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ANALYTICAL METHOD FOR DETERMINING THE STABILITY OF LINEAR RETARDED SYSTEMS WITH TWO DELAYS

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16. Abstract

The stability of the solution of differential-difference equations of the retarded type with constant coefficients and two constant time delays is considered. A new method that makes use of analytical expressions to determine stability boundaries, and hence the stability of the system, is derived. The method is applied to a system represented by a second-order differential equation with constant coefficients and time delays in the velocity and displacement terms. The results obtained are in agreement with those obtained by other investigators.

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SUMMARY

Differential-difference equations occur in various branches of science. The stability of these equations, however, is much more difficult to determine than that of ordinary differential equations. In this paper the stability of differential-difference equations of the retarded type with constant coefficients and two constant time delays is considered. A new method that makes use of analytical expressions to determine stability boundaries, and hence the stability of the equations, is derived. The basis of the method consists in deriving analytical equations for each of the delays which correspond to the purely imaginary roots of the characteristic quasi-polynomial.

The method developed is used to analyze the stability of a second-order differential equation with delays in the velocity and displacement terms. The resulting stability regions are in agreement with those obtained by other investigators.

INTRODUCTION

A differential-difference equation of the retarded type is basically a differential equation in which the highest order derivative of the dependent variable contains no delay in its argument (time), but any of its other derivatives or the dependent variable itself may have delays. Such equations arise when the future state of a system depends not only on its present state but also on part of its past history. (See refs. 1 to 5.)

One of the major obstacles encountered in dealing with systems with delays is the stability analysis. Moreover, it appears that practical applications of such systems have been limited thus far to one and two delays. A convenient method is developed in reference 6 to examine the stability in the delay space for a linear, time invariant differential-difference equation of the retarded type with one constant delay. This method, which is somewhat similar to the method of reference 7, is extended in reference 8 to many constant delays. In particular, stability boundaries in the delay space for a retarded system with two delays can be constructed by using the method of reference 8. However, the points on the boundaries are found by an indirect procedure which involves searching for the roots of transcendental equations. (This is not the case for one delay.)

The purpose of the present study is to develop explicit analytical equations for constructing stability boundaries in the delay space for retarded systems with two constant delays. This report is based on work that has been completed as part of L. Keith Barker's doctoral research program with The George Washington University.

SYMBOLS

A, A_l	$N \times N$ matrices of real constants
a,b,c	real constants
$a_{11}, a_{12}, a_{21}, a_{22}$	real constants
B	$N \times m$ matrix of real constants
$b_{11}, b_{12}, b_{21}, b_{22}$	real constants
C	positive scalar function of ω (see eq. (43))
I	$N \times N$ identity matrix
i	imaginary unit, $\sqrt{-1}$
j,k,l,m,n	integers
K_j	integer associated with θ_j
$L(s), L(s,\theta_j)$	characteristic quasi-polynomial of retarded system
$L_0(s)$	resulting polynomial when delays are made zero in $L(s)$
M	number of delays in system

N	dimension of system
$P(s),Q(s),R(s)$	polynomials in s with real constant coefficients
q,r	angles defined in figure 1
s	complex variable, $\sigma + i\omega$
s^*	root of $L(s,\theta_j)$
t	time or independent variable
$\tilde{t}$	nondimensional time
$\vec{u}$	$m \times 1$ vector forcing function
V_j	integer associated with θ_j
x	scalar function of time
$\vec{x}$	$N \times 1$ state vector
y_1,y_2	defined by equations (14) and (15), respectively
α	argument of $Q(i\omega)\, e^{-i\omega\theta_1}$ (see fig. (1))
β	argument of $P(i\omega)$ (see fig. (1))
γ	argument of $R(i\omega)\, e^{-i\omega\theta_2}$ (see fig. (1))
δ,ϵ	small positive numbers
ζ	damping parameter
θ,θ_1,θ_2	real and constant delays
ρ	radius of small circular contour around any root of $L(s,\theta_j)$

σ real part of complex variable s

$\tau, \tau_1, \tau_2, \tau_l$ real and constant time delays

ω imaginary part of complex variable s

ω_m largest positive real root of equation (20)

Mathematical notations:

$| \; |$ absolute values or magnitude

arg argument

det determinant

min minimum value

Δ small incremental value

Dots over a symbol denote derivatives with respect to time.

ANALYSIS

Retarded Systems

The state equation for a physical system is sometimes written in the form

$$\dot{\vec{x}}(t) = A\,\vec{x}(t) + \sum_{l=1}^{M} A_l\,\vec{x}(t - \tau_l) + B\,\vec{u}(t) \tag{1}$$

where $\vec{x}(t)$ is an $N \times 1$ state vector, $\vec{u}(t)$ is an $m \times 1$ input vector, A and A_l are $N \times N$ constant matrices, B is an $N \times m$ constant matrix, and $\tau_l \geqq 0$ is a constant time delay. Equation (1) is a linear differential-difference equation of the retarded type with constant coefficients and delays. For brevity, equation (1) is referred to as a retarded system. Mathematical treatments of retarded systems can be found in references 1, 9, and 10, for example.

Equation (1) is said to be stable if and only if the corresponding homogeneous equation

$$\dot{\vec{x}}(t) = A\,\vec{x}(t) + \sum_{l=1}^{M} A_l\,\vec{x}(t - \tau_l) \tag{2}$$

is stable; that is, if all the roots $s = \sigma + i\omega$ of the characteristic quasi-polynomial

$$L(s) = \det\left(sI - A - \sum_{l=1}^{M} A_l\,e^{-\tau_l s}\right) \tag{3}$$

have negative real parts. In this case, the solution to the homogeneous equation (2) approaches zero as $t \to \infty$ (ref. 1).

Techniques of generalized harmonic analyses can be applied meaningfully to the retarded system of equation (1) when equation (2) is stable (ref. 11). The stability of equation (2) is also inherently related to the asymptotic stability of some nonlinear control problems (ref. 10). Various methods have been proposed for determining whether any of the roots have nonnegative real parts ($\sigma \geqq 0$); for example, see references 8, 12, 13, and 14.

Class of Retarded Systems

Stability boundaries are examined for the class of retarded systems which has a characteristic quasi-polynomial of the form:

$$L(s) = P(s) + Q(s)\,e^{-\theta_1 s} + R(s)\,e^{-\theta_2 s} \tag{4}$$

where

$$P(s) = \sum_{j=0}^{N} a_j s^j \qquad \left(a_N \neq 0\right)$$

$$Q(s) = \sum_{j=0}^{N-1} b_j s^j$$

$$R(s) = \sum_{j=0}^{N-1} c_j s^j$$

The real nonnegative constants θ_1 and θ_2 will be referred to as delays although they may be some linear combination of the actual time delays τ_l in equation (1). Some examples of retarded systems with characteristic quasi-polynomials of the form of equation (4) are given in appendix A.

The stability condition (stable or unstable) of a retarded system with characteristic quasi-polynomial equation (4) is the same as the same system with zero delays $(\theta_1 = \theta_2 = 0)$, if the values of θ_1 and θ_2 in equation (4) are sufficiently small. This initial stability condition is determined by setting $\theta_1 = \theta_2 = 0$ in equation (4) and then examining the roots of the resulting polynomial

$$L_0(s) = P(s) + Q(s) + R(s) \tag{5}$$

If the degree of the polynomial equation (5) is N and if the delays θ_1 and θ_2 are sufficiently small, then the quasi-polynomial equation (4) has N roots very close to the N roots of equation (5), and the remaining infinity of roots have very large negative real parts (refs. 8 and 13). Additional information on the roots of quasi-polynomials is contained in reference 4 and 15.

As the delays are varied in some continuous manner from essentially zero, the roots of the quasi-polynomial equation (4) move continuously and generate an infinite number of continuous root-locus curves in the complex root plane; that is, in the $\sigma\omega$-plane, $s = \sigma + i\omega$ satisfies $L(s) = 0$. (See appendix B.) Clearly, it is impossible to plot all the root-locus curves of equation (4) for a given continuous variation of the delays. It is possible, however, to determine the number of roots with positive real parts as the delays are varied by examining the behavior of the root-locus curves on the imaginary axis. This approach is used in reference 8.

Partitioning Delay Space Into Stable or Unstable Regions

Root-locus curves are generated by the roots of equation (4) as the delays are varied. If a root-locus curve comes into contact with, or crosses, the imaginary axis at ω (touch point), then by definition, $s = i\omega$ satisfies

$$L(i\omega) = P(i\omega) + Q(i\omega)\, e^{-i\omega\theta_1} + R(i\omega)\, e^{-i\omega\theta_2} = 0 \tag{6}$$

Since the root-locus curves are symmetrical about the real axis, only $\omega \geqq 0$ values are considered.

Equation (6) can be considered as two equations (real and imaginary parts) in two unknowns. The two unknowns which are chosen vary with the different methods which have been used to examine the stability of systems with delays. For example, in reference 6, $\theta_2 = 0$ and the two unknowns are ω and θ_1; in Neimark's method (ref. 14), the delays are held fixed, and the two coefficients are chosen as the unknowns with ω as a coordinating parameter; in reference 16, the unknowns are a delay and a coefficient or gain. In the present study, the two unknowns are θ_1 and θ_2 with ω as a coordinating parameter.

A vector representation of the complex quantities appearing in equation (6) is shown in figure 1. The two distinct solution sets to equation (6) are shown graphically in figure 2. From the geometric properties of a triangle, it follows that a solution to equation (6) exists if and only if the following three relationships simultaneously hold:

$$|P(i\omega)| + |Q(i\omega)| \geqq |R(i\omega)| \tag{7}$$

$$|P(i\omega)| + |R(i\omega)| \geqq |Q(i\omega)| \tag{8}$$

$$|R(i\omega)| + |Q(i\omega)| \geqq |P(i\omega)| \tag{9}$$

These relations express the fact that the sum of the lengths of any two sides of a triangle must be greater than or equal to the length of the remaining side. An equality sign in either of equations (7), (8), or (9) corresponds to collinear vectors.

It follows by using figure 1 that the angles r and q in figure 2(a) are given by

$$r = \pi - \beta + \alpha$$

$$r = \pi - \arg P(i\omega) + \arg Q(i\omega) - \omega\theta_1 \tag{10}$$

and

$$q = \pi - \beta + \gamma$$

$$q = \pi - \arg P(i\omega) + \arg R(i\omega) - \omega\theta_2 \tag{11}$$

Now, solving equation (10) for θ_1 and equation (11) for θ_2 gives

$$\theta_1 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg Q(i\omega) - r\right] \tag{12}$$

and

$$\theta_2 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg R(i\omega) - q\right] \tag{13}$$

The angles r and q are obtained from figure 2(a) by applying the law of cosines as

$$\cos r = \frac{|P(i\omega)|^2 + |Q(i\omega)|^2 - |R(i\omega)|^2}{2|P(i\omega)||Q(i\omega)|} \equiv y_1 \tag{14}$$

and

$$\cos(-q) = \frac{|P(i\omega)|^2 + |R(i\omega)|^2 - |Q(i\omega)|^2}{2|P(i\omega)||R(i\omega)|} \equiv y_2 \tag{15}$$

Choosing $0 \leqq \cos^{-1} y_j \leqq \pi$ and using the geometry in figure 2(a) gives

$$\theta_1 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg Q(i\omega) - \cos^{-1} y_1 + 2\pi K_1\right] \tag{16}$$

and

$$\theta_2 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg R(i\omega) + \cos^{-1} y_2 + 2\pi K_2\right] \tag{17}$$

where K_1 and K_2 are integers.

Using the geometry in figure 2(b) gives

$$\theta_1 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg Q(i\omega) + \cos^{-1} y_1 + 2\pi V_1\right] \tag{18}$$

and

$$\theta_2 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg R(i\omega) - \cos^{-1} y_2 + 2\pi V_2\right] \tag{19}$$

where V_1 and V_2 are integers. A solution to equation (6) exists if and only if θ_1 and θ_2 satisfy the pair of equations (16) and (17) or the pair of equations (18) and (19).

Combination values of θ_1 and θ_2 for which a root-locus curve touches the imaginary axis can be determined by plotting θ_1 and θ_2 against ω. Alternately, θ_2 can be plotted directly against θ_1 for corresponding values of ω. This latter type of figure is actually a partitioning of the delay space into regions of stability (stable or unstable). It should be noted that Neimark's D-partition methods (ref. 14) can be used to construct regions of stability in the plane of two real parameters (gains or coefficients) which occur linearly in the characteristic quasi-polynomial for fixed delays. Other methods, developed within the last decade, are discussed briefly in reference 17.

The stability region for the smallest values of delays may be all that is required to show that the delays are not large enough to make the system unstable or to indicate how much the delays can be increased before the system becomes unstable.

Special Values of ω

There are certain values of ω which are useful in evaluating the equations for θ_1 and θ_2.

Upper bound on ω.- An upper bound on ω can be computed by using equation (6). The dominant power on ω occurs, by definition, in $P(i\omega)$; hence, let $\omega = \omega_m$ be the largest positive real root of the equation

$$|a_N|\omega^N - \sum_{n=0}^{N-1} \left(|a_n| + |b_n| + |c_n|\right)\omega^n = 0 \tag{20}$$

Then, it follows that $\omega \leqq \omega_m$ in equation (6) and in the pertinent pair of equations for θ_1 and θ_2.

Border values of ω.- Partitioning curves are defined only for those values of ω which satisfy equations (7) to (9). These meaningful values of ω are determined by using border values of ω. A border value of ω is defined as a nonnegative real value of ω which satisfies any of the equality relations in equations (7) to (9), which are:

$$|P(i\omega)| + |Q(i\omega)| = |R(i\omega)| \tag{21}$$

$$|P(i\omega)| + |R(i\omega)| = |Q(i\omega)| \tag{22}$$

$$|R(i\omega)| + |Q(i\omega)| = |P(i\omega)| \tag{23}$$

These equality relationships also follow by setting $\cos r = \pm 1$ or $\cos(-q) = \pm 1$ in equation (14) or (15), respectively.

The finite number of border values separates the ω space into different intervals. To determine if a partitioning curve is defined for values of ω in an interval, evaluate $\cos r$ or $\cos(-q)$ at some value of ω within the interval. If, for example, $|\cos r| \leqq 1$, then partitioning curves exist for all values of ω in the interval. On the other hand, if $|\cos r| > 1$, the partitioning curves do not exist for any value of ω in the interval. The maximum border value will be equal to ω_m.

Multiple Touch Points

Multiple touch points occur when more than one root-locus curve touches the imaginary axis for the same values of ω, θ_1, and θ_2. In this case, the points on the partitioning curves correspond to purely imaginary roots of the characteristic quasi-polynomial which have multiplicities greater than unity. Because of its intuitive appeal, a procedure used in reference 8 is adapted later to examine the stability condition of regions established by the partitioning curves. This procedure, however, is presented in reference 8 and in this paper on the assumption that the touch point, where the procedure is applied, is simple. This condition is easily checked by using the partitioning equations.

By construction, $s = i\omega$ is a root of the characteristic quasi-polynomial for any point (θ_1,θ_2) in the delay space which lies on a partitioning curve. The questions is: Is $s = i\omega$ a simple or multiple root?

Differentiating equation (4) with respect to s gives

$$\frac{dL}{ds} = \frac{dP}{ds} + \left(\frac{dQ}{ds} - \theta_1 Q\right) e^{-\theta_1 s} + \left(\frac{dR}{ds} - \theta_2 R\right) e^{-\theta_2 s} \tag{24}$$

The value of dL/ds at a point (θ_1,θ_2) on a partitioning curve is given by equation (24) with $s = i\omega$ and with θ_1 and θ_2 computed by using the partitioning equations. If $dL/ds \neq 0$ at this point, then $s = i\omega$ is a simple root; otherwise, it is not.

Stable and Unstable Regions in Delay Space

The delay space is partitioned into regions which are either stable or unstable by the θ_2 versus θ_1. The stability of each of these regions can be determined by computing the initial stability of the system using equation (5) and evaluating the derivative, on a partitioning curve, of the real part of $s = \sigma + i\omega$ with respect to one of the delays while holding the other delay constant.

Select one of the delays θ_1 or θ_2 to be held fixed. For discussion purposes, denote this delay by θ_l (l = 1 or 2) and the remaining delay by θ_k ($k \neq l$; k = 1 or 2).

It is assumed that the purely imaginary root is simple ($dL/ds \neq 0$) at a point on the partitioning curves. Moreover, suppose that at this point $\partial^n\sigma/\partial\theta_k{}^n = 0$ for $n = 0, 1, \ldots, g - 1$; but that $\partial^g\sigma/\partial\theta_k{}^g \neq 0$. Then, if $\partial^g\sigma/\partial\theta_k{}^g > 0$ and g is an odd integer, the sytem gains a root with positive real part when the point (θ_k,θ_l) is varied from a partitioning curve in the direction of increasing θ_k. Likewise, if $\partial^g\sigma/\partial\theta_k{}^g < 0$ and g is an odd integer, the system loses a root with positive real part. If $\partial^g\sigma/\partial\theta_k{}^g \neq 0$ and g is an even integer, then the root-locus curves are tangent to the imaginary axis, so that the system neither gains nor loses any roots with positive real parts.

For convenience the characteristic quasi-polynomial is written as

$$L = L(s,\theta_l,\theta_k) \tag{25}$$

The characteristic equation is then

$$L = L(s,\theta_l,\theta_k) = 0 \tag{26}$$

Applying implicit differentiation to equation (26) and extracting the real part of $\partial s/\partial\theta_k$ gives

$$\frac{\partial\sigma}{\partial\theta_k} = -R\left(\frac{L_{\theta_k}}{L_s}\right) \tag{27}$$

where the partial derivatives $L_{\theta_k} = \partial L/\partial\theta_k$ and $L_s = \partial L/\partial s$ are obtained by using equation (25) and R denotes the real part of the complex number in parentheses.

The derivative in equation (27) is evaluated on a partitioning curve by setting $s = i\omega$ in equation (27) and using the partitioning equations to obtain θ_l and θ_k. This derivative then gives the change in the real part of the characteristic root when leaving a partitioning curve at (θ_l,θ_k) in the direction of increasing θ_k.

If $\partial\sigma/\partial\theta_k = 0$, then a similar procedure is followed for higher derivatives until $\partial^g\sigma/\partial\theta_k{}^g \neq 0$.

Summary of Stability Procedure

The procedure used to partition the delay space into different regions and to identify the different regions as stable or unstable is summarized in the following steps:

(1) Compute the initial stability of the system by using equation (5).

(2) Compute meaningful values of ω by using equations (21) to (23).

(3) Plot partitioning curves in the delay space by using θ_1 and θ_2 equations as ω varies over the predetermined range of acceptable values in step 2.

(4) If the delays do not cross a partitioning curve (in the plot of step 3) as they are varied in combination from zero to their final desired values, then the retarded system with characteristic quasi-polynomial of the form of equation (4) maintains its initial stability. However, if the delays cannot be varied to their final values without crossing a partitioning curve, compute the multiplicity of the touch points and the derivative of the real part of $s = \sigma + i\omega$ as the curve is crossed. Use equations (24) and (27).

The stability on each side of a partitioning curve is determined by counting the number of roots with positive real parts as the curves are crossed. Stability boundaries are those curves which partition the delay space into stable and unstable regions.

EXAMPLE

To clarify the stability analysis presented in the previous section, the following example is examined:

$$\frac{d^2}{d\tilde{t}^2} x(\tilde{t}) + 2\zeta \frac{d}{d\tilde{t}} x(\tilde{t} - \theta_1) + x(\tilde{t} - \theta_2) = 0 \qquad (28)$$

where $\zeta > 0$ is a real constant damping parameter, and the delays θ_1 and θ_2 are nondimensional. This example is examined in reference 8 in a different manner.

The characteristic quasi-polynomial for equation (28) is

$$L(s) = P(s) + Q(s)\, e^{-\theta_1 s} + R(s)\, e^{-\theta_2 s} \qquad (29)$$

where

$$P(s) = s^2 \qquad Q(s) = 2\zeta s \qquad R(s) = 1$$

The initial stability of the retarded system is found by using equation (5), which is

$$L_0(s) = s^2 + 2\zeta s + 1 \tag{30}$$

Since $\zeta > 0$, the roots of $L_0(s)$ have negative real parts, so that the retarded system is initially stable.

Setting $s = i\omega$ in equation (29) gives

$$P(i\omega) = -\omega^2$$

$$Q(i\omega) = 2\zeta\omega i$$

$$R(i\omega) = 1$$

The first pair of equations for the delays θ_1 and θ_2 (eqs. (16) and (17)) becomes

$$\theta_1 = \frac{1}{\omega}\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{\omega^4 + 4\zeta^2\omega^2 - 1}{4\zeta\omega^3}\right) + 2\pi K_1\right] \tag{31}$$

and

$$\theta_2 = \frac{1}{\omega}\left[\cos^{-1}\left(\frac{\omega^4 - 4\zeta^2\omega^2 + 1}{2\omega^2}\right) + 2\pi K_2\right] \tag{32}$$

The second pair of equations for θ_1 and θ_2 (eqs. (18) and (19)) becomes

$$\theta_1 = \frac{1}{\omega}\left[\frac{\pi}{2} + \cos^{-1}\left(\frac{\omega^4 + 4\zeta^2\omega^2 - 1}{4\zeta\omega^3}\right) + 2\pi V_1\right] \tag{33}$$

and

$$\theta_2 = \frac{1}{\omega}\left[-\cos^{-1}\left(\frac{\omega^4 - 4\zeta^2\omega^2 + 1}{2\omega^2}\right) + 2\pi V_2\right] \tag{34}$$

An Upper Bound and Border Values of ω

An upper bound ω_m.- The upper bound of ω is obtained by using equation (20), which becomes

$$\omega^2 - 2\zeta\omega - 1 = 0 \tag{35}$$

The largest real nonnegative value of ω which satisfies equation (35) is denoted by ω_m. In this case,

$$\omega_m = \zeta + \sqrt{\zeta^2 + 1} \tag{36}$$

Border values of ω.- The border values of ω are obtained by using the relations in equations (21), (22), and (23), which become

$$\omega^2 + 2\zeta\omega - 1 = 0 \tag{37}$$

$$\omega^2 - 2\zeta\omega + 1 = 0 \tag{38}$$

$$\omega^2 - 2\zeta\omega - 1 = 0 \tag{39}$$

Notice that equation (39) is the same as equation (35), so that ω_m is a border value. Other border values are obtained from the solutions of equations (37) and (38) which are, respectively,

$$\omega = -\zeta \pm \sqrt{\zeta^2 + 1} \tag{40}$$

$$\omega = \zeta \pm \sqrt{\zeta^2 - 1} \tag{41}$$

Only the positive radical in equation (40) is of interest since $\omega \geqq 0$. Equation (41) is only of interest when $\zeta \geqq 1$.

Specific Calculations

The border values and upper bound of ω are shown in table I for damping parameter values ζ of 0.2, 0.5, and 1.

TABLE I.- BORDER VALUES AND UPPER BOUND OF ω

Damping parameter, ζ	Border values of ω	Upper bound, ω_m
0.2	0.82	1.22
	1.22	
.5	.62	1.62
	1.62	
1.0	.41	2.41
	1.00	
	2.41	

Figures 3 and 4 show the pairs of the delays θ_1 and θ_2 which result in a root-locus curve coming into contact with, or crossing, the imaginary axis in the complex root plane at ω (touch point) for various values of (K_1,K_2) and (V_1,V_2), respectively. Results are presented for $\zeta = 0.2$, 0.5, and 1.

The consecutive K_j curves in figure 3 and the consecutive V_j curves in figure 4 both differ by $2\pi/\omega$, as shown by equations (31) to (34). Thus, the $V_2 = 0$ curve falls off the scale in figure 4.

The terminal points of the curves in figures 3 and 4 correspond to a zero delay or to a border value in table I. All border values, however, do not necessarily specify a terminal point on the curves. The value $\omega = 1$ is a border value that occurs along the curves in figures 3(c) and 4(c) and is identified by a singularity in the slope of the curves at this value.

Figures 3 and 4 can be used to obtain the touch points ω which occur as the delays are varied in some continuous manner from zero to their final constant values. The touch points clearly depend on the manner in which the delays are varied.

A partitioning of the delay space results when θ_2 is plotted directly against θ_1 with ω as a coordinating parameter, as shown in figure 5. The solid curves correspond to various values of (K_1,K_2) and are generated by using the pair of equations (31) and (32). The dashed curves correspond to various values of (V_1,V_2) and are generated by using the pair of equations (33) and (34). The totality of curves for (K_1,K_2) and (V_1,V_2) partition the delay space into regions. The arrows on the curves denote the direction of increasing ω.

Since the system is initially stable, it will remain stable until the curve $(K_1,K_2) = (0,0)$ in figure 5 is touched. To examine the stability beyond this point requires the determination of the multiplicities of the touch points and the changes in the real parts of the root-locus curves as they pass beyond the touch point. All the regions shown in figure 5 can be entered by crossing one of the solid curves corresponding to (K_1,K_2); hence, only these curves need to be examined further.

It can be shown by setting $s = i\omega$ in equation (24) and using equations (29), (31), and (32) that $dL/ds \neq 0$ on any of the (K_1,K_2) solid curves of figure 5. Hence, the multiplicity of all touch points is unity.

The change in the stability at a touch point is examined by using equation (27) with $\theta_k = \theta_1$ and $\theta_l = \theta_2$. Reference 8 expresses equation (27) for this particular application in a very simple form which is equivalent to

$$C^2 \frac{\partial \sigma}{\partial \theta_1} = \omega^4 + \omega^3 \theta_2 \sin \omega \theta_2 - 1 \tag{42}$$

where

$$C^2 = \frac{1}{4\omega^4} \left| \frac{\partial L(i\omega)}{\partial \theta_1} \right|^2 \tag{43}$$

The scaled derivative $C^2(\partial\sigma/\partial\theta_1)$ has the same sign as $\partial\sigma/\partial\theta_1$. The scaled derivative on the (K_1,K_2) curves in figure 5 is evaluated by using equation (32) to obtain θ_2. Since θ_1 does not appear in equation (42), the scaled derivative is the same for all curves (K_1,K_2) which have the same K_2 value. The scaled derivative is plotted in figure 6 as a function of θ_2 for three values of K_2, which are 0, 1, and 2. The arrows on the curves indicate the direction of increasing ω. The $K_2 = 0$ curve in figure 6 applies to the (0,0), (1,0), (2,0), and (3,0) curves in figure 5; the $K_2 = 1$ curve applies to the (0,1), (1,1), and (2,1) curves; and the $K_2 = 2$ curve applies to the (0,2), (1,2), and (2,2) curves.

Figure 6 is used to determine the stability condition on each side of the partitioning curves in figure 5. The arrows on the curves denote increasing values of ω and are used for corresponding values of the scaled derivative with points on the partitioning curves. For instance, suppose $\theta_2 = 0.2$ on the $(K_1,K_2) = (0,0)$ curve in figure 5, then the corresponding value of scaled derivative in figure 6 is 0.98. This correspondence is established by noting that θ_2 decreases from $\theta_2 = 0.2$ with increasing ω. Now, since $C^2(\partial\sigma/\partial\theta_1) > 0$ on $(K_1,K_2) = (0,0)$ when $\theta_2 = 0.2$, it follows that proceeding off the $(K_1,K_2) = (0,0)$ curve in figure 5, with $\theta_2 = 0.2$, in the direction of θ_1, results in the gain of a root with positive real part. The system, therefore, becomes unstable.

Next, consider the curve $(K_1,K_2) = (1,0)$ in figure 5. It can be shown by using figure 6 that entering the lower triangular region from the left results in the loss of a root with positive real part, so that the system becomes stable again. Upon leaving the triangular region on the right, however, a root with positive real part is gained, and the system becomes unstable.

Entering the upper looped portion of the curve $(K_1,K_2) = (1,0)$ from the left results in the gain of a root with positive real part. So, inside the upper looped region, the system has two roots with positive real parts. Leaving the upper looped portion of the curve on the right results in the loss of a root with positive real part; however, the system had two roots with positive real parts inside the looped portion of the curve, and is, therefore, still unstable.

This type of reasoning is repeated for the other regions of figure 5. It is found that there are only two stable regions in figure 5. These are the initial stability region bounded by the curve $(K_1,K_2) = (0,0)$ and the lower triangular region of the $(K_1,K_2) = (1,0)$ curve.

Figure 7 shows the stability boundaries for the retarded system represented by equation (28) when $\zeta = 0.2$, 0.5, and 1. The hatched lines indicate the stable side of the boundaries. These results are in agreement with those of reference 8.

It is well known that the solution of equation (28) becomes more highly damped as ζ increases if $\theta_1 = \theta_2 = 0$. If $\theta_1 \neq 0$, this may not be the case, as shown in figure 7. For example, let $\theta_2 = 0$ and $\theta_1 = 0.8$. Then, the solution of equation (28) is stable for $\zeta = 0.2$ and unstable for $\zeta = 1$.

CONCLUDING REMARKS

A new method is developed for generating stability boundaries for differential-difference equations of the retarded type with constant coefficients and two constant delays. The basis of the method consists in deriving analytical equations for each of the delays which correspond to the purely imaginary roots of the characteristic quasi-polynomial. These analytical equations are then used to partition the delay space into regions which are stable or unstable.

Each point on a partitioning curve corresponds to a root-locus curve touching or crossing the imaginary axis in the complex root plane. The stability of the different regions, therefore, is examined by computing the initial stability of the equations (zero delays) and counting the number of root-locus curves with positive real parts as the different regions are entered. The analytical equations mentioned previously are used in counting the number of root-locus curves with positive real parts. Specifically, the analytical equations are used in computing the multiplicities of the root-locus curves and the partial derivative of the real part of the root-locus curves with respect to one of the delays.

The method developed is used to analyze the stability of a second-order differential equation with delays in the velocity and displacement terms. The resulting stability regions are in agreement with those obtained by other investigators.

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September 12, 1975

APPENDIX A

EXAMPLES OF RETARDED SYSTEMS WITH TWO DELAYS

The following examples show retarded systems which have characteristic quasi-polynomial equations of the form of equation (4):

Example 1: Consider the following scalar differential equation with two constant real time delays $\tau_1 \geqq 0$ and $\tau_2 \geqq 0$:

$$\sum_{n=0}^{N} a_n x^{(N)}(t) + \sum_{n=0}^{N-1} b_n x^{(n)}(t - \tau_1) + \sum_{n=0}^{N-1} c_n x^{(n)}(t - \tau_2) = f(t) \tag{A1}$$

where $x^{(n)}(t)$ denotes the nth derivative of $x(t)$ and $f(t)$ is a continuous input function of time. The coefficients a_n, b_n, and c_n are real constants. Equation (A1) can be written in the form of equation (1); however, this is unnecessary in obtaining the characteristic quasi-polynomial. The characteristic quasi-polynomial associated with equation (A1) is equation (4) with

$$P(s) = \sum_{n=0}^{N} a_n s^n$$

$$Q(s) = \sum_{n=0}^{N-1} b_n s^n$$

$$R(s) = \sum_{n=0}^{N-1} c_n s^n$$

$$\theta_1 = \tau_1$$

$$\theta_2 = \tau_2$$

Example 2: Let the retarded system be described by two coupled linear equations with one distinct delay as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_1(t - \tau) \\ x_2(t - \tau) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{A2}$$

The characteristic quasi-polynomial associated with equation (A2) is given again by equation (4) with

$$P(s) = s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{21}a_{12}$$

$$Q(s) = -(b_{11} + b_{22})s + a_{11}b_{22} + a_{22}b_{11} - a_{21}b_{12} - a_{12}b_{21}$$

$$R(s) = b_{11}b_{22} - b_{21}b_{12}$$

$$\theta_1 = \tau$$

$$\theta_2 = 2\tau$$

Note that although there is only one time delay τ in equation (A2), there are two delays θ_1 and θ_2 in the associated characteristic quasi-polynomial.

An equation such as equation (A2) occurs in examining airplane stability for the controls-fixed case of reference 5 which is modeled in reference 11. Here, τ accounts for the fact that a vertical gust affects the horizontal tail later than it does the wing. It can be shown also that the characteristic quasi-polynomial for the controls-free case of reference 5 has the form of equation (4) with $\theta_1 = \tau$ and $\theta_2 = 2\tau$.

Example 3: A characteristic quasi-polynomial which occurs in epidemics (ref. 18) is given by equation (4) with

$$P(s) = \tau_2(s + a) \qquad \theta_1 = \tau_1$$

$$Q(s) = -1 \qquad \theta_2 = \tau_1 + \tau_2$$

$$R(s) = 1$$

APPENDIX B

ROOTS OF QUASI-POLYNOMIAL AS CONTINUOUS FUNCTIONS OF DELAYS

Theorem: The roots of the quasi-polynomial $L(s,\theta_j)$, where $j = 1, 2, \ldots, N$, are continuous functions of θ_j.

Proof: The quasi-polynomial $L(s,\theta_j)$ is an analytic function, so that its roots are isolated. Let s^* be any one of the roots of $L(s,\theta_j)$ and let ρ be a positive number such that the only root of $L(s,\theta_j)$ contained inside the contour $|s - s^*| = \rho$ is s^*. A Taylor series expansion of the exponential term $e^{-\theta_j s}$ can be used to show that for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$\left|L(s,\theta_j) - L(s,\theta_j + \Delta\theta_j)\right| < \epsilon$$

for $|s - s^*| \leqq \rho$ whenever $\Delta\theta_j < \delta(\epsilon)$. Choose $\epsilon < \min\left|L(s,\theta_j)\right|$ on $|s - s^*| = \rho$ then

$$\left|L(s,\theta_j) - L(s,\theta_j + \Delta\theta_j)\right| < \epsilon < \left|L(s,\theta_j)\right|$$

for $\Delta\theta_j$ sufficiently small and $|s - s^*| \leqq \rho$. By Rouche's theorem (ref. 19), $L(s,\theta_j)$ and $L(s,\theta_j + \Delta\theta_j)$ have the same number of roots inside the circle $|s - s^*| = \rho$. Since ρ can be made arbitrarily small, it follows that the roots of $L(s,\theta_j)$ must move continuously with θ_j.

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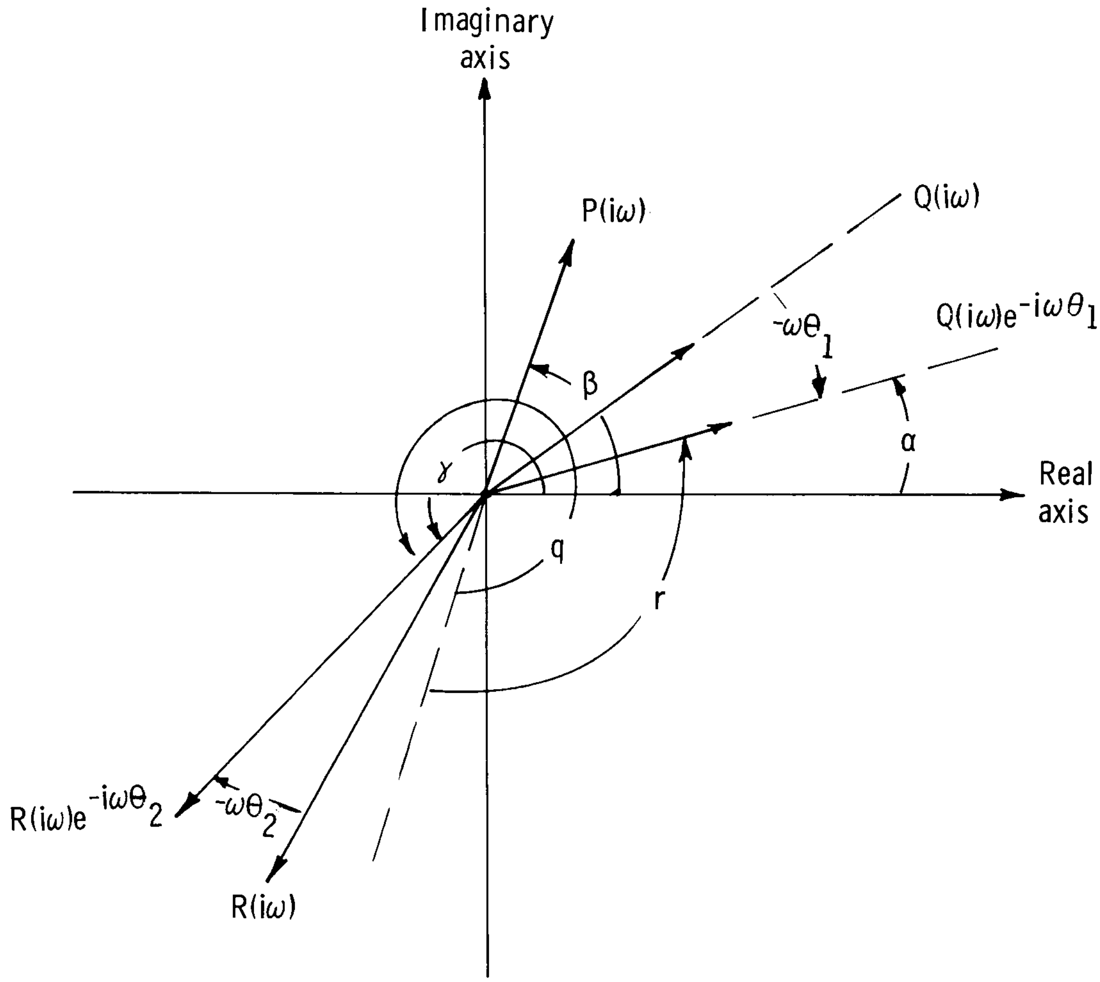
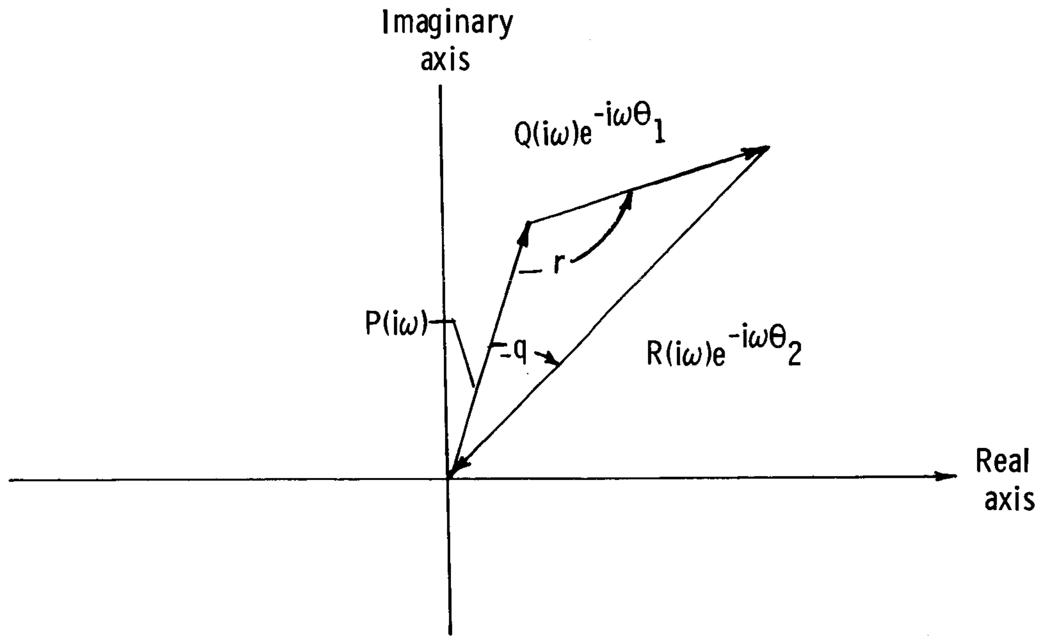
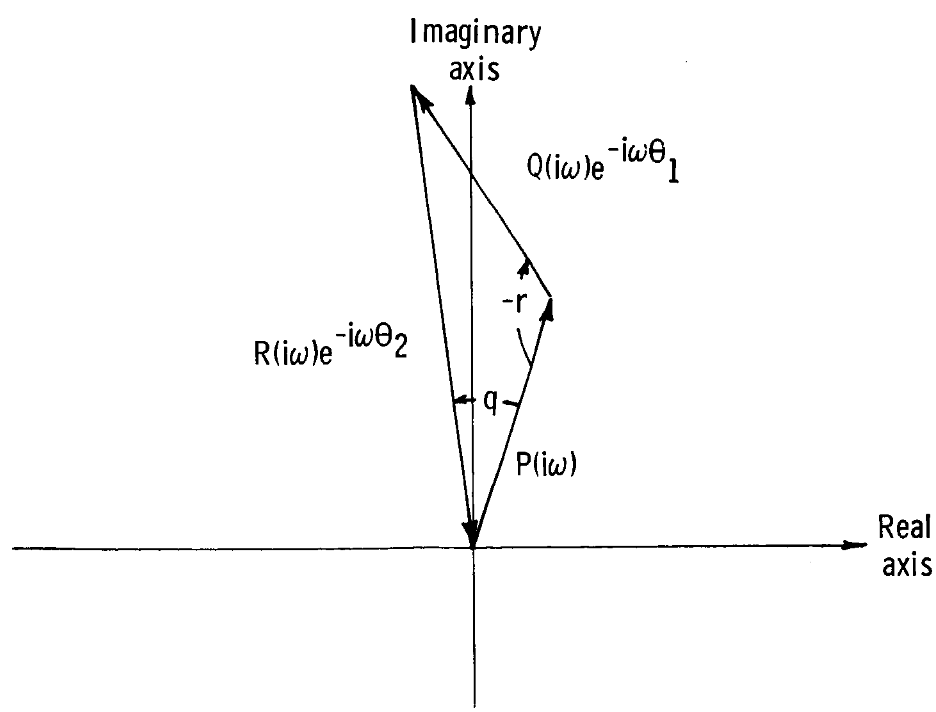


Figure 1.- Complex quantities and angles. Angles are measured positive in the counterclockwise direction.

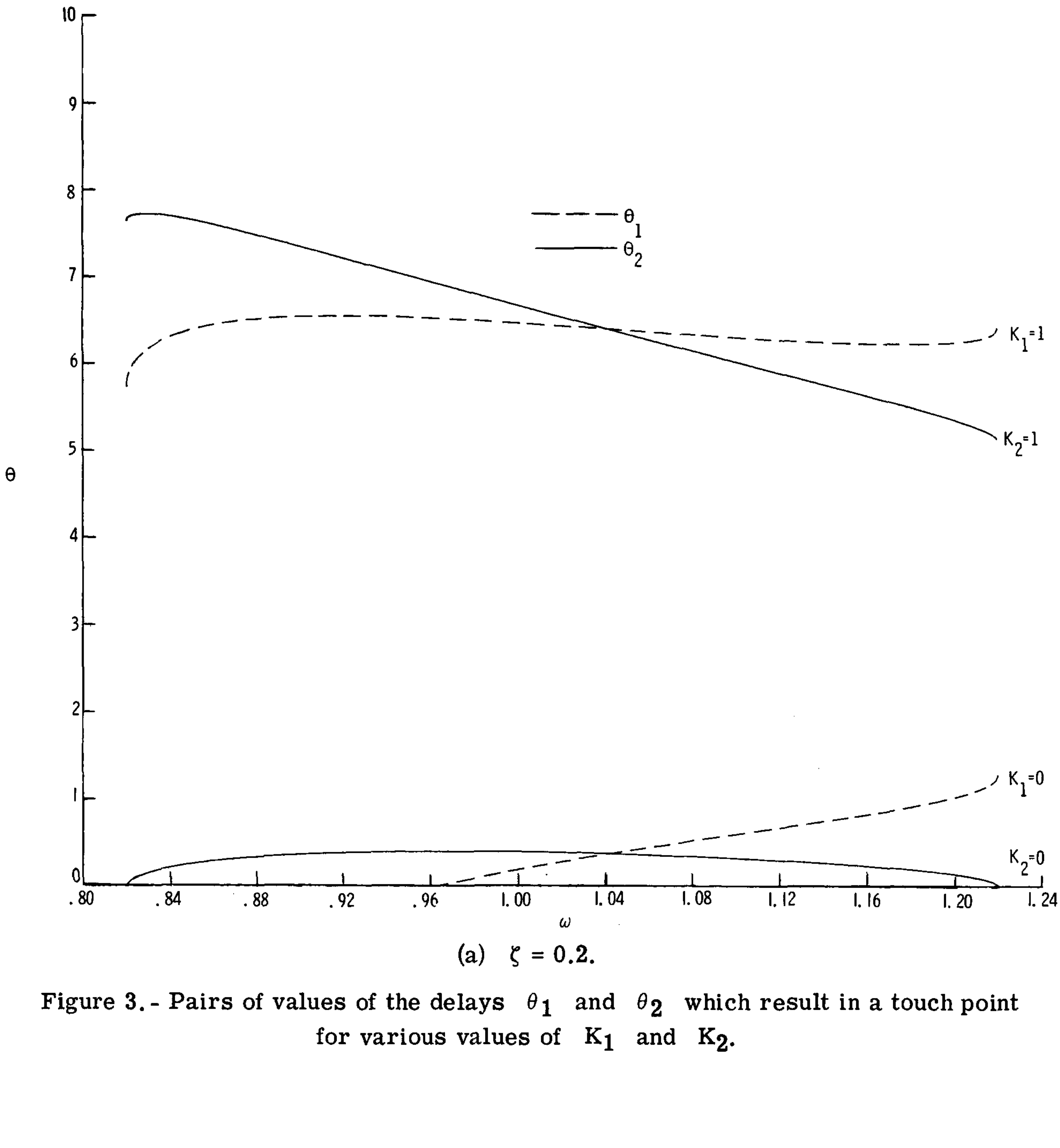


(a) $r \leqq \pi$.



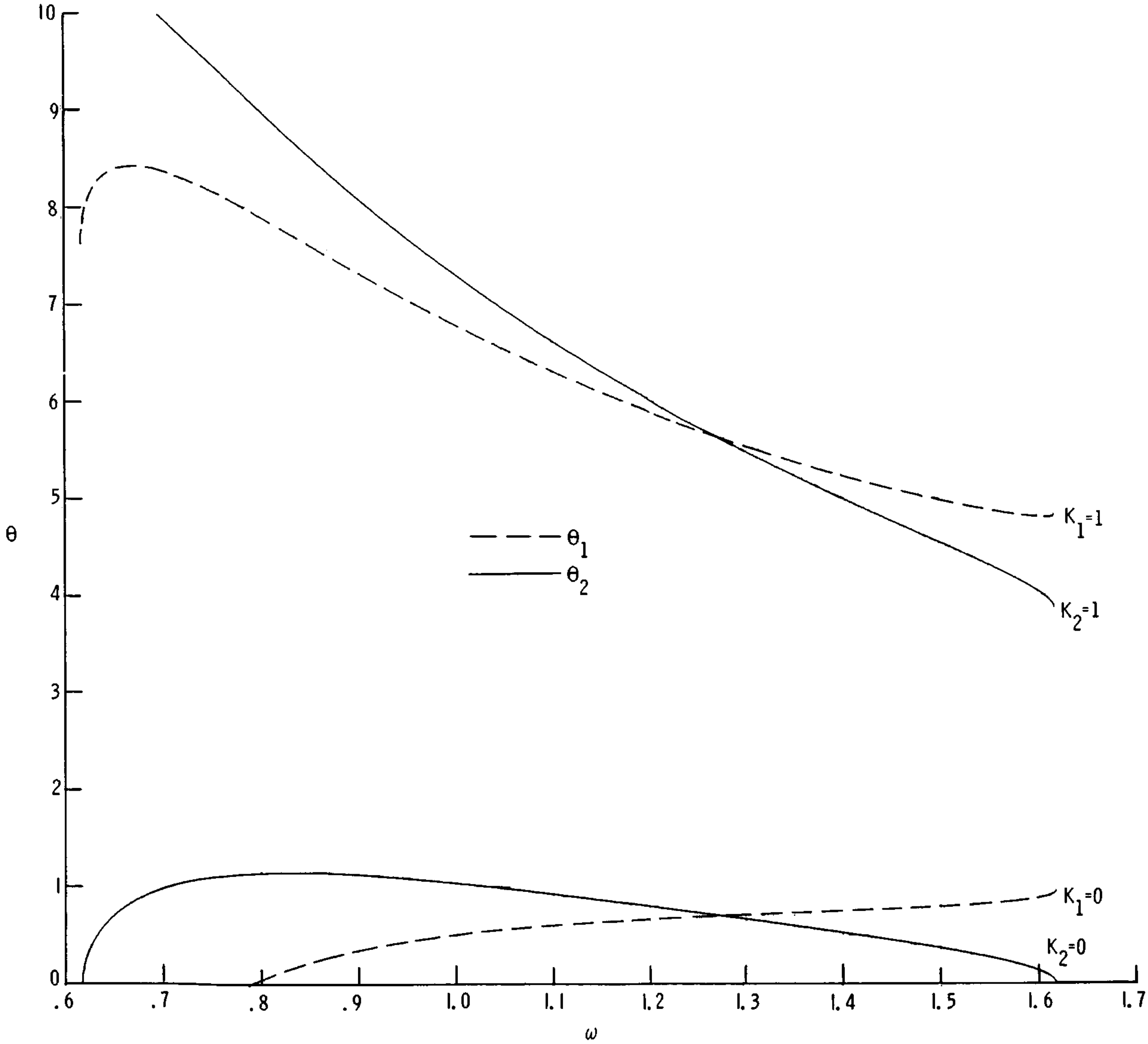
(b) $r \geqq \pi$.

Figure 2.- Graphical representation of the two distinct solution sets for equation (6).



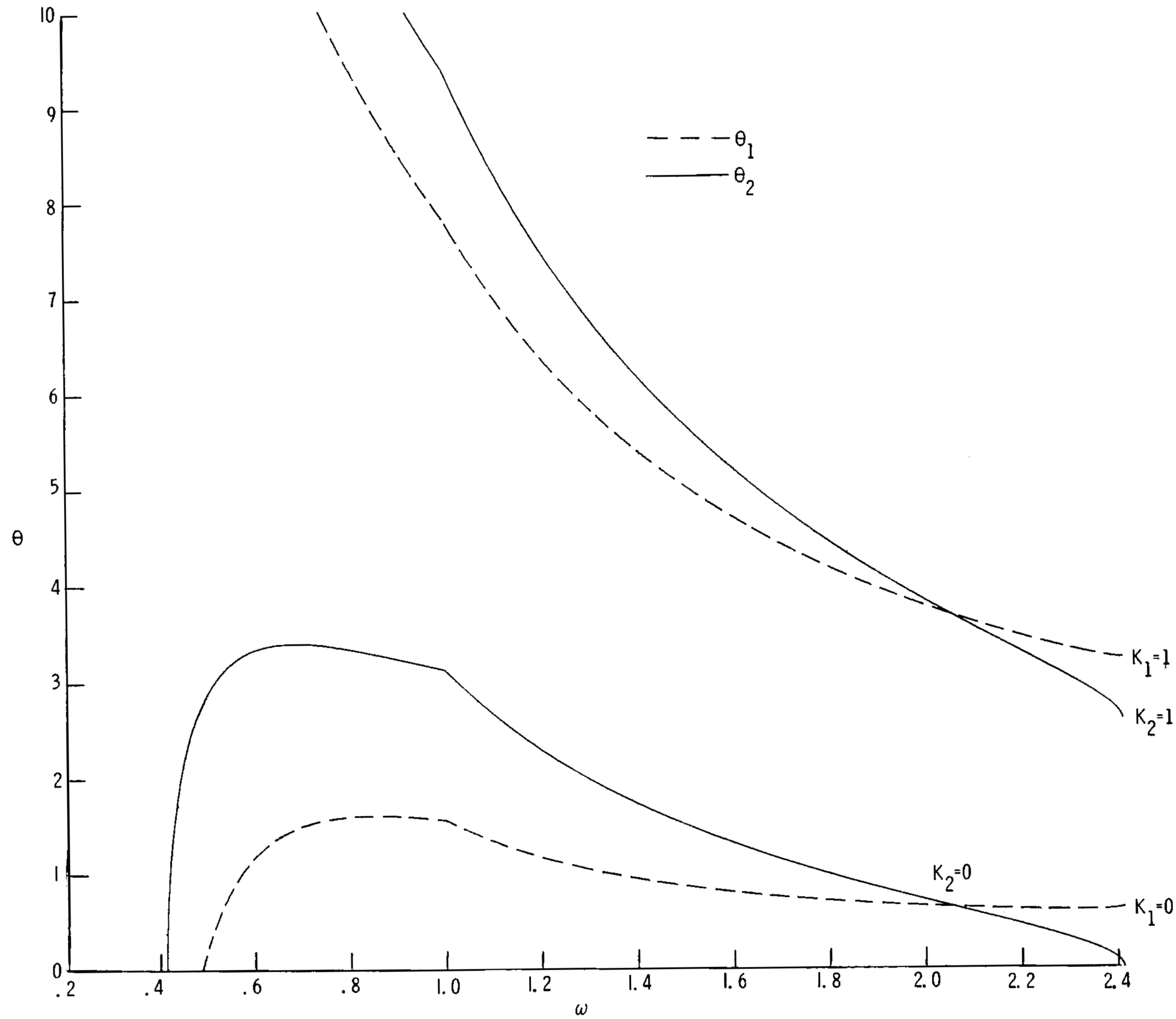
(a) $\zeta = 0.2$.

Figure 3.- Pairs of values of the delays θ_1 and θ_2 which result in a touch point for various values of K_1 and K_2.



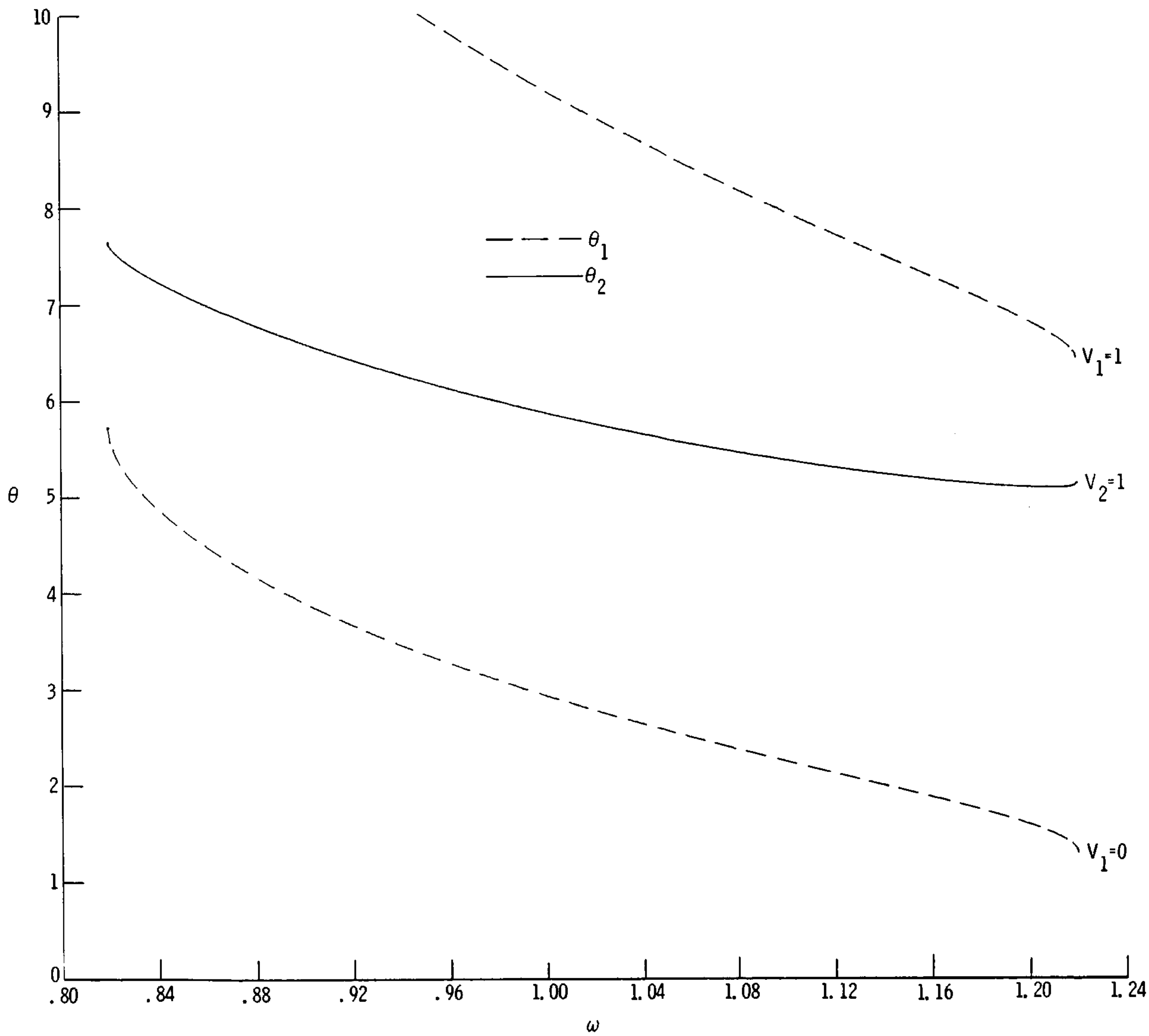
(b) $\zeta = 0.5$.

Figure 3.- Continued.



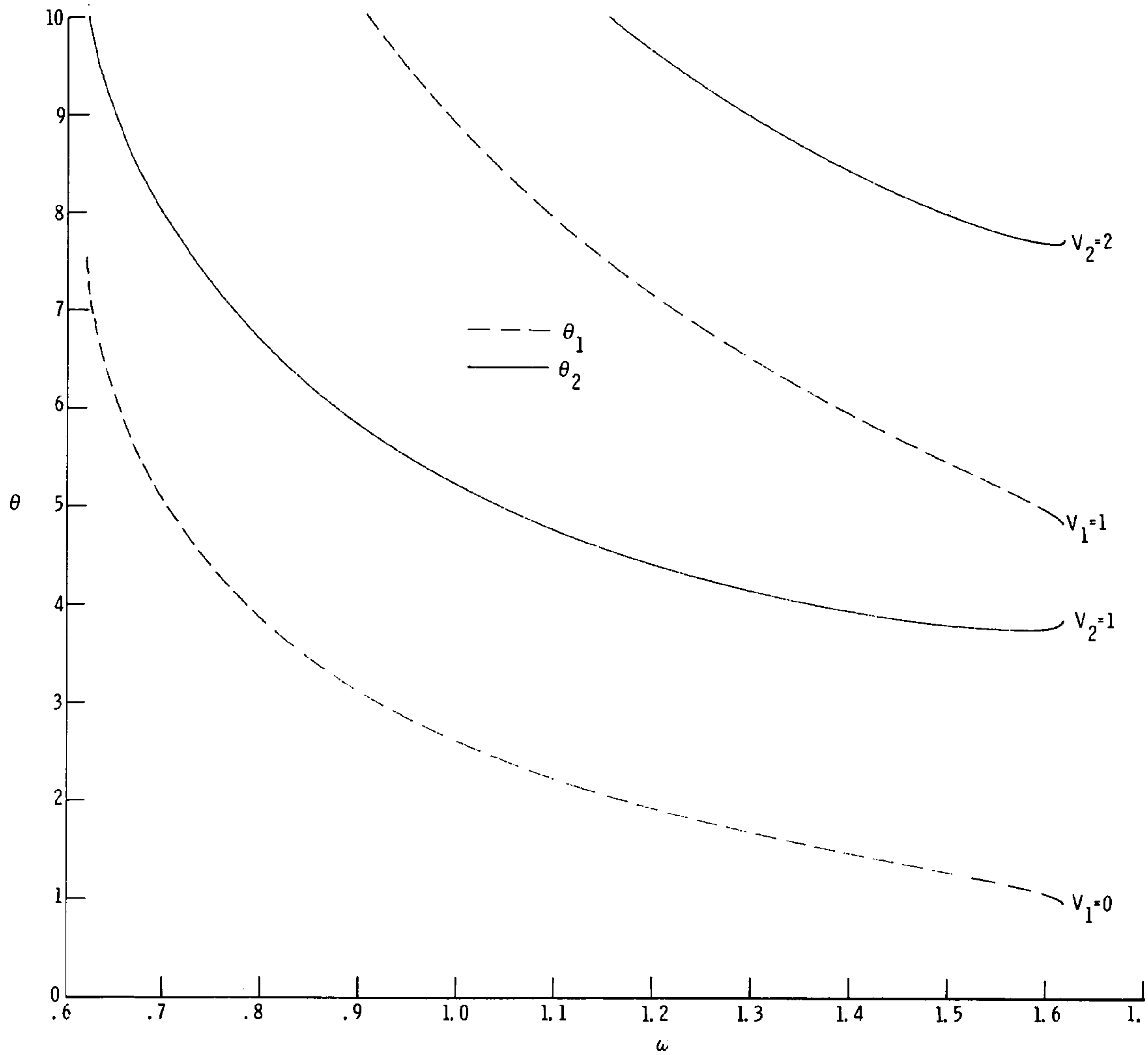
(c) $\zeta = 1.$

Figure 3.- Concluded.



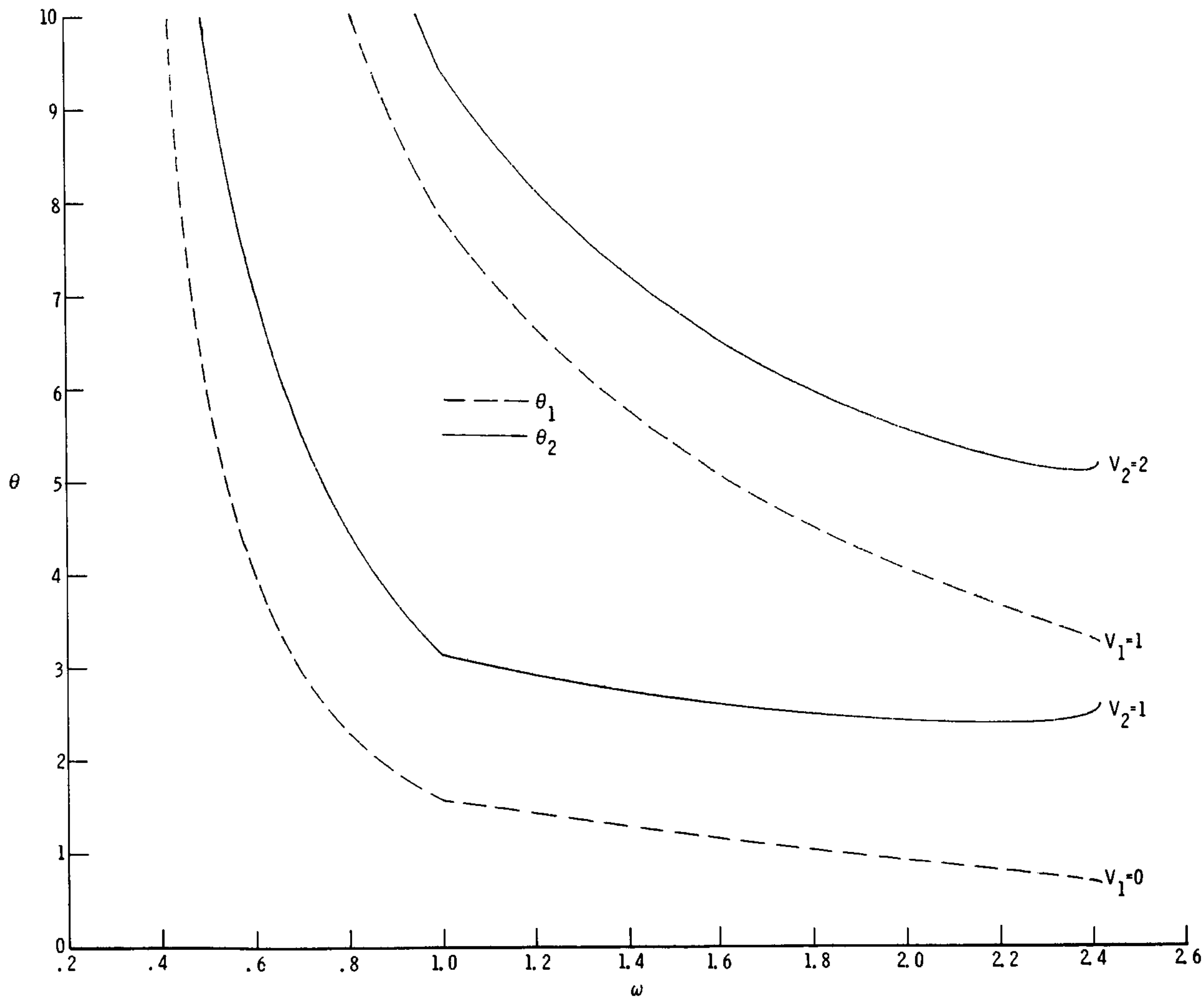
(a) $\zeta = 0.2$.

Figure 4.- Pairs of values of the delays θ_1 and θ_2 which result in a touch point for various values of V_1 and V_2.



(b) $\zeta = 0.5$.

Figure 4.- Continued.



(c) $\zeta = 1$.

Figure 4.- Concluded.

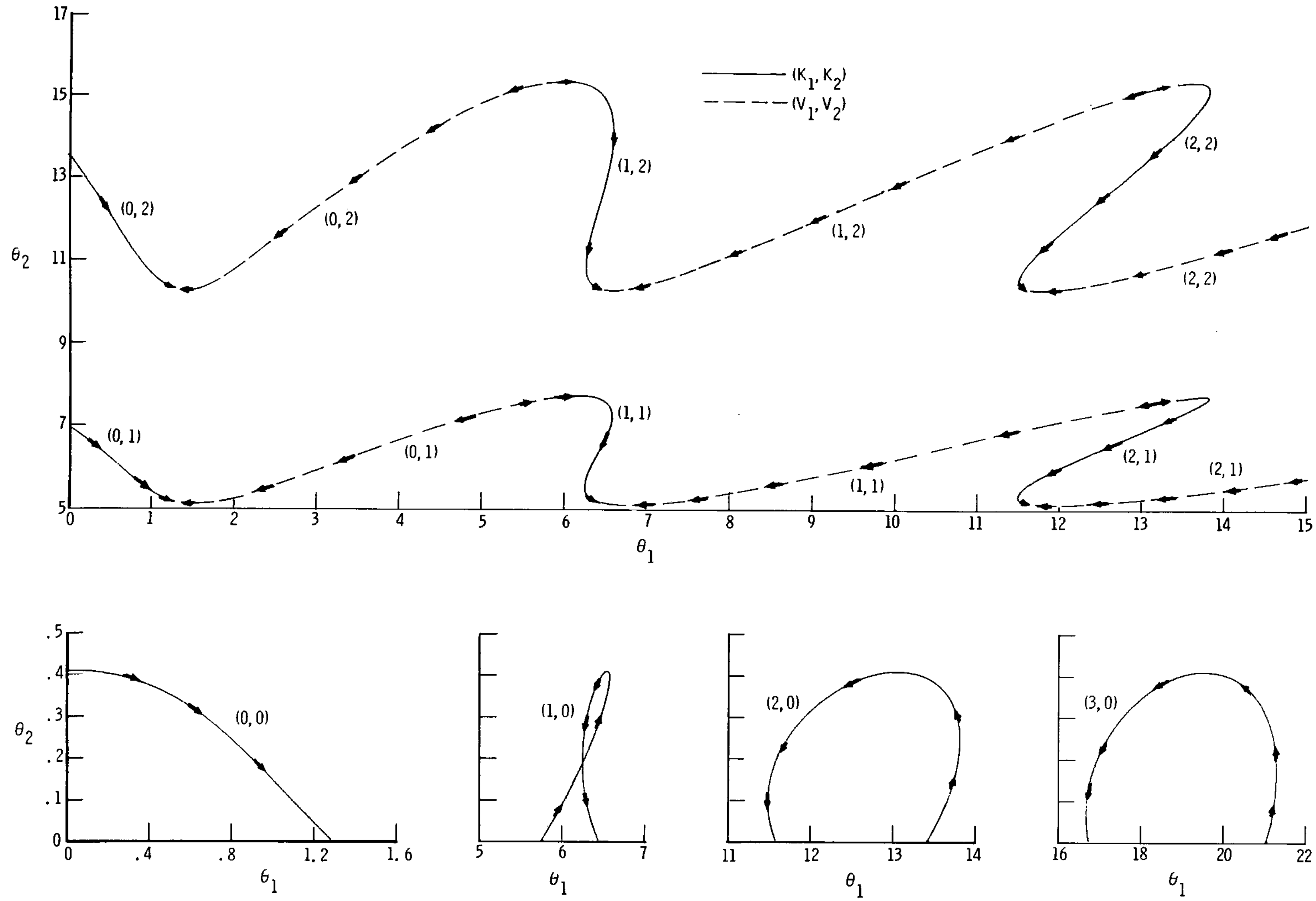


Figure 5.- A partitioning of the delay space by curves generated in the delay space for $\zeta = 0.2$ and various values of (K_1,K_2) and (V_1,V_2). Arrows indicate direction of increasing ω.

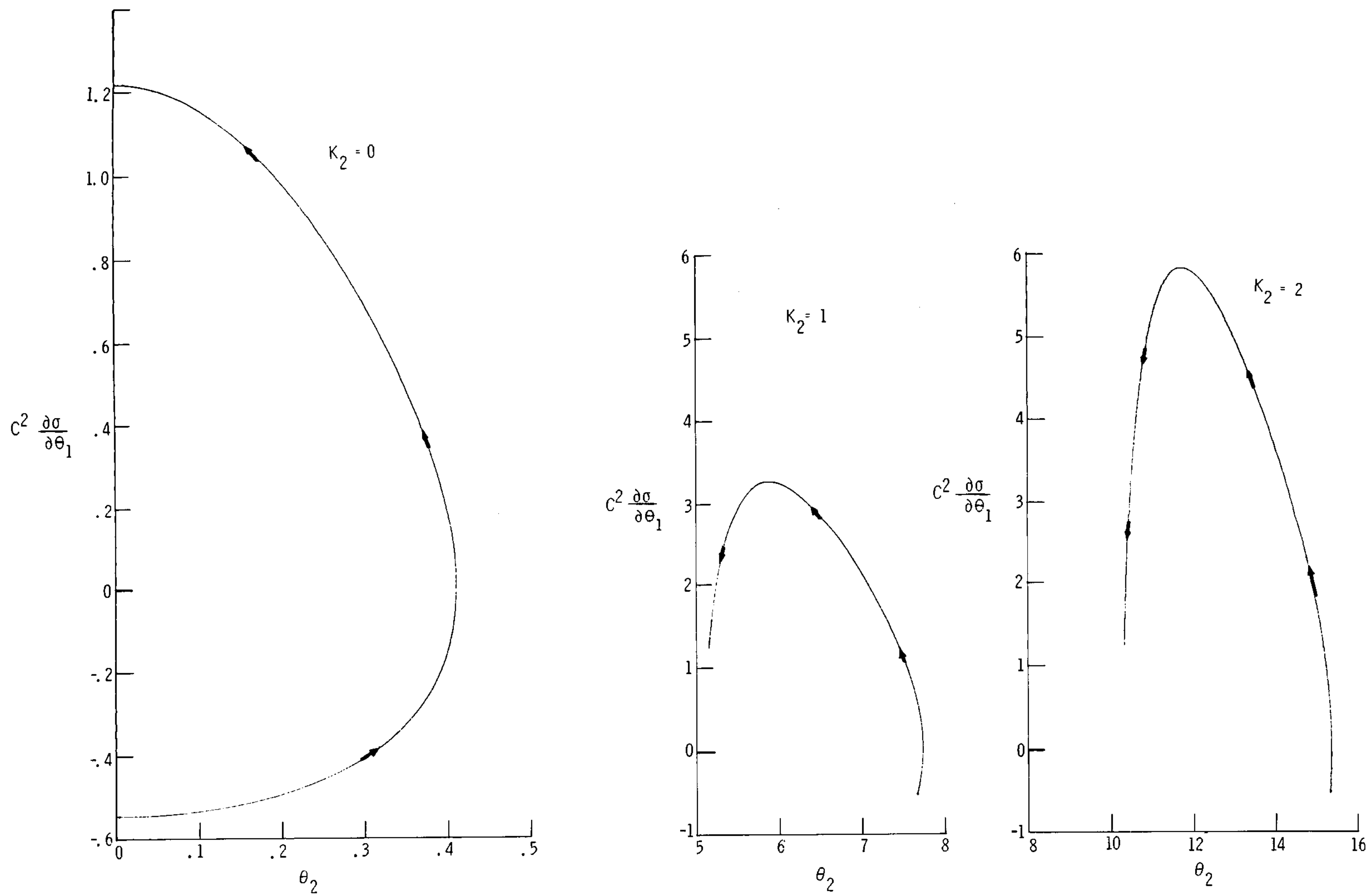


Figure 6.- Scaled derivative of real part of root-locus curve with respect to θ_1 for constant θ_2 along the (K_1,K_2) curves of figure 5. Arrows indicate direction of increasing ω.

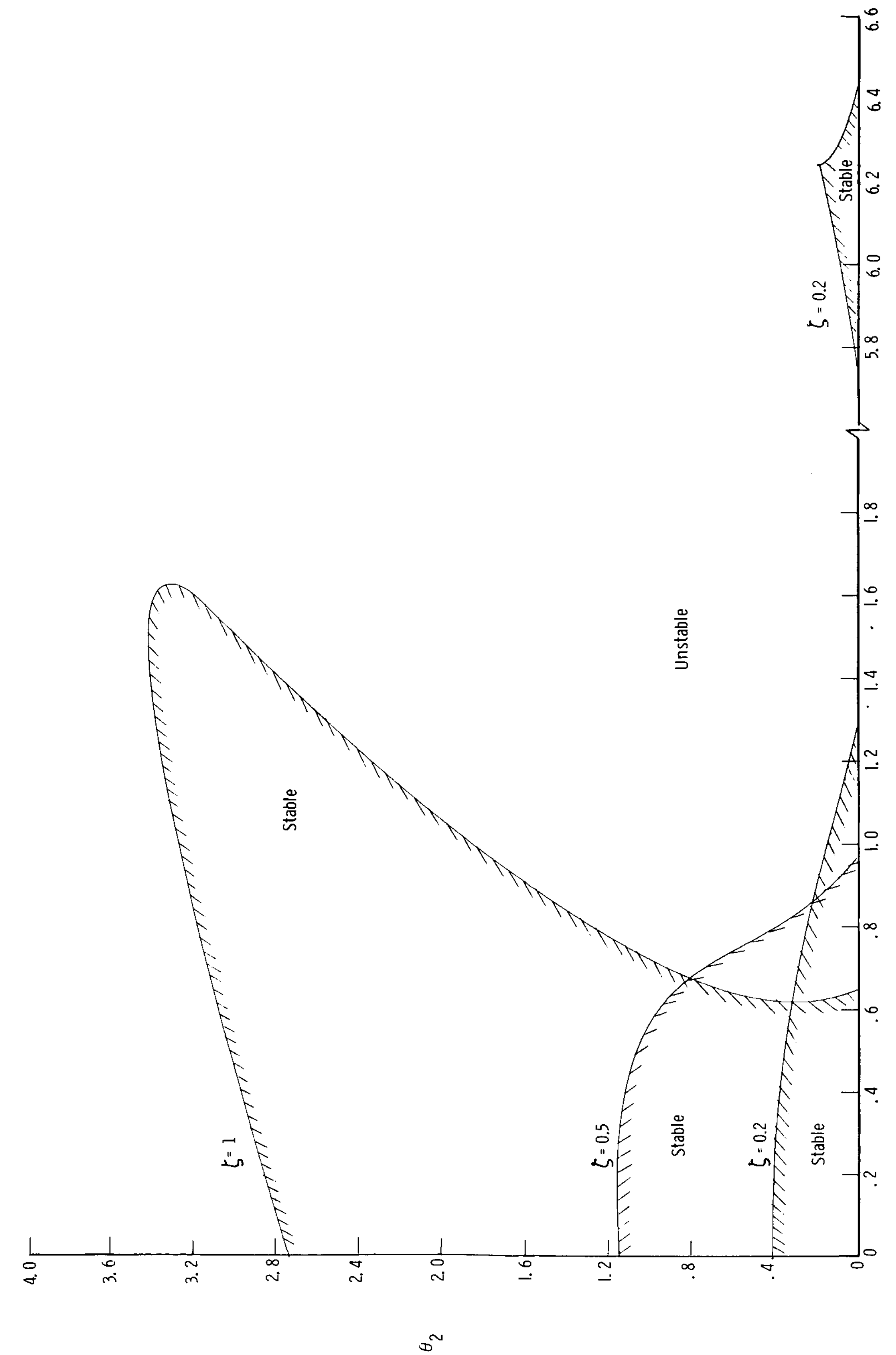


Figure 7.- Stability boundaries for $\frac{d^2}{d\tilde{t}^2}x(\tilde{t}) + 2\zeta\frac{d}{d\tilde{t}}x(\tilde{t} - \theta_1) + x(\tilde{t} - \theta_2) = 0$ for $\zeta = 0.2, 0.5,$ and 1.

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