NASA TECHNICAL REPORT

NASA TR R-453
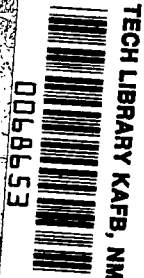
THE PHOTON GAS FORMULATION OF THERMAL RADIATION

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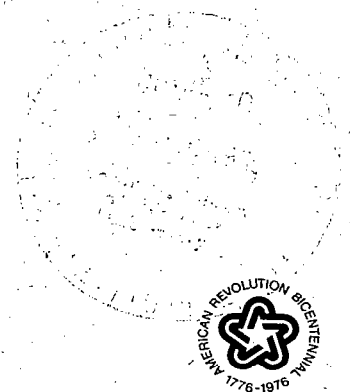
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • NOVEMBER 1975

1. Report No. NASA TR R-453	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle THE PHOTON GAS FORMULATION OF THERMAL RADIATION		5. Report Date November 1975
		6. Performing Organization Code JSC-08803
7. Author(s) Robert C. Ried, Jr.		8. Performing Organization Report No. S-401
		10. Work Unit No. 909-44-39-00-72
9. Performing Organization Name and Address Lyndon B. Johnson Space Center Houston, Texas 77058		11. Contract or Grant No.
		13. Type of Report and Period Covered Technical Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

A statistical consideration of the energy, the linear momentum, and the angular momentum of the photons that make up a thermal radiation field is presented. A nonequilibrium statistical thermodynamics approach toward a macroscopic description of thermal radiation transport is developed generally and then applied to the restricted equilibrium statistical thermostatics derivation of the energy, linear momentum, and intrinsic angular momentum equations for an isotropic photon gas. A brief treatment of a nonisotropic photon gas, as an example of the results produced by the nonequilibrium statistical thermodynamics approach, is given. The relativistic variation of temperature and the invariance of entropy are illustrated, and 12 appendixes are included.

17. Key Words (Suggested by Author(s)) · Transverse Electromagnetic Waves · Nonequilibrium Thermodynamics · Relativistic Temperature · Cosserat Continua · Information Theory · Entropy Invariance · Nonisotropic Gas · Boltzmann Equation	18. Distribution Statement STAR Subject Category: 77 (Thermodynamics and Statistical Physics)

19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 75	22. Price* $4.25

*For sale by the National Technical Information Service, Springfield, Virginia 22161

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THE PHOTON GAS FORMULATION OF THERMAL RADIATION

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SUMMARY

Thermal radiation transport is formulated on the basis of a photon gas and also, independently, on the basis of the corresponding transverse electromagnetic wave formulation. Nonequilibrium statistical thermodynamics is employed in the form of a postulated entropy extremum principle to truncate the photon gas moment equations by way of the most probable distribution function subject to appropriate physical constraints. This approach is applied to an equilibrium statistical thermostatics derivation of the energy, linear momentum, and intrinsic angular momentum equations for an isotropic photon gas. Conservation equations are obtained in terms of a temperature, an energy average velocity, and an intrinsic angular momentum diffusion velocity. Interactions between the radiation and physical media are considered only in a general manner. The relativistic variation of a photon gas temperature is obtained, and the entropy is established as a true scalar.

Major appendixes are included: (1) to illustrate the general form of photon-media interactions, (2) to provide an independent transverse electromagnetic wave formulation of radiation, (3) to establish the photon gas phase space representation by direct quantization of the electromagnetic variables, (4) to summarize integrals required in the statistical formulation, (5) to provide the correspondence between the thermodynamic constraint of momentum and a special relativistic Lorentz transformation, and (6) to illustrate the application of nonequilibrium statistical thermodynamics by the treatment of a nonisotropic photon gas with the subsequent development of a corresponding second-order tensor temperature.

INTRODUCTION

The quantitative description of thermal radiation transport is generally a complex problem inherently involving both the microscopic and macroscopic characteristics of a physical medium. At the microscopic level, classical descriptions of the physical medium and of the electromagnetic field generally are not adequate for the description of mutual interactions. A quantum mechanical treatment of the particles forming the physical medium and a quantum electrodynamic treatment of the transverse electromagnetic waves are required to describe the mutual interaction. These treatments culminate in transition probabilities for emission or absorption by the physical particles of electromagnetic angular momentum quanta having discrete energy and linear momentum, commonly called photons. At the macroscopic level, the radiation emission and absorption characteristics

of the medium involve a statistical consideration of the energy, the linear momentum, and the angular momentum of the physical particles forming the medium. The objective of this report is to provide a comparable statistical consideration of the energy, the linear momentum, and the angular momentum of the photons that make up the thermal radiation field and thereby to provide a more comprehensive basis for the treatment of macroscopic radiation transport.

Thermal radiation transport in astrophysics, physics, and engineering problems has generally been treated by a semimicroscopic approach through the direct application of a radiation transfer equation (refs. 1 and 2). This approach amounts to photon counting without statistical consideration of the photons themselves. The radiation transfer equation can be solved, in general, along any particular direction in space. However, the coupling of these solutions for the infinite number of directions at every point in space is limited to special cases. In an optically thin medium, the coupling of radiation in different directions is relatively weak, and the direct application of the radiation transfer equation provides a reasonable approximation. In one-dimensional radiation transport problems, symmetry implicitly provides a statistical consideration of photon behavior, and the direct application of the radiation transfer equation is possible. In optically thick media, a diffusion approximation (refs. 3 and 4) to the radiation transfer provides a realistic approach to radiation transport within the media; however, this approximation generally breaks down at physical boundaries.

An alternate approach toward obtaining a description of thermal radiation transport is through a moment method in which the Boltzmann equation for photons is expanded in a series of moments of the photon distribution function. Use of this method produces an infinite set of equations in physical space and time that must be truncated for practical application. In practice, these moment equations are normally truncated somewhat arbitrarily for mathematical convenience. This report includes a moment expansion obtained by analogy with molecular gas dynamics (refs. 5 to 13); hence, the photon gas designation. The moment equations are truncated physically by assuming that the general thermodynamic entropy will be a maximum subject to the constraints on the system, an assumption that, in principle, can be applied at any moment level.

In general, this is a nonequilibrium statistical thermodynamics approach toward a macroscopic description of thermal radiation transport. The approach is developed generally and then applied to the restricted equilibrium statistical thermostatics derivation of the energy, linear momentum, and intrinsic angular momentum equations for an isotropic photon gas. Interactions between the photon gas and a medium will be considered only in a general manner. The equations obtained from the photon gas formulation are shown to correspond to an electromagnetic formulation of transverse waves. In turn, the direct quantization of the electromagnetic variables describing transverse waves are related to the phase space representation of the photon gas. The isotropic photon gas relations are shown to be consistent with a relativistic Lorentz transformation of the four-stress tensor; thus, the relativistic nature of the photon gas temperature, by which the photon gas entropy is established as a relativistic invariant, is provided.

A brief treatment of a nonisotropic photon gas is given as an example of the results produced by this nonequilibrium statistical thermodynamics approach.

Studies of radiation have contributed substantially to the development of basic physics in electromagnetics, thermostatics, statistical mechanics, quantum mechanics, and relativity. It now appears that the study of radiation also may aid in the development of nonequilibrium thermodynamics.

SYMBOLS

$\hat{A}$	electromagnetic vector potential (eq. (C1))
A_{nij}, A_{nji}	emission, absorption cross section for species n, between energy states i and j
A_μ, A_ν	arbitrary four-dimensional vectors (appendix G)
a	arbitrary constant (appendix F)
$a_{\mu\nu}, \tilde{a}_{\mu\nu}$	simple Lorentz transformation matrices (eqs. (G7) and (G8))
B_n	scattering cross section for species n
$\hat{B}$	magnetic induction vector (eq. (B2))
$B_{\mu\nu}, B_{\phi\lambda}$	arbitrary four-dimensional tensors (appendix G)
c	speed of light
$c\eta$	frequency
$\dfrac{c\hat{\eta}}{\eta}$	photon velocity vector
$\hat{D}$	electric displacement vector (eq. (B1))
D	interaction operator (eq. (4))
E	electric field
$\hat{E}$	electric field vector

e	energy density (table I), exponential constant (eq. (30))
$\hat{e}$	unit vector in polarization direction (eq. (C2))
$\hat{\hat{F}}$	defined mathematical function (eq. (E11))
f	photon distribution function in phase space (eq. (2))
f^P	Planck photon distribution function
g_{ni}	degeneracy of ith energy level of species n
$\hat{g}$	photon gas momentum (table I), electromagnetic momentum (eq. (B14))
H	magnetic field
$\hat{H}$	magnetic field vector
h, $\hbar$	Planck constant, $\hbar \equiv h/2\pi$
$hc\eta$	energy
$h\hat{\eta}$	momentum
I_η	specific intensity (eq. (8))
k	Boltzmann constant (eq. (25)), gray gas absorption coefficient (eq. (A15))
k_η, k_η^0	relative, absolute absorption coefficients
L	number of dependent variables
$\hat{\ell}$	intrinsic angular momentum density (table I)
$\hat{M}$	magnetization vector (eq. (B2))
$\hat{\hat{m}}$	intrinsic angular momentum density flux (table I)
m	intrinsic angular momentum number density (eq. (14))
N_n	molecular particle number density for species n
n	photon number density (table I), dummy numeral (appendix F)
$\hat{\hat{P}}$	total-pressure tensor (table I)
$\hat{P}$	polarization vector (eq. (B1))

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p	static pressure (eq. (44))
$\hat{\hat{p}}$	static-pressure tensor (eq. (H32))
$\hat{q}$	energy density flux vector (table I)
q_λ	coefficient of infinite series (eq. (C1))
$\hat{r}$	position vector in physical space (x,y,z)
$\hat{S}$	Poynting vector (table II)
S	total entropy (eq. (47))
s	entropy density, length in four space
$\hat{\hat{T}}$	Maxwell stress tensor (table II), tensor temperature (eq. (H10))
T	temperature
t	time
$\hat{U}$	diffusion velocity (eq. (17))
u	dummy variable (appendix F)
$\hat{V}$	average velocity (table I, eq. (H29))
V_μ	four-space velocity (eq. (G3))
V	volume
$\hat{v}$	charged-particle average velocity (eq. (B6))
$\hat{\hat{\hat{w}}}$	stress flux triad (eq. (H24))
x_μ	four-space position vector (eq. (G1))
x	dummy variable (appendix F)
x,y,z	physical space Cartesian coordinates
$\hat{\alpha}$	acceleration vector (eq. (6))
α	defined quantity (eq. (H6))
$\hat{\hat{\beta}}$	Lagrangian multiplier tensor (eq. (H4))
γ	relativistic contraction factor (appendix E)

$\hat{\hat{\delta}}$	unit tensor
δ	relative relativistic contraction factor (eq. (55)), charge density (eq. (B3)), variational operator (eq. (28))
$\delta_{\lambda\mu}$	Dirac delta function
$\hat{\eta}, \eta$	wave-number vector, magnitude
θ	polar-angle coordinate (eq. (H8))
$\hat{\kappa}_\lambda, \kappa_\lambda$	wave-number vector, magnitude (eq. (C2))
λ	wavelength
λ_i	Lagrangian multiplier (eq. (28))
ξ	distance along intensity direction (eq. (9))
σ	Stefan-Boltzmann constant (eq. (41))
$\hat{\sigma}$	entropy density flux (table I)
$\tau_{\mu\nu}$	stress-energy density four tensor (eq. (K10))
Φ	elliptic double integral function (eq. (H16), table H-I)
ϕ	arbitrary function (eqs. (11) and (12), table I)
φ	spherical-angle coordinate (eq. (H8)), dummy variable (appendix F)
$\hat{\Omega}$	unit direction vector

Superscripts:

0	rest condition
+,-	positive, negative angular momentum
'	alternate, moving system
*	complex conjugate

Subscripts:

i,j,k	dummy indices, energy state i,j,k
m	medium
n	species n
T	total, rest
x,y,z	Cartesian components of vector
η	wave-number space variable or operator
λ	per wavelength
μ,ν,φ,λ	dummy indices for four-dimensional tensors (appendix G)
0	rest condition, transverse electromagnetic wave
+,-	positive, negative angular momentum

Mathematical notations:

d^3r	physical space volume element
$d^3\eta$	wave-number space volume element
^	vector, first-order tensor
^^	"tensor," second-order tensor
^^^	triad, third-order tensor
< >	wave-number space average (eq. (11)), macroscopic value of electromagnetic variable

GENERAL FORMULATION

In this section, the Boltzmann equation for photons, the radiation transfer equation, and the equations of change are presented.

Boltzmann Equation for Photons

The quantization of a transverse electromagnetic field is obtained by direct analogy with the quantum mechanics of particle motion (refs. 14 to 16). Maxwell's equations for transverse waves are written in Hamiltonian form with

generalized momenta and coordinates. From this form, the analogy with particle motion leads to a corresponding Schrödinger equation for the transverse field. The solutions of this equation can be put in the form of characteristic energy or momentum functions with photons as a physical interpretation.

To begin a photon gas formulation, it is sufficient to accept the validity of Planck's hypothesis that the momentum of a photon may be considered to be proportional to a vector wave number $\hat{\eta}$. The vector velocity of a photon $c\hat{\eta}/\eta$, where c is the speed of light and η is the vector magnitude, leads to the relativistic relation momentum times velocity equals energy.

$$h\hat{\eta} \cdot c\frac{\hat{\eta}}{\eta} = hc\eta \tag{1}$$

where h is the Planck constant. The statement that a photon has a physical position denoted by $\hat{r}$ and a wave number denoted by $\hat{\eta}$ will not, by definition, imply a specification that is greater than that allowed by Heisenberg's uncertainty principle. Similarly, the statement that a photon has a frequency $c\eta$ at a time t will imply a specification that is less than the maximum allowed by the uncertainty principle. With these implications, the translational state of a photon at a time t is represented by a wave number $\hat{\eta}$ at the position $\hat{r}$. A specification of the spin orientation is also necessary since one of two independent spin states exists for each photon. Because the spin results in a photon having an angular momentum (ref. 14) of $\pm h\frac{\hat{\eta}}{\eta}$, the spin state is denoted by + or -, respectively.

For convenience, the translational state of a photon is represented at a time t as a location in a six-dimensional phase space, or μ-space, having three physical coordinates and three wave-number coordinates. For simplicity, orthogonal Cartesian coordinates $\hat{r} = (x, y, z)$ are used for physical space, and a locally parallel wave-number space $\hat{\eta} = (\eta_x, \eta_y, \eta_z)$ is defined. The motion of a photon in time is represented by a path or trajectory in this μ-space; and, at a given time, each photon in the radiation field is represented by a point in this μ-space. Because photons are indistinguishable, this designation does not represent a unique specification of the radiation field; any number of photon locations can be interchanged without a change in the μ-space representation of the photon gas.

The distribution of the photons in the defined μ-space is represented by densities or distribution functions $f^+(\hat{r}, \hat{\eta}, t)$ and $f^-(\hat{r}, \hat{\eta}, t)$. These distribution functions integrated over μ-space give the total number of photons that make up the radiation field. More explicitly, the number of photons in the μ-space volume element $d^3\eta\, d^3r$ about the point $(\hat{r}, \hat{\eta})$ at time t is the following.

$$\left[f^+(\hat{r}, \hat{\eta}, t) + f^-(\hat{r}, \hat{\eta}, t)\right] d^3\eta\, d^3r \tag{2}$$

The total number of photons in the gas at this time is given by

$$\int_{\hat{\eta}\text{-space}} d^3n \int_{\hat{r}\text{-space}} d^3r \left[f^+(\hat{r},\hat{\eta},t) + f^-(\hat{r},\hat{\eta},t) \right] \tag{3}$$

The Boltzmann equation in phase space is analogous to a continuity equation in physical space. The rates of change of the photon distribution functions are considered for a differential element in μ-space and in time. However, the resulting analysis is valid only when the changes are considered for a finite volume in μ-space and for a finite time increment.

The distribution functions are explicit functions of position, wave number, and time. In general, the behavior of a photon gas will also implicitly depend on parameters that describe the interactions of photons with material particles. It is assumed that changes in the distribution function arising from the statistical effects of interactions are equal to those arising from explicit changes in position, wave number, and time. Because the Boltzmann equation is written as a rate of change with time, this assumption neglects the interaction time with respect to the time between collisions (ref. 17).

The Boltzmann equations for the rate of change of the photon distribution functions are given by

$$\left(\frac{\partial}{\partial t} + \frac{d\hat{r}}{dt} \cdot \nabla + \frac{d\hat{\eta}}{dt} \cdot \nabla_\eta \right) f^\pm(\hat{r},\hat{\eta},t) = D^\pm f^\pm(\hat{r},\hat{\eta},t) \tag{4}$$

where D is the interaction operator. The two equations (positive and negative polarization) in equation (4) are written for a fixed volume element in μ-space during the time dt. The first term on the left side is an explicit rate of change, whereas the second and third terms are the changes because of a changing flux in configuration space and wave-number space, respectively. The right side of equation (4) is an operator representation of the rate of change (due to interactions, i.e., absorption, emission, and scattering) of the distribution function at $\hat{r}$, $\hat{\eta}$, and t. More detailed consideration of this interaction term is provided by St. Pierre (ref. 18) and by others (refs. 14 to 16, 19, and 20). An illustration of the interaction operation between a photon gas and a physical gas is given in appendix A. If the two Boltzmann equations contained in equation (4) are added, a single Boltzmann equation operating on $f^+(\hat{r},\hat{\eta},t) + f^-(\hat{r},\hat{\eta},t)$ is obtained.

The velocity of a photon in configuration space $c\hat{\eta}/\eta$ is equal to the rate of change of photon position with time. That is,

$$\frac{d\hat{r}}{dt} \equiv c\frac{\hat{\eta}}{\eta} \tag{5}$$

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The velocity of a photon in wave-number space is proportional to the photon acceleration in configuration space and is defined as

$$\frac{d\hat{\eta}}{dt} \equiv \frac{\hat{\alpha}^{\pm}}{c} \times \hat{\eta}$$ (6)

This definition is consistent with relativistic requirements (refs. 21 and 22) and retains acceleration vector $\hat{\alpha}^{\pm}(\hat{r},t)$ with the dimensions of an acceleration or force per unit mass. This definition of $\hat{\alpha}^{\pm}$ is meant merely to be representative and not inclusive of the types of external forces that might be included in the specification of photon behavior. In the absence of an external force or its equivalent, $\hat{\alpha}^{+}$ and $\hat{\alpha}^{-}$ are set equal to zero.

By using the expressions for the photon velocity and accelerations, the Boltzmann equations are written as

$$\left[\frac{\partial}{\partial t} + c\frac{\hat{\eta}}{\eta}\cdot\nabla + \left(\frac{\hat{\alpha}^{\pm}}{c} \times \hat{\eta}\right)\cdot\nabla_{\eta}\right]f^{\pm}(\hat{r},\hat{\eta},t) \equiv D^{\pm}f^{\pm}(\hat{r},\hat{\eta},t)$$ (7)

If the effects of interactions on the photon distribution functions are negligible, then the right-hand sides (positive and negative) of equation (7) are set equal to zero. For this case, the distribution functions depend only on the boundary conditions of the μ-space and on the initial conditions.

Radiation Transfer Equation

The radiation transfer equation can be obtained directly from the Boltzmann equation merely by a change in definition (ref. 2). Instead of the photon distribution functions, the intensity of radiation at a given frequency (or wave-number magnitude) traveling in a specific direction is considered. The specific intensity I_η is defined as a scalar energy flux of a given wave number, per steradian, at $\hat{r}$, at time t, and in a direction denoted by $\hat{\Omega}$. Thus, the wave-number vector is replaced by its magnitude and by the unit direction vector $\hat{\Omega}$, and the specific intensity is written as

$$I_\eta(\hat{r},\eta,\hat{\Omega},t) \equiv hc^{2}\eta\left[f^{+}(\hat{r},\eta,\hat{\Omega},t) + f^{-}(\hat{r},\eta,\hat{\Omega},t)\right]$$ (8)

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As is customarily done, the two orthogonal spin states are neglected in this intensity definition. The physical distance along the direction $\hat{\Omega}$ is denoted by ξ. Then, the second operation appearing in the Boltzmann equation is transformed as follows.

$$c\frac{\hat{n}}{\eta} \cdot \nabla \longrightarrow c\frac{d}{d\xi} \tag{9}$$

For illustration, $\hat{\alpha}^+$ and $\hat{\alpha}^-$ are assumed to be zero and η is neither a function of ξ nor an explicit function of time. Thus, adding the Boltzmann equations (eq. (4)) and multiplying by $hc\eta$ yields

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{d}{d\xi}\right)I_\eta\left(\hat{r},\eta,\hat{\Omega},t\right) = \frac{1}{c}D'I_\eta\left(\hat{r},\eta,\hat{\Omega},t\right) \tag{10}$$

The prime on the interaction operator denotes the operation on the specific intensity as opposed to an operation on the distribution functions.

The important philosophical aspect of equation (10) is that it represents a division of a vector energy flux into an essentially infinite number of scalar specific intensities in an infinite number of directions $\hat{\Omega}$. The equation has a general solution for the specific intensity (ref. 1); however, this is not a specification of the radiation field as a whole. The practical application of equation (10) (and its solution) has been limited to one-dimensional problems. To obtain a solution that can satisfy general boundary conditions is difficult, primarily because only one differential equation exists in seven independent variables: three position variables, three wave-number variables, and time. A useful analogy for developing potentially more tractable methods of handling radiation problems is obtained with the techniques used in molecular gas dynamics — hence, the concept of the photon gas.

Equations of Change

In this section, the relation between μ-space variables and macroscopic parameters in physical space is outlined. The general equations of change and the special cases of energy, linear momentum, and angular momentum conservation are derived from the Boltzmann equations. The incompleteness of these equations in the general case is also discussed (ref. 19).

The study of a photon gas as an entity involves consideration of average quantities such as photon gas energy, linear momentum, and angular momentum. These quantities are obtained by integration over wave-number space to obtain average values in physical space. The average values contain an inherent weighting

in wave-number space based on the actual distribution functions. Thus, the μ-space quantity $\phi_{\eta}^{\pm}$ has an average value defined by

$$\langle \phi^{\pm}(\hat{r},t) \rangle \equiv \int_{\eta} \phi_{\eta}^{\pm}(\hat{r},\hat{n},t)\, f^{\pm}(\hat{r},\hat{n},t)\, d^3\eta \tag{11}$$

The average value for the whole photon gas is given by

$$\langle \phi(\hat{r},t) \rangle \equiv \langle \phi^{+}(\hat{r},t) \rangle + \langle \phi^{-}(\hat{r},t) \rangle \tag{12}$$

Physical space variables or average values of interest are defined here as given in table I. Appendix B contains the corresponding electromagnetic transverse wave physical space densities, which are compared to the photon gas quantities in table II (refs. 19, 20, 23, and 24). A verification of this comparison is given in appendix C, in which the electromagnetic quantities are quantized to produce the corresponding photon gas relations for energy momentum and pressure (ref. 14).

It should be noted that

$$\text{trace}\left(\hat{\tilde{P}}^{\pm}\right) = e_{T}^{\pm} \tag{13}$$

where $\hat{\tilde{P}}$ is the total-pressure tensor and e_{T} is the total energy density, that

$$\text{trace}\left(\hat{\tilde{m}}\right) \equiv m = hc\left(n^{+} - n^{-}\right) \tag{14}$$

where $\hat{\tilde{m}}$ is the intrinsic angular momentum density flux, m is the intrinsic angular momentum number density, and n is the photon number density, and that

$$\hat{\ell} = \frac{\hbar}{c}\left(n^{+}\hat{V}^{+} - n^{-}\hat{V}^{-}\right) \tag{15}$$

where $\hat{\ell}$ is the intrinsic angular momentum density, $\hbar \equiv h/2\pi$, and $\hat{V}$ is average velocity. Defining net convective velocity

$$\hat{V} \equiv \frac{1}{2}\left(\hat{V}^{+} + \hat{V}^{-}\right) \tag{16}$$

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and diffusion velocity

$$\hat{U} \equiv \tfrac{1}{2}\left(\hat{V}^{+} - \hat{V}^{-}\right) \tag{17}$$

then

$$n^{+}\hat{V}^{+} + n^{-}\hat{V}^{-} = n\hat{V} + \frac{m}{\hbar c}\hat{U} \tag{18}$$

$$\hat{\ell} = \frac{m}{c^{2}}\hat{V} + \frac{\hbar}{c}n\hat{U} \tag{19}$$

To obtain equations of change, the Boltzmann equations (eq. (7)) are operated on from the left with $\phi_{\hat{n}}^{\pm}$ and integrated over the entire wave-number space. By employing partial integration, by considering c, $\hat{n}$, and t to be independent, and by assuming that any physical quantity will have a regular behavior at the limits of an infinite wave number, the equations of change become

$$\frac{\partial}{\partial t}\langle\phi^{\pm}\rangle + \nabla\cdot\langle c\frac{\hat{n}}{n}\phi_{n}^{\pm}\rangle = \int_{\hat{n}}\phi_{n}^{\pm}D^{\pm}f^{\pm}\,d^{3}n + \langle\left(\frac{\partial\phi_{n}^{\pm}}{\partial t}\right) + \left(c\frac{\hat{n}}{n}\cdot\nabla\phi_{n}^{\pm}\right) + \left(\frac{\hat{a}^{\pm}}{c}\times\hat{n}\right)\cdot\nabla_{n}\phi_{n}^{\pm}\rangle \tag{20}$$

The general equation of change is the sum of the positive and negative cases of equation (20).

$$\frac{\partial}{\partial t}\langle\phi\rangle + \nabla\cdot\langle c\frac{\hat{n}}{n}\phi_{n}\rangle = \int_{\hat{n}}\left(\phi_{n}^{+}D^{+}f^{+} + \phi_{n}^{-}D^{-}f^{-}\right)d^{3}n$$

$$+ \langle\frac{\partial\phi_{n}^{+}}{\partial t} + c\frac{\hat{n}}{n}\cdot\nabla\phi_{n}^{+} + \left(\frac{\hat{a}^{+}}{c}\times\hat{n}\right)\cdot\nabla_{n}\phi_{n}^{+}\rangle$$

$$+ \langle\frac{\partial\phi_{n}^{-}}{\partial t} + c\frac{\hat{n}}{n}\cdot\nabla\phi_{n}^{-} + \left(\frac{\hat{a}^{-}}{c}\times\hat{n}\right)\cdot\nabla_{n}\phi_{n}^{-}\rangle \tag{21}$$

TABLE I.- PHYSICAL SPACE DENSITIES FOR THE PHOTON GAS

$$\left[\langle\phi^{\pm}\left(\hat{r},t\right)\rangle \equiv \int\limits_{\hat{\eta}} \phi_{\eta}^{\pm}\left(\hat{r},\hat{\eta},t\right) f^{\pm}\left(\hat{r},\hat{\eta},t\right) d^{3}\eta\right]$$

$$\left[\langle\phi\left(\hat{r},t\right)\rangle \equiv \langle\phi^{+}\left(\hat{r},t\right)\rangle + \langle\phi^{-}\left(\hat{r},t\right)\rangle\right]$$

Photon gas	$\langle\phi\rangle$	$\langle\phi^{\pm}\rangle$	$\phi_{\eta}^{\pm}$
Number density	n	$n^{\pm}$	1
Number flux	$n^{+}\hat{V}^{+} + n^{-}\hat{V}^{-}$	$n^{\pm}\hat{V}^{\pm}$	$c\dfrac{\hat{\eta}}{\eta}$
Total energy density	e_{T}	$e_{T}^{\pm}$	$hc\eta$
Momentum density	$\hat{g}$	$\hat{g}^{\pm}$	$h\hat{\eta}$
Energy density flux	$\hat{q}$	$\hat{q}^{\pm}$	$hc^{2}\hat{\eta}$
Intrinsic angular momentum density	$\hat{\ell}$	$\hat{\ell}^{\pm}$	$\pm\hbar\dfrac{\hat{\eta}}{\eta}$
Total-pressure tensor	$\hat{\hat{P}}$	$\hat{\hat{P}}^{\pm}$	$hc\dfrac{\hat{\eta}\hat{\eta}}{\eta}$
Intrinsic angular momentum density flux	$\hat{\hat{m}}$	$\hat{\hat{m}}^{\pm}$	$\pm\hbar c\dfrac{\hat{\eta}\hat{\eta}}{\eta^{2}}$
Entropy density	s	$s^{\pm}$	$k\left[\dfrac{\ln\left(1 + f^{\pm}\right)}{f^{\pm}} + \ln\left(\dfrac{1 + f^{\pm}}{f^{\pm}}\right)\right]$
Entropy density flux	$\hat{\sigma}$	$\hat{\sigma}^{\pm}$	$kc\dfrac{\hat{\eta}}{\eta}\left[\dfrac{\ln\left(1 + f^{\pm}\right)}{f^{\pm}} + \ln\left(\dfrac{1 + f^{\pm}}{f^{\pm}}\right)\right]$

TABLE II.- COMPARISON OF PHOTON GAS AND ELECTROMAGNETIC PHYSICAL SPACE DENSITIES[a]

Photon gas	Transverse electromagnetic wave (Gaussian units)[b]
Total energy density: $$e_T = \int hc\eta(f^+ + f^-)d^3\eta$$	Total energy density: $$<e_0> = \frac{1}{8\pi}<E^2 + H^2>$$
Energy flux density: $$\hat{q} = \int hc^2\hat{\eta}(f^+ + f^-)d^3\eta$$	Poynting vector: $$<\hat{S}> = \frac{c}{4\pi}<\hat{E} \times \hat{H}>$$
Momentum: $$\hat{g} = \int h\hat{\eta}(f^+ + f^-)d^3\eta$$	Momentum: $$<\hat{g}_0> = \frac{<\hat{S}>}{c^2} = \frac{1}{4\pi c}<\hat{E} \times \hat{H}>$$
Pressure tensor: $$\hat{\hat{P}} = \int hc\frac{\hat{\eta}\hat{\eta}}{\eta}(f^+ + f^-)d^3\eta$$ trace $(\hat{\hat{P}}) = e_T$	Pressure = minus Maxwell stress tensor: $$-<\hat{\hat{T}}_0> = \frac{1}{8\pi}<E^2 + H^2>\hat{\hat{\delta}}$$ $$-\frac{1}{4\pi}<\hat{E}\hat{E} + \hat{H}\hat{H}>$$ trace $\left(-<\hat{\hat{T}}_0>\right) = <e_0>$

[a]Reference 19.

[b]$\hat{E}$ = electric field vector; $\hat{H}$ = magnetic field vector.

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The energy, linear momentum, and intrinsic angular momentum equations for a photon gas are obtained immediately from equation (21).

Energy:

$$\frac{\partial e_T}{\partial t} + \nabla \cdot \hat{q} = \int_{\hat{\eta}} hc\eta \left(D^+ f^+ + D^- f^- \right) d^3 \eta \tag{22}$$

Linear momentum:

$$\frac{\partial \hat{g}}{\partial t} + \nabla \cdot \hat{\widetilde{P}} = \int_{\hat{\eta}} h\hat{\eta} \left(\hat{D}^+ f^+ + D^- f^- \right) d^3 \eta$$

$$+ \frac{1}{c} \left(\hat{\alpha}^+ \times \hat{g}^+ \right) + \frac{1}{c} \left(\hat{\alpha}^- \times \hat{g}^- \right) \tag{23}$$

Angular momentum:

$$\frac{\partial \hat{\ell}}{\partial t} + \nabla \cdot \hat{\widetilde{m}} = \int_{\hat{\eta}} h\frac{\hat{\eta}}{\eta} \left(D^+ f^+ - D^- f^- \right) d^3 \eta$$

$$+ \frac{1}{c} \left(\hat{\alpha}^+ \times \hat{\ell}^+ \right) + \frac{1}{c} \left(\hat{\alpha}^- \times \hat{\ell}^- \right) \tag{24}$$

In equation (22), $\hat{q}$ is the energy density flux vector; in equation (23), $\hat{g}$ is the momentum density. Equation (24) is the total angular momentum equation minus $\hat{r}$ cross the linear momentum equation since the linear momentum equation is satisfied independently. The photon gas with intrinsic angular momenta is an explicit example of a Cosserat continuum (ref. 25).

Equations (22) to (24) are the conservation equations for a photon gas. Appendix D contains a comparison of the equations of change obtained from either the photon gas formulation or the electromagnetic derivation of appendix B. (See also refs. 19 and 20.) The general equation of change, equation (21), can be used to generate higher order moment equations or constitutive equations such as a momentum flux or stress equation or an angular momentum flux or torsion equation. These higher order moment equations will not be required in the current formulation.

Equations generated from the equation of change, such as conservation equations (22) to (24), are never complete. There are always more dependent variables than equations. For example, the 7 scalar conservation equations are written here

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in terms of 28 scalar components of e_T, $\hat{\hat{P}}$, $\hat{q}$, $\hat{\dot{g}}$, $\hat{\ell}$, and $\hat{\hat{m}}$. The 10 existing algebraic relations (i.e., $\hat{q} = c^2 \hat{g}$, $e_T = \text{trace}\ (\hat{\hat{P}})$, $\hat{\hat{P}}$, and $\hat{\hat{m}}$ are symmetric) reduce the number of unknown variables to 18. To form a complete set of conservation or momentum equations requires additional information, independent of the information obtained from the general equation of change. In this formulation, additional information is obtained from the variational principle of statistical thermodynamics as provided in the next section.

STATISTICAL THERMODYNAMICS

In this section, the most probable distribution function, the physical identification of independent variables, the complete conservation equations, and limiting cases are presented.

Most Probable Distribution Function

The basic principle of statistical thermodynamics is the assumption that the entropy of a system is a maximum subject to the independent constraints on the system (refs. 7 to 13 and 26). Each independent constraint is statistically weighted by a Lagrangian parameter, which then becomes a dependent variable in the equations of change. The most probable distribution function, producing the highest entropy, generally is a function of all these dependent variables. In principle, the "average values" in physical space can be expressed in terms of these dependent variables; therefore, the number of independent equations of change is equal to the number of dependent variables. By this procedure, a complete set of equations for the system is produced. If the state of a system at a given point in space and time can be represented by a specified number of dependent variables at that same point, then this thermodynamic principle can be applied to a single-point distribution function as is carried out here. On the other hand, as pointed out by Kogan (ref. 27), in a rarefied situation, the characteristics of a system at a point are highly dependent on the characteristics of the system at neighboring points. To apply statistical thermodynamics to a rarefied problem requires either consideration of a higher order, multipoint distribution function or acceptance of the dependent variables in a single-point distribution function as being representative averages over a physical space volume or a time period (or both). Therefore, the results of this formulation are not directly applicable to highly rarefied or, in the case of radiation, optically thin situations. The degree of rarefaction for photons must be measured by the interaction operator D, since photon-photon interaction cross sections generally are negligible.

For illustration, photons having positive angular momentum $+\hbar\frac{\hat{n}}{n}$ will be considered first and the more general case later. Since photons obey Bose-Einstein

statistics, the entropy density is, as given in table I,

$$s^+ = k \int_{\hat{\eta}} \left[\ln\left(1 + f^+\right) + f^+ \ln\left(\frac{1 + f^+}{f^+}\right) \right] d^3\eta \qquad (25)$$

where k is the Boltzmann constant. The entropy will be largest if the distribution function is the most probable distribution function possible for the constraints on the system. In this formulation, all constraints are of the form of "average values" at a point in physical space and time.

$$\langle \phi^+(\hat{r},t) \rangle_i = \int_{\hat{\eta}} \phi^+_{\eta_i}(\hat{r},\hat{\eta},t) f^+(\hat{r},\hat{\eta},t) d^3\eta \qquad (26)$$

where i indicates one of a series of constraints. In general, the constraints could be statistical averages over a smaller or greater portion of the μ-space. For example, in classical thermostatics, the constraints on the entropy are of the form

$$\langle\langle \phi^+(t) \rangle\rangle_i \equiv \int_{\hat{r}} \int_{\hat{\eta}} \phi^+_{\eta_i}(\hat{r},\hat{\eta},t) f^+(\hat{r},\hat{\eta},t) d^3\eta \ d^3r \qquad (27)$$

If the constraints are restricted to the form of equation (26), the condition that the entropy density is a maximum subject to a given series of i constraints can be expressed as the variational equation

$$\delta \left[s^+ + \lambda^+_i \left(\langle \phi^+ \rangle_i - \int_{\hat{\eta}} \phi^+_{\eta_i} f^+ d^3\eta \right) \right] = 0 \qquad (28)$$

where the variables λ^+_i are the parameters weighting the constraints and serve as Lagrangian multipliers. If the functions $\phi^+_{\eta_i}$ are not explicit functions of f^+, ∇f^+, $\nabla_\eta f^+$, and so forth, the Euler-Lagrange equation (eq. (28)) is simply

$$\frac{\partial}{\partial f^+} \left\{ k \left[\ln\left(1 + f^+\right) + f^+ \ln\left(\frac{1 + f^+}{f^+}\right) \right] - \lambda^+_i \phi^+_{\eta_i} f^+ \right\} = 0 \qquad (29)$$

This equation is readily solved for the most probable distribution function.

$$f^+ = \frac{1}{e^{\lambda_i^+ \phi_{n_i}^+} - 1} \qquad (30)$$

This most probable distribution function can be substituted back into average value equation (26) to obtain $\phi_i^+ [\lambda_j^+ (\hat{r},t)]$ so that the functions λ_i^+ become the dependent variables for the equation of change.

With a given number, say L, of independent constraints of the form of equation (26), there are, in turn, L Lagrangian multipliers or dependent variables. The entropy of the photon gas is a function of all the constraints on the system or, in general, all the L dependent variables $s^+(\lambda_i^+)$. The total differential of the entropy may be written

$$ds^+ = \frac{\partial s^+}{\partial \lambda_i^+} \, d\lambda_i^+ \qquad (31)$$

Incorporating equations (25) and (30),

$$ds^+ = k\lambda_j^+ \left(\int_{\hat{n}} \phi_{n_j}^+ \frac{\partial f^+}{\partial \lambda_i^+} d^3 n \right) d\lambda_i \qquad (32)$$

Recognizing that equation (32) contains the differential of equation (26),

$$ds^+ = k\lambda_j^+ \, d\left(\langle \phi^+ \rangle_j \right) \qquad (33)$$

This, then, is the differential expression between thermodynamic variables that provides the thermodynamic significance of the Lagrangian multipliers for each value of i.

$$k\lambda_i^{\pm} = \frac{\partial s^{\pm}}{\partial \left(\langle \phi^{\pm} \rangle_i \right)} \qquad (34)$$

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Physical Identification of Dependent Variables

The physical identification of the λ_i^+ requires a selection of particular physical constraints. In this formulation, the constraints are energy and momentum.

$$\left.\begin{array}{l} \phi^+_{\eta_i} = hc\eta \\[2em] \hat{\phi}^+_{\eta_2} = h\hat{\eta} \end{array}\right\} \tag{35}$$

The distribution function (eq. (30)) becomes

$$f^+ = \left(e^{\lambda_1^+ hc\eta + \hat{\lambda}_2^+ \cdot h\hat{\eta}} - 1\right)^{-1} \tag{36}$$

Because of the form of the distribution function, it may be readily shown that

$$\hat{v}^+ \equiv \frac{\int_{\hat{\eta}} c\frac{\hat{\eta}}{\eta} f^+ d^3\eta}{\int_{\hat{\eta}} f^+ d^3\eta} = \frac{-\hat{\lambda}_2^+}{\lambda_1^+} \tag{37}$$

The distribution function becomes

$$f^+ = \left[e^{\lambda_1^+ hc\eta\left(1 - \frac{\hat{v}^+}{c} \cdot \frac{\hat{\eta}}{\eta}\right)} - 1\right]^{-1} \tag{38}$$

If $\hat{v}^+$ is zero, then λ_1^+ is readily identified as $1/kT_+$ when T_+ is the temperature describing the familiar Planck distribution function. Since these parameters are independent

$$\lambda_1^+ \equiv \frac{1}{kT_+} \tag{39}$$

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The distribution function for the case of energy and linear momentum constraints can be written as

$$f^{\pm} = \left[e^{\frac{hc\eta}{kT_{\pm}}\left(1 - \frac{\hat{V}^{\pm}}{c} \cdot \frac{\hat{n}}{\eta}\right)} - 1 \right]^{-1} \tag{40}$$

Incorporating the Stefan-Boltzmann constant,

$$\sigma \equiv \frac{2\pi^5 k^4}{15 h^3 c^2} \tag{41}$$

the average physical values of interest can be obtained in terms of temperature and velocities as given in appendix E. Integrals required for obtaining the relations given in table II are given in appendix F.

The total energy densities can be written as the sum of static energy densities and kinetic energy densities

$$e_T^{\pm} = e^{\pm} + \hat{g}^{\pm} \cdot \hat{V}^{\pm} \tag{42}$$

where

$$e^{\pm} \equiv \frac{2\sigma}{c} T_{\pm}^4 \cdot \gamma_{\pm}^4 \tag{43}$$

is the static energy density and γ is a relativistic contraction factor. Also, the total-pressure tensor $\tilde{P}$ is composed of two parts, a static pressure p and a momentum flux

$$\tilde{P}^{\pm} = p^{\pm}\tilde{\delta} + \hat{g}^{\pm}\hat{V}^{\pm} \tag{44}$$

where $\hat{\hat{\delta}}$ is a unit tensor and

$$p^{\pm} \equiv \frac{2}{3}\frac{\sigma}{c}\gamma_{\pm}^4 T_{\pm}^4 = \frac{e^{\pm}}{3} \tag{45}$$

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The relationships thus obtained are consistent with a simple Lorentz transformation (appendix G and refs. 21 and 28), provided that the static temperature is considered to be a relativistically contracted total temperature (ref. 29); that is,

$$T_{\pm_T} \equiv \gamma_\pm T_\pm \tag{46}$$

Equation (46) is consistent with the invariant or true scalar characteristics of the total entropy

$$S^\pm = s^\pm V_\pm = \frac{8}{3}\frac{\sigma}{c}T_\pm^3 \gamma_\pm^4 V_\pm$$

$$= \frac{8}{3}\frac{\sigma}{c}\left(T_{\pm_T}\right)^4 V_{\pm_T} \tag{47}$$

where the rest or total volumes are given by

$$V_{\pm_T} \equiv \gamma_\pm V_\pm \tag{48}$$

Thus, equation (47), being only a function of the rest conditions and independent of relative motion, is a true constant.

The average values of energy and momentum for a point in physical space and time are sufficient to complete the conservation equations for a photon gas. Appendix H contains an example of a more general constraint in which momentum flux or stress is the quantity of interest as opposed to energy. The relationships obtained for a nonisotropic photon gas are obvious generalizations of the isotropic case through extension of the scalar temperature concept to one of a second-order tensor. Conceptually, this generalization is immediately obvious. The scalar temperature is a statistical weighting parameter for the "conserved" quantity energy. A complex system of a large number of particles has many more degrees of freedom than are accounted for by the single-particle degrees of energy, linear momentum, angular momentum, and mass. The anisotropy in a photon gas can be produced by an anisotropy in the physical media or by constraining boundary conditions. In general, this anisotropy is distinct from relativistic effects alone.

Complete Conservation Equations

The two independent spins of a photon give rise to two independent photon gases. Each gas has an energy equation and a momentum equation. These equations can be coupled to the extent that the physical medium creates or destroys photons of a given spin. It is not possible, however, to obtain a single distribution func-

tion that is descriptive of a photon gas having a given amount of energy, linear momentum, and angular momentum. Both distribution functions are required for this description. A photon gas with energy, linear momentum, and angular momentum can be easily obtained from a combination of the preceding statistical thermodynamics results. The optimization relations for the + and − photon gases are readily combined if the two relative volumes are coincident.

$$V_+ = V_- \equiv V \tag{49}$$

In addition, if only one energy constraint is imposed, then the multipliers for energy must be identical.

$$\mathbf{T}_+ = \mathbf{T}_- \equiv \mathbf{T} \tag{50}$$

The average quantities of interest for a net photon gas are readily obtained and are shown in appendix I.

The familiar thermodynamic relationships hold for the component and for the net photon gas. As an example, for the net gas,

$$\mathbf{e} = \mathbf{s}\mathbf{T} - \mathbf{p} \tag{51}$$

An alternative form of the average quantities of interest is obtained by incorporating a convective velocity $\hat{V}$ and a relative diffusion velocity $\hat{U}$ as in equations (16) and (17).

$$\hat{V} \equiv \frac{1}{2}\left(\hat{V}_+ + \hat{V}_-\right) \tag{52}$$

$$\hat{U} \equiv \frac{1}{2}\left(\hat{V}_+ - \hat{V}_-\right) \tag{53}$$

For convenience, a contraction and a relative contraction, respectively, are defined in terms of the velocities $\hat{V}$ and $\hat{U}$.

$$\gamma^2 \equiv \left(1 - \frac{V^2 + U^2}{c^2}\right)^{-1} = \frac{2\gamma_+^2 \gamma_-^2}{\gamma_+^2 + \gamma_-^2} \tag{54}$$

$$\delta^2 \equiv \frac{2\hat{V}\cdot\hat{U}}{c^2} \gamma^2 = \frac{\gamma_+^2 - \gamma_-^2}{\gamma_+^2 + \gamma_-^2} \tag{55}$$

The relationships in appendix I can be rewritten in terms of $\hat{V}$ and $\hat{U}$, as shown in appendix J.

The total-pressure tensor $\tilde{\tilde{P}}$ contains a hydrostatic component, a linear momentum flux, and an angular momentum diffusion flux.

$$\tilde{\tilde{P}} = p\hat{\tilde{\delta}} + \hat{g}\left(\hat{V} + \delta^2\hat{U}\right) + \left(\frac{18}{5}\frac{kT\gamma^2}{\hbar c}\right)\hat{\ell}\hat{U} \tag{56}$$

The trace of equation (56) gives the total energy density e_T, which also has three parts.

$$e_T = 3p + \hat{g}\cdot\left(\hat{V} + \delta^2\hat{U}\right) + \frac{18}{5}\frac{kT\gamma^2}{\hbar c}\hat{U}\cdot\hat{\ell} \tag{57}$$

The term $\frac{18}{5}\frac{kT\gamma^2}{\hbar c}\hat{U}$ is equivalent to an angular velocity.

The seven scalar equations of motion, equations (22) to (24), can now be written in terms of the seven scalar dependent variables by the use of appendix E. The external forces and the interaction operation (e.g., appendix A) also must be specified. In principle, the solution of a problem also requires coupling with the energy, linear momentum, and angular momentum equations of the medium in which the photon gas exists.

A complete set of equations of change for an isotropic photon gas now has been established. As shown in appendix G, these equations of change are consistent with relativistic requirements. To show this consistency for the angular momentum requires consideration of $\hat{\ell}$ as the antisymmetric part of a second-order tensor (ref. 21). The equations of change are too complex to allow analytical solution other than in simple special cases. The following section on limiting cases

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illustrates some of the simplifying assumptions that make the equations of change for an isotropic photon gas considerably more tractable.

Limiting Cases

Although laser radiation and many astrophysical sources of emission can be highly polarized, thermal radiation is often close to a random polarization state. If a random polarization is assumed, then $\hat{U}$ and δ^2 are zero, as is the solution of the angular momentum equation (eq. (24)). For this case, the physical space dependent variables take on a much simpler form, as given in appendix K.

The limit of V/c approaching unity corresponds to a unidirectional photon flux in which temperature, pressure, and entropy approach zero. In this limit,

$$\lim_{\frac{V}{c} \to 1} \hat{g} \to \frac{e_T}{c^2} \hat{V} \tag{58}$$

and, for a finite momentum,

$$\lim_{\frac{V}{c} \to 1} \gamma^6 T^4 \longrightarrow \text{constant} \tag{59}$$

At the other extreme, if V/c is small compared to 1, the binomial expansion of the variables in appendix K is useful as given in appendix L. This approximation provides for less cumbersome variables and should be applicable in media for which temperature gradients are not extremely severe.

Although the equations are complete in principle, analytical solutions are difficult except for the most elementary cases. In addition, the practical application of the photon gas formulation to situations involving finite boundaries or walls requires a specification of physically realistic boundary conditions. At a wall, however, an optically thick restriction generally is not valid. Thus, the application of this photon gas formulation to finite boundaries requires, in general, restricted equations at the wall or, in analogy with physical gas dynamics, a boundary-layer or slip-layer formulation.

CONCLUSIONS

A statistical treatment of photons, analogous to molecular gas dynamics, has been used to provide equations of motion for a thermal radiation continuum. The photon gas equations have been shown to be identical to equations of motion obtained from an electromagnetic formulation based on Maxwell's equations written in a microscopic form. The correspondence between the two formulations has been established by a quantization of the transverse electromagnetic wave form of energy, momentum, and stress. Nonequilibrium statistical thermodynamic principles have been applied to the photon gas to determine the most probable distribution function or the highest possible entropy subject to the constraints on the system. This application has provided a truncation of the equations of change or the moment equations at an arbitrary level. The maximum entropy principle has been applied to an equilibrium statistical thermostatics derivation of the energy, linear momentum, and intrinsic angular momentum equations for an isotropic photon gas. These conservation equations are obtained in terms of a temperature, an energy average velocity, and an intrinsic angular momentum diffusion velocity. Comparison of these relationships with a Lorentz transformation has provided the relativistic variation of the isotropic photon gas temperature and established the entropy as a true scalar.

An example of extending statistical thermodynamics to a more general non-equilibrium application such as a stress or momentum flux constraint has also been presented. This process results in the definition of a second-order tensor temperature that is descriptive of the nonisotropic gas or of a stress state.

Although this photon gas formulation provides, in principle, a complete set of differential equations, the solution of these equations for a practical problem is quite complex. In addition, the treatment of more general nonequilibrium problems such as radiation in optically thin media, the boundary equations at a medium interface, or the spectral distribution of radiation in a real medium requires extensions of this photon gas formulation by consideration of more general constraints.

The prime application of this work resides in rendering an improved description of thermal radiation transport; potentially, an important application also exists in producing a vivid demonstration of the power of statistical thermodynamics from the vantage point of information theory. Information theory permits a rational truncation of statistical moment equations based on probability concepts rather than on simple particle mechanics. Thus, statistical thermodynamics is applicable to nonequilibrium phenomena and to macroscopic phenomena such as fluid turbulence.

Lyndon B. Johnson Space Center
National Aeronautics and Space Administration
Houston, Texas, February 18, 1975
909-44-39-00-72

APPENDIX A

THE INTERACTION OPERATOR

The right-hand side of the Boltzmann equation represents the net rate of photon production due to interactions with the medium. Thus, $D^{\pm}f^{\pm}(\hat{r},\hat{\eta},t)$ is representative of this net production of photons at the phase space point $\hat{r},\hat{\eta}$ in the volume $d^3r \, d^3\eta$ and between the times of t and $t + dt$. The net production is the difference between the total production and the depletion or extinction. The total production consists of spontaneous emission, induced emission, and scattering into the six-dimensional phase space, or μ-space, volume element. The depletion is due to absorption and scattering from the μ-space volume element.

The relative importance of various interaction processes depends on the physical situation. A comprehensive treatment of photon-media interactions is beyond the scope of appendix A; however, the basic aspects of absorption, emission, and scattering are considered to the extent that these interactions tend to drive the photon gas to a thermodynamic equilibrium with the media (ref. 18).

Consider a molecular gas of several species n, each having a particle concentration of $[N_n]$. Spontaneous emission by these particles may proceed from an energy state i to an energy state j.

$$N_{ni} \longrightarrow N_{nj} + photon(h\hat{\eta})^{\pm} \tag{A1}$$

The discrete energy levels for the particle are obtained from the quantum mechanical description of the possible states for the isolated particle. These states may be degenerate in terms of the same energy. This degeneracy for the energy state i will be denoted by g_{ni}.

Let the probable spontaneous emission rate per particle of species n, in the i energy state, be represented by $cA_{nij}(\hat{\eta},t)$ for photons of energy $hc\eta$. The quantity A_{nij} has the dimensions of a cross section and is directly proportional to the Einstein transition probability of spontaneous emission. The cross-section dimensions are retained since only the emission in one specific direction given by $\hat{\eta}$ is considered. The total probable rate for spontaneous emission at $\hat{r}$, $\hat{\eta}$, and t is then

$$\sum_n \sum_i \sum_j N_{ni}(\hat{r},t) cA_{nij}(\hat{\eta},t) \tag{A2}$$

The summations are only over possible combinations that can produce a photon of wave number $\hat{\eta}$. In addition to this spontaneous emission, there is an induced emission that is proportional to the number of photons present. Thus, the total probable rate of emission is

$$\left[1 + f^{\pm}(\hat{r},\hat{\eta},t)\right]\sum_n\sum_i\sum_j N_{ni}(\hat{r},t)\,cA_{nij}(\hat{\eta},t) \tag{A3}$$

The probable rate of absorption per particle of n in the state j will be represented by the cross section $A_{nji}(\hat{\eta},t)$. This cross section is for the reverse of the reaction shown in equation (A1). The total probable absorption rate at $\hat{r}$, $\hat{\eta}$, and t is

$$f^{\pm}(\hat{r},\hat{\eta},t)\sum_n\sum_i\sum_j N_{nj}(\hat{r},t)\,cA_{nji}(\hat{\eta},t) \tag{A4}$$

The difference of equation (A3) minus equation (A4) is the net rate of photon production due to absorption and emission.

$$\left.D^{\pm}f^{\pm}\right|_{\substack{\text{absorption and}\\\text{emission}}} = \sum_n\sum_i\sum_j c\left[\left(1 + f^{\pm}\right)N_{ni}A_{nij} - f^{\pm}N_{nj}A_{nji}\right] \tag{A5}$$

If the cross section for a particle of species n to scatter a photon of wave number $\hat{\eta}$ to a photon of wave number $\hat{\eta}'$ is defined as $B_n(\hat{\eta},\hat{\eta}')$, then the net rate of photon production due to scattering is

$$\left.D^{\pm}f^{\pm}\right|_{\text{scattering}} = \sum_n c\int d^3\eta'\left[B_n(\hat{\eta}',\hat{\eta})f^{\pm}(\hat{r},\hat{\eta}',t)N_n(\hat{r},t) - B_n(\hat{\eta},\hat{\eta}')f^{\pm}(\hat{r},\hat{\eta},t)N_n(\hat{r},t)\right] \tag{A6}$$

If the molecular gas is assumed to be in a state of local thermodynamic equilibrium at a local temperature $T(\hat{r},t)$, then, by Boltzmann statistics,

$$\frac{N_{ni}}{N_{nj}} = \frac{g_{ni}}{g_{nj}}\,e^{-\frac{hc\eta}{kT}} \tag{A7}$$

where k is the Boltzmann constant. The energy $hc\eta$ is the difference between states i and j. In addition, if the gas is in equilibrium, the principle of detailed balance requires that the probability for a microscopic process must equal the probability for the reverse process. This relationship provides the rationale for the assumption that

$$g_{ni}A_{nij} = g_{nj}A_{nji} \tag{A8}$$

The net rate of photon production for absorption and emission may now be written

$$D^{\pm}f^{\pm}\bigg|_{\substack{\text{absorption and} \\ \text{emission}}} = c \sum_n \sum_i \sum_j N_{nj} A_{nji} \left[\left(1 + f^{\pm}\right) e^{-\frac{hc\eta}{kT}} - f^{\pm} \right] \tag{A9}$$

or

$$D^{\pm}f^{\pm}\bigg|_{\substack{\text{absorption and} \\ \text{emission}}} = ck_{\eta}^0 \left[e^{-\frac{hc\eta}{kT}} - f^{\pm}\left(1 - e^{-\frac{hc\eta}{kT}}\right) \right] \tag{A10}$$

A true absorption coefficient has been defined as

$$k_{\eta}^0 \left(\hat{r}, \hat{\eta}, t\right) \equiv \sum_n \sum_i \sum_j N_{nj}\left(\hat{r}, t\right) A_{nji}\left(\hat{\eta}, t\right) \tag{A11}$$

The induced emission term appearing in equation (A10) appears to be more like a negative absorption. Thus, an effective absorption coefficient is defined as

$$k_{\eta}\left(\hat{r}, \hat{\eta}, t\right) \equiv k_{\eta}^0\left(\hat{r}, \hat{\eta}, t\right)\left(1 - e^{-\frac{hc\eta}{kT}}\right) \tag{A12}$$

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and a Planck photon distribution function as

$$f^P[T(\hat{r},t)] \equiv \left(e^{-\frac{hc\eta}{kT}} - 1\right)^{-1} \tag{A13}$$

These two definitions may be incorporated into equation (A10) to obtain

$$\left. D^{\pm}f^{\pm} \right|_{\substack{\text{absorption and} \\ \text{emission}}} = -ck_{\eta}\left(f^{\pm} - f^P\right) \tag{A14}$$

The physical significance of f^P may be obtained readily by consideration of the case in which the solution of equation (A14) is zero for an arbitrary absorption coefficient. This value implies that the molecular gas and the photon gas are in a state of mutual or complete thermodynamic equilibrium. Thus, the Planck distribution function is the one that would not be altered by the molecular gas. It might be considered as being defined by the temperature T, whereas the temperature $T(\hat{r},t)$ is a local property of the molecular gas.

As a special case, it can be assumed that the effective absorption coefficient is independent of $\hat{\eta}$.

$$k_{\eta}(\hat{r},\hat{\eta},t) \longrightarrow k(\hat{r},t) \tag{A15}$$

In equation (A15), k is the gray gas absorption coefficient, and this assumption is commonly referred to as the gray gas assumption. In general, it is not a very realistic assumption; however, it retains the basic interactions of emission and absorption and greatly simplifies the equations.

The physical features of the scattering interaction term are best realized by considering the equations of change. The prime effect of the scattering term is a dissipation of the net relative momentum between the medium and the photon gas.

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APPENDIX B

ELECTROMAGNETIC FORMULATION OF RADIATION

In this appendix, Maxwell's equations, electromagnetic energy, electromagnetic momentum, and an electromagnetic phase space are considered.

MAXWELL'S EQUATIONS

In comparing the equations for a photon gas with those for electromagnetic waves, it must be noted that the two approaches are complementary but distinct (ref. 19). The comparison in this appendix is in terms of macroscopic or statistical average quantities. Brackets < > are used to denote statistical averages of the enclosed quantities. In general, these averages have a variation with position and time $(\hat{r}, t)$. This average is analogous to the statistical average for photon transport as presented in appendix C.

The macroscopic electric and magnetic field intensities are defined as $\langle \hat{E} \rangle$ and $\langle \hat{H} \rangle$, respectively. In addition, the displacement vector and the magnetic induction vector are defined as $\langle \hat{D} \rangle$ and $\langle \hat{B} \rangle$, respectively, in Gaussian units.

$$\langle \hat{D} \rangle = \langle \hat{E} \rangle + 4\pi \langle \hat{P} \rangle \tag{B1}$$

and

$$\langle \hat{B} \rangle = \langle \hat{H} \rangle + 4\pi \langle \hat{M} \rangle \tag{B2}$$

In equation (B1), $\langle \hat{P} \rangle$ is the polarization vector of the medium; in equation (B2), $\langle \hat{M} \rangle$ is the medium magnetization vector. In free space, $\langle \hat{P} \rangle$ and $\langle \hat{M} \rangle$ are zero.

The familiar form of Maxwell's equations (in Gaussian units) (refs. 23 and 24) is

$$\nabla \cdot \langle \hat{D} \rangle = 4\pi\delta \tag{B3}$$

$$\nabla \cdot \langle \hat{B} \rangle = 0 \tag{B4}$$

$$\nabla \times \langle \hat{E} \rangle + \frac{1}{c}\frac{\partial}{\partial t}\langle \hat{B} \rangle = 0 \tag{B5}$$

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and

$$\nabla \times \langle \hat{H} \rangle - \frac{1}{c} \frac{\partial}{\partial t} \langle \hat{D} \rangle = \frac{4\pi}{c} \delta \hat{v} \tag{B6}$$

In equations (B3) and (B6), the current is the charge density δ times the average charge velocity $\hat{v}$.

The macroscopic form of Maxwell's equations cannot be used for a comparison with the photon transport approach. It is necessary to revert to the microscopic form of Maxwell's equations, transform this form to equations of motion, and then find the rate of change of the average values (ref. 24). Thus, the macroscopic energy and momentum equations for the electromagnetic field cannot, in general, be obtained from the macroscopic forms of Maxwell's equations. This restriction is probably one of the major sources of difficulty encountered in attempts to include absorption and emission coefficients in electromagnetic field approaches to radiation transport. The microscopic form of Maxwell's equations is effectively the same as the equations obtained when the brackets are removed from all the terms in equations (B3) to (B6).

ELECTROMAGNETIC ENERGY

The energy density of an electromagnetic field is assumed to have the form

$$\langle e(\hat{r},t) \rangle = \frac{1}{8\pi} \langle \hat{E} \cdot \hat{D} + \hat{H} \cdot \hat{B} \rangle \tag{B7}$$

To compare this electromagnetic energy density with the energy density of a photon gas, two components

$$\langle e \rangle = \langle e_0 \rangle + \langle e_m \rangle \tag{B8}$$

where

$$\langle e_0 \rangle \equiv \frac{1}{8\pi} \langle E^2 + H^2 \rangle \tag{B9}$$

and

$$\langle e_m \rangle \equiv \frac{1}{2} \langle \hat{E} \cdot \hat{P} + \hat{H} \cdot \hat{M} \rangle \tag{B10}$$

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are defined. In general, the energy density $<e_0>$ contains more than just a transverse electromagnetic wave energy. It is assumed that, for a basically neutral and dilute gas, $<e_0>$ may be considered as the transverse wave energy of the field. On this basis, $<e_0>$ is compared to the total photon energy density e_T. If this assumption is not made, it is necessary to divide $<e_0>$ into two additional parts for a comparison. The energy $<e_m>$ is the field energy due to the polarization and magnetization of the material particles. On the average, for a neutral gas, $<e_m>$ is negligible except for localized fluctuations in the immediate vicinity of a material particle. In the absence of a medium, $<e_m>$ is zero, and $<e_0>$ should be identically equal to the total photon energy density e_T (table I).

It is assumed that the Poynting vector

$$<\hat{S}> \equiv \frac{c}{4\pi} <\hat{E} \times \hat{H}> \tag{B11}$$

represents a transverse wave energy flux. Although some question exists as to the general interpretation of $<\hat{S}>$ as a physical flux of energy, the vector $<\hat{S}>$ appears to be a reasonable assumption for a transverse wave field. As with $<e_0>$, it is assumed that the Poynting vector contains only contributions from the transverse waves, or that the energy flux associated with longitudinal waves is negligible. If these assumptions are valid, then the Poynting vector should be comparable to the photon energy flux vector $\hat{q}$. This definition of the Poynting vector differs from the conventional use of the "basic" magnetic field B (ref. 23). This difference is made intentionally to show analogy with the photon gas vacuum energy flux.

The energy equation is obtained from a scalar combination of the microscopic forms of equations (B5) and (B6). By employing the microscopic form of equations (B3) and (B4), performing a statistical average, and using the definitions (eqs. (B8) and (B11)), the following is obtained.

$$\frac{\partial}{\partial t} <e_0> + \nabla \cdot <\hat{S}> = - <\hat{E} \cdot \frac{\partial \hat{P}}{\partial t} + \hat{H} \cdot \frac{\partial \hat{M}}{\partial t}> - <\delta \hat{E} \cdot \hat{v}> \tag{B12}$$

Equation (B12) is to be compared to the photon gas energy equation (eq. (22) and table II). In the absence of a medium, the right-hand sides are equal to zero, and the quantities on the left-hand sides should have a one-to-one correspondence. When a dilute medium is present, this correspondence depends on the assumptions noted previously. It is also noted that if photon scattering processes are conservative, they should not appear in equation (B12). However, absorption and emission are represented on the right-hand side of equation (B12).

ELECTROMAGNETIC MOMENTUM

The momentum of an electromagnetic field is assumed to have the form

$$\langle \mathbf{g}(\hat{r},t) \rangle = \frac{1}{4\pi c} \langle \hat{\mathbf{D}} \times \hat{\mathbf{B}} \rangle \qquad (B13)$$

It is desirable to separate this momentum into two components.

$$\langle \hat{\mathbf{g}} \rangle = \frac{1}{c^2} \langle \hat{\mathbf{S}} \rangle + \frac{1}{c} \langle \hat{\mathbf{E}} \times \hat{\mathbf{M}} + \hat{\mathbf{P}} \times \hat{\mathbf{B}} \rangle \qquad (B14)$$

In the absence of a medium, the momentum of the field is the Poynting vector divided by c^2.

When the Poynting vector is examined for a simple plane monochromatic transverse wave, it is noted that $\hat{\mathbf{S}}$ is not constant. Rather, the value of $\hat{\mathbf{S}}$ oscillates with time and periodically becomes zero. Even this behavior, however, is not inconsistent with the similar but separate representation of $\hat{q}$ for one photon. It is recalled that the energy of a photon must also be specified over a finite period. The Poynting vector $\langle \hat{\mathbf{S}} \rangle$ is an average over an assumed large number of small transverse waves. Similarly, $\hat{q}$ is the average energy flux for a large number of photons.

To obtain a momentum equation, the vector product of $\hat{\mathbf{D}}$ with a microscopic form of equation (B5) is added to the vector product of $\hat{\mathbf{B}}$ with a microscopic form of equation (B6). Vector identities are used to rearrange the result, which is then integrated over wave-number space to obtain

$$\frac{\partial}{\partial t} \langle \hat{\mathbf{g}} \rangle - \nabla \cdot \langle \widetilde{\mathbf{T}}_0 \rangle = -\frac{1}{4\pi} \langle \hat{\mathbf{E}} \left(\nabla \cdot \hat{\mathbf{E}} \right) + \hat{\mathbf{H}} \left(\nabla \cdot \hat{\mathbf{H}} \right) \rangle$$

$$- \langle \delta \left(\frac{\hat{\mathbf{v}}}{c} \times \hat{\mathbf{B}} \right) + \hat{\mathbf{P}} \times \left(\nabla \times \hat{\mathbf{E}} \right) + \hat{\mathbf{M}} \times \left(\nabla \times \hat{\mathbf{H}} \right) \rangle \qquad (B15)$$

The Maxwell stress tensor for a pure electromagnetic field has been defined as

$$\langle \widetilde{\mathbf{T}}_0 (\hat{r},t) \rangle \equiv \frac{1}{4\pi} \langle \hat{\mathbf{E}}\hat{\mathbf{E}} + \hat{\mathbf{H}}\hat{\mathbf{H}} - \frac{1}{2} \widetilde{\delta} \left(E^2 + H^2 \right) \rangle$$

$$= \frac{1}{4\pi} \langle \hat{\mathbf{E}}\hat{\mathbf{E}} + \hat{\mathbf{H}}\hat{\mathbf{H}} \rangle - \widetilde{\delta} \langle e_0 \rangle \qquad (B16)$$

where $\hat{\hat{\delta}}$ is the unit second-order tensor. The physical significance of this second-order tensor is a stress due to the relative momentum flux of the vacuum components of the field. The transverse portion of the tensor should correspond to the negative of the photon gas relative pressure tensor $\hat{\hat{P}}$ (table II).

In further development of the momentum equation, the relationships for $\hat{D}$ and $\hat{B}$ (eqs. (B3) and (B4)) are included to obtain

$$\frac{1}{c^2} \frac{\partial}{\partial t} \langle \hat{S} \rangle - \nabla \cdot \langle \hat{\hat{T}}_0 \rangle = -\langle \delta \left(\hat{E} + \frac{\hat{v}}{c} \times \hat{H} \right) \rangle$$

$$+ \langle \hat{E} \left(\nabla \cdot \hat{P} \right) + \hat{H} \left(\nabla \cdot \hat{M} \right) \rangle$$

$$- \langle \frac{1}{c} \hat{E} \times \frac{\partial \hat{M}}{\partial t} + \frac{1}{c} \frac{\partial \hat{P}}{\partial t} \times \hat{H} \rangle \qquad (B17)$$

Equation (B17) will be taken as the final form of the momentum equation and should be compared to the photon gas momentum equation (eq. (23) and table II).

As in the energy equations, in the absence of a medium and any external forces, the right-hand side of equation (B17) is equal to zero. The right-hand side of equation (B17) can be manipulated into many forms. The first term is the normal form of the Lorentz force, the second term is the photon scattering, and the last term is the emission and absorption.

AN ELECTROMAGNETIC PHASE SPACE

The discussion of electromagnetic phenomena was initiated with the macroscopic form of Maxwell's equations. The statistical or bracket average used to define the macroscopic values of the fields is a classical sum, or integration, over an independent wave-number space. The requirement for this statistical average becomes apparent when a comparison is made with the photon gas formulation.

The general transverse electromagnetic field cannot be discussed without considering it as being divided into component waves. This division is physically reasonable because the field cannot, in general, be associated with the motion of one large charge. Because the component waves can have various directions and frequencies, it is convenient to consider a wave-number vector. The analogies with the photon gas are straightforward. For example, a wavelength greater than the extent of the physical volume element under consideration is difficult to observe. However, the wave number, the position, and the time must all be independent. It is not certain that $\hat{E}$ and $\hat{H}$ are the best parameters to describe the transverse waves in this space; however, they do offer advantages and will suffice for this discussion.

The effect of the wave-number space average on the form of Maxwell's equations is negligible, since the equations are linear in terms of the fields. However,

the energy and momentum equations are not linear in the fields, and their forms are definitely affected by the wave-number space average. Explicitly,

$$\langle \hat{E} \rangle \langle \hat{E} \rangle \neq \langle \hat{E}\hat{E} \rangle \tag{B18}$$

This difference can be illustrated by the following general tensor identity, which is derived for an explicit relation between Maxwell's stress tensor and the Poynting vector.

$$\tilde{\tilde{T}}_0 \cdot \tilde{\tilde{T}}_0 - \frac{1}{c^2}\hat{S}\hat{S} = \hat{\tilde{\delta}}\left(e_0^{\,2} - \frac{1}{c^2}S^2\right) \tag{B19}$$

This algebraic tensor equation only holds in what might be termed an electromagnetic phase space. In general, equation (B19) does not hold when written in terms of the corresponding statistical or macroscopic variables. In the special case in which $\hat{S}$ is equal to zero, no net energy flux occurs in any direction. For simplicity, it is assumed that the symmetric tensor $\tilde{T}_0$ has been diagonalized. Under these restrictions, equation (B19) produces three relations:

$$T_{0,xx}^{\,2} = T_{0,yy}^{\,2} = T_{0,zz}^{\,2} = e_0^{\,2} \tag{B20}$$

However, because the trace of $-\tilde{T}_0$ is e_0, equation (B10) is identically equal to zero, which indicates the absence of radiation. If this physical interpretation of the Poynting vector is correct, equation (B20) is not valid for the macroscopic quantities. However, the analogies drawn in this study tend to support the physical interpretation given to $\langle e_0 \rangle$, $\langle \hat{S} \rangle$, and $\langle \tilde{T}_0 \rangle$. Equation (B19) is also valid for the six-dimensional phase space, or μ-space, quantities associated with the photon gas parameters e_T, $\hat{q}$, and $\tilde{P}$. In general, the electromagnetic phase space is a six-dimensional μ-space. For example,

$$\hat{E} = \hat{E}\left(\hat{r},\hat{\eta}',t\right) \tag{B21}$$

The prime on the wave-number vector distinguishes the vector from a direct association with a photon wave number.

The Poynting vector and other quantities familiar to physical space are not necessarily fundamental to the μ-space. For example,

$$\hat{S}\left(\hat{r},\hat{\eta}',t\right) = \frac{c}{4\pi}\left[\hat{E}\left(\hat{r},\hat{\eta}',t\right) \times \hat{H}\left(\hat{r},\hat{\eta}',t\right)\right] \tag{B22}$$

is a double-valued function of $\hat{\eta}'$ for the two polarization components.

The most important result of this statistical treatment is that the stress tensor in a macroscopic sense cannot, in general, be represented by the microscopic form. However, this result does show that the macroscopic electromagnetic formulation is incomplete in the same sense in which the photon gas formulation is incomplete without the application of statistical thermodynamics. The algebraic relation of equation (B19) is valid for the macroscopic quantities for a one-dimensional physical space. For this case, equation (B19) is merely a scalar identity.

Although a classical electromagnetic phase space is somewhat abstract, the quantum mechanical concept is straightforward. As shown in appendix C, the quantized forms of e_0, $\hat{\tilde{T}}_0$, and $\hat{S}$ are precisely the same as those of the photon gas quantities.

The comparison of the left-hand sides of the energy equations and the momentum equation (table II) is reassuring, both with respect to the physical quantities and with respect to the mathematical operations. A comparison of the right-hand sides of these equations requires a more extensive study.

APPENDIX C

QUANTIZATION OF ELECTROMAGNETIC VARIABLES

Correspondence between the variables and the equations of change for a photon gas and a statistical (yet classical) form of the electromagnetic formulation is given in appendix B. The statistical average or bracket notation, however, is a quantum mechanical concept. To establish the correspondence between the bracket average and the photon gas wave-number average requires a quantization of the electromagnetic variables associated with transverse waves as given for example by Heitler (ref. 14). It should be noted that the quantization and the statistical averaging are conceptually and mathematically distinct. For a clarification of this point, refer to reference 30.

The electromagnetic vector potential is expanded by the use of a complex series of plane waves.

$$\hat{A} = \sum_{\lambda} \left(q_\lambda^+ \hat{A}_\lambda^+ + q_\lambda^{+*} \hat{A}_\lambda^{+*} + q_\lambda^- \hat{A}_\lambda^- + q_\lambda^{-*} \hat{A}_\lambda^{-*} \right) \tag{C1}$$

$$\hat{A}_\lambda^\pm = 2c \sqrt{\pi}\ \hat{e}_\lambda^\pm e^{i\hat{\kappa}_\lambda \cdot \hat{r}} \tag{C2}$$

$$q_\lambda^\pm = \left| q_\lambda^\pm \right| e^{i\frac{\kappa_\lambda t}{c}} \tag{C3}$$

Here, $\hat{\kappa}_\lambda$ is a wave-number vector for the plane wave, $\hat{e}_\lambda$ is a unit vector in the direction of the wave polarization, and * designates complex conjugate. For transverse waves,

$$\nabla \cdot \hat{A}_\lambda^\pm = 0 \tag{C4}$$

which makes the $\hat{e}_\lambda^\pm$ always perpendicular to $\hat{\kappa}_\lambda$. Equation (C2) is a result of the normalization over the volume integral.

$$\int_V \hat{A}_\lambda^\pm \cdot \hat{A}_\mu^{\pm *} dV = 4\pi c^2 \delta_{\lambda\mu} \tag{C5}$$

where $\delta_{\lambda\mu}$ is the Dirac delta function. The scalar coefficients in equation (C3) are normalized to the photon density

$$q_{\lambda}^{\pm *} q_{\mu}^{\pm} = \frac{\hbar}{4\kappa_{\lambda}c}\, n_{\lambda}^{\pm}\, \delta_{\lambda\mu} \tag{C6}$$

where the photon number may be equated with the photon gas density

$$n^{\pm} = \sum_{\lambda} n_{\lambda}^{\pm} = \int f^{\pm} d^3\eta \tag{C7}$$

The electromagnetic field strengths are obtained by the operator representation of the vector potential.

$$\hat{E} = -\frac{1}{c}\frac{\partial \hat{A}}{\partial t} \longrightarrow i\sum_{\lambda} \kappa_{\lambda}\left(q_{\lambda}^{+}\hat{A}_{\lambda}^{+} - q_{\lambda}^{+*}\hat{A}_{\lambda}^{+*} + q_{\lambda}^{-}\hat{A}_{\lambda}^{-} - q_{\lambda}^{-*}\hat{A}_{\lambda}^{-*}\right) \tag{C8}$$

$$\hat{H} = \nabla \times \hat{A} \longrightarrow i\sum_{\lambda}\left[q_{\lambda}^{+}\left(\hat{\kappa}_{\lambda} \times \hat{A}_{\lambda}^{+}\right) - q_{\lambda}^{+*}\left(\hat{\kappa}_{\lambda} \times \hat{A}_{\lambda}^{+*}\right)\right.$$

$$\left. + q_{\lambda}^{-}\left(\hat{\kappa}_{\lambda} \times \hat{A}_{\lambda}^{-}\right) - q_{\lambda}^{-*}\left(\hat{\kappa}_{\lambda} \times \hat{A}_{\lambda}^{-*}\right)\right] \tag{C9}$$

where the scalar coefficient time derivatives are replaced with the operator representation

$$\frac{\partial}{\partial t}q_{\lambda}^{\pm} \longrightarrow -i\frac{\kappa_{\lambda}}{c}q_{\lambda}^{\pm} \tag{C10}$$

$$\frac{\partial}{\partial t}q_{\lambda}^{\pm *} \longrightarrow +i\frac{\kappa_{\lambda}}{c}q_{\lambda}^{\pm *} \tag{C11}$$

The bracket operation for energy, for momentum, and for the total-pressure tensor is obtained directly from the operators in equations (C8) and (C9).

Energy density:

$$\langle e_0 \rangle = \langle \tfrac{1}{2}\left(E^2 + H^2\right) \rangle$$

$$= \sum_\lambda \tfrac{1}{2}\left(E_\lambda^{+2} + E_\lambda^{-2} + H_\lambda^{+2} + H_\lambda^{-2}\right)$$

$$= \sum_\lambda \left(q_\lambda^+ q_\lambda^{+*}\hat{A}_\lambda^+ \cdot \hat{A}_\lambda^{+*} + q_\lambda^- q_\lambda^{-*}\hat{A}_\lambda^- \cdot \hat{A}_\lambda^{-*}\right)$$

$$= \sum_\lambda hc\kappa_\lambda\left(n_\lambda^+ + n_\lambda^-\right) = \int hcn\left(f^+ + f^-\right)d^3n \qquad (C12)$$

Momentum density:

$$\frac{1}{c^2}\langle \hat{S} \rangle = \langle \left(\hat{E} \times \hat{H}\right) \rangle$$

$$= \sum_\lambda \left(\hat{E}_\lambda^+ \times \hat{H}_\lambda^+ + \hat{E}_\lambda^- \times \hat{H}_\lambda^-\right)$$

$$= \sum_\lambda \kappa_\lambda \left\{ q_\lambda^+ q_\lambda^{+*}\left[\hat{A}_\lambda^+ \times \left(\hat{\kappa}_\lambda \times \hat{A}_\lambda^{+*}\right)\right] \right.$$

$$\left. + q_\lambda^- q_\lambda^{-*}\left[\hat{A}_\lambda^- \times \left(\kappa_\lambda \times \hat{A}_\lambda^{-*}\right)\right]\right\} \qquad (C13)$$

Since

$$\hat{\kappa}_\lambda \cdot \hat{A}_\lambda^{\pm} = \hat{\kappa}_\lambda \cdot \hat{A}_\lambda^{\pm*} = 0 \qquad (C14)$$

then

$$\frac{1}{c^2} \langle \hat{S} \rangle = \sum_\lambda \kappa_\lambda \left(q_\lambda^+ q_\lambda^{+*} \hat{A}_\lambda^+ \cdot \hat{A}_\lambda^{+*} + q_\lambda^- q_\lambda^{-*} \hat{A}_\lambda^- \cdot \hat{A}_\lambda^{-*} \right) \hat{\kappa}_\lambda$$

$$= \sum_\lambda h \hat{\kappa}_\lambda \left(n_\lambda^+ + n_\lambda^- \right) = \int h \hat{\eta} \left(f^+ + f^- \right) d^3\eta \qquad (C15)$$

Total-pressure tensor:

$$- \langle \hat{\tilde{T}}_0 \rangle = \langle e_0 \rangle \hat{\tilde{\delta}} - \langle \hat{E}\hat{E} + \hat{H}\hat{H} \rangle$$

$$= \langle e_0 \rangle \hat{\tilde{\delta}} - \sum_\lambda \left(\hat{E}_\lambda^+ \hat{E}_\lambda^+ + \hat{E}_\lambda^- \hat{E}_\lambda^- + \hat{H}_\lambda^+ \hat{H}_\lambda^+ + \hat{H}_\lambda^- \hat{H}_\lambda^- \right) \qquad (C16)$$

$$\hat{E}_\lambda^\pm \hat{E}_\lambda^\pm = \kappa_\lambda^2 q_\lambda^\pm q_\lambda^{\pm*} \hat{A}_\lambda^\pm \hat{A}_\lambda^{\pm*}$$

$$= \kappa_\lambda^2 q_\lambda^\pm q_\lambda^{\pm*} \left(4\pi c^2 \right) \hat{e}_\lambda^\pm \hat{e}_\lambda^\pm$$

$$= hc\kappa_\lambda n_\lambda^\pm \hat{e}_\lambda^\pm \hat{e}_\lambda^\pm \qquad (C17)$$

In a similar manner, since $\hat{e}_\lambda^\pm$ and $\hat{\kappa}_\lambda$ are orthogonal,

$$\hat{H}_\lambda^\pm \hat{H}_\lambda^\pm = hc\kappa_\lambda n_\lambda^\pm \left(\hat{\tilde{\delta}} - \hat{e}_\lambda^\pm \hat{e}_\lambda^\pm - \frac{\hat{\kappa}_\lambda \hat{\kappa}_\lambda}{\kappa_\lambda^2} \right) \qquad (C18)$$

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Therefore

$$
\begin{aligned}
-\langle \widetilde{T}_0 \rangle &= \langle e_0 \rangle \, \widetilde{\widehat{\delta}} - \sum_\lambda \hbar c \kappa_\lambda \left(n_\lambda^+ + n_\lambda^- \right) \left(\widetilde{\widehat{\delta}} - \frac{\widehat{\kappa}_\lambda \widehat{\kappa}_\lambda}{\kappa_\lambda{}^2} \right) \\
&= \sum_\lambda \hbar c \frac{\widehat{\kappa}_\lambda \widehat{\kappa}_\lambda}{\kappa_\lambda} \left(n_\lambda^+ + n_\lambda^- \right) \\
&= \int_{\widehat{\eta}} \hbar c \frac{\widehat{\eta}\,\widehat{\eta}}{\eta} \left(f^+ + f^- \right) d^3\eta \equiv \widehat{\widehat{P}}
\end{aligned}
\tag{C19}
$$

Thus, the quantization of the transverse electromagnetic field variables substantiates the photon gas representation.

APPENDIX D

COMPARISON OF ENERGY AND MOMENTUM EQUATIONS OBTAINED FROM THE PHOTON GAS AND AN ELECTROMAGNETIC FORMULATION

In this appendix, energy and momentum equations obtained from the photon gas and an electromagnetic formulation (ref. 19) are stated for comparison.

ENERGY

Photon Gas

$$\frac{\partial e_T}{\partial t} + \nabla \cdot \hat{q} = \int_{\hat{\eta}} hc\eta \left(D^+ f^+ + D^- f^- \right) d^3\eta \tag{22}$$

Electromagnetic

$$\frac{\partial}{\partial t} \langle e_0 \rangle + \nabla \cdot \langle \hat{S} \rangle = - \langle \hat{E} \cdot \left(\frac{\partial \hat{P}}{\partial t} + \delta \hat{v} \right) + \hat{H} \cdot \frac{\partial \hat{M}}{\partial t} \rangle \tag{D1}$$

MOMENTUM

Photon Gas

$$\frac{\partial \hat{g}}{\partial t} + \nabla \cdot \tilde{P} = \int_{\hat{\eta}} h\hat{\eta} \left(D^+ f^+ + D^- f^- \right) d^3\eta$$

$$+ \frac{1}{c} \left(\hat{\alpha}^+ \times \hat{g}^+ \right) + \frac{1}{c} \left(\hat{\alpha}^- \times \hat{g}^- \right) \tag{23}$$

Electromagnetic

$$\frac{\partial}{\partial t} \left(\frac{\langle \hat{S} \rangle}{c^2} \right) - \nabla \cdot \langle \tilde{T}_0 \rangle = -\frac{1}{c} \langle \hat{E} \times \frac{\partial \hat{M}}{\partial t} + \left(\frac{\partial \hat{P}}{\partial t} + \delta \hat{v} \right) \times \hat{H} \rangle$$

$$+ \langle \hat{E} (\nabla \cdot \hat{P} - \delta) + \hat{H} (\nabla \cdot \hat{M}) \rangle \tag{D2}$$

APPENDIX E

ISOTROPIC PHOTON GAS PHYSICAL SPACE DENSITIES

In this appendix, terms of isotropic photon gas physical space densities are defined on the basis of the photon distribution function in phase space f.

$$f^{\pm} = \left[e^{\frac{hcn}{kT_{\pm}}\left(1 - \frac{\hat{V}^{\pm}}{c} \cdot \frac{\hat{n}}{n}\right)} - 1 \right]^{-1} \tag{40}$$

Photon number:

$$n^{\pm} = \frac{20}{27}\frac{\sigma}{kc} T_{\pm}^{3}\gamma_{\pm}^{4} \tag{E1}$$

Total energy:

$$e_{\pm_{T}} = 2\frac{\sigma}{c} T_{\pm}^{4}\gamma_{\pm}^{4} + \frac{8}{3}\frac{\sigma}{c} T_{\pm}^{4}\gamma_{\pm}^{6} \frac{V_{\pm}^{2}}{c^{2}} \tag{E2}$$

Linear momentum:

$$\hat{g}_{\pm} = \frac{\hat{q}_{\pm}}{c^{2}} = \frac{8}{3}\frac{\sigma}{c} T_{\pm}^{4}\gamma_{\pm}^{6}\hat{V}_{\pm} \tag{E3}$$

Intrinsic angular momentum:

$$\hat{\ell}_{\pm} = \pm\frac{20}{27}\frac{\hbar}{kc^{2}} \sigma T_{\pm}^{3}\gamma_{\pm}^{4} \hat{V}_{\pm} \tag{E4}$$

Total pressure:

$$\tilde{\hat{P}}_{\pm} = \frac{2}{3} \frac{\sigma}{c} T_{\pm}^{4}\gamma_{\pm}^{4}\tilde{\hat{\delta}} + \frac{8}{3}\frac{\sigma}{c}T_{\pm}^{4}\gamma_{\pm}^{6} \frac{\hat{V}_{\pm}\hat{V}_{\pm}}{c^{2}} \tag{E5}$$

Intrinsic angular momentum flux:

$$\hat{\hat{m}}_{\pm} = \pm\frac{10}{27}\frac{\hbar}{k}\sigma T_{\pm}^{3}\left[\hat{\hat{\delta}}\left(\gamma_{\pm}^{2} - \frac{1}{3}\right) + \gamma_{\pm}^{2}\left(2\gamma_{\pm}^{2} - 1\right)\frac{\hat{V}_{\pm}\hat{V}_{\pm}}{c^{2}} + \hat{\hat{F}}\left(\frac{\hat{V}_{\pm}}{c}\right)\right]$$ (E6)

Entropy:

$$s_{\pm} = \frac{8}{3}\frac{\sigma}{c}T_{\pm}^{3}\gamma_{\pm}^{4}$$ (E7)

Entropy flux:

$$\hat{\sigma} = s^{\pm}V_{\pm}$$ (E8)

where

$$\gamma_{\pm} \equiv \left(1 - \frac{V_{\pm}^{2}}{c^{2}}\right)^{-\frac{1}{2}}$$ (E9)

$$\hat{\hat{F}}(\hat{A}) \equiv \left(3\hat{A}\hat{A} - A^{2}\hat{\hat{\delta}}\right)\sum_{n=2}^{\infty}\frac{\left(A^{2}\right)^{n-2}}{2n + 1}$$ (E10)

$$\text{trace}\left(\hat{\hat{F}}\right) = 0$$ (E11)

APPENDIX F

INTEGRALS REQUIRED FOR FORMULATION

     The evaluation of integrals over wave-number space is simplified by application of the following mathematical relationships. In this appendix, x, u, and φ are dummy variables, n is a dummy numeral, and a is an arbitrary constant.

$$\left(e^x - 1\right)^{-1} = \sum_{n=1}^{\infty} e^{-nx} \tag{F1}$$

$$\ln\left(1 - e^{-x}\right) = \sum_{n=1}^{\infty} \frac{1}{n} e^{-nx} \tag{F2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \tag{F3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{\pi^3}{25.79436\ \ldots} \simeq \frac{\pi^4}{81} \tag{F4}$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \tag{F5}$$

$$\int_0^{\pi} \frac{\sin\varphi\,d\varphi}{\left(1 + a\cos\varphi\right)^4} = \int_{-1}^{+1} \frac{du}{\left(1 + au\right)^4} = \frac{2\left(1 + \frac{1}{3}a^2\right)}{\left(1 - a^2\right)^3} \tag{F6}$$

$$\int_{-1}^{+1} \frac{u\,du}{\left(1 + au\right)^4} = \frac{-8a}{3\left(1 - a^2\right)^3} \tag{F7}$$

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$$\int_{-1}^{+1} \frac{du}{(1 + au)^3} = \frac{2}{\left(1 - a^2\right)^2} \tag{F8}$$

$$\int_{-1}^{+1} \frac{u\ du}{(1 + au)^3} = \frac{-2a}{\left(1 - a^2\right)^2} \tag{F9}$$

Also, the Stefan-Boltzmann constant is defined as

$$\sigma \equiv \frac{2\pi^5 k^4}{15h^3 c^2} \tag{41}$$

An example of the application of these integrals to the evaluation of the photon gas densities is provided in appendix H for the nonisotropic photon gas.

APPENDIX G

THE LORENTZ TRANSFORMATION

For consideration of the relativistic aspects of a photon gas, a four-dimensional space of position and time is required. The position vector in this space (ref. 21) is

$$
x_\mu = \left\{ \begin{array}{c} \hat{r} \\ ict \end{array} \right\} = \left\{ \begin{array}{c} x \\ y \\ z \\ ict \end{array} \right\}
\tag{G1}
$$

An element of length in this space is an invariant scalar

$$
ds \equiv -\sqrt{c^2\,dt^2 - dr^2}
\tag{G2}
$$

The generalized velocity is a four-space vector

$$
V_\mu \equiv \frac{dx_\mu}{ds} = \gamma \left\{ \begin{array}{c} \dfrac{\hat{V}}{c} \\ i \end{array} \right\}
\tag{G3}
$$

where $\hat{V} \equiv \dfrac{d\hat{r}}{dt}$ and $\gamma \equiv \left(1 - \dfrac{V^2}{c^2}\right)^{-\frac{1}{2}}$. The velocity is a true vector in a four-dimensional space; therefore, its magnitude is a true scalar.

$$
V_\mu V_\mu = \gamma^2\left(\frac{V^2}{c^2} - 1\right) = -1
\tag{G4}
$$

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The stress-energy density tensor in this four-dimensional space for the photon gas and transverse electromagnetic waves (refs. 14, 22, and 28) is

$$
\tau_{\mu\nu} \equiv
\begin{bmatrix}
-\hat{\hat{P}} & -ic\hat{g} \\
-\dfrac{i}{c}\hat{q} & +e_T
\end{bmatrix}
=
\begin{bmatrix}
\langle\hat{\hat{T}}_0\rangle & -ic\langle\hat{g}_0\rangle \\
-\dfrac{i\langle S\rangle}{c} & +\langle e_0\rangle
\end{bmatrix}
\tag{G5}
$$

This four tensor has an invariant magnitude

$$
\tau_{\mu\mu} = e_T - \text{trace}\left(\hat{\hat{P}}\right) = 0
\tag{G6}
$$

To transform four vectors or four tensors from one coordinate system to another (prime ') moving at a constant velocity $\hat{V}$ requires the application of the Lorentz transformation matrices (ref. 21).

$$
a_{\mu\nu} \equiv
\begin{bmatrix}
\left(\hat{\hat{\delta}} + \dfrac{\gamma^2}{\gamma+1}\dfrac{\hat{V}\hat{V}}{c^2}\right) & -\dfrac{i\gamma\hat{V}}{c} \\
+i\gamma\dfrac{\hat{V}}{c} & \gamma
\end{bmatrix}
\tag{G7}
$$

$$
\tilde{a}_{\mu\nu} \equiv
\begin{bmatrix}
\left(\hat{\hat{\delta}} + \dfrac{\gamma^2}{\gamma+1}\dfrac{\hat{V}\hat{V}}{c^2}\right) & +i\gamma\dfrac{\hat{V}}{c} \\
-i\gamma\dfrac{\hat{V}}{c} & +\gamma
\end{bmatrix}
\tag{G8}
$$

The transformation for four vectors and four tensors is obtained by

$$
\left.
\begin{aligned}
A_\mu &= a_{\mu\nu}A_\nu' & A_\nu' &= \tilde{a}_{\nu\mu}A_\mu \\
B_{\mu\nu} &= a_{\mu\phi}B_{\phi\lambda}'\tilde{a}_{\lambda\nu} & B_{\phi\lambda}' &= \tilde{a}_{\phi\mu}B_{\mu\nu}a_{\nu\lambda}
\end{aligned}
\right\}
\tag{G9}
$$

If a photon gas is at a complete thermodynamic equilibrium and at rest in a given coordinate system, the stress-energy density four tensor is

$$\overset{0}{\tau}_{\phi\lambda} = \frac{4}{3}\frac{\sigma}{c}T_0^4 \begin{bmatrix} -\overset{\approx}{\delta} & 0 \\ 0 & +3 \end{bmatrix} \tag{G10}$$

To an observer moving with a velocity $\hat{V}$ with respect to this static photon gas, the stress-energy density four tensor is

$$\tau_{\mu\nu} = a_{\mu\phi}\overset{0}{\tau}_{\phi\lambda}\tilde{a}_{\lambda\nu}$$

$$= \frac{4}{3}\frac{\sigma}{c}T_0^4 \begin{bmatrix} -\overset{\approx}{\delta} - 4\gamma^2\dfrac{\hat{V}\hat{V}}{c^2} & -i4\gamma^2\dfrac{\hat{V}}{c} \\ & \\ -i4\gamma^2\dfrac{\hat{V}}{c} & +3 + 4\gamma^2\dfrac{V^2}{c^2} \end{bmatrix} \tag{G11}$$

This tensor is identical to the stress-energy density four tensor obtained through statistical thermodynamics for the photon gas with both an energy and a momentum constraint (appendix K) providing

$$T_0 = \gamma T \tag{G12}$$

Therefore, T is the apparent or static temperature seen by a general observer, whereas T_0 is the total or rest temperature of the photon gas. On the other hand, the distribution function is not affected by relative motion even though the temperature and wave number are. The distribution functions in rest and moving frames, respectively, are

$$f_0 = \left(e^{\frac{hcn_0}{kT_0}} - 1\right)^{-1} \tag{G13}$$

$$f = \left[e^{\frac{hcn}{kT}\left(1-\frac{\hat{V}}{c}\cdot\frac{\hat{n}}{n}\right)} - 1\right]^{-1} \tag{G14}$$

If $f = f_0$ and $T_0 = \gamma T$, then

$$\eta_0 = \eta \left(1 - \frac{\hat{\mathbf{v}}}{c} \cdot \frac{\hat{\mathbf{n}}}{\eta}\right) \gamma \qquad \text{(G15)}$$

which is the correct Doppler relationship between the relative wave-number magnitudes for a given photon (ref. 14). Thus, the characteristics of a photon gas moving with a velocity can be obtained either through statistical thermodynamics with a momentum constraint or by a Lorentz transformation of a static photon gas.

APPENDIX H

THE NONISOTROPIC PHOTON GAS

Statistical thermodynamics is classically restricted to the determination of the most probable distribution function, producing the highest entropy, subject to the constraint of energy. The extension to constraints involving other conserved quantities, such as linear momentum and angular momentum, is straightforward as given, for example, in this report. However, since the concepts of statistical thermodynamics can be obtained from information theory, the principle should be applicable to general independent constraints beyond the simple body parameters that are conserved (refs. 7 to 13 and 26). For example, a photon gas may have the highest entropy possible under the constraints of the initial conditions, the boundary conditions, or the medium and not be isotropic. This case is considered as a demonstration of the concept. For simplicity in notation, a randomly polarized photon gas is assumed. The extension to the case of intrinsic angular momentum is obvious but cumbersome.

Consider the constraints on the photon gas to be the total-pressure tensor and the momentum density

$$\tilde{P} = 2 \int_{\hat{\eta}} hc \, \frac{\hat{\eta}\hat{\eta}}{\eta} f \, d^3\eta \tag{H1}$$

$$\hat{g} = 2 \int_{\eta} h\hat{\eta} f \, d^3\eta \tag{H2}$$

where

$$f^+ = f^- \equiv f \tag{H3}$$

The most probable distribution function under these constraints is obtained from the variational equation for the maximum entropy (eq. (28)).

$$\delta \left\{ 2k \int_{\hat{\eta}} \left[\ln (1 + f) + f \ln \left(\frac{1 + f}{f} \right) \right] d^3\eta \right.$$

$$\left. + \tilde{\beta} : \left[\tilde{P} - 2 \int_{\hat{\eta}} hc\frac{\hat{\eta}\hat{\eta}}{\eta} f \, d^3\eta + \hat{V} \cdot \left(\hat{g} - 2 \int_{\hat{\eta}} h\hat{\eta} f \, d^3\eta \right) \right] \right\} = 0 \tag{H4}$$

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Here, $\hat{\hat{\beta}}$ and $\hat{\hat{\beta}} \cdot \hat{V}$ are the tensor and vector multipliers for the total-pressure tensor and momentum, respectively. Equation (H4) produces the distribution function for a nonisotropic, randomly polarized photon gas with momentum.

$$f = \left[e^{hc\eta\hat{\hat{\beta}}:\left(\frac{\hat{n}}{\eta} - \frac{\hat{V}}{c}\right)\frac{\hat{n}}{\eta}} - 1 \right]^{-1} \tag{H5}$$

The unknown Lagrangian multipliers are, in principle, obtained through equations (H1) and (H2). In this case, however, it is simpler to first obtain the entropy in terms of $\hat{\hat{\beta}}$ and $\hat{V}$. Defining

$$\alpha \equiv \hat{\hat{\beta}}:\left(\frac{\hat{n}}{\eta} - \frac{\hat{V}}{c}\right)\frac{\hat{n}}{\eta} \tag{H6}$$

the entropy integral, based on equations (F1) and (F2), becomes

$$s = 2k \sum_{n=1}^{\infty} \int_{\hat{n}} \left(\frac{1}{n} + hc\eta\alpha\right) e^{-nhc\alpha\eta} d^3\eta \tag{H7}$$

If spherical wave-number coordinates are used

$$\int_{\hat{n}} d^3\eta \longrightarrow \int_0^{2\pi} d\theta \int_0^\eta \sin\varphi \, d\varphi \int_0^\infty \eta^2 d\eta \tag{H8}$$

Using equations (F5), (F3), and (41), the entropy becomes

$$s = 16k\left(\sum_{n=1}^{\infty} \frac{1}{n^4}\right) \int_0^{2\pi} d\theta \int_0^\pi \frac{\sin\varphi \, d\varphi}{(hc\alpha)^3}$$

$$= \frac{16}{3}\frac{\sigma}{k^3}\left(\frac{1}{4\pi}\int_0^{2\pi} d\theta \int_0^\pi \frac{\sin\varphi \, d\varphi}{\alpha^3}\right) \tag{H9}$$

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To describe the nonisotropic state, a tensor temperature is defined.

$$\tilde{\tilde{T}} \equiv \frac{1}{k}\left(\tilde{\tilde{\beta}}^{-1}\right) \tag{H10}$$

where

$$\tilde{\tilde{T}} \cdot \tilde{\tilde{\beta}} = \frac{1}{k} \tilde{\tilde{\delta}} \tag{H11}$$

$$\left|\tilde{\tilde{T}}\right| = \frac{1}{k^3 \left|\tilde{\tilde{\beta}}\right|} \tag{H12}$$

The scalar temperature with its classical significance is retained.

$$T \equiv \frac{1}{3} \text{ trace } \left(\tilde{\tilde{T}}\right) \tag{H13}$$

The entropy for a nonisotropic photon gas may be written

$$s = \frac{16}{3}\frac{\sigma}{c}\gamma^4\left|\tilde{\tilde{T}}\right| \Phi \tag{H14}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{H15}$$

and a dimensionless number that is only a function of the rest principle direction temperatures $(T_{22}/T_{11}, \ T_{33}/T_{11})$ has been defined

$$\Phi \equiv \frac{\left|\tilde{\tilde{\beta}}\right|}{4\pi\gamma^4} \int\limits_0^{2\pi} d\theta \int\limits_0^{\pi} \frac{\sin\varphi \ d\varphi}{\left[\tilde{\tilde{\beta}}:\left(\dfrac{\hat{n}}{\eta} - \dfrac{\hat{v}}{c}\right)\dfrac{\hat{n}}{\eta}\right]^3} \tag{H16}$$

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In general, Φ is an elliptic integral. This integral has been computed numerically and is tabulated for the rest condition together with an approximate expression Φ' in table H-I. It is obvious from this table that Φ is a very weak function of the deviations from an isotropic temperature condition.

The entropy for a nonisotropic situation is always lower than the isotropic entropy for the same energy, a condition verified by the fact that

$$\left[\frac{1}{3}\ \text{trace}\ (\hat{\hat{T}})\right]^3 \geq \left|\hat{\hat{T}}\right|\Phi \tag{H17}$$

The randomly polarized, nonisotropic photon gas variables can be expressed in the integral forms

$$n \equiv 2\!\int\! f\ d^3\eta = \frac{4}{(hc)^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^3}\right)\int_0^{2\pi}d\theta\int_0^{\pi}\frac{\sin\varphi\ d\varphi}{\alpha^3} \tag{H18}$$

$$n\hat{V} = 2\!\int\! c\frac{\hat{n}}{n}\ f\ d^3\eta = \frac{4c}{(hc)^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^3}\right)\int_0^{2\pi}d\theta\int_0^{\pi}\frac{\hat{n}}{n}\frac{\sin\varphi\ d\varphi}{\alpha^3} \tag{H19}$$

$$s \equiv 2k\!\int\!\left[\ln\ (1+f) + f\ln\left(\frac{1+f}{f}\right)\right]d^3\eta$$

$$= \frac{16k}{(hc)^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\int_0^{2\pi}d\theta\int_0^{\pi}\frac{\sin\varphi\ d\varphi}{\alpha^3} \tag{H20}$$

$$\hat{\sigma} \equiv 2kc\int_{\hat{\eta}}\frac{\hat{n}}{n}\left[\ln\ (1+f) + f\ln\left(\frac{1+f}{f}\right)\right]d^3\eta$$

$$= \frac{16kc}{(hc)^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\int_0^{2\pi}d\theta\int_0^{\pi}\frac{\hat{n}}{n}\frac{\sin\varphi\ d\varphi}{\alpha^3} \tag{H21}$$

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$$\hat{\mathbf{g}} \equiv 2 \int_{\hat{\eta}} h\hat{\eta}f \ d^3\eta = \frac{12}{c\,(hc)^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\int_0^{2\pi} d\theta \int_0^{\pi} \frac{\hat{\eta}}{\eta}\frac{\sin\varphi \ d\varphi}{\alpha^4} \tag{H22}$$

$$\tilde{\mathbf{P}} \equiv 2 \int_{\hat{\eta}} hc\,\frac{\hat{\eta}\hat{\eta}}{\eta}\,f \ d^3\eta = \frac{12}{(hc)^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\int_0^{2\pi} d\theta \int_0^{\pi} \frac{\hat{\eta}\hat{\eta}}{\eta^2}\frac{\sin\varphi \ d\varphi}{\alpha^4} \tag{H23}$$

$$\tilde{\mathbf{w}} \equiv 2 \int_{\hat{\eta}} hc^2\,\frac{\hat{\eta}\hat{\eta}\hat{\eta}}{\eta^2}f \ d^3\eta = \frac{12}{(hc)^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)\int_0^{2\pi} d\theta \int_0^{\pi} \frac{\hat{\eta}\hat{\eta}\hat{\eta}}{\eta^3}\frac{\sin\varphi \ d\varphi}{\alpha^4} \tag{H24}$$

The third-order tensor $\hat{\hat{\mathbf{w}}}\hat{}$ appears in the third moment or stress equation, which, from equation (21), is

$$\frac{\partial \hat{\hat{\mathbf{P}}}}{\partial t} + \nabla\cdot\tilde{\mathbf{w}} = 2\int_{\hat{\eta}} hc\frac{\hat{\eta}\hat{\eta}}{\eta}\,Df \ d^3\eta \tag{H25}$$

for $D^+ = D^- = D$ and $\hat{\alpha}^+ = \hat{\alpha}^- = 0$. Reductions in $\hat{\hat{\mathbf{w}}}\hat{}$ are $\hat{q}$ (in index notation).

$$w_{ijj} = w_{jij} = w_{jji} = q_i \tag{H26}$$

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In addition, the trace of equation (H25) is the energy equation. Noting the pattern in equations (H18) to (H24), the following interrelationships may be obtained.

$$n = \frac{40}{27} \frac{\sigma}{ck} \left| \overset{\approx}{T} \right| \gamma^4 \phi \tag{H27}$$

$$s = \frac{18}{5} kn \tag{H28}$$

$$\frac{\hat{V}}{c} = \frac{\int \frac{\hat{\eta}}{\eta} f \, d^3\eta}{\int f \, d^3\eta} \tag{H29}$$

$$\hat{\sigma} = s\hat{V} \tag{H30}$$

$$\hat{g} = \frac{1}{4c} \overset{\approx}{T} \cdot \frac{\partial s}{\partial \hat{V}} \tag{H31}$$

$$\overset{\approx}{P} - \hat{V}\hat{g} = -\frac{1}{4k} \frac{\partial s}{\partial \hat{\beta}} \equiv \overset{\approx}{p} \tag{H32}$$

where $\overset{\approx}{p}$ is the static-pressure tensor.

$$\overset{\approx}{w} = -\frac{1}{4k} \frac{\partial}{\partial \overset{\approx}{\beta}} \hat{\sigma} + \hat{V}\overset{\approx}{P}$$

$$= \overset{\approx}{P}\hat{V} + \hat{V}\overset{\approx}{P} - \hat{V}\hat{g}\hat{V}$$

$$= \overset{\approx}{p}\hat{V} + \hat{V}\overset{\approx}{p} + \hat{V}\hat{V}\hat{g} \tag{H33}$$

The expressions containing derivatives with respect to the Lagrangian multipliers are, for example,

$$\frac{\partial s}{\partial \widetilde{\widetilde{\beta}}} \equiv \begin{bmatrix} \frac{\partial s}{\partial \beta}_{11} & \frac{\partial s}{\partial \beta}_{12} & \frac{\partial s}{\partial \beta}_{13} \\[2mm] \frac{\partial s}{\partial \beta}_{21} & \frac{\partial s}{\partial \beta}_{22} & \frac{\partial s}{\partial \beta}_{23} \\[2mm] \frac{\partial s}{\partial \beta}_{31} & \frac{\partial s}{\partial \beta}_{32} & \frac{\partial s}{\partial \beta}_{33} \end{bmatrix} \tag{H34}$$

Equations (H33) and (H26) may be used to relate the momentum and velocity.

$$\hat{g} = \frac{\gamma^2}{c^2}\left(3p\hat{V} + \hat{V}\cdot\widetilde{p}\right) \tag{H35}$$

It is noted that, since

$$\widetilde{\widetilde{\beta}}:\frac{\partial s}{\partial \widetilde{\widetilde{\beta}}} = -3s \tag{H36}$$

$$\widetilde{\widetilde{\beta}}:\frac{\partial}{\partial \widetilde{\widetilde{\beta}}}\left(\ln \phi\right) = 0 \tag{H37}$$

the thermodynamic relationship between entropy and the local nonisotropic pressure is

$$s = \frac{4}{3}k\widetilde{\widetilde{\beta}}:\left(\widetilde{\widetilde{P}} - \hat{V}\hat{g}\right) = \frac{4}{3}k\widetilde{\widetilde{\beta}}:\widetilde{\widetilde{p}} \tag{H38}$$

Equation (H36) is readily obtained through the indicated operation on equation (H20). In a similar manner, equation (H37) follows from the same operation on equation (H16).

The momentum and pressure for a nonisotropic photon gas may be written in a more convenient form.

$$\hat{g} = \frac{4\sigma}{3c^3} \left|\overset{\approx}{T}\right| \gamma^4 \phi \overset{\approx}{T} \cdot \left[4\lambda^2 \hat{V} + c^2 \frac{\partial}{\partial \hat{V}}\left(\ln \phi\right)\right] \tag{H39}$$

$$\widetilde{p} = \frac{4\sigma}{3c} \left|\overset{\approx}{T}\right| \gamma^4 \phi \left[\overset{\approx}{T} - \frac{1}{k}\frac{\partial}{\partial \overset{\approx}{\beta}}\left(\ln \phi\right)\right] \tag{H40}$$

$$\overset{\approx}{p} = \frac{9}{10} nk\overset{\approx}{T} - \frac{9}{10} n\frac{\partial}{\partial \overset{\approx}{\beta}} \ln \phi \tag{H41}$$

Equation (H25) and the momentum equation form a complete set of nine equations in terms of the nine dependent variables, $\hat{V}$ and $\hat{T}$ (i.e., $\hat{T}$ is symmetric). The solution of these equations in general would be quite difficult. The first obvious simplification is to approximate $\hat{g}$ and p by the dominant terms in equations (H39) and (H40).

$$\hat{g} \simeq \frac{16}{3} \frac{\sigma}{c^3}\left|\overset{\approx}{T}\right| \gamma^6 \phi \overset{\approx}{T} \cdot \hat{V} \tag{H42}$$

$$\widetilde{p} \simeq \frac{4\sigma}{3c}\left|\overset{\approx}{T}\right| \gamma^4 \phi \overset{\approx}{T} \tag{H43}$$

$$\therefore \quad \overset{\approx}{w} \simeq \frac{4\sigma}{3c}\left|\overset{\approx}{T}\right| \gamma^4 \phi \left(\overset{\approx}{T}\hat{V} + \hat{V}\overset{\approx}{T} + 4\gamma^2 \hat{V}\hat{V}\hat{V}\right) \tag{H44}$$

The relationships obtained for a nonisotropic photon gas are obvious generalizations of the isotropic case through the generalization of the scalar temperature concept to one of a second-order tensor. Conceptually, this generalization is immediately obvious. The scalar temperature is a statistical weighting parameter for the "conserved" quantity energy. A complex system of a large number of particles has many more degrees of freedom than are accounted for by the single-particle degrees of energy, linear momentum, angular momentum, and mass.

TABLE H-I.- ELLIPTIC INTEGRAL Φ [a,b]

$$\Phi_0 = \frac{4\pi \int_0^{2\pi} d\theta \int_0^{\pi} \left(\hat{\beta} : \frac{\hat{n}\hat{n}}{\eta^2}\right)^{-3} \sin\varphi \, d\varphi}{\left[\int_0^{2\pi} d\theta \int_0^{\pi} \left(\hat{\beta} : \frac{\hat{n}\hat{n}}{\eta^2}\right)^{-\frac{3}{2}} \sin\varphi \, d\varphi\right]^2} \geq 1$$

$$\Phi_0 \simeq \Phi_0' \equiv \frac{\sqrt{\frac{1}{3} \text{ trace} (\hat{\hat{T}} \cdot \hat{\hat{T}})}}{\left|\hat{\hat{T}}\right|^{\frac{1}{3}}}$$

T_{11}/T_{22}	T_{11}/T_{33}	Φ_0	Φ_0'/Φ_0	$\ln \Phi_0$
1	1	1.00000	1.00	0.0000
1	2	1.08904	1.00	.0853
2	2	1.1087	1.01	.1032
2	4	1.3139	1.005	.2730
2	8	1.70088	.962	.53115
2	20	2.55385	.867	.9376
1	4	1.3440	.978	.2957
1	10	1.9613	.904	.6736
2	10	1.86882	.936	.6253
4	4	1.61757	1.020	.4809
4	10	2.12456	.963	.7536
10	10	2.9469	.916	1.8675
5	10	2.2728	--	.8210
10	10	2.9469	--	1.8075
10	20	3.9966	--	1.3854
10	40	5.5388	--	1.7118
10	100	8.6490	--	2.157

[a] Defined in equation (H16).

[b] $\Phi_0 \equiv \Phi$ when $\hat{V}/c = 0$.

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APPENDIX I

ISOTROPIC PHOTON GAS WITH LINEAR MOMENTUM AND

INTRINSIC ANGULAR MOMENTUM

In this appendix, terms for an isotropic photon gas with linear momentum and intrinsic angular momentum are defined.

Photon number density:

$$n = \frac{20}{27} \frac{\sigma}{kc} T^3 \left(\gamma_+^{\ 4} + \gamma_-^{\ 4} \right) \tag{I1}$$

Total energy density:

$$e_T = \frac{2\sigma}{c} T^4 \left[\gamma_+^{\ 4} + \gamma_-^{\ 4} + \frac{4}{3} \left(\gamma_+^{\ 6} \frac{V_+^{\ 2}}{c^2} + \gamma_-^{\ 6} \frac{V_-^{\ 2}}{c^2} \right) \right]$$

$$= \frac{2\sigma}{3c} T^4 \left[4 \left(\gamma_+^{\ 6} + \gamma_-^{\ 6} \right) - \gamma_+^{\ 4} - \gamma_-^{\ 4} \right] \tag{I2}$$

Momentum density:

$$\hat{g} = \frac{8}{3} \frac{\sigma}{c^3} T^4 \left(\gamma_+^{\ 6} \hat{V}_+ + \gamma_-^{\ 6} \hat{V}_- \right) \tag{I3}$$

Intrinsic angular momentum density:

$$\hat{\ell} = \frac{20}{27} \frac{\hbar\sigma}{c^2 k} T^3 \left(\gamma_+^{\ 4} \hat{V}_+ - \gamma_-^{\ 4} \hat{V}_- \right) \tag{I4}$$

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Total pressure:

$$\widetilde{P} = \frac{2}{3}\frac{\sigma}{c}T^4\left[\left(\gamma_+^4 + \gamma_-^4\right)\widetilde{\delta} + 4\left(\gamma_+^6\frac{\hat{V}_+\hat{V}_+}{c^2} + \gamma_-^6\frac{\hat{V}_-\hat{V}_-}{c^2}\right)\right]$$

$$= p\widetilde{\delta} + \hat{g}_+\hat{V}_+ + \hat{g}_-\hat{V}_- \tag{I5}$$

Local pressure:

$$p = \frac{e}{3} = \frac{2\sigma T^4}{3c}\left(\gamma_+^4 + \gamma_-^4\right) \tag{I6}$$

Intrinsic angular momentum density flux:

$$\widetilde{m} = \frac{10}{27}\frac{\hbar}{k}\sigma T^3\left[\left(\gamma_+^2 - \gamma_-^2\right)\widetilde{\delta} + \frac{\hat{V}_+\hat{V}_+}{c^2}\gamma_+^2\left(2\gamma_+^2 - 1\right) - \frac{\hat{V}_-\hat{V}_-}{c^2}\gamma_-^2\left(2\gamma_-^2 - 1\right)\right.$$

$$\left. + \widehat{F}\left(\frac{\hat{V}_+}{c}\right) - F\left(\frac{\hat{V}_-}{c}\right)\right] \tag{I7}$$

Intrinsic angular momentum number density:

$$m = \text{trace}\left(\widetilde{m}\right) = \frac{20}{27}\frac{\hbar}{k}\sigma T^3\left(\gamma_+^4 - \gamma_-^4\right) \tag{I8}$$

Entropy density:

$$s = \frac{8}{3}\frac{\sigma}{c}T^3\left(\gamma_+^4 + \gamma_-^4\right) \tag{I9}$$

Entropy density flux:

$$\hat{\sigma} = \frac{8}{3}\frac{\sigma T^3}{c}\left(\gamma_+^4\hat{V}_+ + \gamma_-^4\hat{V}_-\right) \tag{I10}$$

APPENDIX J

ISOTROPIC PHOTON GAS WITH LINEAR MOMENTUM AND INTRINSIC ANGULAR MOMENTUM IN TERMS OF THE NET AND DIFFUSION VELOCITIES

In this appendix, terms of an isotropic photon gas with linear momentum and intrinsic angular momentum are defined in terms of the net and diffusion velocities.

Photon number density:

$$n = \frac{40}{27} \frac{\sigma}{kc} T^3 \gamma^4 \left[\frac{1 + \delta^4}{\left(1 - \delta^4\right)^2} \right] \tag{J1}$$

Total energy density:

$$e_T = \frac{4\sigma}{3c} T^4 \gamma^4 \left[\frac{4\gamma^2 \left(1 + 3\delta^4\right)}{\left(1 - \delta^4\right)^3} - \frac{1 + \delta^4}{\left(1 - \delta^4\right)^2} \right] \tag{J2}$$

Momentum density:

$$\hat{g} = \frac{\hat{q}}{c^2} = \frac{16}{3} \frac{\sigma}{c^3} \frac{T^4 \gamma^6}{\left(1 - \delta^4\right)^3} \left[\left(1 + 3\delta^4\right) \hat{V} + \delta^2 \left(3 + \delta^4\right) \hat{U} \right] \tag{J3}$$

Intrinsic angular momentum density:

$$\hat{\ell} = \frac{n\hbar}{c} \hat{U} + m\hat{V} = \frac{40}{27} \frac{\hbar}{c^2} \frac{\sigma}{k} \frac{T^3 \gamma^4}{\left(1 - \delta^4\right)^2} \left[2\delta^2 \hat{V} + \left(1 + \delta^4\right) \hat{U} \right] \tag{J4}$$

Total pressure:

$$\tilde{P} = p\tilde{\delta} + \frac{16}{3}\frac{\sigma}{c^3}\frac{T^4\gamma^6}{\left(1 - \delta^4\right)^3}\left[\left(1 + 3\delta^4\right)\left(\hat{V}\hat{V} + \hat{U}\hat{U}\right) + \delta^2\left(3 + \delta^4\right)\left(\hat{V}\hat{U} + \hat{U}\hat{V}\right)\right] \quad (J5)$$

Local pressure:

$$p = \frac{e}{3} = \frac{4}{3}\frac{\sigma T^4}{c}\frac{\gamma^4}{\left(1 - \delta^4\right)^2}\left(1 + \delta^4\right) = \frac{9}{10}nkT \quad (J6)$$

Intrinsic angular momentum density flux:

$$\tilde{m} = \frac{20}{27}\frac{\hbar}{k}\sigma T^3 \left\{ \frac{\gamma^2}{1 - \delta^4}\left[\delta^2\tilde{\delta} + \frac{\hat{V}\hat{U} + \hat{U}\hat{V}}{c^2}\left(\frac{2\gamma^2\left(1 + \delta^4\right)}{1 - \delta^4} - 1\right) \right. \right.$$
$$\left. \left. + \delta^2\left(\frac{\hat{V}\hat{V} + \hat{U}\hat{U}}{c^2}\right)\left(\frac{4\gamma^2}{1 - \delta^4} - 1\right)\right] + \tilde{F}\left(\frac{\hat{V} + \hat{U}}{c}\right) - \tilde{F}\left(\frac{\hat{V} - \hat{U}}{c}\right)\right\} \quad (J7)$$

Intrinsic angular momentum number density:

$$m = \frac{80}{27}\frac{\hbar}{k}\frac{\sigma T^3\gamma^4\delta^2}{\left(1 - \delta^4\right)^2} \quad (J8)$$

Entropy density:

$$s = \frac{16}{3}\frac{\sigma}{c}T^3\gamma^4\frac{\left(1 + \delta^4\right)}{\left(1 - \delta^4\right)^2} = \frac{18}{5}nk \quad (J9)$$

Entropy density flux:

$$\hat{\sigma} = \frac{16}{3}\frac{\sigma}{c}\frac{T^3\gamma^4}{\left(1 - \delta^4\right)^2}\left[\left(1 + \delta^4\right)\hat{V} + 2\delta^2\hat{U}\right] \quad (J10)$$

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APPENDIX K

PHYSICAL SPACE VARIABLES FOR RANDOM POLARIZATION

Physical space variables for random polarization are defined in this appendix.

Photon number density:

$$n = \frac{40}{27} \frac{\sigma}{kc} T^3 \gamma^4 \tag{K1}$$

Total energy density:

$$e_T = \frac{4\sigma}{3c} T^4 \gamma^4 \left(4\gamma^2 - 1\right) \tag{K2}$$

Momentum density:

$$\hat{g} = \frac{\hat{q}}{c^2} = \frac{16}{3} \frac{\sigma}{c^3} T^4 \gamma^6 \hat{V} \tag{K3}$$

Intrinsic angular momentum density:

$$\hat{\ell} = 0 \tag{K4}$$

Total pressure:

$$\widetilde{P} = p\widetilde{\hat{\delta}} + \hat{g}\hat{V} \tag{K5}$$

Local pressure:

$$p = \frac{e}{3} = \frac{4}{3} \frac{\sigma}{c} T^4 \gamma^4 = \frac{9}{10} nkT \tag{K6}$$

Intrinsic angular momentum density flux:

$$\widetilde{\hat{m}} = 0 \tag{K7}$$

Intrinsic angular momentum number density:

$$m = 0 \tag{K8}$$

Entropy density:

$$s = \frac{16}{3} \frac{\sigma}{c} T^3 \gamma^4 = \frac{18}{3} n k \tag{K9}$$

Entropy density flux:

$$\hat{\sigma} = s\hat{V} \tag{H30}$$

Energy-stress density four tensor:

$$\tau_{\mu\nu} \equiv \begin{bmatrix} -\tilde{\tilde{P}} & -ic\hat{g} \\ -ic\hat{g} & +e_T \end{bmatrix} = \frac{4}{3} \frac{\sigma}{c} T^4 \gamma^4 \begin{bmatrix} -\left(\tilde{\tilde{\delta}} + 4\gamma^2 \frac{\hat{V}\hat{V}}{c^2}\right) & -4i\gamma^2 \frac{\hat{V}}{c} \\ -4i\gamma^2 \frac{\hat{V}}{c} & +\left(3 + 4\gamma^2 \frac{V^2}{c^2}\right) \end{bmatrix} \tag{K10}$$

APPENDIX L

PHYSICAL SPACE VARIABLES FOR RANDOM POLARIZATION, V/c << 1

Physical space variables for random polarization where V/c << 1 are defined in this appendix.

Photon number density:

$$n \simeq \frac{40}{27} \frac{\sigma}{kc} T^3 \left(1 + 2\frac{V^2}{c^2} + \ldots \right)$$ (L1)

Total energy density:

$$e_T \simeq \frac{4}{3} \frac{\sigma}{c} T^4 \left(3 + 10\frac{V^2}{c^2} + \ldots \right)$$ (L2)

Momentum density:

$$\hat{g} \simeq \frac{16}{3} \frac{\sigma T^4}{c^3} \hat{V} \left(1 + 3\frac{V^2}{c^2} + \ldots \right)$$ (L3)

Intrinsic angular momentum density:

$$\hat{\ell} = 0$$ (K4)

Total pressure:

$$\widetilde{P} \simeq \frac{4}{3} \frac{\sigma}{c} T^4 \left[\widetilde{\hat{\delta}} \left(1 + 2\frac{V^2}{c^2} + \ldots \right) \right.$$

$$\left. + 4\frac{\hat{V}\hat{V}}{c^2} \left(1 + 2\frac{V^2}{c^2} + \ldots \right) \right]$$ (L4)

Local pressure:

$$p \simeq \frac{4}{3}\frac{\sigma}{c}T^4\left(1 + 2\frac{V^2}{c^2} + \ldots\right)$$

(L5)

Intrinsic angular momentum density flux:

$$\hat{\hat{m}} = 0$$

(K7)

Intrinsic angular momentum number density:

$$m = 0$$

(K8)

Entropy density:

$$s \simeq \frac{16}{3}\frac{\sigma}{c}T^3\left(1 + 2\frac{V^2}{c^2} + \ldots\right)$$

(L6)

Entropy density flux:

$$\hat{\sigma} = s\hat{V}$$

(H30)

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