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REPORT

to

THE NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

(NASA)

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I. Introduction:

To gain information about the distribution, from statistics of a sample from that distribution, remains one of the important goals of investigation in Mathematical Statistics.

The investigation initiated here is concerned with drawing conclusions about the characteristics of the unknown part of a certain collection, from knowledge of some characteristics of a suitable selected part of the collection of elements.

We note here in general discussion classical results obtained in the area in which we are concerned without giving precise and due credit to those great scholars whose products we use. This is done mainly because the contributions to a basic idea are numerous and it is difficult to either list all of them for a specific idea or to list the principal person responsible.

We are interested in different characteristics of the elements of the population under investigation.

When we say that the population has the distribution $F(x)$, we mean that we are investigating a characteristic X of elements of this population and that this characteristic X is a random variable with the distribution function $F(x)$.

Let the characteristic X of elements of a population be a random variable with the distribution function $F(x)$. This suggests a need for the definitions that are essential in what follows. We use the classical concept of Fisz.

In general, let e denote an elementary event of a set E of elementary events. On set E we define a single-valued real function $X(e)$

such that the probability that this function will assume certain values
is defined.

For a precise formulation of the conditions which are to be
satisfied by X(e) we introduce the notion of inverse image.

Let X(e) be a single-valued real function defined on the set E
of elementary events. The set A of all elementary events to which the
function X(e) assigns values in a given set S of real numbers is called
the inverse image of the set S.

It is clear that the inverse image of the set R of all real
numbers is the whole set E.

A single-valued real function X(e) defined on the set E of ele-
mentary events is called a random variable if the inverse image of every
interval I on the real axis of the form $(-\infty, x)$ is a random event.

We shall set the probability $P^{(x)}(I)$ that the random variable
X(e) takes on a value in the interval I equal to the probability $P(A)$ of
the inverse image A of I.

(The notion of a random variable corresponds in the theory of
real functions to the notion of a function measurable with respect to the
field of sets being considered.)

It follows that if a random event A is the inverse image of a point
x, the probability that the random variable X takes on the value x equals
the probability of the event A, $P^{(x)}(X = x) = P(A)$.

The function $P^{(x)}(S)$ giving the probability that a random variable X
takes on a value belonging to S, where S is an arbitrary Borel set on the
real axis, is called the probability function of X. $P^{(x)}(S) = P^{(x)}(X \in S)$.

The function $F(x) = P(X < x)$ is entitled the distribution func-
tion of the random variable X. It now follows that the probability distri-
bution of a random variable X is determined by its distribution function.

(We note that a random variable is a function and not a variable
in the sense which is usually understood in mathematical analysis.)

II. Characteristic function: (Ch. fu.)

For X a random variable and F(x) its distribution function, the function

$$f(t) = E(e^{itX}),$$

where t is a real number and i is the imaginary unit, is called the characteristic function of the random variable X.

$$f(t) = \int_{-\infty}^{\infty} e^{itX} \, dF(x).$$

We list without discussion some classical theorems about ch. fu.

(a) If the jth moment of a random variable exists, it is expressed by

$$m_j = \frac{f^{(J)}(0)}{i^f}$$

Where $f^J(0)$ is the jth derivative of the ch. fu. of this random variable at $t = 0$.

(b) If $f(t, u)$, $f_1(t)$ and $f_2(u)$ denote the ch. fu. of the random variables (X,Y), X and Y, respectively, then the random variables X and Y are independent iff the equation

$$f(t,u) = f_1(t) \, f_2(u)$$

holds for all real t and μ.

(c) Convergence of ch. fu. implies convergence of the corresponding distribution functions, i.e., distribution functions "depend continuously" on their ch. fu.

(d) Two distribution functions $F_1(x)$ and $F_2(x)$ are identical if, and only if, their ch. fu. $f_1(t)$ and $f_2(t)$ are identical.

(e) If a ch. fu. $f(t)$ is absolutely integrable over $(-\infty, \infty)$ then the corresponding distribution function $F(x)$ is absolutely continuous and

$$F^1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \, f(t) \, dt$$

(f) Let f(t) be an arbitrary ch. fu. For every real x the limit

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx} f(t) dt \quad \text{exists}$$

and is equal to the saltus of the distribution function of f(t) @ the point x.

(g) Let $\left\{ f_n(t) \right\}$ be a sequence of characteristic functions and suppose that this sequence converges for all values of t to a limit function f(t). Assume that f(t) is continuous @ t = 0; then f(t) is also a ch.fu.-

(h) A complex-valued function of a real variable t is a ch. fu. if, and only if, (i) f(t) is non-negative definite and (ii) f(0) = 1.

(i) Let f(t) be a real-valued and continuous function which is defined for all real t and which satisfies the following conditions:

$\quad$ **(i)** f(o) = 1

$\quad$ **(ii)** f(-t) = f(t)

$\quad$ **(iii)** f(t) convex for t > 0, i.e. $\dfrac{f(t_1 + t_2)}{2} \leq \dfrac{f(t_1) + f(t_2)}{2}$

$$\text{for all } t_1 > 0, t_2 > 0.$$

$\quad$ **(iv)** $\lim_{t \to \infty} f(t) = 0.$

Then f(t) is the ch. fu. of an absolutely continuous distribution F(x).

III. The Quadratic Regression Problem

The concern is with two random variables X and Y with the conditional expectation of Y given X is given by

(1) $\quad E(Y/X) = B_0 + B_1 x + \cdots + B_k x^k$ holding almost everywhere.

We say that Y has polynomial regression of order K on S.

We make assumptions that the first moment of Y and the moment of order K of X exist.

It follows from the definition of expected values and (1) that

$$E(Y) = B_0 + B_1 E(X) + \cdots + B_k E(x^k)$$

i.e. $E(E\,(Y/X)) = E(Y)$.

For the proper values of B'_o and K, we use the terms quadratic regression and linear regression.

We use the term constant regression of Y on X if $E(Y/X) = E(Y)$. a.e.

Now, for an n-sample of independently and identically distributed random variable from population with distribution function $F(x)$ -- we symbolize the statistics

$$A = \sum_{i=1}^{n} X_v = n\,\overline{X}$$

$$S = S(X_1 \cdots x_n) \text{ some other statistic.}$$

We are concerned with the task of finding a statistic which has constant regression on A. And conversely, this property will sometime determine the population.

Consider

(2)
$$Q = \sum_{v=1}^{k} \sum_{j=1}^{k} A_{ij}\, X_i\, X_j + \sum_{j=1}^{k} b_j\, X_j$$

and some population which has the property that Q has quadratic regression on A.

We must necessarily distinguish cases defined in terms of relations between coefficients A_{ij} and b_j of Q and the regression coefficient B_i.

The Demonstration is that the population is characterized by the property indicated.

We need

Lemma: A   If

i) X, Y are random variables

ii) $E(Y)$, $E(X^k)$ exist for $K \geqq 0$ (K is an integer),

then Y has polynomial regression of order K on X if, and only if,

(3)
$$E(Ye^{itx}) = \sum_{s=1}^{k} B\, E(X^s e^{itx})$$

hold for all real t.

Proof: of

Necessity:

Assume Y has polynomial regression of order K on X. i.e.

$$E(Y/X) = B_0 + B_1 x + \cdots B_k Y^k. \qquad \text{a.e.}$$

multiply (1) by e^{itx}

$$e^{itx} E(Y/X) = e^{itx} \left[B_0 + B_1 X + \cdots B_k X^k \right]$$

Take Expectation of both sides:

$$E\left\{ e^{itx} E(Y/X) \right\} = E\left\{ e^{itx} \left[\sum_{s=1}^{K} B_s X_s^s \right] \right\}$$

$$E(e^{itx}) E\left[E(Y/X) \right] = E\left\{ e^{itx} \sum_{s=0}^{K} B_s X_s^s \right\}$$

$$E(e^{itx}) E(Y) = \sum_{s=0}^{K} B_s E\left\{ X^s e^{itx} \right\}$$

$$E(Ye^{itx}) = \sum_{s=0}^{K} B_s E\left\{ X^s e^{itX} \right\}.$$

To prove Sufficiency: Assume that (3) is valid for all t.

Then:

$$E\left\{ e^{itx} \left[Y - \sum_{s=0}^{K} B_s X^s \right] \right\} = 0,$$

This may be written as:

$$(4) \qquad \int_{-\infty}^{\infty} e^{itx} E\left(Y - \sum_{s=0}^{K} B_s X^s \right) X \, d F_1(x) = 0$$

where $F_1(x)$ is the marginal distribution of X.

Let $P_x(A)$ be the probability function of the random variable X. This is a set function defined on all borel sets of R_1. Then we can write for equation (4):

$$\int_{R_1} e^{itx} E\left(Y - \sum_{s=0}^{K} B_s x^s \,\middle/\, x \right) d P_x = 0$$

Let $\mathcal{U}(A) = \int_A E(Y - \sum_{s=0}^{K} B_s X^s \mid x) \, dP_x$. This is a function of bounded variation which is defined on all Borel sets A of R_1. We see that

$$\int_{R_1} e^{itx} d\,\mathcal{U} = 0$$

Since the uniqueness theorem for ch. functions is valid for the Fourier transforms of functions of bounded variation we conclude that

$$\mathcal{U}(A) = \mathcal{U}(R_1) = 0 \text{ for all Borel sets A.}$$

This is only possible if

$$E(Y - \sum_{s=0}^{K} B_s X^s \mid x) = 0 \qquad \text{a.e.}$$

IV. The Differential Equation for the Ch. Fu.

To return to the tone of the introduction, here we add these comments:

Characterizations of both discrete and continuous distributions are obtainable in a variety of ways. As an example of this for a discrete distribution, it is known that the relationship of equality of mean and variance characterizes the Poisson distribution.

It is possible also to characterize distributions by reason of one linear statistics. And the independence of two statistics can be used to characterize distributions.

Also, various populations have been characterized by using the property that one suitably chosen statistic has constant or quadratic regression on another. Much work in this field has been done by Lukacs and by Laha.

We are here concerned with a case of quadratic on linear.

Let $x_1 \, x_2 \, \text{---} \, x_n$ be a sample of size n from a population with distribution function $F(x)$. Assume that the second moment of $F(x)$ exists.

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$$

is the ch. fu. of $F(x)$.

Consider the quadratic statistic:

$$Q = \sum_{i,j=1}^{K} a_{ij} X_i X_j + \sum_{j=1}^{K} b_j x_j$$

and suppose that Q has quadratic regression on $A = \sum_{j=0}^{K} X_j i$.

i.e. (5) $E(Q/A) = B_0 + B_1 A + B_2 A^2$

If we multiply both sides of (5) by e^{itA} and recall this step in the proof of the Lemma, we have

(6) $\qquad E(Qe^{itA}) = B_0 E(e^{itA}) + B_1 E(Ae^{itA}) + B_2 E(A^2 e^{itA})$

To simplify computation we let

$$a_1 = \sum_{i,j=1}^{K} a_{ij}$$

$$a_2 = \sum_{i,j=1}^{K} a_{i \neq j}$$

$$b = \sum_{j=1}^{K} b_j \; ; \text{ thus changing the symbols for coefficients}$$

in the value of Q. We also note for convenience:

$$f(t) = E(e^{itx})$$

$$f'(t) = i\, E(xe^{itx})$$

$$f''(t) = -E(X^2 e^{itx})$$

$$f'''(t) = -i\, E(X^3 e^{itx})$$

$$E(Xe^{itx}) = \frac{1}{i}\, f'(t)$$

$$E(X^2 e^{itx}) = -f''(t)$$

$$E(X^3 e^{itx}) = i\, f'''(t)$$

From a well-known property of ch. fu., in some neighborhood of the origin $f(t)$ is different from zero so that we can write

$$\phi(t) = \ln f(t)$$

We then have

(7) $\dfrac{f'(t)}{f(t)} = \phi^1(t)$ and $\dfrac{f''(t)}{f(t)} = \phi''(t) + \left[\phi^1(t)\right]^2$

The left-hand side of (6) is:

$$E(Qe^{itA}) = E\left[\left(\sum_{i,j=1}^{K} a_{ij} X_i X_y + \sum_{j=1}^{K} b_j X_j\right) e^{itA}\right] =$$

$$E\left\{\left[\sum_{i=y=1}^{K} A_j X_j^2 + \sum_{i=1}^{K}\sum_{j=1}^{K} a_{ij} X_i X_j + \sum_{j=1}^{K} b_j X_j\right] e^{itA}\right\}.$$

Set the result $= E_1 + E_2 + E_3$. Then $E_1 = \sum_{j=1}^{K} a_j E(X_j^2 e^{itx_j}) E(e^{it\sum_{i \neq j}^{K} x_i})$

$$= a_1 \left(-f''(t)\right)\left[f(t)\right]^{n-1}$$

$$E_2 = \sum_{i=1}^{K}\sum_{j=1}^{K} a_{ij} E(X_i X_j) E\left(e^{it\sum_{i=1}^{K} X_i}\right)$$

$$= \sum_{i \neq}\sum_{j} a_{ij} E\left(X_i e^{it\sum X_i}\right) E\left(X_j e^{it\sum X_j}\right)$$

$$= a_2 \left(-\left[f'(t)\right]^2\right)\left[f(t)\right]^{n-2}$$

$$E_3 = b\left(-i f'(t)\right)\left[f(t)\right]^{n-1}$$

Thus: the left-hand side of (6) is $E_1 + E_2 + E_3 =$

$$-\left[f(t)\right]^n \left[a_1 \frac{f''(t)}{f(t)} + a_2 \frac{\left[f^1(t)\right]^2}{\left[f(t)\right]^2} + ib \frac{f^1(t)}{f(t)}\right]$$

and this expression, by (7) is

(8) $-\left[f(t)\right]^n \left\{a_1 \phi''(t) \quad (a_1 + a_2) \left[\phi^1(t)\right]^2 + ib \phi^1(t)\right\}$

The right-hand side of (6):

$$B_0 \cdot E(e^{itA}) + B_1\, E(Ae^{itA}) + B_2 E(A^2\, e^{itA})$$

Recalling that $A = \sum_{i=1}^{K} X_i$,

$$B_0\, E(e^{itA}) = B_0 \left[f(t) \right]^n$$

$$B_1\, E(Ae^{itA}) = B_1\, E(Ae^{it \sum x_\nu}) = B_1 \sum_\nu E\left[X_\nu\, e^{itx_\nu}\, e^{it \sum_{j \neq i} x} \right]$$

$$= -i\, B_1\, n\, f^1(t) \left[f(t) \right]^{n-1}$$

$$B_2 E(A^2 e^{itA}) = B_2\, E\left(\sum_\nu X_\nu^2\, e^{itx_\nu}\, e^{it \sum_j x_j} \right) +$$

$$B_2\, E\left(\sum_{i \neq j} X_i X_j\, e^{itx_\nu}\, e^{itx_j}\, e^{it \sum_{\nu \neq j} x} \right)$$

$$= B_2 \left\{ - n\, f''(t) \left[f(t) \right]^{n-1} + n(n-1) \left[-i\, F^1(t) \right]^2 \left[f(t) \right]^{(n-2)} \right\}$$

$$= - \left[f(t) \right]^n B_2 \left\{ n \left[\frac{f''(t)}{f(t)} - \frac{\left[f^1(t) \right]^2}{\left[f(t) \right]^2} + n^2 \left[\frac{f^1(t)}{f(t)} \right]^2 \right] \right\}$$

$$= - B_2 \left[f(t) \right]^n \left\{ n\, \phi''(t) + n^2 \left[\phi^1(t) \right]^2 \right\}.$$

So the right-hand side is:

$$- \left[f(t) \right]^n \left\{ n\, B_2\, \phi''(t) + n^2 B_2 \left[\phi^1(t) \right]^2 + n\, B_1\, i\, \phi^1(t) - B_0 \right\}$$

The equation (6) is now:

$$a_1\, \phi''(t) + (a_1 + a_2) \left[\phi^1(t) \right]^2 + i\, b\, \phi^1(t) =$$

$$n\, B_2\, \phi''(t) + n^2\, B_2 \left[\phi^1(t) \right]^2 + i n\, B_1\, \phi^1(t) - B_0$$

We can solve for B_0

$$B_0 = (n\, B_2 - a_1)\, \phi''(t) + (n^2 B_2 - a_1 - a_2) \left[\phi^1(t) \right]^2 + i\, (nB_1 - b)\, \phi^1(t)$$

And if we substitute convenient symbols for the coefficients--we have:

(9) $\quad d_1 \phi''(t) + d_2 \left[\phi^1(t)\right]^2 + i\, d_3\, \phi^1(t) = B_0$

Introduce the function:

$$s(t) = \frac{1}{i}\, \phi^1(t)$$

Equation (9) above becomes:

$i\, d_1\, s^1(t) - d_2 (s(t))^2\, d_3\, s(t) = B_0 \quad$ or

(10) $\quad i\, d_1\, \dfrac{ds}{dt} = d_2\, s^2 + d_3\, s + B_0$

So our function $S(t)$ is such that

$$S(0) = \frac{1}{i}\, \phi^1(0) = \mu \quad \text{and}$$

(11) $$\left.\dfrac{ds}{dt}\right|_{t=0} = \frac{1}{i}\, \phi''(0) = i\, \sigma^2$$

Here, μ and σ^2 are the mean and variance respectively of the distribution $F(x)$.

Now, from equation (10),

We get, by substituting the values above gotten by putting $t = 0$, an equation (1) a relation between the coefficients of Q and the regression coefficient; i.e.,

(12) $$-\sigma^2 d_1 = d_2\, \mu^2 + d_3\, \mu + B_0$$

We see, from (12) that if $d_1 = 0$, while at least one of d_2 and d_3 is different from 0 then $S(t)$ and $\phi^1(t)$ are constants so that we obtain a degenerate distribution. We, thus, may assume without loss of generality that $d_1 \neq 0$. Let $d_2 = 0$, $d_1 \neq 0$, $d_3 = 0$.

These conditions will support this theorem:

Let $x_1\, x_2\, \text{---} x_n$ be a sample of size n taken from a population which has a finite variance σ^2. Now consider the quadratic statistics:

$$Q = \sum_{i,j}^{n} a_{ij} X_i X_j + \sum_{j=1}^{n} b_j X_j \qquad \text{with}$$

$$a = a_1(n-1) - a_2 \neq 0$$

$$a_1 = \sum_{i=1}^{n} a_{ii} \qquad a_2 = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij}$$

Let B_1 and B_2 be two real constants:

$$B_1 = \sum_{j=1}^{n} b_j$$

$$B_2 = \frac{1}{n^2}(a_1 + a_2).$$

The relation:

$$E(Q/A) = B_0 + B_1 A + B_2 A^2$$

holds a.e. iff the following conditions hold:

(1) $\quad B_0 = \dfrac{\sigma^2}{n}\left\{ a_1(n-1) - a_2 \right\}$

(2) The population is normal.

The relations: (i) $a_1(n-1) - a_2 \neq 0$

(ii) $B_1 = \dfrac{\sum_i b_i}{n}$

(iii) $B_2 = \dfrac{1}{n^2}(a_1 + a_2)$

are equivalent to the conditions:

$$\lambda_1 \neq 0 \quad \lambda_2 = 0, \quad \lambda_3 = 0.$$

(13) i) $\dfrac{ds}{dt} = \lambda_2 S^2 + \lambda_3 S + B_0 \quad$ reduces to

(14) ii) $\dfrac{ds}{dt} = B_0 .\qquad$ Put $t = 0$ and use

$$\left. \frac{ds}{dt} \right|_{t=0} = \frac{1}{i}\phi''(0) = i\sigma^2$$

and we have: $\qquad B_0 = -\lambda_1 \sigma^2.$

and since $\partial_1 = n\,B_2 - a_1$

$$B_0 = (a_1 - \eta\,B_2)\,\sigma^2 \quad \text{and since} \quad B_2 = \frac{a_1 + a_2}{n^2}$$

$$B_0 = (a_1 - \frac{a_1 + a_2}{n}\,\sigma^2 = \frac{\sigma^{-2}}{n}\left[(a_1\,(n-1) - a_2\right]$$

.And this is condition (1) above.

We integrate: (14)

(15)
$$i\,\partial_1\,\frac{ds}{dt} = B_0$$

$$\text{for } t = 0, \quad \frac{ds}{dt} = i\,\sigma^2$$

$$\text{So } B_0 = -\partial\,\sigma^2$$

$$i\partial_1\,\frac{ds}{dt} = -\partial_1\,\sigma^2 = B_0$$

$$i\,\frac{ds}{dt} = -\sigma^2$$

$$i\,S(t) = -\sigma^2 t + k$$

$$\phi^1(t) = -\sigma^2 t + i\,\mu$$

$$\phi(t) = -\frac{1}{2}\,\sigma^2 t^2 + i\,\mu\,t$$

$$\phi(t) = \frac{1}{2}\,\sigma^2 t^2 + i\,\mu\,t.$$

and this is the ch. fu. of a normal distribution. We next prove sufficiency
of the assumption.

   We continue this technique of assigning different relationships be-
tween the coefficients of Q and the regression coefficients; we get *different*
distribution**s**.

ADMINISTRATIVE ASSISTANT

Miss Bernice L. Spigner served as administrative assistant. Her duties were:

> to secure copies of reference articles from area libraries,
>
> to type material as compiled,
>
> to retype material per additions, deletions, and changes,
>
> to edit the material

Research Activity

Student Assistants

Two upper level mathematics students in the College were
research assistants on the project. The students are:

Ms. Mary Murray

Mr. Kendrick Henderson

The students were assigned to ascertain definitions and
properties of the following transforms:

Fourier Cosine

Fourier Sine

Laplace

Inverse Laplace

Mellins

Inverse Mellins

Stielties

Hilbert

In my opinion, the investigation should be continued. The subject is quite broad and the many items of reference articles gathered to support the inquiry have not been utilized so far to any appreciable degree.

We view the next step in a broad investigation to be that of extending Quadratic regression to cubic to quartic, etc. The number of relations, as utilized here, will necessarily increase.

The size of the sample, as demonstrated here, can be increased with the computational format make use of a computer program.

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