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A FINITE-DIFFERENCE PROGRAM FOR STRESSES IN ANISOTROPIC, LAYERED PLATES IN BENDING

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16. ABSTRACT

Results from the initial phase of a study of the interlaminar stresses induced in a layered laminate that is bent into a cylindrical surface are given. The laminate is modeled as a continuum, and the resulting elasticity equations are solved using the finite-difference method. The report sets forth the mathematical framework, presents some preliminary results, and provides a listing and explanation of the computer program. Significant among the results are apparent symmetry relationships that will reduce the numerical size of certain problems and an interlaminar stress behavior having a sharp rise at the free edges.

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LIST OF SYMBOLS

Symbol	Definition
A	laminate configuration; coefficient matrix [equation (22)]
B	laminate configuration; load vector [equation (22)]
B'_{ij}	constitutive matrix (Appendix A)
B_u,B_v	laminate load constants [equation (7)]
C_i	laminate load constants [equation (5)]
c'_{ij}	elastic coefficients with respect to x′,y′,z′
c_{ij}	elastic coefficients with respect to x, y, z [equation (1)]
C,D	load values [equation (33)]
D_v	laminate load constant [equation (7)]
D'_{ij}	constitutive matrix (Appendix A)
E_{ii}	Young's moduli
G_{ij}	shear moduli
h_i	node spacing (Fig. 2)
I,J	nodal coordinates (Figs. 2 and 3)
M,M_i	applied moments [equation (4a)]
m	layer number (Fig. 1)
U,V,W	displacement functions [equation (6)]
u,v,w	displacements with respect to x, y, z [equations (3) and (8)]
x,y,z	laminate coordinate axes (Fig. 1)
x′,y′,z′	lamina orthotropic axes (Fig. 1)

LIST OF SYMBOLS (Concluded)

Symbol	Definition
X	unknown vector [equation (22)]
γ_{ij}	shear strains [equation (2)]
ϵ_i	normal strains [equation (2)]
θ	lamina orientation angle (Fig. 1)
σ_i	normal stress [equation (1)]
τ_{ij}	shear stress [equation (1)]
ν_{ij}	Poisson's ratio

Symbols appearing in the computer program are defined in the subsection entitled "The Mesh Simulation."

A FINITE-DIFFERENCE PROGRAM FOR STRESSES IN ANISOTROPIC, LAYERED PLATES IN BENDING

INTRODUCTION

One critical feature associated with structural composites of laminated construction, using materials or geometrical arrangements that exhibit different elastic properties from layer to layer, is the possibility that the glued layers will separate or delaminate. This was undoubtedly realized from the outset of their use, and a brief historical sketch of the American scene is presented by Pipes [1]. However, the earliest serious investigation into the cause of delamination-type failure, namely the interlaminar stress problem, was apparently done in Japan by Hayashi [2,3], who reported that the maximum interlaminar shearing stresses occurred at the free edge of a laminate under tension. Hayashi used a plane stress model for the layers and approximated the interlaminar shears by a strain-averaging technique. Using a similar model, Puppo and Evensen [4] likewise discovered a sharp rise in the interlaminar stresses near a free edge. Notably, the use of the above models ignored the interlaminar normal stress. In two publications, Pipes and Pagano [5,6] developed a finite-difference program to solve the exact elasticity equations for a long laminate in uniaxial extension. In their development, the stresses are assumed independent of the axial coordinate and include all six components. The results of this investigation show that a sharp rise in both the interlaminar shear stresses and the normal stress occurs near the free edge. Thereafter, Oplinger [7] did an analysis of angle ply laminates in tension using a model similar to that of References 2 through 4. His approach allows a large number of layers to be considered. Indeed it was discovered that a singularity in the interlaminar shear occurs at the free edge of a laminate of one particular type of construction. An alternative solution to that employed in the above references is used by Rybicki [8] who applied a three-dimensional finite element formulation. His results agree with References 5 and 6.

The present report marks the initial phase of a study of the interlaminar stresses induced in a layered laminate by bending. Following the approach used by Pipes [5], the laminate is modeled as a continuum and the resulting elasticity equations are solved using the finite-difference method. This solution technique is made possible by assuming that the laminate is bent into a cylindrical surface such that the stresses are independent of the axial coordinate. The objective of this report is to set forth the mathematical framework, present some preliminary results, and to avail the computer program to others. The results reveal a simplifying symmetry relationship in the displacements that will allow significant reduction in the size of certain numerical problems. In addition, trends in the interlaminar stress distribution are somewhat similar to those found for stretching problems, in that a sharp rise occurs at the free edge.

PROBLEM FORMULATION

Laminate Description

The laminated composites considered in this report consist of rectangular laminae symmetrically stacked with respect to a midplane and bonded together to form a flat laminate. The bonding is assumed to provide perfect adhesion between the laminae, which nullifies the possibility of slip between adjacent laminae thus establishing the conditions of continuous displacements and tractions at each interface. Each individual lamina is considered to be elastic, homogeneous, and orthotropic (i.e., each lamina possesses three planes of elastic symmetry). The assumption of homogeneity eliminates micromechanical effects such as those involving fibers or matrix. The geometry of a typical lamina and laminate is illustrated in Figure 1. One may note that the orthotropic coordinate axes (x′,y′,z) of a lamina are referred through a clockwise rotation about z to the fixed coordinate axes (x, y, z) of the laminate. The laminae are stacked along z to form a laminate whose sides are normal to x, y, and z. Each lamina is given a layer number m.

Limiting the analysis to linear elastic materials, the constitutive relation for each lamina referred to the x, y, z coordinate system is

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ & c_{22} & c_{23} & 0 & 0 & c_{26} \\ & & c_{33} & 0 & 0 & c_{36} \\ \text{(symmetric)} & & & c_{44} & c_{45} & 0 \\ & & & & c_{55} & 0 \\ & & & & & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} , \qquad (1)$$

where the elastic constants c_{ij} are related to the nine orthotropic constants c'_{ij} through the well known transformation equations of References 9 and 10.[1] By associating the displacements u, v, and w with x, y, and z, respectively, the strains for each lamina are defined as

1. In using the transformation equations in References 9 and 10 substitute $-\theta$ for $+\theta$ since here the constants are referred to the unprimed coordinate axes of the laminate.

$$\epsilon_x{}^m = u,_x{}^m \qquad \epsilon_y{}^m = v,_y{}^m \qquad \epsilon_z{}^m = w,_z{}^m$$

$$\gamma_{yz}{}^m = w,_y{}^m + v,_z{}^m \qquad \gamma_{xz}{}^m = w,_x{}^m + u,_z{}^m \qquad \gamma_{xy}{}^m = v,_x{}^m + u,_y{}^m \quad , \tag{2}$$

where the comma denotes partial differentiation.

Loading and Field Quantities

Consider a laminate loaded by bending about y at the ends x = constant. Assuming that the laminate is long enough in the x-direction and that Saint-Venant's principle holds for a laminate, the resulting stress distribution will be independent of x in regions sufficiently removed from the areas of loading. Using this assumption and following Lekhnitskii [11], the elastic strain-stress relations can be integrated to yield displacements for each lamina of the form

$$u^m = (C_1 y + C_2 z + C_3) x + U^m(y, z)$$

$$v^m = -\frac{1}{2} C_1 x^2 + C_4 xz + V^m(y, z)$$

$$w^m = -\frac{1}{2} C_2 x^2 - C_4 xy + W^m(y, z) \quad , \tag{3}$$

where U^m, V^m, and W^m are unknown functions of y, z. The layer number, m, is left off the constants C_i because it results that each C_i must be the same for every lamina in order to satisfy the displacement continuity conditions at the interfaces. Thus, the C_i are found to be properties of the entire laminate. The displacement equations (3) represent the full three-dimensional elasticity solution that holds for all points in the laminate.

To evaluate the C_i, the scheme is as follows. Since equations (3) hold for all points in the laminate, they must converge to the plane stress solution, which is an exact solution, in the interior region of the laminate. Integrating the relation [10,12]

$$e_i = B'_{ij} M_j + z D'_{ij} M_j \quad ; \quad i, j = 1, 2, 6 \tag{4a}$$

for the case where $M_1 = -M$ and $M_2 = M_6 = 0$, the plane stress displacements are found to be

$$u_{ps} = (-D'_{11}Mz - B'_{11}M)x - B'_{61}My - \frac{1}{2}D'_{16}Myz + f(z)$$

$$v_{ps} = -\frac{1}{2}D'_{16}Mxz - (B'_{21}M + D'_{12}Mz)y + g(z)$$

$$w_{ps} = \frac{1}{2}D'_{11}Mx^2 + \frac{1}{2}D'_{16}Mxy + \frac{1}{2}D'_{12}My^2 + f^*(x) + g^*(y) \quad , \qquad (4b)$$

where B'_{ij} and D'_{ij} are laminate properties defined in Appendix A, and M is the applied moment. Comparing equations (3) and (4b) leads to the results:

$$C_1 = 0 \qquad C_2 = -D'_{11}M$$
$$C_3 = -B'_{11}M \qquad C_4 = -\frac{1}{2}D'_{16}M \qquad (5)$$

and

$$U^m(y, z) \rightarrow B_u y + C_4 yz + U^m(y, z)$$

$$V^m(y, z) \rightarrow B_v y + D_v yz + V^m(y, z)$$

$$W^m(y, z) \rightarrow -\frac{1}{2}D_v y^2 + W^m(y, z) \quad , \qquad (6)$$

where[2]

$$B_u = -B'_{61}M \quad , \quad B_v = -B'_{21}M \quad , \quad \text{and} \quad D_v = -D'_{12}M \quad . \qquad (7)$$

---

2. The extended forms (6) for U^m, V^m, and W^m are not necessary to the solution.

Substituting the results (6) into equations (3) yields displacements of the following functional form for each layer

$$u^m = (C_2 z + C_3)x + (B_u + C_4 z)y + U^m(y, z)$$

$$v^m = C_4 xz + (B_v + D_v z)y + V^m(y, z)$$

$$w^m = -\frac{1}{2} C_2 x^2 - C_4 xy - \frac{1}{2} D_v y^2 + W^m(y, z) \quad , \tag{8}$$

where C_i, B_i, and D_v are defined by equations (5) and (7). The strains are found by substituting the displacements (8) into the strain relations (2). The stresses then follow directly using the constitutive relation (1).

It is of interest to examine the strain $\epsilon_x{}^m$ which is

$$\epsilon_x{}^m = C_2 z + C_3 \quad . \tag{9}$$

Should the laminate be a balanced composite, i.e., the laminae are symmetrically stacked, according to composition and orientation with respect to the midplane z = 0, then B'_{ij} = 0 and from equations (5) C_3 = 0, which results in a case of pure bending. For the opposite case, an unbalanced composite exhibits an extensional strain, C_3, in bending. Such coupling effects are common to laminated composites.

Field Equations and Boundary Conditions

In regions sufficiently removed from the load planes, the nonboundary points must satisfy the reduced equilibrium equations

$$\tau^m_{xy,y} + \tau^m_{xz,z} = 0$$

$$\sigma^m_{y,y} + \tau^m_{yz,z} = 0$$

$$\tau^m_{yz,y} + \sigma^m_{z,z} = 0 \quad , \tag{10}$$

where the stresses exhibit no x-dependence, which conforms to an earlier assumption. Substituting for the stresses in terms of displacements yields the field equations for each lamina

$$c_{66}^m U_{,yy}^m + c_{55}^m U_{,zz}^m + c_{26}^m V_{,yy}^m + c_{45}^m V_{,zz}^m + (c_{36}^m + c_{45}^m) W_{,yz}^m = 0$$

$$c_{26}^m U_{,yy}^m + c_{45}^m U_{,zz}^m + c_{22}^m V_{,yy}^m + c_{44}^m V_{,zz}^m + (c_{23}^m + c_{44}^m) W_{,yz}^m = 0$$

$$(c_{36}^m + c_{45}^m) U_{,yz}^m + (c_{23}^m + c_{44}^m) V_{,yz}^m + c_{44}^m W_{,yy}^m + c_{33}^m W_{,zz}^m$$

$$= -(c_{13}^m C_2 + c_{23}^m D_V + 2c_{36}^m C_4) \quad . \tag{11}$$

The boundary conditions on the free surfaces normal to y are

$$\sigma_y{}^m = \tau_{xy}^m = \tau_{yz}^m = 0 \tag{12}$$

and on the free surfaces normal to z are

$$\sigma_z{}^m = \tau_{xz}^m = \tau_{yz}^m = 0 \quad . \tag{13}$$

For continuity at the interfaces, the boundary conditions are:

$$(u^m, v^m, w^m) = (u^{m+1}, v^{m+1}, w^{m+1})$$

and (14)

$$(\sigma_z{}^m, \tau_{xz}^m, \tau_{yz}^m) = (\sigma_z{}^{m+1}, \tau_{xz}^{m+1}, \tau_{yz}^{m+1}) \quad ,$$

respectively.

It is noted that the corner conditions are ambiguous in that there are five possible conditions out of which only three can be employed at any one time. The remaining two may or may not be satisfied by the solution. Thus, combinations may be tried until some satisfying results are achieved.

FINITE-DIFFERENCE SIMULATION

Function Representation

The mathematical basis for the finite-difference method is Taylor's Series. Referring to Figure 2, the Taylor Series expansion for a function f at some point y, z about the point (or node) I, J is

$$f(y, z) = f(I, J) + y f_{,y}(I, J) + z f_{,z}(I, J)$$

$$+ \frac{1}{2} y^2 f_{,yy}(I, J) + \frac{1}{2} z^2 f_{,zz}(I, J) + yz f_{,yz}(I, J) + \ldots \quad . \qquad (15)$$

Thus, for the specific node I-1, J, the expansion is

$$f(I - 1, J) = f(I, J) - h_1 f_{,y} + \frac{1}{2} h_1{}^2 f_{,yy} - \ldots \quad . \qquad (16)$$

Writing similar expansions for the remaining seven points neighboring the node I, J and simultaneously solving expansions for the first and second derivatives yields the finite-difference approximations for these derivatives. All but the last of these expressions, given below, are taken from Forsythe and Wasow [13]. They are

$$f_{,y}(I, J) = \frac{1}{h_1 + h_2}\left[\frac{h_1}{h_2} f(I + 1, J) - \frac{h_2}{h_1} f(I - 1, J)\right] + \frac{h_2 - h_1}{h_1 h_2} f(I, J) + 0(h^2)$$

$$f_{,z}(I, J) = \frac{1}{2h_3}\left[f(I, J + 1) - f(I, J - 1)\right] + 0(h^2)$$

$$f_{,yy}(I, J) = \frac{2}{h_1 + h_2}\left[\frac{1}{h_2} f(I + 1, J) + \frac{1}{h_1} f(I - 1, J)\right] - \frac{2}{h_1 h_2} f(I, J) + 0(h^2)$$

$$f_{,zz}(I, J) = \frac{1}{h_3{}^2}\left[f(I, J + 1) + f(I, J - 1) - 2f(I, J)\right] + 0(h^2)$$

$$f_{,yz}(I, J) = \frac{1}{2h_3(h_1 + h_2)}\left[f(I + 1, J + 1) - f(I - 1, J + 1) - f(I + 1, J - 1)\right.$$

$$\left. + f(I - 1, J - 1)\right] + 0(h^2) \quad , \qquad (17)$$

where h is an order of magnitude equal to h_1, h_2, or h_3. The difference equations (17) are "central" differences.

At boundaries and interfaces it is convenient to use "forward" and "backward" differences. Such difference equations are one-sided in that they express a boundary point in terms of neighboring points interior to the boundary. For the present problem, only first derivatives are of concern.

To derive such difference equations, expand two points, both lying on one side of the reference point I, J, by using equation (15) in conjunction with Figure 2. For example, a forward expansion yields

$$f(I+1, J) = f(I, J) + h_2 f_{,y}(I, J) + \frac{1}{2} h_2^2 f_{,yy}(I, J) + 0(h_2^3)$$

$$f(I+2, J) = f(I, J) + 2h_2 f_{,y}(I, J) + \frac{1}{2}(4h_2^2)\, f_{,yy}(I, J) + 0(h_2^3) \quad . \tag{18}$$

Subtracting one expression from the other to eliminate the second derivative leads to the difference equation for the first derivative. Thus, the forward differences are

$$f_{,y}(I, J) = \frac{1}{2h_2}\left[4f(I+1, J) - 3f(I, J) - f(I+2, J)\right] - 0(h_2^2)$$

$$f_{,z}(I, J) = \frac{1}{2h_3}\left[4f(I, J+1) - 3f(I, J) - f(I, J+2)\right] - 0(h_3^2) \quad . \tag{19}$$

Similarly, the backward differences are

$$f_{,y}(I, J) = \frac{1}{2h_1}\left[3f(I, J) + f(I-2, J) - 4f(I-1, J)\right] + 0(h_1^2)$$

$$f_{,z}(I, J) = \frac{1}{2h_3}\left[3f(I, J) + f(I, J-2) - 4f(I, J-1)\right] + 0(h_3^2) \quad . \tag{20}$$

It should be pointed out that more simplified, but less accurate, forward and backward expressions can be written; however, the present application requires all the accuracy that it is possible to attain near the free boundaries. Thus, the higher order difference was chosen. In addition, this choice yields a magnitude of error equal to that found in equations (17).

Using the representations just obtained, equations (11) through (14) can be transformed into difference equations characterizing the problem. For example, the last equation in (11) becomes

$$\frac{h_1 h_2}{2h_3(h_1 + h_2)} \Big\{ (c_{36}^m + c_{45}^m) [U(I + 1, J + 1) - U(I - 1, J + 1) - U(I + 1, J - 1)$$

$$+ U(I - 1, J - 1)] + (c_{23}^m + c_{44}^m) [V(I + 1, J + 1)$$

$$- V(I - 1, J + 1) - V(I + 1, J - 1) + V(I - 1, J - 1)] \Big\}$$

$$+ \frac{2h_1}{h_1 + h_2} c_{44}^m \left[W(I + 1, J) + \frac{h_2}{h_1} W(I - 1, J) \right]$$

$$+ \frac{h_1 h_2}{h_3^2} c_{33}^m [W(I, J + 1) + W(I, J - 1)]$$

$$- 2(c_{44}^m + \frac{h_1 h_2}{h_3^2} c_{33}^m) W(I, J) = -h_1 h_2 [c_{13}^m C_2 + c_{23}^m D_V$$

$$+ 2c_{36}^m C_4] \quad , \tag{21}$$

where the layer number, m, is left off U, V, and W since their location is determined by the node I, J.

Developing the Matrix Equation

In this section, the difference equations, like (21), are transformed into a linear matrix equation of the form

$$[A] \quad [X] = [B] \quad , \tag{22}$$

where A is an N X N coefficient matrix (N being the number of unknowns or equations), X is the solution vector, and B is the load or input vector. To accomplish this, the three unknowns (U, V, and W) must be uniquely collapsed into the single unknown X so that at each node three unique equations in X will be created. For instance, let

$$\left.\begin{array}{l} U \rightarrow X(1) \\ V \rightarrow X(2) \\ W \rightarrow X(3) \end{array}\right\} \text{ at Node 1} \qquad \left.\begin{array}{l} U \rightarrow X(4) \\ V \rightarrow X(5) \\ W \rightarrow X(6) \end{array}\right\} \text{ at Node 2} \quad . \tag{23}$$

It remains to generalize such a transformation for all nodes.

It is convenient to follow Pipes [1] and his notation is adopted. If LAT is the number of nodes in one column along the vertical axis (LAminate Thickness direction), then the nodes, unknowns, and equations can be identified by a unique number in terms of the nodal position (I, J). If

$$JJ1 = 3[LAT(I-1) + J] - 2 \quad , \tag{24}$$

then

$$\begin{aligned} NODE &= LAT(I-1) + J \\ U(I, J) &= X(JJ1) \\ V(I, J) &= X(JJ1 + 1) \\ W(I, J) &= X(JJ1 + 2) \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\text{Number the 1st equation: } JJ1 \\ &\text{Number the 2nd equation: } JJ1 + 1 \\ &\text{Number the 3rd equation: } JJ1 + 2 \quad . \end{aligned} \tag{26}$$

Letting I = 1 and J = 1, 2 consecutively generates the results in (23).

Since the finite-difference equations involve unknowns at nodes neighboring the JJ1 node, it is necessary to develop transformation relations like (24) in order to number unknowns at these nodes as well. For example, using I, J as the reference node, a transformation relation for an unknown at the node I - 1, J + 1 is found by letting $I \rightarrow I - 1$ and $J \rightarrow J + 1$ in (24) and giving the result a unique name, for example JJ7. Thus,

$$JJ7 = 3[LAT(I - 2) + J] + 1 \quad . \tag{27}$$

Using Table 1, which identifies all the unknowns at nodes neighboring I, J, and following the above procedure yields the transformation relations that uniquely number each unknown. In summary, all of these transformations are

$$
\begin{aligned}
JJ1 &= 3*(LAT*I1 + J) - 2 \\
JJ2 &= 3*(LAT*I2 + J) - 2 \\
JJ3 &= 3*(LAT*I2 + J) - 5 \\
JJ4 &= 3*(LAT*I + J) - 2 \\
JJ5 &= 3*(LAT*I + J) + 1 \\
JJ6 &= 3*(LAT*I1 + J) + 1 \\
JJ7 &= 3*(LAT*I2 + J) + 1 \\
JJ8 &= 3*(LAT*I1 + J) - 5 \\
JJ9 &= 3*(LAT*I + J) - 5 \\
JJ10 &= 3*(LAT*I1 + J) - 8 \\
JJ11 &= 3*(LAT*(I + 1) + J) - 2 \\
JJ12 &= 3*(LAT*I1 + J) + 4 \\
JJ13 &= 3*(LAT*(I - 3) + J) - 2 \quad ,
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
I1 &= I - 1 \\
I2 &= I - 2 \quad .
\end{aligned} \tag{29}
$$

TABLE 1. NODE IDENTIFICATION

Node	U	V	W
I, J	X(JJ1)	X(JJ1 + 1)	X(JJ1 + 2)
I - 1, J	X(JJ2)	X(JJ2 + 1)	X(JJ2 + 2)
I - 1, J - 1	X(JJ3)	X(JJ3 + 1)	X(JJ3 + 2)
I + 1, J	X(JJ4)	X(JJ4 + 1)	X(JJ4 + 2)
I + 1, J + 1	X(JJ5)	X(JJ5 + 1)	X(JJ5 + 2)
I, J + 1	X(JJ6)	X(JJ6 + 1)	X(JJ6 + 2)
I - 1, J + 1	X(JJ7)	X(JJ7 + 1)	X(JJ7 + 2)
I, J - 1	X(JJ8)	X(JJ8 + 1)	X(JJ8 + 2)
I + 1, J - 1	X(JJ9)	X(JJ9 + 1)	X(JJ9 + 2)
I, J - 2	X(JJ10)	X(JJ10 + 1)	X(JJ10 + 2)
I + 2, J	X(JJ11)	X(JJ11 + 1)	X(JJ11 + 2)
I, J + 2	X(JJ12)	X(JJ12 + 1)	X(JJ12 + 2)
I - 2, J	X(JJ13)	X(JJ13 + 1)	X(JJ13 + 2)

Generation of the matrix equation (22) now remains. To do this, straightforward substitution for U, V, and W, using Table 1, into equations (11) through (14) yields the desired results in equation form. For example, equation (21) becomes

$$\frac{h_1 h_2}{2h_3(h_1 + h_2)} \Big\{ (c^m_{36} + c^m_{45}) \, [X(JJ5) - X(JJ7) - X(JJ9) + X(JJ3)]$$

$$+ (c^m_{23} + c^m_{44}) \, [X(JJ5 + 1) - X(JJ7 + 1) - X(JJ9 + 1)$$

$$+ X(JJ3 + 1)] \Big\} + \frac{2h_1}{h_1 + h_2} c^m_{44} [X(JJ4 + 2) + \frac{h_2}{h_1} X(JJ2 + 2)]$$

$$+ \frac{h_1 h_2}{{h_3}^2} c^m_{33} [X(JJ6 + 2) + X(JJ8 + 2)]$$

$$- 2(c^m_{44} + \frac{h_1 h_2}{{h_3}^2} c^m_{33}) \, X(JJ1 + 2)$$

$$= -h_1 h_2 [c^m_{13} C_2 + c^m_{23} D_V + 2c^m_{36} C_4] \quad . \tag{30}$$

To assure non-zero diagonal terms in the A-matrix, an appropriate equation number for (30) is JQ2 (in this case there is only one possibility) where

$$JQ2 = JJ1 + 2 \quad . \tag{31}$$

Now, from equation (30), the only nonzero elements for the JQ2 row in the A-matrix are

$$A(JQ2, JJ5) = A(JQ2, JJ3) = C$$

$$A(JQ2, JJ7) = A(JQ2, JJ9) = -C$$

$$A(JQ2, JJ5 + 1) = A(JQ2, JJ3 + 1) = D$$

$$A(JQ2, JJ7 + 1) = A(JQ2, JJ9 + 1) = -D$$

$$A(JQ2, JJ4 + 2) = 2h_1 c^m_{44}/(h_1 + h_2)$$

$$A(JQ2, JJ2 + 2) = (h_2/h_1) \cdot 2h_1 c^m_{44}/(h_1 + h_2)$$

$$A(JQ2, JJ6 + 2) = A(JQ2, JJ8 + 2) = h_1 h_2 c^m_{33}/{h_3}^2$$

$$A(JQ2, JJ1 + 2) = -2(c^m_{44} + h_1 h_2 c^m_{33}/{h_3}^2) \quad , \tag{32}$$

where

$$C = h_1 h_2 (c_{36}^m + c_{45}^m)/2h_3(h_1 + h_2)$$

$$D = h_1 h_2 (c_{23}^m + c_{44}^m)/2h_3(h_1 + h_2) \quad . \tag{33}$$

Note that the material constants c_{44}^m and c_{33}^m are non-zero ensuring a non-zero diagonal element A(JQ2, JJ1 + 2). In addition to this, the load vector is

$$B(JQ2) = -h_1 h_2 [c_{13}^m C_2 + c_{23}^m D_V + 2c_{36}^m C_4] \quad . \tag{34}$$

Of course, these results only apply to node numbers where the third equilibrium equation in (11) holds. The computer program logically connects appropriate equations with each node. The matrix elements for the remaining equations (11) through (14) are generated in a similar fashion.

The Mesh Simulation

The continuum is to be simulated by a number of nodal points that form a finite-difference mesh. The mesh is distributed over a cross section of the laminate as shown in Figure 3. The mesh is defined by the following parameters:

NLAY: the number of laminae

LAT: the number of nodes along one column in the LAminate Thickness direction

LAW: the number of nodes along one row in the LAminate Width direction

FSW1: the first change in nodal spacing termed Fine Simulation Width

K: magnification factor of the fine simulation width

H: the fine simulation width

Given these parameters, the following parameters can be determined:

INF(M): values of J at the upper INterFace of the mth layer

FSW2: the second change in nodal spacing

KH: the coarse simulation width (K = 1, 2, 3, ...)

JQMAX = 3*LAT*LAW: the number of unknowns or equations

IBW = 2*(3*LAT + 1): the half bandwidth

NBAND = 2*IBW + 1: the full band

The bandwidth of the coefficient matrix is found by considering that the maximum number of nodes involved in the difference equations is three, as can be seen from expressions (19) and (20), and calculating the maximum number of consecutive elements on both sides of the diagonal to and including the last off-diagonal non-zero element.

Selecting equations representing the conditions to be imposed at each node remains to be accomplished. Because of the arbitrariness of the corner conditions, a number of choices are possible. Those selected for this program are illustrated in Figure 4.

A user's guide and a more detailed description of the computer program are presented in Appendix C. A program listing is provided also in Appendix C.

RESULTS

The results given below were obtained using a square mesh, magnification factor K = 1, of size (LAW, LAT) = (13, 9). A complete mesh description, taken from the program output, is displayed in Table 2. It is seen that these dimensions represent a beam rather than a plate. The program was run on an IBM 370 computer utilizing virtual storage.

A single material having properties typical of a high modulus graphite-epoxy was chosen for the above mesh. Using standard notation,

$$E_{11} = 20.0 \times 10^6 \text{ psi} \quad , \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.21$$

$$E_{22} = E_{33} = 2.1 \times 10^6 \text{ psi}$$

$$G_{12} = G_{13} = G_{23} = 0.85 \times 10^6 \text{ psi} \quad ,$$

TABLE 2. MESH DESCRIPTION TAKEN FROM PROGRAM OUTPUT

```
*** UNIFORM BENDING OF A LAMINATED PLATE ***

*** INPUT DATA ***

        NUMBER OF LAYERS IN CROSS SECTION, NLAY =   4
        NUMBER OF NODES ON VERTICAL AXIS, LAT   =  13
        NUMBER OF NODES ON HORIZONTAL AXIS, LA  =   9

        CHANGE IN MESH WIDTH (FSW1) AT I   =   3
        CHANGE IN MESH WIDTH (FSW2) AT I   =   7
        MESH WIDTH MAGNIFICATION FACTOR, K =   1

        LAYER NO.   1    INTERFACE AT J =   4
        LAYER NO.   2    INTERFACE AT J =   7
        LAYER NO.   3    INTERFACE AT J =  10
        LAYER NO.   4    INTERFACE AT J =  13

        FINE SIMULATION WIDTH, H = 0.00167
```

where the subscript "1" refers to the fiber direction. The two laminate configurations which are considered are

$$A = [\theta, 0, 0, \theta]$$

and

$$B = [0, \theta, \theta, 0]$$

with θ as in Figure 1 such that 0 degree $\leqslant \theta \leqslant$ 90 degrees. Typical laminate data and load constants are displayed in Table 3.[3] Here the additional constant MT is the resulting moment required to produce a specified maximum strain which, for the present analysis, is $\epsilon_x = 1.0 \times 10^{-3}$ inch/inch (see Appendix B).

A sample of the results for the displacement functions U, V, and W is presented in Table 4. Examination of their variation with respect to z reveals the apparent symmetry relations,

U, V	antisymmetric in z
W	symmetric in z

within an accuracy of two digits.

Symmetries with respect to y are evident for the strains within three-digit accuracy. Samples of these results are plotted in Figures 5 and 6. Coupling these apparent symmetries with the strain relations (2) in an expanded form yields

U, V	antisymmetric in y
W	symmetric in y .

The displacement results verify this precisely for U (to four places), but show some deviation in V and W.[4]

To illustrate the effect of bending on the stress distribution, Figures 7 through 19 are presented. Although convergence to the exact values has yet to be demonstrated, the results do have qualitative merit. The following cases result from a bending strain of $\epsilon_x = 1.0 \times 10^{-3}$ inch/inch prescribed at the bottom surface.

Of principal interest are the interlaminar stresses illustrated in Figures 7 through 12. We note that laminates composed of 30 degree or 45 degree layers produce the greatest stress rise in σ_z at the free edge with a more pronounced effect occurring if the angle plies are on the outside, i.e., system A = $[\theta, 0, 0, \theta]$. A similar effect is seen in the shear stress τ_{yz}, although the rise in stress is sharply blunted by the requirement of zero

3. The thermal problem is neglected in this preliminary analysis even though expansion coefficients appear in the program.

4. It is interesting to note that the y-symmetries for V and W are verified precisely using the coarser mesh (LAW, LAT) = (8, 9) which decreases the relative size of the bandwidth.

TABLE 3. TYPICAL LAMINATE DATA AND LOAD CONSTANTS TAKEN FROM PROGRAM OUTPUT

*** MATERIAL DATA ***

LAYER	E11	E22	E33	E12	E13	E23	NU12	NU13	NU23
1	20.000E+06	2.100E+06	2.100E+06	0.850E+06	0.850E+06	0.850E+06	0.21	0.21	0.21
2	20.000E+06	2.100E+06	2.100E+06	0.850E+06	0.850E+06	0.850E+06	0.21	0.21	0.21
3	20.000E+06	2.100E+06	2.100E+06	0.850E+06	0.850E+06	0.850E+06	0.21	0.21	0.21
4	20.000E+06	2.100E+06	2.100E+06	0.850E+06	0.850E+06	0.850E+06	0.21	0.21	0.21

*** STIFFNESS MATRICES ***

LAYER/THETA	X-Y-Z MATRIX						X-Y-Z PRIME MATRIX					
1	6.745E+06	5.045E+06	5.210E+05	0.0	0.0	4.506E+06	2.024D+07	5.648D+05	5.648D+05	0.0	0.0	0.0
		6.745E+06	5.210E+05	0.0	0.0	4.506E+06		2.213D+06	4.771D+05	0.0	0.0	0.0
			2.213E+06	0.0	0.0	4.387E+04			2.213D+06	0.0	0.0	0.0
45.0				8.500E+05	0.0	0.0				8.500D+05	0.0	0.0
					8.500E+05	0.0					8.500D+05	0.0
						5.330E+06						8.500D+05
2	2.024E+07	5.648E+05	5.648E+05	0.0	0.0	0.0	2.024D+07	5.648D+05	5.648D+05	0.0	0.0	0.0
		2.213E+06	4.771E+05	0.0	0.0	0.0		2.213D+06	4.771D+05	0.0	0.0	0.0
			2.213E+06	0.0	0.0	0.0			2.213D+06	0.0	0.0	0.0
0.0				8.500E+05	0.0	0.0				8.500D+05	0.0	0.0
					8.500E+05	0.0					8.500D+05	0.0
						8.500E+05						8.500D+05
3	2.024E+07	5.648E+05	5.648E+05	0.0	0.0	0.0	2.024D+07	5.648D+05	5.648D+05	0.0	0.0	0.0
		2.213E+06	4.771E+05	0.0	0.0	0.0		2.213D+06	4.771D+05	0.0	0.0	0.0
			2.213E+06	0.0	0.0	0.0			2.213D+06	0.0	0.0	0.0
0.0				8.500E+05	0.0	0.0				8.500D+05	0.0	0.0
					8.500E+05	0.0					8.500D+05	0.0
						8.500E+05						8.500D+05

TABLE 3. (Concluded)

4	6.745E+06	5.345E+06	5.210E+05	0.0	0.0	4.596E+06	2.024D+07	5.648D+05	5.648D+05	0.0	0.0	0.0
		6.745E+06	5.210E+05	0.0	0.0	4.596E+06		2.213D+06	4.771D+05	0.0	0.0	0.0
			2.213E+06	0.0	0.0	4.387E+04			2.213D+06	0.0	0.0	0.0
45.0				8.500E+05	0.0	0.0				8.500D+05	0.0	0.0
					8.500E+05	0.0					8.500D+05	0.0
						5.330E+06						8.500D+05

*** COEFFICIENTS OF THERMAL EXPANSION ***

LAYER	THETA	AL1	AL2	AL3	AL6	AL1P	AL2P	AL3P
1	45.0	0.600E-05	0.600E-05	0.120E-04	-0.120E-04	0.0	0.120E-04	0.120E-04
2	0.0	0.0	0.120E-04	0.120E-04	0.0	0.0	0.120E-04	0.120E-04
3	0.0	0.0	0.120E-04	0.120E-04	0.0	0.0	0.120E-04	0.120E-04
4	-45.0	0.600E-05	0.600E-05	0.120E-04	-0.120E-04	0.0	0.120E-04	0.120E-04

*** THE LAMINATE LOAD CONSTANTS ***

C2 = -1.000E-01 C3 = 0.0 C4 = -1.311E-01 BU = 0.0 BV = 0.0 DV = -2.908E-01 MT = -5.537E-01

ERROR CONDITION OF SOLVER ROUTINE IS 0.0 RANK IS 351.0 DETERMINANT = 1.00

NOTE: MT is the resulting moment required to produce the specified maximum strain.

TABLE 4. DISPLACEMENT FUNCTION RESULTS TAKEN FROM PROGRAM OUTPUT FOR LAMINATE DESCRIBED IN TABLES 2 AND 3

*** GRID POINT DISPLACEMENT FUNCTIONS ***

NODE	J-DISPLACEMENT	V-DISPLACEMENT	W-DISPLACEMENT
1	0.161561D-04	0.264636D-04	-0.909039D-05
2	0.149580D-04	0.219831D-04	-0.936804D-05
3	0.125381D-04	0.173176D-04	-0.951939D-05
4	0.953594D-05	0.127014D-04	-0.953590D-05
5	0.611696D-05	0.318937D-05	-0.940002D-05
6	0.304395D-05	0.403364D-05	-0.924686D-05
7	0.487189D-09	0.250769D-08	-0.916589D-05
8	-0.304291D-05	-0.403918D-05	-0.925084D-05
9	-0.611575D-05	-0.319432D-05	-0.937698D-05
10	-0.926689D-05	-0.126308D-04	-0.952892D-05
11	-0.125354D-04	-0.173211D-04	-0.954418D-05
12	-0.149550D-04	-0.219380D-04	-0.937161D-05
13	-0.161503D-04	-0.264808D-04	-0.909891D-05

stress at the free edge, and here the stress in system B = [0, θ, θ, 0] is slightly more pronounced than that in A. The largest stress rise, an order of magnitude greater than σ_z and τ_{yz}, is created in the A-system in τ_{xz}. Again it is the 30 degree laminate incurring the sharpest stress rise, but here the 15 degree laminate overshadows the 45 degree laminate. In summary, the laminates containing 15 degree through 45 degree layers located adjacent to 0 degree layers have the largest interlaminar stresses for the cases considered; i.e., 0 degree $\leqslant \theta \leqslant$ 90 degrees taken in 15 degree intervals.

Some results peculiar to the numerical method of solution should be pointed out. Referring to Figure 9, we note a sharp rise in the stress σ_z at the midpoint node (I, J) = (5, 7). This is a result of fixing the displacements at I = 5 and 6, J = 7 in the program in order to zero-out rigid body motion and drift in the solution routine. However averaging the values for σ_z just above and just below the interface (at J = 7, m = 2 and m = 3) yields a more plausible result. Since the tractions must be continuous at the interface anyway, this averaging technique was also applied at the free edges where the free surface conditions were adopted in lieu of the continuity conditions. This technique had varying success as illustrated by the 75 degree and 90 degree configurations in Figures 10 and 11.

The in-plane stresses are illustrated in Figures 13 through 19. In Figure 13, we find that σ_x in the 0 degree layers is independent of the orientation of the adjacent layer when the maximum strain is specified.[5] This facilitates the presentation of both systems A and B in one figure. It is interesting to note in Figure 14 that σ_x rises at the free edge if the 0 degree layers are outside the laminate and drops if these layers are inside the laminate.

Observation of Figures 15 and 17 for the distribution of σ_y and τ_{xy} with respect to z reveals that the off-axis layers, particularly again for 15 degrees through 45 degrees, serve as stress raisers with the effect considerably more pronounced if the 0 degree layers are inside.

Typical distributions of σ_y and τ_{xy} with respect to y are shown in Figures 18 and 19. The disturbing feature of these plots is that the stresses just above an interface do not approach zero at the free surface. One cause of this problem is the placement of nodes directly on the interface, which requires their occupation by both layers. Then at the corners, as stated previously, the multitude of boundary conditions cannot be satisfied.[6] However this problem is confined to the free surface nodes and one line of

5. In agreement with the beam theory approximation.

6. Placing the interface between two nodal lines may alleviate this problem.

interior nodes. To see this, one may examine the curves for the A-system at J = 4- and J = 10+ and note that they are reflections of each other within the range $3 \leqslant I \leqslant 7$. Since, from above, σ_y and τ_{xy} appear, in general, to be antisymmetric in z, the correct values at J = 10+ are recovered within this range if we accept the values at J = 4-.

CONCLUSIONS

Although only two types of laminate systems were considered, namely A = [θ, 0, 0, θ] and B = [0, θ, θ, 0], it is reasonable to assume from these results and from physical considerations that the following symmetry relations hold for balanced (B_{ij} = 0) composites:

U, V antisymmetric in y and z

W symmetric in y and z ,

where U, V, and W are displacement functions of y and z. Based on the stress results, laminates containing layers oriented within the range 15 degrees $\leqslant \theta \leqslant$ 45 degrees produce the largest interlaminar stresses out of the cases studied, 0 degrees $\leqslant \theta \leqslant$ 90 degrees taken in 15 degree intervals. In fact this same group of laminates produces high values in the in-plane stresses as well, with the effect considerably more pronounced for the A-system. Although some deviations in stress occur in the numerical solution, they are localized to a double line of nodes at the boundary. This is a disconcerting feature of the solution in that the boundary region stresses appear to be critically involved in delamination-type failure, which makes their accurate determination desirable.

This study provides a base for future work in this area. Using the present program coupled with an out-of-core equation solver routine, unbalanced laminates may be studied. Using the symmetry relations discussed above, the present computer program may be modified to more efficiently handle balanced laminates (B_{ij} = 0).

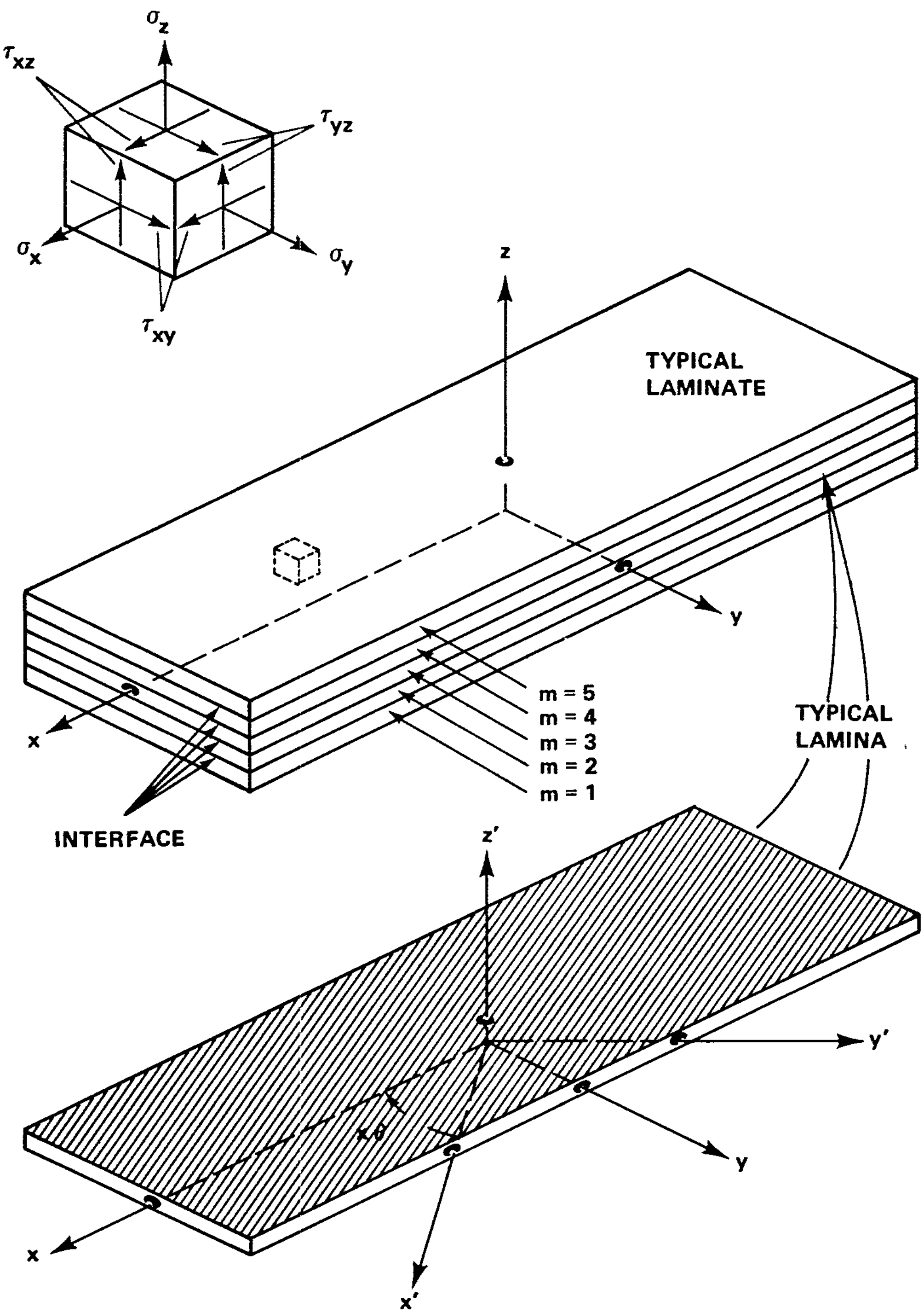


Figure 1. Laminate geometry.

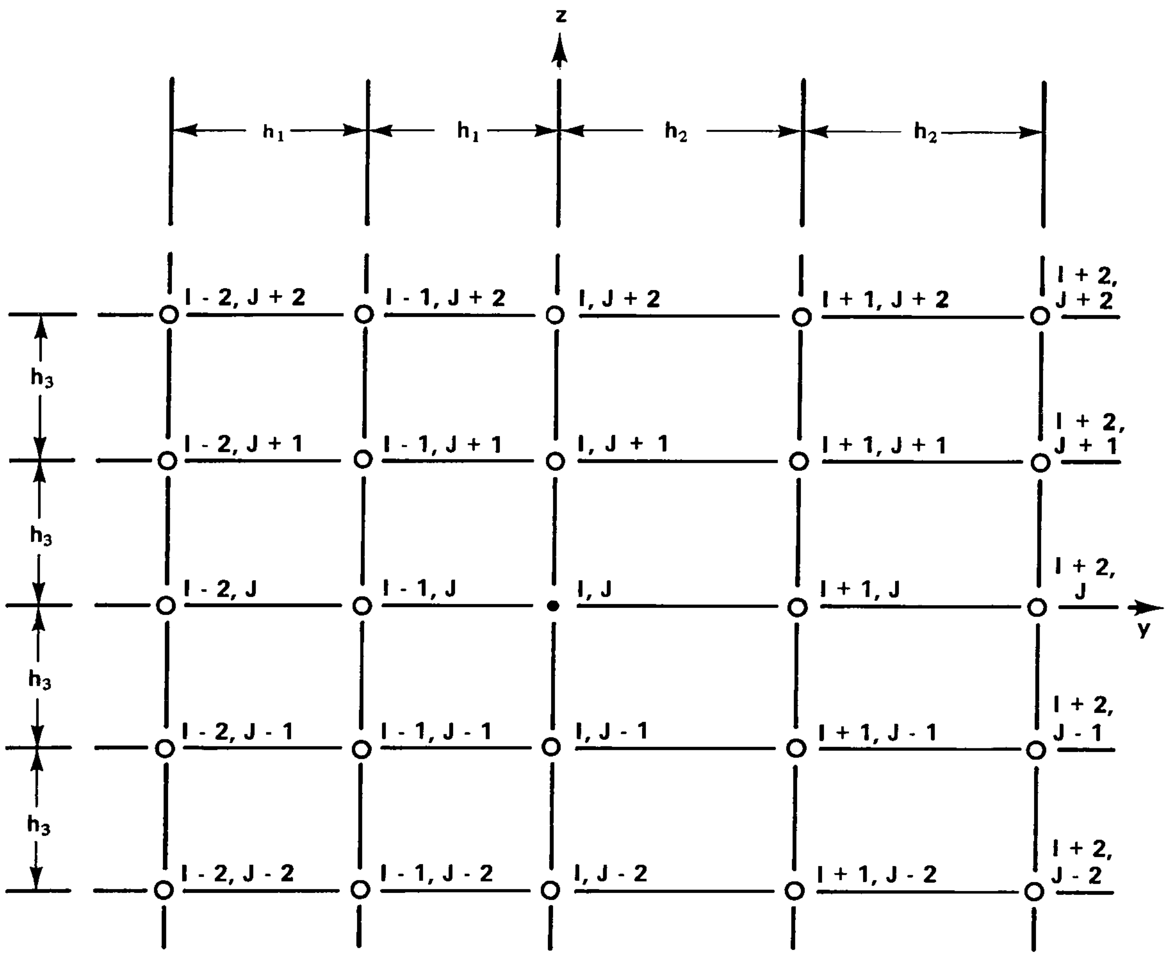


Figure 2. Finite-difference mesh.

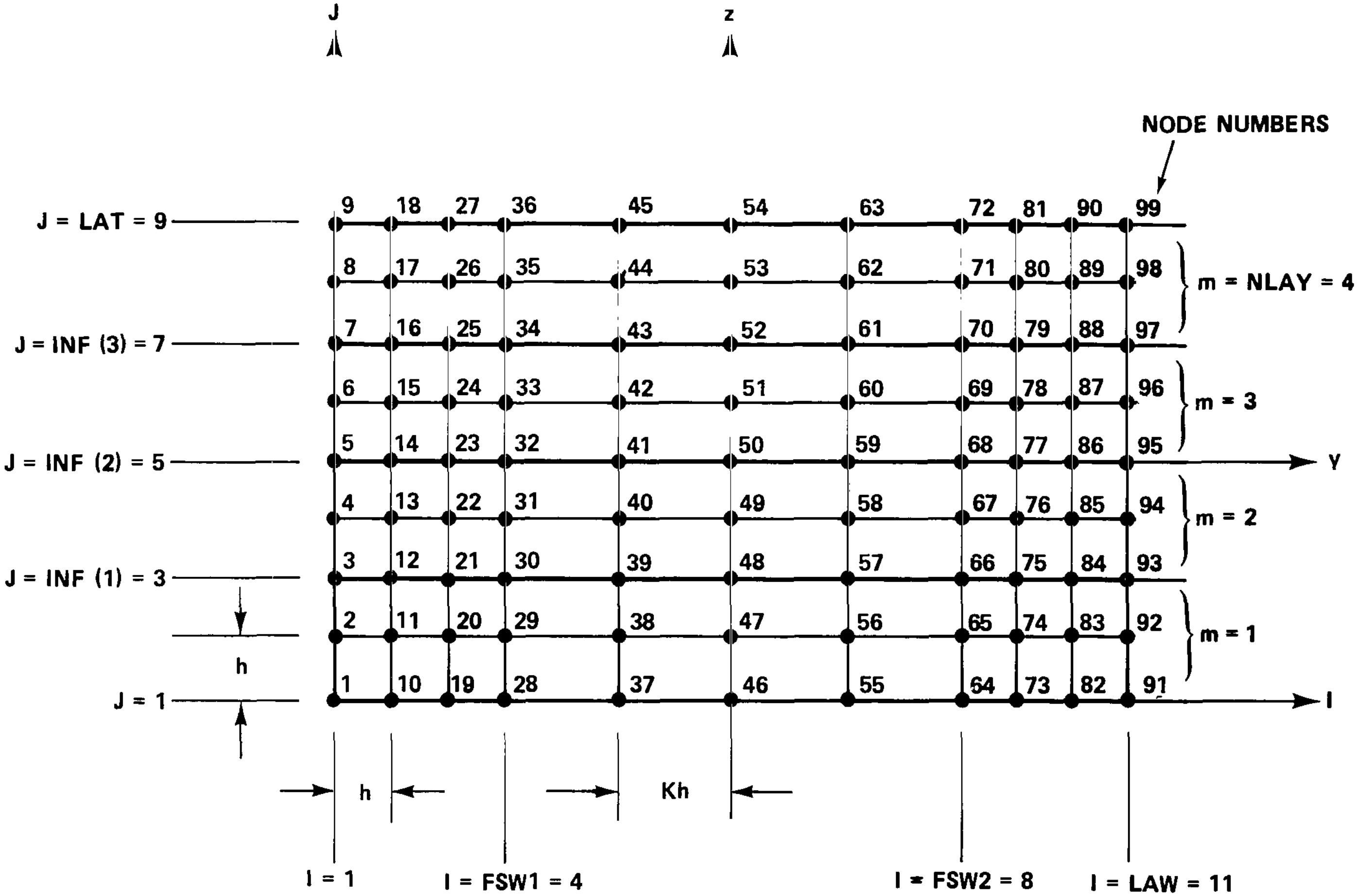
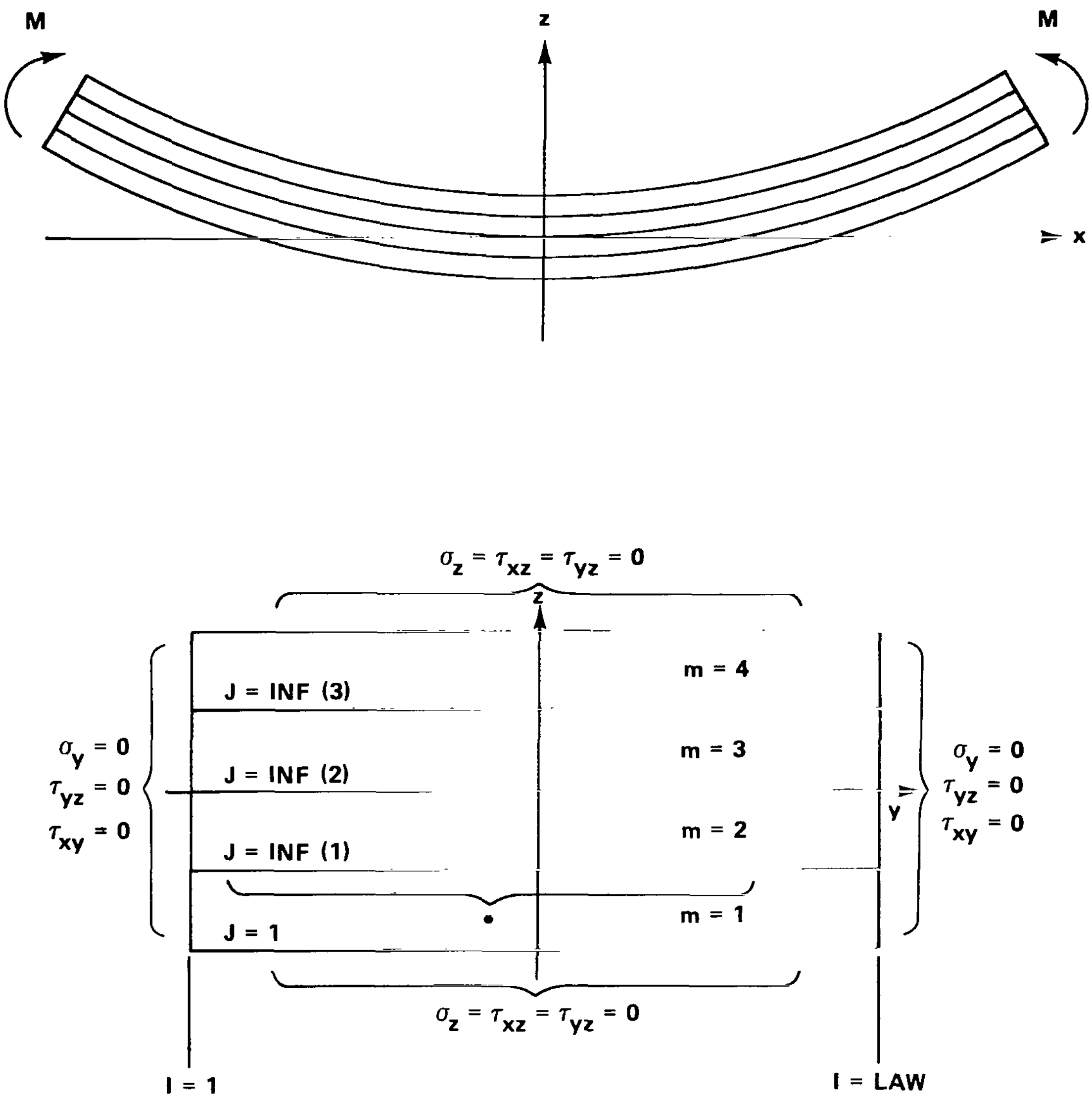


Figure 3. A typical laminate mesh.



*AT INF(m) WHERE 1 < I < LAW AND 1 ⩽ m < NLAY:

$$[u^m, v^m, w^m] = [u^{m+1}, v^{m+1}, w^{m+1}]$$

$$[\sigma_z^m, \tau_{yz}^m, \tau_{xz}^m] = [\sigma_z^{m+1}, \tau_{yz}^{m+1}, \tau_{xz}^{m+1}]$$

● STATIC EQUILIBRIUM IS IMPOSED AT ALL INTERIOR POINTS

Figure 4. Equations selected for each node.

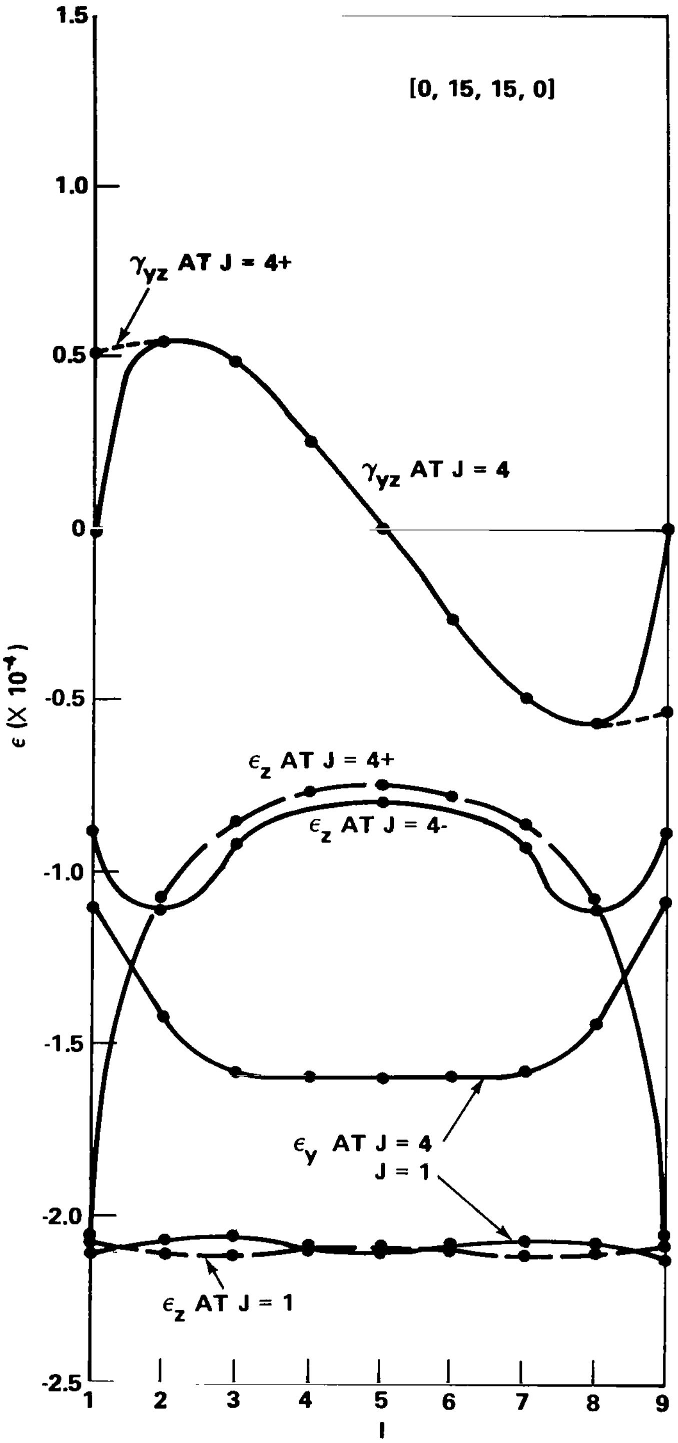


Figure 5. Variation of strain with y.

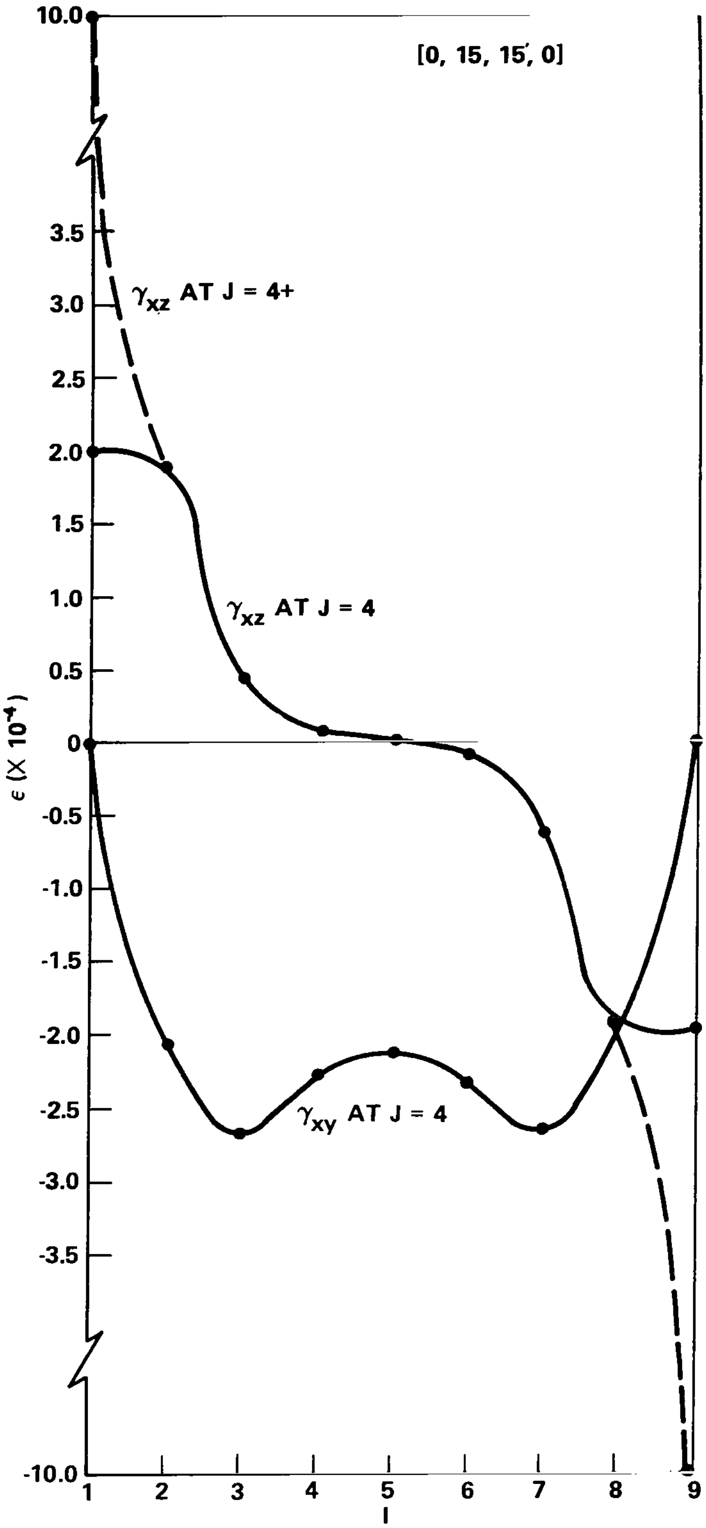


Figure 6. Variation of shear strain with y.

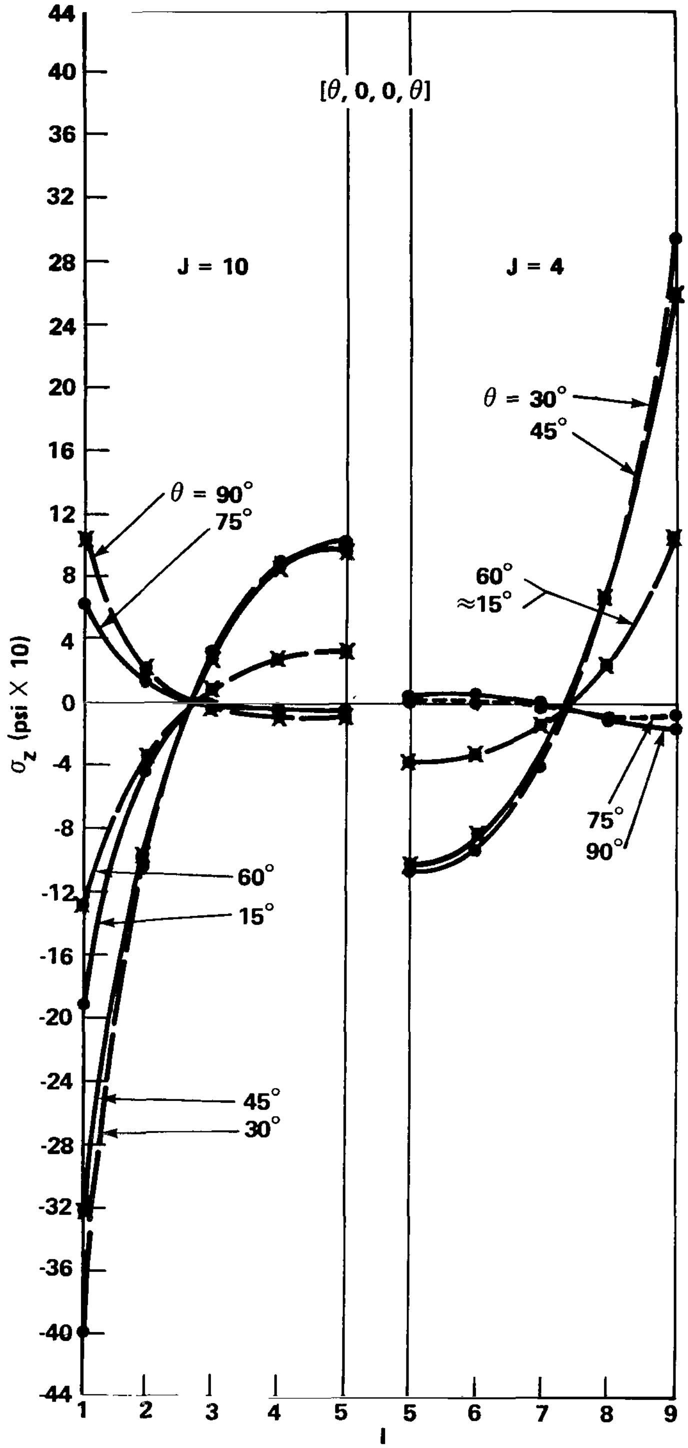


Figure 7. Variation of the normal stress σ_z (symmetric in y) with y for a [θ, 0, 0, θ] laminate.

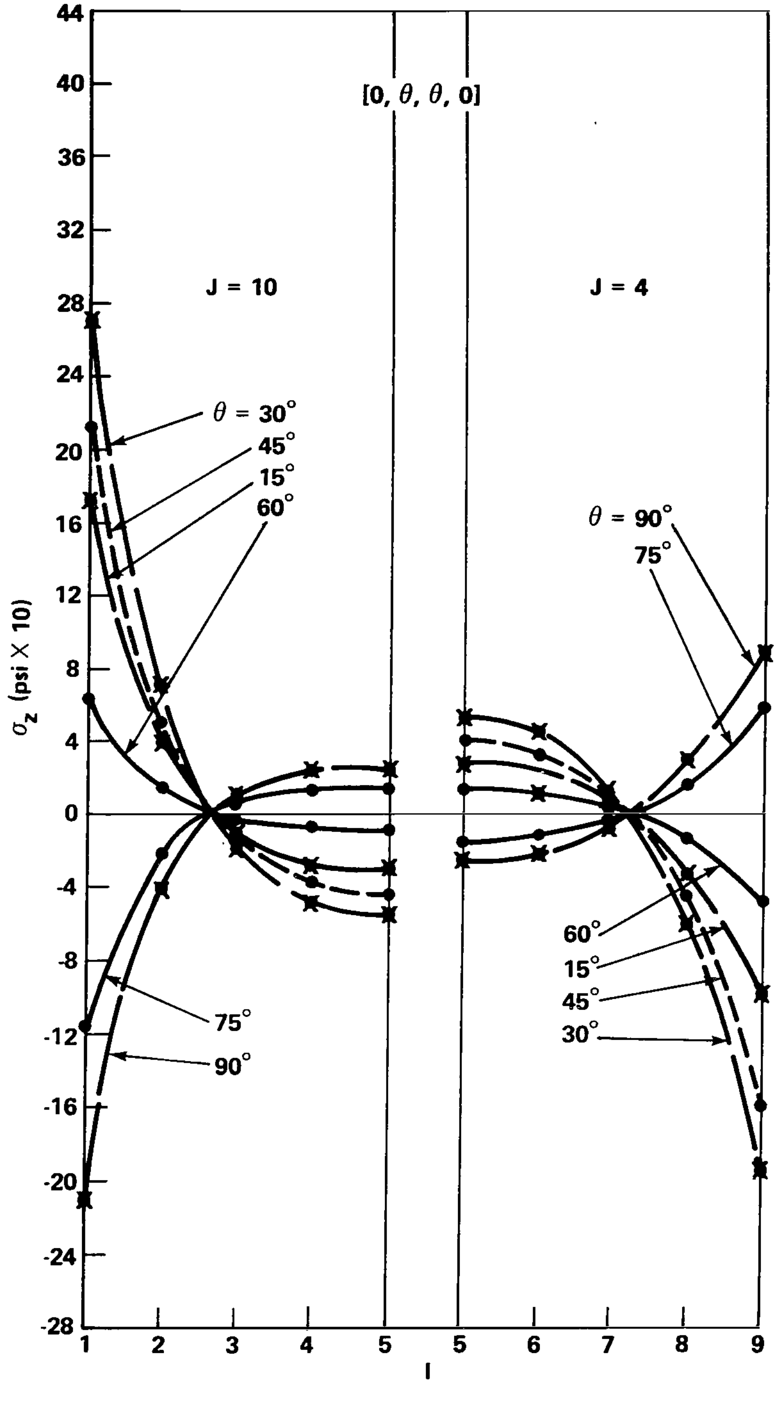


Figure 8. Variation of the normal stress σ_z (symmetric in y) with y for a [0, θ, θ, 0] laminate.

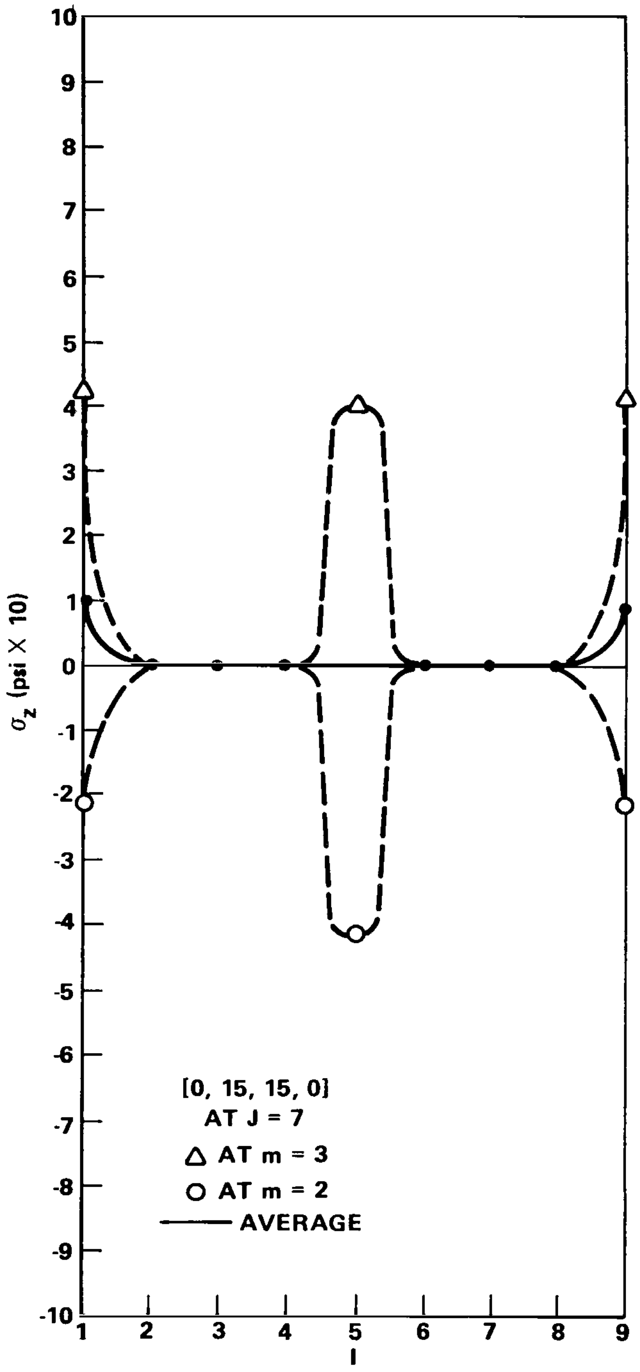


Figure 9. Numerical peculiarities in the normal stress σ_z.

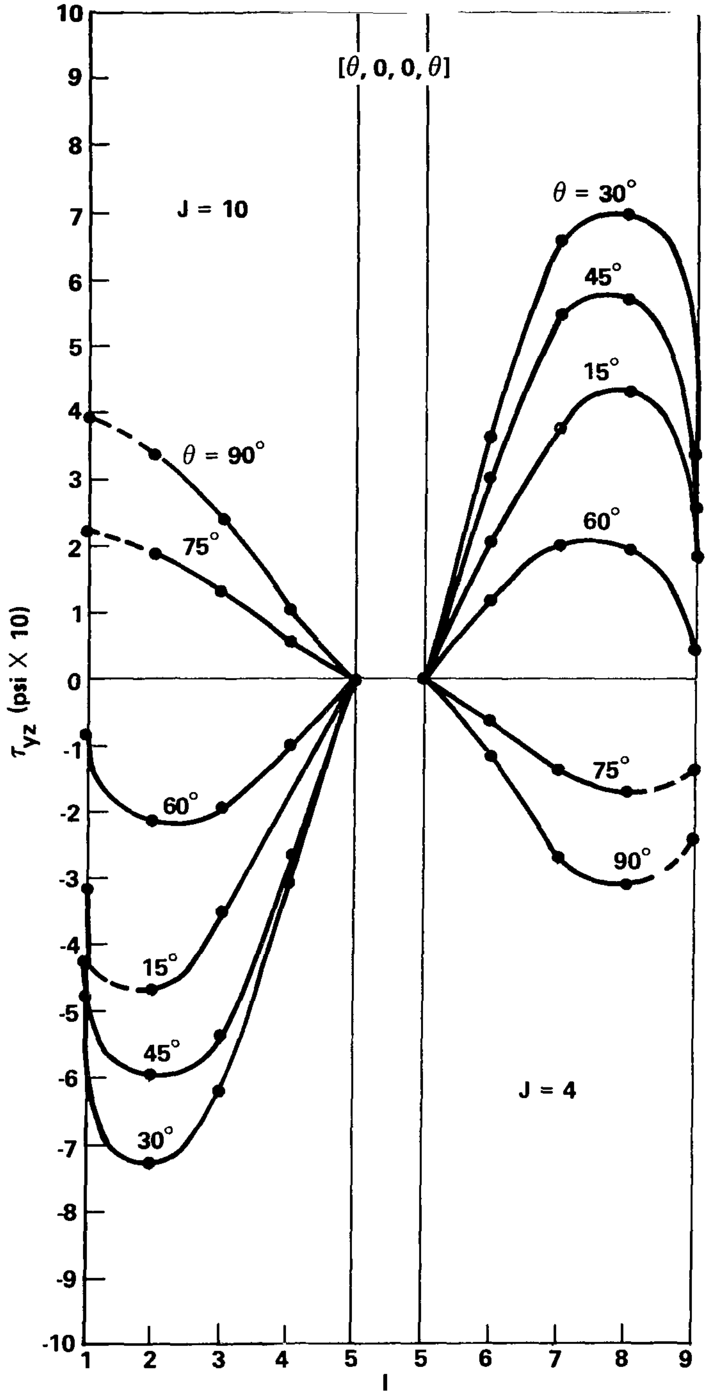


Figure 10. Variation of the shear stress τ_{yz} (antisymmetric in y) with y for a [θ, 0, 0, θ] laminate.

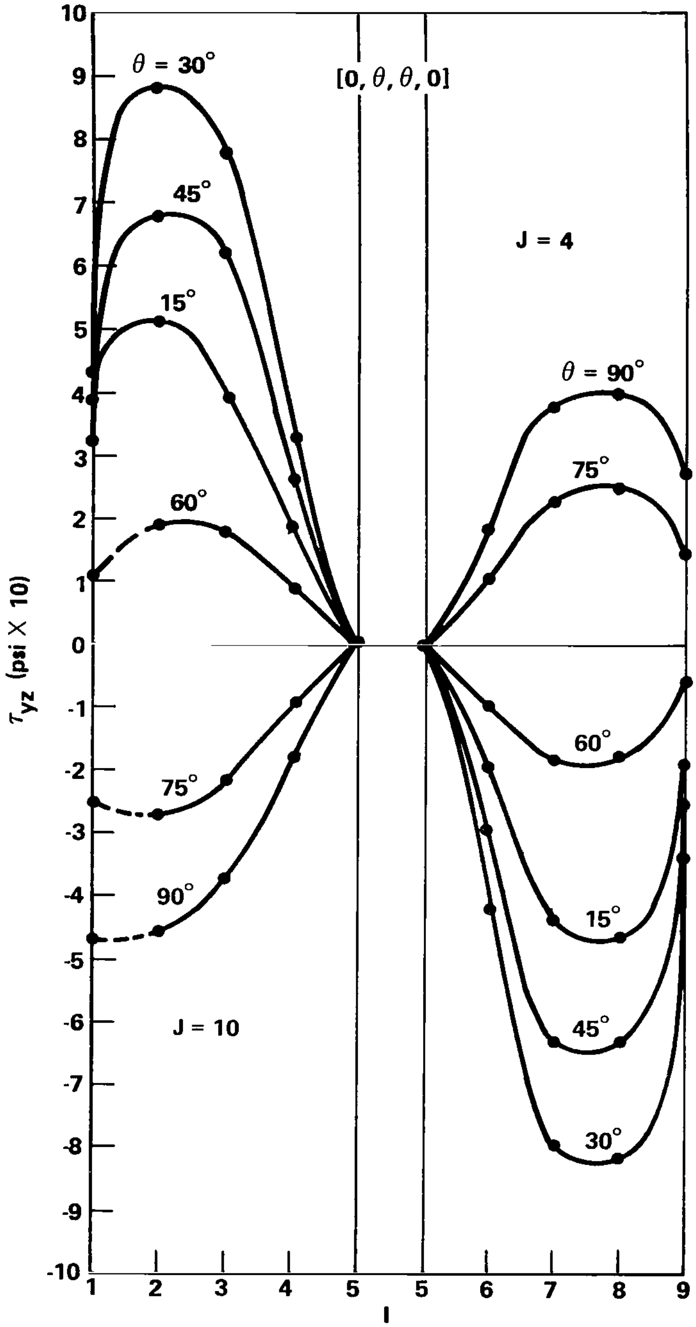


Figure 11. Variation of the shear stress τ_{yz} (antisymmetric in y) with y for a [0, θ, θ, 0] laminate.

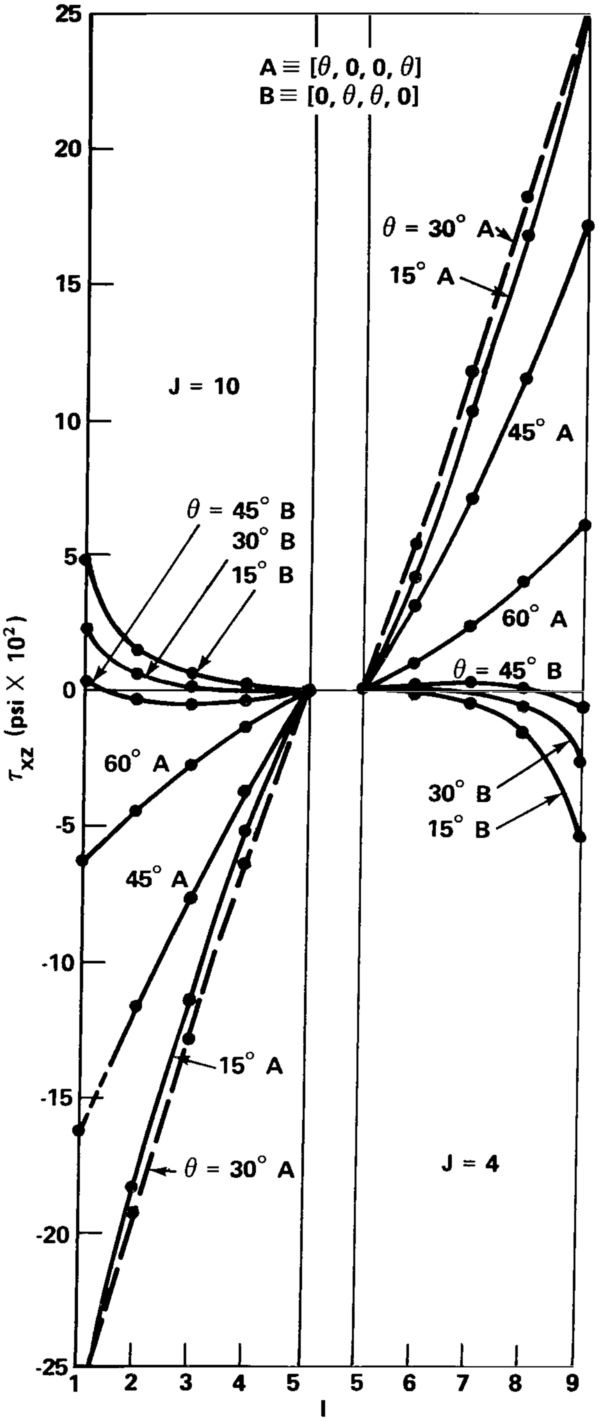


Figure 12. Variation of the shear stress τ_{xz} (antisymmetric in y) with y.

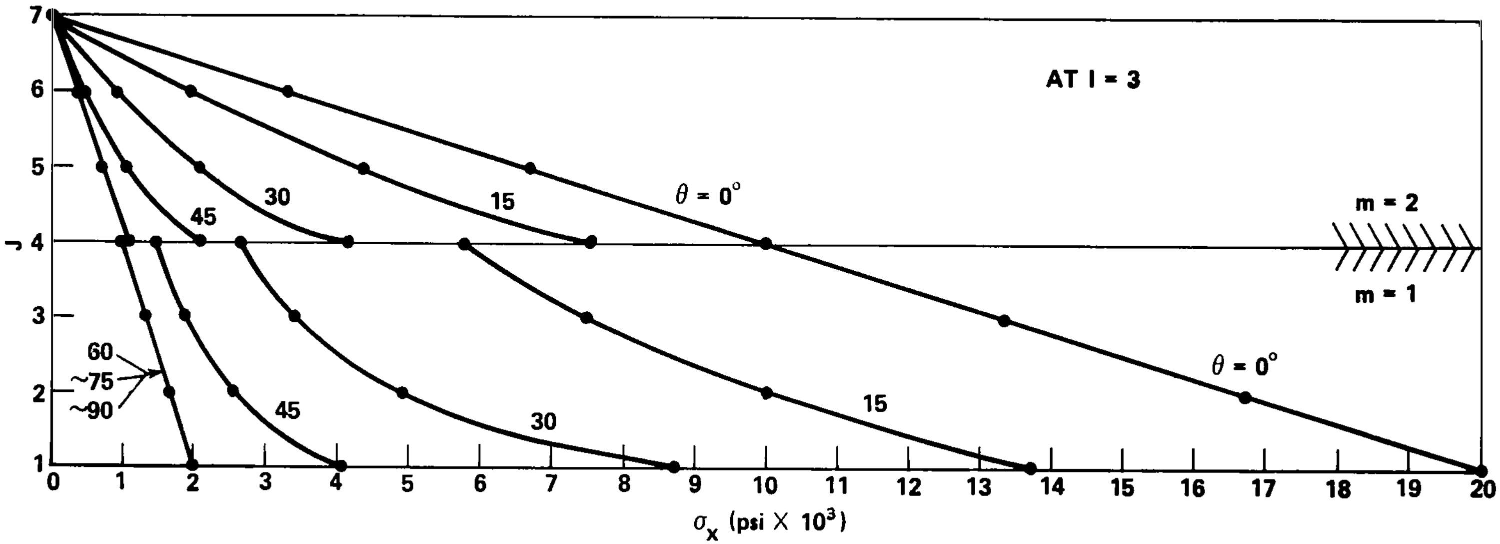


Figure 13. Variation of the normal stress σ_x (antisymmetric in z) with z for each layer with respect to position where the adjacent layer is oriented at θ = 0 degree.

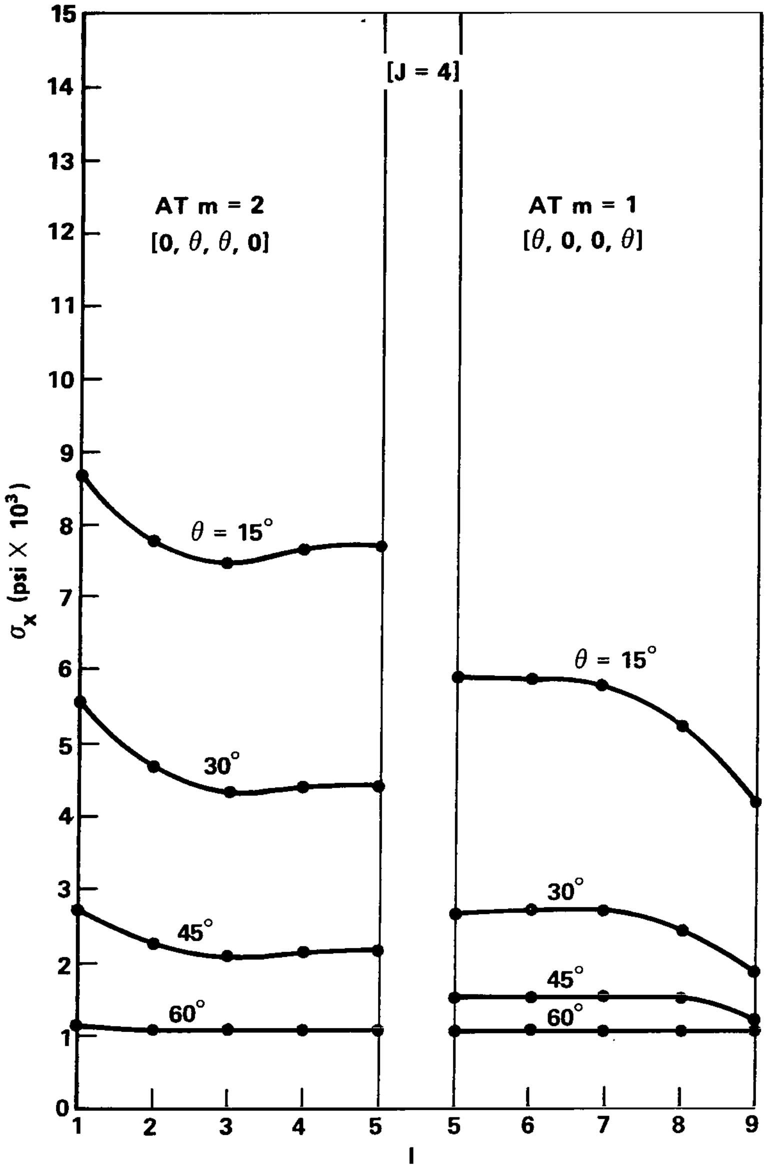


Figure 14. Variation of the normal stress σ_x (symmetric in y) with y.

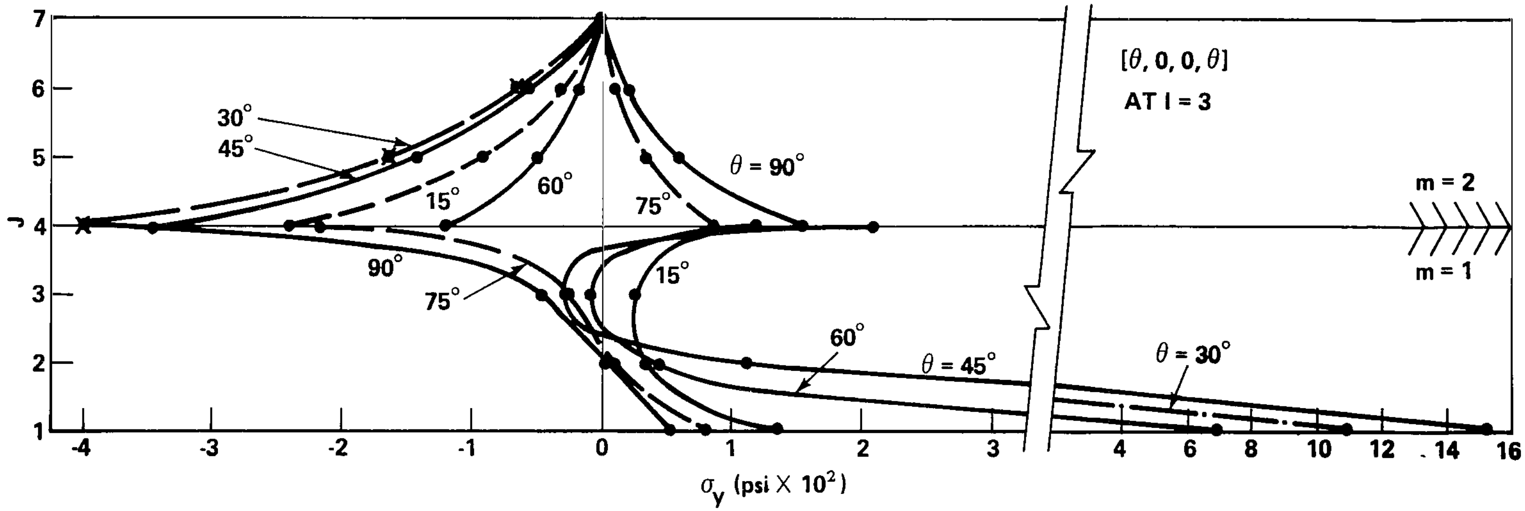


Figure 15. Variation of the normal stress σ_y (antisymmetric in z) with z for a [θ, 0, 0, θ] laminate.

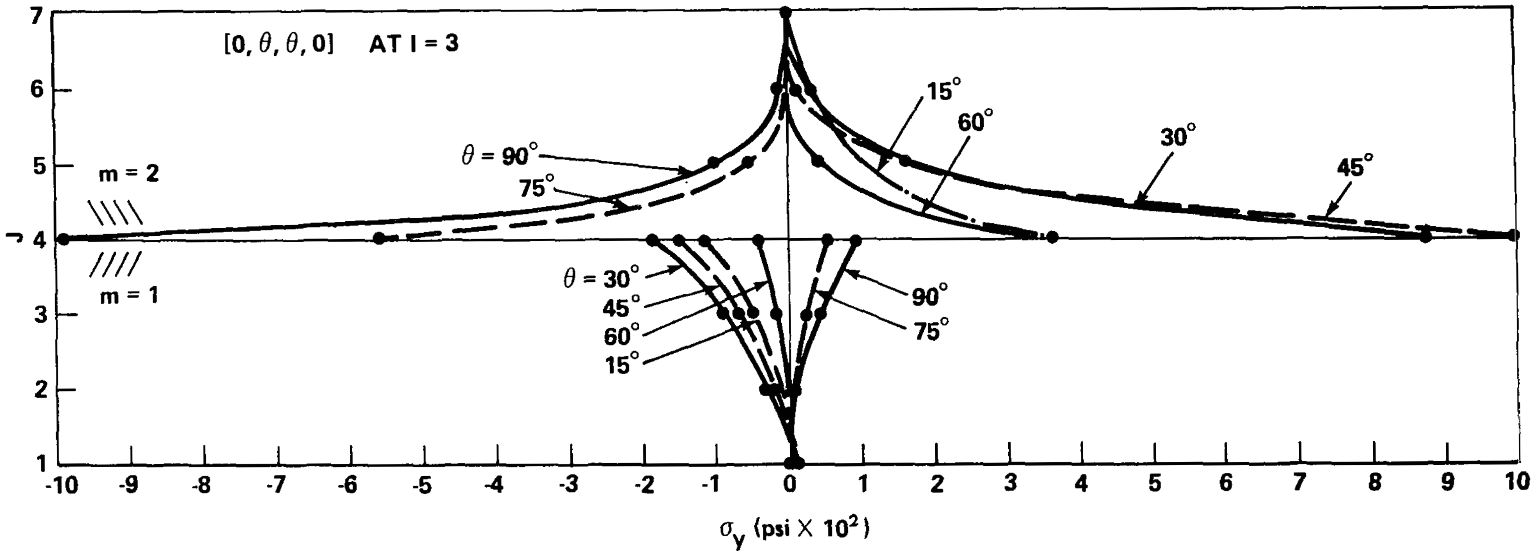


Figure 16. Variation of the normal stress σ_y (antisymmetric in z) with z for a [0, θ, θ, 0] laminate.

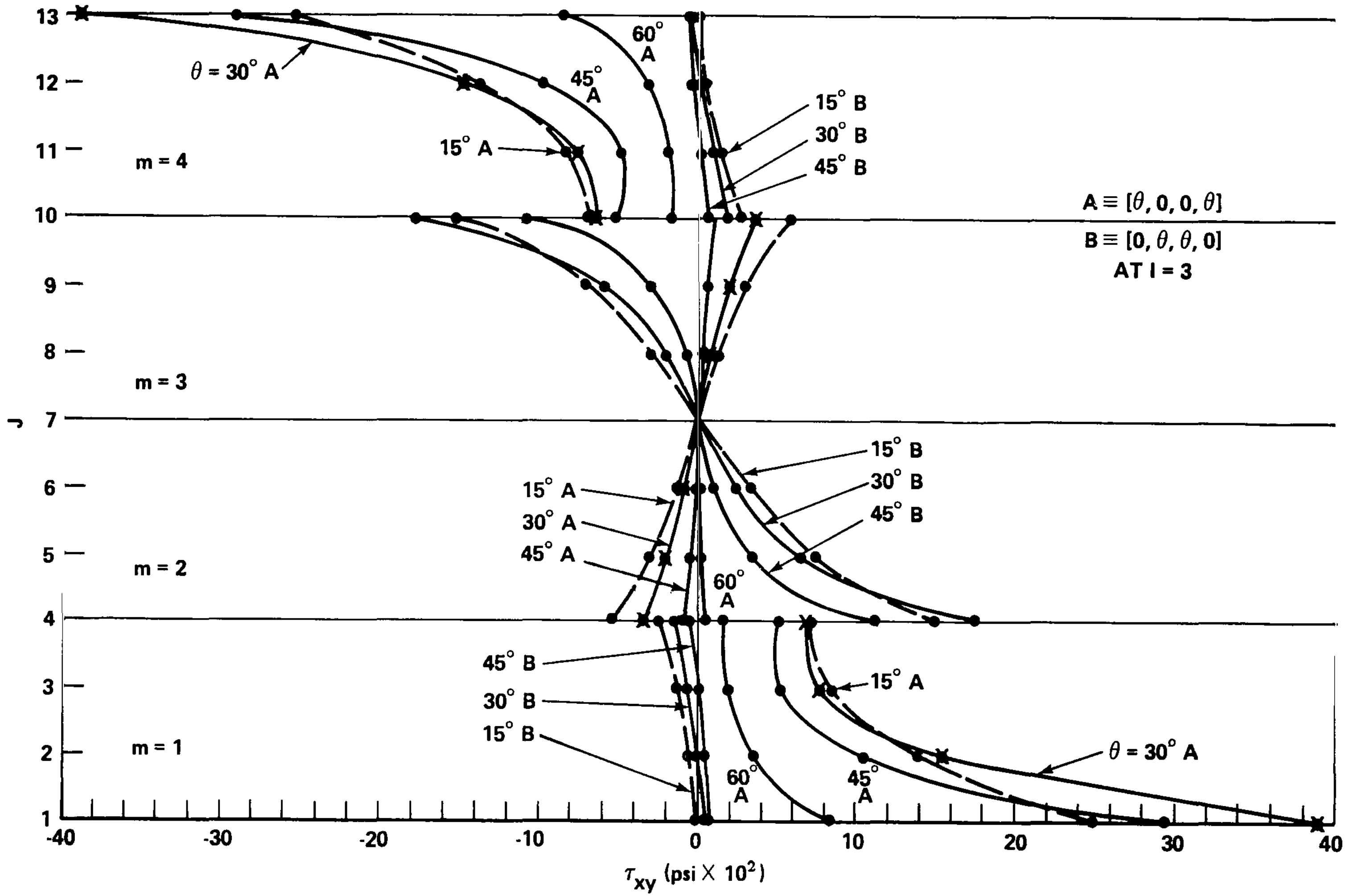


Figure 17. Variation of the shear stress τ_{xy} with z.

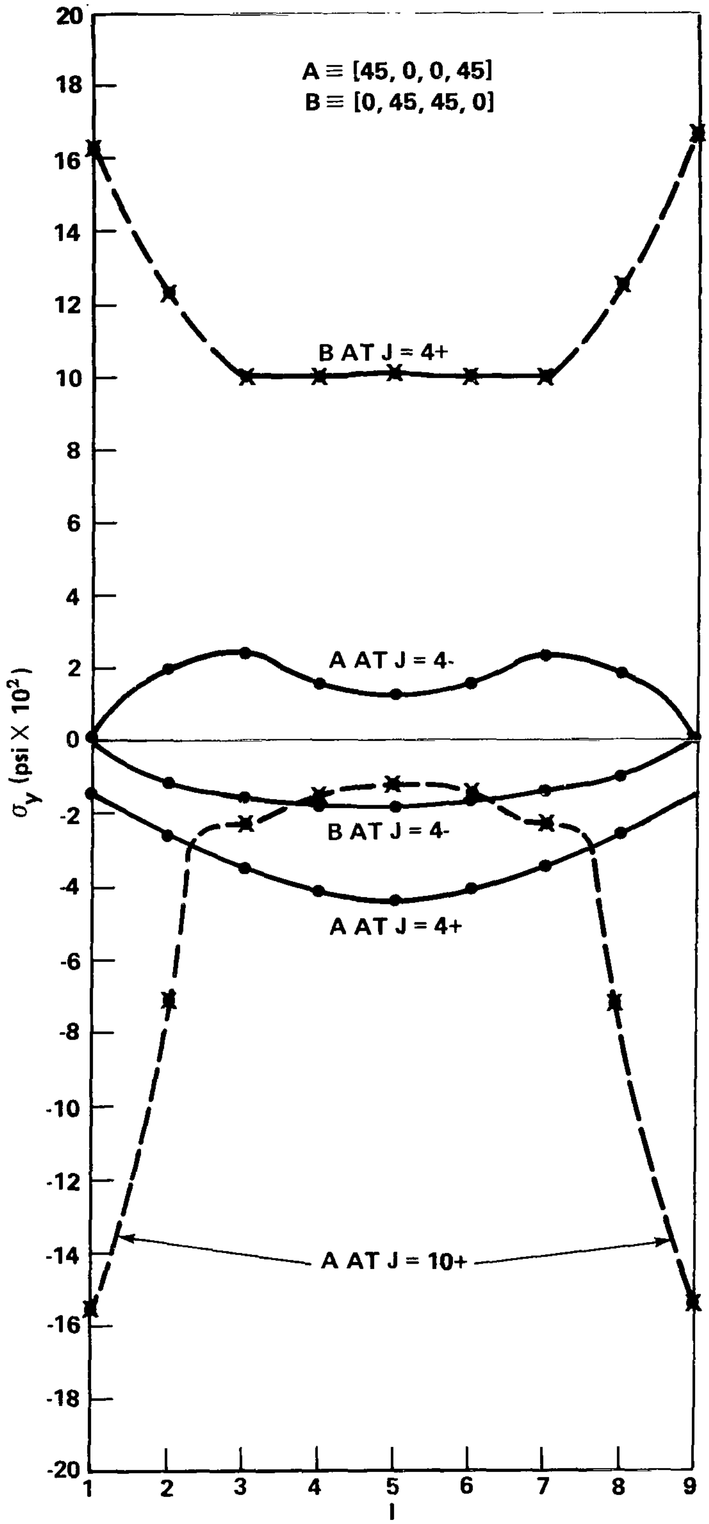


Figure 18. Variation of the normal stress σ_y with y.

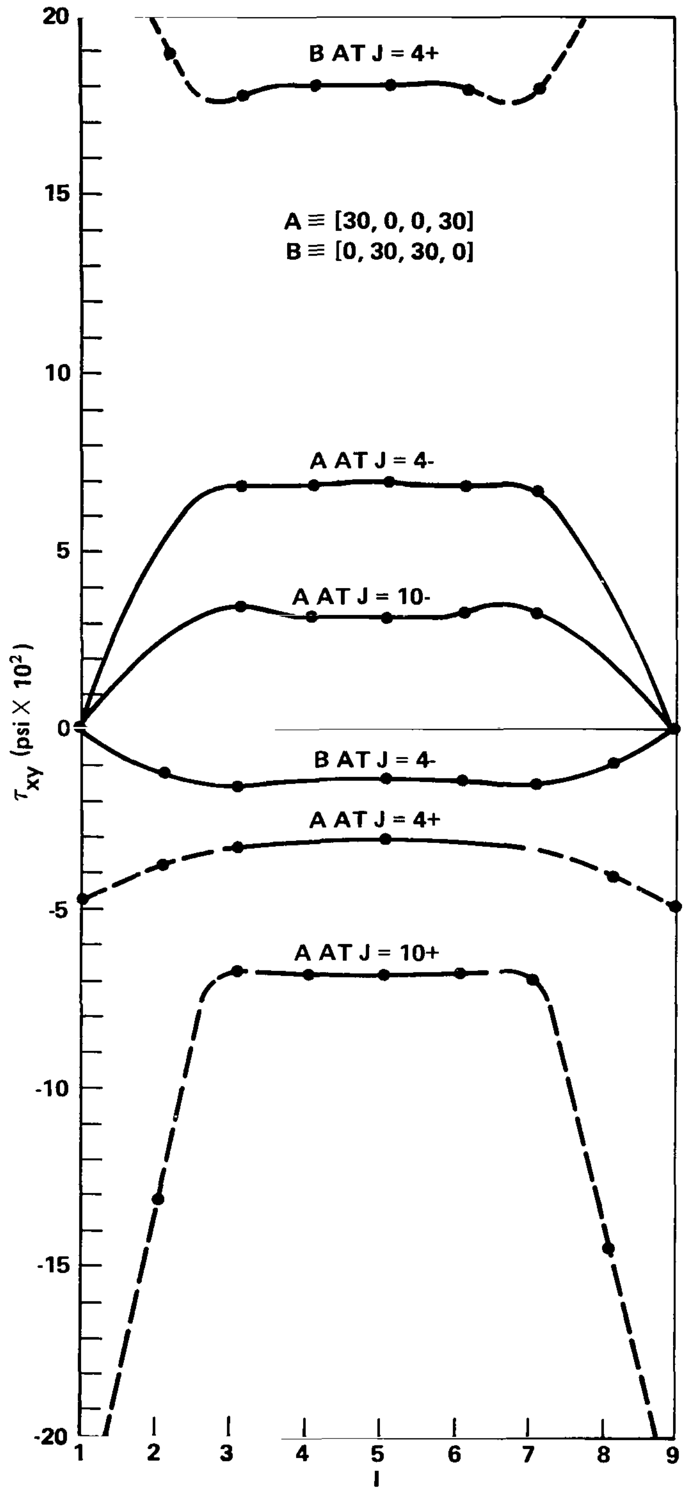


Figure 19. Variation of the shear stress τ_{xy} with y.

APPENDIX A

LAMINATE CONSTANTS

Following Reference 9 or 10, define

$$Q_{ij}^m = c_{ij}^m - \frac{c_{i3}^m c_{j3}^m}{c_{33}^m} \quad ; \quad i, j = 1, 2, 6$$

and let t be the half-thickness of the laminate, h_0 the thickness of a lamina, and N the total number of layers; then

$$A_{ij} = h_0 \sum_{m=1}^{N} Q_{ij}^m$$

$$B_{ij} = \frac{h_0^2}{2} \left\{ \sum_{m=1}^{N} Q_{ij}^m (2m - 1) - N \sum_{m=1}^{N} Q_{ij}^m \right\}$$

$$D_{ij} = \frac{h_0^3}{3} \left\{ \sum_{m=1}^{N} Q_{ij}^m (3m^2 - 3m + 1) - \frac{3}{2} N \sum_{m=1}^{N} Q_{ij}^m (2m - 1) + \frac{3}{4} N^2 \sum_{m=1}^{N} Q_{ij}^m \right\}$$

with i, j = 1, 2, 6. Finally, let

$$A^* = A^{-1} \quad , \quad B^* = -A^{-1}B \quad , \quad \text{and} \quad D^* = D - BA^{-1}B$$

where the letters symbolize 3 X 3 matrices. Then,

$$B' = B^*(D^*)^{-1}$$

and

$$D' = (D^*)^{-1} \quad .$$

Considerable simplification is attained if the laminate is balanced, which implies $B_{ij} = B'_{ij} = 0$.

APPENDIX B

STRAIN SPECIFICATION

Rather than prescribe the laminate loading as end moments, the maximum strain, ϵ_x^{max}, at the top and bottom surfaces, $z = \pm z^{max}$, will be prescribed. From equation (9), we have

$$\epsilon_x^{max} = C_2 z^{max} + C_3 \quad .$$

Now from equations (5),

$$C_3 = -B'_{11} M = \frac{B'_{11}}{D'_{11}} C_2 \quad ,$$

so that

$$\epsilon_x^{max} = C_2 \left(z^{max} + \frac{B'_{11}}{D'_{11}} \right)$$

and, thus,

$$C_2 = \frac{D'_{11} \, \epsilon_x^{max}}{B'_{11} + D'_{11} \, z^{max}} \quad .$$

In the computer program, we set $\epsilon_x^{max} = -1.0 \times 10^{-3}$ inch/inch at the top surface $z = +z^{max}$ to evaluate the constant C_2 which represents the inverse bending radius.

APPENDIX C

THE COMPUTER PROGRAM

Program Description

The computer program is an in-core program and is not overlayed. It is felt that a flow chart of the program would be no less complicated than the presentation of a listing with an accompanying explanation, so the latter choice will be followed. Certain statements in the program are extraneous to the problem in this report because the program is in steady transition to handle more general problems. A part-by-part description follows.

Part I. Part I contains a brief definition of terms and an explanation of the order and format of the data cards. The dimensions of the data are: H is in inches, E is material constants in psi (the shear moduli G_{12}, etc., are read into the E12, etc., arrays), ALPHA is the coefficient of expansion in inches/inch/°F, and THETA is the lamina orientation in angular degrees. Precision and dimension statements are then established, data are read in, and mesh parameters are calculated. The letter M refers to the layer number. In the loop, DO 9000, IRAN counts each laminate layup from one to IRUN (only changes in lamina orientation are allowed for within this loop).

Part II. Part II calculates the anisotropic stiffness matrix. BETA is in radians. CP11, etc., are the orthotropic elastic constants in the primed coordinate system. C11(M), etc., are the anisotropic elastic constants for the Mth lamina in the x, y, z coordinate system. AL1P(M), etc., are the coefficients of thermal expansion in the primed coordinate system and AL1(M) are those coefficients in the x, y, z coordinate system, both the the Mth lamina. Finally, the subroutine MATCON, which calculates the laminate MATerial CONstants, is called.

Part III. Part III calculates the coefficient matrix for the difference equations. The loops DO 100 and DO 101 count through the mesh node-by-node. DO 3000 zeroes out the A-matrix.

The logic that associates the various field conditions with each node and correctly fills out the A-matrix is contained in DO 102. First the node I, J is tested to determine the proper layer number, M. Then the node is checked to see if it lies on a boundary, along J equals a constant, or lies at some select position (in this case, IMID or JMID). If it does, the program is routed to the statement number that contains the non-zero matrix elements satisfying the conditions imposed at this node. Should the node not lie at any of these preselected locations, the program passes through the IF statements on J to statement number 193, which initiates a series of checks to see if the node lines on selected values of I. These values include the boundaries I = 1 or I = LAW, the changes in nodal spacing I = FSW1 or I = FSW2, and all points in the region between FSW1 and FSW2. Should the node not lie at any of these locations, the program passes through the IF statements on I and evaluates the non-zero coefficients for the only remaining possibility, the equilibrium terms for a square mesh.

When a node does lie on some select location, say J equals LAT, then the logic in that statement series, say the series starting from statement number 202, guides the program through the checks on selected values of I in a fashion similar to that above. The logic is easily understood by reading directly from the listing.

Upon reaching statement number 102, the A-matrix (3 × JQMAX) is full. The elements of the A-matrix lying within the bandwidth are then stored in the banded matrix AX. The loops DO 100 or DO 101 then continue for the next node, if any. The previous A-matrix is destroyed and regenerated for the new node until the loop DO 100 is satisfied.

At rewind 9, the matrix AX and the load vector X are stored for later use. The loop DO 107 stores the load vector X(I) in AX(I, NBD). Then a series of WRITE statements (listed as comments) will output the coefficient matrix AX and load vector X should they be desired. Finally, the solver routine, TRMSTR, is called.[7]

Part IV. Part IV outputs the functional displacements and provides an accuracy check. Just below statement number 4006, the STOP 1 statement will terminate the program if the coefficient matrix AX is singular. (Such an occurrence probably indicates an error.) The loop DO 108 stores the solution vector AX(I, 1) in X(I). Then the original values of the matrix AX and load vector X are read back into the AX array and R vector, respectively.

The loops DO 11 and DO 12 output the values for the functions U(y, z), V(y, z), and W(y, z) which occur in the displacements u, v, and w, respectively.

The series of statements from the one above 9950 to 9990 outputs the accuracy results. These results provide the difference between the original load vector, now stored in the R-array, with the calculated load vector, which is found by substituting the appropriate solution vectors, X(I), into each matrix equation. In addition to giving the accuracy of each equation, an average accumulated accuracy is provided.

Part V. Part V outputs the strains and stresses. The logic is similar to that in Part III. Knowing that the finite-difference relations for the strains differ for various mesh locations, the strains are split into terms dependent upon the value of I and terms dependent upon the value of J. The strain SX, which represents ϵ_x, depends upon neither the value of I nor the value of J and is determined prior to any logical branching.

---

7. Actually the AX-matrix stores a transposed A-matrix; i.e., instead of storing row elements crosswise or in a row, they are stored in the AX-matrix vertically or in a column. The result is a drastic reduction in "wall-time" on the IBM 370. This necessitated a slight revision in the solver routine, TRIMSS, as written by Billy Gibbs, U.S. Army, Redstone Arsenal [14]. So here it is called TRMSTR or TRIMSS transposed.

First, the node is checked to determine its location with respect to I, and I-dependent strains (or the partial strain, SYZI) are calculated. Then the loop D0 392 establishes the correct layer number, M, in order to check if J lies on the interface, INF(M). Upon determining the correct location of the node with respect to J, the J-dependent strains (or the partial strain, SYZJ) are calculated. Statement number 391 totals the partial strains to obtain SYZ. The stresses are then calculated in a straight forward manner using equation (1). Note that the stresses are calculated twice at interface nodes, once for the material below the interface and again for the material above.

Part VI. The subroutine MATCON calculates the MATerial CONstants C_j, BU, BV, and DV as defined earlier in the text.

Part VII. The subroutine MAMULT is a MAtrix MULTiplier and is easily understood from the listing.

Part VIII. The subroutine MATIN4 is a MATrix INversion routine which is described in Reference 14.

Part IX. The subroutine TRMSTR is the equation solver which is described in the listing.

Part X. The subroutine RITE is used to wRITE out a matrix or vector.

Program Listing

The complete listing of the program is contained in the following pages.

```
          C  JQMAX IS THE NUMBER OF UNKNOWNS OR EQUATIONS TO BE SOLVED.
          C  A IS A FULL MATRIX (3 X JQMAX) REPRESENTING EACH NODE.
          C  AX IS THE BANDED MATRIX (NBAND+1 X JQMAX).
          C  X IS THE LOAD VECTOR. AFTER TRIMSS X BECOMES THE SOLUTION VECTOR.
          C
          C  IF THE NUMBER OF LAYERS EXCEED 6, THE COMMON /MC/ AND DIMENSION (E11,
          C  E22, ETC.) STATEMENTS MUST BE REDIMENSIONED TO AGREE WITH LAT.
          C  REMEMBER TO PLACE A COMMON /MC/ STATEMENT IN SUBROUTINE MATCON.
          C
          C  USE THE FOLLOWING ORDER FOR DATA CARDS
          C
          C  DATA CARD NO.                DATA                              FORMAT
          C      1          NLAY, LAT, LAW, FSW1, K                         5I10
          C      2          H                                               G12.5
          C      3          E11, E22, E33, E12, E13, E23                    8G12.5
          C      4          NU12, NU13, NU23                                8G12.5
          C      5          ALPHA 1 PRIME, ALPHA 2 PRIME, ALPHA 3 PRIME     8G12.5
          C  NOTE, REPEAT CARDS OF THE TYPE 3, 4, 5 FOR EACH ADDITIONAL LAYER
          C      6          SXMAX, C3E                                     10G10.3
          C      7          IRUN                                            5I10
          C      8          THETA(1), THETA(2), THETA(3), ETC.             10G10.3
          C  NOTE, REPEAT CARD 8 FOR EACH ADDITIONAL LAYUP.
          C
          C
0001            INTEGER P, FSW1, FSW2
0002            DOUBLE PRECISION TEST, R, ERR, AVE, DT
0003            DOUBLE PRECISION AX, X
0004            DOUBLE PRECISION THETA, BETA
0005            DOUBLE PRECISION  CM, CN, CM4, CN4, CM3N, CN3M, CM2, CN2, GNU21,
               1                  GNU31, GNU32, DET, CP11, CP22, CP33, CP12, CP13,
               2                  CP23, CP44, CP55, CP66
          C
0006            DIMENSION AX(162,351),A(3,351), X(351), R(351)
          C
0007            COMMON /MC/ C11(6),C12(6),C16(6),C22(6),C26(6),C66(6),C13(6),
               1       C23(6),C36(6),C44(6),C45(6),C55(6),C33(6),AL1(6),AL2(6),
               2       AL3(6),AL6(6),C2,C3,C3E,C4,BU,DU,BV,DV,H,SXMAX,NLAY,INF(6)
          C
0008            DIMENSION E11(6),E22(6),E33(6),E12(6),E13(6),E23(6),GNU12(6),
               1          GNU13(6),GNU23(6),THETA(6), AL1P(6), AL2P(6), AL3P(6)
          C
0009            TEMP = 0.0
          C
0010            WRITE(6,600)
0011            READ(5,601)NLAY,LAT,LAW,FSW1,K
          C
0012            FSW2=LAW-FSW1+1
0013            JQMAX = 3*LAW*LAT
0014            IBW = 2*(3*LAT+1)
0015            IBW1 = IBW+1
0016            NBAND = 2*IBW+1
          C
0017            WRITE(6,602)NLAY,LAT,LAW,FSW1,FSW2,K
0018            LAT1=LAT-1
0019            IMID = (LAW+1)/2
0020            JMID = (LAT+1)/2
          C
0021            DO 501 M=1, NLAY
```

```
0022          INF(M)=1+M*LAT1/NLAY
0023          WRITE(6,608)M,INF(M)
0024      501 CONTINUE
        C
        C  NOTE THAT INF(NLAY) EQUALS LAT AND IS NOT AN ACTUAL INTERFACE.
        C
0025          READ(5,603) H
0026          WRITE(6,607) H
0027          HSQ = H**2
        C
0028          WRITE(6,604)
        C
0029          DO 500 M=1,NLAY
0030          READ(5,603)E11(M),E22(M),E33(M),E12(M),E13(M),E23(M)
0031          READ(5,603)GNU12(M),GNU13(M),GNU23(M)
0032          WRITE(6,605) M, E11(M), E22(M), E33(M), E12(M), E13(M), E23(M),
             1             GNU12(M), GNU13(M), GNU23(M)
0033          READ(5,603)AL1P(M), AL2P(M), AL3P(M)
0034      500 CONTINUE
        C
0035          READ(5,606) SXMAX, C3E
0036          READ(5,601) IRUN
        C
0037          DO 9000 IRAN = 1, IRUN
0038          READ(5,606) (THETA(M),M=1,NLAY)
        C
        C***********************************************************************
        C
        C CALCULATION OF ANISOTROPIC STIFFNESS MATRIX TERMS REFERRED TO X,Y,Z
        C
        C***********************************************************************
        C
0039          WRITE(6,613)
0040          XX = 0.0
0041          DO 3001 M=1,NLAY
0042          BETA = .0174532925199433D0*THETA(M)
0043          CM=DCOS(BETA)
0044          CN=DSIN(BETA)
0045          IF(DABS(CM).LT.1.E-08) CM = 0.
0046          IF(DABS(CN).LT.1.E-08) CN = 0.
0047          CM4=CM**4
0048          CN4=CN**4
0049          CM3N=CM**3*CN
0050          CN3M=CN**3*CM
0051          CM2=CM**2
0052          CN2=CN**2
0053          GNU21=GNU12(M)*E22(M)/E11(M)
0054          GNU31=GNU13(M)*E33(M)/E11(M)
0055          GNU32=GNU23(M)*E33(M)/E22(M)
0056          DET=1.-GNU12(M)*GNU21-GNU23(M)*GNU32-GNU13(M)*GNU31
             1-2.*GNU12(M)*GNU23(M)*GNU31
0057          CP11=E11(M)*(1.-GNU23(M)*GNU32)/DET
0058          CP22=E22(M)*(1.-GNU13(M)*GNU31)/DET
0059          CP33=E33(M)*(1.-GNU12(M)*GNU21)/DET
0060          CP12=E11(M)*(GNU21+GNU23(M)*GNU31)/DET
0061          CP13=E11(M)*(GNU31+GNU21*GNU32)/DET
0062          CP23=E22(M)*(GNU32+GNU12(M)*GNU31)/DET
0063          CP44=E23(M)
```

```
0064          CP55=E13(M)
0065          CP66=E12(M)
0066          C11(M)=CM4*CP11+2.*CM2*CN2*CP12+CN4*CP22+4.*CM2*CN2*CP66
0067          C12(M)=CM2*CN2*CP11+(CM4+CN4)*CP12+CM2*CN2*CP22-CM2*CN2*4.*CP66
0068          C16(M)=CM3N*CP11-(CM3N-CN3M)*CP12-CN3M*CP22-2.*(CM3N-CN3M)*CP66
0069          C22(M)=CN4*CP11+2.*CM2*CN2*CP12+CM4*CP22+4.*CM2*CN2*CP66
0070          C26(M)=CN3M*CP11-(CN3M-CM3N)*CP12-CM3N*CP22-2.*(CN3M-CM3N)*CP66
0071          C66(M)=CM2*CN2*CP11-2.*CM2*CN2*CP12+CM2*CN2*CP22+(CM2-CN2)**2*CP66
0072          C13(M)=CM2*CP13+CN2*CP23
0073          C23(M)=CN2*CP13+CM2*CP23
0074          C36(M)=CM*CN*(CP13-CP23)
0075          C44(M)=CM2*CP44+CN2*CP55
0076          C45(M)=CM*CN*(CP55-CP44)
0077          C55(M)=CN2*CP44+CM2*CP55
0078          C33(M)=CP33
      C
      C
      C*********************************************************************
      C
      C CALCULATION OF THE COEF. OF THERMAL EXPANSION REFERRED TO X,Y,Z
      C
      C*********************************************************************
      C
0079          AL1(M)=CM2*AL1P(M)+CN2*AL2P(M)
0080          AL2(M)=CN2*AL1P(M)+CM2*AL2P(M)
0081          AL3(M)=AL3P(M)
0082          AL6(M)=2.*CM*CN*(AL1P(M)-AL2P(M))
      C
0083          WRITE(6,620) M, C11(M), C12(M), C13(M), XX, XX, C16(M), CP11,
             1            CP12, CP13, XX, XX, XX, C22(M), C23(M), XX, XX,
             2            C26(M), CP22, CP23, XX, XX, XX, C33(M), XX, XX,
             3            C36(M), CP33, XX, XX, XX, THETA(M), C44(M), C45(M),
             4            XX, CP44, XX, XX, C55(M), XX, CP55, XX, C66(M), CP66
      C
0084   3001 CONTINUE
      C
0085          WRITE(6,611)
      C
0086          DO 503 M=1,NLAY
0087          WRITE(6,614) M, THETA(M), AL1(M), AL2(M), AL3(M), AL6(M),
             1             AL1P(M), AL2P(M), AL3P(M)
0088    503 CONTINUE
      C
0089          CALL MATCON
      C
      C
      C ********************************************************************
      C
      C CALCULATION OF THE COEFFICIENT MATRIX FOR THE DIFFERENCE EQUATIONS
      C
      C ********************************************************************
      C
0090          KJ1 = 1
0091          KQ1 = KJ1 + 1
0092          KQ2 = KJ1 + 2
      C
0093          DO 100 I=1,LAW
0094          DO 101 J=1, LAT
```

```
                C
0095                  DO 3000 IM = KJ1, KQ2
0096                  DO 3000 JM = 1, JQMAX
0097                  A(IM,JM) = 0.
0098             3000 CONTINUE
                C
0099                  I1=I-1
0100                  I2=I-2
0101                  Z = (FLOAT(J)-(FLOAT(LAT)+1.)/2.)*H
0102                  NODE = LAT*I1+J
0103                  JJ1 = 3*(LAT*I1+J)-2
0104                  JJ2 = 3*(LAT*I2+J)-2
0105                  JJ3 = 3*(LAT*I2+J)-5
0106                  JJ4 = 3*(LAT*I+J)-2
0107                  JJ5 = 3*(LAT*I+J)+1
0108                  JJ6 = 3*(LAT*I1+J)+1
0109                  JJ7 = 3*(LAT*I2+J)+1
0110                  JJ8 = 3*(LAT*I1+J)-5
0111                  JJ9 = 3*(LAT*I+J)-5
0112                  JJ10 =3*(LAT*I1+J)-8
0113                  JJ11 =3*(LAT*(I+1)+J)-2
0114                  JJ12 =3*(LAT*I1+J)+4
0115                  JJ13 =3*(LAT*(I-3)+J)-2
                C
0116                  JQ1 = JJ1+1
0117                  JQ2 = JJ1+2
                C
0118                  DO 102 M=1, NLAY
0119                  IF(M.EQ.1.AND.J.GT.INF(1)) GO TO 102
0120                  IF(M.EQ.1) GO TO 192
0121                  IF(J.LE.INF(M-1).OR.J.GT.INF(M)) GO TO 102
0122              192 IF(J.EQ.1) GO TO 200
0123                  IF(I.EQ.IMID.AND.J.EQ.JMID) GO TO 203
0124                  IF(I.EQ.IMID+1.AND.J.EQ.JMID) GO TO 203
0125                  IF(J.EQ.LAT) GO TO 202
0126                  IF(J.EQ.INF(M)) GO TO 201
                C
                C SHOULD J EQUAL NONE OF THE ABOVE, CONTINUE ON BELOW TO STATEMENT 193
                C
0127              193 IF(I.EQ.1) GO TO 194
0128                  IF(I.EQ.FSW1.OR.I.EQ.FSW2) GO TO 195
0129                  IF(I.LT.FSW2.AND.I.GT.FSW1) GO TO 197
0130                  IF(I.EQ.LAW) GO TO 198
                C
                C EQUILIBRIUM MATRIX TERMS FOR A SQUARE MESH, H1=H2=H3=H
                C
0131                  A(KJ1,JJ1) =-8.*(C66(M)+C55(M))
0132                  A(KJ1,JJ2) = 4.*C66(M)
0133                  A(KJ1,JJ4) = 4.*C66(M)
0134                  A(KJ1,JJ6) = 4.*C55(M)
0135                  A(KJ1,JJ8) = 4.*C55(M)
0136                  A(KJ1,JJ1+1) = -8.*(C26(M)+C45(M))
0137                  A(KJ1,JJ2+1) = 4.*C26(M)
0138                  A(KJ1,JJ4+1) = 4.*C26(M)
0139                  A(KJ1,JJ6+1) = 4.*C45(M)
0140                  A(KJ1,JJ8+1) = 4.*C45(M)
                C
0141                  C = C36(M)+C45(M)
```

```
                      C
0142                        A(KJ1,JJ3+2) = C
0143                        A(KJ1,JJ5+2) = C
0144                        A(KJ1,JJ7+2) = -C
0145                        A(KJ1,JJ9+2) = -C
                      C
0146                        X(JJ1) = 0.
                      C
0147                        A(KQ1,JJ1) = -8.*(C26(M)+C45(M))
0148                        A(KQ1,JJ2) = 4.*C26(M)
0149                        A(KQ1,JJ4) = 4.*C26(M)
0150                        A(KQ1,JJ6) = 4.*C45(M)
0151                        A(KQ1,JJ8) = 4.*C45(M)
0152                        A(KQ1,JJ1+1) = -8.*(C22(M)+C44(M))
0153                        A(KQ1,JJ2+1) = 4.*C22(M)
0154                        A(KQ1,JJ4+1) = 4.*C22(M)
0155                        A(KQ1,JJ6+1) = 4.*C44(M)
0156                        A(KQ1,JJ8+1) = 4.*C44(M)
                      C
0157                        D = C23(M)+C44(M)
                      C
0158                        A(KQ1,JJ3+2) = D
0159                        A(KQ1,JJ5+2) = D
0160                        A(KQ1,JJ7+2) = -D
0161                        A(KQ1,JJ9+2) = -D
                      C
0162                        X(JQ1) = 0.
                      C
0163                        A(KQ2,JJ3) = C
0164                        A(KQ2,JJ5) = C
0165                        A(KQ2,JJ7) = -C
0166                        A(KQ2,JJ9) = -C
0167                        A(KQ2,JJ3+1) = D
0168                        A(KQ2,JJ5+1) = D
0169                        A(KQ2,JJ7+1) = -D
0170                        A(KQ2,JJ9+1) = -D
0171                        A(KQ2,JJ1+2) = -8.*(C44(M)+C33(M))
0172                        A(KQ2,JJ2+2) = 4.*C44(M)
0173                        A(KQ2,JJ4+2) = 4.*C44(M)
0174                        A(KQ2,JJ6+2) = 4.*C33(M)
0175                        A(KQ2,JJ8+2) = 4.*C33(M)
                      C
0176                        X(JQ2) = -4.*(C13(M)*C2 + C23(M)*DV + 2.*C36(M)*C4)*HSQ
0177                        GO TO 102
                      C
                      C FREE SURFACE MATRIX TERMS FOR I=1 AND J NOT EQUAL TO 1, INF OR LAT
                      C
0178                    194 A(KJ1,JJ1) = -3.*C66(M)
0179                        A(KJ1,JJ4) = 4.*C66(M)
0180                        A(KJ1,JJ11) = -C66(M)
0181                        A(KJ1,JJ1+1) = -3.*C26(M)
0182                        A(KJ1,JJ4+1) = 4.*C26(M)
0183                        A(KJ1,JJ11+1) = -C26(M)
0184                        A(KJ1,JJ6+2) = C36(M)
0185                        A(KJ1,JJ8+2) = -C36(M)
                      C
0186                        A(KQ1,JJ1) = -3.*C26(M)
0187                        A(KQ1,JJ4) =4.*C26(M)
```

```
0188              A(KQ1,JJ11) = -C26(M)
0189              A(KQ1,JJ1+1) = -3.*C22(M)
0190              A(KQ1,JJ4+1) = 4.*C22(M)
01                A(KQ1,JJ11+1) = -C22(M)
01                A(KQ1,JJ6+2) = C23(M)
0193              A(KQ1,JJ8+2) = -C23(M)
            C
0194              A(KQ2,JJ6) = C45(M)
0195              A(KQ2,JJ8) = -C45(M)
0196              A(KQ2,JJ6+1) = C44(M)
0197              A(KQ2,JJ8+1) = -C44(M)
0198              A(KQ2,JJ1+2) = -3.*C44(M)
0199              A(KQ2,JJ4+2) = 4.*C44(M)
0200              A(KQ2,JJ11+2) = -C44(M)
            C
0201              CY1 = C12(M)*C3 + C22(M)*BV + C26(M)*BU
0202              CY2 = C12(M)*C2 + C22(M)*DV + 2.*C26(M)*C4
0203              CXY1 = C16(M)*C3 + C26(M)*BV + C66(M)*BU
0204              CXY2 = C16(M)*C2 + C26(M)*DV + 2.*C66(M)*C4
            C
0205              X(JJ1) = -2.*H*(CXY1 + CXY2*Z)
0206              X(JQ1) = -2.*H*(CY1 + CY2*Z)
0207              X(JQ2) = 0.
0208              GO TO 102
            C
0209          195 H1 = H
0210              H2 = FLOAT(K)*H
0211              H3 = H
            C
0212              IF(I.NE.FSW2) GO TO 196
0213              H1 = FLOAT(K)*H
0214              H2 = H
            C
0215          196 CONTINUE
0216              HH = H2/H1
0217              HR = HH/(1.+HH)
0218              HH1 = H1/H3
0219              HH2 = H2/H3
0220              HH3 = H1*H2
0221              HMU = HH1*HH2
0222              GO TO 199
            C
0223          197 H1 = FLOAT(K)*H
0224              H2 = H1
0225              H3 = H
0226              GO TO 196
            C
            C FREE SURFACE MATRIX TERMS FOR I=LAW AND J NOT EQUAL TO 1, INF OR LAT
            C
0227          198 A(KJ1,JJ1) = 3.*C66(M)
0228              A(KJ1,JJ2) = -4.*C66(M)
0229              A(KJ1,JJ13) = C66(M)
0230              A(KJ1,JJ1+1) = 3.*C26(M)
0231              A(KJ1,JJ2+1) = -4.*C26(M)
0232              A(KJ1,JJ13+1) = C26(M)
0233              A(KJ1,JJ6+2) = C36(M)
0234              A(KJ1,JJ8+2) = -C36(M)
            C
```

```
0235          A(KQ1,JJ1) = 3.*C26(M)
0236          A(KQ1,JJ2) = -4.*C26(M)
0237          A(KQ1,JJ13) = C26(M)
0238          A(KQ1,JJ1+1) = 3.*C22(M)
0239          A(KQ1,JJ2+1) = -4.*C22(M)
0240          A(KQ1,JJ13+1) = C22(M)
0241          A(KQ1,JJ6+2) = C23(M)
0242          A(KQ1,JJ8+2) = -C23(M)
        C
0243          A(KQ2,JJ6) = C45(M)
0244          A(KQ2,JJ8) = -C45(M)
0245          A(KQ2,JJ6+1) = C44(M)
0246          A(KQ2,JJ8+1) = -C44(M)
0247          A(KQ2,JJ1+2) = 3.*C44(M)
0248          A(KQ2,JJ2+2) = -4.*C44(M)
0249          A(KQ2,JJ13+2) = C44(M)
        C
0250          CY1 = C12(M)*C3 + C22(M)*BV + C26(M)*BU
0251          CY2 = C12(M)*C2 + C22(M)*DV + 2.*C26(M)*C4
0252          CXY1 = C16(M)*C3 + C26(M)*BV + C66(M)*BU
0253          CXY2 = C16(M)*C2 + C26(M)*DV + 2.*C66(M)*C4
        C
0254          X(JJ1) = -2.*H*(CXY1 + CXY2*Z)
0255          X(JQ1) = -2.*H*(CY1 + CY2*Z)
0256          X(JQ2) = 0.
0257          GO TO 102
        C
        C EQUILIBRIUM MATRIX TERMS FOR A VARIABLE MESH, H1, H2 , H3 INDEPENDENT
        C
0258      199 A(KJ1,JJ1) = -2.*(C66(M)+HMU*C55(M))
0259          A(KJ1,JJ2) = 2.*HR*C66(M)
0260          A(KJ1,JJ4) = 2.*C66(M)/(1.+HH)
0261          A(KJ1,JJ6) = HMU*C55(M)
0262          A(KJ1,JJ8) = HMU*C55(M)
0263          A(KJ1,JJ1+1) =-2.*(C26(M)+HMU*C45(M))
0264          A(KJ1,JJ2+1) = 2.*HR*C26(M)
0265          A(KJ1,JJ4+1) = 2.*C26(M)/(1.+HH)
0266          A(KJ1,JJ6+1) = HMU*C45(M)
0267          A(KJ1,JJ8+1) = HMU*C45(M)
        C
0268          C = HH1*HR*(C36(M)+C45(M))/2.
        C
0269          A(KJ1,JJ3+2) = C
0270          A(KJ1,JJ5+2) = C
0271          A(KJ1,JJ7+2) = -C
0272          A(KJ1,JJ9+2) = -C
        C
0273          A(KQ1,JJ1) = -2.*(C26(M)+HMU*C45(M))
0274          A(KQ1,JJ2) = 2.*HR*C26(M)
0275          A(KQ1,JJ4) = 2.*C26(M)/(1.+HH)
0276          A(KQ1,JJ6) = HMU*C45(M)
0277          A(KQ1,JJ8) = HMU*C45(M)
0278          A(KQ1,JJ1+1) = -2.*(C22(M)+HMU*C44(M))
0279          A(KQ1,JJ2+1) = 2.*HR*C22(M)
0280          A(KQ1,JJ4+1) = 2.*C22(M)/(1.+HH)
0281          A(KQ1,JJ6+1) = HMU*C44(M)
0282          A(KQ1,JJ8+1) = HMU*C44(M)
        C
```

```
0283              D = HH1*HR*(C23(M)+C44(M))/2.
            C
0284              A(KQ1,JJ3+2) = D
0285              A(KQ1,JJ5+2) = D
0286              A(KQ1,JJ7+2) = -D
0287              A(KQ1,JJ9+2) = -D
            C
0288              A(KQ2,JJ3) = C
0289              A(KQ2,JJ5) = C
0290              A(KQ2,JJ7) = -C
0291              A(KQ2,JJ9) = -C
0292              A(KQ2,JJ3+1) = D
0293              A(KQ2,JJ5+1) = D
0294              A(KQ2,JJ7+1) = -D
0295              A(KQ2,JJ9+1) = -D
0296              A(KQ2,JJ1+2) = -2.*(C44(M)+HMU*C33(M))
0297              A(KQ2,JJ2+2) = 2.*HR*C44(M)
0298              A(KQ2,JJ4+2) = 2.*C44(M)/(1.+HH)
0299              A(KQ2,JJ6+2) = HMU*C33(M)
0300              A(KQ2,JJ8+2) = HMU*C33(M)
            C
0301              X(JJ1) = 0.
0302              X(JQ1) = 0.
0303              X(JQ2) = -HH3*(C13(M)*C2 + C23(M)*DV + 2.*C36(M)*C4)
0304              GO TO 102
            C
0305          200 IF(I.EQ.1) GO TO 210
0306              IF(I.EQ.LAW) GO TO 211
            C
            C FREE SURFACE MATRIX TERMS FOR I BETWEEN 1 AND LAW AND J=1.
            C
0307              A(KJ1,JJ1) = -3.*C55(M)
0308              A(KJ1,JJ6) = 4.*C55(M)
0309              A(KJ1,JJ12) = -C55(M)
            C
0310              A(KJ1,JJ1+1) = -3.*C45(M)
0311              A(KJ1,JJ6+1) = 4.*C45(M)
0312              A(KJ1,JJ12+1) = -C45(M)
            C
0313              A(KQ1,JJ1) = -3.*C45(M)
0314              A(KQ1,JJ6) = 4.*C45(M)
0315              A(KQ1,JJ12) = -C45(M)
            C
0316              A(KQ1,JJ1+1) = -3.*C44(M)
0317              A(KQ1,JJ6+1) = 4.*C44(M)
0318              A(KQ1,JJ12+1) = -C44(M)
            C
0319              A(KQ2,JJ1+2) = -3.*C33(M)
0320              A(KQ2,JJ6+2) = 4.*C33(M)
0321              A(KQ2,JJ12+2) = -C33(M)
            C
0322              CZ1 = C13(M)*C3 + C23(M)*BV + C36(M)*BU
0323              CZ2 = C13(M)*C2 + C23(M)*DV + 2.*C36(M)*C4
            C
0324              X(JJ1) = 0.
0325              X(JQ1) = 0.
0326              X(JQ2) = -2.*H*(CZ1 + CZ2*Z)
            C
```

```
0327           IF(I.EQ.FSW1) GO TO 206
0328           IF(I.EQ.FSW2) GO TO 206
0329           IF(I.GT.FSW1.AND.I.LT.FSW2) GO TO 209
      C
      C IF I IS BETWEEN 1 AND FSW1 OR BETWEEN FSW2 AND LAW, CONTINUE BELOW
      C
0330           A(KJ1,JJ2+2) = -C45(M)
0331           A(KJ1,JJ4+2) = C45(M)
      C
0332           A(KQ1,JJ2+2) = -C44(M)
0333           A(KQ1,JJ4+2) = C44(M)
      C
0334           A(KQ2,JJ2) = -C36(M)
0335           A(KQ2,JJ4) = C36(M)
0336           A(KQ2,JJ2+1) = -C23(M)
0337           A(KQ2,JJ4+1) = C23(M)
0338           GO TO 102
      C
      C CASE WHERE I=FSW1 OR FSW2 AND J=1
      C
0339       206 XK = FLOAT(K)
0340           D1 = 2.*(XK-1.)/XK
0341           D2 = 2.*XK/(XK+1.)
0342           D3 = 2./((XK+1.)*XK)
      C
0343           IF(I.EQ.FSW2) GO TO 207
      C
0344           A(KJ1,JJ1+2) = D1*C45(M)
0345           A(KJ1,JJ2+2) = -D2*C45(M)
0346           A(KJ1,JJ4+2) = D3*C45(M)
      C
0347           A(KQ1,JJ1+2) = D1*C44(M)
0348           A(KQ1,JJ2+2) = -D2*C44(M)
0349           A(KQ1,JJ4+2) = D3*C44(M)
      C
0350           A(KQ2,JJ1) = D1*C36(M)
0351           A(KQ2,JJ2) = -D2*C36(M)
0352           A(KQ2,JJ4) = D3*C36(M)
      C
0353           A(KQ2,JJ1+1) ■ D1*C23(M)
0354           A(KQ2,JJ2+1) = -D2*C23(M)
0355           A(KQ2,JJ4+1) = D3*C23(M)
0356           GO TO 102
      C
0357       207 A(KJ1,JJ1+2) = -D1*C45(M)
0358           A(KJ1,JJ2+2) = -D3*C45(M)
0359           A(KJ1,JJ4+2) = D2*C45(M)
      C
0360           A(KQ1,JJ1+2) = -D1*C44(M)
0361           A(KQ1,JJ2+2) = -D3*C44(M)
0362           A(KQ1,JJ4+2) = D2*C44(M)
      C
0363           A(KQ2,JJ1) = -D1*C36(M)
0364           A(KQ2,JJ2) = -D3*C36(M)
0365           A(KQ2,JJ4) = D2*C36(M)
      C
0366           A(KQ2,JJ1+1) = -D1*C23(M)
0367           A(KQ2,JJ2+1) = -D3*C23(M)
```

```
0368             A(KQ2,JJ4+1) = D2*C23(M)
0369             GO TO 102
           C
           C CASE WHERE I IS BETWEEN FSW1 AND FSW2 AND J=1
           C
0370         209 XK = FLOAT(K)
0371             A(KJ1,JJ2+2) = -C45(M)/XK
0372             A(KJ1,JJ4+2) = C45(M)/XK
           C
0373             A(KQ1,JJ2+2) = -C44(M)/XK
0374             A(KQ1,JJ4+2) = C44(M)/XK
           C
0375             A(KQ2,JJ2) = -C36(M)/XK
0376             A(KQ2,JJ4) = C36(M)/XK
           C
0377             A(KQ2,JJ2+1) = -C23(M)/XK
0378             A(KQ2,JJ4+1) = C23(M)/XK
0379             GO TO 102
           C
           C  FREE SURFACE MATRIX TERMS FOR I=J=1
           C
0380         210 A(KJ1,JJ1) = -3.*C66(M)
0381             A(KJ1,JJ4) = 4.*C66(M)
0382             A(KJ1,JJ11) = -C66(M)
0383             A(KJ1,JJ1+1) = -3.*C26(M)
0384             A(KJ1,JJ4+1) = 4.*C26(M)
0385             A(KJ1,JJ11+1) = -C26(M)
0386             A(KJ1,JJ1+2) = -3.*C36(M)
0387             A(KJ1,JJ6+2) = 4.*C36(M)
0388             A(KJ1,JJ12+2) = -C36(M)
           C
0389             A(KQ1,JJ1) = -3.*C26(M)
0390             A(KQ1,JJ4) = 4.*C26(M)
0391             A(KQ1,JJ11) = -C26(M)
0392             A(KQ1,JJ1+1) = -3.*C22(M)
0393             A(KQ1,JJ4+1) = 4.*C22(M)
0394             A(KQ1,JJ11+1) = -C22(M)
0395             A(KQ1,JJ1+2) = -3.*C23(M)
0396             A(KQ1,JJ6+2) = 4.*C23(M)
0397             A(KQ1,JJ12+2) = -C23(M)
           C
0398             A(KQ2,JJ1) = -3.*C45(M)
0399             A(KQ2,JJ6) = 4.*C45(M)
0400             A(KQ2,JJ12) = -C45(M)
0401             A(KQ2,JJ1+1) = -3.*C44(M)
0402             A(KQ2,JJ6+1) = 4.*C44(M)
0403             A(KQ2,JJ12+1) = -C44(M)
0404             A(KQ2,JJ1+2) = -3.*C44(M)
0405             A(KQ2,JJ4+2) = 4.*C44(M)
0406             A(KQ2,JJ11+2) = -C44(M)
           C
0407             CY1 = C12(M)*C3 + C22(M)*BV + C26(M)*BU
0408             CY2 = C12(M)*C2 + C22(M)*DV + 2.*C26(M)*C4
0409             CXY1 = C16(M)*C3 + C26(M)*BV + C66(M)*BU
0410             CXY2 = C16(M)*C2 + C26(M)*DV + 2.*C66(M)*C4
           C
0411             X(JJ1) = -2.*H*(CXY1 + CXY2*Z)
0412             X(JQ1) = -2.*H*(CY1 + CY2*Z)
```

```
0413          X(JQ2) = 0.
0414          GO TO 102
        C
        C FREE SURFACE MATRIX TERMS FOR J=1 AND I=LAW
        C
0415      211 A(KJ1,JJ1) = 3.*C66(M)
0416          A(KJ1,JJ2) = -4.*C66(M)
0417          A(KJ1,JJ13) = C66(M)
0418          A(KJ1,JJ1+1) = 3.*C26(M)
0419          A(KJ1,JJ2+1) = -4.*C26(M)
0420          A(KJ1,JJ13+1) = C26(M)
0421          A(KJ1,JJ1+2) = -3.*C36(M)
0422          A(KJ1,JJ6+2) = 4.*C36(M)
0423          A(KJ1,JJ12+2) = -C36(M)
        C
0424          A(KQ1,JJ1) = 3.*C26(M)
0425          A(KQ1,JJ2) = -4.*C26(M)
0426          A(KQ1,JJ13) = C26(M)
0427          A(KQ1,JJ1+1) = 3.*C22(M)
0428          A(KQ1,JJ2+1) = -4.*C22(M)
0429          A(KQ1,JJ13+1) = C22(M)
0430          A(KQ1,JJ1+2) = -3.*C23(M)
0431          A(KQ1,JJ6+2) = 4.*C23(M)
0432          A(KQ1,JJ12+2) = -C23(M)
        C
0433          A(KQ2,JJ1) = -3.*C45(M)
0434          A(KQ2,JJ6) = 4.*C45(M)
0435          A(KQ2,JJ12) = -C45(M)
0436          A(KQ2,JJ1+1) = -3.*C44(M)
0437          A(KQ2,JJ6+1) = 4.*C44(M)
0438          A(KQ2,JJ12+1) = -C44(M)
0439          A(KQ2,JJ1+2) = 3.*C44(M)
0440          A(KQ2,JJ2+2) = -4.*C44(M)
0441          A(KQ2,JJ13+2) = C44(M)
        C
0442          CY1 = C12(M)*C3 + C22(M)*BV + C26(M)*BU
0443          CY2 = C12(M)*C2 + C22(M)*DV + 2.*C26(M)*C4
0444          CXY1 = C16(M)*C3 + C26(M)*BV + C66(M)*BU
0445          CXY2 = C16(M)*C2 + C26(M)*DV + 2.*C66(M)*C4
        C
0446          X(JJ1) = -2.*H*(CXY1 + CXY2*Z)
0447          X(JQ1) = -2.*H*(CY1 + CY2*Z)
0448          X(JQ2) = 0.
0449          GO TO 102
        C
0450      201 P = M+1
0451          IF(I.EQ.1) GO TO 220
0452          IF(I.EQ.FSW1) GO TO 221
0453          IF(I.LT.FSW2.AND.I.GT.FSW1) GO TO 222
0454          IF(I.EQ.FSW2) GO TO 221
0455          IF(I.EQ.LAW) GO TO 223
        C
        C MATRIX TERMS AT INTERFACE FOR I BETWEEN 1 AND FSW1 OR FSW2 AND LAW
        C
0456          A(KJ1,JJ1) = 3.*(C55(M)+C55(P))
0457          A(KJ1,JJ6) = -4.*C55(P)
0458          A(KJ1,JJ8) = -4.*C55(M)
0459          A(KJ1,JJ10) = C55(M)
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0460              A(KJ1,JJ12) = C55(P)
            C
0461              A(KJ1,JJ1+1) = 3.*(C45(M)+C45(P))
0462              A(KJ1,JJ6+1) = -4.*C45(P)
0463              A(KJ1,JJ8+1) = -4.*C45(M)
0464              A(KJ1,JJ10+1) = C45(M)
0465              A(KJ1,JJ12+1) = C45(P)
            C
0466              A(KJ1,JJ2+2) = C45(P)-C45(M)
0467              A(KJ1,JJ4+2) = C45(M)-C45(P)
            C
0468              A(KQ1,JJ1) = 3.*(C45(M)+C45(P))
0469              A(KQ1,JJ6) = -4.*C45(P)
0470              A(KQ1,JJ8) = -4.*C45(M)
0471              A(KQ1,JJ10) = C45(M)
0472              A(KQ1,JJ12) = C45(P)
            C
0473              A(KQ1,JJ1+1) = 3.*(C44(M)+C44(P))
0474              A(KQ1,JJ6+1) = -4.*C44(P)
0475              A(KQ1,JJ8+1) = -4.*C44(M)
0476              A(KQ1,JJ10+1) = C44(M)
0477              A(KQ1,JJ12+1) = C44(P)
            C
0478              A(KQ1,JJ2+2) = C44(P)-C44(M)
0479              A(KQ1,JJ4+2) = C44(M)-C44(P)
            C
0480              A(KQ2,JJ2) = C36(P)-C36(M)
0481              A(KQ2,JJ4) = C36(M)-C36(P)
            C
0482              A(KQ2,JJ2+1) = C23(P)-C23(M)
0483              A(KQ2,JJ4+1) = C23(M)-C23(P)
            C
0484              A(KQ2,JJ1+2) = 3.*(C33(M)+C33(P))
0485              A(KQ2,JJ6+2) = -4.*C33(P)
0486              A(KQ2,JJ8+2) = -4.*C33(M)
0487              A(KQ2,JJ10+2) = C33(M)
0488              A(KQ2,JJ12+2) = C33(P)
            C
0489              CZ1 = (C13(P)-C13(M))*C3 + (C23(P)-C23(M))*BV + (C36(P)-C36(M))*BU
0490              CZ2 = (C13(P)-C13(M))*C2+(C23(P)-C23(M))*DV+2.*(C36(P)-C36(M))*C4
            C
0491              X(JJ1) = 0.
0492              X(JQ1) = 0.
0493              X(JQ2) = 2.*H*(CZ1 + CZ2*Z)
0494              GO TO 102
            C
            C FREE SURFACE MATRIX TERMS AT ANY INTERFACE WHERE I=1 AND J=INF OR AT
            C THE FREE SURFACE POINT I=1, J=LAT
            C
0495          220 A(KJ1,JJ1) = -3.*C66(M)
0496              A(KJ1,JJ4) = 4.*C66(M)
0497              A(KJ1,JJ11) = -C66(M)
            C
0498              A(KJ1,JJ1+1) = -3.*C26(M)
0499              A(KJ1,JJ4+1) = 4.*C26(M)
0500              A(KJ1,JJ11+1) = -C26(M)
            C
0501              A(KJ1,JJ1+2) = 3.*C36(M)
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0502                A(KJ1,JJ8+2) = -4.*C36(M)
0503                A(KJ1,JJ10+2) = C36(M)
          C
0504                A(KQ1,JJ1) = -3.*C26(M)
0505                A(KQ1,JJ4) =4.*C26(M)
0506                A(KQ1,JJ11) = -C26(M)
          C
0507                A(KQ1,JJ1+1) = -3.*C22(M)
0508                A(KQ1,JJ4+1) = 4.*C22(M)
0509                A(KQ1,JJ11+1) = -C22(M)
          C
0510                A(KQ1,JJ1+2) = 3.*C23(M)
0511                A(KQ1,JJ8+2) = -4.*C23(M)
0512                A(KQ1,JJ10+2) = C23(M)
          C
0513                A(KQ2,JJ1) = 3.*C45(M)
0514                A(KQ2,JJ8) = -4.*C45(M)
0515                A(KQ2,JJ10) = C45(M)
          C
0516                A(KQ2,JJ1+1) = 3.*C44(M)
0517                A(KQ2,JJ8+1) = -4.*C44(M)
0518                A(KQ2,JJ10+1) = C44(M)
          C
0519                A(KQ2,JJ1+2) = -3.*C44(M)
0520                A(KQ2,JJ4+2) = 4.*C44(M)
0521                A(KQ2,JJ11+2) = -C44(M)
          C
0522                CY1 = C12(M)*C3 + C22(M)*BV + C26(M)*BU
0523                CY2 = C12(M)*C2 + C22(M)*DV + 2.*C26(M)*C4
0524                CXY1 = C16(M)*C3 + C26(M)*BV + C66(M)*BU
0525                CXY2 = C16(M)*C2 + C26(M)*DV + 2.*C66(M)*C4
          C
0526                X(JJ1) = -2.*H*(CXY1 + CXY2*Z)
0527                X(JQ1) = -2.*H*(CY1 + CY2*Z)
0528                X(JQ2) = 0.
0529                GO TO 102
          C
          C MATRIX TERMS AT THE INTERFACE FOR J=INF AND I=FSW1 OR I=FSW2
          C
0530            221 XK = FLOAT(K)
0531                D1 = (XK-1.)/XK
0532                D2 = XK/(XK+1.)
0533                D3 = 1./((XK+1.)*XK)
          C
0534                A(KJ1,JJ1) = 3.*(C55(M)+C55(P))
0535                A(KJ1,JJ6) = -4.*C55(P)
0536                A(KJ1,JJ8) = -4.*C55(M)
0537                A(KJ1,JJ10) = C55(M)
0538                A(KJ1,JJ12) = C55(P)
          C
0539                A(KJ1,JJ1+1) = 3.*(C45(M)+C45(P))
0540                A(KJ1,JJ6+1) = -4.*C45(P)
0541                A(KJ1,JJ8+1) = -4.*C45(M)
0542                A(KJ1,JJ10+1) = C45(M)
0543                A(KJ1,JJ12+1) = C45(P)
          C
0544                A(KQ1,JJ1) = 3.*(C45(M) + C45(P))
0545                A(KQ1,JJ6) = -4.*C45(P)
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0546                 A(KQ1,JJ8) = -4.*C45(M)                                              00007550
0547                 A(KQ1,JJ10) = C45(M)                                                 00007560
0548                 A(KQ1,JJ12) = C45(P)                                                 00007570
           C                                                                              00007580
0549                 A(KQ1,JJ1+1) = 3.*(C44(M)+C44(P))                                    00007590
0550                 A(KQ1,JJ6+1) = -4.*C44(P)                                            00007600
0551                 A(KQ1,JJ8+1) = -4.*C44(M)                                            00007610
0552                 A(KQ1,JJ10+1) = C44(M)                                               00007620
0553                 A(KQ1,JJ12+1) = C44(P)                                               00007630
           C                                                                              00007640
0554                 A(KQ2,JJ1+2) = 3.*(C33(M)+C33(P))                                    00007650
0555                 A(KQ2,JJ6+2) = -4.*C33(P)                                            00007660
0556                 A(KQ2,JJ8+2) = -4.*C33(M)                                            00007670
0557                 A(KQ2,JJ10+2) = C33(M)                                               00007680
0558                 A(KQ2,JJ12+2) = C33(P)                                               00007690
           C                                                                              00007700
0559                 CZ1 = (C13(P)-C13(M))*C3 + (C23(P)-C23(M))*BV + (C36(P)-C36(M))*BU00007710
0560                 CZ2 = (C13(P)-C13(M))*C2+(C23(P)-C23(M))*DV+2.*(C36(P)-C36(M))*C4 00007720
           C                                                                              00007730
0561                 X(JJ1) = 0.                                                          00007740
0562                 X(JQ1) = 0.                                                          00007750
0563                 X(JQ2) = 2.*H*(CZ1 + CZ2*Z)                                          00007760
           C                                                                              00007770
0564                 C=C45(M)-C45(P)                                                      00007780
0565                 D=C44(M)-C44(P)                                                      00007790
0566                 E=C23(M)-C23(P)                                                      00007800
0567                 CC=C36(M)-C36(P)                                                     00007810
           C                                                                              00007820
0568                 IF(I.EQ.FSW2) GO TO 227                                              00007830
           C                                                                              00007840
0569                 A(KJ1,JJ1+2) = 2.*D1*C                                               00007850
0570                 A(KJ1,JJ2+2) = -2.*D2*C                                              00007860
0571                 A(KJ1,JJ4+2) = 2.*D3*C                                               00007870
           C                                                                              00007880
0572                 A(KQ1,JJ1+2) = 2.*D1*D                                               00007890
0573                 A(KQ1,JJ2+2) = -2.*D2*D                                              00007900
0574                 A(KQ1,JJ4+2) = 2.*D3*D                                               00007910
           C                                                                              00007920
0575                 A(KQ2,JJ1) = 2.*D1*CC                                                00007930
0576                 A(KQ2,JJ2) = -2.*D2*CC                                               00007940
0577                 A(KQ2,JJ4) = 2.*D3*CC                                                00007950
           C                                                                              00007960
0578                 A(KQ2,JJ1+1) = 2.*D1*E                                               00007970
0579                 A(KQ2,JJ2+1) = -2.*D2*E                                              00007980
0580                 A(KQ2,JJ4+1) = 2.*D3*E                                               00007990
0581                 GO TO 102                                                            00008000
           C                                                                              00008010
0582             227 A(KJ1,JJ1+2) = -2.*D1*C                                              00008020
0583                 A(KJ1,JJ2+2) = -2.*D3*C                                              00008030
0584                 A(KJ1,JJ4+2) = 2.*D2*C                                               00008040
           C                                                                              00008050
0585                 A(KQ1,JJ1+2) = -2.*D1*D                                              00008060
0586                 A(KQ1,JJ2+2) = -2.*D3*D                                              00008070
0587                 A(KQ1,JJ4+2) = 2.*D2*D                                               00008080
           C                                                                              00008090
0588                 A(KQ2,JJ1) = -2.*D1*CC                                               00008100
0589                 A(KQ2,JJ2) = -2.*D3*CC                                               00008110
0590                 A(KQ2,JJ4) = 2.*D2*CC                                                00008120
```

```
              C
 0591               A(KQ2,JJ1+1) = -2.*D1*E
 0592               A(KQ2,JJ2+1) = -2.*D3*E
 0593               A(KQ2,JJ4+1) = 2.*D2*E
 0594               GO TO 102
              C
              C MATRIX TERMS AT AN INTERFACE FOR J=INF AND I BETWEEN FSW1 AND FSW2
              C
 0595           222 XK = FLOAT(K)
 0596               A(KJ1,JJ1) = 3.*(C55(M)+C55(P))
 0597               A(KJ1,JJ6) = -4.*C55(P)
 0598               A(KJ1,JJ8) = -4.*C55(M)
 0599               A(KJ1,JJ10) = C55(M)
 0600               A(KJ1,JJ12) = C55(P)
              C
 0601               A(KJ1,JJ1+1) = 3.*(C45(M)+C45(P))
 0602               A(KJ1,JJ6+1) = -4.*C45(P)
 0603               A(KJ1,JJ8+1) = -4.*C45(M)
 0604               A(KJ1,JJ10+1) = C45(M)
 0605               A(KJ1,JJ12+1) = C45(P)
              C
 0606               A(KJ1,JJ2+2) = (C45(P)-C45(M))/XK
 0607               A(KJ1,JJ4+2) = (C45(M)-C45(P))/XK
              C
 0608               A(KQ1,JJ1) = 3.*(C45(M)+C45(P))
 0609               A(KQ1,JJ6) = -4.*C45(P)
 0610               A(KQ1,JJ8) = -4.*C45(M)
 0611               A(KQ1,JJ10) = C45(M)
 0612               A(KQ1,JJ12) = C45(P)
              C
 0613               A(KQ1,JJ1+1) = 3.*(C44(M)+C44(P))
 0614               A(KQ1,JJ6+1) = -4.*C44(P)
 0615               A(KQ1,JJ8+1) = -4.*C44(M)
 0616               A(KQ1,JJ10+1) = C44(M)
 0617               A(KQ1,JJ12+1) = C44(P)
              C
 0618               A(KQ1,JJ2+2) = (C44(P)-C44(M))/XK
 0619               A(KQ1,JJ4+2) = (C44(M)-C44(P))/XK
              C
 0620               A(KQ2,JJ2) = (C36(P)-C36(M))/XK
 0621               A(KQ2,JJ4) = (C36(M)-C36(P))/XK
              C
 0622               A(KQ2,JJ2+1) = (C23(P)-C23(M))/XK
 0623               A(KQ2,JJ4+1) = (C23(M)-C23(P))/XK
              C
 0624               A(KQ2,JJ1+2) = 3.*(C33(M)+C33(P))
 0625               A(KQ2,JJ6+2) = -4.*C33(P)
 0626               A(KQ2,JJ8+2) = -4.*C33(M)
 0627               A(KQ2,JJ10+2) = C33(M)
 0628               A(KQ2,JJ12+2) = C33(P)
 0629               X(JQ1) = 0.
              C
 0630               CZ1 = (C13(P)-C13(M))*C3 + (C23(P)-C23(M))*BV + (C36(P)-C36(M))*BU
 0631               CZ2 = (C13(P)-C13(M))*C2+(C23(P)-C23(M))*DV+2.*(C36(P)-C36(M))*C4
              C
 0632               X(JJ1) = 0.
 0633               X(JQ2) = 2.*H*(CZ1 + CZ2*Z)
 0634               GO TO 102
```

```
                C
                C FREE SURFACE MATRIX TERMS AT ANY INTERFACE WHERE I=LAW, J=INF OR AT
                C THE FREE SURFACE POINT I=LAW, J=LAT
                C
0635              223 A(KJ1,JJ1) = 3.*C66(M)
0636                  A(KJ1,JJ2) = -4.*C66(M)
0637                  A(KJ1,JJ13) = C66(M)
                C
0638                  A(KJ1,JJ1+1) = 3.*C26(M)
                C
0644                  A(KQ1,JJ1) = 3.*C26(M)
0645                  A(KQ1,JJ2) = -4.*C26(M)
0646                  A(KQ1,JJ13) = C26(M)
                C
0647                  A(KQ1,JJ1+1) = 3.*C22(M)
0648                  A(KQ1,JJ2+1) = -4.*C22(M)
0649                  A(KQ1,JJ13+1) = C22(M)
                C
0650                  A(KQ1,JJ1+2) = 3.*C23(M)
0651                  A(KQ1,JJ8+2) = -4.*C23(M)
0652                  A(KQ1,JJ10+2) = C23(M)
                C
0653                  A(KQ2,JJ1) = 3.*C45(M)
0654                  A(KQ2,JJ8) = -4.*C45(M)
0655                  A(KQ2,JJ10) = C45(M)
                C
0656                  A(KQ2,JJ1+1) = 3.*C44(M)
0657                  A(KQ2,JJ8+1) = -4.*C44(M)
0658                  A(KQ2,JJ10+1) = C44(M)
                C
0659                  A(KQ2,JJ1+2) = 3.*C44(M)
0660                  A(KQ2,JJ2+2) = -4.*C44(M)
0661                  A(KQ2,JJ13+2) = C44(M)
                C
0662                  CY1 = C12(M)*C3 + C22(M)*BV + C26(M)*BU
0663                  CY2 = C12(M)*C2 + C22(M)*DV + 2.*C26(M)*C4
0664                  CXY1 = C16(M)*C3 + C26(M)*BV + C66(M)*BU
0665                  CXY2 = C16(M)*C2 + C26(M)*DV + 2.*C66(M)*C4
                C
0666                  X(JJ1) = -2.*H*(CXY1 + CXY2*Z)
0667                  X(JQ1) = -2.*H*(CY1 + CY2*Z)
0668                  X(JQ2) = 0.
0669                  GO TO 102
                C
                C  MATRIX TERMS TO FIX THE RIGID TRANSLATIONS
                C
0670              203 A(KJ1,JJ1) = 1.0
0671                  A(KQ1,JJ1+1) = 1.0
0672                  A(KQ2,JJ1+2) = 1.0
                C
0673                  X(JJ1) = 0.
0674                  X(JQ1) = 0.
```

```
0675            X(JQ2) = 0.
0676            GO TO 102
          C
0677        202 IF(I.EQ.1) GO TO 220
0678            IF(I.EQ.LAW) GO TO 223
          C
          C FREE SURFACE MATRIX TERMS FOR I BETWEEN 1 AND LAW AND J=LAT
          C
0679            A(KJ1,JJ1) = 3.*C55(M)
0680            A(KJ1,JJ8) = -4.*C55(M)
0681            A(KJ1,JJ10) = C55(M)
          C
0682            A(KJ1,JJ1+1) = 3.*C45(M)
0683            A(KJ1,JJ8+1) = -4.*C45(M)
0684            A(KJ1,JJ10+1) = C45(M)
          C
0685            A(KQ1,JJ1) = 3.*C45(M)
0686            A(KQ1,JJ8) = -4.*C45(M)
0687            A(KQ1,JJ10) = C45(M)
          C
0688            A(KQ1,JJ1+1) = 3.*C44(M)
0689            A(KQ1,JJ8+1) = -4.*C44(M)
0690            A(KQ1,JJ10+1) = C44(M)
          C
0691            A(KQ2,JJ1+2) = 3.*C33(M)
0692            A(KQ2,JJ8+2) = -4.*C33(M)
0693            A(KQ2,JJ10+2) = C33(M)
          C
0694            CZ1 = C13(M)*C3 + C23(M)*BV + C36(M)*BU
0695            CZ2 = C13(M)*C2 + C23(M)*DV + 2.*C36(M)*C4
          C
0696            X(JJ1) = 0.
0697            X(JQ1) = 0.
0698            X(JQ2) = -2.*H*(CZ1 + CZ2*Z)
          C
0699            IF(I.EQ.FSW1) GO TO 231
0700            IF(I.EQ.FSW2) GO TO 231
0701            IF(I.GT.FSW1.AND.I.LT.FSW2) GO TO 234
          C
          C IF I IS BETWEEN 1 AND FSW1 OR BETWEEN FSW2 AND LAW, CONTINUE BELOW
          C
0702            A(KJ1,JJ2+2) = -C45(M)
0703            A(KJ1,JJ4+2) = C45(M)
          C
0704            A(KQ1,JJ2+2) = -C44(M)
0705            A(KQ1,JJ4+2) = C44(M)
          C
0706            A(KQ2,JJ2) = -C36(M)
0707            A(KQ2,JJ4) = C36(M)
          C
0708            A(KQ2,JJ2+1) = -C23(M)
0709            A(KQ2,JJ4+1) = C23(M)
0710            GO TO 102
          C
          C CASE WHERE I=FSW1 OR FSW2 AND J=LAT
          C
0711        231 XK = FLOAT(K)
0712            D1 = 2.*(XK-1.)/XK
```

```
0713              D2 = 2.*XK/(XK+1.)
0714              D3 = 2./((XK+1.)*XK)
            C
0715              IF(I.EQ.FSW2) GO TO 232
            C
0716              A(KJ1,JJ1+2) = D1*C45(M)
0717              A(KJ1,JJ2+2) = -D2*C45(M)
0718              A(KJ1,JJ4+2) = D3*C45(M)
            C
0719              A(KQ1,JJ1+2) = D1*C44(M)
0720              A(KQ1,JJ2+2) = -D2*C44(M)
0721              A(KQ1,JJ4+2) = D3*C44(M)
            C
0722              A(KQ2,JJ1) = D1*C36(M)
0723              A(KQ2,JJ2) = -D2*C36(M)
0724              A(KQ2,JJ4) = D3*C36(M)
            C
0725              A(KQ2,JJ1+1) = D1*C23(M)
0726              A(KQ2,JJ2+1) = -D2*C23(M)
0727              A(KQ2,JJ4+1) = D3*C23(M)
0728              GO TO 102
            C
0729          232 A(KJ1,JJ1+2) = -D1*C45(M)
0730              A(KJ1,JJ2+2) = -D3*C45(M)
0731              A(KJ1,JJ4+2) = D2*C45(M)
            C
0732              A(KQ1,JJ1+2) = -D1*C44(M)
0733              A(KQ1,JJ2+2) = -D3*C44(M)
0734              A(KQ1,JJ4+2) = D2*C44(M)
            C
0735              A(KQ2,JJ1) = -D1*C36(M)
0736              A(KQ2,JJ2) = -D3*C36(M)
0737              A(KQ2,JJ4) = D2*C36(M)
            C
0738              A(KQ2,JJ1+1) = -D1*C23(M)
0739              A(KQ2,JJ2+1) = -D3*C23(M)
0740              A(KQ2,JJ4+1) = D2*C23(M)
0741              GO TO 102
            C
            C CASE WHERE I IS BETWEEN FSW1 AND FSW2 AND J=LAT
            C
0742          234 XK = FLOAT(K)
0743              A(KJ1,JJ2+2) = -C45(M)/XK
0744              A(KJ1,JJ4+2) = C45(M)/XK
            C
0745              A(KQ1,JJ2+2) = -C44(M)/XK
0746              A(KQ1,JJ4+2) = C44(M)/XK
            C
0747              A(KQ2,JJ2) = -C36(M)/XK
0748              A(KQ2,JJ4) = C36(M)/XK
            C
0749              A(KQ2,JJ2+1) = -C23(M)/XK
0750              A(KQ2,JJ4+1) = C23(M)/XK
            C
0751          102 CONTINUE
            C
            C  FORM THE NONSYMETRIC BANDED MATRIX AX
            C
```

```
0752              IL = KJ1+3*(NODE-1)
0753              IN = IL+2
            C
0754              DO 103 IK=IL, IN
0755              II = IK-IL+1
            C
0756              DO 104 JK=1,NBAND
0757              JJ = IK+JK-IBW-1
0758              IF(IK.LE.IBW1) JJ = JK
0759              IF(JJ.GT.JQMAX) GO TO 105
0760              AX(JK,IK) = A(II,JJ)
0761              GO TO 104
0762          105 AX(JK,IK) = 0.0
0763          104 CONTINUE
0764          103 CONTINUE
0765          101 CONTINUE
0766          100 CONTINUE
            C
0767              REWIND 9
0768              WRITE(9) ((AX(J,I),J=1,NBAND),I=1,JQMAX)
0769              WRITE(9) (X(I),I=1,JQMAX)
0770              END FILE 9
0771              REWIND 9
            C
0772              NBD = NBAND+1
0773              DO 107 I=1, JQMAX
0774              AX(NBD,I) = X(I)
0775          107 CONTINUE
            C
            C     WRITE(6,4000)
            C4000 FORMAT(1H1,' EQUATION', 35X, 'THE BANDED MATRIX TERMS AX(I,J)' //)
            C     CALL RITE(1, JQMAX, NBD, JQMAX, NBD, AX)
            C     WRITE(6,4003)
            C4003 FORMAT(1H1, 45X, '*** THE LOAD VECTOR X(I) ***' /// )
            C     WRITE(6,4004) (X(I), I=1, JQMAX)
            C4004 FORMAT(28(2X, 10D12.3 / ))
            C
0776              CALL TRMSTR(AX, JQMAX, NBD  , IBW, IBW, NBAND, DT, RT, ET)
            C
0777              WRITE(6,4006) ET, RT, DT
0778         4006 FORMAT(/// ' ERROR CONDITION OF SOLVER ROUTINE IS ', F4.1, 5X,
                 1 'RANK IS ', F6.1, 5X, 'DETERMINANT = ', G10.3)
0779              IF(ET.EQ.1.) STOP 1
            C
0780              DO 108 I=1,JQMAX
0781              X(I) = AX(1,I)
0782          108 CONTINUE
            C
0783              READ(9) ((AX(J,I),J=1,NBAND),I=1,JQMAX)
0784              READ(9) (R(I),I=1,JQMAX)
            C
            C
            C    **********************************************************************
            C
            C    OUTPUT OF THE NODAL DISPLACEMENTS, U, V, W
            C
            C    **********************************************************************
            C
```

```
      WRITE(6,650)
      J = 1
      DO 12 IK = 1, LAW
      DO 11 JK = 1, LAT
      WRITE(6,651) J, X(3*J-2), X(3*J-1), X(3*J)
      J = J+1
   11 CONTINUE
      WRITE(6,653)
   12 CONTINUE
C
      WRITE(6,9950)
 9950 FORMAT(1H1, 5X, 'EQUATION', 5X, '*** THE ACCURACY TEST, TEST-R(I)
     1 ***', 10X, '*** THE AVERAGE ABSOLUTE ERROR ***' ///)
      ERR = 0.D0
      DO 9990 I=1,JQMAX
      TEST = 0.D0
      DO 9960 J=1,NBAND
      IC = I+J-IBW-1
      IF(I.LE.IBW1) IC = J
      IF(IC.GT.JQMAX) GO TO 9970
      TEST = TEST+AX(J,I)*X(IC)
 9960 CONTINUE
 9970 TEST = TEST-R(I)
      ERR = ERR+DABS(TEST)
      AVE = ERR/I
      WRITE(6,9980) I, TEST, AVE
 9980 FORMAT(5X, I4,10X, G15.8, 32X, G15.8)
 9990 CONTINUE
C
C *******************************************************************
C
C     CALCULATION OF THE STRAIN (S) AND STRESS (T)
C
C *******************************************************************
C
      SXM = SXMAX * 1.E06
      SXE = C3E * 1.E06
      WRITE(6,670) SXM, SXE
      WRITE(6,671)
      HR = 1./(2.*H)
      XK = FLOAT(K)
C
      DO 399 I=1, LAW
      DO 398 J=1, LAT
C
      I1=I-1
      I2=I-2
      NODE = LAT*I1+J
      JJ1 = 3*(LAT*I1+J)-2
      JJ2 = 3*(LAT*I2+J)-2
      JJ3 = 3*(LAT*I2+J)-5
      JJ4 = 3*(LAT*I+J)-2
      JJ5 = 3*(LAT*I+J)+1
      JJ6 = 3*(LAT*I1+J)+1
      JJ7 = 3*(LAT*I2+J)+1
      JJ8 = 3*(LAT*I1+J)-5
      JJ9 = 3*(LAT*I+J)-5
      JJ10 =3*(LAT*I1+J)-8
```

```
0832              JJ11 =3*(LAT*(I+1)+J)-2
0833              JJ12 =3*(LAT*I1+J)+4
0834              JJ13 =3*(LAT*(I-3)+J)-2
            C
0835              Z = (FLOAT(J)-(FLOAT(LAT)+1.)/2.)*H
0836              SX = C2*Z + C3
            C
0837              IF(I.EQ.1) GO TO 385
0838              IF(I.EQ.LAW) GO TO 386
0839              IF(I.GT.FSW1.AND.I.LT.FSW2) GO TO 382
0840              IF(I.EQ.FSW1.OR.I.EQ.FSW2) GO TO 383
            C
0841              H1 = H
0842              H2 = H1
0843              GO TO 384
            C
0844          382 H1 = XK*H
0845              H2 = H1
0846              GO TO 384
            C
0847          383 H1 = H
0848              H2 = XK*H
0849              IF(I.EQ.FSW1) GO TO 384
0850              H1 = XK*H
0851              H2 = H
            C
0852          384 H12 = H1/H2
0853              H21 = H2/H1
0854              HRD = (H2-H1)/(H1*H2)
0855              HRS = 1./(H1+H2)
            C
0856              SY = HRS*(H12*X(JJ4+1)-H21*X(JJ2+1))+HRD*X(JJ1+1) + DV*Z + BV
0857              SXY = HRS*(H12*X(JJ4)-H21*X(JJ2)) + HRD*X(JJ1) + 2.*C4*Z + BU
0858              SYZI = HRS*(H12*X(JJ4+2)-H21*X(JJ2+2))+HRD*X(JJ1+2)
0859              GO TO 387
            C
0860          385 SY = HR*(4.*X(JJ4+1)-3.*X(JJ1+1)-X(JJ11+1)) + DV*Z + BV
0861              SXY = HR*(4.*X(JJ4)-3.*X(JJ1)-X(JJ11)) + 2.*C4*Z + BU
0862              SYZI = HR*(4.*X(JJ4+2)-3.*X(JJ1+2)-X(JJ11+2))
0863              GO TO 387
            C
0864          386 SY = HR*(3.*X(JJ1+1)+X(JJ13+1)-4.*X(JJ2+1)) + DV*Z + BV
0865              SXY = HR*(3.*X(JJ1)+X(JJ13)-4.*X(JJ2)) + 2.*C4*Z + BU
0866              SYZI = HR*(3.*X(JJ1+2)+X(JJ13+2)-4.*X(JJ2+2))
            C
0867          387 DO 392 M=1, NLAY
0868              IF(M.EQ.1.AND.J.GT.INF(1)) GO TO 392
0869              IF(M.EQ.1) GO TO 388
0870              IF(J.LE.INF(M-1).OR.J.GT.INF(M)) GO TO 392
0871          388 IF(J.EQ.1) GO TO 389
0872              IF(J.EQ.INF(M).OR.J.EQ.LAT) GO TO 390
            C
0873              SZ = HR*(X(JJ6+2)-X(JJ8+2))
0874              SYZJ = HR*(X(JJ6+1)-X(JJ8+1))
0875              SXZ = HR*(X(JJ6)-X(JJ8))
0876              GO TO 391
            C
0877          389 SZ = HR*(4.*X(JJ6+2)-3.*X(JJ1+2)-X(JJ12+2))
```

```
0878              SYZJ = HR*(4.*X(JJ6+1)-3.*X(JJ1+1)-X(JJ12+1))
0879              SXZ = HR*(4.*X(JJ6)-3.*X(JJ1)-X(JJ12))
0880              GO TO 391
            C
0881          390 SZ = HR*(3.*X(JJ1+2)+X(JJ10+2)-4.*X(JJ8+2))
0882              SYZJ = HR*(3.*X(JJ1+1)+X(JJ10+1)-4.*X(JJ8+1))
0883              SXZ = HR*(3.*X(JJ1)+X(JJ10)-4.*X(JJ8))
            C
0884          391 SYZ = SYZI + SYZJ
            C
            C     CALCULATION OF THE STRESS (T)
            C
0885              TX = C11(M)*SX + C12(M)*SY + C13(M)*SZ + C16(M)*SXY
0886              TY = C12(M)*SX + C22(M)*SY + C23(M)*SZ + C26(M)*SXY
0887              TZ = C13(M)*SX + C23(M)*SY + C33(M)*SZ + C36(M)*SXY
            C
0888              TYZ = C44(M)*SYZ + C45(M)*SXZ
0889              TXZ = C45(M)*SYZ + C55(M)*SXZ
0890              TXY = C16(M)*SX + C26(M)*SY + C36(M)*SZ + C66(M)*SXY
            C
0891              WRITE(6,672) NODE, TX, TY, TZ, TYZ, TXZ, TXY, SY, SZ, SYZ, SXZ,SXY
0892              WRITE(6,397) SX
            C
            C  STRESS AND STRAINS JUST ABOVE AN INTERFACE
            C
0893              IF(J.NE.INF(M).OR.J.EQ.LAT) GO TO 392
0894              P = M+1
0895              SZ = HR*(4.*X(JJ6+2)-3.*X(JJ1+2)-X(JJ12+2))
0896              SYZJ = HR*(4.*X(JJ6+1)-3.*X(JJ1+1)-X(JJ12+1))
0897              SXZ = HR*(4.*X(JJ6)-3.*X(JJ1)-X(JJ12))
0898              SYZ = SYZI + SYZJ
            C
0899              TX = C11(P)*SX + C12(P)*SY + C13(P)*SZ + C16(P)*SXY
0900              TY = C12(P)*SX + C22(P)*SY + C23(P)*SZ + C26(P)*SXY
0901              TZ = C13(P)*SX + C23(P)*SY + C33(P)*SZ + C36(P)*SXY
            C
0902              TYZ = C44(P)*SYZ + C45(P)*SXZ
0903              TXZ = C45(P)*SYZ + C55(P)*SXZ
0904              TXY = C16(P)*SX + C26(P)*SY + C36(P)*SZ + C66(P)*SXY
            C
0905              WRITE(6,672) NODE, TX, TY, TZ, TYZ, TXZ, TXY, SY, SZ, SYZ, SXZ,SXY
            C
0906          392 CONTINUE
0907          398 CONTINUE
0908              WRITE(6,652)
0909          399 CONTINUE
0910         9000 CONTINUE
            C
            C ********************************************************************
            C
            C     FORMATS
            C
            C ********************************************************************
            C
0911          397 FORMAT(14X,1P1E11.3/)
0912          600 FORMAT(1H1, 44X, 44H*** UNIFORM BENDING OF A LAMINATED PLATE ***)
0913          601 FORMAT(5I10)
            C
```

```
0914         602 FORMAT(//// 5X, 18H*** INPUT DATA *** ///
                1        18X, 'NUMBER OF LAYERS IN CROSS SECTION, NLAY =', I4 //
                2        18X, 'NUMBER OF NODES ON VERTICAL AXIS, LAT    =', I4 //
                3        18X, 'NUMBER OF NODES ON HORIZONTAL AXIS, LAW =', I4 ///
                4            18X, 37HCHANGE IN MESH WIDTH (FSW1) AT I    = , I4 //
                5            18X, 37HCHANGE IN MESH WIDTH (FSW2) AT I    = , I4 //
                6            18X, 37HMESH WIDTH MAGNIFICATION FACTOR, K = , I4 /)
           C
0915         603 FORMAT(8G12.5)
           C
0916         604 FORMAT(1H1, 55X, 21H*** MATERIAL DATA *** //// 2X, 5HLAYER, 7X,
                1        3HE11, 9X, 3HE22, 9X, 3HE33, 9X, 3HE12, 9X, 3HE13, 9X,
                2        3HE23, 8X, 4HNU12, 4X, 4HNU13, 4X, 4HNU23 // )
           C
0917         605 FORMAT(3X, I2, 6X, 2PE10.3, 2(2X, 1PE10.3), 3(2X, 0PE10.3),
                1        3(3X, F5.2) / )
           C
0918         606 FORMAT(10G10.3)
           C
0919         607 FORMAT(/// 18X, 26HFINE SIMULATION WIDTH, H = ,F8.5)
           C
0920         608 FORMAT(// 18X, 9HLAYER NO., 2X, I3, 5X, 17HINTERFACE AT J = ,I3)
           C
0921         611 FORMAT(// 45X, 41H*** COEFFICIENTS OF THERMAL EXPANSION ***, ///
                1        1X, 5HLAYER, 8X, 5HTHETA, 12X, 3HAL1, 12X, 3HAL2, 12X,
                2        3HAL3, 12X, 3HAL6, 12X, 4HAL1P, 11X, 4HAL2P, 11X, 4HAL3P
                3        /// )
           C
0922         613 FORMAT(// 53X, 26H*** STIFFNESS MATRICES *** /// 1X,
                1        11HLAYER/THETA, 21X, 12HX-Y-Z MATRIX, 44X,
                2        18HX-Y-Z PRIME MATRIX /// )
           C
0923         614 FORMAT(2X, I2, 9X, F5.1, 5X, 7(5X, E10.3))
           C
0924         620 FORMAT(2X, I2, 5X, 1P12E10.3 // 19X, 5E10.3, 10X, 5E10.3 // 29X,
                1        4E10.3, 20X, 4E10.3 // 1X,0PF5.1, 33X, 1P3E10.3, 30X,
                2        3E10.3 // 49X, 2E10.3, 40X, 2E10.3 // 59X, E10.3, 50X,
                3        E10.3 /// )
           C
0925         650 FORMAT(1H1 // 10X, '*** GRID POINT DISPLACEMENT FUNCTIONS ***' ///
                1        16X, 5H NODE, 5X, 14HU-DISPLACEMENT, 6X, 14HV-DISPLACEMENT,
                2        6X, 14HW-DISPLACEMENT /// )
           C
0926         651 FORMAT(10X, I10, 3E20.6 // )
0927         652 FORMAT(// 12H ********** // )
0928         653 FORMAT(// 10X, 12H ********** // )
           C
0929         670 FORMAT(1H1, 10X, 77H*** OUTPUT STRESSES AND STRAINS FOR A MAXIMUM
                1LONGITUDINAL BENDING STRAIN OF , F6.0, 22H MICRO-INCHES/INCH AND /
                2 48X, 40H AN APPLIED AXIAL EXTENSIONAL STRAIN OF , F6.0,
                3 19H MICRO-INCHES/INCH. // 10X, 'NOTE:  INTERFACE NODES ARE REPEAT
                4ED WITH VALUES GIVEN BELOW AND ABOVE THE INTERFACE RESPECTIVELY.'
                5 //// )
           C
0930         671 FORMAT(1X,5HNODE , 5X, 5HSIG-X, 6X, 5HSIG-Y, 6X, 5HSIG-Z, 6X,
                1        6HTAU-YZ, 5X, 6HTAU-XZ, 5X, 6HTAU-XY, 5X, 5HEPS-Y, 6X,
                2        5HEPS-Z, 6X, 6HEPS-YZ, 5X, 6HEPS-XZ, 5X, 6HEPS-XY / 17X,
                3        5HEPS-X /// )
           C
0931         672 FORMAT(1X, I3, 4X, 1P11E11.3 /)
           C
0932             STOP
0933             END
```

```
0001             SUBROUTINE MATCON
            C
            C ********************************************************************
            C
            C  CALCULATION OF LAMINATE LOAD CONSTANTS FOR A FULL CROSS SECTION
            C
            C ********************************************************************
            C
            C  THIS SUBROUTINE IS GOOD FOR BENDING OF AN ARBITRARILY LAID UP
            C  LAMINATE WHICH IS SYMMETRIC OR NONSYMMETRIC ABOUT THE MIDPLANE.
            C
            C  THE CONSTANTS ARE  C2 = INVERSE BENDING RADIUS
            C                     C3E = APPLIED UNIFORM EXTENSIONAL STRAIN
            C                     C3 = EXTENSIONAL COUPLING DUE TO BENDING PLUS C3E
            C                     C4 = IN-PLANE SHEAR COUPLING
            C
            C                     BU OCCURS IN THE FCTN. U(Y,Z)
            C                     BV AND DV OCCUR IN THE FCTN. V(Y,Z)
            C
            C  SXMAX (EFFECTIVELY THE LOAD INPUT) IS A MAXIMUM STRAIN
            C
0002             INTEGER ORDER
            C
0003             COMMON /MC/ C11(6),C12(6),C16(6),C22(6),C26(6),C66(6),C13(6),
               1        C23(6),C36(6),C44(6),C45(6),C55(6),C33(6),AL1(6),AL2(6),
               2       AL3(6),AL6(6),C2,C3,C3E,C4,BU,DU,BV,DV,H,SXMAX,NLAY,INF(6)
            C
0004             DIMENSION A(3,3), B(3,3), D(3,3), QM(3,3)
            C
0005             DOUBLE PRECISION A, B, D
            C
0006             ORDER = 3
            C
0007             LAY = INF(1)-1
0008             HL = H*FLOAT(LAY)
0009             HL2 = HL**2/2.
0010             HL3 = HL**3/3.
0011             RN = FLOAT(NLAY)
0012             RN2 = RN**2
            C
0013             DO 20 I=1,3
0014             DO 20 J=1,3
0015             A(I,J) = 0.D0
0016             B(I,J) = 0.D0
0017             D(I,J) = 0.D0
0018          20 CONTINUE
            C
0019             DO 30 I=1,3
0020             DO 30 J=1,3
0021             DO 30 M=1,NLAY
0022             QM(1,1) = C11(M)-C13(M)*C13(M)/C33(M)
0023             QM(1,2) = C12(M)-C13(M)*C23(M)/C33(M)
0024             QM(1,3) = C16(M)-C13(M)*C36(M)/C33(M)
0025             QM(2,1) = QM(1,2)
0026             QM(2,2) = C22(M)-C23(M)*C23(M)/C33(M)
0027             QM(2,3) = C26(M)-C23(M)*C36(M)/C33(M)
0028             QM(3,1) = QM(1,3)
0029             QM(3,2) = QM(2,3)
```

```
0030                QM(3,3) = C66(M)-C36(M)*C36(M)/C33(M)
          C
          C  NOTE THAT THE SUBSCRIPT 3 IN QM REPLACES A 6 IN STANDARD NOTATION.
          C  THE SAME IS TRUE BELOW IN A(I,J), B(I,J), D(I,J), ETC.
          C
0031                M1 = 2*M-1
0032                M2 = 3*M*(M-1)+1
          C
0033                A(I,J) = A(I,J) + HL*QM(I,J)
0034                B(I,J) = B(I,J) + HL2*QM(I,J)*(M1-RN)
0035                D(I,J) = D(I,J) + HL3*QM(I,J)*(M2-1.5*RN*M1+.75*RN2)
0036             30 CONTINUE
          C
          C  INVERT (A). STORE IN (A).
          C
0037                CALL MATIN4 (A,ORDER)
          C
          C  MULTIPLY (A)INVERSE * (B). STORE IN A.
          C
0038                CALL MAMULT (A,B,ORDER,A)
          C
          C  MULTIPLY (B) * (A)INVERSE * (B). STORE IN B.
          C
0039                CALL MAMULT (B,A,ORDER,B)
          C
0040                DO 40 I=1,3
0041                DO 40 J=1,3
0042                A(I,J) = -1.*A(I,J)
0043                D(I,J) = D(I,J) - B(I,J)
0044             40 CONTINUE
          C
          C  INVERT NEW MATRIX (D). THE RESULT IS D-PRIME. STORE IN D.
          C
0045                CALL MATIN4 (D,ORDER)
          C
          C  MULTIPLY -(A)INVERSE * B * D-PRIME WHICH YIELDS B-PRIME. STORE IN B.
          C
0046                CALL MAMULT (A,D,ORDER,B)
          C
          C  DETERMINE THE LOAD CONSTANTS. MINUS C2 IMPLIES A SMILING PLATE.
          C
0047                ZMAX = RN*HL/2.
0048                C2 = -D(1,1)*SXMAX/(B(1,1) +D(1,1)*ZMAX)
0049                RATIO = C2/D(1,1)
          C
0050                C3 = B(1,1)*RATIO + C3E
0051                C4 = .5*D(1,3)*RATIO
0052                BU = B(3,1)*RATIO
0053                BV = B(2,1)*RATIO
0054                DV = D(1,2)*RATIO
          C         RATIO = -RATIO
0055                WRITE(6,50)
0056             50 FORMAT(///// 48X, 35H*** THE LAMINATE LOAD CONSTANTS *** /// )
0057                WRITE(6,60) C2, C3, C4, BU, BV, DV, RATIO
0058             60 FORMAT(' C2 = ', 1PE10.3, 4X, 'C3 = ', E10.3, 4X, 'C4 = ',E10.3,
                1         4X, 'BU = ', E10.3, 4X, 'BV = ', E10.3, 4X, 'DV = ', E10.3,
                2         4X, 'MT = ', E10.3 )
0059                RETURN
0060                END
```

```
FORTRAN IV G1  RELEASE 2.0           MATCON           DATE = 75082         19/49/20

 *OPTIONS IN EFFECT*  NOTERM,NOID,EBCDIC,SOURCE,NOLIST,NODECK,LOAD,NOMAP,NOTEST
 *OPTIONS IN EFFECT*  NAME = MATCON   , LINECNT =        60
 *STATISTICS*    SOURCE STATEMENTS =        60,PROGRAM SIZE =      2060
 *STATISTICS*  NO DIAGNOSTICS GENERATED
```

```
FORTRAN IV G1  RELEASE 2.0           MAMULT           DATE = 75082         19/49/20

0001                SUBROUTINE MAMULT(B,C,N,A)                                   00014550
              C                                                                  00014551
              C  MAMULT POSTMULTIPLIES MATRIX (B) BY MATRIX (C) AND STORES THE    00014552
              C  RESULT IN MATRIX (A) WHERE N IS THE ORDER OF THE MATRICES.       00014553
              C                                                                  00014554
0002                DOUBLE PRECISION A,B,C,SUM                                   00014560
0003                DIMENSION A(N,N), B(N,N), C(N,N)                             00014570
0004                DO 1 I=1,N                                                   00014580
0005                DO 1 J=1,N                                                   00014590
0006                SUM = 0.                                                     00014600
0007                DO 2 K=1,N                                                   00014610
0008                SUM = SUM + B(I,K)*C(K,J)                                    00014620
0009              2 CONTINUE                                                     00014630
0010                A(I,J) = SUM                                                 00014640
0011              1 CONTINUE                                                     00014650
0012                RETURN                                                       00014660
0013                END                                                          00014670
```

```
FORTRAN IV G1  RELEASE 2.0           MAMULT           DATE = 75082         19/49/20

 *OPTIONS IN EFFECT*  NOTERM,NOID,EBCDIC,SOURCE,NOLIST,NODECK,LOAD,NOMAP,NOTEST
 *OPTIONS IN EFFECT*  NAME = MAMULT   , LINECNT =        60
 *STATISTICS*    SOURCE STATEMENTS =        13,PROGRAM SIZE =       702
 *STATISTICS*  NO DIAGNOSTICS GENERATED
```

```
FORTRAN IV G1  RELEASE 2.0           MATIN4           DATE = 75082         19/49/20

0001                SUBROUTINE MATIN4(ARRAY,N)                                   00014680
              C                                                                  00014681
              C  MATIN4 INVERTS THE MATRIX (ARRAY) WHICH IS OF ORDER N.           00014682
              C                                                                  00014683
0002                DIMENSION ARRAY(N,N)                                         00014690
0003                DOUBLE PRECISION ARRAY                                       00014700
0004                DO 604 I=1,N                                                 00014710
0005                STORE = ARRAY(I,I)                                           00014720
0006                ARRAY(I,I) = 1.                                              00014730
0007                DO 601 J=1,N                                                 00014740
0008            601 ARRAY(I,J) = ARRAY(I,J)/STORE                                00014750
0009                DO 604 K=1,N                                                 00014760
0010                IF(K-1)602,604,602                                           00014770
0011            602 STORE = ARRAY(K,I)                                           00014780
0012                ARRAY(K,I) = 0.                                              00014790
0013                DO 603 J=1,N                                                 00014800
0014            603 ARRAY(K,J) = ARRAY(K,J) - STORE*ARRAY(I,J)                   00014810
0015            604 CONTINUE                                                     00014820
0016                RETURN                                                       00014830
0017                END                                                          00014840
```

```
0001              SUBROUTINE TRMSTR(A,N,ND,NLD,NRD,NED,D,R,E)
            C
            C  TRMSTR IS THE SUBROUTINE TRIMSS WITH MATRIX A TRANSPOSED.
            C         THE SIMULTANEOUS SOLUTIONS IS GAUSSIAN ELIMINATION,
            C         MODIFIED TO TAKE ADVANTAGE OF THE REDUCED MATRIX. THE
            C         ROUTINE ALSO USES PARTIAL PIVOTING TO REDUCE ROUNDOFF ERROR.
            C     INPUT
            C         1  A          FIRST LOCATION OF COEFFICIENT MATRIX,I.E. A(1,1).
            C                       THE BAND ELEMENTS IN EACH ROW MUST BE LEFT
            C                       JUSTIFIED AND EXTEND TO THE RIGHT M PLACES
            C                       (M=MIN(N,NLD+NRD+1). IF IN ANY PARTICULAR ROW
            C                       THERE ARE ONLY K BAND ELEMENTS AND K IS LESS
            C                       THAN M, THEN THE M-K RIGHT MOST ELEMENTS OF THAT
            C                       ROW WILL BE SET TO ZERO. THE ROW WHOSE LEFT
            C                       MOST COLUMN IN THE FULL BLOWN MATRIX CONTAINS
            C                       A NON-ZERO ELEMENT MUST BE THE FIRST ROW OF THE
            C                       REDUCED MATRIX AND ETC. THE COLUMN TO THE
            C                       IMMEDIATE RIGHT OF THE REDUCED MATRIX (FORMED AS
            C                       ABOVE) MUST CONTAIN THE RIGHT HAND SIDE OF THE
            C                       EQUATION SET IN QUESTION. IT SHOULD NOW BE
            C                       OBVIOUS THAT AN N X N+1 FULL BLOWN SYSTEM WOULD
            C                       BE REDUCED BY THE ABOVE METHOD TO AN N X M+1
            C                       SYSTEM.
            C         2  N          NUMBER OF SIMULTANEOUS EQUATIONS TO BE SOLVED.
            C         3  ND         VARIABLE DIMENSION INTEGER. MUST BE EQUAL TO
            C                       ROW DIMENSION OF A IN CALLING PROGRAM.
            C         4  NLD        MAXIMUM NUMBER OF BAND ELEMENTS TO THE LEFT
            C                       OF PRINCIPAL DIAGONAL IN ANY ROW OF SYSTEM TO
            C                       BE DETERMINED.
            C         5  NRD        MAXIMUM NUMBER OF BAND ELEMENTS TO THE RIGHT
            C                       OF PRINCIPAL DIAGONAL IN ANY ROW OF SYSTEM TO
            C                       BE DETERMINED.
            C         6  NED        NED=MIN(N,NLD+NRD+1)
            C     OUTPUT
            C         1  A          THE FIRST COLUMN OF A CONTAINS THE SOLUTION
            C                       VECTOR.
            C         2  D          CONTAINS DETERMINANT OF A.
            C         3  R          CONTAINS RANK OF A.
            C         4  E          E=0., SOLUTION O.K. E=1., A SINGULAR.
            C                       E=2., SOLUTION ATTEMPTED, BUT A ILL CONDITIONED
            C                       OR SINGULAR. IN THIS CASE SOLUTIONS SHOULD BE
            C                       CHECKED TO ASSURE VALIDITY.
            C
            C     SUBROUTINE TRMSTR(A,N,ND,NLD,NRD,NED,D,R,E)
0002              DIMENSION A(ND,1)
0003              DOUBLE PRECISION A,D,Y,W,S
0004              X1 = 1.
0005              L1 = 1
0006              E=0.
0007              R = 0.
0008              D=1.
0009              ND1=NED+1
0010              M=NLD
0011              NM1=N-1
0012              DO 1 I=1,NM1
0013              IF(I.GT.(N-NLD))M=M-1
0014              NN=I+M-1
0015              DO 2 II=I,NN
```

```
          IF(DABS(A(1,I)).GE.DABS(A(1,II+1))) GO TO 2
          D=-D
          DO 3 J=1,ND1
          Y=A(J,I)
          A(J,I)=A(J,II+1)
        3 A(J,II+1)=Y
        2 CONTINUE
C         D=D*A(1,I)
          IF(A(1,I) .EQ. 0.) GO TO 10
          GO TO (5,13),L1
       13 IF(DABS(DABS((X1-A(1,I))/X1)-1.).LT.1.E-07) E=2.
          X1 = A(1,I)
        5 R = R + 1.
          L1 = 2
          DO 4 J=2,ND1
        4 A(J,I)=A(J,I )/ A(1,I)
          K=I+1
          NN=I+M
          DO 1 II=K,NN
          W=A(1,II)
          DO 6 J=1,NED
        6 A(J,II)=A(J+1,II)-A(J+1,I)*W
          A(ND1,II)=A(NED,II)
        1 A(NED,II)=0.
          IF(A(1,N).EQ.0.)GO TO 10
          IF(DABS(DABS((X1-A(1,N))/X1)-1.).LT.1.E-07) E=2.
        9 R = R + 1.
          A(1,N)=A(ND1,N)/A(1,N)
          K=NM1
          NN=2
        8 IF(NN.GT.NED)NN=NED
          J=K+1
          S=0.
          DO 7 I=2,NN
          S=S+A(1,J)*A(I,K)
        7 J=J+1
          A(1,K)=A(ND1,K)-S
          NN=NN+1
          K=K-1
          IF(K.NE.0)GO TO 8
          RETURN
       10 E=1.
          RETURN
          END
```



```
*OPTIONS IN EFFECT*  NOTERM,NOID,EBCDIC,SOURCE,NOLIST,NODECK,LOAD,NOMAP,NOTEST
*OPTIONS IN EFFECT*  NAME = TRMSTR  , LINECNT =       60
*STATISTICS*    SOURCE STATEMENTS =        58,PROGRAM SIZE =      2294
*STATISTICS*  NO DIAGNOSTICS GENERATED
```

```
      SUBROUTINE RITE(IDUM,NR,NC,MR,MC,A)
      DOUBLE PRECISION A
      DIMENSION A(MR,MC)
      IPRINT= 12
      IF(IDUM.NE.1) IPRINT= 30
      IPR= IPRINT-1
      DO 35 K=1,NC,IPRINT
      MAX= K+IPR
      IF(MAX.GT.NC) MAX=NC
      IF(K.NE.1) WRITE(6,103)
   45 WRITE(6,102) (I,I=K,MAX)
      DO 40 J=1,NR
   40 WRITE(6,105) J,(A(J,I),I=K,MAX)
   35 CONTINUE
      RETURN
  101 FORMAT(6X,30I4)
  102 FORMAT(6X,12I10)
  103 FORMAT('1')
  104 FORMAT(' ',I5,30I4)
  105 FORMAT(' ',I5,12G10.3)
      END
```

FORTRAN IV G1 RELEASE 2.0   RITE   DATE = 75007   08/16/07

```
*OPTIONS IN EFFECT*  NOTERM,NOID,EBCDIC,SOURCE,NOLIST,NODECK,LOAD,NOMAP,NOTEST
*OPTIONS IN EFFECT*  NAME = RITE    , LINECNT =       60
*STATISTICS*    SOURCE STATEMENTS =        21,PROGRAM SIZE =       864
*STATISTICS*  NO DIAGNOSTICS GENERATED

*STATISTICS*  NO DIAGNOSTICS THIS STEP
```

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