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THE EXTERIOR TIDAL POTENTIAL ACTING ON A SATELLITE

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ABSTRACT

The high accuracy of contemporary observational techniques now requires a better physical and mathematical understanding of the form of the tidal potential acting on the satellite and a gradual reshaping of the computational methods currently in use. The dependence of elastic responses and tidal lags on the geographic position and on tidal frequency is introduced into the tidal potential on the surface of the Earth. By making use of Dirichlet's theorem we continue the tidal potential from the surface of the Earth into outer space. The final form of the disturbing tidal potential acting on the satellite is obtained in terms of the mean elements of the satellite and in terms of the standard arguments of the lunar theory. This form of the tidal potential is suitable for performing the analytical integration and for computation of the tidal effects over a long interval of time, because all agruments are linear or very nearly linear with respect to time. We interpreted the tidal effects in the motion of the satellite as perturbations in the orbital elements and as the small periodic variations in the coefficients of the spherical harmonics in the expansion of the geopotential. In transforming the exterior tidal potential to a form containing the orbital elements it was convenient to resort to the expansion in terms of the generalized spherical harmonics familiar from the theory of angular momentum in quantum mechanics.

BASIC NOTATIONS

G - gravitational constant,

M - the mass of the Earth,

R - the radius of the Earth, considered as a sphere,

$\mathbf{r}'$ - position vector of the Moon relative to the center of the Earth,

$\mathbf{r}'^0$ - the unit vector in the direction of $\mathbf{r}'$,

p', q', s' - the equatorial components of $\mathbf{r}'^0$,

m' - the mass of the Moon,

a' - the mean distance of the Moon from the Earth, defined in such a manner that the constant part in the expansion of a'/r' is equal to 1,

ℓ, ℓ', F, D and Γ - the standard arguments of Brown's lunar theory,

ϕ', λ' - the geographic colatitude and the longitude of the Moon,

α', δ' - the right ascension and the declination of the Moon,

Ω' - the lunar tidal disturbing function,

$\cos \gamma' = \mathbf{r}^0 \cdot \mathbf{r}'^0$,

$\mathbf{r}''$ - position vector [illegible] Sun relative to the center of the Earth,

$r'' = |\mathbf{r}''|$,

$\mathbf{r}''^0$ - the unit vector in the direction of $\mathbf{r}''$,

p'', q'', s'' - the equatorial components of $\mathbf{r}''^0$,

m'' - the mass of the Sun,

a'' - the mean distance of the Sun from the Earth, defined in such a manner that the constant part in the expansion of a''/r'' is equal to 1,

ϕ'', λ'' - the geographic colatitude and longitude of the Sun,

α'', δ'' - the right ascension and the declination of the Sun,

Ω - the solar tidal disturbing function,

$\cos \gamma'' = \mathbf{r}^0 \cdot \mathbf{r}''^0$,

θ - the sidereal time,

ϕ, λ - the geographic colatitude and the longitude of a point on the Earth surface,

$\mathbf{r}$ - position vector of the Satellite, relative to the center of the Earth,

$r = |\mathbf{r}|$,

$\mathbf{r}^0$ - the unit vector in the direction of $\mathbf{r}$,

α, δ - the right ascension and declination of the Satellite,

a, e, ω, Ω, I - the elliptic elements of the instantaneous orbit of the satellite,

f - the true anomaly of the satellite,

$\mathbf{R}$ - the unit vector normal to the instantaneous orbital plane of the satellite.

INTRODUCTION

In the present work we investigate the form of the exterior tidal potential acting on a satellite. The tidal potential can be defined as small periodic variations in the geopotential caused by the formation of the tidal buldge and by the redistribution of masses in the Earth's interior under the influence of lunisolar tidal attraction. In other words, the tidal effects in the motion of the satellite can be interpreted as the result of tidal periodic oscillations in the coefficients of the spherical harmonics in the expansion of the exterior geopotential. Of considerable interest and importance are the long period oscillations caused by the regression of the lunar node along the ecliptic. The existence of such oscillations in the oblateness coefficient C_{20} was first emphasized by Kozai (1965). At the present time the influence of the long period tidal effects on the coefficients in the expansion of geopotential lies on the limit of observability. However, with the further amelioration of the observational techniques and extension of observations over a longer interval of time we shall be obliged to take these effects into consideration. The variations in geopotential produce in turn small tidal perturbations in the motion of the satellite. They can be clearly detected by the modern observational techniques (Newton, 1968), (Smith et al., 1973). The high accuracy of contemporary techniques now requires a better physical and mathematical understanding of the phenomena and a gradual reshaping of the computational methods currently in use. The programs currently in use permit one to determine the average elastic properties of the Earth from the magnitude of the tidal effects in the motion of satellites.

By adjusting numerical values of parameters of the Earth's elastic response and of the tidal lags one can achieve a satisfactory representation of the satellites motior It was found, however, that the numerical values of the mean elastic parameters, as determined from the motion of satellites, are different for different orbital inclinations (Smith et al., 1973), (Douglas et al., 1974). This clearly indicates the existence of lateral inhomogeneities and, possibly, asymmetry in the Earth's internal structure and of their influence on the tidal perturbations of a satellite. Until the problem of the influence of lateral inhomogeneities on the tidal oscillations is resolved, we are forced to use the information available on the Earth's surface, the parameters of the elastic response (Love coefficients) and tidal lags, and by making use of Dirichlet's theorem continue the tidal potential analytically from the surface of the Earth into outer space.

Thus, for the present any further refinement in the theory of tidal perturbations in the motion of satellites is tied, to considerable extent, to the measurements on a global scale of tides or tidal elastic responses on the surface of the Earth, directly or by remote sensing. In other words, the dependence of elastic responses and tidal lags on the geographic position and on tidal frequency shall be introduced into the tidal potential.

Original Love elastic parameters (Love, 1909) are the numerical coefficients attached to Legendre polynomials in the expansion of the tidal potential. Thus, Love formulation assumes that the elastic responses are the same for all tidal constituents associated with a given Legendre polynomial, irrespectively of the tidal argument. Kaula (1969) and Balmino (1973) assumed that Love coefficients depend on geographic position and are sufficiently smooth functions on the Earth surface so that they can be expanded into a series of spherical harmonics. Balmino assumed in addition a dependence of Love coefficients on the tidal frequency. This last assumption is in conformity with the results by Alterman et al. (1959). In the present exposition we follow Balmino and assume the dependence of Love numbers on the geographic position as well as on the tidal frequency. We found it convenient to use complex "Love numbers," because they carry the combined influence of elastic responses and lags on the motion of the satellite.

It is necessary to remark that the introduction of Love coefficients presupposes the absence of non-linearities, i.e. the absence of noticeable (in the motion of a satellite) interactions between the tidal constituents (Kaula, 1969).

Only the influence of long period tidal effects can be detected in the elements of artificial satellites. As a consequence the form of the exterior tidal potential can be greatly simplified, because all terms which involve the sidereal time or the mean anomaly of the satellite can be omitted from the expansion. This means also that the complex Love coefficients can be averaged along the parallels. The remaining dependence of the averaged Love numbers on latitude will produce the primary and the secondary tidal effects in the motion of the satellite. The secondary effects are of the same form and magnitude as some primary ones (Musen and Felsentreger, 1973).

Care must be taken against oversimplification of the theory and computations by neglecting essential secondary effects or by confusing them with primary ones. In transforming the tidal potential to a form containing the orbital elements it is convenient to resort to expansion in terms of special and surface generalized spherical harmonics. These harmonics are familiar from the theory of angular momentum in quantum-mechanics (Edmonds, 1960) (Rose, 1957), (Vilenkin, 1965). They also can be profitably applied in the theory of motion of celestial bodies and in theoretical seismology (Burridge, 1969), (Phinney and Burridge, 1973), (Smith, 1974).

The F-coefficients in Kaula's (1969) elegant expansion of the tidal potential are intimately related to the generalized associated Legendre functions which are being used in the present work.

Numerous recursive relations between the generalized spherical functions and their derivatives greatly facilitate the process of expansion of the tidal potential. Using the recursive formulas and an analytic programming language one can obtain a purely analytic expansion by computer or, by substituting the numerical value of the inclination (and of the eccentricity) at the very outset, one can obtain a semi-analytical expansion of the tidal perturbations, in the form of periodic series with purely numerical coefficients.

The first option gives the solution in a most general form. The second one requires less computing time and makes the programming work easier.

We propose, as in our previous work (Musen and Estes, 1972), to discard the expansion of the tidal potential in terms of the elements of the Moon, equatorial or ecliptical. In particular, the equatorial elements of the Moon are not linear with respect to time and therefore are not quite suitable for the integration in analytical or semi-analytical form, nor for the investigation of the tidal effects over a long interval of time. We propose instead a Doodson (1922) type expansion of the tidal potential in terms of the standard arguments of the lunar theory with purely numerical coefficients. In Doodson type expansion all arguments are very nearly linear with respect to time. This facilitates integration of differential equations for the variation of elements and the search for the critical terms with low frequencies. In performing this expansion we can make use of the developments of the lunar rectangular coordinates directly available from Brown's lunar theory (with some modern corrections). All necessary programs to produce the semi-analytical expansions of the tidal potential and of the tidal perturbations in the elements of the satellites were prepared by R. Estes. (Musen and Ester, 1972).

TIDAL DISTURBING POTENTIAL

We start from the standard expansion on the surface of the Earth of the lunar

$$\Omega' = Gm' \sum_{\ell=2}^{+\infty} \frac{R^{\ell}}{r'^{\ell+1}} P_{\ell}(\cos \gamma') \tag{1}$$

and solar

$$\Omega'' = Gm'' \sum_{\ell=2}^{+\infty} \frac{R^{\ell}}{r''^{\ell+1}} P_{\ell}(\cos \gamma'') \tag{2}$$

tidal disturbing functions, respectively, with the Love elastic coefficients and tidal lags temporarily omitted. Introducing normalized surface harmonics

$$Y_{\ell m}(\phi, \lambda) = \sqrt{\frac{(\ell - m)!}{(\ell + m)!}} P_{\ell}^{m}(\cos \phi) \exp(im\lambda), \tag{3}$$

$$Y_{\ell, -m}(\phi, \lambda) = (-1)^m Y_{\ell m}^{*}(\phi, \lambda) \tag{4}$$

with the normalization

$$\frac{1}{4\pi} \int Y_{\ell m}(\phi, \lambda) Y_{jk}^{*}(\phi, \lambda)\, dS = \frac{1}{2\ell + 1} \delta_{\ell j} \delta_{mk}, \tag{5}$$

$$dS = \sin\phi\, d\phi\, d\lambda,$$

over the unit sphere, and making use of the addition theorem:

$$P_{\ell}(\cos \gamma') = \sum_{m=-\ell}^{m=+\ell} Y_{\ell m}^{*}(\phi', \lambda') Y_{\ell m}(\phi, \lambda) \tag{6}$$

and with

$$\lambda' = \alpha' - \theta, \tag{7}$$

we obtain

$$\Omega' = Gm' \sum_{\ell=2}^{+\infty} \frac{R^{\ell}}{a'^{\ell+1}} \sum_{m=-\ell}^{m=+\ell} \exp(im\theta) Y_{\ell m} T'_{\ell m}, \tag{8}$$

where we set:

$$T'_{\ell m} = \left(\frac{a'}{r'}\right)^{\ell+1} Y_{\ell m}^{*}(\phi', \alpha') \tag{9}$$

and a similar expression:

$$\Omega'' = Gm'' \sum_{\ell=2}^{+\infty} \frac{R^{\ell}}{a''^{\ell+1}} \sum_{m=-\ell}^{m=+\ell} \exp(im\theta)\, Y_{\ell_m}(\phi, \lambda)\, T''_{\ell_m}, \tag{10}$$

$$T''_{\ell_m} = \left(\frac{a''}{r''}\right)^{\ell+1} Y^{*}_{\ell_m}(\phi'', \alpha''), \tag{11}$$

for the solar tidal disturbing function. The expressions (9) and (10) depend on the equatorial coordinates of the Moon and Sun, respectively. The lunar and solar theories currently in use give the expansions of coordinates, rectangular or polar, of both bodies in form of trigonometric series with the standard arguments ℓ, ℓ', F, D and Γ, and purely numerical coefficients. The accuracy of the expansions is more than sufficient for the computation of tidal effects in the motion of the satellite.

The arguments ℓ, ℓ', F, D and Γ are very nearly linear with respect to time and their use facilitates the integration of the differential equations for variation of elements and the long time prediction of tidal effects in the motion of a satellite (Musen and Estes, 1972). To facilitate the programming and the expansion into periodic series, we can express the "lunar" spherical functions $Y_{\ell_m}(\phi', \alpha')$ in Maxwellian form as polynomials in the equatorial components p', q', s' of the unit vector $\mathbf{r}'^{0}$ directed toward the Moon from the center of the Earth. We have:

$$Y_{\ell m}(\phi', \alpha') = \frac{1}{2^{\ell}} \sqrt{\frac{(\ell - m)!}{(\ell + m)!}} (p' + iq')^m \cdot$$

$$\sum_{k=0}^{[(\ell - m)/2]} (-1)^k \frac{(2\ell - 2k)!}{k!\,(\ell - k)!\,(\ell - m - 2k)!} s'^{\ell - m - 2k} \tag{12}$$

for $m \geq 0$. We can use (4) to obtain the expansion for $m < 0$. Expansions of p', q', s', a'/r' and then of $T'_{\ell m}$ can be obtained easily from Browns series (1919) of the lunar coordinates and parallax (with modern corrections) using existing computer programs (Musen and Estes, 1972).

The resulting expansion of $T'_{\ell m}$ is the sum of the expressions of the form

$$T'_{\ell m \nu} = N_{\ell m \nu} \exp(i \nu \cdot \tau), \tag{13}$$

$$T'_{\ell m} = \sum_{\nu} T'_{\ell m \nu}, \tag{14}$$

where

$$\nu = (j_1, j_2, j_3, j_4, j_5)$$

is a vector with integral components and

$$\tau = (\ell, \ell', F, D, \Omega)$$

and $N_{\ell m \nu}$ are the numerical coefficients. It is convenient at this point to introduce Love coefficients and their dependence on geographic position and frequency. We replace (13) in the expansion of the tidal disturbing function by the expression

$$\kappa_{\ell m \nu}(\phi, \lambda) T'_{\ell m \nu},$$

where

$$\kappa_{\ell m \nu}(\phi, \lambda) = k_{\ell m \nu}(\phi, \lambda) \exp[-i \epsilon_{\ell m \nu}(\phi, \lambda)], \tag{15}$$

and $k_{\ell m \nu}$, $\epsilon_{\ell m \nu}$ are the standard Love coefficients and tidal lags, respectively.

The lunar tidal disturbing function on the surface of the Earth is the sum of expressions of the form:

$$\Omega'_{\ell m\nu} = Gm' \frac{R^{\ell}}{a'^{\ell+1}} \exp(im\theta)\, \kappa_{\ell m\nu}(\phi, \lambda)\, Y_{\ell m}(\phi, \lambda)\, T'_{\ell m\nu}. \tag{16}$$

For the sake of compactness we shall temporarily suppress the dependence on ν and write simply:

$$\Omega'_{\ell m} = Gm' \frac{R^{\ell}}{a'^{\ell+1}} \exp(im\theta)\, \kappa_{\ell m}(\phi, \lambda)\, Y_{\ell m}(\phi, \lambda)\, T'_{\ell m}. \tag{17}$$

Substituting the expansion

$$\kappa_{\ell m} Y_{\ell m} = \sum_{p=0}^{+\infty} \sum_{q=-p}^{q=+p} K_{\ell m p q} Y_{pq}(\phi, \lambda), \tag{18}$$

where

$$K_{\ell mpq} = \frac{2p+1}{4\pi} \int \kappa_{\ell m} Y_{\ell m} Y^{*}_{pq} dS, \tag{19}$$

into (17), we obtain:

$$\Omega'_{\ell m} = Gm' \frac{R^{\ell}}{a'^{\ell+1}} \exp(im\theta)\, T'_{\ell m} \sum_{p=0}^{+\infty} \sum_{q=-p}^{q=+p} K_{\ell mpq} Y_{pq} \tag{20}$$

Making use of (4) we have:

$$K_{\ell mpq} = (-1)^{q} \frac{2p+1}{4\pi} \int \kappa_{\ell m} Y_{\ell m} Y_{p,-q} dS. \tag{21}$$

Below we make use of theorems and formulas familiar from the theory of the representation of the rotation group and from the theory of angular momentum in quantum-mechanics. We found it convenient in our work to follow Vilenkin's (1965) notation and symbolism as close as possible.

Let χ, ϕ and ψ be Euler's angles defining a rotation g. The elements of the representation of order ℓ of g(the generalized spherical harmonics) have the form:

$$t^{\ell}_{mn}(\chi, \phi, \psi) = \exp[-i(m\chi + n\psi)] \; P^{\ell}_{mn}(\cos\phi), \tag{22}$$

$$m, \; n = -\ell, \ldots, +\ell,$$

where P^{ℓ}_{mn} $(\cos\phi)$ are the generalized associated Legendre functions. We define the angles χ and ψ in such a manner that P^{ℓ}_{mn} $(\cos\phi)$ are always real. We have the relation:

$$Y_{\ell m}(\phi, \lambda) = t^{\ell}_{-m,0}(\chi, \phi, -) \tag{23}$$

with

$$\chi = \lambda + \pi. \tag{24}$$

From (23), (4) and making use of the Clebsch-Gordan series:

$$t^{\ell_1}_{j_1 k_1}(\chi, \phi, \psi) \cdot t^{\ell_2}_{j_2 k_2}(\chi, \phi, \psi) = \tag{25}$$

$$\sum_{s=|\ell_2-\ell_1|}^{s=+|\ell_2+\ell_1|} C(\ell_1, \ell_2, s; j_1, j_2)\, C(\ell_1, \ell_2, s; k_1, k_2)\, t^{s}_{j_1+j_2,\, k_1+k_2}(\chi, \phi, \psi),$$

where the symbol C designates Clebsch-Gordan coefficients (Rose, 1957), we derive:

$$Y_{\ell m} Y_{p,-q} = (-1)^{m-q} \sum_{s=|\ell-p|}^{s=|\ell+p|} C(\ell, p, s; -m, +q)\, C(\ell, p, s; 0, 0)\, Y^{*}_{s,q-m}. \tag{26}$$

Substituting (26) into (21) we obtain:

$$K_{\ell_{mpq}} = (-1)^m \sum_{s=|\ell-p|}^{s=|\ell+p|} C(\ell, p, s; -m, +q)\, C(\ell, p, s; 0, 0)\, \frac{2p+1}{2s+1}\, \kappa_{\ell m s, q-m} \tag{27}$$

where symbols $K_{\ell_{msj}}$ designate the coefficients in the expansion of the Love coefficient κ_{ℓ_m} into a series of spherical harmonics,

$$\kappa_{\ell_{msj}} = \frac{2s+1}{4\pi} \int \kappa_{\ell_m} Y^*_{sj}\, dS, \tag{28}$$

$$\kappa_{\ell_m} = \sum_{s=0}^{+\infty} \sum_{j=-s}^{j=+s} \kappa_{\ell_{msj}} Y_{sj}. \tag{29}$$

In particular we have:

$$\kappa_{\ell_{ms0}} = \frac{2s+1}{2} \int_0^{\pi} \left\langle \kappa_{\ell_m} \right\rangle P_s(\cos\phi)\, \sin\phi\, d\phi, \tag{30}$$

where

$$\left\langle \kappa_{\ell_m} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \kappa_{\ell_m}\, d\lambda \tag{31}$$

is the value of κ_{ℓ_m} averaged along the parallel. The analytical continuation of the tidal potential (20) into extraterrestrial space is a harmonic function and therefore has the form:

$$\Omega'_{\ell_m} = \frac{Gm'}{R} \beta'^{\ell+1} \exp(im\theta)\, T'_{\ell_m} \cdot \sum_{p=0}^{+\infty} \sum_{q=-p}^{q=+p} K_{\ell_{mpq}} \left(\frac{R}{r}\right)^{p+1} Y_{pq}(\phi, \lambda), \tag{32}$$

where β' is the lunar parallactic factor,

$$\beta' = \frac{R}{a'}.$$

In application to the motion of a satellite we set in (32):

$$\lambda = \alpha - \theta, \qquad \phi = \frac{\pi}{2} - \delta,$$

and we obtain

$$\Omega'_{\ell m} = \frac{Gm'}{R}\,\beta'^{\,\ell+1}\,T'_{\ell m}\;\cdot \tag{33}$$

$$\sum_{p=0}^{+\infty}\;\sum_{q=-p}^{q=+p} K_{\ell mpq}\,\beta^{p+1}\,\exp\left[i\,(m-q)\,\theta\right]\left(\frac{a}{r}\right)^{p+1} Y_{pq}(\phi,\ \alpha),$$

where β is the satellite's parallactic factor,

$$\beta = \frac{R}{a}.$$

We are not interested in the tidal effects with period of one day or less. Therefore only the terms for which $m = q$ are of any practical importance and shall be kept in (33). All remaining terms can be discarded and we can set:

$$\Omega'_{\ell m} = \sum_{p=|m|}^{+\infty} \Omega'_{\ell mp}, \tag{34}$$

where

$$\Omega'_{\ell mp} = \frac{Gm'}{R}\,\beta'^{\,\ell+1}\,\beta^{p+1}K_{\ell mpm}\cdot T'_{\ell m}\cdot\left(\frac{a}{r}\right)^{p+1} Y_{pm}(\phi,\ \alpha), \tag{35}$$

and from (27) we have:

$$K_{\ell mpm} = (-1)^m \sum_{s=|\ell-p|}^{|\ell+p|} C(\ell,\ p,\ s;\ -m,\ +m)\,C(\ell,\ p,\ s;\ 0,\ 0)\,\frac{2p+1}{2s+1}\,\kappa_{\ell ms0}, \tag{36}$$

and $\kappa_{\ell ms0}$ is given by the combination of (30) and (31).

The form

$$\Omega'_{\ell mp} = \left[\frac{GM}{r} \cdot \frac{m'}{M} \beta'^{\ell+1} K_{\ell mpm} T'_{\ell m}\right] \cdot \left(\frac{R}{r}\right)^{p} Y_{pm}(\phi, \alpha)$$

of (35) indicates that

$$\delta C_{\ell pm} = \frac{m'}{M} \beta'^{\ell+1} K_{\ell mpm} T'_{\ell m} \exp(im\theta)$$

can be interpreted as perturbations in the coefficient C_{pm} of the spherical harmonic

$$\left(\frac{R}{r}\right)^{p+1} Y_{pm}(\phi, \lambda)$$

in the expansion of the exterior geopotential.

With this interpretation the coefficients in the geopotential become changing periodic functions of time. For perturbations in the coefficient of the zonal harmonic $(R/r)^{p+1}$ Y_{p0} we have:

$$\delta C_{\ell p0} = \frac{m'}{M} \beta'^{\ell+1} K_{\ell 0p0} T'_{\ell 0}.$$

In particular for the perturbations in the oblateness-coefficient C_{20} we have:

$$\delta C_{\ell 20} = \frac{m'}{M} \beta'^{\ell+1} K_{\ell 0p0} T'_{\ell 0}.$$

An estimate of the perturbations in C_{20} is given by the formula:

$$\delta C_{20} = \frac{m'}{M} \beta'^{3} k_2 \left(\frac{a'}{r'}\right)^{3} P_2(s')$$

where $k_2 \cong 0.3$, the approximate value of the Love number, and $(a'/r')^3\ P_2(s')$ shall be expanded into a periodic series with the arguments of the lunar theory. The terms in $T'_{\ell 0}$ which contain only the longitude of the lunar ascending node $\ell' - F + D + \Gamma$ produce "very long period" effects with the periods of approximately 18, 9, . . . years.

At the present time the influence of such "very long period" terms, with the periods of the order of years, on the coefficients in the expansion of the geopotential (or on the motion of satellites) lies on the limit of observability, but with the further improvement of the observational techniques and extension of observations over a longer interval of time the importance of these terms might increase and we will be obliged to take them into consideration, especially in C_{20}. Application of the addition theorem

$$t^{\ell}_{mn}(g_1 g_2) = \sum_{k=-\ell}^{k=+\ell} t^{\ell}_{mk}(g_1)\ t^{\ell}_{kn}(g_2) \tag{37}$$

for the generalized spherical harmonics, as defined by the rotations g_1 and g_2 respectively, probably represents the shortest way to obtain the expansion of the tidal disturbing function (35) in terms of the angular elements of the satellite's orbit. The transition from the frame as defined by the unit vectors $(\mathbf{r}, \mathbf{R} \times \mathbf{r}^0, \mathbf{R})$ to the system associated with the equator involves the rotations:

$$g_1: \quad \chi = \Omega + \frac{\pi}{2}, \quad \phi = I, \quad \psi = u + \frac{\pi}{2} = f + \omega + \frac{\pi}{2},$$

$$g_2: \quad \chi = \pi, \quad \phi = \frac{\pi}{2}$$

$$g_1 g_2: \quad \chi = \alpha + \pi, \quad \phi = \frac{\pi}{2} - \delta.$$

From (37) and taking (22)-(23) into account we obtain:

$$Y_{pm} = \sum_{k=-p}^{k=+p} t^p_{k0}\left(\pi, \frac{\pi}{2}, -\right) \cdot \exp\left\{i\left[m\Omega - ku + (m-k)\frac{\pi}{2}\right]\right\}. \tag{38}$$

$$\cdot P^p_{-m,k}(\cos I).$$

From (22) and making use of relations

$$P^p_{-k,0}(x) = (-1)^k P^p_{k,0}(x),$$

$$P^p_{k0}(x) = \sqrt{\frac{(p-k)!}{(p+k)!}}\, P^k_p(x)$$

We obtain after some easy transformations:

$$t^p_{k0}\left(\pi, \frac{\pi}{2}, -\right) = \frac{(-1)^{(p+k)/2}}{2^{(p-|k|)/2}} \sqrt{\frac{(p-|k|)!}{(p+|k|)!}} \cdot \frac{(p+|k|-1)!!}{\left(\frac{p-|k|}{2}\right)!}, \quad \text{for } p-k \text{ even}$$

$$= 0, \qquad \text{for } p-k \text{ odd.}$$

It follows from (39) that (35) can be represented in turn as a sum of the terms of the form:

$$\Omega'_{\ell mpq} = \frac{Gm'}{R} \beta'^{\ell+1} \beta^{p+1} K_{\ell mpm} Q_{pq} T'_{\ell m} \left(\frac{a}{r}\right)^{p+1} \cdot \tag{40}$$

$$\exp\left\{i\left[m\Omega - (p-2q)(f+\omega) + (m-p+2q)\frac{\pi}{2}\right]\right\} \cdot P^{p}_{-m,p-2q}(\cos I),$$

$$q = 0, 1, 2, \ldots p$$

where we set

$$Q_{pq} = t^{p}_{p-2q,0}\left(\pi, \frac{\pi}{2}\right). \tag{41}$$

Using Hansen's notation:

$$X^{-p-1,-p+2q} = \left(\frac{a}{r}\right)^{p+1} \exp[i(2q-p)f],$$

we re-write (40) as:

$$\Omega'_{\ell mpq} = \frac{Gm'}{R} \beta'^{\ell+1} \beta^{p+1} K_{\ell mpm} Q_{pq} T'_{\ell m} X^{-p-1,-p+2q}. \tag{42}$$

$$\exp\left\{i\left[m\Omega - (p-2q)\omega + (m-p+2q)\frac{\pi}{2}\right]\right\} \cdot P^{p}_{p-m,p-2q}(\cos I).$$

From the last expression we eliminate all terms with the period equal to the period of revolution of the satellite or less by performing the averaging over the satellite's instantaneous orbit. Making use of Hansen's formulas (Tisserand, 1889) we obtain:

$$\Omega'_{\ell mpq} = \frac{Gm'}{R} \beta'^{\ell+1} \beta^{p+1} K_{\ell mpm} Q_{pq} T'_{\ell m} \cdot \tag{43}$$

$$\exp\left\{i\left[m\Omega - (p-2q)\omega + (m-p+2q)\frac{\pi}{2}\right]\right\} \cdot P^{p}_{-m,p-2q}(\cos I)\, X_0^{-p-1,p-2q}(e).$$

where

$$X_0^{-p-1,\,p-2q} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^{p+1}\exp i\,(p-2q)\,f\cdot dM \tag{44}$$

$$= \binom{p-1}{|p-2q|}\left(\frac{e}{2}\right)^{|p-2q|}(1-e^2)^{-p+1/2}\times$$

$$F\left(\frac{|p-2q|-p+1}{2},\ \frac{|p-2q|-p+2}{2},\ |p-2q|+1;\ e^2\right)$$

The hypergeometric series in the right hand side of (44) is always a polynomial in e^2.

Designating the modulus of $N_{\ell m}\,K_{\ell mpm}\,Q_{pq}$ by $A_{\ell mpq}$ and its argument by $\alpha_{\ell mpq}$ and keeping only the real part of the disturbing function we have:

$$\Omega'_{\ell mpq} = \frac{Gm'}{R}\,\beta'^{\,\ell+1}\beta^{p+1}A_{\ell mpq}\,P^{p}_{-m,\,p-2q}(\cos\ I)\cdot X_0^{-p-1,\,p-2q} \tag{43'}$$

$$\cdot\cos\left[\nu\cdot\tau + m\Omega - (p-2q)\,\omega + (m-p+2q)\frac{\pi}{2} - \alpha_{\ell mpq}\right],$$

and the dependence on ν as before is not indicated explicitly. The disturbing functions (43') are to be used in the actual computations of the tidal effects. Some particular cases of (43') deserve special attention. It follows from (44) that

$$X_0^{-p-1,\,\pm p} = 0, \tag{45}$$

and from (43) and (45):

$$\Omega'_{\ell mp0} = \Omega'_{\ell mpp} = 0,$$

and, consequently,

$$\Omega'_{\ell mp} = \sum_{q=1}^{\nu-1} \Omega'_{\ell mpq}, \tag{46}$$

The terms of the type

$$\Omega'_{\ell m \ell q} = \frac{Gm'}{R} (\beta' \beta)^{\ell+1} A_{\ell m \ell q} P^{\ell}_{-m, \ell - 2q}(\cos I) \cdot \tag{47}$$

$$X_0^{-\ell-1, \ell-2q} \cos \left[\nu \cdot \tau + m\Omega - (\ell - 2q)\,\omega + (m - \ell + 2q)\frac{\pi}{2} - a_{\ell m \ell q}\right]$$

are the only terms which remain in the expansion if Love coefficients are assumed to be constant. All other terms represent "cross-effects" between the terms of the type (47) and deviation of Love coefficients from their mean values. For this reason the terms of the type (47) can be designated as "primary" ones.

Among the primary terms of a special interest are the terms of the type $\Omega'_{2s,m,2s,s}$ (m = -2s, . . . , + 2s). They do not contain the argument of the satellite's perigee) and, as a consequence, they do not change the eccentricity and the shape of the orbit. The same can be said about the terms $\Omega'_{\ell,m,2s,s}$ of a slightly more general type. Terms $\Omega'_{\ell 0pq}$ do not contain the right ascension of the satellite's ascending node and, as a consequence, do not change the orbital inclination. Finally, the terms

$$\Omega'_{\ell,0,2s,s} = \frac{Gm'}{R}\,\beta'^{\ell+1}\beta^{2s+1}A_{\ell,0,2s,s}P_{2s}(\cos I)\cdot \tag{48}$$

$$X_0^{-2s-1,0}\cos(\nu\cdot\tau - \alpha_{\ell,0,2s,s})$$

$$X_0^{-2s-1,0} = (1-e^2)^{-2s+1/2}\,F\left(\frac{1}{2}-s,\ 1-s,\ 1;\ e^2\right),$$

contain only the periodic terms associated with the motion of the Moon. In particular, they carry very long period tidal effects as produced by the regression of the lunar node along the ecliptic.

Setting $m = q = 0$ and taking

$$Q_{00} = C(\ell,\ 0,\ \ell;\ 0,\ 0) = +1$$

into consideration, we have from (40)

$$\Omega'_{\ell 000} = \frac{GM}{r}\cdot\frac{m'}{M}\beta'^{\ell+1}\cdot\frac{\kappa_{\ell 0}\kappa_{\ell 0}}{2\ell+1}\cdot T'_{\ell 0}\cdot \tag{49}$$

the terms additive to the first term GM/r in the geopotential. Theoretically, if not taken into account over a long interval of time, they will cause spurious small oscillations in GM, roughly one order higher than the tidal perturbations in the oblateness coefficient C_{20}. In the motion of the satellite they can produce very small perturbations with the periods approximately equal to the period of revolution of the satellite.

DIFFERENTIAL EQUATIONS FOR VARIATION OF ELEMENTS

Substituting

$$\Omega'_{\ell mpq} = n^2 a^2 BPX \cos W, \tag{50}$$

where we set for brevity

$$B = \frac{m'}{M} \beta'^{\ell+1} \beta^p A_{\ell mpq}, \tag{51}$$

$$P = P_{-m, p-2q} (\cos I), \tag{52}$$

$$X = \Lambda_0^{-p-1, p-2q}(e) \tag{53}$$

$$W = \nu \cdot \tau + m\Omega - (p - 2q)\,\omega + (m - p + 2q)\frac{\pi}{2} - \alpha_{\ell mpq} \tag{54}$$

for the disturbing function Ω in the differential equations

$$\frac{d\delta\Omega}{dt} = + \frac{1}{na^2\sqrt{1 - e^2}\sin I} \cdot \frac{\partial\Omega}{\partial I}, \tag{55}$$

$$\frac{d\delta I}{dt} = + \frac{\cos I}{na^2\sqrt{1 - e^2}\sin I} \cdot \frac{\partial\Omega}{\partial\omega} - \frac{1}{na^2\sqrt{1 - e^2}\sin I} \cdot \frac{\partial\Omega}{\partial\Omega}, \tag{56}$$

$$\frac{d\delta\omega}{dt} = - \frac{\cos I}{na^2\sqrt{1 - e^2}\sin I} \cdot \frac{\partial\Omega}{\partial I} + \frac{\sqrt{1 - e^2}}{na^2 e} \cdot \frac{\partial\Omega}{\partial e} \tag{57}$$

$$\frac{d\delta e}{dt} = - \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial\Omega}{\partial\omega}, \tag{58}$$

$$\frac{d\delta M}{dt} = - \frac{1 - e^2}{na^2 e} \frac{\partial\Omega}{\partial e} - \frac{2}{na} \frac{\partial\Omega}{\partial a} \tag{59}$$

We obtain

$$\frac{d\delta\Omega}{dt} = + \frac{n}{\sqrt{1-e^2}\sin I} B \frac{dP}{dI} X_0 \cos W, \tag{60}$$

$$\frac{d\delta I}{dt} = \frac{n}{\sqrt{1-e^2}\sin I} [(p-2q)\cos I + m] BPX \sin W, \tag{61}$$

$$\frac{d\delta\omega}{dt} = n\left(-\frac{\cos I}{\sqrt{1-e^2}\sin I} X \frac{dP}{dI} + \frac{\sqrt{1-e^2}}{e} P \frac{dX}{de}\right) B\cos W \tag{62}$$

$$\frac{d\delta e}{dt} = -\frac{n\sqrt{1-e^2}}{e}(p-2q) BPX \sin W, \tag{63}$$

$$\frac{d\delta M}{dt} = n\left[-\frac{1-e^2}{e}\frac{dX}{de} + 2(p+1)X\right] BP\cos W \tag{64}$$

Computation of generalized associate Legendre functions $P^{\ell}_{mn}(\cos I)$ and of their derivatives with respect to I can be accomplished using numerous recursive relations. The main relations are (Vilenkin, 1965):

$$\begin{aligned}
&[(\ell+n)(\ell-n+1)]^{1/2} P^{\ell}_{m,n-1}(\cos I) \\
&\quad + \frac{2}{\sin I}(n\cos I - m) P^{\ell}_{mn}(\cos I) \\
&\quad + [(\ell-n)(\ell+n+1)]^{1/2} P^{\ell}_{m,n+1}(\cos I) = 0,
\end{aligned} \tag{65}$$

$$\begin{aligned}
\frac{dP^{\ell}_{mn}(\cos I)}{dI} &= + \frac{n\cos I - m}{\sin I} P^{\ell}_{mn}(\cos I) \\
&\quad + [(\ell-n)(\ell+n+1)]^{1/2} P^{\ell}_{m,n+1}(\cos I).
\end{aligned} \tag{66}$$

$$\frac{dP^{\ell}_{mn}(\cos I)}{dI} = -\frac{n\cos I - m}{\sin I} P^{\ell}_{mn}(\cos I) \tag{67}$$

$$- [(\ell+n)(\ell-n+1)]^{1/2} P^{\ell}_{m,n-1}(\cos I).$$

In addition:

$$P^{\ell}_{\ell n}(\cos I) = (-1)^{\ell}\sqrt{\frac{(2\ell)!}{(\ell-n)!\,(\ell+n)!}}\sin^{\ell-n}\frac{I}{2}\cos^{\ell+n}\frac{I}{2}, \tag{68}$$

in particular

$$P^{\ell}_{\ell\ell}(\cos I) = (-1)^{\ell}\cos^{2\ell}\frac{I}{2}, \tag{69}$$

$$P^{\ell}_{\ell,-\ell}(\cos I) = (-1)^{\ell}\sin^{2\ell}\frac{I}{2} \tag{70}$$

$$P^{\ell}_{\ell 0}(\cos I) = (-1)^{\ell}\frac{\sqrt{(2\ell)!}}{2^{\ell}\ell!}\sin^{\ell} I, \tag{71}$$

and

$$P^{\ell}_{m,-n}(\cos I) = (-1)^{m-n} P^{\ell}_{-m,n}(\cos I), \tag{72}$$

$$P^{\ell}_{mn}(\cos I) = (-1)^{m+n} P^{\ell}_{-m,-n}(\cos I), \tag{73}$$

$$P^{\ell}_{mn}(\cos I) = P^{\ell}_{nm}(\cos I), \tag{74}$$

$$P^{\ell}_{m,0}(\cos I) = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P^{m}_{\ell}(\cos I), \tag{75}$$

$$P^{\ell}_{mn}(-\cos I) = (-1)^{\ell-m} P_{m,-n}(\cos I). \tag{76}$$

CONCLUSION

Balmino analysis (1974) points out the existence of the influence of the lateral inhomogeneities in the structure of the Earth on the tidal perturbations in the motion of a satellite. The same conclusion follows from the investigations by Douglas et al (1974). In our opinion, in addition to analysis of perturbations of the satellite, a future planning should include the tidal observations on the Earth surface. With them it would be easier to obtain a proper value for the exterior tidal potential. The present theory points out the existence of several tidal long period and "cross effects" in the coefficients in the expansion of geopotential and in the motion of satellites. How long can we continue without including these effects and what are in fact the "average" elastic parameters which are being presently used to represent the observations of satellites? These questions constitute topics for a future research.

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