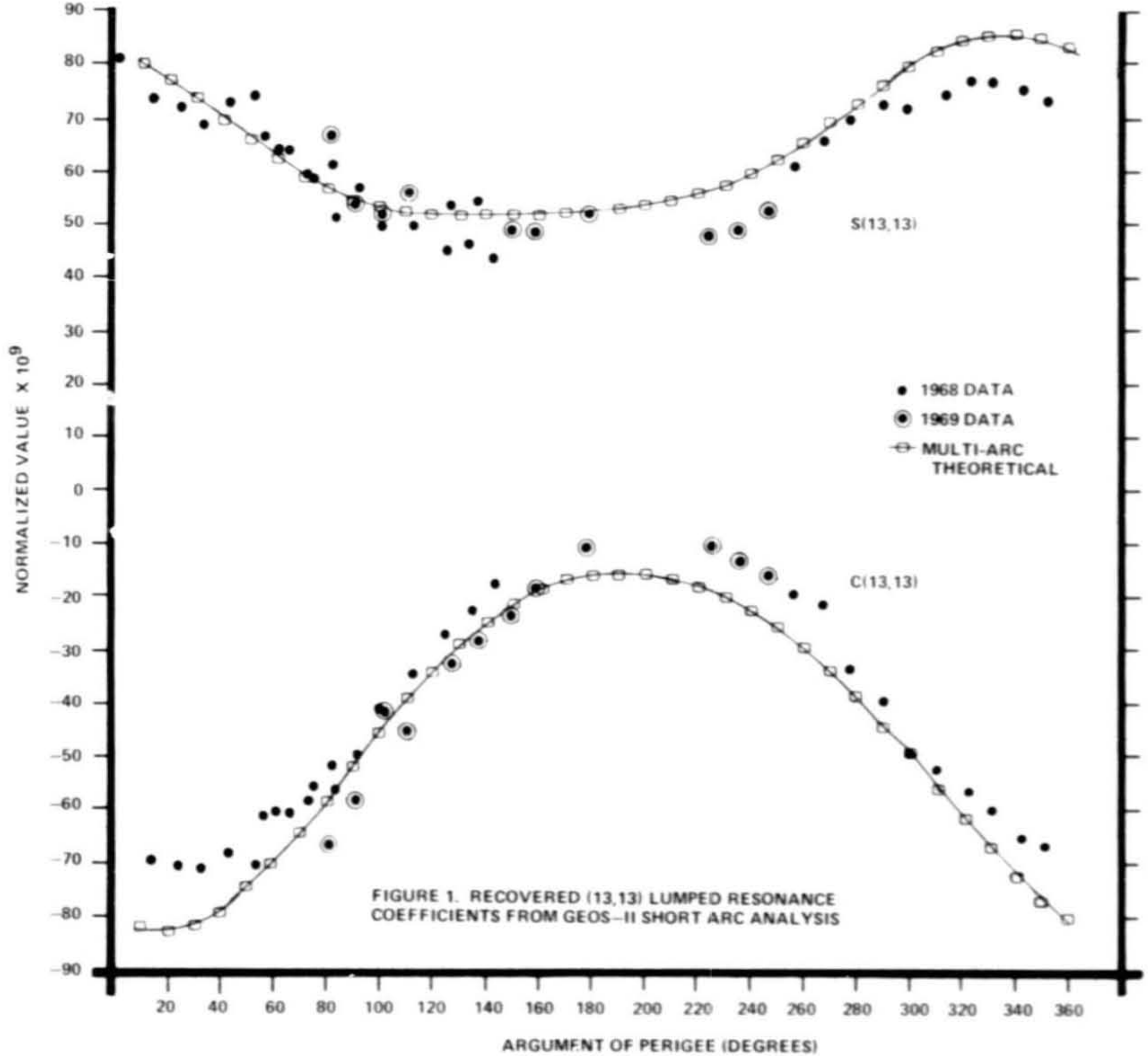

FIGURE 1. RECOVERED (13,13) LUMPED RESONANCE COEFFICIENTS FROM GEOS-II SHORT ARC ANALYSIS

TABLE 6. GEOS–II LUMPED COEFFICIENTS FROM INDIVIDUAL ARCS

ARC	C LUMPED $\times 10^{-9}$	S LUMPED $\times 10^{-9}$	ARG OF PERIGEE DEGREES	FORMAL C SIGMA $\times 10^{-11}$	FORMAL S SIGMA $\times 10^{-11}$	WEIGHTED RMS OF FIT TO DATA
1	−56.12	51.19	83.33	0.28	0.44	1.55
2	−55.58	58.68	75.06	0.51	0.78	0.87
3	−60.95	64.34	65.75	0.28	0.36	0.98
4	−61.16	66.82	56.10	0.26	0.36	0.95
5	−68.87	72.53	43.26	0.16	0.24	0.94
6	−71.07	69.11	33.71	0.31	0.48	0.90
7	−70.79	71.75	24.97	0.23	0.32	0.87
8	−69.88	73.69	15.15	0.43	0.31	1.60
9	−72.77	80.85	1.43	0.17	0.22	1.95
10	−67.11	73.35	351.74	0.14	0.22	1.01
11	−65.07	75.13	342.41	0.15	0.19	0.99
12	−60.24	76.50	331.83	0.48	0.44	0.90
13	−57.35	76.80	323.41	0.18	0.22	1.00
14	−52.97	74.09	312.42	0.27	0.31	0.95
15	−49.07	71.35	299.93	0.34	0.34	1.26
16	−39.73	72.60	290.07	0.23	0.22	0.91
17	−33.55	69.75	276.73	0.21	0.14	0.97
18	−21.46	65.80	266.56	0.23	0.17	1.00
19	−19.84	60.92	256.41	0.25	0.15	1.11
20	−17.92	43.31	143.55	0.36	0.25	0.91
21	−23.61	46.01	134.27	1.26	0.84	0.88
22	−26.80	44.45	125.13	0.22	0.23	0.90
23	−34.69	49.96	112.70	0.65	0.48	0.96
24	−41.62	49.78	100.87	0.26	0.29	0.94
25	−49.24	53.42	91.49	0.24	0.33	1.05
26	−51.94	61.26	81.96	0.27	0.37	0.88
27	−58.78	59.33	72.14	0.52	1.09	0.95
28	−60.25	64.44	61.66	0.40	0.46	0.95
29	−70.10	73.51	52.40	0.40	0.35	1.14
30	−16.19	52.44	246.74	0.28	0.15	1.14
31	−13.15	48.80	234.92	0.13	0.11	1.64
32	−10.40	47.02	225.58	0.36	0.15	1.39
33	−17.94	48.97	159.35	0.18	0.31	0.90
34	−23.11	48.42	149.70	0.15	0.22	1.01
35	−28.20	54.89	137.01	0.18	0.22	1.34
36	−33.05	53.32	126.92	0.16	0.24	0.86
37	−10.91	52.79	178.18	0.30	0.25	1.15
38	−45.09	56.17	111.52	0.18	0.13	1.01
39	−41.70	51.93	101.09	0.33	0.23	1.26
40	−58.17	56.28	91.74	0.62	0.37	0.77
41	−66.80	66.61	81.98	0.65	0.58	0.77

$$C_0' = -40.0086 \times 10^{-9}$$

$$S_0' = 61.2619 \times 10^{-9}$$

$$S_c' = 15.6332 \times 10^{-9} \qquad (23)$$

$$C_c' = -29.0172 \times 10^{-9}$$

$$C_s' = -11.3765 \times 10^{-9}$$

$$S_s' = -\ 2.8247 \times 10^{-9}$$

Since no other resonant term is modeled the $(C,S)_{13,13}$ values themselves can be considered the lumped coefficients in in equation (10). We have labeled these C', S'. In equations (11-20), the full model for these coefficients are (arbitrarily) normalized with respect to the (15,13) term which has the greatest influence.

The assumption here is that the resonant information is entirely along track (equations 16-20). Therefore, according to the convention in these equations (C^*, S^*) $= .872\ (S', -C')$. The terms of these lumped resonant coefficients are thus:

$$S_0 = 34.922 \times 10^{-9}$$

$$-C_0 = 53.473 \times 10^{-9}$$

$$S_c = 25.328 \times 10^{-9} \qquad (24)$$

$$-C_c = 13.646 \times 10^{-9}$$

$$S_s = 9.930 \times 10^{-9}$$

$$-C_s = -2.466 \times 10^{-9}$$

These coefficients can then be used in equations (16-18) to produce any three resonant coefficient sets modeling the three distinct frequencies for the q=-1,0 and +1 terms. Such a set is presented in Table 7. The $q=\pm 2$ frequencies were barely detectable in the lumped coefficients and were not successfully recovered from this data.

This same analysis can be performed using the Δa constraints found in equations (11), (12) and (13). The same resonant coefficient set for Δa are also presented in Table 7. Since the two sets of constraint equations are nearly the same, these sets are also.

TABLE 7
RESONANT COEFFICIENTS
FROM GLOBAL * GEOS–II TRACKING DATA
(SINGLE ARC ANALYSIS)

	from (Δ Along Track)	from (Δ a)
C(15,13)	34.922×10^{-9}	35.438×10^{-9}
S(15,13)	-53.473×10^{-9}	-54.263×10^{-9}
C(14,13)	27.976×10^{-9}	28.880×10^{-9}
S(14,13)	47.408×10^{-9}	48.204×10^{-9}
C(16,13)	27.228×10^{-9}	29.330×10^{-9}
S(16,13)	-29.845×10^{-9}	-27.204×10^{-9}

*Data over a full rotation of perigee.

ERROR ANALYSIS

The ORAN program (C.F. Martin, 1970) was used to perform a comprehensive error analysis of our resonant harmonic determination. In particular we wanted to know why the single arc lumped harmonics in 1969 were systematically displaced from those in 1968 (see Figure 1). The ORAN program calculates the effect of unsolved-for (and poorly determined) parameters on the resonant determination. ORAN does this (without lengthy simulations) by computing numerical measurement partials with respect to a large number of unrecovered effects. Three kinds of problems were investigated:

1. ORAN was used to perform a classical error analysis and gave both accuracy assessments and located the dominant error sources affecting the recovered resonance terms. The modeled errors included:

   - atmospheric drag at 40% error in a ballistic coefficient, C_D:

   - tesseral and zonal harmonic errors at 25% of the difference between two independent gravity models (Martin and Roy, 1970):

   - tracking station coordinate uncertainties in the adopted set of positions employed for these solutions (Marsh et. al., 1973):

   - an error in μ of 1 ppm.

The results of this error analysis are summarized in Table 8. This analysis indicated that atmospheric drag was a large error source and could substantially bias the resonance recovery from each arc. This was surprising since the overall effect of drag on the rather high GEOS II orbit is small. Greater effects were seen, however, in the high solar cycle years of 1968-69. In fact, a large part of the discrepancy between the 1968-1969 data in Figure 1 could be due to drag error. The non-resonant geopotential error is probably overestimated in Table 8 since the field used for the GEOS arcs (GEM-1) was determined with much of the same optical data. Also, the gravity model error magnitude was scaled to the SAO Standard Earth II Gravity Model (Gaposchkin and Lambeck, 1970) which has been shown by Klosko and Krabill (1974) to yield approximately twice the orbital errors as that of GEM-1 on GEOS II orbits. Attempts were made to determine a drag coefficient, C_D, in each of the 6.5 day orbital solutions, but excessively high correlations between C_D and the resonance terms (at times exceeding .90) prevented our having much confidence in the results.

2. The ORAN program was also used · verify the analytical development and pos ation of the previous sections of this report. Values for all of the 13th (through degree 21) order resonance terms were modeled as error sources at the magnitudes listed in Table 9 as α and thereby had their perturbations propagate into the recovered values for $(13,13)_{C,S}$ at the magnitude listed as β. By comparing quantities α and β we were able to numerically determine values for

TABLE 8. ESTIMATED ERROR IN RECOVERED $(13,13)_{C,S}$
VALUES FROM ERROR ANALYSIS FOR TWO SAMPLE GEOS–II ARCS
(NORMALIZED VALUE X 10^{-9})

		ERROR SOURCES (Magnitude)				
ARC EPOCH		μ (1 ppm)	DRAG (40% OF C_D)	GRAVITY MODEL ERROR (.25 APL–SAO M–1)	STATION COORDINATES (AT MARSH ESTIMATED UNCERTAINTY)	RSS TOTAL
680307	C(13,13)	.015	3.068	0.771	0.334	3.181
	S(13,13)	–.011	0.464	1.238	0.929	1.616
690207	C(13,13)	.027	–1.660	3.073	0.747	3.572
	S(13,13)	.029	2.579	–0.415	0.265	2.625

TABLE 9
ORAN CALIBRATION OF GEOS-II SHALLOW RESONANCE STUDY

CO, SO	(α) VALUE PROPAGATED (NORMALIZED) $\times 10^{-9}$	(β) VALUE OF COEFF. ABSORBED BY (13,13) $\times 10^{-9}$	ORAN ESTIMATED SENSITIVITY $\left(\frac{\beta}{\alpha}\right)$	SENSITIVITY VALUE PREDICTED FROM THEORY FOR q = −1,0,+1 TERMS: SEMI-MAJOR AXIS	ASCEND. NODE	ECC.	ALONG TRACK
C(15,13)	4.444	−5.174	−1.164	−1.129	1.564	−1.129	−1.146
S(15,13)	4.444	−5.144	−1.158				
C(17,13)	3.460	−1.840	− .532	− .515	−1.182	− .515	− .530
S(17,13)	3.460	−1.842	− .532				
C(19,13)	2.770	+ .087	.031	.022	−1.726	.022	.023
S(19,13)	2.770	+ .076	.027				
C(21,13)	2.628	+ .514	.227	.204	−0.954	.205	.217
S(21,13)	2.628	+ .507	.223				
[CC, SC] · cos ω							
−S(14,13)	51.020	4.160	.292	.162	− .000	−102.83	.165
C(14,13)	51.020	4.106	.288				
−S(16,13)	39.063	+ 0.292	− .027	− .011	.266	−122.50	− .009
C(16,13)	39.063	−0.195	− .018				
−S(18,13)	3.086	+ 0.086	− .099	− .050	.133	− 26.71	− .052
C(18,13)	3.086	−0.081	− .094				
−S(20,13)	2.500	+ 0.046	− .066	− .035	− .032	43.95	− .037
C(20,13)	2.500	−0.047	− .068				
[CS, SS] · sin ω							
C(14,13)	51.020	−7.636	− .156	− .134	.233	1150.3	− .188
S(14,13)	51.020	−6.454	− .132				
C(16,13)	39.063	−7.539	− .201	− .204	− .162	− 75.32	− .208
S(16,13)	39.063	−7.530	− .201				
C(18,13)	3.086	−.1373	−.0463	− .044	− .605	−357.2	− .042
S(18,13)	3.086	−.1597	−.0539				
C(20,13)	2.500	+.1756	+.0731	.073	− .542	−244.9	.080
S(20,13)	2.500	+.1600	+.0667				

the sensitivity coefficients, checking the analytic results. Table 9 presents these results. As anticipated, the along track constraint best represents the relative sensitivity of the 13^{th} order terms for the q=0, and composite q=±1 frequencies for GEOS-II. The fringe terms are less well modeled than the dominant constant C_o and S_o terms, especially the small cosω terms for this sample $\omega \approx 75°$ orbit.

3. ORAN simulations were also performed in a manner similar to the preceding, but simulating the adjustment of more than one pair of 13^{th} order terms. This analysis was done to assess how much resonance information was contained within each 6.5 day data set. By propagating the effects of the unsolved for and therefore neglected resonance terms into the orbit, and getting an estimated satellite positional error, we found that when two even and two odd degree pairs of coefficients were recovered, the remaining orbital error from all other resonance terms was estimated to be less than 50 cm. When two even and a single odd pair of coefficients are recovered, the estimated orbital errors were at times, 2 meters. With a single odd pair recovered, the estimated orbital errors due to neglected resonance, at times were 10's of meters. We therefore deduced that two even and two odd pairs of coefficients could recover the total resonance information for GEOS-II.

The problem of atmospheric drag errors was still present and we therefore decided to recover a single set of two odd and two even pairs of resonance coefficients in a multiple-arc solution using 14 6.5 day orbits well distributed over the apsidal period. Two pairs of 26^{th} order ($\sim 3\frac{1}{2}$ day period) terms were also estimated. In each of these arcs, a C_D was independently estimated. However, since all 14 arcs contributed information to the resonance recovery, the correlation between the recovered C_D's and the resonance terms was satisfactorily reduced to no more than 0.6, and seldom exceeded 0.3.

It is these results which we have adopted as best for this report. These 13th order coefficients are presented in Table 10 and are used for the theoretical resonance values plotted in Figure 1.

Comparisons with Comprehensive Gravity Model Solutions

There are many comprehensive gravity models which have been produced using satellite tracking data. Some used data from GEOS-II, while others did not. One can get some estimate of the consistency between various models and also compare the results inferred from these models for GEOS-II with the results we have obtained using our numerical analytical technique.

By taking equations (16) through (20) derived from $(\Delta\omega + \Delta M + \cos i \cdot \Delta\Omega)$ and substituting coefficient values from a given gravity model, one can compute a value of the lumped coefficients for each of the gravity models. Table 11 presents these results.

Table 11 indicates very good agreement between our GEOS-II multi-arc analysis and the results obtained from comprehensive geopotential solutions which had a strong presence of GEOS-II data. The lumped coefficients for the comprehensive models are remarkedly consistent (on the whole) in spite of a fairly wide divergence of actual coefficients (see Figure 2 and Table 12). In Table 12 we have listed 6 satellite only solutions (and their GEOS-

TABLE 10. GEOS–II RESONANCE HARMONIC RECOVERY USING A MULTI–ARC SOLUTION

COEFFICIENT	NORMALIZED VALUE x 10^{-9}	ALONG TRACK CONSTRAINT LUMPED VALUES
C(13,13)	−65.166	
S(13,13)	70.009	S_O = 38.1376
C(14,13)	27.054	C_O = +55.9037
S(14,13)	53.582	C_C = +13.3597
C(15,13)	−18.744	S_C = +27.9889
S(15,13)	5.205	S_S = 8.5028
C(16,13)	20.516	C_S = −6.8818
S(16,13)	−11.993	C_{2C} = 2.7169
C(26,26)	−63.160	S_{2C} = 2.2488
S(26,26)	23.582	C_{2S} = − 2.3294
C(27,26)	− 3.035	S_{2S} = 2.3611
S(27,26)	30.581	

TABLE 11. GRAVITY MODEL COMPARISON FOR RECOVERED LUMPED GEOS–II RESONANCE COEFFICIENTS (NORMALIZED X 10^9)

LUMPED COEF ($\times 10^9$)		GEOS–II WAGNER KLOSKO ANALYSIS (MULTI–ARC)	GEOS–II WAGNER KLOSKO INDIVIDUAL ARCS	MODELS NOT USING GEOS–II: RAPP (1967)	AFL 5.0 (1973)	YIONOULIS (1968)	MODELS USING GEOS–II: DOUGLAS• (1969)	SAO II (1970)	SAO III (1974)	GEM1 (1971)	GEM4 (1972)	GEM5 (1974)	GEM6 (1974)	PGS 62 (1974)	DOPPLER
S_O		38.138	34.92	17.41	21.67	18.10	18.10 •	37.78	29.96	37.52	40.06	39.93	40.03	38.71	37.83
C_O		55.904	53.47	21.41	62.87	68.04	68.04 •	56.57	62.48	55.10	55.78	55.02	55.02	55.32	54.54
S_C	Cos ω	27.989	25.33	14.82	2.32		48.18	4.77	27.54	29.87	27.48	28.33	28.16	27.22	27.07
C_C	Cos ω	13.360	13.65	10.37	4.02		4.22	18.06	16.29	11.32	13.59	12.97	12.94	15.22	16.64
S_S	Sin ω	8.503	9.93	3.39	9.35		1.38	11.43	7.01	9.64	5.59	5.98	6.06	7.77	8.24
C_S	Sin ω	−6.882	−2.47	−4.65	−2.95		−15.77	−7.89	−5.41	−5.26	−2.43	−3.53	−3.50	−3.74	−3.52
C_{C_2}	Cos 2ω	2.717	—	—	—	—	—	3.57	3.43	3.05	2.81	2.74	2.80	2.49	3.05
S_{C_2}	Cos 2ω	2.249	—	—	—	—	—	2.84	2.37	1.65	1.28	1.70	1.66	1.79	2.00
C_{S_2}	Sin 2ω	2.329	—	—	—	—	—	−4.47	−3.35	−1.20	−1.18	−1.31	−1.35	−1.72	−3.01
S_{S_2}	Sin 2ω	2.361	—	—	—	—	—	4.56	3.97	3.34	1.80	2.49	2.55	2.13	2.25

•USED YIONOULIS (1968) ODD DEGREE TERMS

+FROM ALONG TRACK CONSTRAINTS

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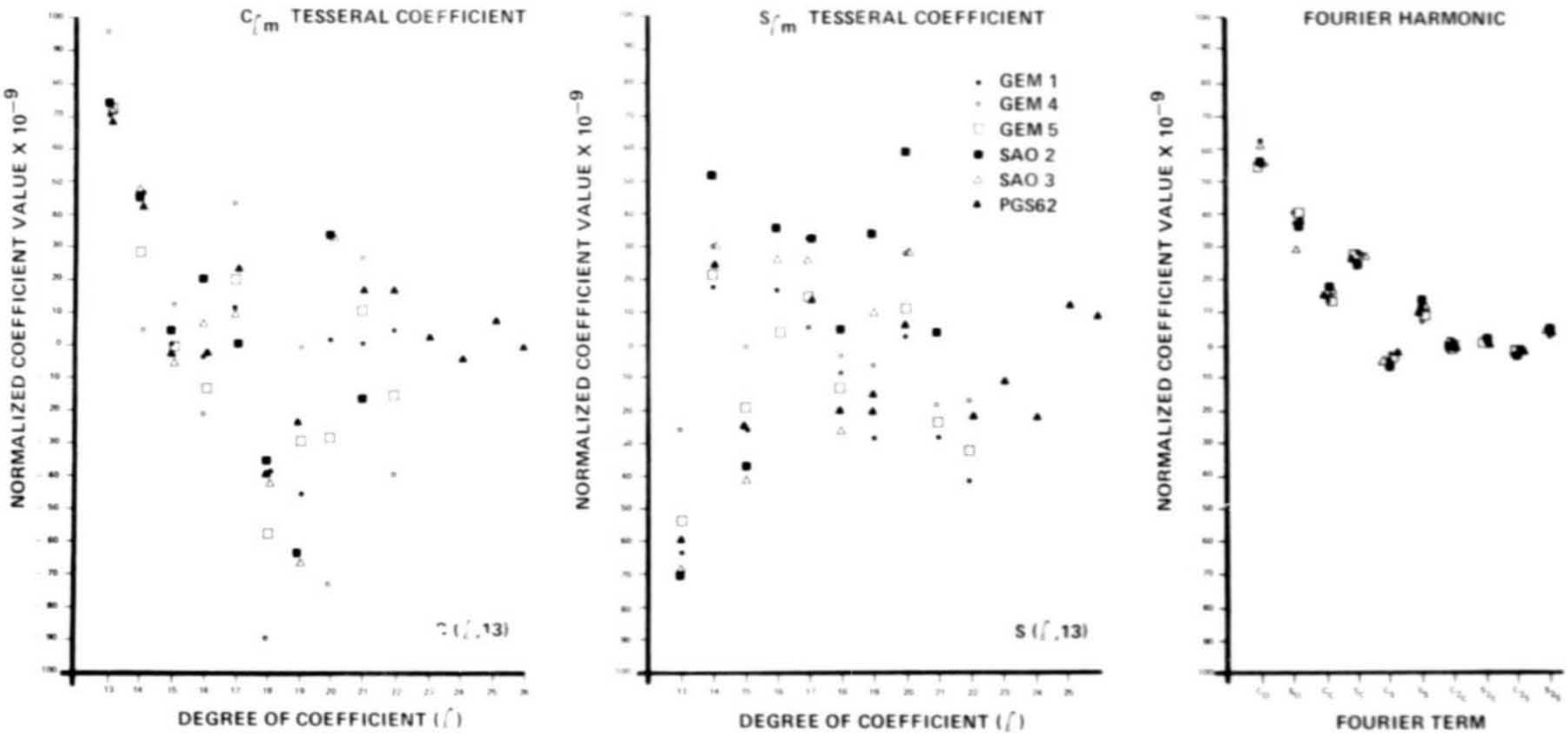
TABLE 12. 13Th ORDER SATELLITE SOLUTIONS: UNITS; 10^{-9}

	GEM 1	GEM 3	PGS 62	DOPPLER	RIEGBER	GEM 5	$10^{-5}/\ell^2$	AVG.	RMS	(%) $\frac{RMS}{10^{-5}/\ell^2}$
13C	−63.3	−26.2	−59.6	−62	−44.4	−59.8	59.2	−52.6	13.3	22.5
13S	70.4	95.1	68.3	70	58.1	68.9	59.2	71.8	11.2	19.0
14C	17.4	30.2	24.8	32	35.3	14.5	51.0	25.7	7.6	14.9
14S	46.4	4.5	42.0	48	46.6	26.4	51.0	35.6	15.7	30.8
15C	−26.4	− 1.4	−25.1	−31	−13.1	−22.6	44.4	−19.9	9.9	22.3
15S	− 0.5	12.0	− 2.5	−11	− 8.4	− 2.2	44.4	− 1.9	7.4	16.6
16C	16.3	4.4	5.5	18	37.4	0.9	39.1	13.8	12.3	31.4
16S	− 3.4	−21.9	− 2.5	− 4	−17.8	−13.2	39.1	−10.5	7.6	19.5
17C	5.6	32.4	13.1	19	10.4	10.4	34.6	15.1	8.7	25.1
17S	11.1	43.6	23.8	27	19.0	18.8	34.6	23.9	10.1	29.2
18C	− 8.7	− 3.9	−20.3	−10	6.4	−22.5	30.9			
18S	−38.0	−89.7	−39.8	−29	−20.1	−61.9	30.9			
19C	−28.5	− 6.2	−15.3	5	−62.5	−24.4	27.7			
19S	−45.4	− 0.3	−23.3	−77	−20.0	−30.0	27.7			
20C	2.1	27.7	5.9	19	36.6	− 2.0	25.0			
20S	1.5	−72.5	1.1	14	2.1	−31.7	25.0			
21C	−29.0	−18.8	−25.3	− 9	−46.6	−27.0	22.7			
21S	1.0	26.3	16.2	− 1	19.2	10.8	22.7			
22C	−41.3	−18.0	−22.5	−21		−41.2	18.9			
22S	5.0	−39.8	16.7	20		−15.9	18.9			
23C			−11.4	−19						
23S			2.4	−17						
24C			−22.8	10						
24S			− 4.2	− 1						
25C			11.5	−13						
25S			7.8	−19						
26C			8.5	11						
26S			− 0.5	− 4						
27C			− 3.6	−16						
27S			12.8	2						
28C			−55.1							
28S			− 8.7							
29C			−15.4							
29S			−15.9							
30C			29.0							
30S			2.6							
C_O	55.10	55.75	55.32	54.54	53.51	55.02	70.2	54.9	.71	1.0
S_O	37.52	40.09	38.71	37.88	40.48	39.93	70.2	39.1	1.13	1.6
C_C	11.32	12.57	15.22	16.64	11.85	12.97	27.0	13.6	1.85	6.9
S_C	29.87	27.45	27.22	27.07	27.57	28.33	27.0	27.9	.96	3.6
C_S	− 5.26	− 2.50	− 3.74	− 3.52	− 3.71	− 3.53	11.9	− 3.7	.81	6.8
S_S	9.64	5.59	7.70	8.24	10.67	5.98	11.9	8.0	1.82	15.3

FIGURE 2.
COMPARISON OF GLOBAL GEOPOTENTIAL SOLUTIONS
FOR 13th ORDER TESSERAL HARMONICS
COMPARED TO FOURIER TERMS



lumped coefficients) with strong GEOS-II tracking. GEM1 [Smith et.al, 1973] contains optical data only. GEM3 [Lerch et.al., 1972] has electronic (Tranet Doppler, C and S-band radar and laser) and additional optical data added to the GEM1 tracking. The data is employed at full weight (according to the accuracies of the systems as judged by the arc residuals). Thus, GEOS-II tracking dominates the GEM3 geopotential, but this heavy tracking had some deficiencies which were remedied in GEM5 and later solutions. The chief deficiency was the poorly known Doppler stations. In GEM5 [Lerch et.al., 1974] the electronic and optical data for the GEOS (I and II) orbits were downweighted to reduce the effect of these processing errors. With PGS62 (F.J. Lerch, Private communication, 1974), the station positions and data biases were resolved and the GEOS-II data (with additional Doppler and laser tracking) had full weight again.

In Table 12 the Doppler solution uses only Doppler data on 9 distinct satellite orbits with heavy GEOS-II coverage. The Riegber and Ilk (1975, Table 2, Col. 3) solution uses optical and laser data reduced to 36 (resonance) condition equations on 6 orbits including GEOS-II.

In spite of these differences in observation and data reduction, the GEOS-II lumped harmonics are relatively stable compared to the geopotential itself. [In Table 12, we compare the RMS of coefficient variation with Kaula's rule ($10^{-5}/\ell^2$). This is a uniform measure of precision over all degrees. For the lumped coefficients the estimate ($10^{-5}/\ell^2$) is actually the root sum of squares of all terms in each lumped component.] Except for S_s, the lumped coefficients have a precision about an order of magnitude greater than the geopotential harmonics. The greater scatter of S_s values may be due (subtley) to the scarcity of GEOS-II observations around ω=190°. This comparison is presented graphically in Figure 2. In general the lumped coefficients appear to represent excellent constraints for the GEOS-II tracking record.

Comparison with Douglas, Marsh and Williamson (1969)

Douglas, Marsh and Williamson (1969) performed an analysis to recover resonance terms from timing errors (along track errors) in the Rosman GRARR data over a 5 day arc of GEOS-II. They modeled the odd degree, 13th order terms with the values of Yionoulis (1968) and attempted to enhance this set of coefficients by solving for a value of (14,13) which would combine to properly model the GEOS-II resonance perturbations. They assumed all the timing error was taken up in the resonance perturbation of the mean anomaly. This is the second term of the expression for M in equation (3). A more complete expression for the along track error [$\Delta\omega+\Delta m+\cos i\cdot\Delta\Omega$] which we use, differs from theirs by about 10%. In addition the Douglas-Yionoulis values apply only to the limited argument of perigee during their 5-day arc (~350°). Nevertheless a comparison of our lumped coefficient with theirs at this perigee for GEOS-II is reasonably close.

Using the constraint for along track information only [Table 11 in (10) for our (multi-arc) solution; equations (16-18) in (10) for the Douglas-Yionoulis set]; this comparison is presented in Table 13. The 6 'global' constraints themselves, computed from the Douglas-Yionoulis field are much poorer. But GEOS-II data is only represented in that field by a single satellite arc (see Table 1).

Comparison with Riegber's (1973) Constraint

In his 1973 paper Riegber has shown how it is possible to derive resonance (or other periodic) variations from a 'Fourier' solution of the satellite's motion as a boundary value problem. Instead of the 'natural' frequencies ($\dot{\psi}$) of

TABLE 13
COMPARISON OF DOUGLAS–YIONOULIS AND GEOS–II
GLOBAL RESONANCE ANALYSIS FOR
LUMPED COEFFICIENTS AT
$\omega = 349^{o}.65$

	$\underline{C \times 10^9}$	$\underline{S \times 10^9}$
Douglas, et al, 1969	75.02	65.25
GEOS–II Resonance Analysis (this report)	67.81	64.14

the orbital perturbations, Riegber uses twice the period of the (analyzed) arc as fundamental, and all necessary subharmonics of this to describe the variations of 'Kepler' element combinations. Using these (arbitrary) frequencies, he calculates (by quadrature) theoretical amplitudes for the element combinations in the style of a Fourier analysis. The advantage of the method is that it (apparently) separates effects perfectly; only orthogonal functions are determined. The disadvantage is that the natural frequencies are not all simply related to (even) a well chosen arc length. Therefore, a large number of terms may have to be evaluated to resolve a variation of a few close frequencies. More serious may be the restriction that only the amplitudes are determined by the method. Geopotential phase information is lost in defining two boundary values for each element from the orbit data. Nevertheless, impressive results have been achieved with this method (Riegber, 1974,1975) and an agreement with our analysis can be demonstrated.

Riegber (1973) analyzed optical data in a GEOS-II arc (5.8 days long) for the amplitude of the resonant variation of $\omega+M+\Omega$. The measured amplitude is related to a condition equation (a calculated quantity) involving all 13^{th} order harmonics. Unfortunately, no direct check can be made with our influence coefficients since ours are for sine and cosine terms independently. They contain phase as well as amplitude information. However, the amplitudes of our lumped coefficients $[(C^*)^2+(S^*)^2]^{1/2}$ can be compared to Riegber's.

In Table 14 we make this comparison for 3 fields:

1. Our GEOS-II field from individual arcs (Table 7, along track)

2. The SAO SE2

TABLE 14. COMPARISON OF GEOS–2 RESULTS WITH A RIEGBER CONSTRAINT

DATA ARC IS 5.8 DAYS LONG

			CALCULATED AMPLITUDE (10^{-7} RADIANS)		
'FOURIER' PERIOD (DAYS)	ω^{o}	OBSERVED AMPLITUDE (10^{-7} RADIANS)	SAO SE 2	OUR INDIVIDUAL ARC SOLUTION	RIEGBER ADJUSTED SOLUTION
5.8	261.2	300 ± 16	303	303	327

CALCULATED AMPLITUDES |C*,S*| AND (RATIO) |C*,S*| /CALCULATED RIEGBER AMPLITUDE

			SAO SE 2	OUR INDIVIDUAL ARC SOLUTION	RIEGBER ADJUSTED SOLUTION
			55.95	56.91	62.32
			(5.4)	(5.4)	(5.3)

3. Riegber's (1973) corrected with 10^9 $(17,13)_{C,S}$ = (60,-4.5)

It is noted that our field agrees with Riegber's observation as well as his which was 'fit' to this data.

The computed lumped coefficients for ω+M+Ω by our analysis are very close to those for the along track since it is dominated by ω+M which is the same for both. But the correspondence of the amplitudes of our lumped values (using the various fields) with those computed from Riegber's condition equations is close but not exact. It is gratifying that the comparison is good.

Verification of Results Using GEOS-II Data

Another obvious means for assessing the accuracy of the derived values for the m=13 resonance coefficients found in this analysis is to use these coefficients and compare orbit determinations with them and other sets of coefficients. Five epochs were selected having a wide range of argument of perigee. The data reductions were repeated three different ways:

- a solution was performed using the GEM6 (Lerch, 1974) Gravity Model without any m=13 terms. This was used as a basis for assessing the total resonance information in the arcs.

- a solution was performed using the GEM6 Gravity Model complete to (22,14). Three pairs of resonance coefficients were adjusted - $(14,13)_{C,S}$, $(15,13)_{C,S}$ and $(16,13)_{C,S}$. These solutions were used with those above to gauge the total resonance information in the arcs.

- and lastly, the same arc was reduced using GEM6 without its m=13 coefficients, but added to GEM6 were the resonance coefficients recovered from the multi-arc analysis (presented in Table 10). When this solution is compared to the one immediately above, one can get a fairly accurate measure of resonance modeling obtained using the constraints derived in this analysis.

Table 15 summarizes these results.

It is realized that this form of verification has certain limitations. When one introduces six additional degrees of freedom to any data reduction, other errors (i.e., drag, solar radiation pressure, low degree and order geopotential) will be partially accommodated. Therefore, the solution using the complete GEM6 with three pairs of coefficients adjusting may yield results which include accommodation to these other error sources. Nevertheless, the level of resonance modeling obtained from our analysis can be inferred (pessimistically) from the results presented in Table 15. Our global solution models all but about 1.7% of the 13th order resonance information in the GEOS-II orbit.

Table 15 uses weighted RMS as its measure of the quality of fit to the data. The actual data weights employed were:

camera observations:	2"
range (laser) observations:	5m

The laser range data was also sampled so that only 50 pts/pass was used in this analysis.

TABLE 15.

RESONANCE VALIDATION USING GEOS–II DATA

		RMS OF FIT TO THE DATA					
EPOCH OF ARC	ARG OF PERIGEE (EPOCH)	GEM 6 W/OUT ANY (m=13) RES TERMS ①	GEM 6 ADJ. (14, 13) (15, 13) (16, 13) ②	GEM 6 USING DERIVED (m=13) RES TERMS ③	④ TOTAL RESONANCE CONTRIBUTION ① − ②	⑤ ESTIMATED DEGRADATION USING DERIVED VALUES ③ − ②	PERCENT PERFORMANCE OF DERIVED MODEL ⑤ ÷ ④
680509	337°	10.616	0.840	0.876	9.776	.036	0.0%
680407	30°	9.224	0.776	0.823	8.448	.047	0.6%
680907	140°	3.348	0.857	0.885	2.491	.028	1.1%
690207	252°	14.739	0.930	1.360	13.809	.430	3.1%
680611	282°	9.666	0.871	1.026	8.795	.155	1.8%
						RMS PERFORMANCE	1.7%

CONCLUSIONS

Gravitational constraint equations have been derived from GEOS-II data and a detailed analysis of the shallow resonance problem. These equations follow from elementary perturbation theory and the along-track constraint derived from them accounts for all but about 2% of the 13th order resonant information in the tracking data. The equations are also in good agreement with recent comprehensive gravity models of SAO, GSFC and from Doppler data only, which use substantial amounts of GEOS-II data.

The goal of the analysis has been met; to derive simple constraints which account for nearly all of the shallow resonant information in satellite orbits. The method makes it feasible to reduce comprehensive field models to the lumped coefficients for orbits or orbital arcs used in their solutions. The proper combination and extension of this constraint data should be straightforward, but is a task left for the future.

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