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TWO-DIMENSIONAL CONVOLUTE INTEGERS FOR OPTICAL IMAGE DATA PROCESSING AND SURFACE FITTING

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16. ABSTRACT

Two-dimensional convolute integers for optical image data processing and surface fitting represent an approach toward low-pass, high-pass and band-pass filtering. These convolution coefficients possess the filtering speed associated with a moving smoothing average without suffering a loss of resolution. Resolution is retained because these coefficients represent the equivalence of applying high order two-dimensional regression calculations to an image without considering the time-consuming summations associated with the usual normal equations. The smoothing (low-pass) and roughing (high-pass) aspects of the filters are a result of being derived from regression theory. The coefficients are universal integer values completely described by filter size and surface order, and possess a number of symmetry properties.

Double convolution leads to a single set of coefficients with an expanded mask which can yield band-pass filtering and the surface normal. For low order surfaces (0, 1), the two-dimensional convolute integers are equivalent to a moving smoothing average.

Sufficient calculations are presented to indicate the manner by which these coefficients are derived for any order surface and any mask size.

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INTRODUCTION

Two-dimensional convolute integers are sets of convolution coefficients of integer value which can be used in optical image data processing and surface fitting problems. Applying these coefficients to a set of data is equivalent to doing two-dimensional regression calculations: i.e., fitting a surface by least squares, without considering the time-consuming summations associated with the usual normal equations. The only requirement for use of these coefficients is that the data points on the surface be either equidistant along their orthogonal axis, or be spaced an integer multiple of a common factor.

A moving smoothing average is one of the fastest algorithms which can be applied for noise filtering in an optical image, but it tends to reduce resolution. The coefficients in such a filter are convolution coefficients. Two-dimensional convolute integers utilize the fast logic of the moving smoothing average without sacrificing resolution. From the definition of these coefficients, ten properties appear. These properties indicate the existence and universal aspect of the filter numbers. As a consequence of being derived from regression theory, the filters have low-pass and high-pass filtering properties. This is equivalent to smoothing and generating the partial derivatives associated with surface order. The number of unique integers per filter represents the minimum number of multiplications per image point and significantly affects the filtering time. A zero order or first order surface is identical to nearest neighbor averaging. Double convolution, which represents band-pass filtering and can generate surface normals, can be accomplished by a single set of integers using an expanded filter mask.

The appendices contain detailed calculations for a 25-point filter, 5 by 5 mask, second order surface. The normal equations for a first and third order are also presented.

MOVING SMOOTHING AVERAGE

The concept of a moving smoothing average represents one of the simplest and fastest filters that can be applied to noise reduction in a two-dimensional image. The filter can be considered a mask, m rows by n columns, which when applied to the data in a two-dimensional image causes a nearest neighbor interaction.

The interaction results from adding the intensities of all the points surrounding a particular location, including the value of the intensity at this location in the sum, and then dividing by the total number of points considered. In this way, nearest neighbors have an opportunity to influence the new average value, which shall be considered the filter data point.

This m by n mask can be passed over all the data points in an image, with the exception of a border of points left unfiltered. Requiring an odd number of rows and columns unambiguously determines the center point of the filter mask, which can then be shifted column by column, advancing one row at a time when a border column is reached.

In particular, each data point in the mask can be thought of as multiplied by a coefficient C_{ij} [Eq. (1)], added with all other points in the mask, the sum divided by a normalizer (Norm), and the resulting new filtered data point put in place of the raw data point. By this method, the raw data array is transformed into a filtered data array.

For a moving smoothing average, all the C_{ij} terms equal unity, and Norm is equal to the number of points in the array.

$$z'_{rc}(x,y) = \sum_{i=-m}^{m} \sum_{j=-n}^{n} C_{ij} z_{ij}(x,y)/\text{Norm} \quad , \tag{1}$$

where

z = array of raw data points,

z' = array of filter data points,

ij represents a point in the $(2m + 1)$ by $(2n + 1)$ filter, and rc

represents a raw data point in an image at row r and column c and is the center point of the filter mask.

Such a filter is nonrecursive and, therefore, requires raw data points in the Z vector at all times. Unfortunately, this type of filtering, though simple and fast, tends to blur an image by decreasing the overall sharpness, or resolution.

The problem which then arises requires the development of a set of weighting coefficients C_{ij} and Norm which filter with the speed of Eq. (1) and yet do not decrease resolution.

CONVOLUTION

The coefficients C_{ij} in Eq. (1) are convolution coefficients for the following reason. The filter can be considered an operator which forms the filtered data by integrating the raw data over a weighting function $\Omega(\alpha,\beta)$:

$$z'(x,y) = \int \Omega(\alpha,\beta)\, z(x-\alpha,\ y-\beta)\, d\alpha d\beta \quad . \tag{2}$$

This integral is defined as the convolution of $z(x-\alpha,\ y-\beta)$ with $\Omega(\alpha,\beta)$. In a digital filter, the weighting function is of the form

$$\Omega(\alpha,\beta) = \sum_{i=-m}^{m} \sum_{j=-n}^{n} C_{ij}\delta(\alpha + i,\ \beta + j)/\text{Norm} \quad , \tag{3}$$

where δ is the Dirac delta function representing the discrete sampling of the data. Using this in Eq. (2) gives

$$z'(x,y) = \int \left[\sum_{i=-m}^{m} \sum_{j=-n}^{n} C_{ij}\delta(\alpha + i,\ \beta + j)/\text{Norm} \right] z(x-\alpha,\ y-\beta)\, d\alpha d\beta$$

$$= \sum_{i=-m}^{m} \sum_{j=-n}^{n} C_{ij} z(x + i,\ y + j)/\text{Norm} \quad . \tag{4}$$

When z(x + i, y + j) is equated with $z_{ij}(x,y)$, Eq. (4) is seen to be identical to Eq. (1). Thus, the C_{ij}'s and Norm are known as convolution coefficients.

TWO-DIMENSIONAL REGRESSION

Two-dimensional regression calculations are equivalent to fitting a polynomial surface to a set of data by the method of least squares [1]. A general expression for a two-dimensional surface is

$$z'_k = \sum_{i=0}^{m} \sum_{j=0}^{n} A_{ij} x_k^i y_k^j \quad . \tag{5}$$

Equation (5) represents the intensity z'_k at the point k on the surface of order m + n described by the regression coefficients A_{ij}.

Let z_k represent the actual intensity at the kth position on an image. Fitting the surface to the data set $z_k(x,y)$ by the residual sum of squares leads to Eq. (6),

$$\delta^2 = \sum_{k=1}^{np} \left(z_k - \sum_{i=0}^{m} \sum_{j=0}^{n} A_{ij} x_k^i y_k^j \right)^2 \quad . \tag{6}$$

Selecting the best fit regression coefficients,

$$\partial \delta^2 / \partial A_{uv} = 2 \sum_{k=1}^{np} \left(z_k - \sum_{i=0}^{m} \sum_{j=0}^{n} A_{ij} x_k^i y_k^j \right) x_k^u y_k^v = 0 \quad , \tag{7}$$

leads to the normal equation

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=1}^{np} A_{ij} x_k^{i+u} y_k^{j+v} = \sum_{k=1}^{np} z_k x_k^u y_k^v \quad , \tag{8}$$

where $u = 0, m$ and $v = 0, n$. Matrix representation of Eq. (8) results in the more familiar form of the normal equations.

$$\mathbf{XY}^T \cdot \mathbf{XY} \cdot A = \mathbf{XY}^T \cdot Z \quad , \tag{9}$$

where Z is a column vector of raw data points, np by 1; A is a column vector of coefficients representing the surface, nt by 1; $\mathbf{XY}$ is a matrix of cross product terms, np by nt; and

$$nt = (m+n+1) \cdot (m+n+2)/2 \quad ,$$

number of rows and columns in the symmetric $\mathbf{XY}^T \cdot \mathbf{XY}$ matrix.

TWO-DIMENSIONAL CONVOLUTE INTEGERS

Definition

Rewriting Eq. (1) in matrix form allows an identification of the regression coefficients, A, with the convolution coefficients C.

$$z'_{rc} = C \cdot Z/\text{Norm} \tag{10}$$

(Note that C is a row vector of double-subscripted elements, Z is a column vector of double subscripted elements, and both have np elements.) Solving the normal equations for A, which is equivalent to obtaining intensity values on the fitted surface, yields

$$A = \left(\mathbf{XY}^T \cdot \mathbf{XY}\right)^{-1} \cdot \mathbf{XY}^T \cdot Z \quad . \tag{11}$$

Recognizing the similarity in form between Eq. (10) and (11) (the A's and Z's are both linear combinations of the Z's) permits a definition of the A's in terms of the C's,

$$A_{ij} = C \cdot Z/\text{Norm} \quad . \tag{12}$$

Evaluation of the C's and Norm associated with each A_{ij} is readily accomplished by applying Cramer's Rule for obtaining the A's in Eq. (9) and identifying the coefficients of Z as C/Norm.

Properties

The convolution coefficients have ten properties which can be seen by solving Eq. (11) and identifying the C's and Norm. These properties, listed in Table 1, will be discussed in detail and lead to the application of the sets of C's and Norm's in optical image data processing and surface fitting.

TABLE 1. PROPERTIES OF THE TWO-DIMENSIONAL CONVOLUTION COEFFICIENTS

1. Existence – Number of points on surface must be greater than surface order.
2. Universal Numbers – Convolution coefficients depend only on surface order and filter mask size.
3. Antisymmetry – The cross product matrix is sparse.
4. Equal Interval – Convolution coefficients are integral numbers.
5. Regression – Convolution coefficients derived from regression theory are filters.
6. Surface Fitting – Convolution coefficients represent surface partial derivatives.
7. Equivalences – Convolution coefficients for adjacent surface orders are identical, as are the transpose of complementary roughing filters of the same order.
8. Symmetries – Convolution coefficients have a minimum unique set of values per filter.
9. Plane Surface – Convolution coefficients for a first order surface represent a moving smoothing average.
10. Double Convolution – Convolution coefficients for band-pass filtering and surface normals represent double convolution.

Existence. For each A_{ij} associated with a surface of order m+n, there exist a set of C's and a value for Norm so long as

$$\mathbf{XY} \cdot \mathbf{XY}^T \neq 0 \quad . \tag{13}$$

This nonsingularity requirement indicates that the number of points on the surface, np, must be greater than the order of the surface, m+n.

Universal Numbers. The values of C and the value of Norm associated with a surface of order m+n and np points are independent of the values of the intensities, z_k, on the surface. Thus, the convolution coefficients are universal numbers completely described by two numbers, m+n and np.

Antisymmetry. An antisymmetric filter mask has an odd number of rows and columns, not necessarily equal, which, as previously stated, unambiguously establishes the center point as 0, 0. Such a mask reduces the elements in the symmetric cross product matrix to zero where

$$\sum_{k=1}^{np} x_k^p y_k^q = 0 \quad , \tag{14}$$

when either p or q is odd. Equation (14) is true since the single sum over all the data points in the filter mask can be replaced by a double summation, Eq. (1) or Eq. (15).

$$\sum_{k=1}^{np} = \sum_{x=-r}^{r} \sum_{y=-c}^{c} \quad . \tag{15}$$

where

$$np = (2r+1) \cdot (2c+1) \quad .$$

Substitution of this expression in Eq. (14) leads to the sparseness for antisymmetric mask since each sum on the right-hand side of Eq. (16) is zero.

$$\sum_{k=1}^{np} x_k^p y_k^q = \sum_{x=-r}^{r} x^p \sum_{y=-c}^{c} y^q = 0 \quad , \tag{16}$$

when either p or q is odd. Because of the sparseness, the regression coefficients A_{ij} can be partitioned into matrices of lower order (see Appendix A).

Equal Interval. An equal interval filter mask has the spacing between data points an integer multiple of a common factor. The common factor between rows Δr need not equal the common factor between columns Δc,

$$\Delta r \neq \Delta c \quad , \tag{17}$$

and the integers need not be consecutive but must remain fixed as the mask moves over an image (otherwise the C's and Norm change). As a consequence, the terms in the cross product matrix $\mathbf{XY}$ are integer values, exclusive of the common factors Δr and Δc, and accordingly the C's and Norm are integer values.

Factoring common integers from the C's and Norm leads to a universal set of integers associated with each A_{ij} which can be described as two-dimensional convolute integers [2]. Note that the antisymmetry and equal interval properties are mutually independent.

Regression. As a result of being derived from regression theory, sets of C's and Norm's satisfy two filtering principles. The A_{ij} terms can be obtained from the normal equations (see Appendix A) by Cramer's Rule. Now the coefficients of z on the right-hand side of the normal equations are identical to the first column of the $\mathbf{XY} \cdot \mathbf{XY}^T$ matrix (see Appendices A and B). Solving the normal equations for A_{ij} and factoring the Σz terms from the i+j column of the numerator clearly identifies the C's and Norm (see Appendix A). Summing the C's over all the data points in the mask and considering the determinant of the numerator leads to two filtering properties [3].

(1) Smoothing: The set of C's and Norm associated with A_{00} satisfy

$$\sum_{k=1}^{np} C_k/\text{Norm} = 1 \quad , \tag{18}$$

low-pass filtering.

(2) Roughing: The set of C's and Norm associated with A_{ij}, where $i, j \neq 0$, satisfy

$$\sum_{k=1}^{np} C_k/\text{Norm} = 0 \quad , \tag{19}$$

band-pass or high-pass filtering.

Surface Fitting. The two-dimensional convolution filters generate the various partial derivatives for a surface of order m+n fitted to a data set by regression. This can be seen by identifying all the regression coefficients in Eq. (5), evaluated at the center point of the mask.

The constant term, A_{00}, is the fitted value of the intensity or a smoothed data point (the zero partial derivative). Thus, the C's and Norm associated with A_{00} represent smoothing filters.

Expanding Eq. (5), assuming an antisymmetric equal interval ($\Delta r \neq \Delta c$) mask, taking partial derivatives evaluated at the central point ($x, y = 0$), leads to Eqs. (20) through (23).

$$\frac{\partial^{i+j} z_k}{\partial r^i \partial c^j} = \frac{1}{\Delta r^i \Delta c^j} \frac{\partial^{i+j} z_k}{\partial x^i \partial y^j} \tag{20}$$

where

$$\begin{aligned} r &= x \cdot \Delta r + \alpha \quad , && \text{row spacing;} \\ c &= y \cdot \Delta c + \beta \quad , && \text{column spacing;} \end{aligned} \tag{21}$$

and x and y are antisymmetric integer values. At the center point in the filter mask,

$$\frac{\partial^{i+j} z_{00}}{\partial x^i \, \partial y^j} = (i+j)!A_{ij} \quad . \tag{22}$$

The fitted partial derivatives of the surface at x,y = 0 are

$$A_{ij} = \frac{\Delta r^i \, \Delta c^j}{(i+j)!} \; \frac{\partial^{i+j} z_{00}}{\partial r^i \, \partial c^j} \quad . \tag{23}$$

Thus, the C's and Norm's associated with A_{ij} where $i, j \neq 0$ represent partial derivative (roughing) filters. Note that Eq. (23) indicates that A_{00} terms are independent of the row column spacing (Δr, Δc).

Equivalences. As a result of the antisymmetric properties of the filter mask, a number of equivalences exist between convolution coefficients of adjacent surface orders and between the transpose of complementary roughing filters of the same surface order. The effect of these equivalences is to reduce the number of calculations required to present all the filters associated with the various surface orders and mask sizes.

Table 2 lists the two types of equivalences in the convolution coefficients: $C^{\ell}(ij)$, where ℓ represents surface order and ij the appropriate partial derivative. The proof of these properties is presented in Appendix B, which shows the normal equations for a first order and third order surface.

Symmetries. Some general symmetry properties exist for the set of C's associated with each A_{ij}. These properties depend on ij odd or even and result from the symmetry associated with the normal equations, the antisymmetry of the mask, and the equal interval between data points. The symmetry properties are expressed in Table 3. An alternate expression for the convolution coefficients leads to these properties, Appendix B.

Coefficients which lie on a zero row or column have a two-fold symmetry. All other coefficients have at least a four-fold symmetry; and for a symmetric mask, nondiagonal, nonzero, row or column terms possess eight-fold symmetry.

TABLE 2. EQUIVALENCES[a,b,c]

Regression Coefficient	Surface Order	
A(ij)	ℓ odd	ℓ even
i+j odd	$C^{\ell} = C^{\ell+1}$	$C^{\ell} = C^{\ell-1}$
i+j even	$C^{\ell} = C^{\ell-1}$	$C^{\ell} = C^{\ell+1}$
Complementary Transpose	Identical Surface Order	
ji → ij	$C(ji) = C(ij)^T$	

a. ℓ represents the order of the surface.

b. ij represents the appropriate partial derivative.

c. T represents transpose.

TABLE 3. SYMMETRY PROPERTIES[a b]

Mask Symmetry

$i = j$ Symmetric Filter

$i \neq j$ Nonsymmetric Filter

Coefficient Symmetry

$$C_{-r-c} = (-1)^{i+j} C_{rc}$$

$$C_{-rc} = (-1)^{j} C_{rc}$$

$$C_{r-c} = (-1)^{i} C_{rc}$$

Redundancy

$i \neq j$		$i = j$
4 fold	$r, c \neq 0$	4 fold
2 fold	$r, c = 0$	2 fold
4 fold	$r \neq c \neq 0$	8 fold

Zero Terms

$C_{ro} = 0$	i odd
$C_{oc} = 0$	j odd

[a] r, c represent row, column location in the filter mask.

[b] i, j represent the appropriate partial derivative.

These properties allow the number of multiplications per filter to be considerably reduced. Table 4 indicates the number of multiplications per filter. In general this table gives the maximum number of multiplications. Specific filters are capable of having fewer unique numbers; e.g., multiplications. These properties sufficiently affect the time required to filter an image.

A Plane Surface. A satisfying, though possibly trivial result, is the equivalence to be found between regression calculation filtering by a plane surface and nearest neighbor averaging, which was discussed under the heading Moving Smoothing Average. Appendix B indicates the reduced form of the normal equations for a first order surface. For smoothing, the C's are clearly unity and Norm = np, the number of points considered. Smoothing an image by applying the two-dimensional convolute integers for a first order surface is identical to a moving smoothing average.

Double Convolution. The convolute integers generated so far have satisfied low-pass and high-pass filtering criteria. Combining these filters will result in band-pass filtering, or double convolution.

Let z_k^p represent the filtered value of a point in an image as a result of convolving the raw data with a pth order, rth row and cth column filter.

$$z_k^p = \frac{(i+j)!}{\Delta r^i \Delta c^j} \sum_{m=-r}^{r} \sum_{n=-c}^{c} C_{mn}^p z_{mn} / \text{Norm}_p \quad . \tag{24}$$

This may be a smoothing $(i, j = 0)$ or high-pass filter $(i, j \neq 0)$.

Assuming the data needs additional filtering and the mask spacing $(\Delta r, \Delta c)$ remains fixed, let z_k^{qp} represent convolving the already filtered data with a surface of qth order.

$$z_k^{qp} \quad \frac{(i' + j')!}{\Delta r^{i'} \Delta c^{j'}} \sum_{m'=-r'}^{r'} \sum_{n'=-c'}^{c'} C_{m'n'}^q z_{m'n'}^q / \text{Norm}_q \tag{25}$$

TABLE 4. MULTIPLICATIONS PER FILTER[a][b]

Mask	No. of Multiplications
Symmetric,	
$i = j$	
even, even	$(np+1)(np+3)/8$
odd, odd	$(np-1)(np+1)/8$
Nonsymmetric,	
$i \neq j$	
even, even	
$i, j <$ order	$\left[(np-1)(np+1)/4\right] + 1$
$i, j =$ order $(j, i = 0)$	$(np+1)/2$ or $(np-1)/2$
odd, odd	
$i, j <$ order	$(np-1)/2$
$i, j =$ order[c]	
even, odd	
$i, j <$ order	$[(np-1)/2]^2$ or $(np-1)(np+1)/4$
$i, j =$ order $(j.i = 0)$	$(np-1)/2$

a. np represents the number of data points in a row or column of the filter mask.

b. i, j represents the appropriate partial derivative.

c. i, j combination not possible.

$$z_k^{qp} = \frac{(i' + j')!\,(i + j)!}{\Delta r^{i+i'}\ \Delta c^{j+j'}} \sum_{m'=-r'}^{r'} \sum_{n'=-c'}^{c'} \sum_{m=-r}^{r} \sum_{n=-c}^{c} C^q_{m'n'}$$

$$\cdot\ C^p_{mn}\ z_{m+m',\ n+n'} / \text{Norm}_p\ \text{Norm}_q \qquad (26)$$

Recognizing that the factorial terms and the powers of Δr and Δc are merely gain factors (which in no way affect the filtering), the double convolution can be combined into a single convolution where

$$z_k^{qp} = \sum_{\alpha=-(r+r')}^{r+r'} \sum_{\beta=-(c+c')}^{c+c'} C_{\alpha\beta}\ z_{\alpha\beta} / \text{Norm} \quad , \qquad (27)$$

where

$$C_{\alpha\beta} = \sum_{m'=-r'}^{r'} \sum_{n'=-c'}^{c'} \sum_{m=-r}^{r} \sum_{n=-c}^{c} C_{m'n'}\ C_{mn} \quad , \qquad (28)$$

and where the sums are constrained $\alpha = m'+m$ and $\beta = n'+n$, and where

$$\text{Norm} = \text{Norm}_p\ \text{Norm}_q \quad . \qquad (29)$$

The $C_{\alpha\beta}$'s and Norm show that double convolution can be accomplished by a new single set of two-dimensional convolute integers using a $[2(r+r') + 1]$ by $[2(c+c') + 1]$ expanded mask. A considerable savings in time results from applying the results of Eqs. (28) and (29). Indeed, double convolution represents the means by which a single filter can be constructed for obtaining the normal to the surface, at points in an image, or on a two-dimensional surface.

CONCLUSION

The theory of two-dimensional convolute integers has been discussed emphasizing the filtering aspects of the coefficients for optical image data processing. The coefficients generate the various partial derivatives associated with the surface order. Band-pass filtering and surface normals can be obtained from a single set of integers with an expanded mask representing double convolution.

The coefficients have the filtering speed of a moving smoothing average and retain resolution since they represent two-dimensional regression calculations. Being derived from regression theory, the coefficients satisfy the criteria for low-pass and high-pass filtering.

Based upon a few simple concepts — (1) antisymmetry, (2) equal interval, and (3) mask rows equal columns —, a number of equivalences exist between surface orders and among filter coefficients. All these properties tend to reduce the number of unique coefficients per filter; e.g., the number of multiplications per filter point, or enhance the speed of the filter. The integer nature of the coefficients also tends to speed the filtering operation. Each filter is completely described by two numbers, mask size and surface order.

Detailed calculations have shown the surface order and coefficient equivalences between first, second, and third order surfaces and indicate the manner by which two-dimensional convolute integers for any size mask and any order surface can be obtained.

A companion report is planned which will deal with the software and hardware aspects of the filters and provide extensive tabulated data on filter coefficients.

APPENDIX A

TWO-DIMENSIONAL CONVOLUTE INTEGERS

SECOND ORDER SURFACE

5 BY 5 MASK

The normal equations for the quadratic surface are

$$\begin{array}{c|cccccc||c|c|c|c}
m, n & 00 & 01 & 02 & 10 & 11 & 20 & & & & uv \\
\hline
 & np & \Sigma y & \Sigma y^2 & \Sigma x & \Sigma xy & \Sigma x^2 & A_{00} & & \Sigma z & 00 \\
 & \Sigma y & \Sigma y^2 & \Sigma y^3 & \Sigma xy & \Sigma xy^2 & \Sigma x^2y & A_{01} & & \Sigma zy & 01 \\
 & \Sigma y^2 & \Sigma y^3 & \Sigma y^4 & \Sigma xy^2 & \Sigma xy^3 & \Sigma x^2y^2 & A_{02} & = & \Sigma zy^2 & 02 \\
 & \Sigma x & \Sigma xy & \Sigma xy^2 & \Sigma x^2 & \Sigma x^2y & \Sigma x^3 & A_{10} & & \Sigma zx & 10 \\
 & \Sigma xy & \Sigma xy^2 & \Sigma xy^3 & \Sigma x^2y & \Sigma x^2y^2 & \Sigma x^3y & A_{11} & & \Sigma zxy & 11 \\
 & \Sigma x^2 & \Sigma x^2y & \Sigma x^2y^2 & \Sigma x^3 & \Sigma x^3y & \Sigma x^4 & A_{20} & & \Sigma zx^2 & 20
\end{array}$$

For brevity, all summations are sums over all the raw data points $k = 1, np$; and $m + n$ is constrained to be ≤ 2.

The sparse coefficient matrix for an antisymmetric mask is

$$\left|\begin{array}{cccccc}
np & 0 & \Sigma y^2 & 0 & 0 & \Sigma x^2 \\
 & \Sigma y^2 & 0 & 0 & 0 & 0 \\
 & & \Sigma y^4 & 0 & 0 & \Sigma x^2y^2 \\
\text{Symmetric} & & & \Sigma x^2 & 0 & 0 \\
 & & & & \Sigma x^2y^2 & 0 \\
 & & & & & \Sigma x^4
\end{array}\right|
\left|\begin{array}{c} A_{00} \\ A_{01} \\ A_{02} \\ A_{10} \\ A_{11} \\ A_{20} \end{array}\right|
=
\left|\begin{array}{c} \Sigma z \\ \Sigma zy \\ \Sigma zy^2 \\ \Sigma zx \\ \Sigma zxy \\ \Sigma zx^2 \end{array}\right|$$

This leads to the following reduced matrices.

$$\begin{vmatrix} np & \Sigma y^2 & \Sigma x^2 \\ \Sigma y^2 & \Sigma y^4 & \Sigma x^2 y^2 \\ \Sigma x^2 & \Sigma x^2 y^2 & \Sigma x^4 \end{vmatrix} \begin{vmatrix} A_{00} \\ A_{02} \\ A_{20} \end{vmatrix} = \begin{vmatrix} \Sigma z \\ \Sigma zy^2 \\ \Sigma zx^2 \end{vmatrix}$$

$$\Sigma y^2 \;.\; A_{01} = \Sigma zy$$

$$\Sigma x^2 \;.\; A_{10} = \Sigma zx$$

$$\Sigma x^2 y^2 \;.\; A_{11} = \Sigma zxy$$

The evaluation of matrix terms leads to the following values.

$$np = (2r + 1)(2c + 1) = 25$$

$$\sum_{k=1}^{np} y_k^2 = \sum_{k=1}^{np} x_k^2 = \sum_{x=-2}^{2} \sum_{y=-2}^{2} y^2 = 5 \sum_{y=-2}^{2} y^2 = 50$$

$$\sum_{k=1}^{np} x_k^4 = \sum_{k=1}^{np} y_k^4 = \sum_{x=-2}^{2} \sum_{y=-2}^{2} y^4 = 5 \sum_{y=-2}^{2} y^4 = 170$$

$$\sum_{k=1}^{np} x_k^2 \, y_k^2 = \sum_{x=-2}^{2} \sum_{y=-2}^{2} x^2 y^2 = 100$$

Evaluation of the A_{00} regression coefficient by Cramer's Rule is

$$A_{00} = \frac{\begin{vmatrix} \Sigma z & 50 & 50 \\ \Sigma zy^2 & 170 & 100 \\ \Sigma zx^2 & 100 & 170 \end{vmatrix}}{\begin{vmatrix} 25 & 50 & 50 \\ 50 & 170 & 100 \\ 50 & 100 & 170 \end{vmatrix}} \;.$$

Factoring common terms from rows, expanding the numerator, and evaluating the denominator results in

$$A_{00} = \left[27 \sum_{k=1}^{np} z - 5 \left(\sum_{k=1}^{np} zy^2 + \sum_{k=1}^{np} zx^2 \right) \right] \div \text{Norm},$$

where Norm = 175. Recalling that

$$\sum_{k=1}^{np} = \sum_{x=-r}^{r} \sum_{y=-c}^{c} \quad ,$$

and combining coefficients of like terms, lead to the convolute integers

$$\begin{aligned} A_{00} = \Big(& - 13z_{-2-2} + 2z_{-2-1} + 7z_{-20} + 2z_{-21} - 13z_{-22} \\ & + 2z_{-1-2} + 17z_{-1-1} + 22z_{-10} + 17z_{-11} + 2z_{-12} \\ & + 7z_{0-2} + 22z_{0-1} + 27z_{00} = 22z_{01} + 7z_{02} \\ & + 2z_{1-2} + 17z_{1-1} + 22z_{10} + 17z_{11} + 2z_{12} \\ & - 13z_{2-2} + 2z_{2-1} + 7z_{20} + 2z_{21} - 13z_{22} \Big) \div 175 \quad . \end{aligned}$$

Identifying and expressing the C_{ij} coefficients and Norm results in the following matrix form for two-dimensional convolute integers for smoothing, using a 5 by 5 filter on a second order surface.

r \ c	-2	-1	0	1	2
-2	-13	2	7	2	-13
-1	2	17	22	17	2
0	7	22	27	22	7
1	2	17	22	17	2
2	-13	2	7	2	-13

Normalizer 175

APPENDIX B

NORMAL EQUATIONS

FIRST AND THIRD ORDER SURFACES

The normal equations for a plane surface constrain $m + n \leq 1$ and are

$$\begin{array}{l|ccc||c|c|c|l}
mn & 00 & 01 & 10 & & & & uv \\
 & np & \Sigma y & \Sigma x & A_{00} & & \Sigma z & 00 \\
 & \Sigma y & \Sigma y^2 & \Sigma xy & A_{10} & = & \Sigma zy & 01 \\
 & \Sigma x & \Sigma xy & \Sigma x^2 & A_{01} & & \Sigma zx & 10
\end{array} .$$

For an antisymmetric mask, the normal equations lead to the separable form

$$\begin{vmatrix} np & 0 & 0 \\ 0 & \Sigma y^2 & 0 \\ 0 & 0 & \Sigma x^2 \end{vmatrix} \begin{vmatrix} A_{00} \\ A_{10} \\ A_{01} \end{vmatrix} = \begin{vmatrix} \Sigma z \\ \Sigma zy \\ \Sigma zx \end{vmatrix} ,$$

where

$$A_{00} = \Sigma z/np \quad ,$$

and

$$A_{10} = \Sigma zy/\Sigma y^2 \quad ,$$

and

$$A_{01} = \Sigma zx/\Sigma x^2 \quad .$$

The following normal equations for a cubic surface constrain $m + n \leq 3$. Note that the arrangement of these equations is different from that previously considered.

ℓ	0	1		2			3				Surface Order				
i+j	even	odd		even			odd								
mn	00	01	10	02	11	20	03	12	21	30					uv
	np	Σy	Σx	Σy^2	Σxy	Σx^2	Σy^3	Σxy^2	Σx^2y	Σx^3		A_{00}		Σz	00
	Σy	Σy^2	Σxy	Σy^3	Σxy^2	Σx^2y	Σy^4	Σxy^3	Σx^2y^2	Σx^3y		A_{01}		Σzy	01
	Σx	Σxy	Σx^2	Σxy^2	Σx^2y	Σx^3	Σxy^3	Σx^2y^2	Σx^3y	Σx^4		A_{10}		Σzx	10
	Σy^2	Σy^3	Σxy^2	Σy^4	Σxy^3	Σx^2y^2	Σy^5	Σxy^4	Σx^2y^3	Σx^3y^2		A_{02}		Σzy^2	02
	Σxy	Σxy^2	Σx^2y	Σxy^3	Σx^2y^2	Σx^3y	Σxy^4	Σx^2y^3	Σx^3y^2	Σx^4y	$\cdot$	A_{11}	=	Σzxy	11
	Σx^2	Σx^2y	Σx^3	Σx^2y^2	Σx^3y	Σx^4	Σx^2y^3	Σx^3y^2	Σx^4y	Σx^5		A_{20}		Σzx^2	20
	Σy^3	Σy^4	Σxy^3	Σy^5	Σxy^4	Σx^2y^3	Σy^6	Σxy^5	Σx^2y^4	Σx^3y^3		A_{03}		Σzy^3	03
	Σxy^2	Σxy^3	Σx^2y^2	Σxy^4	Σx^2y^3	Σx^3y^2	Σxy^5	Σx^2y^4	Σx^3y^3	Σx^4y^2		A_{12}		Σzxy^2	12
	Σx^2y	Σx^2y^2	Σx^3y	Σx^2y^3	Σx^3y^2	Σx^4y	Σx^2y^4	Σx^3y^3	Σx^4y^2	Σx^5y		A_{21}		Σzx^2y	21
	Σx^3	Σx^3y	Σx^4	Σx^3y^2	Σx^4y	Σx^5	Σx^3y^3	Σx^4y^2	Σx^5y	Σx^6		A_{30}		Σzx^3	30

The antisymmetric property leads to the following sparseness:

$$\begin{vmatrix} np & 0 & 0 & \Sigma y^2 & 0 & \Sigma x^2 & 0 & 0 & 0 & 0 \\ 0 & \Sigma y^2 & 0 & 0 & 0 & 0 & \Sigma y^4 & 0 & \Sigma x^2y^2 & 0 \\ 0 & 0 & \Sigma x^2 & 0 & 0 & 0 & 0 & \Sigma x^2y^2 & 0 & \Sigma x^4 \\ \Sigma y^2 & 0 & 0 & \Sigma y^4 & 0 & \Sigma x^2y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Sigma x^2y^2 & 0 & 0 & 0 & 0 & 0 \\ \Sigma x^2 & 0 & 0 & \Sigma x^2y^2 & 0 & \Sigma x^4 & 0 & 0 & 0 & 0 \\ 0 & \Sigma y^4 & 0 & 0 & 0 & 0 & \Sigma y^6 & 0 & \Sigma x^2y^4 & 0 \\ 0 & 0 & \Sigma x^2y^2 & 0 & 0 & 0 & 0 & \Sigma x^2y^4 & 0 & \Sigma x^4y^2 \\ 0 & \Sigma x^2y^2 & 0 & 0 & 0 & 0 & \Sigma x^2y^4 & 0 & \Sigma x^4y^2 & 0 \\ 0 & 0 & \Sigma x^4 & 0 & 0 & 0 & 0 & \Sigma x^4y^2 & 0 & \Sigma x^6 \end{vmatrix} \cdot \begin{vmatrix} A_{00} \\ A_{01} \\ A_{10} \\ A_{02} \\ A_{11} \\ A_{20} \\ A_{03} \\ A_{12} \\ A_{21} \\ A_{30} \end{vmatrix} = \begin{vmatrix} \Sigma z \\ \Sigma zy \\ \Sigma zx \\ \Sigma zy^2 \\ \Sigma zxy \\ \Sigma zx^2 \\ \Sigma zy^3 \\ \Sigma zxy^2 \\ \Sigma zx^2y \\ \Sigma zx^3 \end{vmatrix}$$

The surface order equivalences can now be obtained by considering the orthogonal nature of various groups of columns, which leads to the following separable forms:

$$\begin{vmatrix} np & \Sigma y^2 & \Sigma x^2 \\ \Sigma y^2 & \Sigma y^4 & \Sigma x^2y^2 \\ \Sigma x^2 & \Sigma x^2y^2 & \Sigma x^4 \end{vmatrix} \begin{vmatrix} A_{00} \\ A_{02} \\ A_{20} \end{vmatrix} = \begin{vmatrix} \Sigma z \\ \Sigma zy^2 \\ \Sigma zx^2 \end{vmatrix} ,$$

$$\begin{vmatrix} \Sigma y^2 & \Sigma y^4 & \Sigma x^2y^2 \\ \Sigma y^4 & \Sigma y^6 & \Sigma x^2y^4 \\ \Sigma x^2y^2 & \Sigma x^2y^4 & \Sigma x^4y^2 \end{vmatrix} \begin{vmatrix} A_{01} \\ A_{03} \\ A_{21} \end{vmatrix} = \begin{vmatrix} \Sigma zy \\ \Sigma zy^3 \\ \Sigma zx^2y \end{vmatrix} ,$$

$$\begin{vmatrix} \Sigma x^2 & \Sigma x^2y^2 & \Sigma x^4 \\ \Sigma x^2y^2 & \Sigma x^2y^4 & \Sigma x^4y^2 \\ \Sigma x^4 & \Sigma x^4y^2 & \Sigma x^6 \end{vmatrix} \begin{vmatrix} A_{10} \\ A_{12} \\ A_{30} \end{vmatrix} = \begin{vmatrix} \Sigma zx \\ \Sigma zxy^2 \\ \Sigma zx^6 \end{vmatrix} ,$$

$$\Sigma x^2y^2 \,.\, A_{11} = \Sigma zxy \quad .$$

Comparing the same partial derivative regression coefficients for adjacent surface orders indicates the equivalence to be found in the convolution coefficients. The A_{00}, A_{02}, and A_{20} values are equivalent for second and third order surfaces; the A_{01} and A_{10} terms are equivalent for first and second order surfaces. Table 2 represents a general expression for the equivalences to be found in the convolution coefficients of adjacent surface orders.

An alternate definition of the filter coefficients can be expressed as

$$C^{\ell}_{ij} = \frac{\sum\limits_{\alpha=0}^{\ell} \sum\limits_{\beta=0}^{\ell-\alpha} x^{\alpha} y^{\beta} B^{\ell}_{\alpha\beta,\, ij}}{\det(D^{\ell})}$$

where $\det(D^\ell)$ represents the determinant of the cross product matrix of order ℓ,

$$D^\ell = \mathbf{XY}^T \cdot \mathbf{XY} \quad ,$$

and $B^\ell_{\alpha\beta,ij}$ represents the cofactor of the element in the $\alpha\beta$ row, ij column of D^ℓ, as expressed for the second order surface in Appendix A.

The complementary transpose of an element of a roughing filter can be expressed as

$$C^\ell_{ji}(x,y) = C^\ell_{ij}(y,x) = [C^\ell_{ij}(x,y)]^T \quad .$$

From the preceding definition of the convolution coefficients, the above is so when

$$B^\ell_{\alpha\beta,ji} = B^\ell_{\alpha\beta,ij} \,,$$

equality between the cofactors of elements in complementary columns. Since the filter masks are symmetric, y is identical with x. Either ignore the numerical factors associated with the partial derivatives, Eq. (23), or assume equal row column mask spacing, $\Delta r = \Delta c$. Now the normal equations indicate that for such masks, the cofactors of elements in complementary columns are identical. This leads to the equivalences expressed in the lower section of Table 2.

Furthermore, for these masks

$$B_{\alpha\beta,ij} = 0$$

where either $\alpha + i$ or $\beta + j$ is odd. The convolution coefficients are thus concerned with those cofactors where $\alpha + i$ and $\beta + j$ are even. This requires the parity of α and i, β and j to be the same. Now

$$C_{ij}(-x,y) = \frac{\sum_{\alpha=0}^{l} \sum_{\beta=0}^{l-\alpha} (-1)^{\alpha} x^{\alpha} y^{\beta} B_{\alpha\beta,ij}}{\det(D)} \quad .$$

Since only parity affects $(-1)^{\alpha}$, this can be replaced by $(-1)^{i}$; α and i equal parity. This leads to

$$C_{ij}(-x,y) = (-1)^{i} C_{ij}(x,y) \quad .$$

Table 3 represents a general expression for the symmetries to be found in symmetric filter masks.

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APPROVAL

TWO-DIMENSIONAL CONVOLUTE INTEGERS FOR OPTICAL IMAGE DATA PROCESSING AND SURFACE FITTING

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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