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A GENERAL SIMULATOR USING STATE ESTIMATION FOR A SPACE TUG NAVIGATION SYSTEM

Prepared By

GUIDANCE AND CONTROL STUDY GROUP

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FINAL TECHNICAL REPORT

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FORWARD

The Auburn University Engineering Experiment Station submitted a proposal which resulted in Contract NAS8-29852 being awarded in June 1973. The contract was awarded to the Engineering Experiment Station by the George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama, and was active until November 19, 1975.

This report is a technical summary of the progress made by the Electrical Engineering Department, Auburn University, Auburn, Alabama in the performance of this contract.

SUMMARY

A general simulation program (GSP) involving nonlinear state estimation for space vehicle flight navigation systems is developed and used as a basis for evaluating the performance of a Space Tug navigation system. A complete explanation of the iterative guidance mode (IGM) guidance law, derivation of the dynamics, coordinate frames, and state estimation routines are given so as to fully clarify the assumptions and approximations involved so that simulation results can be placed in their proper perspective.

So as to make the simulation program as useful as possible, a complete set of computer acronyms and their definitions as well as explanations of the subroutines used in the GSP simulator are included. To facilitate input/output, a complete set of compatable numbers, with units, are included to aid in data development. Format specifications, output data phrase meanings and purposes, and computer card data input are clearly spelled out.

A large number of simulation and analytical studies are used to determine the validity of the simulator itself as well as various data runs. Included in the studies are (1) covariance initialization, (2) initial offset vector parameter study, (3) propagation time vs. accuracy, (4) measurement noise parametric study, (5) deterministic vs. filtered run terminal error, (6) reduction in computational burden of an on-board implementable scheme, and many others.

From the results of these studies, conclusions and recommendations concerning future areas of important practical and theoretical purpose are presented. These use as a basis an extension of the general GSP simulator developed to date.

PERSONNEL

The following staff members of Auburn University were active participants in the research for this report:

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A. Definition of Space Tug and Statement of Project Task

The basic Space Tug vehicle will be used to extend the capabilities of the Space Shuttle orbit vehicle. The missions of the Shuttle will be limited to low earth orbit, severely limiting mission flexibility. The Space Tug concept was devised to extend the mission capability to geo-synchronous orbit, defined as a 19,323 nautical mile - 0° circular orbit. For maximum flexibility the Space Tug will fly in the cargo bay of the Space Shuttle as payload into low earth orbit. Departure of the Tug from the Shuttle will then take place in an approximate 160 n. mi. x 160 n. mi. low earth orbit. This Space Tug with its own payload will then fly its desired mission and return to the Shuttle for transportation back to earth.

The primary mission of the Space Tug is to deliver and retrieve automated payloads from low earth orbit to geo-synchronous orbit and return. Other mission assignments for the Space Tug are as follows:

1. Astronomy - Depart from a 160 n.mi. 28.5° orbit and transfer to a 39,000 n.mi. 28.5° circular orbit.

2. Atmospheric Physics - Depart from a 160 n.mi. 28.5° orbit and transfer to an escape trajectory.

3. Earth Observation - Depart from a 100 n.mi. 90° orbit and transfer to a 900 n.mi. 90° circular orbit.

4. Planetary - Depart from 160 n.mi. 28.5° orbit and transfer to an escape trajectory.

5. Communication and Navigation - Depart from 160 n.mi. 28.5° orbit and transfer to 19,323 n.mi. 0° circular orbit (geo-synchronous orbit). Depart from 205 n.mi. 103° orbit (sun synchronous orbit) [3].

The design configuration of the Space Tug assumed for this project is that the vehicle be unmanned, ground based control, reusable and launched with payload on a single shuttle flight. It is desired that the Space Tug be a low cost vehicle. Under this assumption, existing hardware should be incorporated into design and a minimal amount of ground support tracking should be employed, necessitating a large burden on an on-board inertial navigation-guidance system.

As stated previously, the primary mission of the Space Tug, and the one used as a base mission for this project, is the flight of the vehicle from low earth orbit to geo-synchronous orbit. The exact definition of this mission is: Depart from 160 n.mi. 28.5° orbit and transfer to a 19,323 n.mi. 0° circular orbit [3]. This mission flight is depicted in Figure I-1. The guidance scheme for this orbit transfer had previously been adopted by NASA to be a modified Iterative Guidance Mode Law (IGM) which is optimal in the sense of minimum fuel [1]. To achieve the orbit transfer with minimum fuel, the guidance law will employ a three burn-three coast maneuver, instead of a long duration burn. IGM is of the modified form in the sense that for the Space Tug the terminal orbit end conditions can either be fixed or floating. Another specification of the Space Tug flight profile is that the capability of making trajectory changes during flight is required. To meet this design specification the guidance and navigation scheme needs to have an accurate fix on its position and velocity in an inertial coordinate frame at all times. A navigation system

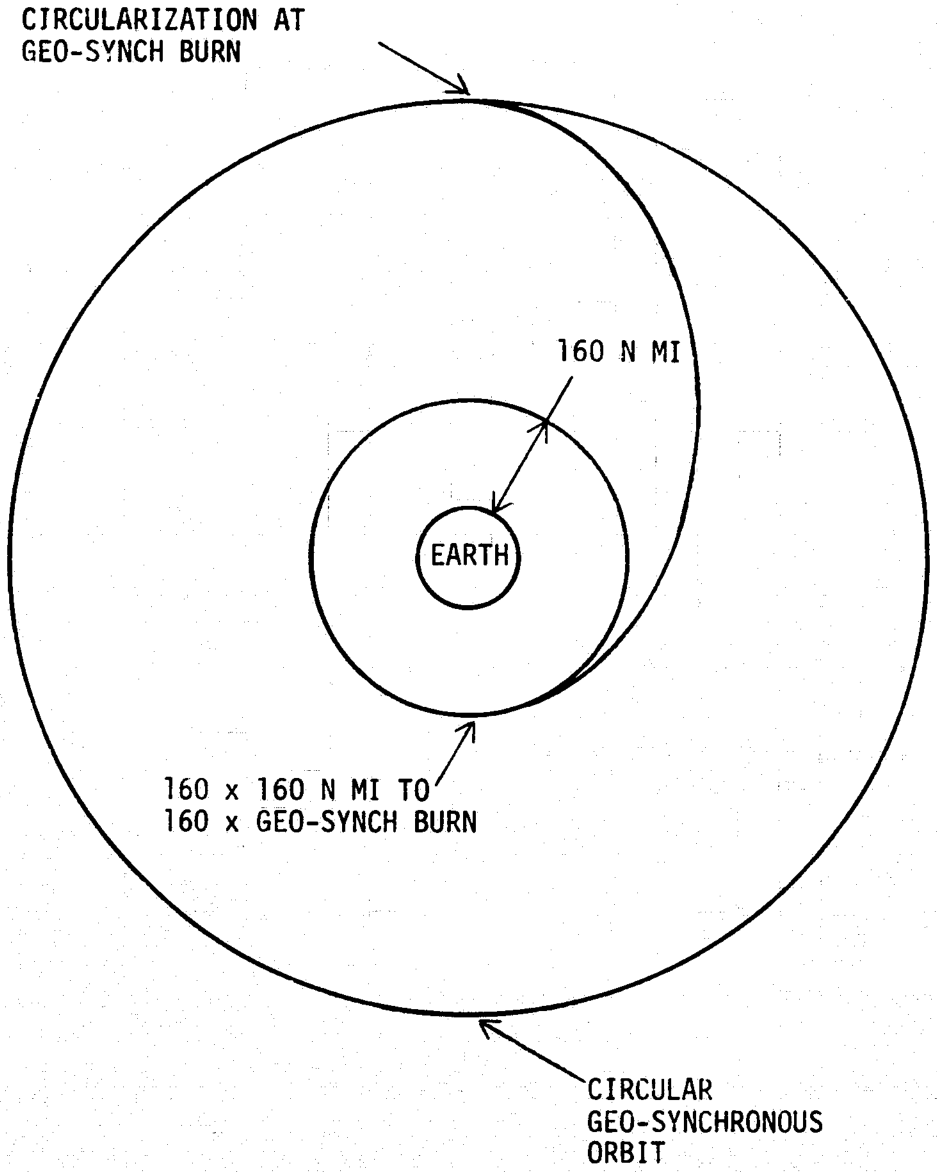


Figure I-1. BURNS TO CIRCULAR GEO-SYNCHRONOUS ORBIT with guidance parameters supplied by IGM could not accomplish this task, since IGM does not produce the position and velocity vectors of the Space Tug, using them instead as inputs. This report describes a simulator and navigation system for estimation of the position and velocity state vector in an inertial coordinate frame using an extended Kalman filter type approach. The overall problem of mid-course guidance and correction is involved and for the present purpose it is assumed IGM functions properly [2].

B. Approach to Problem Solution

The three position and three velocity coordinates in an inertial reference frame make up the 6 x 1 state vector of interest in this problem. Because of the nonlinear nature of the vehicle dynamics, which include the effects of gravity and aerodynamic drag, the linear Kalman filter is not applicable. Therefore, the extended Kalman filter technique, which requires linearization of the vehicle dynamics about the nominal state estimate, was used. The state estimator scheme uses the linearized model in the Kalman Filter to find the best or optimal estimate of the state vector at some time, t. The Space Tug's terminal position vector, R_T, terminal velocity vector, V_T, and terminal range angle, θ_T, are used as inputs to the system.

The state estimation process uses measurement data supplied from various measurement configurations to form a measurement state vector. The state estimator then forms a measurement residual vector, which is the difference of the actual measurement and an estimated measurement, to be used in the nonlinear state estimator to update the state vector.

The different measurement configurations used for the estimation procedure should be kept at a minimum to reduce overall cost, storage, and computation time in the on-board computer. However, the measurement systems must be capable of supplying enough measurement data to the state estimator so that the state estimates meet the error requirements placed on the position and velocity vectors. Due to the nature of the Space Tug's flight, different measurement configurations during the burn and coast phase are considered.

Three axis accelerometer measurements are made during the burn phase of flight. The model of this measurement used in the state estimator program is given in (I-1).

$$\underline{\ddot{X}}_s = \begin{bmatrix} \ddot{X}_s \\ \ddot{Y}_s \\ \ddot{Z}_s \end{bmatrix} = \begin{bmatrix} \ddot{X}_a \\ \ddot{Y}_a \\ \ddot{Z}_a \end{bmatrix} + \underline{\nu} \qquad \text{(I-1)}$$

In (I-1) $\underline{\ddot{X}}_s$ is the sensed acceleration vector, $\underline{\ddot{X}}_a$ is the actual acceleration vector of the vehicle and $\underline{\nu}$ is the additive noise vector used to model measurement error due to the accelerometer equipment. A problem arises as to how the acceleration measurement can be used in the Kalman filter. To use a Kalman filtering approach to state estimation it is necessary that the measurements taken be a linear combination of the states. Since acceleration is not part of the state vector this requirement could not be fulfilled in a straightforward manner. It was decided that the acceleration measurements could be used to construct a measurement vector that is a linear combination of the states. The form of the constructed position and velocity measurements are given in (I-2) and (I-3).

$$Z_x = \hat{X} + \ddot{X}_s \cdot \frac{\Delta t^2}{2} + \dot{\hat{X}} \cdot \Delta t \tag{I-2}$$

$$\dot{Z}_x = \dot{\hat{X}} + \ddot{X}_s \cdot \Delta t \tag{I-3}$$

In (I-2) and (I-3) $\hat{X}$ and $\dot{\hat{X}}$ denote the previous position and velocity estimates of the state X. More detail as to how the constructed measurement vector was applied and modified will be given in a later chapter of this report.

Since no measurement data can be derived from the accelerometers during the Space Tug's coast phase of flight, a separate measurement system is needed. From an investigation of many types of measurement devices it was decided to use a horizon sensor and sun seeker during the coast phase to obtain the measurements needed by the state estimator. The basic geometry for the horizon sensor describing the relation of the earth and the Space Tug is shown in Figure I-2. Using the horizon sensor a measurement of the angle ϕ is made. Figure I-3 shows the geometry for the sun seeker. The sun seeker is a line of sight device with two gimballed axes, from which the angles α and β are supplied as measurement data. The measurement vector used during the coast phase of flight is given in (I-4).

$$\underline{Z} = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \csc\phi \end{bmatrix} \tag{I-4}$$

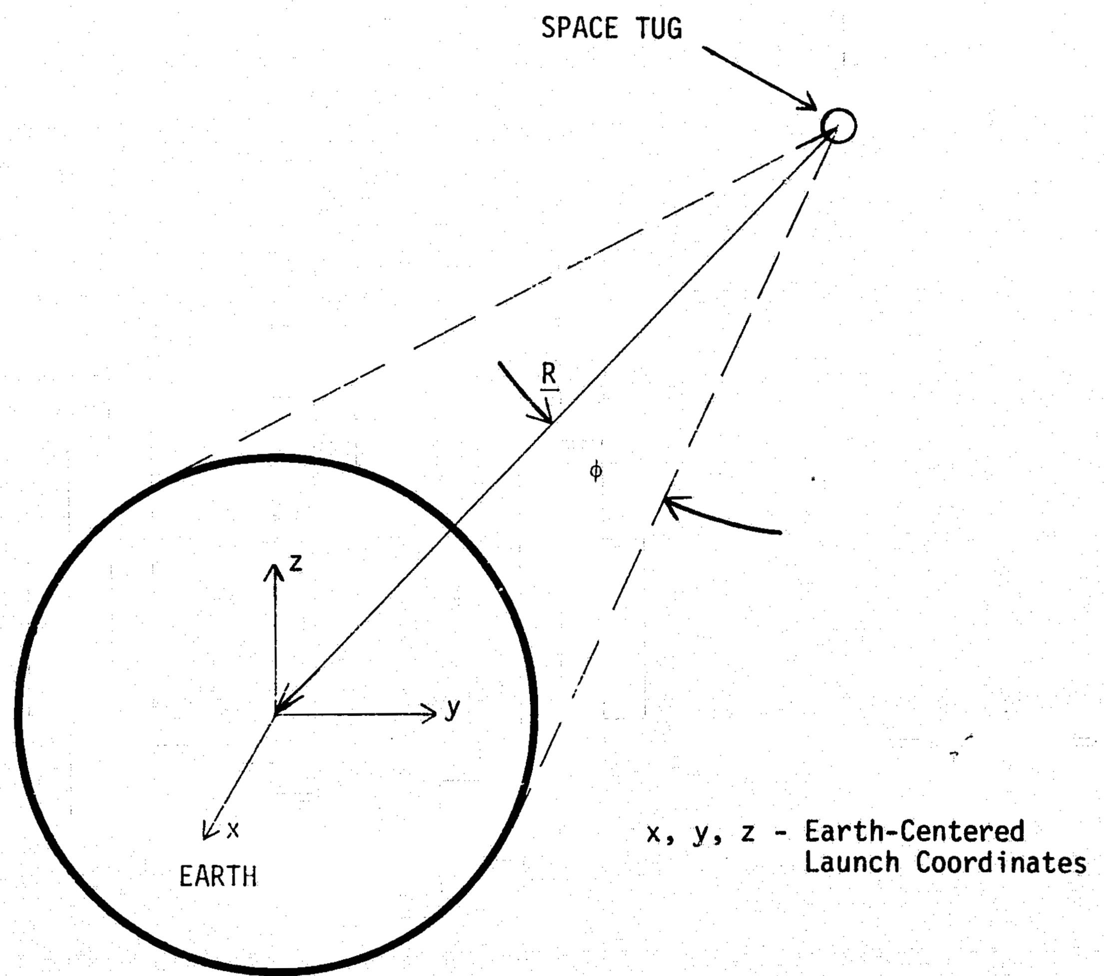


Figure I-2. HORIZON SENSOR GEOMETRY

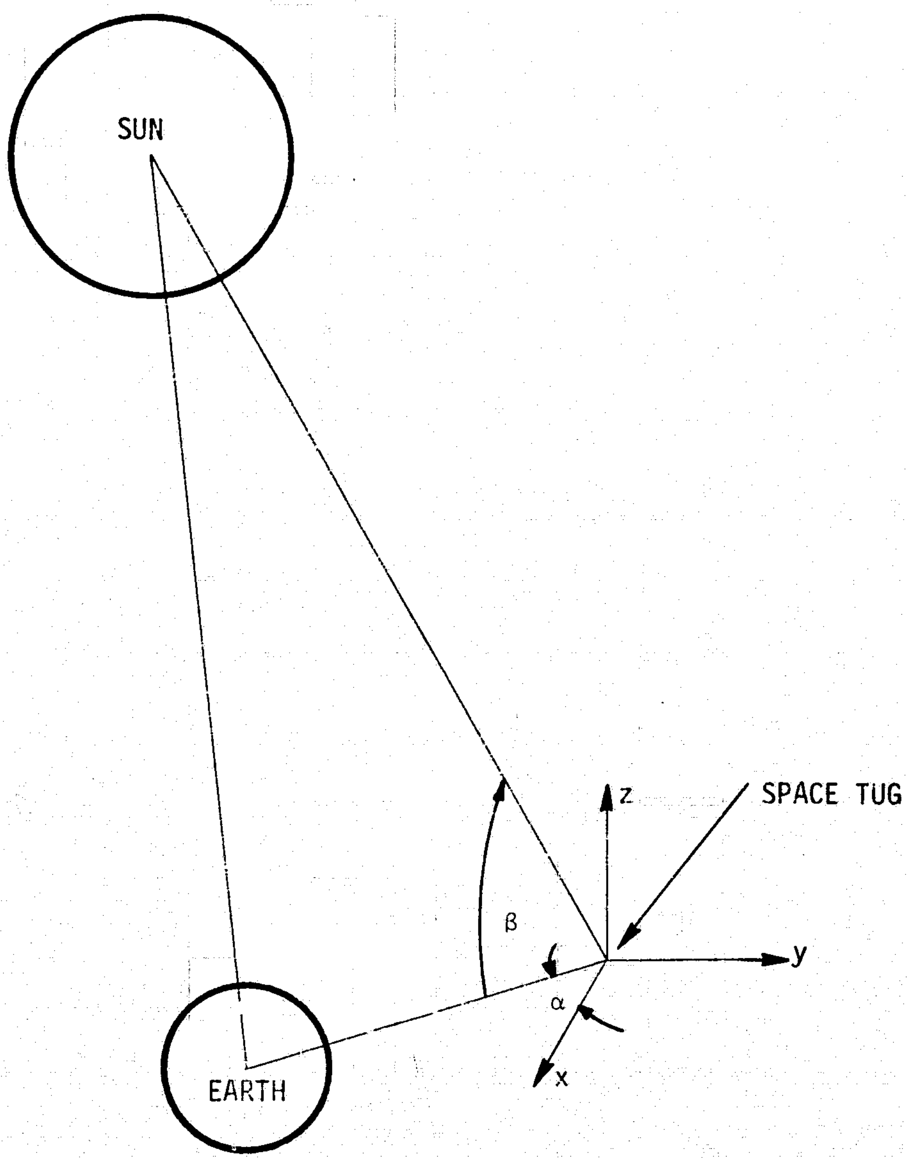


Figure I-3. SUN SEEKER GEOMETRY

Unlike the measurements during the burn, the measurements during the coast can be related directly to the position states of the state vector. Since no information concerning the velocity states is given by these measurements, the dynamics of the system must be used to reconstruct velocity states during the coast. From (I-4) it can be seen that the coast measurements are nonlinear, therefore a linearization of the measurements must be made. The linearization procedure along with analytic expressions relating the measurement angles to the states will be given in a later chapter.

C. General Simulator Package

In order to demonstrate the capability of the estimation algorithm, an entire simulation package of the Space Tug guidance and navigation system was developed. By having the entire simulation package available, the state estimator can be tested and results verified in detail. The general simulator package is a FORTRAN computer program, constructed in subroutine form, which simulates all properties of the Space Tug guidance and navigation which have an effect on the operation of the state estimator. Even though the general simulator package is based on the IGM guidance law of the Space Tug, the simulator is general in the sense that guidance and navigation systems under the influence of nonlinear dynamics can be simulated by the program with only slight modifications. The program is constructed so that the initial state vector and the terminal orbit conditions are supplied as inputs. These parameters can be changed by changing only a few data cards in the program. A detailed explanation of the general simulator package will be given in a later chapter.

D. Report Outline

This report is divided into five body chapters with four additional supplementary appendices. Chapter II details the development, purpose, and operation of the simulator package. Included is a general discussion of the IGM guidance law including its input/output characteristics. Chapter III introduces the overall nonlinear dynamical problem investigated in the Space Tug study and the development of a general navigation state-estimation package using the dynamics. Chapter IV introduces and discusses the measurement systems for both the burn and the coast, the relationship between the measurements and the state vector, and the linearization approach used to implement the measurement process in the state estimator. Chapter V displays results of simulation studies, along with discussions and analyses of the results. Chapter VI covers the findings and conclusions from the work performed, and in addition makes recommendations for future work.

The appendices cover supplementary but important material pertinent to a full understanding of the GSP system--Space Tug flight work. Appendix A shows the substitutions and partial derivative operations required in order to develop the general partial derivative statements to be used in $\phi(k + 1, k)$ and $H(k + 1)$. Appendix B covers input/output data handling and an explanation of the GSP output. Appendix C tabulates overall Tug burn-coast tracking accuracies, initialization errors, etc. Appendix D includes a list of symbols and their definitions which appear in the GSP computer printout listing.

II. SIMULATOR DESCRIPTION

In order to allow for flexible studies with a minimum of programming burden, a general, nonlinear state estimation type simulator was developed. Referred to as GSP (General Simulation Program), in its present form the simulator is a complete description of the gravity and drag forces impinging on a space vehicle, as well as the guidance. navigation, measurement. and state estimation operations which are involved in space vehicle flight. As it is presently implemented there are three distinct modes of operation available, (1) deterministic, (2) passenger mode, (3) full filter mode. These three modes encompass the major conditions of interest in most navigation system studies. For greater accuracy, the program is written entirely in double precision.

The deterministic mode means that the simulator and all operations with it (i.e. navigation, guidance. tracking, etc.) are supplied with exact position and velocity data. No noise and/or position and velocity offsets occur. This condition represents the "ideal case," the best that is possible. The second mode, a "passenger" mode. is one in which deterministic data is used to run the simulation as in the first mode, but, in addition, a complete state estimator is operated and a 'one step propogation' state estimate carried along as a "passenger" for inspection of the accuracy of the estimation process. The third, or "full filter" mode, uses noisy signals and initially biased position and velocity coordinate data to simulate a real-world navigation problem. In this mode a non-linear state estimation process is used to provide position and velocity data to the guidance law. The full-filter mode simulates the case of an actual flight.

In the remainder of this chapter details of the overall program, the flow of data, input/output data definition, units of data, and external user operation of such a simulator package are covered. Available upon request from the Department of Electrical Engineering, Auburn University is a complete listing of the GSP package. A punched deck utilizing an IBM compatable 029 punch is also available on request. In Appendix II is a complete tabulation of input/output, including an example listing, details as to data entry by cards, etc.

A. General Simulator Operation

A basic flowchart of the GSP simulator is given in Figure II-1. Shown are the major operations the program is designed to automatically handle.

The general flow of data is shown by the arrows in Figure II-1. First, an initial state estimate $\hat{\underline{X}}(0/0)$ is input to the program. This is the information which is available to the on-board navigation system. For the deterministic and passenger mode runs this estimate is the actual initial position and velocity coordinates of the vehicle. But, for the full filter mode, $\hat{\underline{X}}(0/0)$ represents an error value of initial state, i.e.

$$\underline{X}(0/0) = \underline{X}(0) + \underline{\varepsilon}(0)$$

where $\underline{X}(0)$ is the actual initial vehicle coordinates and $\underline{\varepsilon}(0)$ is the initial misalignment error. This error in practice is due to tracking

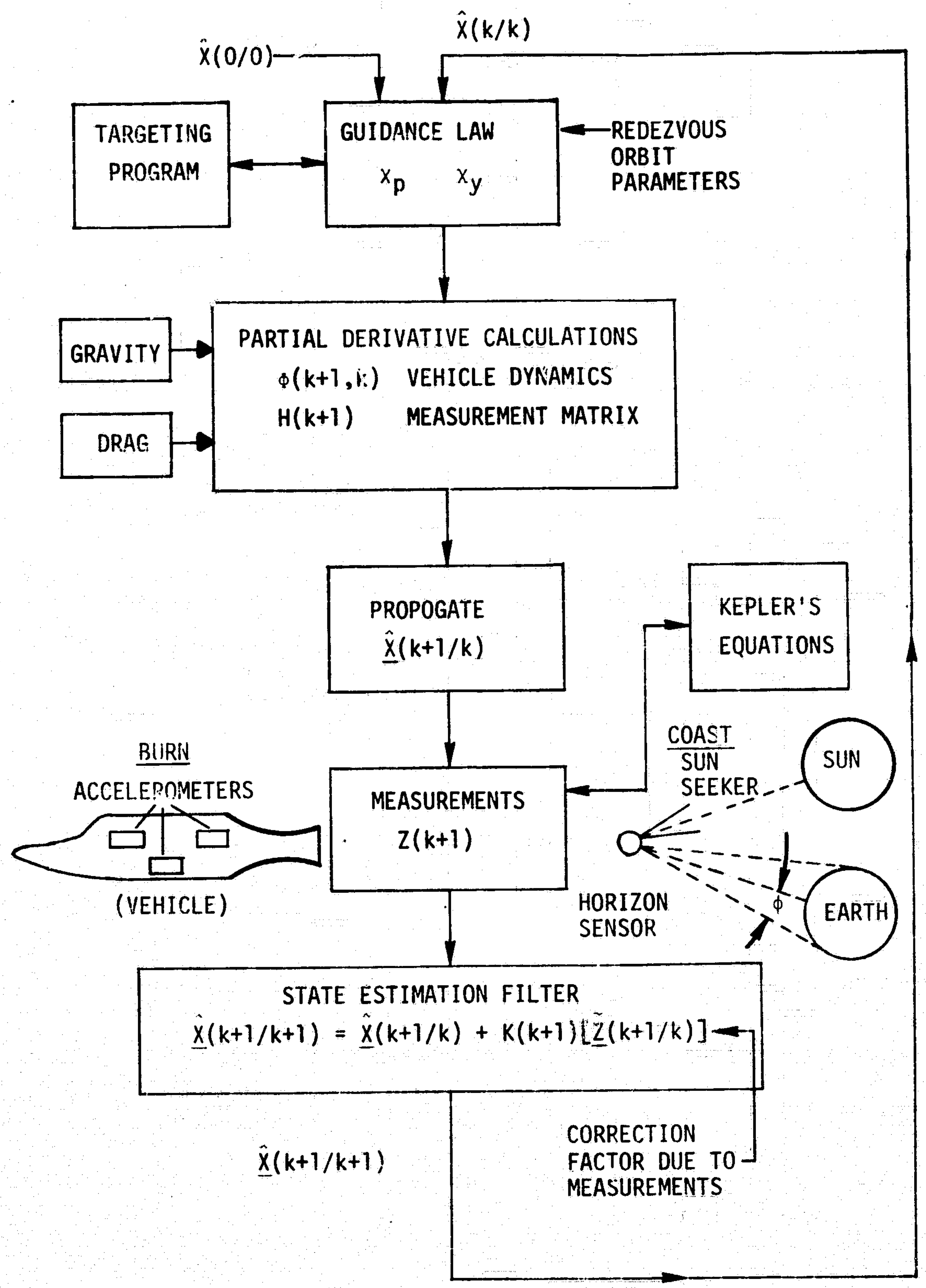


Figure II-1. NAVIGATION SIMULATION FLOW CHART.

modeling inaccuracies of the ground-based tracking network. Bounds on the allowed size of the elements of $\underline{\varepsilon}$ are given in Appendix III.

This initial state is then input to the targeting program calculations along with the rendezvous orbital parameters to determine whether a coast or burn phase should occur first. The targeting program is not actually included in GSP. Instead, only the various initial-final trajectory parameters for a multi-burn mission to get to geosynchronous orbit are available to GSP, the data having been assumed to have come from the targeting program.

In any event, either a burn or coast will occur first. If a coast is called for, gravity, drag information is used to develop, on-board and in real-time, an equivalent set of state transition and observation matrices. The state transition matrix $\Phi(k+1,k)$ is a piece-wise constant 6x6 matrix which is needed to propagate the state vector forward in time, for use in obtaining a cleaned-up, estimated value of the state at a number of seconds later after using observation information. During the coast the measurements assumed available were (1) a star tracker, (2) a sun seeker, (3) a horizon sensor. The star tracker is used to help align the vehicle reference platform by providing angular data. The sun seeker provides tracking information in the form of elevation and azimuthal angles. The horizon sensor physically measures the half angle ϕ between the earth's horizon and the centerline from the space vehicle through the center of the earth. This particular angle information actually determines the vehicle's altitude from the earth's center. This "radius vector" plus the two sun angles, is sufficient to yield a unique set of position coordinates of the vehicle.

Using knowledge of the geometry of the various measurement processes, equivalent measurement estimates are made using an equation of the form

$$\underline{Z}(k+1/k) = H(k+1)\ \hat{\underline{X}}(k+1/k)$$

where the H(k+1) matrix is used to take state information and construct measurements $\underline{Z}$. Both the $\Phi(k+1,k)$ and H(k+1) matrices are determined by numerical evaluation of analytical partial derivatives programmed into the computer. The numerical evaluation is effected by using $\hat{\underline{X}}(k+1/k)$ values to compute the constants entering $\Phi(k+1,k)$ and H(k+1).

Next, information using $\Phi(k+1/k)$ and H(k+1), along with the measurement data, are input to a nonlinear state estimation process, the end result being a "best estimate" (in a mean-square sense) of what the state vector is some time Δt after the last available value. This set of calculations involves covariance matrix computations and state update. The output of the filter is a new state estimate $\hat{\underline{X}}(k+1/k+1)$.

This new estimate is the only information the vehicle has as to what its position is and at what velocities it is traveling. This information is then used <u>in place of</u> the original $\hat{\underline{X}}(0/0)$ as a new "initial condition", only this time at some time instant greater than zero and the process repeats. This closed-loop cyclic operation is clearly depicted in Figure II-1, where the output of the filter becomes the input to the guidance law (which is bypassed during the coast phase).

At the appropriate time, main engine ignition occurs and the burn phase begins. The initial state value at the beginning of the coast is used in the guidance law (a form of IGM) to compute initial burn angle information, where the output of the guidance law are two angles, χ_p, χ_y, pitch and yaw angles respectively. These are used to command the main engine gimbals to properly direct the thrust vector.

Similar to the coast phase, a set of matrices $\Phi(k+1,k)$, $H(k+1)$ are then developed, based on a series of analytical partial derivatives programmed into the navigation system and evaluated at the state value. $\Phi(k+1,k)$ is evaluated at $\hat{\underline{X}}(k/k)$ and $H(k+1)$ at $\hat{\underline{X}}(k+1/k)$. The $\Phi(k+1,k)$ matrix is made up of terms from gravity, drag, and the guidance law. It is the complexity of the guidance law which greatly adds to the computational burden. A flow chart of the IGM guidance law used is shown in Figure II-2. Suffice it to say that the volume of calculations involved is very great. A later section of the chapter discusses in detail the use, operations, and calculations of the IGM guidance law.

During the burn a different set of measurements are used than during the coast. The measurements in this ıse are only relative measurements of states, namely the vehicle acceleration in the vehicle position coordinates. They are only relative measurements because integrating acceleration gives only a relative <u>change</u> of velocity, but it never says what the actual velocity is. Integrating the acceleration twice gives a relative position <u>change</u>, but never says what the position actually is. Using these relative position and velocity changes, a measurement is "constructed" from past position and velocity data plus the relative changes as determined by the acceleration data. In fact, the acceleration data includes not only acceleration due to thrusting, but

INPUT

I. VEHICLE AND IGM PARAMETERS

$F,\ I_{sp},\ m,\ T_g,\ \delta_T,\ \overline{X},\ \dot{\overline{X}},\ \ddot{\overline{X}}_g$

II. TARGET PARAMETERS $A_z,\ \phi_L,\ i,\ \theta_N$, and $R_T,\ V_T,\ \gamma_T$ FOR FIXED END CONDITIONS OR $P_T,\ e_T,\ \omega_T$ FOR FLOATING END CONDITIONS

III. GUIDANCE OPTIONS

IOP1 = 1 FIXED END CONDITIONS

IOP1 = 2 FLOATING END CONDITIONS

NFK6 = 0	FK6 = 0
NFK6 = 1	FK6 = QUADRATIC CURVE FIT
NCYL	NUMBER OF CYCLES THROUGH TERMINAL RANGE ANGLE COMPUTATION

↓

PRECOMPUTATION

$$V_{E_X} = G_o I_{sp},\ \dot{m} = -F/V_{EX},\ T = -m/\dot{m}$$

$$[A] = \begin{bmatrix} \cos\phi_L & \sin\phi_L \sin A_z & -\sin\phi_L \cos A_z \\ -\sin\phi_L & \cos\phi_L \sin A_z & -\cos\phi_L \cos A_z \\ 0 & \cos A_z & \sin A_z \end{bmatrix}$$

$$[B] = \begin{bmatrix} \cos\theta_N & 0 & \sin\theta_N \\ \sin\theta_N \sin i & \cos i & -\cos\theta_N \sin i \\ -\sin\theta_N \cos i & \sin i & \cos\theta_N \cos i \end{bmatrix}$$

$$[G] = [B][A],\ G_T = -\mu/R_T^2$$

↓

FIGURE II-2. SIMPLIFIED FLOW CHART OF THE IGM GUIDANCE LAW USED WITH THE SPACE TUG

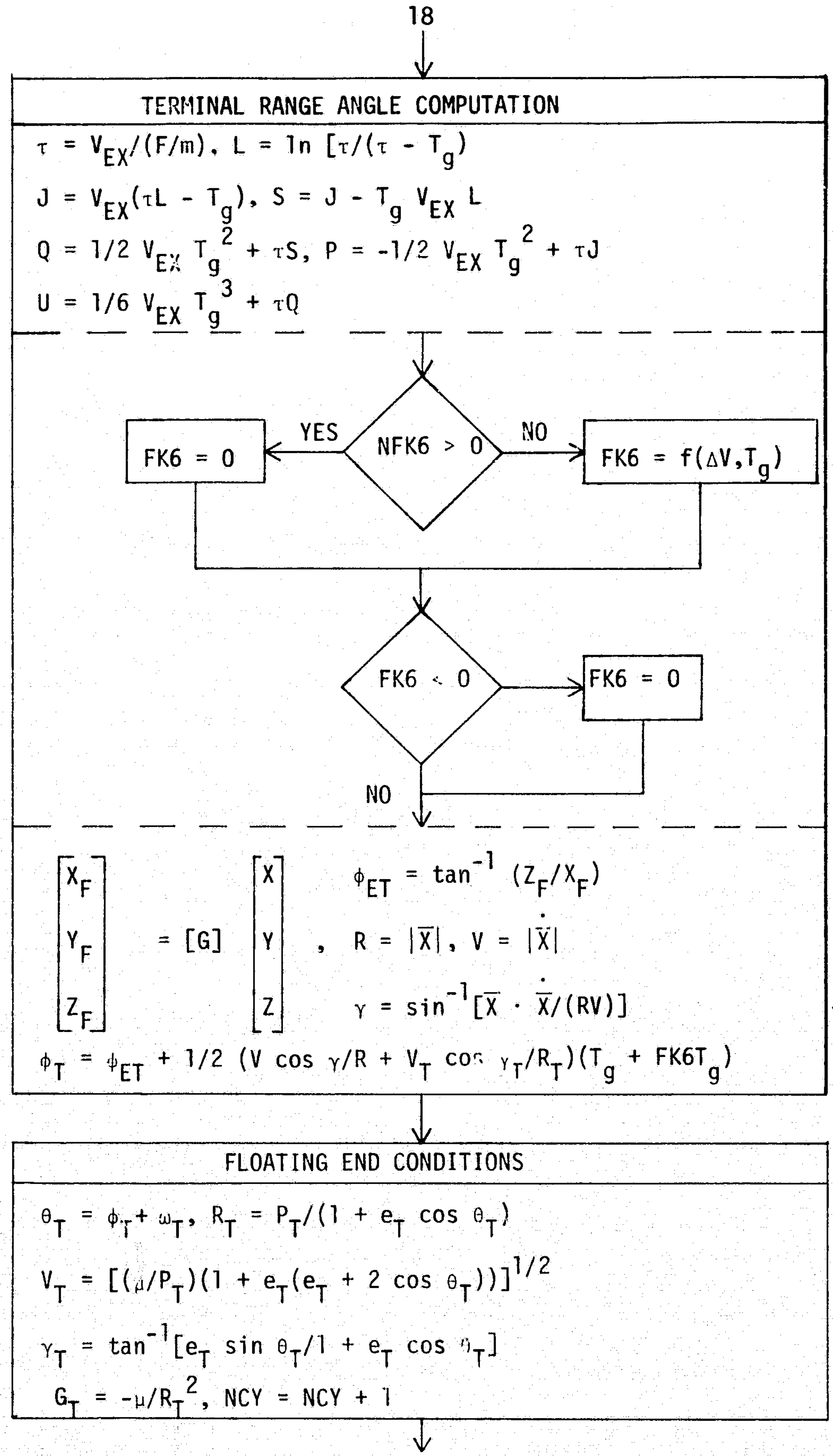


FIGURE II-2. SIMPLIFIED FLOW CHART OF THE IGM GUIDANCE LAW USED WITH THE SPACE TUG (Continued)

COORDINATE ROTATIONS

$$\xi_T = R_T,\quad \eta_T = \zeta_T = 0$$

$$\dot{\xi}_T = V_T \sin \gamma_T,\quad \dot{\eta}_T = 0,\quad \dot{\zeta}_T = V_T \cos \gamma_T$$

$$\ddot{\xi}_{GT} = G_T,\quad \ddot{\eta}_{GT} = \ddot{\zeta}_{GT} = 0$$

$$[\phi_T] = \begin{bmatrix} \cos \phi_T & 0 & \sin \phi_T \\ 0 & 1 & 0 \\ -\sin \phi_T & 0 & \cos \phi_T \end{bmatrix},\quad [K] = [\phi_T][G]$$

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = [K] \begin{bmatrix} X \\ Y \\ Z \end{bmatrix},\quad \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{bmatrix} = [K] \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\xi}_g \\ \ddot{\eta}_g \\ \ddot{\zeta}_g \end{bmatrix} = \tfrac{1}{2} \left\{ \begin{bmatrix} \ddot{\xi}_{GT} \\ \ddot{\eta}_{GT} \\ \ddot{\zeta}_{GT} \end{bmatrix} + [K] \begin{bmatrix} \ddot{X}_g \\ \ddot{Y}_g \\ \ddot{Z}_g \end{bmatrix} \right\}$$

TIME-TO-GO COMPUTATION

$$\Delta\dot{\xi}^* = \dot{\xi}_T - \dot{\xi} - \ddot{\xi}_g T_g$$

$$\Delta\dot{\eta}^* = \dot{\eta}_T - \dot{\eta} - \ddot{\eta}_g T_g$$

$$\Delta\dot{\zeta}^* = \dot{\zeta}_T - \dot{\zeta} - \ddot{\zeta}_g T_g$$

$$G = \tfrac{1}{2} \left[\frac{(\Delta\dot{\xi}^*)^2 + (\Delta\dot{\eta}^*)^2 + (\Delta\dot{\zeta}^*)^2}{V_{EX}\, L} - V_{EX}\, L \right]$$

$$\Delta T = G(\tau - T_g)/V_{EX},\quad T_g = T_g + \Delta T$$

FIGURE II-2. SIMPLIFIED FLOW CHART OF THE IGM GUIDANCE LAW USED WITH THE SPACE TUG (Continued)

↓

PITCH AND YAW STEERING PARAMETERS

$$\Delta\dot{\xi} = \Delta\dot{\xi}^* - \ddot{\xi}_g \Delta T, \quad \Delta\dot{\eta} = \Delta\dot{\eta}^* - \ddot{\eta}_g \Delta T$$

$$\Delta\dot{\zeta} = \Delta\dot{\zeta}^* - \zeta_g \Delta T, \quad \tilde{\chi}_p = \tan^{-1}(\Delta\dot{\xi}/\Delta\dot{\zeta})$$

$$\tilde{\chi}_y = \tan^{-1}[\Delta\dot{\eta}/(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)^{1/2}]$$

$$A_y = V_{EX}\, L, \quad B_y = J + T_g\, V_{EX}\, L$$

$$C_y = S \cos \tilde{\chi}_y, \quad D_y = Q \cos \tilde{\chi}_y$$

$$E_y = \eta + \dot{\eta}\, T_g + \tfrac{1}{2}\, \ddot{\eta}_g\, T_g^{\,2} - S \sin \tilde{\chi}_y$$

$$K_3 = B_y\, E_y/(A_y D_y - B_y C_y), \quad K_4 = K_3\, A_y/B_y$$

$$M = \cos \tilde{\chi}_y + K_3 \sin \tilde{\chi}_y, \quad N = K_4 \sin \tilde{\chi}_y$$

$$A_p = MA_y - NB_y, \quad B_p = MB_y - NP$$

$$C_p = (MS - NQ) \cos \tilde{\chi}_p, \quad D_p = (MQ - NU) \cos \tilde{\chi}_p$$

$$E_p = \xi - \xi_T + \dot{\xi} T_g + 1/2\, \ddot{\xi}_g\, T_g^{\,2} - (MS - NQ) \sin \tilde{\chi}_p$$

$$K_1 = B_p\, E_p/(A_p D_p - B_p C_p), \quad K_2 = K_1 A_p/B_p$$

↓

IGM STEERING ANGLES

$$\chi_p' = \tilde{\chi}_p - K_1 + K_2\, \delta_T$$

$$\chi_y' = \tilde{\chi}_y - K_3 + K_4\, \delta_T$$

$$\begin{bmatrix} DCX \\ DCY \\ DCZ \end{bmatrix} = [K]^{-1} \begin{bmatrix} \sin \chi_p' \cos \chi_y' \\ \sin \chi_y' \\ \cos \chi_p' \cos \chi_y' \end{bmatrix}$$

$$\chi_p = \tan^{-1}[DCX/DCZ]$$

$$\chi_y = -\tan^{-1}[DCY/(1 - DCY^2)^{1/2}]$$

↓

ENGINE GIMBAL

FIGURE II-2. SIMPLIFIED FLOW CHART OF THE IGM GUIDANCE LAW USED WITH THE SPACE TUG (Continued)

also acceleration due to gravity and drag. This is accomplished by using models of gravity and drag, which in actual practice would be carried in an on-board computer. The accelerations are then added together and the relative positional and velocity changes determined. The reason drag and gravity models must be used is that on-board accelerometers can measure only relative thrusting, no measuring devices are available on-board to measure the accelerations due to these external forces.

During the burn the measurement matrix H(k+1) is assumed constant, equal to the identity matrix, I_6. This is used to form a propagated estimate of the measurements, and together, the measurement estimates and the measurements are used in the state estimator to come up with a one step "cleaned-up" estimate of $\hat{\underline{X}}(k+1/k+1)$. This updated, 'optimal' estimate is then used by the guidance law and a closed-loop cycle has begun.

For all three operating modes, i.e. deterministic, passenger, and full filter, exact state vector information is available from a fourth-order Runge Kutta integration routine. This information is then available for comparison with the estimated data so as to determine tracking accuracy and observation sightings during the coast phase. The means by which this is performed in GSP is shown schematically in Figure II-3. The upper flow graph represents the on-board type equations used in the navigation system, with gravity, drag, and guidance information going into Φ, the data based on $\hat{\underline{X}}$ information. The arrows indicate direction of information (either one way or two way). The lower line of flow represents the exact integration carried out. It is this process that is not available in the real problem. It shows that after integrating $\underline{X}(k)$

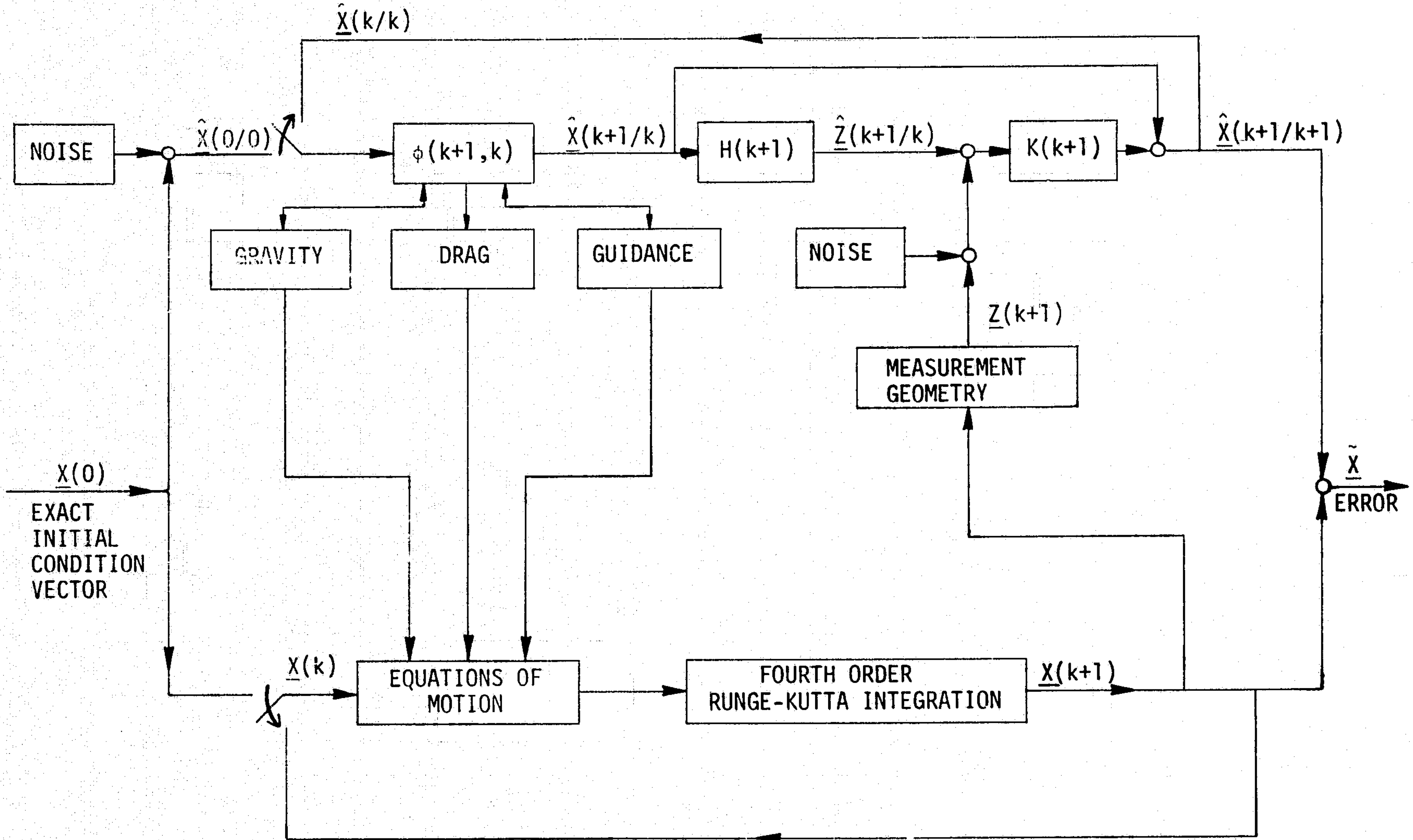


Figure II-3. SIMULATION SCHEMATIC OF GSP ACCURACY EVALUATION TECHNIQUE.

to get $\underline{X}(k+1)$, the geometry of the measurement process, a nonlinear function of the state vector, is used to determine a measurement vector, to which noise is added. This noisy measurement is then used with the on-board filter to determine a new best estimate of state. In order to allow for flexible operation, a general gaussian, random noise generator with ten independent channels is available in GSP.

B. Description of the IGM Guidance Law

Shown in Figure II-2 is a flow chart of the Iterative Guidance Mode (IGM) law being considered for use on the Space Tug as supplied by Northrop Services, Inc. This section discusses in detail the purpose, plan, and operation of this version of IGM and how it fits into the GSP simulator.

The version of IGM shown in Figure II-2 is a modification of that used with the Saturn V SII and SIV-B steering systems. It was selected over a Cross Product Steering (CPS) law (as used in the Apollo spacecraft CSM and LM orbit transfer steering systems) due to its greater flexibility and optimality. It is designed to guide the Tug from low-earth orbit to geosynch and back again through a series of phasing orbit plane changes. Since basic IGM loses effectiveness when burn times are over 500 seconds, a modification was provided which slightly biases the time-to-burn as a function of the difference (Δa) between the semimajor axis of the initial parking orbit and the semimajor axis of the phasing orbit. This allows for greater accuracy with burn arcs of over 500 seconds.

The IGM law is an explicit guidance method which differs greatly from previous techniques in that nominal mission profile data is not stored on-board. Instead the IGM law is an iterative step-by-step closed-loop solution which enforces the desired end conditions at end-of-burn. It is based upon the idea of a linear steering law of the form

$$\chi = a + bt \tag{II-1}$$

χ = a burn angle
a,b constants
t = running time with respect to (WRT) a fixed reference

The law makes use of a flat earth assumption, an average gravity force magnitude and direction, and small angle approximations. Performance losses (i.e. additional fuel burden) are due to the previous factors plus the errors in the predictions of the burn arc and time to go (burn time).

In Table II-1 is a list of symbols, with appropriate definitions of those variables appearing in Figure II-2. Upon entering IGM, the following variables must be known in order to complete the computations: F, I_{sp}, m, T_g, $\underline{X}$, $\dot{\underline{X}}$, $\ddot{\underline{X}}_g$ and either A_Z, ϕ_L, θ_N, R_T, V_T, γ_T for fixed end conditions or P_T, e_T, ω_T for floating end conditions. In addition, in both cases an estimate of the burn-time-to-go is required even though it changes in flight.

A series of time-varying coefficients are computed and a large number of coordinate rotations effected to project present conditions to various final coordinate frames and use the differences to drive the vehicle. The reference coordinate frame used is the Appolo 13 earth-centered launch coordinates (ECLC).

Table II-1. List of Symbols for the IGM Guidance Law

SYMBOL	DEFINITION
F, THRUST	Vehicle thrust
V_{ex}	Exit velocity
I_{SP}, SISP	Specific impulse
m	Mass of vehicle
T_g, TGØ	Time-to-go
δt	Computation cycle time
$\underline{X}$, $\dot{\underline{X}}$	Instantaneous position and velocity vector, Apollo 13 coordinates (z-axis downrange, x along negative gravity vector at launch site, y completes right hand system)
$\ddot{\underline{X}}_g$	Instantaneous gravity vector
A_z	Launch azimuth
ϕ_L, PHIL	Launch latitude
$\underline{i}$	Desired terminal inclination
$\underline{\theta_N}$	Right ascention of desired terminal descending node
R_T	Desired terminal radius
V_T	Desired terminal velocity
γ_T	Desired terminal flight-path angle
P_T	Desired terminal semilatus rectum
E_T, e_T	Desired terminal eccentricity
ω_T	Desired terminal argument of perigee
$\dot{m}$	Mass flow rate
τ	Tau (mass/mass flow rate)

SYMBOL	DEFINITION
ϕ_{ET}	Instantaneous range angle
γ	Flight path angle
θ_T	Desired terminal true anomaly
G_T	Terminal gravity magnitude
χ_p	Pitch gimbal engine burn angle
χ_y	Yaw gimbal engine burn angle
ΔT	Time-to-go correction due to velocity errors

A terminal range angle ϕ_T is found from

$$G = \begin{bmatrix} \cos\theta_N & 0 & \sin\theta_N \\ \sin\theta_N \sin i & \cos i & -\cos\theta_N \sin i \\ -\sin\theta_N \cos i & \sin i & \cos\theta_n \cos i \end{bmatrix} \cdot \begin{bmatrix} \cos\phi_L & \sin\phi_L \sin A_Z & -\sin\phi_L \cos A_Z \\ -\sin\phi_L & \cos\phi_L \sin A_Z & -\cos A_Z \cos\phi_L \\ 0 & \cos A_Z & \sin A_Z \end{bmatrix} \tag{II-2}$$

$$\begin{bmatrix} X_F \\ Y_F \\ Z_F \end{bmatrix} = G\,\underline{X} \tag{II-3}$$

$$\phi_{ET} = \tan^{-1} Z_F/X_F \tag{II-4}$$

$$R = \sqrt{X^2 + Y^2 + Z^2} \tag{II-5}$$

$$V = \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2} \tag{II-6}$$

$$\gamma = \sin^{-1}((\underline{X} \cdot \dot{\underline{X}})/RV) \tag{II-7}$$

$$\phi_T = \phi_{ET} + 1/2\,(\frac{V}{R}\cos\gamma + \frac{V_T}{R_T}\cos\gamma_T)(1 + FK6)T_g \tag{II-8}$$

and γ is the flight path angle, referenced to the local horizontal. This on-line computed angle is a function of the instantaneous range angle, ϕ_{ET}, where the average rate of change of the range angle is

$$\dot{\phi}_{av} = 1/2\,(\frac{V}{R}\cos\gamma + \frac{V_T}{R_T}\cos\gamma_T). \tag{II-9}$$

$\dot{\phi}_{av}$ times T_g gives an approximation to the additional angle traversed. The FK6 is a small adjustment value

$$|FK6| < 1 \tag{II-10}$$

which is used to improve overall targeting, hence

$$\phi(\text{final}) = \phi(\text{present}) + \dot{\phi}_{av}(t_{final} - t_{present}) \cdot (1 + FK6) \tag{II-11}$$

A users guide to the development and operation of FK6 is available [4]. For a detailed description of IGM concepts and derivation, see [5,6,7]. An averaged acceleration vector is found by rotating the present acceleration values into a terminal frame and comparing it to an equivalent terminal gravity vector,

$$\begin{bmatrix} \ddot{\xi}_g \\ \ddot{\eta}_g \\ \ddot{\zeta}_g \end{bmatrix} = 1/2 \left\{ K \begin{bmatrix} \ddot{X}_g \\ \ddot{Y}_g \\ \ddot{Z}_g \end{bmatrix} + \begin{bmatrix} \ddot{\xi}_{g_T} \\ \ddot{\eta}_{g_T} \\ \ddot{\zeta}_{g_T} \end{bmatrix} \right\} \tag{II-12}$$

where K is a rotation matrix defined by

$$K = \Psi\ G \tag{II-13}$$

$$\Psi = \begin{bmatrix} \cos\phi_T & 0 & \sin\phi_T \\ 0 & 1 & 0 \\ -\sin\phi_T & 0 & \cos\phi_T \end{bmatrix} \tag{II-14}$$

and ξ, η, ζ represent terminal coordinate frame variables. The ξ-ζ plane is contained in the desired orbit plane, with ξ measured from the center of the earth and passing through the orbit injection point. η is then normal to the orbit plane resulting in

$$\eta_T = \dot{\eta}_T = 0 \tag{II-15}$$

This is shown in Figure II-4.

Because of the definitions of ξ, η, ζ and their earth orientation,

$$\ddot{\xi}_{g_T} = -\mu/R_T^2 \quad , \quad \ddot{\eta}_{g_T} = \ddot{\zeta}_{g_T} = 0 \tag{II-16}$$

$$\dot{\xi}_T = V_T \sin \gamma_T \quad , \quad \dot{\eta}_T = 0 \quad , \quad \dot{\zeta}_T = V_T \cos \gamma_T \tag{II-17}$$

$$\xi_T = R_T \quad , \quad \eta_T = \zeta_T = 0 \tag{II-18}$$

Using the definitions, an incremental correction factor ΔT, representing a lumped equivalent of burn time corrections over the cycle time δt of the guidance law (it iterates every δt seconds), is determined. Terminal frame velocity errors are computed assuming the averaged acceleration value over the remaining flight from (II-12) is accurate. This is effected as follows:

$$\underline{V}_{avg}(\text{terminal frame}) \Big|_{\text{end-of-burn}} = \underline{V}(\text{terminal frame}) \Big|_{\text{present time}} + \underline{a}_{avg}(\text{terminal frame}) \cdot \underbrace{(T_g + \Delta T)}_{\text{estimated remaining burn time}} \tag{II-19}$$

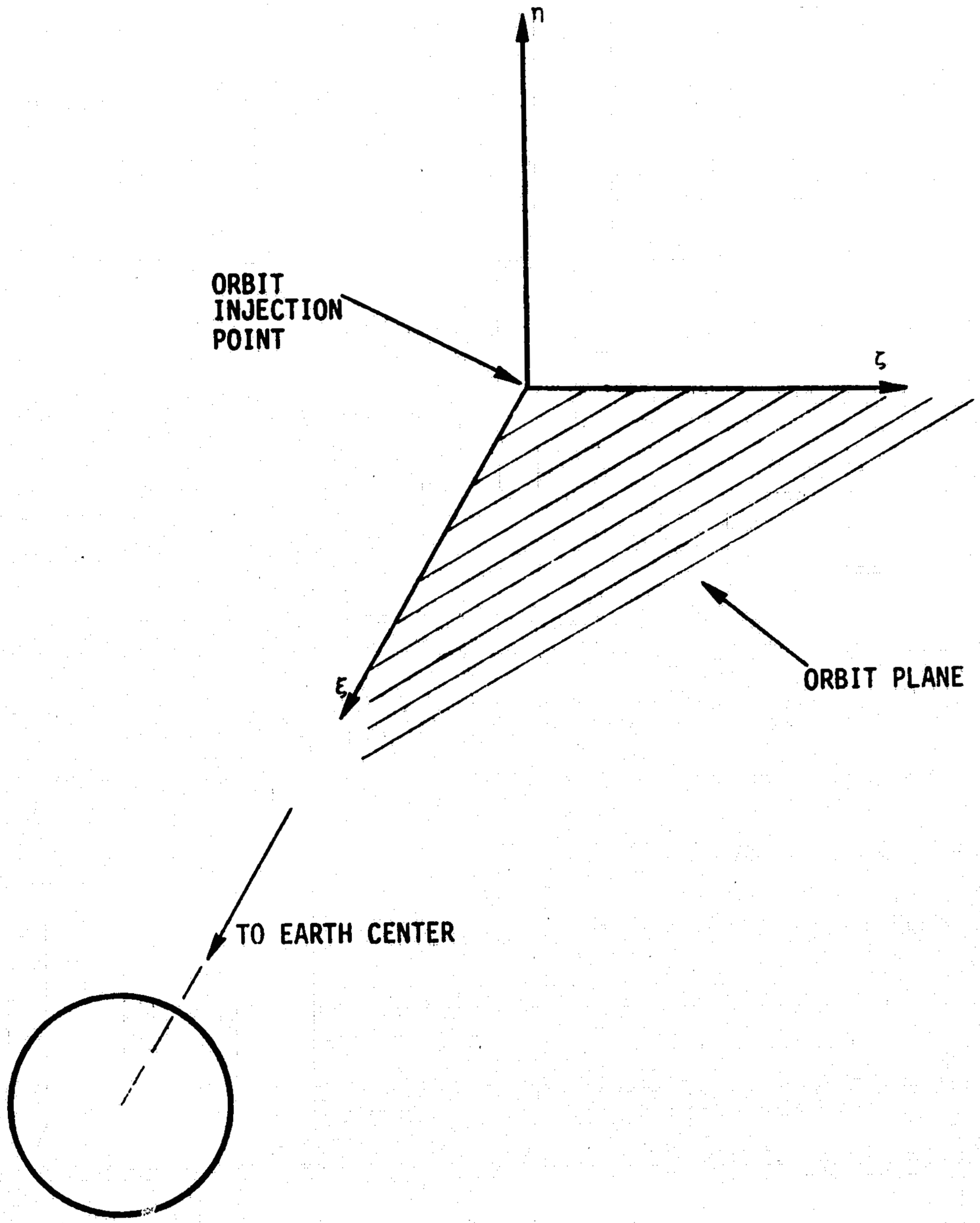


Figure II-4. SPACE TUG TERMINAL COORDINATE FRAME.

Using (II-19), velocity errors are computed as

$$\Delta\dot{\xi} = \xi_T - \dot{\xi}_{est} \text{ (end-of-burn)} \tag{II-20}$$

where $\dot{\xi}_{est}$ (end-of-burn) is the first entry of (II-19). Similar results occur for $\Delta\dot{\eta}$, $\Delta\dot{\zeta}$. The velocity error predictions at end-of-burn are really only extrapolated estimates of what type of terminal velocity errors will occur if the present thrusting history were to continue and if the approximations of (1) averaged gravitational acceleration, and (2) $T_g + \Delta T$ actually yields the burn time-to-go. It is true that as end-of-burn approaches, all of the approximations become very good and terminal homing conditions prevail (small angle approximations hold, average acceleration values approach instantaneous, errors tend to zero, extrapolation equations become accurate, etc.).

This extrapolated estimate of the velocity errors at end-of-burn in the terminal coordinate frame (ideally the velocity errors should be zero) is used to determine relative thrust angles $\tilde{\chi}_p$, $\tilde{\chi}_y$, the p and y representing pitch and yaw respectively. These are computed from

$$\tilde{\chi}_p = \tan^{-1} (\Delta\dot{\xi}/\Delta\dot{\zeta}) \tag{II-21}$$

$$\tilde{\chi}_y = \tan^{-1} [\Delta\dot{\eta}/ \sqrt{(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2})} \tag{II-22}$$

These angles are shown in Figure II-5. A number of parameters A_y, B_y, C_y, D_y, E_y, A_p, B_p, C_p, D_p, E_p, which are functions of V_{ex}, T_g, $\tilde{\chi}_p$, $\tilde{\chi}_y$, ξ, η, ζ, ξ_T, η_T, ζ_T, $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, $\ddot{\xi}_g$, $\ddot{\eta}_g$ are computed. The end result of interest here is that using the listed parameters, steering angle parameters K_1, K_2, K_3, K_4 are computed. Their purpose is to provide a linear steering

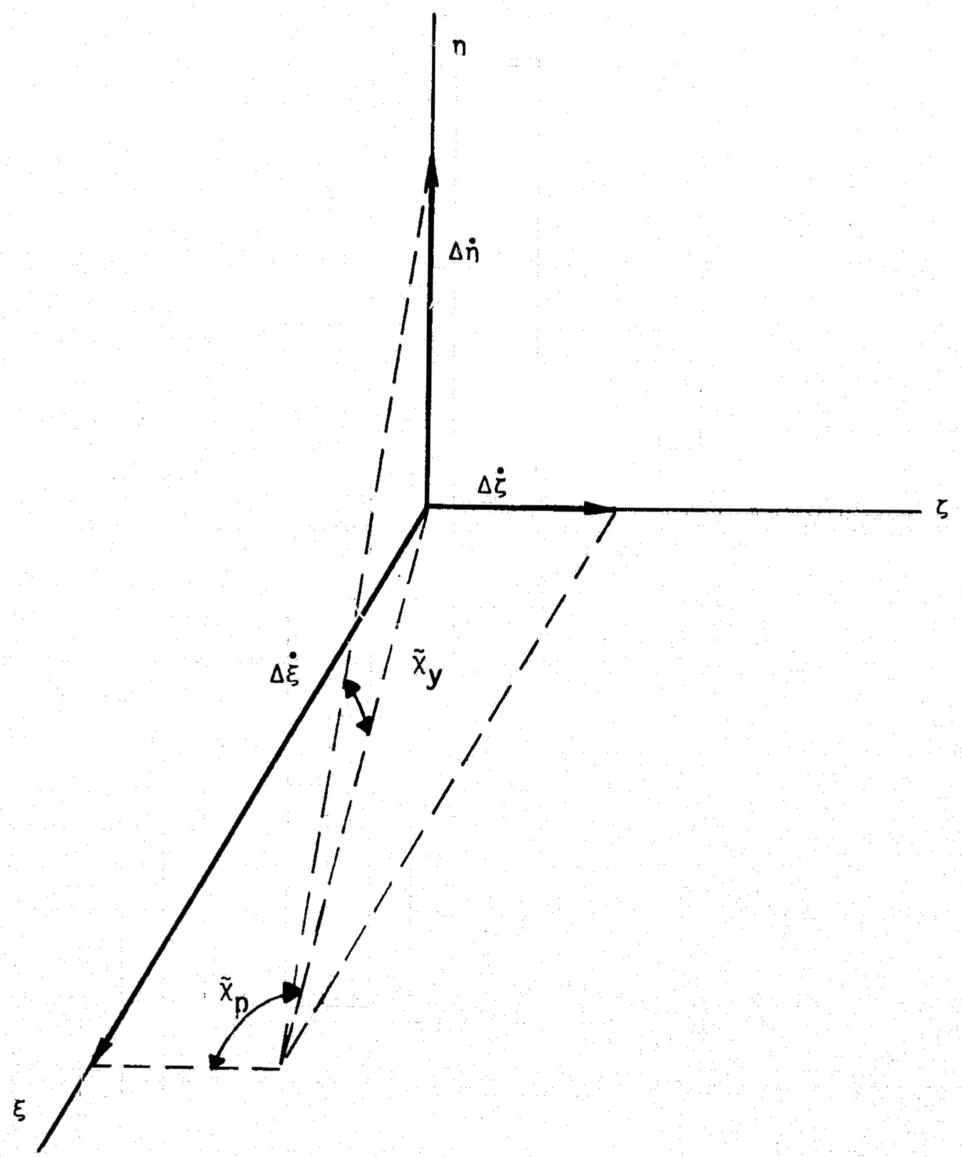


Figure II-5. DEFINITION OF $\tilde{\chi}_p$ AND $\tilde{\chi}_y$ ANGLES.

law form as in (II-1). In theory they should be constants, but due to the large number of approximations used in the development of IGM, they are updated every cycle (δt seconds) so as to form essentially a series of piece-wise linear steering laws of the form

$$\chi_p' = \tilde{\chi}_p - K_1 + K_2\ \delta t \tag{II-23}$$

$$\chi_y' = \tilde{\chi}_y - K_3 + K_4\ \delta t \tag{II-24}$$

(the signs on K_i are unimportant as they are a function of the particular defining variables). The angles are measured in the terminal (ξ,η,ζ) coordinate frame and as such must be converted to angle information in ECLC so that the gimballed thruster can be properly oriented to drive the vehicle in the proper direction.

This last set of operations is effected by using χ_p', χ_y' to determine the direction cosines u, v, w, in the ξ, η, ζ frame,

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \sin\chi_p' \cos\chi_y' \\ \sin\chi_y' \\ \cos\chi_p' \cos\chi_y' \end{bmatrix} \tag{II-25}$$

Using the K rotation matrix from (II-13), the corresponding direction cosines DCX, DCY, DCZ, in ECLC are found from

$$\begin{bmatrix} DCX \\ DCY \\ DCZ \end{bmatrix} = K^{-1} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{II-26}$$

It should be noted that since K is an orthogonal matrix, then $K^{-1} = K^T$. This greatly simplifies the computer programming burden. The burn angles χ_p, χ_y can then be computed to be

$$\chi_p = \tan^{-1} (DCX/DCZ) \tag{II-27}$$

$$\chi_y = -\tan^{-1} [DCY/ \sqrt{DCX^2 + DCZ^2}] \tag{II-28}$$

which is similar to the previous ξ-η-ζ case. These two angles are the output which go to the gimballed thruster servos to orient the thrusters properly.

This entire procedure, ending with angles χ_p and χ_y, is recomputed every δt seconds. The case just described is one in which fixed end conditions are assumed and the terminal point is fixed (time-invariant). A similar procedure occurs for the case of floating end conditions. Here, P_T, e_T, ω_T are fixed but R_T, V_T, γ_T are variable insofar as they are able to cause the proper orbit intercept; they must be calculated on-line, however, since the remaining calculations require these values.

Since this mode essentially results in more unknowns than there are equations, a sub-iterative loop within the IGM loop itself is set up to match parameters as best as possible. The number of iterations is controlled by an externally read in data card variable NCYL. Because of the discrete-continuous nature of the IGM equations, the resulting partial derivatives for use in the navigation-state estimation process are only approximations to the true partial derivatives. The iterative cycle with the floating end conditions creates a problem because each iteration adds incremental values to T_g, velocity differences, etc. so that if more than one cycle is run through in the sub-loop, the partial derivatives in the state estimator portion of GSP will not correspond to actual parameter values. Both sets of data are required and both must be compatible if proper estimation is to occur. This problem is alleviated by setting NCYL to a value allowing only a single cycle to occur. The shortcoming of this approach is that terminal intercept accuracy can be lost to some degree. At present, however, this appears to be the only viable solution.

C. Basic IGM Optimum Trajectory Concept

In essence, the IGM guidance law is a technique for meeting guidance objectives with a minimum expenditure of fuel. All equations are based on Newtonian physics and a point mass vehicle in a vacuum. Two different simplified two-dimensional derivations of optimum steering laws for a minimum fuel system will be briefly presented and discussed with respect to the full IGM law in Section B so as to obtain a clearer picture of the reasoning behind IGM.

1. Flat Earth Equations

For short range flights of approximately constant altitude, the earth can be assumed to be flat and to have a constant gravity vector. A simplified model of the 2-D flight problem is given in Figure II-6. Referring to this figure the equations of motion are:

$$\dot{X}_1 = X_3 \quad , \quad X_1(0) = 0 \tag{II-29}$$

$$\dot{X}_2 = X_4 \quad , \quad X_2(0) = h_1,\ X_2(t_F) = h_2, \tag{II-30}$$

$$\dot{X}_3 = a_F(t)\cos\theta \quad . \quad X_3(0) = v_{R1},\ X_3(t_F) = v_{R2}, \tag{II-31}$$

and

$$\dot{X}_4 = a_F(t)\sin\theta - g, \quad X_4(0) = v_{h_1},\ X_4(t_F) = v_{h2}, \tag{II-32}$$

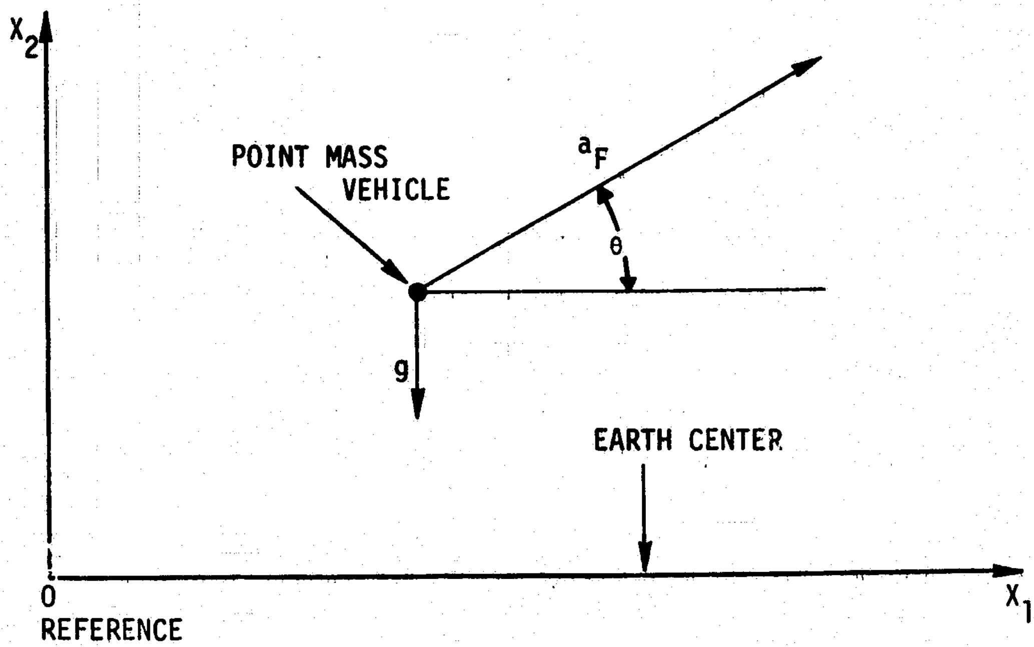


Figure II-6. CARTESIAN COORDINATE SYSTEM FOR A FLAT EARTH.

where $a_F(t)$ is the magnitude of the acceleration on the vehicle due to engine thrusting. The auxiliary, or costate equations formed from (II-29) through (II-32) are

$$\dot{\psi}_1 = 0 \quad , \qquad \psi_1(t_F) = 0, \tag{II-33}$$

$$\dot{\psi}_2 = 0 \quad , \tag{II-34}$$

$$\dot{\psi}_3 = -\psi_1, \tag{II-35}$$

and

$$\dot{\psi}_4 = -\psi_2. \tag{II-36}$$

These are obtained by forming the Hamiltonian [8]. Taking the partial derivative of H and setting it to zero yields

$$\theta = \tan^{-1}(\psi_4/\psi_3) \tag{II-37}$$

Equations (II-33) - (II-36) are readily integrated, resulting in

$$\begin{aligned} \psi_1(t) &= 0 \\ \psi_2(t) &= \psi_2(0) \\ \psi_3(t) &= \psi_3(0) \\ \psi_4(t) &= \psi_4(0) - \psi_2(0)t \end{aligned} \tag{II-38}$$

Substitution of (II-38) into (II-37) yields

$$\theta = \tan^{-1}(C_1 - C_2 t) \tag{II-39}$$

where

$$C_1 = \frac{\psi_4(0)}{\psi_3(0)} \qquad\qquad C_2 = \frac{\psi_2(0)}{\psi_3(0)} \tag{II-40}$$

Now (II-39) can be written as

$$\tan\theta = C_1 - C_2 t \tag{II-41}$$

and for small angles $\tan\theta \approx \theta$, so by selecting the flight geometry so the angle θ tends toward zero near the end of burn flight, the control law is linear as in (II-1). Since gravity is constant, so are C_1 and C_2. The resulting control law from (II-41),

$$\theta \approx C_1 - C_2 t \tag{II-42}$$

is similar in form to (II-23) and (II-24) in IGM. The differences come from the fact that in IGM the guidance algorithm is updated every δt seconds, representing the "life span" of the particular constants K_1-K_4, and also because of the discretized nature of the IGM terms.

2. Round Earth Equations

In this section a more realistic approximation of the actual flight of a space vehicle is studied. It is assumed that the vehicle is flying about a round earth and that an inverse square law gravity field is present. The basic flight geometry is shown in Figure II-7. The equations governing flight under these conditions can now be written:

$$\dot{X}_1 = X_3, \tag{II-43}$$

$$\dot{X}_2 = X_4, \tag{II-44}$$

$$\dot{X}_3 = X_1 X_4^2 - \frac{\mu}{X_1^2} + a_F(t)\sin\theta, \tag{II-45}$$

and

$$\dot{X}_4 = -\frac{2X_3X_4}{X_1} + a_F(t)\cos\theta, \tag{II-46}$$

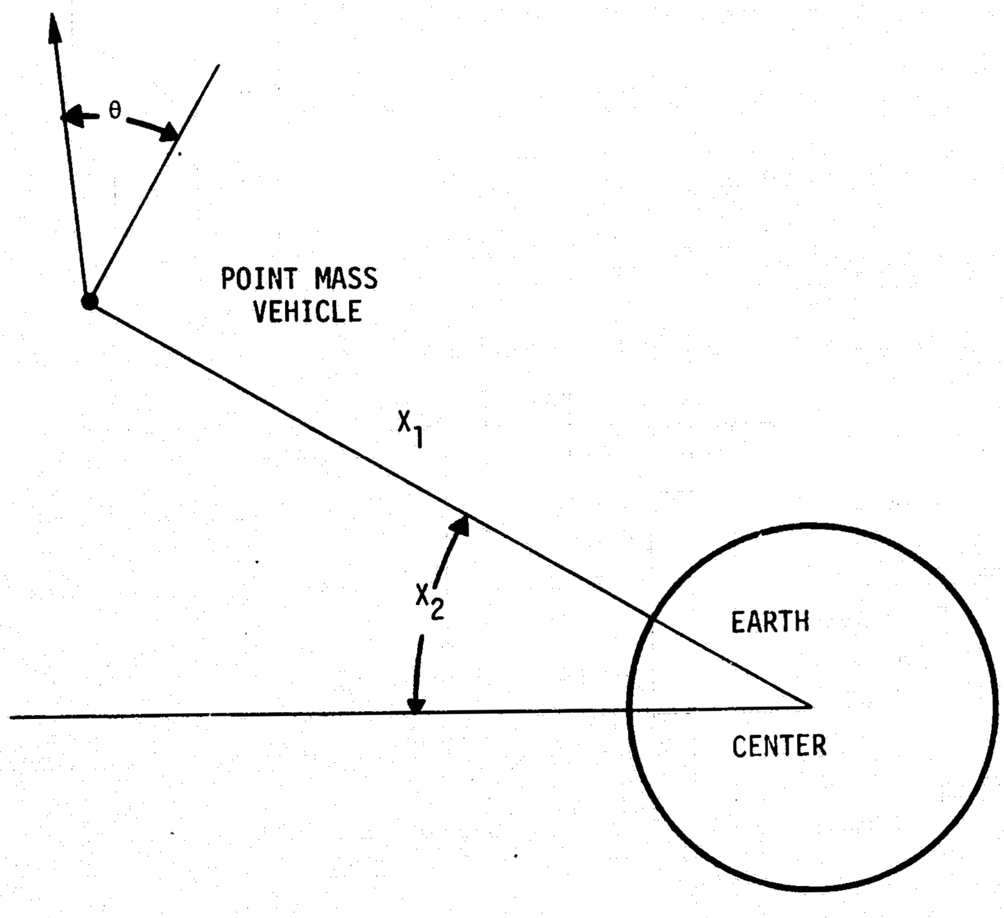


Figure II-7. COORDINATE SYSTEM FOR A ROUND EARTH.

where X_1 represents the distance of the space vehicle from the earth's center, and X_2 is the range angle defined by Figure II-6. As before, the costate equations can be determined using calculus of variations (COV) principles to be

$$\dot{\psi}_1 = -\psi_3({X_4}^2 + \frac{2\mu}{{X_1}^3}) - \psi_4 (\frac{2X_3X_4}{{X_1}^2} - \frac{a_F(t)}{{X_1}^2} \cos\theta), \tag{II-47}$$

$$\dot{\psi}_2 = 0, \tag{II-48}$$

$$\dot{\psi}_3 = -\psi_1 + \psi_4 \frac{2X_4}{X_1}, \tag{II-49}$$

$$\dot{\psi}_4 = -\psi_2 - \psi_3(2X_1X_4) + \psi_4(2\frac{X_3}{X_1}), \tag{II-50}$$

and

$$\dot{\psi}_5 = a_F(t)\ (\psi_3 \sin\theta + \frac{\psi_4}{X_1}\cos\theta) \tag{II-51}$$

where the boundary conditions $X_1(0)$, $X_2(0)$, $X_3(0)$, $X_4(0)$, $X_1(t_f)$, $X_3(t_f)$, $X_4(t_f)$, $\psi_2(t_f)$ and $\psi_5(t_f)$ are known. Again taking the partial derivative of H WRT θ yields

$$\theta = \tan^{-1}\left[\frac{X_1\ \psi_3}{\psi_4}\right] \tag{II-52}$$

which can be put in the form

$$\tan\theta = \frac{X_1\ \psi_3}{\psi_4} \tag{II-53}$$

To obtain a simple expression for θ in terms of X_1, X_2 would be very difficult due to the complex forms appearing in (II-47) - (II-51). This clearly shows that, just for the simplified 2-D problem, adding the freedom of a spherical earth with an inverse square gravity law as opposed to a flat earth, constant gravity model has added greatly to the complexity of the optimum solution form. Considering the full 3-D problem with general gravity terms compounds the problem even more severely; hence the reason for an on-line, iteratively implemented, piece-wise linear type guidance law such as IGM.

D. GSP Subroutine Operations

The simulation routine GSP is made up of a number of function subprograms and subroutines in conjunction with a large number of COMMON blocks for ease of data transfer. A series of routines is used so that each can be optimized to perform a certain repetitive chore with a minimum of computer cards and hence computer storage and compile load.

In this section the subroutines are listed, along with the data input to each routine, the resulting data output, and the purpose behind each routine. It is hoped that this will give the user a better understanding of the flexibility of GSP as well as afford a clear enough understanding to be able to correctly modify the package without interrupting the proper flow of data.

MAIN

input

reads e, E, n, T_r, R of each WRT sun
(for sun coordinates)

$\underline{X}$, $\dot{\underline{X}}$, Thrust, I_{sp}, m, T_f, T_{start}

X_{offset}

Δt_C Δt_B print increment coast
print increment burn

A_Z, ϕ_L, - launch coordinates

ICHECK - mode selection
1 deterministic
2 state estimator
3 passenger

ISW = burn or coast initialization

NCYL = # cycles through sub-loop if only final orbit (not position) is specified

NFK6 = whether or not the correction factor FK6 is used
0 means no
1 means yes

Initializes (1) state covariance
(2) noise covariance
(3) measurement covariance
(4) $\hat{X}$ vector

T COAST - time coast is to end; added to be used as an ignition equation. This idea suggested by B. Williams when routines supplied by NASA failed

output

none-intermediate step

Purpose

(1) read in start data
(2) initialize states, covariance matrices
(3) Run GSP
(4) control output print cycle

NPRINT

input

updated simulation variable values

States, velocities, range angle, accelerations, gravity, time to burn, etc.

output

prints out values

Purpose

Data output routine

CONIC

input

present flight conditions

output

orbit parameters

purpose

as open-loop computation of orbit parameter equivalents

GRAV

input - present vehicle position

output - 3-D gravity value

purpose - supply gravity values to simulation

Fischer gravity model
see NASA TN D-2691 March 1965 by Fischer

IFUN

input

position, velocity vector, time

output

accelerations due to thrusting and gravity

purpose

to compute accelerations on vehicle as integration routine steps through fractional integration intervals. This is accomplished (for simplicity) by computing an approximate angle rate,

$$\dot{\chi}_p(k+1) \approx \frac{\chi_p(k+1) - \chi_p(k)}{\Delta t}$$

and then assuming that between time (k+1) and (k+2) that $\dot{\chi}_p$ is constant. With 4th order Runge-Kutta intervals of β_1, β_2, β_3, β_4, all less than the integration step size Δt, the corresponding new burn angles are approximated by

$$\chi_p(k+1+\beta_i) = \chi_p(k+1) + \dot{\chi}_p(k+1) \cdot \beta_i$$

ARTAN

input

sine, cosine of burn angles, integer selector ISW

output

normalized angle data

purpose

if ISW = 1, angle normalized between 0 and 2π
if ISW = -1, angle normalized between $-\pi$ and π

VMAG

input

three dimensional vector

output

magnitude of vector

purpose

compute magnitude of 3-D vector

VSUB

input

two 3-D vectors

output

one 3-D vector

purpose

compute difference of two 3-D vectors

VDOT

input

two 3-D vectors

output

one 3-D vector

purpose

compute the dot product of two 3-D vectors

VUNIT

input

3-D vector

output

3-D vector

purpose

normalize a 3-D vector

$$\underline{X}_{new} = \underline{X}_{old} / |\underline{X}_{old}|$$

VCROSS

input

two 3-D vectors

output

one 3-D vector

purpose

compute the cross product of two vectors

IRUN

input

positions, velocities, accelerations, time, integration step size

output

updated positions, velocities, time

purpose

Exact integration of system dynamics so that (1) true estimation error can be determined, (2) observation sightings of celestial objects can be determined for the measurement system simulation

MATMV

input

A - a 3x3 matrix
B - a 3x1 vector

output

C - 3x1 vector

purpose

$\underline{C} = A \cdot \underline{B}$ matrix multiplication

ATNA

input

sin and cos of an angle χ

output

angle χ normalized to be between 0 and 2π

purpose

normalize burn angle to be between 0 and 2π

IGM

input

old burn angles, gravity, positions, velocities, time, terminal or rendevous conditions

output

gimballed thruster burn angles χ_p, χ_y, for pitch and yaw commands

purpose

determine optimum burn angles for main engine

ATNB

input

sin and cos of an angle

output

normalized burn angle

purpose

normalize an angle so it lies between $\pm\pi$.

ASIN

input

sin of an angle χ

output

angle χ normalized to $\pm\frac{\pi}{2}$

purpose

normalize an angle χ, given sin χ, to be between $\pm\frac{\pi}{2}$

ACOS

input

cos of an angle χ

output

angle χ normalized to $\pm\frac{\pi}{2}$

purpose

normalize an angle χ, given cos χ, to be between $\pm\frac{\pi}{2}$

YDOT

input

state variable dynamical information; system states

output

derivative of state variables

purpose

compute $\dot{\underline{y}}$ of $\dot{\underline{y}} = A\underline{y} + B\underline{u}$. Can be used as part of a separate, on-board routine for on-line integration of the burn accelerometer data

RUNGE

input

system order, derivative of state vector, present state vector, present time, integration interval

output

new state vector at updated time, new time

purpose

use in conjunction with YDOT as part of an on-board routine for on-line integration of the burn accelerometer data.

MMUL3

input

matrix A 6x6
vector B 6x1

output

vector C 6x1

purpose

matrix multiply a matrix times a vector $\underline{C} = A \cdot \underline{B}$

MMUL

input

matrix X 6x6
matrix Y 6x6

output

matrix Z 6x6
$Z = X \cdot Y$

purpose

matrix multiply two 6x6 matrices

MATINV

input

matrix, system order

output

matrix

purpose

matrix inversion

MEASUR

input

vehicle position, measurement observations, noise generator signals

output

perfect and noisy measurement observations

purpose

to generate accelerometer and/or optical sighting signals of celestial bodies and to make the signals noisy.

FILTER

input

present system state, noisy measurements, filter partial derivative data from IGM, covariance matrix data, mode of operation (Coast or Burn)

output

updated state covariance, updated state estimate

purpose

nonlinear state estimation

PROPGT

input

present system state estimate $\hat{\underline{X}}(k/k)$, linearized state matrix, gravity

output

one step propagated state estimate, $\hat{\underline{X}}(k+1/k)$

purpose

perform the one step state vector propagation

SUN

input

exact state vector, eccentricity, eccentric anamoly, mean daily motion, reference time (time of pericenter passage), semi-major axis dimension

output

coordinates of sun WRT the earth in a set of coordinates with the plane of the ecliptic lying in one of the coordinate planes.

purpose

compute sun coordinates for use in sun seeker tracking measurements

ATTUDE

input

at present, none

output

three attitude angles

purpose

This routine is to represent the attitude control system and inertial platform alignment of the Tug. It is properly called in the program cycle; at present, the routine is set up for the platform to be aligned.

PARTL1

input

$\hat{\underline{X}}(k/k)$, $\hat{\underline{X}}(k+1/k)$, all data from IGM calculations, mode type (Coast or Burn), gravity

output

parts of calculations for $\Phi(k+1,k)$, $H(k+1)$

purpose

To evaluate intermediate expressions which make up the partial derivative calculations. There is a PARTL1 and PARTL2, and PARTL3, all performing calculations. Three separate routines are desired because of Roll Size limitations of the IBM 370/155 system at Auburn University.

PARTL2

input

same as PARTL1

output

same as PARTL1

purpose

same as PARTL1

PARTL3

input

same as PARTL1

output

final calculations for $\phi(k+1,k)$, $H(k+1)$

purpose

same as PARTL1

Output

input

output data from PARTL1, PARTL2, SUN, PROPGT, FILTER, MEASUR, RUNGE (if used), IGM, IRUN, IFUN, GRAV, CONIC, MAIN, PARTL3

output

partial derivative intermediate calculations, $\Phi(k+1,k)$, $H(k+1)$, state vector estimate, exact state, exact and noisy observations, covariance matrix, state and measurement error residuals

purpose

make computation results available; also to compare and analyze filter accuracy as a function of time.

GRAND

input

mean value, standard deviation, integer channel selection number between 1 and 10

output

A guassian random (Gaussian RANDom) number with characteristics of each mean and standard deviation for each channel.

purpose

Noise generator for measurement process.

URAND

input

integer representing a channel number between 1 and 10

output

a single number

purpose

Uniform RANDom distribution function

BLK DATA

input

none

output

none

purpose

provide initialization values for the random number generators.

CONOBS

input

matrices A, B, H and their respective dimensions of the form

$$\dot{\underline{X}} = A\underline{x} + B\underline{u}$$

$$\underline{Z} = H\underline{x}$$

output

statement as to controllability and/or observability or the lack thereof, and a scalar representing the determinant of a matrix result based on Kalman's criteria for linear, time-invariant systems

purpose

determine controllability and/or observability or the lack thereof of a linear (or the equivelant), time-invariant dynamical system in state variable form.

MATRIX

input

none

output

none

purpose

Actually a package of ENTRY statements which perform operations of determining eigenvalues of a matrix, multiply, subtraction, inversion, normalization, etc.

CHARD

input

An nxn matrix A and an integer print control character

output

Eigenvalues of a real matrix

purpose

compute the eigenvalues of a real matrix, symmetric or nonsymmetric

DINVER

input

A square matrix and its dimensions

output

a matrix and a scalar representing the determinant of the input matrix

purpose

compute the double precision inverse (Double precision INVERse) of a matrix

POLYRF

input

root of a polynomial, coefficients of polynomial

output

reduced order root, normalized polynomial coefficients

purpose

divide out first order factor from polynomial in order to obtain a reduced order polynomial

POLYEV

input

root of a polynomial, coefficients of a polynomial

output

reduced order polynomial coefficients

purpose

used in conjunction with POLYRF and LEMBRT to find all roots of a polynomial

LEMBRT

input

coefficients of a polynomial and the degree of the polynomial

output

one root of the remaining reduced-order polynomial

purpose

this routine systematically finds the roots of a polynomial one at a time using a simple caging scheme based on D'Alembert's Lemma

E. GSP Input/Output

In this section a discussion is given concerning the input/output data, the units employed, any constants needed, units of the output data, and what the output data means. As the GSP program is now structured, all major data is read in from cards or parameters initialized from ENTRY statements from commands at the beginning of MAIN, the cycling control routine.

Following is a list of the variables read in, their definition, units required (where applicable), and representative values used at the present. In Appendix B details of computer card formats and examples of what the output looks like are given. The variables read in are:

ECC - eccentricity of the Earth's orbit about the sun; unitless. Present value = .16727

ECCAML - eccentric anamoly of the Earth; .0083013

ANM - mean daily motion of the Earth; rad/sec. Present value = 1.9908 x 10^{-7} rad/sec.

TREF - time of Earth's pericenter passage; seconds. Due to lack of definitive data, arbitrarily chosen as 0.0.

AMAJOR - dimension of semi-major axis of Earth's orbit about the sun; meters. Present value = 1.4947×10^{12} meters.

ISW - program mode switch; 1 means in Coast, 2 means in Burn. The initial value is not really important as the program alters it to fit the necessary flight conditions.

NCYL - integer counter which controls the number of sub-loops performed in IGM if floating end conditions are used. NCYL = 0 is used, indicating one cycle is performed.

NFK6 - integer mode switch; if 0 it means FK6 = 0 and if 1 it means the guadratic curve fit for FK6 is employed.

THRUST - vehicle main engine thrust. Units of pounds. Present value = 15,000 pounds.

SISP - engine-fuel specific impulse. Units of seconds. Present value = 440 seconds.

WTSTAG - initial weight of Space Tug at simulation initialization time. Units of pounds. Present value = 58,500 pounds.

TF - freeze time before engine cut-off. Used to control certain IGM parameters near end-of-burn. Units of seconds. Present value = 10 seconds.

TIME - simulation running time referenced to Space Shuttle Earth launch time. Units of seconds. Present value = 41975.59 seconds.

$\underline{X}$ - 3-D position vector in ECLC. Units of meters. Present values =
-87143.37 -241404.6 -6673876.0

$\underline{DX}$ - 3-D velocity vector in ECLC. Units of meters/second. Present values =
7730.2 -17.55 -105.50

XOFFST - initial navigation state vector error. Six elements; first three are position errors in meters and second three are velocity errors in meters/second.

DTC - Coast propagation time increment. Units of seconds. Present value = 60 seconds.

DPRTC - Coast print increment. Automatically adjusted to be the nearest multiple of DTC possible. Units of seconds. Present values 60 to 500 seconds.

DTB - Burn propogation time increment. Units of seconds. Values between .5 and 2 seconds.

DPRTB - Burn print increment. Automatically adjusted to be the nearest multiple of DTB possible. Units of seconds. Present values 2 to 10 seconds.

AZ - launch azimuth on Earth. Units of degrees. Present value = 83.39 degrees.

PHIL - launch attitude. Units of degrees. Present value = 28.6 degrees

TPHIT - not used at present. Use a big number, i.e. 1.0×10^8

TCHIT - first (of two) characteristics freeze times in IGM. Units of seconds. Present value = 15 seconds.

ICHECK - program mode selection switch. If 1 then a deterministic run is made; 2 the full nonlinear state estimator is employed; 3 the passenger mode is run.

XINTL (entry statement) - located near the end of FILTER subroutine. Used to initialize $\hat{X}(k+1/k+1)$ by adding the offset vector, $\overline{XOFFST}$.

COVRNC (entry statement) - located near the end of FILTER subroutine. Used to initialize the state covariance matrix.

Only diagonal elements are initially defined. Since the diagonal entries represent variances, ${\sigma_1}^2 \rightarrow {\sigma_3}^2$ have units of $meter^2$ and ${\sigma_4}^2 \rightarrow {\sigma_6}^2$ of $(meter/second)^2$. Present values are ${\sigma_1}^2 \rightarrow {\sigma_3}^2 = 10^8$, ${\sigma_4}^2 \rightarrow {\sigma_6}^2 = 10^6$.

QMATX (entry statement) - located near the end of FILTER subroutine. Used to initialize the process noise covariance matrix Q. Only diagonal elements are defined. Units of $meter^2$ and $(meter/second)^2$. Present values of ${\sigma_1}^2 \rightarrow {\sigma_6}^2 = 2500.0$.

RT - terminal radius scalar from the center of the Earth at end-of-burn. Units of meters. Present value = 6,695,842.0 meters.

VT - terminal velocity scalar at end-of-burn. Units of meters/second. Present value = 8360.4 meters/second.

THT - γ_T, desired terminal flight path angle. Units of degrees. Present value = 1.254 degrees.

PO2 - semi-latus rectum of orbit the burn is to inject Space Tug into. Units of meters. Present value = 7,858,094.0 meters.

E2 - eccentricity of orbit at end-of-burn. Unitless. Present value = .17547.

ARGPER - argument of the perigee (of the orbit at end-of-burn). Units of degrees. Present value = 180.689 degrees.

DINCL - inclination of the desired burn orbit. Units of degrees. Present value = 29.28 degrees.

DNODE - longitude of the ascending node of the desired burn orbit. Units of degrees. Present value = 99.4 degrees.

TGB - initial estimate of time-to-go in the burn. Units of seconds. Present value = 242.3 seconds.

RANT - range angle for burn orbit. Units of degrees. Present value = 171.028 degrees.

WDROP - mass flow rate of the fuel during the burn. Results in a decrease in Space Tug weight.

RMATX (entry statement) - located near the end of FILTER subroutine. Initializes the measurement noise covariance matrix R. Only diagonal elements are defined.

During burn, units are $\sigma_1^2 \rightarrow \sigma_3^2$ meters^2, $\sigma_4^2 \rightarrow \sigma_6^2$ (meters/second)^2. Present values are $\sigma_1^2 \rightarrow \sigma_6^2 = 10.0$.

RCOAST (entry statement) - located near the end of FILTER subroutine. Initializes the measurement noise covariance matrix R during the Coast phase. Only diagonal elements are defined. Unitless. Present values are $\sigma_1^2 \rightarrow \sigma_3^2 = 10^{-4}$.

TCOAST - is a replacement method for use as an ignition equation. It is the time the coast is to end. Physically this is to be accomplished by comparing rangle angle measurements. Units of seconds. Present value = 0.0 (start off with a burn).

III. FLIGHT DYNAMICS OF SPACE TUG AND THE NAVIGATION SYSTEM

In this chapter the basic dynamical equation considerations are investigated and the general nonlinear dynamics of the Space Tug in powered and unpowered flight are developed. The equations and derivations tacitly assume that the vehicle is a point mass. At this point only the 3 degree-of-freedom translational problem is considered, although an inertial platform with provisions for an attitude control system package are included in GSP.

A. Problem Formulation

The basic dynamics of the Space Tug flight problem arise from Newtonian force considerations, i.e.

$$\underline{F} = m\underline{a} \qquad \text{(III-1)}$$

For the Tug, assuming solar pressure is negligible, gravity and drag are the only external force terms acting on the vehicle. The gravity model employed was a 5^{th} order polynomial approximation used on the Saturn V Apollo flights [9]. The guidance law is a modified version of IGM, which was also used in the Apollo program. It is a practical form of a minimum fuel optimal control law with internal terminal position and velocity rendezvous control. A basic flow chart of the computational operation of IGM was given in Figure II-2.

Under the previous conditions then, the system dynamics may be written in a fixed, dextral, orthogonal, cartesian frame as

$$\begin{bmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{bmatrix} = \begin{bmatrix} T/m \;\; DCX + \ddot{X}_g + \ddot{X}_d \\ T/m \;\; DCY + \ddot{Y}_g + \ddot{Y}_d \\ T/m \;\; DCZ + \ddot{Z}_g + \ddot{Z}_d \end{bmatrix} \qquad \text{(III-2)}$$

where

$\left.\begin{matrix} DCX \\ DCY \\ DCZ \end{matrix}\right\}$ engine gimball angle direction cosines

$$DCX = C_{11} \sin \chi_p' \cos \chi_y' + C_{12} \sin \chi_y' + C_{13} \cos \chi_p' \cos \chi_y'$$

$$DCY = C_{21} \sin \chi_p' \cos \chi_y' + C_{22} \sin \chi_y' + C_{23} \cos \chi_p' \cos \chi_y'$$

$$DCZ = C_{31} \sin \chi_p' \cos \chi_y' + C_{32} \sin \chi_y' + C_{33} \cos \chi_p' \cos \chi_y'$$

χ_p = engine pitch burn angle; in radians.

χ_y = engine yaw burn angle; in radians.

C_{ij} = elements of inverse rotation matirx (K^{-1})

T = vehicle thrust, in pounds.

$\ddot{X}_g$, $\ddot{Y}_g$, $\ddot{Z}_g$ = accelerations due to gravity in cartesian coordinates; units of ft/sec^2

m = vehicle mass; units of weight/gravity

For analysis purposes the vehicle thrust is assumed constant, although in practice a stochastic thrust variation can be expected. The vehicle mass is assumed time-varying with a constant mass flow rate due to fuel expenditure, although, again, in practice a certain amount of random variation can be expected.

In order to develop a state estimator, (III-2) must be expressed in terms of state equations. Since (III-2) represents 3 second-order systems, a sixth order state vector results. Since the direction cosines, gravity, and drag terms are nonlinear functions of X, Y, Z, $\dot{X}$, $\dot{Y}$, $\dot{Z}$ the overall state equations for (III-2) can be written in the general form

$$\underline{\dot{X}} = \underline{f}(\underline{X}) \tag{III-3}$$

where

$\underline{X}^T = (X\ Y\ Z\ \dot{X}\ \dot{Y}\ \dot{Z})$

$\underline{f}$ = nonlinear dynamic vector

A discrete form of extended Kalman filtering is to be used to process the on-board data to obtain the state estimates for the guidance law, on-board gravity and drag models. For real-time operation this requires the use of a set of linear, discrete equations which approximate (III-3), at least over a short time interval.

To obtain the linearized equations, the function vector $\underline{X}(t + \Delta t)$, representing a propogation disturbance, is expanded in a Taylor series about the operating point $\underline{X}(t)$. Such an approach was used by Mehra [10] for a nonlinear tracking problem for an Earth re-entry vehicle acted upon by gravity and drag forces. Results obtained by such an expansion are

$$\underline{X}(t + \Delta t) = \underline{X}(t) + \underline{\dot{X}}\,\frac{\Delta t}{1!} + \underline{\ddot{X}}\,\frac{\Delta t^2}{2!} + \text{H.O.T.} \tag{III-4}$$

where H.O.T. represents the higher order partial derivative terms. Using (k) to represent time (t) and (k + 1) to represent time (t + Δt), (III-4) can be written as

$$\underline{X}(k+1) = \underline{X}(k) + \dot{\underline{X}}(k)\Delta t + \frac{\ddot{\underline{X}}(k)}{2!}\Delta t^2 + \underline{n}(k) \qquad \text{(III-5)}$$

where $\underline{n}(k)$ is a random noise vector which is to represent a lumping of the H.O.T. of (III-4). But

$$\dot{\underline{X}}(k) = \underline{f}(\underline{X}(k)) \qquad \text{(III-6)}$$

and

$$\ddot{\underline{X}}(k) = \left.\frac{\partial \underline{f}}{\partial \underline{X}}\right|_k \cdot \left.\frac{dx}{dt}\right|_k = \left.\frac{\partial \underline{f}}{\partial \underline{X}}\right|_k \cdot \underline{f}(\underline{X}(k)) \qquad \text{(III-7)}$$

Using (III-6) and (III-7) with (III-5) results in

$$\underline{X}(k+1) = \underline{X}(k) + \left[A(\underline{X})\frac{\Delta t}{2} + I\right] \underline{f}(\underline{X}(k))\,\Delta t + \underline{n}(k) \qquad \text{(III-8)}$$

where

$$A(\underline{X}) = \frac{\partial \underline{f}}{\partial \underline{X}}, \text{ the Jacobian matrix.}$$

Defining the state transition matrix as

$$\Phi(k+1, k) = A(\underline{X})\frac{\Delta t}{2} + I \qquad \text{(III-9)}$$

then (III-8) becomes

$$\underline{X}(k+1) = \underline{X}(k) + \Phi(k+1, k)[\underline{f}(\underline{X})\Delta t] + \underline{n}(k) \qquad \text{(III-10)}$$

Note that $\underline{f}(\underline{X})\Delta t$ represents an estimate of the change in the state vector $\underline{X}$ over the time period Δt since $\underline{f}$ represents the derivative vector of $\underline{X}$, as noted in (III-6).

The actual analytical computation of $A(\underline{X})$ is extraordinarily complex and will not be given here due to the length. In Appendix A is detailed the analysis and substitution approach used to compute the partial derivatives in $A(\underline{X})$, and an Addendum is available listing all the partial derivatives. The form, however, is straightforward and is given below

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \left[\frac{\partial DCX}{\partial X} \cdot \frac{T}{m} + \frac{\partial \ddot{X}_g}{\partial X} + \frac{\partial \ddot{X}_d}{\partial X}\right] & \left[\frac{\partial DCX}{\partial Y} \cdot \frac{T}{m} + \frac{\partial \ddot{X}_g}{\partial Y} + \frac{\partial \ddot{X}_d}{\partial Y}\right] & \cdots\cdots\cdots\cdot & & & \\ \left[\frac{\partial DCY}{\partial X} \cdot \frac{T}{m} + \frac{\partial \ddot{Y}_g}{\partial X} + \frac{\partial \ddot{Y}_d}{\partial X}\right] & \left[\frac{\partial DCY}{\partial Y} \cdot \frac{T}{m} + \frac{\partial \ddot{Y}_g}{\partial Y} + \frac{\partial \ddot{Y}_d}{\partial Y}\right] & \cdots\cdots\cdots\cdot & & & \\ \left[\frac{\partial DCZ}{\partial X} \cdot \frac{T}{m} + \frac{\partial \ddot{Z}_g}{\partial X} + \frac{\partial \ddot{Z}_d}{\partial X}\right] & \left[\frac{\partial DCZ}{\partial Y} \cdot \frac{T}{m} + \frac{\partial \ddot{Z}_g}{\partial Y} + \frac{\partial \ddot{Z}_d}{\partial Y}\right] & \cdots\cdots\cdots\cdot & & & \end{bmatrix} \qquad \text{(III-10)}$$

$$\Phi(k+1, k) = (I + A\,\tfrac{\Delta t}{2})$$

Δt = propogation time

Similar to the dynamics in (III-3) there is a set of measurement observation equations to be considered. These are of the form

$$\underline{Z}(k+1) = \underline{h}(\underline{X}(k+1)) \qquad \text{(III-11)}$$

where

$\underline{Z}(k)$ is the actual measurement at time k

$\underline{h}(\underline{X})$ is the nonlinear vector relationship between the state vector $\underline{X}$ and the observation. It is assumed there are no dynamics involved in this operation.

Equation (III-11) can be expanded in a Taylor series and be placed in the form

$$\underline{Z}(k+1) = \underline{Z}(k) + H(k+1)\Big|_{\underline{X}(k+1)} \cdot \underline{X}(k+1) + \underline{v}(k+1) \qquad \text{(III-12)}$$

where

H(k + 1) is the linearized Jacobian-type matrix of partial derivatives

$\underline{v}(k+1)$ represents the random round-off effect plus general noise in the actual measurement process.

The actual form of H depends on whether the Tug is in the burn or coast mode. Details on the measurement process are covered in Chapter IV.

Using (III-9) and (III-12), a basic linear state estimation process can be formulated [11, 12] as follows:

Basic dynamics:

$$\underline{X}(k+1) = \underline{f}_1(\underline{X}(k))$$

One step propogation:

$$\hat{\underline{X}}(k+1/k) = \hat{\underline{X}}(k/k) + \Phi(k+1, k)\ \underline{f}(\hat{\underline{X}}(k/k))\Delta t \qquad \text{(III-13)}$$

where the ^ refers to estimated information

Measurement:

$$\underline{Z}(k+1) = \underline{Z}(k) + H\underline{X}(k+1) + \underline{v}(k+1) \qquad \text{(III-14)}$$

Estimated measurement:

$$\hat{Z}(k+1/k) = \hat{Z}(k) + H\hat{\underline{X}}(k+1/k) \qquad \text{(III-15)}$$

Measurement residual:

$$\tilde{\underline{Z}}(k+1/k) = Z(k+1) - \hat{\underline{Z}}(k+1/k) \qquad \text{(III-16)}$$

State covariance matrix one-step propogation:

$$P(k+1/k) = \Phi(k+1, k)P(k/k)\Phi^T(k+1, k) + Q(k) \qquad \text{(III-17)}$$

where

$$P(k/k) = E\left\{\tilde{\underline{X}}(k)\ \tilde{\underline{X}}(k)^T\right\} \qquad \text{(III-18)}$$

$$\tilde{\underline{X}} = \underline{X}(k) - \hat{\underline{X}}(k/k) \qquad \text{(III-19)}$$

$$Q(k) = E\left\{\underline{n}(k)\ \underline{n}(k)^T\right\} \quad \text{process noise covariance matrix} \qquad \text{(III-20)}$$

State estimator measurement weighting matrix:

$$K(k+1) = P(k+1/k)H^T[HP(k+1/k)H^T + R(k+1)]^{-1} \qquad \text{(III-21)}$$

where

$$R(k+1) = E\left\{\underline{v}(k+1)\ \underline{v}(k+1)^T\right\} \quad \text{measurement error covariance matrix} \qquad \text{(III-22)}$$

Filtered state covariance matrix:

$$P(k + 1/k + 1) = [I - K(k + 1)H]P(k + 1/k) \qquad \text{(III-23)}$$

These filter equations are then iterated along with the process dynamics so as to obtain an on-line optimal estimate of the states for the guidance law and the drag and gravity models. The overall equations of flight are similar to those in [13].

B. Simplifications of Partial Derivative Computations

Of considerable interest with the development of GSP is the reduction of computation burden imposed upon the implementable navigation system. In particular, the large number of intermediate calculations required for the partial derivatives in $\Phi(k + 1, k)$ and $H(k + 1)$ suggest some sort of sensitivity and/or simulation analysis. Upon original completion of the IGM and measurement partials, a study was undertaken to determine which computations could be easily eliminated without noticeable degradation in tracking accuracy. First, a number of the final parameter calculations in IGM such as K_1-K_4 were set to zero. Results were very good in that very little noticeable degradation of tracking accuracy resulted. A number of other terms were investigated but most proved to be too sensitive when widely varying mission profiles were considered. Partials near the end of IGM are the ones which it is desired to eliminate, for by eliminating them, many intermediate computations (of which there are many) can be avoided.

A detailed investigation of this area of study is reported in Chapter V. The importance of this work is great, for it is only if a simplified set of calculations requiring a minimum amount of storage are available and can be processed in "real time" (with respect to guidance calculations) that a practical navigation package can be used on-board with the Space Tug.

C. Simulator Flight Modes

Figure III-1 represents the concept of the position tracking errors during the burn phase of the flight. T_a and T_e correspond to the actual trajectory of the flight and the estimated trajectory of the flight respectively, while T_d represents a desired or nominal trajectory from deterministic considerations. This nominal trajectory was obtained by using perfect data in the simulation model. T_a and T_e are both corrupted by noise due to inherent inaccuracies in the measurement system and propagation calculations. Also note that while T_a and T_d originate at the same point, T_e is initially offset to represent the incomplete knowledge of the initial state vector value.

During the initial phase of the burn, T_e and T_a tracked the nominal trajectory while staying relatively close to one another. T_e and T_a remaining close results in a small normed position error, given by

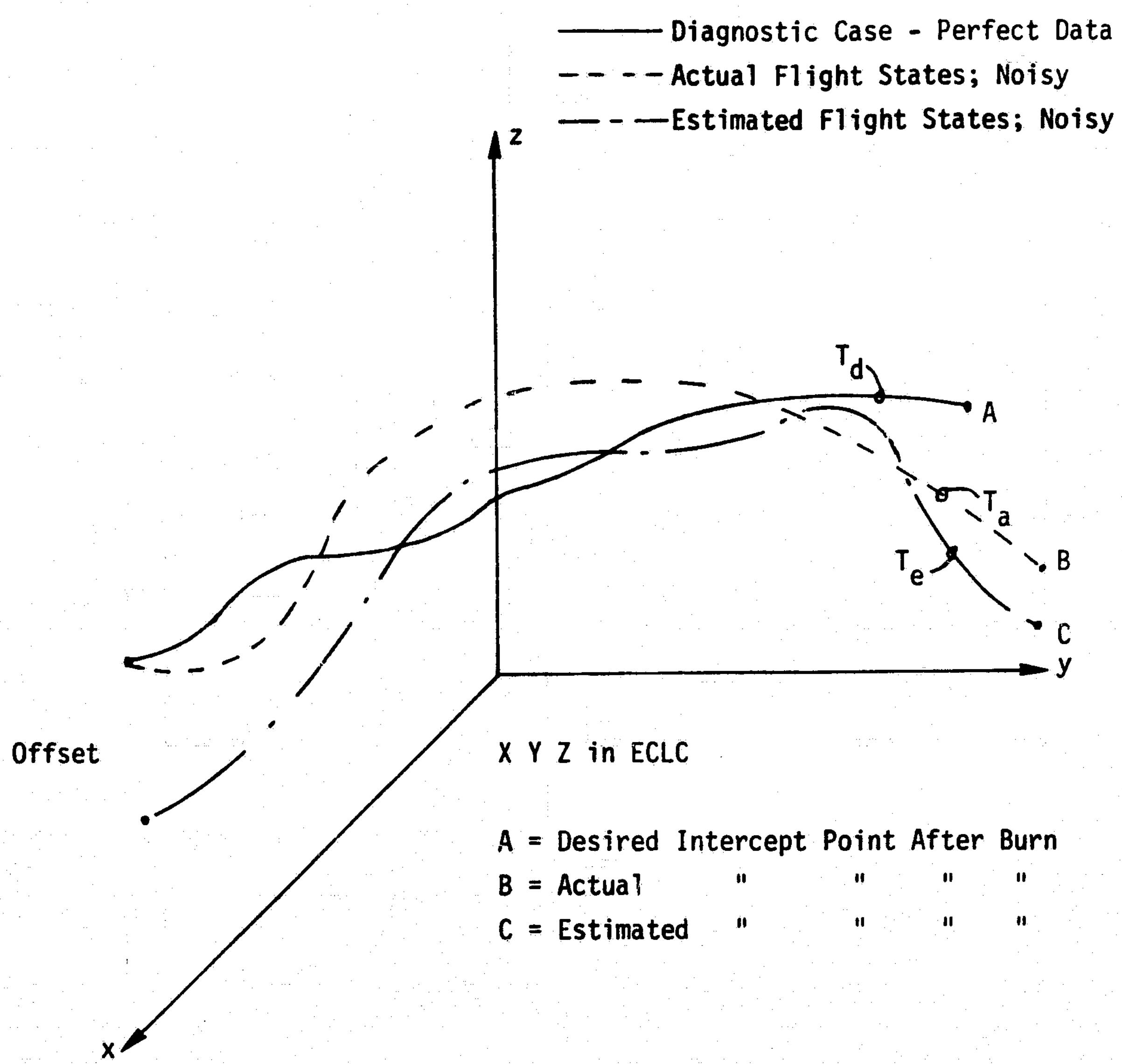
$$||\underline{X}(t')|| = \sqrt{[X_e(t) - X_a(t)]^2 + [Y_e(t) - Y_a(t)]^2 + [Z_e(t) - Z_a(t)]^2}\,\Bigg|_{t = t'} \qquad \text{(III-24)}$$

where t' is any instant of time.

Equation (III-24) is noted to be time variant and is never expected to be zero owing to the fluctuating nature of T_e and T_a as the flight progresses in time. The actual state position vector, $[X_a(t), Y_a(t), Z_a(t)]$, and the estimated state position vector, $[X_e(t), Y_e(t), Z_e(t)]$, are both in ECLC.

Figure III-1 also shows the desired, actual, and estimated intercept points, A, B, and C respectively, after the burn phase, evaluated at $t = t_{end\ burn}$. Note that even though the normed error is "small", this does not guarantee that the actual intercept point, B, is close to the desired intercept point, A. However, this error is the only information available during an actual flight. For this reason, the flight is broken down to three "burn-coast" phases in order to minimize the error between the desired end point and the actual end point.

Since the desired goal is not to follow the desired trajectory exactly, but to reach the terminal point with minimum fuel cost, it is intuitive that one should try to keep the normed error "small" and therefore keep the terminal error between A and B small.



T_d = ———— = $(x_d(t), y_d(t), z_d(t))$

T_a = - - - - - - = $(x_a(t), y_a(t), z_a(t))$

T_e = —— · —— = $(x_e(t), y_e(t), z_e(t))$

Figure III-1. FLIGHT PROBLEM UNDER STUDY

IV. MEASUREMENT SYSTEM

In this chapter is presented the details and derivations of the observation system used at present for the on-board inertial guidance system for the Space Tug. The development of the measurement equations relating the states to the actual measurements are presented, along with the appropriate partial derivative calculations for the development of H(k + 1).

A. Development of Observation Equations

As in all Kalman filtering schemes, there must be at least one measurement taken for implementation in the filter. These measurements, while they may not be actual states, should be able to be manipulated to yield state information, and they must be causally related to the states. How many and what kind of measurements to be taken plays an integral part of any state estimation program.

Actually, the measurements are not used in their raw form but rather are used to make up the "measurement residual". In equation form,

$$\underline{\tilde{Z}} = \underline{Z} - \underline{\hat{Z}} \tag{IV-1}$$

where $\underline{\tilde{Z}}$ = measurement residual

$\underline{Z}$ = actual measurements

$\underline{\hat{Z}}$ = predicted measurements

The development of the predicted measurements, which parallels closely that of the actual measurements will be discussed later.

The measurement system can best be discussed if divided into two areas, (1) the burn and (2) the coast phases of the flight. Each makes use of different numbers and types of measurements.

During the burn, strapdown accelerometers are used to obtain the actual measurements. These devices measure accelerations along the three principal axes of the tug centered coordinate system. The governing equations are:

$$Z(1) = \ddot{X}_x + \gamma_x$$
$$Z(2) = \ddot{X}_y + \gamma_y \qquad \text{(IV-2)}$$
$$Z(3) = \ddot{X}_z + \gamma_z$$

where $\ddot{X}_x$, $\ddot{X}_y$, $\ddot{X}_z$ = accelerations along the three principal axes of the tug centered system

γ_x, γ_y, γ_z = noise

Before being used, these measurements must be transformed into Earth Centered Launch Coordinates. A description of the different coordinate systems and the rotations required between them will be given in the discussion of the coast phase measurement scheme.

As is often the case, the states are not measured directly. Therefore, they must be reconstructed from the available measurements, i.e. the measurement must be transformed into the state estimate. During the burn, the state estimates are obtained by integrating each accelerometer measurement.

$$\underline{\dot{X}}_{AVAIL} = \int_0^t (\underline{\ddot{X}} + \gamma)dt + \hat{\underline{\dot{X}}}_0 \qquad \text{(IV-3)}$$

$$\underline{X}_{AVAIL} = \int_0^t (\underline{\dot{X}}_{AVAIL})dt + \hat{\underline{X}}_0 \qquad \text{(IV-4)}$$

where $\underline{\dot{X}}_{AVAIL}$ = available velocity measurement

$\underline{X}_{AVAIL}$ = available position measurement

$\hat{\underline{\dot{X}}}_0$, $\hat{\underline{X}}_0$ = available initial condition estimates

Since the integration yields only the incremental change in the state value, the initial conditions, $\hat{\underline{\dot{X}}}_0$ and $\hat{\underline{X}}_0$, must be added. This in itself presents a problem as only the estimates for these quantities are available.

The dependence on previous estimates in the current measurements is a totally undesirable situation that for the moment must be tolerated. It in fact violates the Kalman filter assumption that the measurements must be independent of one another. A further discussion of the problem of these indirect measurements will be given at the end of this chapter.

Because the accelerometers no longer give useful information, the coast phase must make use of a completely different measurement scheme. Instead of state measurements, only indirect positional data can be obtained using devices to measure angles between the tug and various reference points. The basic geometry of the coast measurements is shown in Figure IV-1, where $\underline{R}_{BT}$, $\underline{R}_{ET}$, and $\underline{R}_{EB}$ are measured in star tracker coordinates [14].

The desired vector in the configuration is $\underline{R}_{EB}$. To find it, however, will require some knowledge of all three vectors.

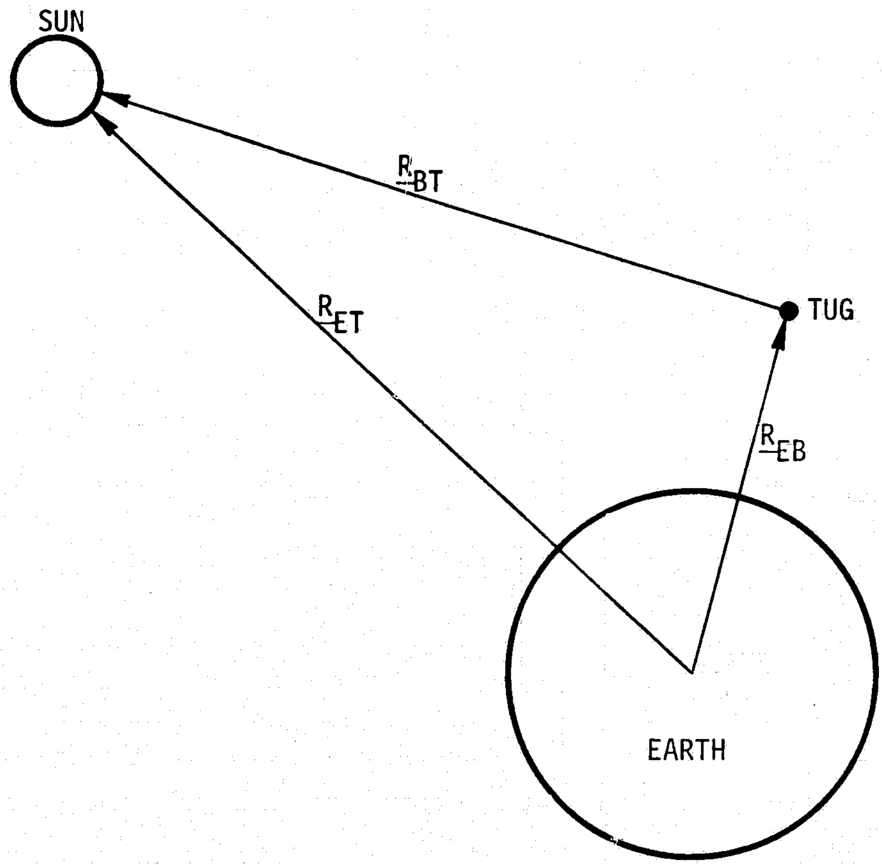


Figure IV-1. CELESTIAL GEOMETRY

Even though all of the measurements are taken in the same coordinates, they must be transformed to Earth Centered Launch Coordinates to be used in the filter. This is accomplished by a series of coordinate rotations.

There are four basic rotations needed to transform the information in successive stages between the five coordinate systems used. In addition, all but one will be time-varying. The first coordinate system is the one in which the measurements are actually taken, the star-tracker coordinates [15, 16]. The second coordinate system is centered at the platform upon which the measurement devices are mounted. As the vehicle flies, the measurement devices must be able to move to remain sighted on the proper object. Since the platform is in a fixed position, this rotation will be time-varying. Next comes a fixed rotation from the platform to a tug centered system. The three axes of this system coincide with the three principal axes of the tug. This rotation accounts for the fact that the axes of the platform system are not aligned with those of the tug centered system. From there, the measurements must be rotated through the Euler angles, Ψ, θ, and ϕ, to an earth centered inertial system. The final rotation puts the measurements into Earth Centered Launch Coordinates (ECLC) [17, 18]. In this coordinate system X points along the negative gravity vector at the launch size, Z points downrange, and Y completes the right-handed, Cartesian set.

The rotations can be lumped together into the following matrices:

$$\underline{X}_{\substack{\text{TRACKER}\\\text{SYSTEM}}} = [\alpha][\beta][R][A]\underline{X}_S \qquad \text{(IV-5)}$$

where α = measurement device to platform rotation

β = platform to tug centered rotation

R = tug centered to inertial earth rotation (contains the Euler angles)

A = inertial earth to ECLC rotation

For the problem at present, the measurement system will consist of rigidly mounted devices whose reference axes coincide with those of the tug centered system. Thus, the first two rotations are no longer required, and α and β become identity matrices.

The rotation to the inertial earth frame is really three separate rotations involving the three coordinate axes. For purposes of this study, it will be assumed that a 2-3-1 rotation sequence should be used. The matrices needed to carry out the rotations are given below.

$$R_\theta = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad \text{(IV-6)}$$

$$R_\Psi = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(IV-7)}$$

$$R_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \quad \text{(IV-8)}$$

where R_θ corresponds to rotation 2, R_Ψ to 3, and R_ϕ to rotation 1.

A 2-3-1 rotation requires that the matrices be multiplied in reverse order, i.e., 1-3-2. Performing the indicated multiplication yields the following matrix

$$\begin{bmatrix} \cos\Psi & \sin\Psi & -\sin\theta\cos\Psi \\ -\cos\phi\sin\Psi\cos\theta + \sin\phi\sin\Psi & \cos\phi\cos\Psi & \cos\phi\sin\Psi\sin\theta + \sin\phi\cos\theta \\ \sin\Psi\cos\theta\sin\phi + \cos\phi\sin\theta & -\sin\phi\cos\Psi & -\sin\phi\sin\Psi\sin\theta + \cos\phi\cos\theta \end{bmatrix} \qquad \text{(IV-9)}$$

The final rotation to Earth Centered Launch Coordinates is carried out through the elevation angle, ϕ_L, and the azimuth angle, A_Z, as shown in Figure IV-2. The two rotation matrices required are shown below

$$R\phi_L = \begin{bmatrix} \cos\phi_L & \sin\phi_L & 0 \\ -\sin\phi_L & \cos\phi_L & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(IV-10)}$$

$$R_{A_Z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin A_Z & -\cos A_Z \\ 0 & \cos A_Z & \sin A_Z \end{bmatrix} \qquad \text{(IV-11)}$$

Multiplying the matrices in the order shown yields the proper matrix denoted by A in (IV-5)

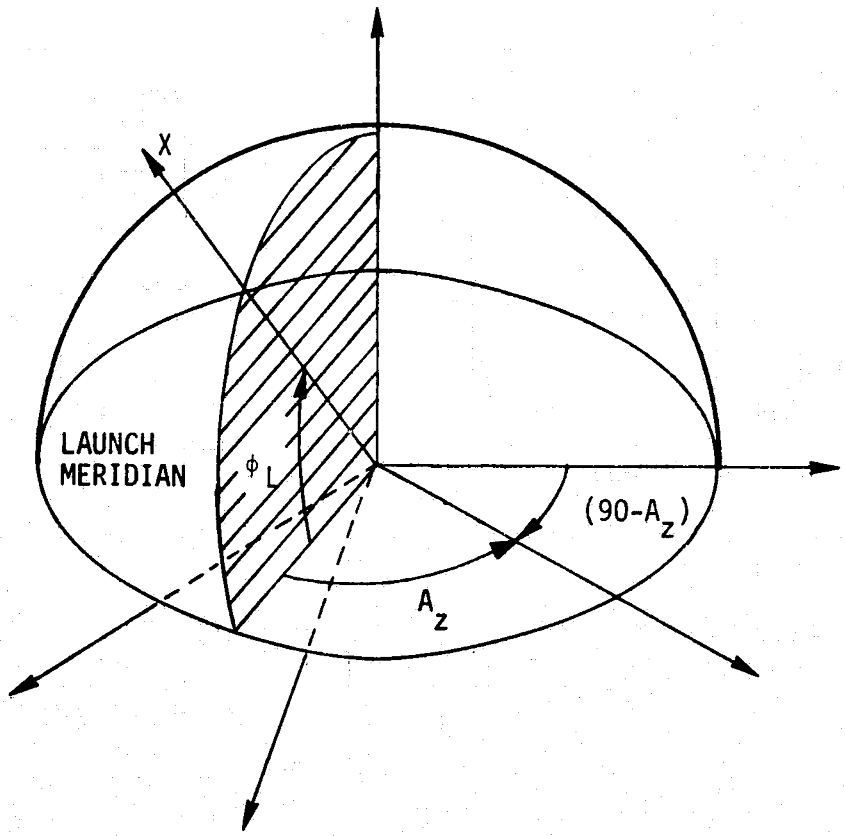


Figure IV-2. DEFINITION OF ϕ_L AND A_z

$$A = \begin{bmatrix} \cos \phi_L & \sin \phi_L \sin A_z & -\sin \phi_L \cos A_z \\ -\sin \phi_L & \cos \phi_L \sin A_z & -\cos \phi_L \cos A_z \\ 0 & \cos A_z & \sin A_z \end{bmatrix} \qquad \text{(IV-12)}$$

Premultiplying (IV-12) by (IV-9) yields the total rotation matrix, T^1

$$
T^{1} = \begin{bmatrix}
\begin{array}{l} \cos\Psi \cos\theta \cos\phi_L \\ -\sin\phi_L \sin\Psi \end{array} &
\begin{array}{l} \cos\Psi \cos\theta \sin\phi_L \\ \sin A_z + \sin\Psi \cos\phi_L \sin A_z \\ -\cos A_z \sin\theta \cos\Psi \end{array} &
\begin{array}{l} -\cos\Psi \cos\theta \sin\phi_L \\ \cos A_z - \sin\Psi \cos\phi_L \\ \cos A_z - \sin A_z \sin\theta \cos\Psi \end{array} \\[2ex]
\begin{array}{l} -\cos\phi \sin\Psi \cos\theta \\ \cos\phi_L + \sin\phi \sin\phi \\ \cos\phi_L - \sin\phi_L \cos\phi \\ \cos\Psi \end{array} &
\begin{array}{l} -\cos\phi \sin\Psi \cos\Theta \sin\phi_L \sin A_z \\ + \sin\phi_L \sin A_z \sin\phi \sin\theta + \\ \cos\phi \cos\Psi \cos\phi_L \sin A_z \\ + \cos A_z \cos\phi \sin\Psi \sin\theta \\ + \cos A_z \sin\phi \cos\theta \end{array} &
\begin{array}{l} - \sin\phi_L \cos A_z \cos\phi \sin\Psi \cos\theta \\ - \sin\phi_L \cos A_z \sin\phi \sin\theta \\ - \cos\phi_L \cos A_z \cos\phi \cos\Psi \\ + \cos\phi \sin\Psi \sin\theta \sin A_z \\ + \sin A_z \sin\phi \cos\theta \end{array} \\[2ex]
\begin{array}{l} \cos\phi_L \sin\Psi \cos\Theta \\ \sin\phi + \cos\phi_L \cos\phi \\ \sin\theta + \sin\phi_L \sin\phi \\ \cos\Psi \end{array} &
\begin{array}{l} \sin\phi_L \sin A_z \sin\Psi \cos\theta \sin\phi \\ + \cos\phi \sin\theta \sin\phi_L \sin A_z \\ - \sin\phi \cos\Psi \cos\phi_L \sin A_z \\ - \sin\phi \sin\Psi \cos A_z \\ + \cos\phi \cos\theta \cos A_z \end{array} &
\begin{array}{l} - \sin\Psi \cos\theta \sin\phi \sin\phi_L \cos A_z \\ - \sin\phi_L \cos A_z \cos\phi \sin\theta \\ + \sin\phi \cos\Psi \cos\phi_L \cos A_z \\ - \sin A_z \sin\phi \sin\Psi \sin\theta \\ + \cos\phi \cos\theta \sin A_z \end{array}
\end{bmatrix}
\qquad \text{(IV-13)}
$$

Having defined T^1, (IV-5) can be rewritten

$$\underline{X}_{\substack{TRACKER\\SYSTEM}} = T^1\ \underline{X}_{ECLC} \tag{IV-14}$$

From Figure (IV-1), it can be seen that

$$\underline{R}_{BT} = \underline{R}_{ET} - \underline{R}_{EB} \tag{IV-15}$$

Writing $\underline{R}_{BT}$ in its major components yields

$$\underline{R}_{BT} = X_{BT}\hat{I} + Y_{BT}\hat{J} + Z_{BT}\hat{K} \tag{IV-16}$$

where $\hat{I}$, $\hat{J}$, $\hat{K}$ are the unit vectors of the star tracker system.

These unit vectors are related to the Earth Centered Launch Coordinates by the transformation described above. In equation form

$$\begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} = T^1 \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \tag{IV-17}$$

where $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors in ECLC.

By using the relationship in (IV-17), the elements of $\underline{R}_{BT}$ can further be expanded as

$$X_{BT} = t_{11}[X_{ET} - X_{EB}] + t_{12}[Y_{ET} - Y_{EB}] + t_{13}[Z_{ET} - Z_{EB}] \tag{IV-18}$$

$$Y_{BT} = t_{21}[X_{ET} - X_{EB}] + t_{22}[Y_{ET} - Y_{EB}] + t_{23}[Z_{ET} - Z_{EB}] \tag{IV-19}$$

$$Z_{BT} = t_{31}[X_{ET} - X_{EB}] + t_{32}[Y_{ET} - Y_{EB}] + t_{33}[Z_{ET} - Z_{EB}] \tag{IV-20}$$

where the t_{ij}'s are elements of the transformation matrix T^1

X_{ET}, Y_{ET}, Z_{ET} are the components of $\underline{R}_{ET}$

X_{EB}, Y_{EB}, Z_{EB} are the components of $\underline{R}_{EB}$

The calculation of X_{ET}, Y_{ET}, and Z_{ET}, the earth to sun coordinates, will be shown later after a discussion of the measurement hardware.

The three measurement devices used during the coast are the horizon sensor, the star tracker, and the sun seeker. Each device contributes at least one non-redundant angle measurement to be used to compute a position fix.

The horizon sensor measurement is shown in Figure I-2.

Defining $Z_3 = \csc \phi$ yields

$$Z_3 = R/r_e \qquad \text{(IV-21)}$$

where r_e = radius of the earth

$$R = \sqrt{X_{EB}^2 + Y_{EB}^2 + Z_{EB}^2}$$

Since r_e is known, by measuring ϕ, R can be computed. Thus the horizon sensor is used mainly to obtain a radius fix.

The sun seeker and star tracker provide essentially the same angle measurements. The only difference is the actual object sighted upon, either the sun or a star. The data is manipulated in the same way, i.e., the same transformations are used. Only the sun seeker will be shown here.

The sun seeker provides the additional two angles, elevation and azimuth, necessary to reconstruct the position vector in the filter. Together with the angle measurement from the horizon sensor, they make up the measurement vector. The angles that are measured are shown in Figure I-3, where α is the azimuth angle and β is the elevation angle.

Letting $Z_1 = \cos \alpha$

$$Z_2 = \cos \beta \qquad \text{(IV-22)}$$

yields

$$Z_1 = \frac{X_{BT}}{\sqrt{X_{BT}^2 + Y_{BT}^2}} \tag{IV-23}$$

$$Z_2 = \frac{\sqrt{X_{BT}^2 + Y_{BT}^2}}{\sqrt{X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2}} \tag{IV-24}$$

In summary, the measurement vector during the burn is

$$Z(k + 1) = [X\ Y\ Z\ \ \dot{X}\ \dot{Y}\ \dot{Z}]^T \tag{IV-25}$$

and during the coast it is

$$Z(k + 1) = [\cos\alpha \ \ \cos\beta \ \ \csc\phi]^T \tag{IV-26}$$

It should be remembered that the measurement vector is not composed of the actual hardware measurements but, rather, quantities derived from them in a manner previously discussed. To review, the actual measurements during the burn are accelerations while during the coast they are the angles themselves and not the trigonometric functions of them.

Returning now to the previous discussion, the components of the earth to sun vector, $\underline{R}_{ET}$, are generated by using Kepler's equation

$$n(t - T) = E - e \sin E \tag{IV-27}$$

where n = mean daily motion of the earth

E = eccentric anomaly

e = eccentricity

T = time of pericenter passage

t = running time variable

An examination of equations (IV-18)-(IV-20) will show that the measurements are now in terms of the desired states, X_{EB}, Y_{EB}, and Z_{EB}, with only those states as unknowns. This is the desired configuration for use in the filter.

An expression for the predicted measurement, which was mentioned at the outset of this section, can now be obtained. This predicted measurement is, as one would think, the value of the measurement at time k + 1 based on the predicted state at time k + 1 given data through time k. In equation form,

$$\hat{\underline{Z}}(k + 1/k) = H(k + 1)\hat{\underline{X}}(k + 1/k) + E[\underline{V}(k + 1)] \qquad \text{(IV-28)}$$

where $\hat{\underline{Z}}(k + 1/k)$ = predicted measurement

$\hat{X}(k + 1/k)$ = predicted state at time k + 1 based on data through time k

$E[\underline{V}(k + 1)]$ = expected value of the measurement noise (assumed to be 0)

B. DEVELOPMENT OF H(k + 1)

In this section the measurement equation approach of the last section is used to develop the measurement matrix, H, which is required in the state estimation process. The detailed partial derivative computations are fully explained so that a clear understanding of the approach utilized may be obtained.

H(k + 1) is the matrix that relates the measurements to the states, i.e., it transforms the state vector into a set of measurements. It is of dimension m x n where m is the number of measurements and n is the number of states.

The derivation of the H(k + 1) matrix can now be shown in light of the recent insight into the measurement system's operation. During the burn phase, the calculation of H(k + 1) is a simple matter. Since the elements of the measurement vector are states, the predicted measurement vector must also be the predicted states. Therefore, $\hat{Z}(k + 1/k)$ equals $\hat{X}(k + 1/k)$ and from equation (IV-28), H(k + 1) must be the identity matrix

$$H(k + 1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_6 \qquad \text{(IV-29)}$$

During the coast, however, the calculation of H(k + 1) is not so direct. The measurement vector in this case consists of the cosines or cosecants of the angles sighted by the horizon sensor and sun seeker (star tracker). Some means must be found to transfer the predicted states into the set of predicted measurements.

The first step is to recall the fact that given equations (IV-18)-(IV-20) and using Kepler's equation to generate X_{ET}, Y_{ET}, and Z_{ET}, the measurements are now in terms of the states with only those states as unknowns. One approximation for the predicted measurements can be obtained by expanding the expression for $\hat{\underline{Z}}(k + 1/k)$ in a Taylor series yielding

$$\hat{\underline{Z}}(k+1/k) = \hat{\underline{Z}}(k) + \left.\frac{\partial \hat{\underline{Z}}(k)}{\partial \underline{X}}\right|_{X = \hat{X}(k+1/k)} (\hat{X}(k+1/k)-\hat{X}(k/k))+H.O.T. \qquad \text{(IV-30)}$$

where H.O.T. are the second and higher order terms hereafter neglected as insignificant

$\hat{\underline{Z}}(k)$ = predicted measurement at time k

Thus, the predicted measurement will be of the form

$$\underline{Z}_{NEW} = \underline{Z}_{OLD} + \Delta\underline{Z} \qquad \text{(IV-31)}$$

Relating (IV-28) to (IV-30), it can be seen that

$$H(k + 1) = \frac{\partial \hat{\underline{Z}}(k)}{\partial \underline{X}} \qquad \text{(IV-32)}$$

Performing the indicated partial derivatives on (IV-21), (IV-23), and (IV-24) yields

$$\frac{\partial Z_1}{\partial X} = \frac{\partial X_{BT}}{\partial X} [X_{BT}{}^2 + Y_{BT}{}^2]^{-\frac{1}{2}} - X_{BT}\left(X_{BT} \frac{\partial X_{BT}}{\partial X} + Y_{BT} \frac{\partial Y_{BT}}{\partial X}\right)\left(X_{BT}{}^2 + Y_{BT}{}^2\right)^{-3/2}$$

(IV-33)

$$\frac{\partial Z_1}{\partial Y} = \frac{\partial X_{BT}}{\partial Y} [X_{BT}{}^2 + Y_{BT}{}^2]^{-\frac{1}{2}} - X_{BT}\left(X_{BT} \frac{\partial X_{BT}}{\partial Y} + Y_{BT} \frac{\partial Y_{BT}}{\partial Y}\right)\left(X_{BT}{}^2 + Y_{BT}{}^2\right)^{-3/2}$$

(IV-34)

$$\frac{\partial Z_1}{\partial Z} = \frac{\partial X_{BT}}{\partial Z} [X_{BT}{}^2 + Y_{BT}{}^2]^{-\frac{1}{2}} - X_{BT}\left(X_{BT} \frac{\partial X_{BT}}{\partial Z} + Y_{BT} \frac{\partial Y_{BT}}{\partial Z}\right)\left(X_{BT}{}^2 + Y_{BT}{}^2\right)^{-3/2}$$

(IV-35)

C,2

$$\frac{\partial Z_1}{\partial \dot{X}} = \frac{\partial Z_1}{\partial \dot{Y}} = \frac{\partial Z_1}{\partial \dot{Z}} = 0$$

$$\frac{\partial Z_2}{\partial X} = \frac{[X_{BT} \frac{\partial X_{BT}}{\partial X} + Y_{BT} \frac{\partial Y_{BT}}{\partial X}]}{\sqrt{X_{BT}^2 + Y_{BT}^2} \sqrt{X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2}} - \left[\frac{(X_{BT} \frac{\partial X_{BT}}{\partial X} + Y_{BT} \frac{\partial Y_{BT}}{\partial X} + Z_{BT} \frac{\partial Z_{BT}}{\partial X})(X_{BT}^2 + Y_{BT}^2)^{\frac{1}{2}}}{(X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}} \right] \quad \text{(IV-36)}$$

$$\frac{\partial Z_2}{\partial Y} = \frac{[X_{BT} \frac{\partial X_{BT}}{\partial Y} + Y_{BT} \frac{\partial X_{BT}}{\partial Y}]}{\sqrt{X_{BT}^2 + Y_{BT}^2} \sqrt{X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2}} - \left[\frac{(X_{BT} \frac{\partial X_{BT}}{\partial Y} + Y_{BT} \frac{\partial Y_{BT}}{\partial Y} + Z_{BT} \frac{\partial Z_{BT}}{\partial Y}) \ (X_{BT}^2 + Y_{BT}^2)^{\frac{1}{2}}}{(X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}} \right] \quad \text{(IV-37)}$$

$$\frac{\partial Z_2}{\partial Z} = \frac{(X_{BT} \frac{\partial X_{BT}}{\partial Z} + Y \frac{\partial Y_{BT}}{\partial Z})}{\sqrt{X_{BT}^2 + Y_{BT}^2} \sqrt{X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2}} - \left[\frac{(X_{BT} \frac{\partial X_{BT}}{\partial Z} + Y_{BT} \frac{\partial Y_{BT}}{\partial Z} + Z_{BT} \frac{\partial Z_{BT}}{\partial Z}) \ (X_{BT}^2 + Y_{BT}^2)^{\frac{1}{2}}}{(X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}} \right] \quad \text{(IV-38)}$$

$$\frac{\partial Z_2}{\partial \dot{X}} = \frac{\partial Z_2}{\partial \dot{Y}} = \frac{\partial Z_2}{\partial \dot{Z}} = 0$$

$$\frac{\partial Z_3}{\partial X} = \frac{1}{r_e} \frac{X_{BT}}{\sqrt{X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2}} \tag{IV-39}$$

$$\frac{\partial Z_3}{\partial Y} = \frac{1}{r_e} \frac{Y_{BT}}{\sqrt{X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2}} \tag{IV-40}$$

$$\frac{\partial Z_3}{\partial Z} = \frac{1}{r_e} \frac{Z_{BT}}{\sqrt{X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2}} \tag{IV-41}$$

$$\frac{\partial Z_3}{\partial \dot{X}} = \frac{\partial Z_3}{\partial \dot{Y}} = \frac{\partial Z_3}{\partial \dot{Z}} = 0$$

Putting (IV-33) over a common denominator

$$\frac{\partial Z_1}{\partial X} = \frac{\frac{\partial X_{BT}}{\partial X}(X_{BT}^2 + Y_{BT}^2) - X_{BT}\left(X_{BT}\frac{\partial X_{BT}}{\partial X} + Y_{BT}\frac{\partial Y_{BT}}{\partial X}\right)}{(X_{BT}^2 + Y_{BT}^2)^{3/2}} \tag{IV-42}$$

$$= \frac{X_{BT}^2\frac{\partial X_{BT}}{\partial X} + Y_{BT}^2\frac{\partial X_{BT}}{\partial X} - X_{BT}^2\frac{\partial X_{BT}}{\partial X} - X_{BT}Y_{BT}\frac{\partial Y_{BT}}{\partial X}}{(X_{BT}^2 + Y_{BT}^2)^{3/2}} \tag{IV-43}$$

$$= \frac{Y_{BT}^2\frac{\partial X_{BT}}{\partial X} - X_{BT}Y_{BT}\frac{\partial Y_{BT}}{\partial X}}{(X_{BT}^2 + Y_{BT}^2)^{3/2}} \tag{IV-44}$$

From (IV-18) and (IV-19)

$$\frac{\partial X_{BT}}{\partial X} = -t_{11} \tag{IV-45}$$

$$\frac{\partial Y_{BT}}{\partial X} = -t_{21} \tag{IV-46}$$

Substituting (IV-45) and (IV-46) into (IV-44)

$$\frac{\partial Z_1}{\partial X} = \frac{X_{BT}\ Y_{BT}\ t_{21} - Y_{BT}^{2}\ t_{11}}{(X_{BT}^{2} + Y_{BT}^{2})^{3/2}} \tag{IV-47}$$

Using the same manipulations (IV-34) and (IV-35) can be written

$$\frac{\partial Z_1}{\partial Y} = \frac{Y_{BT}^{2}\ \frac{\partial X_{BT}}{\partial Y} - X_{BT}\ Y_{BT}\ \frac{\partial Y_{BT}}{\partial Y}}{(X_{BT}^{2} + Y_{BT}^{2})^{3/2}} \tag{IV-48}$$

$$\frac{\partial Z_1}{\partial Z} = \frac{Y_{BT}^{2}\ \frac{\partial X_{BT}}{\partial Z} - X_{BT}\ Y_{BT}\ \frac{\partial Y_{BT}}{\partial Z}}{(X_{BT}^{2} + Y_{BT}^{2})^{3/2}} \tag{IV-49}$$

From (IV-18) and (IV-19)

$$\frac{\partial X_{BT}}{\partial Y} = -t_{12} \tag{IV-50}$$

$$\frac{\partial X_{BT}}{\partial Z} = -t_{13} \tag{IV-51}$$

$$\frac{\partial Y_{BT}}{\partial Y} = -t_{22} \tag{IV-52}$$

$$\frac{\partial Y_{BT}}{\partial Z} = -t_{23} \qquad \text{(IV-53)}$$

Substituting (IV-50)-(IV-53) into (IV-48) and (IV-49)

$$\frac{\partial Z_1}{\partial Y} = \frac{t_{22}\, X_{BT}\, Y_{BT} - t_{12}\, Y_{BT}^2}{(X_{BT}^2 + Y_{BT}^2)^{3/2}} \qquad \text{(IV-54)}$$

$$\frac{\partial Z_1}{\partial Z} = \frac{t_{23}\, X_{BT}\, Y_{BT} - t_{13}\, Y_{BT}^2}{(X_{BT}^2 + Y_{BT}^2)^{3/2}} \qquad \text{(IV-55)}$$

Putting (IV-36) over a common denominator

$$\frac{\partial Z_2}{\partial X} = \frac{(X_{BT}\frac{\partial X_{BT}}{\partial X} + Y_{BT}\frac{\partial Y_{BT}}{\partial X})(X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)}{(X_{BT}^2 + Y_{BT}^2)^{\frac{1}{2}}\ (X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}}$$

$$- \frac{[(X_{BT}^2 + Y_{BT}^2)(X_{BT}\frac{\partial X_{BT}}{\partial X} + Y_{BT}\frac{\partial Y_{BT}}{\partial X} + Z_{BT}\frac{\partial Z_{BT}}{\partial X})]}{(X_{BT}^2 + Y_{BT}^2)^{\frac{1}{2}}\ (X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}} \qquad \text{(IV-56)}$$

Expanding the numerator and cancelling terms

$$X_{BT}^3\frac{\partial X_{BT}}{\partial X} + X_{BT}\, Y_{BT}^2\frac{\partial X_{BT}}{\partial X} + X_{BT}\, Z_{BT}^2\frac{\partial X_{BT}}{\partial X} + Y_{BT}\, X_{BT}^2\frac{\partial Y_{BT}}{\partial X}$$

$$+ Y_{BT}^3\frac{\partial Y_{BT}}{\partial X} + Y_{BT}\, Z_{BT}^2\frac{\partial Y_{BT}}{\partial X} - X_{BT}^3\frac{\partial X_{BT}}{\partial X} - X_{BT}^2\, Y_{BT}\frac{\partial Y_{BT}}{\partial X} \qquad \text{(IV-57)}$$

$$- X_{BT}^2\, Z_{BT}\frac{\partial Z_{BT}}{\partial X} - Y_{BT}^2\, X_{BT}\frac{\partial X_{BT}}{\partial X} - Y_{BT}^3\frac{\partial Y_{BT}}{\partial X} - Y_{BT}^2\, Z_{BT}\frac{\partial Z_{BT}}{\partial X}$$

Combining (IV-56) and (IV-57)

$$\frac{\partial Z_2}{\partial X} = \frac{[X_{BT}\frac{\partial X_{BT}}{\partial X} + Y_{BT}\frac{\partial Y_{BT}}{\partial X}]Z_{BT}^2 - [X_{BT}^2 + Y_{BT}^2]Z_{BT}\frac{\partial Z_{BT}}{\partial X}}{(X_{BT}^2 + Y_{BT}^2)^{\frac{1}{2}}\ (X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}} \qquad \text{(IV-58)}$$

From (IV-20)

$$\frac{\partial Z_{BT}}{\partial X} = -t_{31} \qquad \text{(IV-59)}$$

Combining (IV-45), (IV-46), (IV-58), and (IV-59) yields

$$\frac{\partial Z_2}{\partial X} \quad \frac{(-t_{11}\ X_{BT} - t_{21}\ Y_{BT})Z_{BT}^2 + (X_{BT}^2 + Y_{BT}^2)Z_{BT}(t_{31})}{(X_{BT}^2 + Y_{BT}^2)^{\frac{1}{2}}\ (X_{BT} + Y_{BT} + Z_{BT})^{3/2}} \qquad \text{(IV-60)}$$

By analagous reasoning (IV-37) and (IV-38) can be written

$$\frac{\partial Z_2}{\partial Y} = \frac{(X_{BT}\frac{\partial X_{BT}}{\partial Y} + Y_{BT}\frac{\partial Y_{BT}}{\partial Y})Z_{BT}^2 - (X_{BT}^2 + Y_{BT}^2)Z_{BT}\frac{\partial Z_{BT}}{\partial Y}}{(X_{BT}^2 + Y_{BT}^2)^{\frac{1}{2}}\ (X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}} \qquad \text{(IV-61)}$$

$$\frac{\partial Z_2}{\partial Z} = \frac{(X_{BT}\frac{\partial X_{BT}}{\partial Z} + Y_{BT}\frac{\partial Y_{BT}}{\partial Z})Z_{BT}^2 - (X_{BT}^2 + Y_{BT}^2)Z_{BT}\frac{\partial Z_{BT}}{\partial Z}}{(X_{BT}^2 + Y_{BT}^2)^{\frac{1}{2}}\ (X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}} \qquad \text{(IV-62)}$$

From (IV-20)

$$\frac{\partial Z_{BT}}{\partial Y} = -t_{32} \qquad \text{(IV-63)}$$

$$\frac{\partial Z_{BT}}{\partial Z} = -t_{33} \tag{IV-64}$$

Substituting (IV-50)-(IV-53), (IV-63), (IV-64) into (IV-61) and (IV-62)

$$\frac{\partial Z_2}{\partial Y} = \frac{[-t_{12}X_{BT}-t_{22}Y_{BT}]Z_{BT}^2 - (X_{BT}^2 + Y_{BT}^2)(t_{32})Z_{BT}}{(X_{BT}^2 + Y_{BT}^2)^{1/2} \quad (X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}} \tag{IV-65}$$

$$\frac{\partial Z_2}{\partial Z} = \frac{[-t_{13}X_{BT}-t_{23}Y_{BT}]Z_{BT}^2 - (X_{BT}^2 + Y_{BT}^2)(t_{33})Z_{BT}}{(X_{BT}^2 + Y_{BT}^2)^{1/2} \quad (X_{BT}^2 + Y_{BT}^2 + Z_{BT}^2)^{3/2}} \tag{IV-66}$$

All of the partial derivatives that are necessary to implement H(k + 1) in the filter have now been derived.

Expanding equation (IV-32)

$$H(k+1) = \begin{bmatrix} \frac{\partial Z_1}{\partial X} & \frac{\partial Z_2}{\partial X} & \frac{\partial Z_3}{\partial X} & 0 & 0 & 0 \\ \frac{\partial Z_1}{\partial Y} & \frac{\partial Z_2}{\partial Y} & \frac{\partial Z_3}{\partial Y} & 0 & 0 & 0 \\ \frac{\partial Z_1}{\partial Z} & \frac{\partial Z_2}{\partial Z} & \frac{\partial Z_3}{\partial Z} & 0 & 0 & 0 \end{bmatrix} \tag{IV-67}$$

An examination of the H(k + 1) matrix, or the partial derivatives, will reveal a problem peculiar to the coast phase. All the partial derivatives with respect to the velocities are zero, indicating that the measurements have yielded no velocity information. In other words, no velocity measurements, either direct or indirect, have been taken.

Thus, the accuracy of the velocities rely solely on the accuracy of the partial derivatives from the dynamics.

The concept of direct and indirect measurements, which was mentioned briefly in the preceeding paragraphs, deserves some consideration. Actually the concept is self-explanatory; it is its effect on the state estimator that is of interest here. A direct measurement is one in which the measurements are directly related to the actual states. A single transformation matrix would at most be required to transform the data into a usable form for the filter. An indirect measurement means that no such direct state information is available. The value of the state measurements must be inferred from a combination of the present measurements, past measurements, and an initial estimate. An examination of equations (IV-3), (IV-4), and (IV-16) shows that, because of the initial conditions contained therein, all of the measurements are indirect ones.

The effect of indirect measurement on the normed error can be seen without much difficulty. If at the start of a cycle, either burn or coast, the initial conditions are in error, then the predicted and actual measurement will also be wrong by at least the amount of error in the initial conditions. Of course, any errors introduced by the measurement system itself will only add to the overall error. Since the initial conditions for the $(k + 1)^{st}$ measurements are the measurements at time k, each measurement will contain the initial error. Thus, under these constraints, the best that a filtering scheme can do is to prevent the growth of the error. It appears that it cannot be completely eliminated.

C. System Observability

Having designed the measurement system, there must be an assurance that this system will be of use. This assurance comes from considering the observability of the system, that is, whether or not the desired states can be inferred by observing the output of the system. A necessary and sufficient condition for the system to be observable is that the n x mn matrix

$$[H^T : FH^T : F^2H^T : \dots : F^{(n-1)}H^T] \tag{IV-68}$$

has rank n.

In equation (IV-68)

n = order of the system (6)

H = observation matrix defined in this section

F = state transition matrix defined in Chapter II

Performing the indicated multiplication for the H matrix used during the burn will yield a 6 x 36 matrix. To see if its rank is the necessary 6, it is postmultiplied by its transpose and the determinant of the resulting 6 x 6 matrix is found. For the system to be observable, this determinant must be non-zero. The procedure for the coast phase is essentially the same with the exception that application of (IV-68) will yield a 6 x 18 matrix. The results of this test showed that, for both phases, the determinant is non-zero and thus the system is observable.

Simply because the system is observable at one point in the flight does not mean that it will always be so. This is due to the time-varying nature of both the F and H matrices. Therefore, the observability condition must be checked at each state update. Thus far, no problem with a non-observable system has been encountered at any stage of the flight.

D. Propagation Time Considerations

The effect of ΔT, the propagation time, on the measurement system will be the final topic considered in this section. The value of ΔT during the coast is not a very critical factor because of the slowly varying nature of the measurements. The effect during the burn, however, while arising due to a rather subtle reason, is much more marked.

The reason lies in the measurement devices themselves. The accelerometers measure on-board accelerations only, those provided by the engine. The acceleration due to gravity, which is a function only of position, must be added. As will be recalled from the chapter dealing with the vehicle dynamics, the accuracy of the value obtained from the gravity model is a direct function of the propagation time. Thus the proper choice of ΔT is necessary to achieve accurate measurement results.

E. Other Measurement Techniques

The burn and coast measurement combinations are not the only ones possible or even contemplated. Some of the other possibilities include electron beam imagers [19], landmark tracking [20], tracking and data relay satellite system (TDRSS) [21] among others. Each requires accuracy, cost, and reliability trade-offs as well as depending on the type of mission (low earth, geosynch, etc.). The accelerometers-only during burn and horizon sensor-sun seeker during coast are reasonable combinations, however.

V. FLIGHT SIMULATION ANALYSIS AND RESULTS

In this chapter actual simulation results and an analysis of them are supplied in order to yield an indication of what type of overall response-tracking characteristics could be expected to be encountered. In most cases data is supplied in terms of lumped position and velocity results with one axis of either x and $\dot{x}$, y and $\dot{y}$, or z and $\dot{z}$ also included. It should be kept in mind that even though one axis (x, y, or z) may exhibit favorable response, the other axes may not and this is the purpose of the lumped position and velocity tracking results.

A. Basic Mission Description

The primary mission for the Space Tug is given in Chapter I. This mission profile was chosen as a base mission for the project. In the studies which follow in this chapter the first burn and first coast phases of the Space Tug's maneuver from low earth to geosynchronous orbit are investigated. It was decided that these two phases of flight would cause the most difficulty for the estimation scheme and other burns and coasts would be an extension of the first.

B. Filter Checkout

The logical first stage in checking the entire simulation program was a verification of the operation of the filter itself. This checkout was accomplished in three steps in order to verify that the simulator was functioning properly.

The first of these tests was a deterministic one in which actual (perfect) state information was made available as measurement data. In addition to the perfect measurements, the initial estimated states were made equal to the actual ones.

Next, offsets were added to the initial states but the measurements were kept perfect. This was to take into account the practical problem of initial state indeterminacy. Finally, noise was added along with the initial offsets to simulate the combined effect of system noise and measurement errors. The noise was computed during each cycle by using a pseudo-random number generator with mean of zero and standard deviation (σ) a percentage of the actual states. By using the actual states as measurements, the problem of initial error propagation discussed in Chapter IV in association with indirect measurements was avoided. This is not the method to be used in the final program, but suffices to check the filter by assessing its sensitivity to noise. The variable standard deviation allows some parametric data to be gathered regarding the amount of noise tolerable in the system in order to meet design specifications.

The results of these tests are shown in Figure V-1 through V-3. Each figure contains plots to show the results of each test on the normed position and velocity errors. These error norms were calculated by using the equations below

$$X_e(t) = \sqrt{(X-\hat{X})^2 + (Y-\hat{Y})^2 + (Z-\hat{Z})^2} \qquad \text{(V-1)}$$

$$\dot{X}_e(t) = \sqrt{(\dot{X}-\dot{\hat{X}})^2 + (\dot{Y}-\dot{\hat{Y}})^2 + (\dot{Z}-\dot{\hat{Z}})^2} \qquad \text{(V-2)}$$

where X_e and $\dot{X}_e$ are the normed position and velocity errors respectively

$X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}$ are the actual states

$\hat{X}, \hat{Y}, \hat{Z}, \hat{\dot{X}}, \hat{\dot{Y}}, \hat{\dot{Z}}$ are the estimated states

In addition, each run was performed with a propagation time (Δt) of two seconds, and P, Q, and R matrices defined as follows:

$$P = \begin{bmatrix} 10^8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10^8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10^6 \end{bmatrix} \quad \text{(V-3)}$$

$$Q = \begin{bmatrix} 2500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2500 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2500 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2500 \end{bmatrix} \quad \text{(V-4)}$$

$$R = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix} \quad \text{(V-5)}$$

The results for the deterministic run showed that both the normed position and velocity errors remained very nearly zero throughout the first burn. No significant deviation was encountered.

Adding an initial error to the estimated states resulted in Figure V-1. These tests show that the filter converged regardless of the magnitude of the offsets. The offset had been filtered out after one, or at the most two, cycles.

The results of adding noise to the measurements and errors to the initial state estimate are shown in Figures V-2 and V-3. The various curves represent different values of the standard deviation of the noise ranging from .01% of the actual states in (a) to .3% in (d). Once again, the amount of offset was not a critical factor. From these curves it appears that a noise of up to .1% can be tolerated and still fulfill the design requirements.

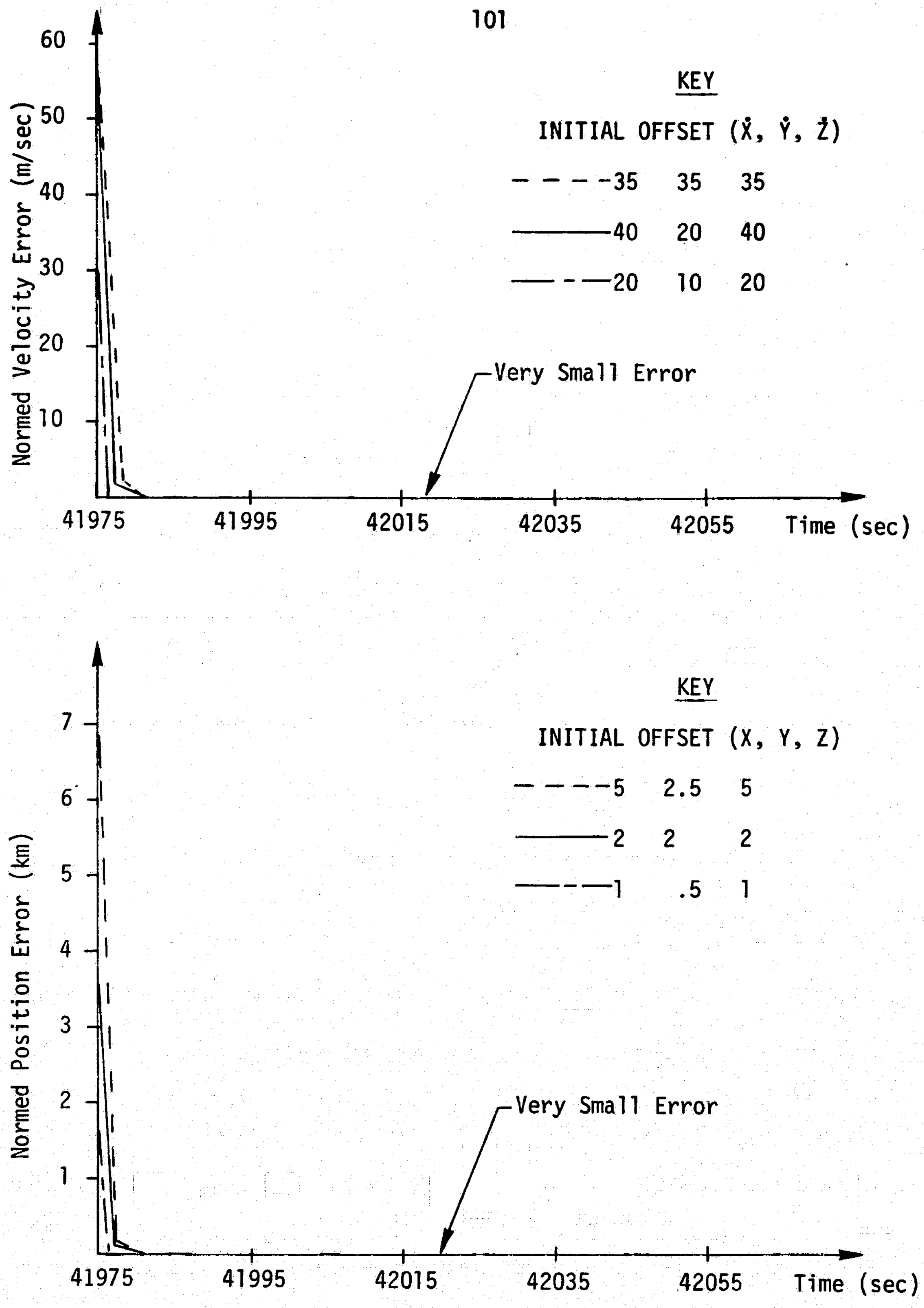


Figure V-1. NORMED POSITION AND VELOCITY ERRORS FOR RUN WITH INITIAL OFFSETS AND NO NOISE.

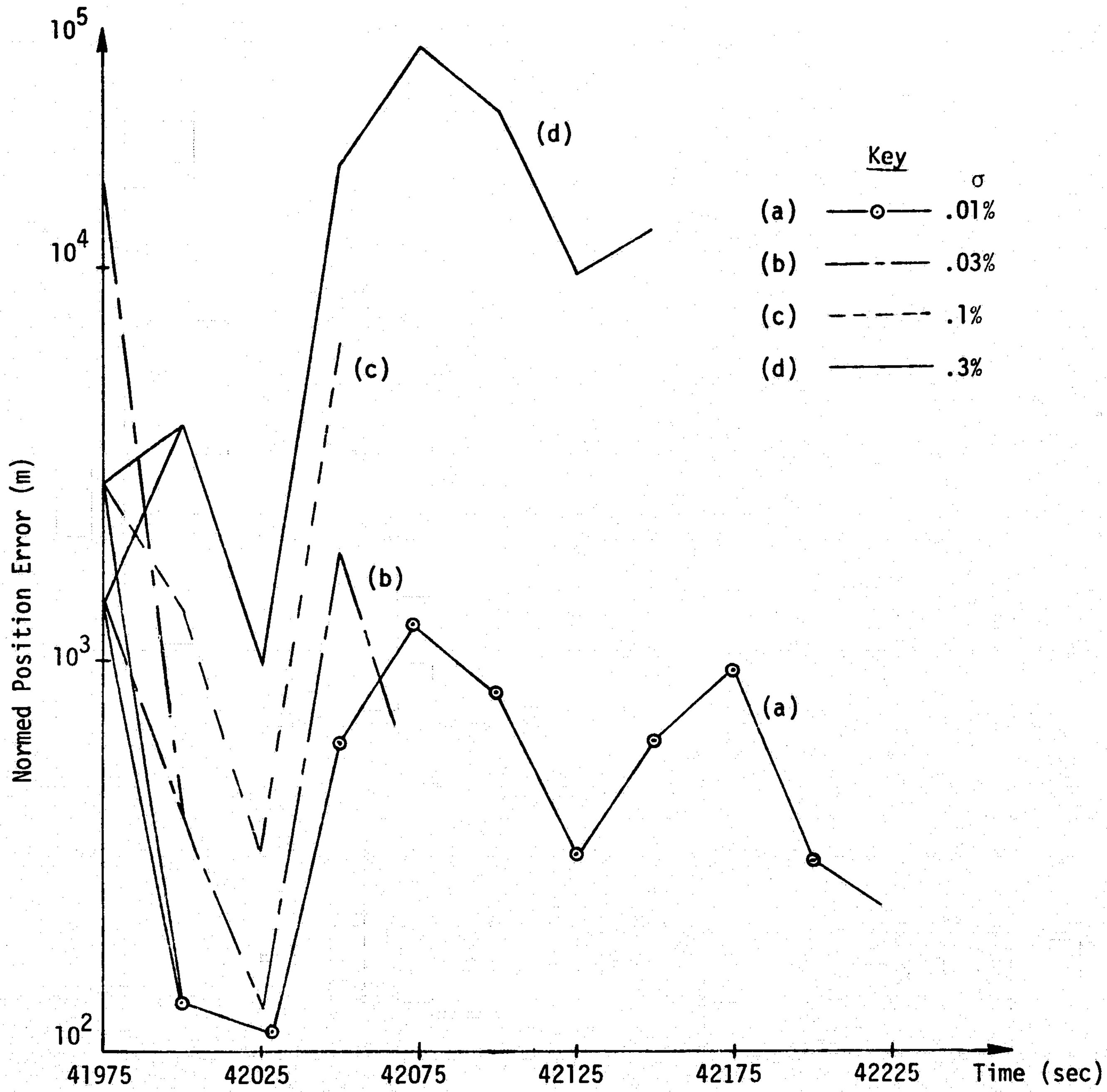


Figure V-2. NORMED POSITION ERROR FOR RUN WITH VARIOUS INITIAL OFFSETS AND NOISE.

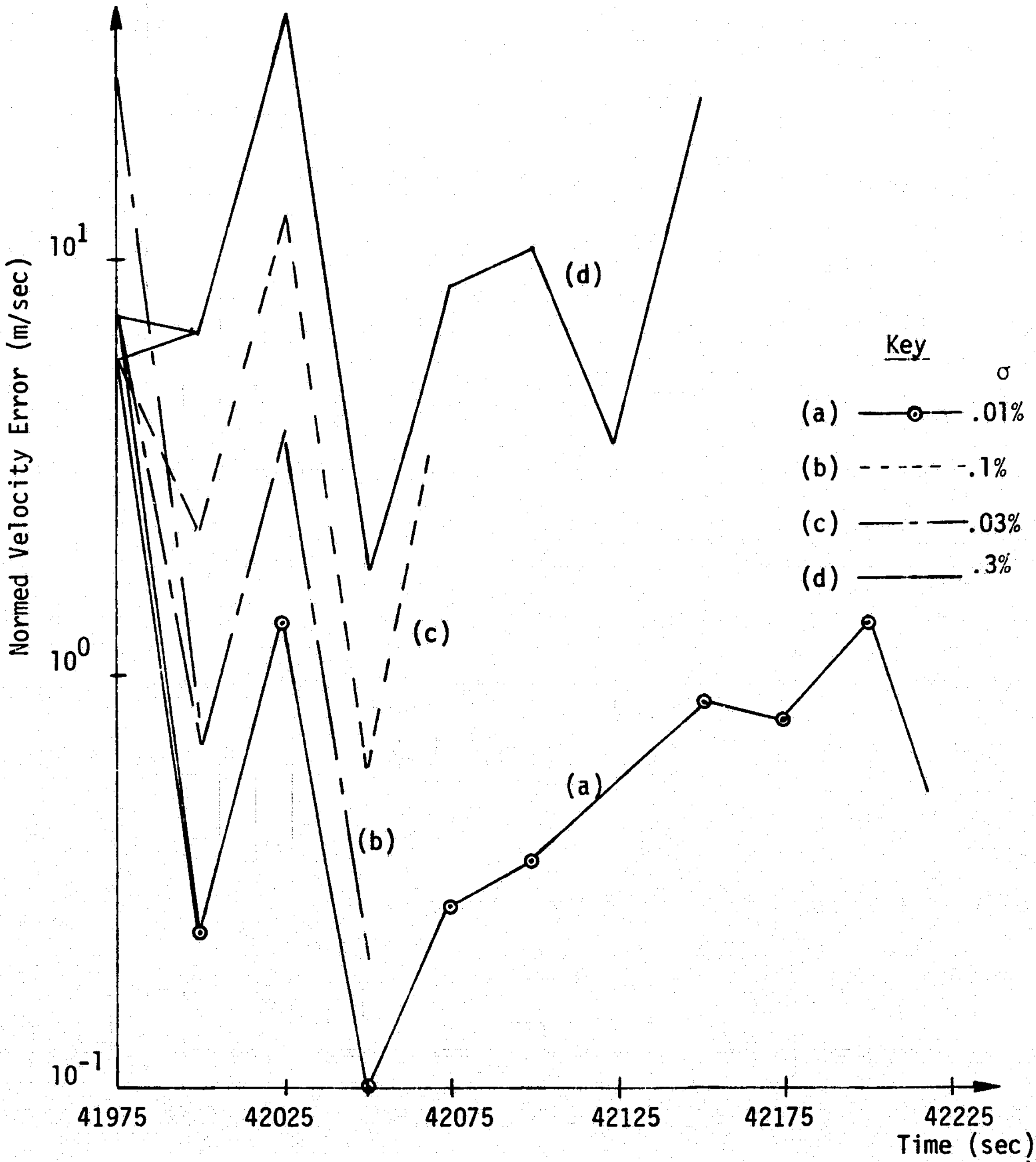


Figure V-3. NORMED VELOCITY ERROR FOR RUN WITH INITIAL OFFSETS AND NO NOISE.

C. Accuracy of $\Phi(k+1,k)$ Propagation

Figure V-4 shows position and velocity error norms during the first burn phase. These error norms are based on the difference between the actual and the estimated state vector and calculated by (V-1) and (V-2). The actual state vector, $\underline{X}$, is obtained by direct integration of the state vector at the previous time instant while the estimated state vector, $\hat{\underline{X}}$, is obtained by inputing the present actual state information into the filter to get an estimated update of the state vector. This tells the accuracy of the one step propagation.

Since the actual information is provided as perfect data, i.e. no initial offsets or noise, this comparison between the two sets of state information allows an indication of how well the filter is operating. This mode of operation is termed as the "passenger" mode since it carries the filter only as a passenger for each one step propagation.

As seen from the figure, both error norms decrease almost logarithmically during the first half of the ≈240 second burn period. During the final half of the burn phase both show a moderate increase. However, during the entire burn they both stay well within tolerable limits.

These results show that the dynamic model incorporated in the filter is an acceptable representation of the actual system's dynamics. However, since the filtered estimates do not actually drive the filter in this mode of operation, these errors norms give no indication about overall simulator package accuracy.

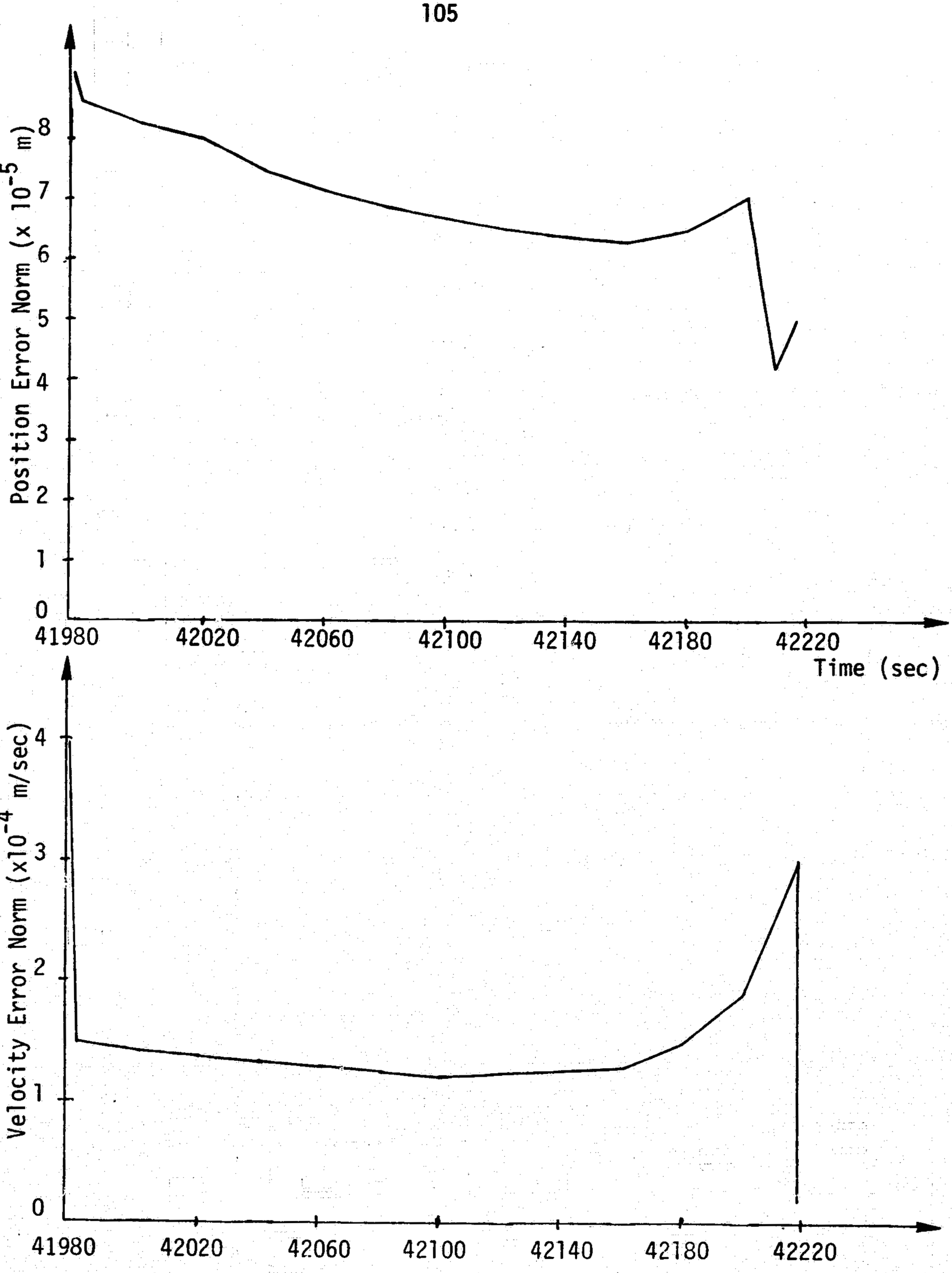


Figure V-4. POSITION AND VELOCITY ERROR NORMS OF ONE-STEP PROPAGATION.

D. Covariance Initialization

In theory, the diagonal elements of the covariance matrix P are directly related to the uncertainty in the filter's output. The question of covariance initialization arises from the non-linear nature of the system. Kalman filter theory states that, for a linear system, the elements of the covariance matrix will get smaller as the estimated values approach the actual ones. However, the theory does not always hold for non-linear systems. The problem here is that the elements of P may get unjustifiably small meaning that they no longer reflect the error in the state vector. This causes the filter to use the one-step propagation at the expense of the measurements to arrive at the estimated state. In other words, the filter believes it is closer to the actual state than is actually the case. The amount of correction is therefore small and filter divergence can take place. The proper initial values for P must be chosen so that they reflect the initial uncertainty and yet be large enough to prevent divergence of the form just described.

Although there is no straightforward analytical way to find these initial values, a starting point can be found by using the estimated uncertainty in the initial states. These uncertainties are approximately 1000 meters in each of the X, Y, and Z directions and 1 meter/second for the velocity components. Since these values are standard deviations and the elements of the covariance matrix are variances, each must be squared. In an attempt to prevent divergence, the elements were initially multiplied by a factor of ten to yield the initial simulation test data.

Several runs were made with various initial values for the covariance matrix. The tests served a dual purpose; to find proper initial covariance values, and to check the sensitivity of the system to changes in those values. Runs were made using values along the main diagonal ranging from 10^2 to 10^{14} with each position state and each velocity state uncertainty equal although the two groups were not necessarily equal to one another.

The results showed that the system is not particularly sensitive to the covariance initialization. Although divergence did occur for some of the small values, the system response was essentially the same for initial values several orders of magnitude apart. As a result of these tests, the initial covariance matrix was chosen to be

$$P = \begin{bmatrix} 10^8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10^8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10^8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10^6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10^6 \end{bmatrix} \tag{V-6}$$

E. Parametric Data Using Constructed States

The tests run in this section were much akin to the ones run in the section dealing with the initial filter checkout, i.e., the filter was tested to see its response to various initial offsets and measurement noises. The difference was that here, states that were used as the measurement data during the burn were constructed from actual accelerometer data as opposed to the previous use of actual state information. The measurements were obtained by propagating the vehicle's acceleration to determine velocity and position data. The defining equations are:

$$\dot{X} = (\ddot{X} + \gamma)\,\Delta t + \dot{X}_o \qquad \text{(V-7)}$$

$$X = (\ddot{X} + \gamma)\left(\frac{\Delta t}{2}\right)^2 + \dot{X}\,\Delta t + X_o \qquad \text{(V-8)}$$

where $\dot{X}$ = vehicle's new velocity

X = vehicle's new position

$\ddot{X}$ = vehicle's acceleration at time of measurement

Δt = propagation time

$\dot{X}_o$ = vehicle's initial velocity

X_o = vehicle's initial position

γ = noise

A more detailed explanation of these equations is in order. There are two acceleration effects present here; an acceleration provided by the engines and one due to gravity. The on-board accelerometers only measure the first of these. The second must be accounted for by using the gravity model previously described, with the next state estimate as its input. The $\ddot{X}$ term includes both effects. Once again, the noise term was added to take care of both system noise and measurement errors.

The filter was checked through the first two burn and coast phases, the first one of which is plotted here. To account for errors in the coast measurement, a noise was added to the coast measurements. Due to the angular nature of the coast measurements, the standard deviation of the noise was expressed in radians. When used in this manner, the simulation program is now in the form to be implemented on an acutal flight.

As previously stated, the main objective of these tests was to determine the effect of noise on the filter. However, an important sidelight was to verify the predicted effect of initial state errors on the indirect measurement scheme used during the burn. Recall the discussion at the end of Chapter IV of these indirect measurements. It was stated there that since each new measurement depends on the previous one (through the initial condition factor), all measurements will include any initial error. Therefore, in general, the elimination of the error does not appear possible. It is enough of a problem to minimize it's growth.

All tests were carried out with a propagation time of two seconds during a burn and sixty seconds during a coast. Also the P, Q, and R matrices were the same as before. The plots shown are the errors in R, V, X, and $\dot{X}$ for the first burn and coast phases. Standard deviations were chosen on the basis of device specifications as given in [3, 4, 22, 23, 24].

The plots for the first burn, shown in Figures V-5 and V-6 reveal two interesting features. Contrary to the results of actual states from Figure V-1, the initial offsets were the biggest factor in determining the response, while the noise, at least for reasonable values, had little effect. In fact for several cases, the plots for the same offsets and different noise were so close together as to be indistinguishable. This situation clearly demonstrates the effect of initial measurement error. While the growth of the error is not at all rapid and indeed was reduced slightly in some cases, the presence of that initial error can be felt throughout this phase.

The estimation scheme during the coast fared better. Here the reduction of the error can be seen in Figures V-7 and V-8. Even relatively large noises did not deter the estimator from converging. This situation is attributable to the direct measurements that are available during the coast phase.

In summary, these tests showed that the amount of the initial state error is a crucial factor during the burn but not the coast. These results were predicted using the defining equations (V-7) and (V-8). The slow growth of the error during a burn and its subsequent reduction during a coast leads to the conclusion that, given good initial estimates, the filter will perform as required on an actual mission.

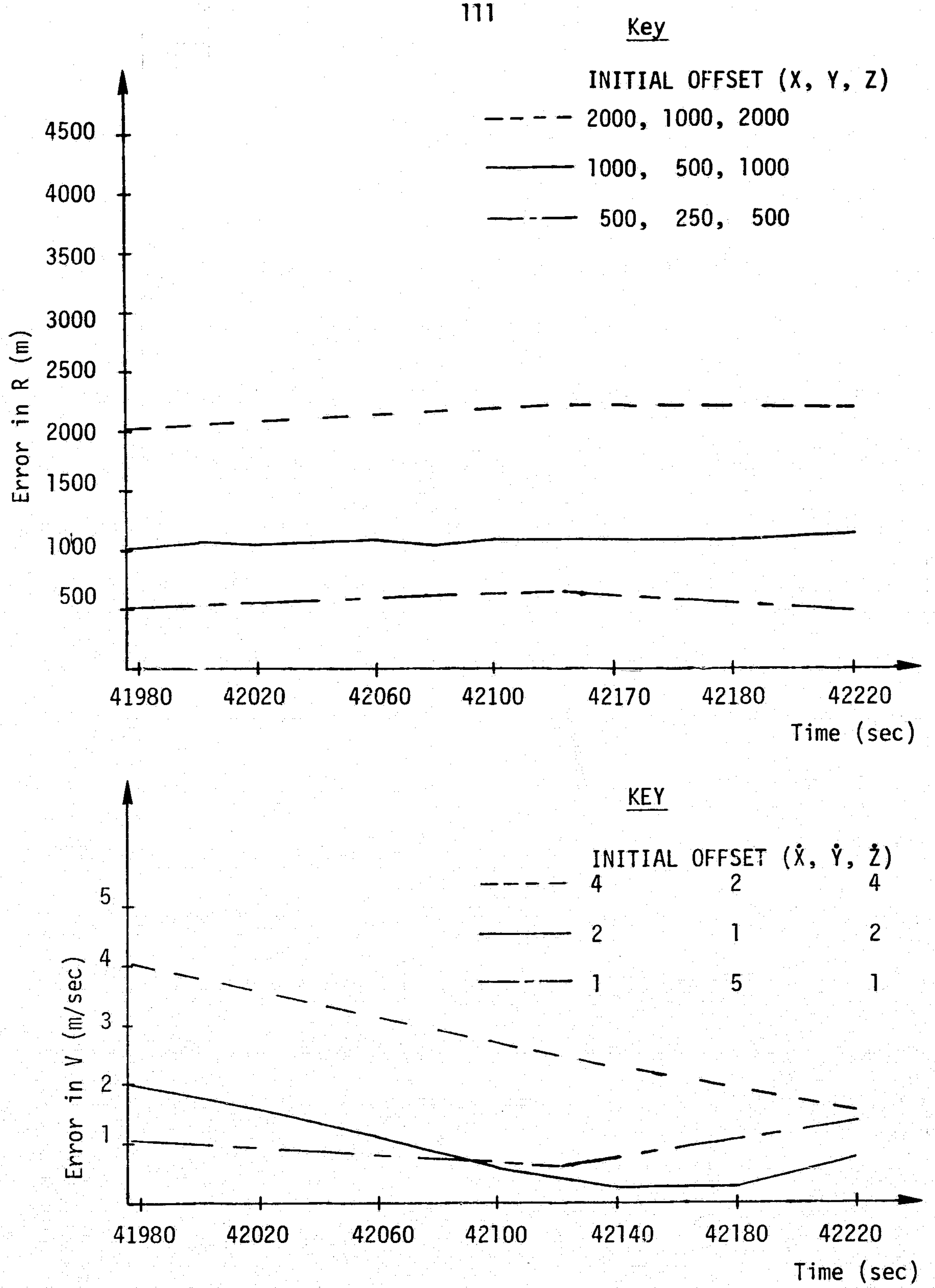


Figure V-5. PLOT OF R AND V ERROR DURING FIRST BURN USING CONSTRUCTED STATES.

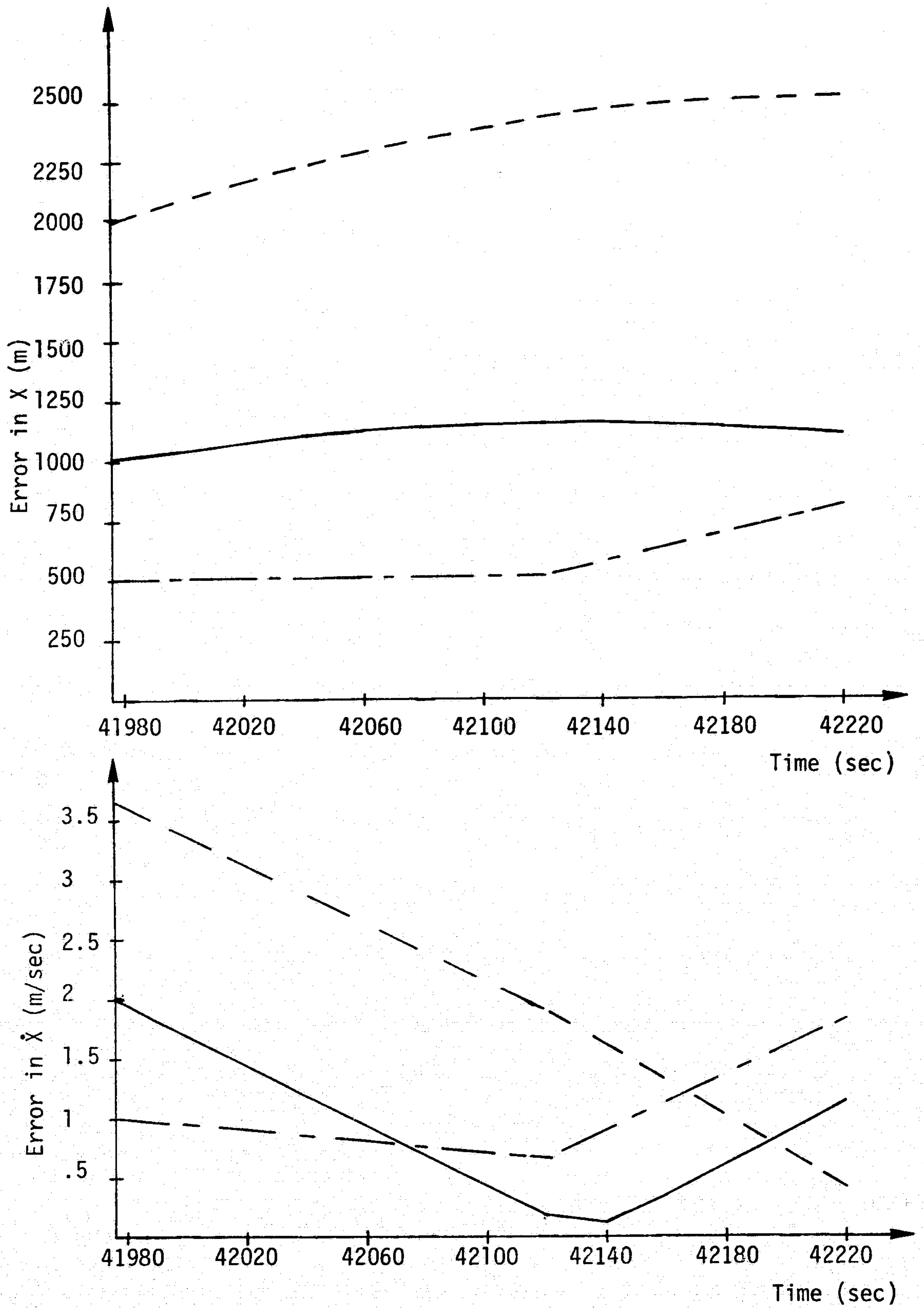


Figure V-6. PLOT OF X AND Ẋ ERROR DURING FIRST BURN USING CONSTRUCTED STATES.

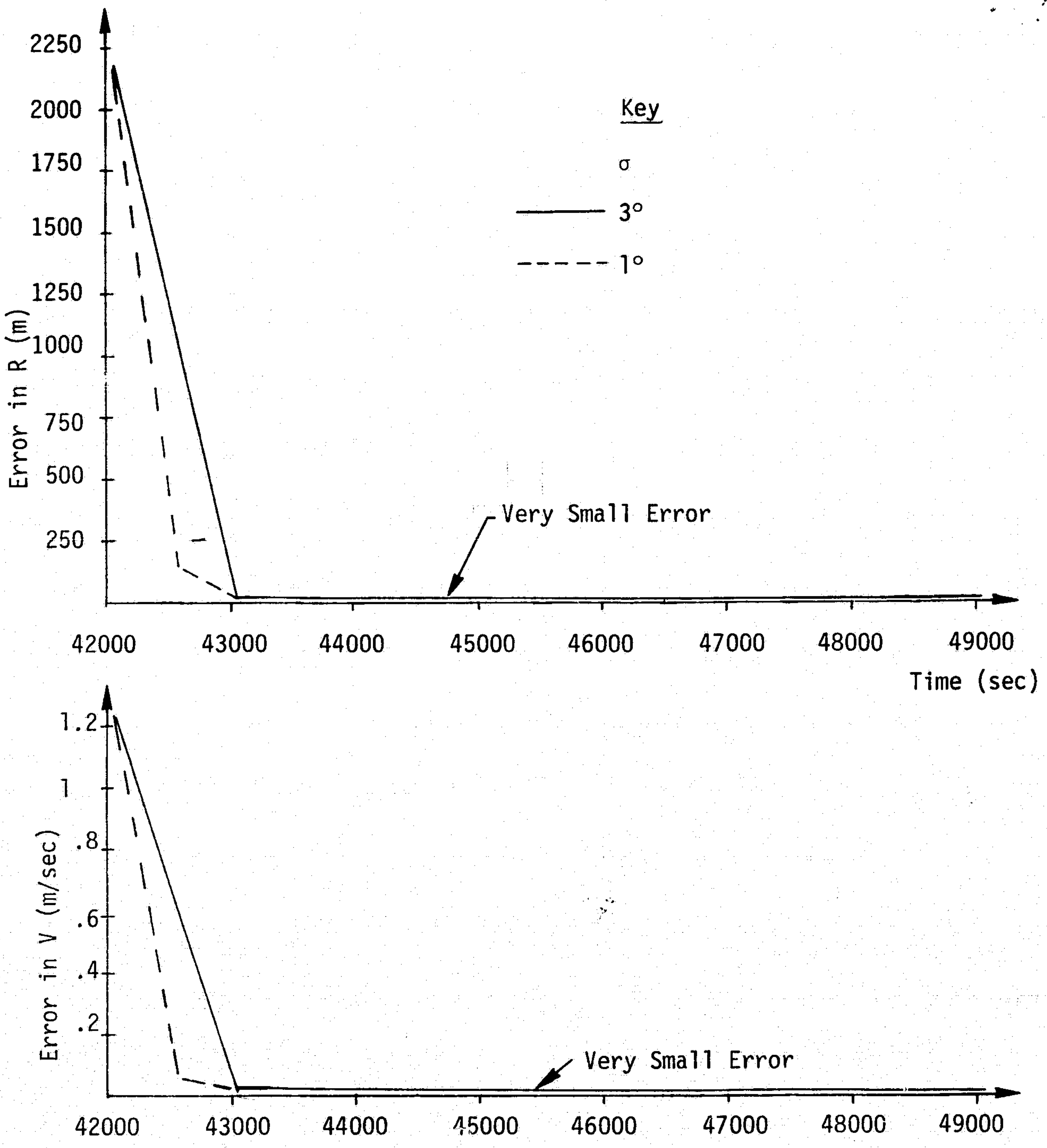


Figure V-7. PLOT OF ERROR IN R AND V DURING FIRST COAST.

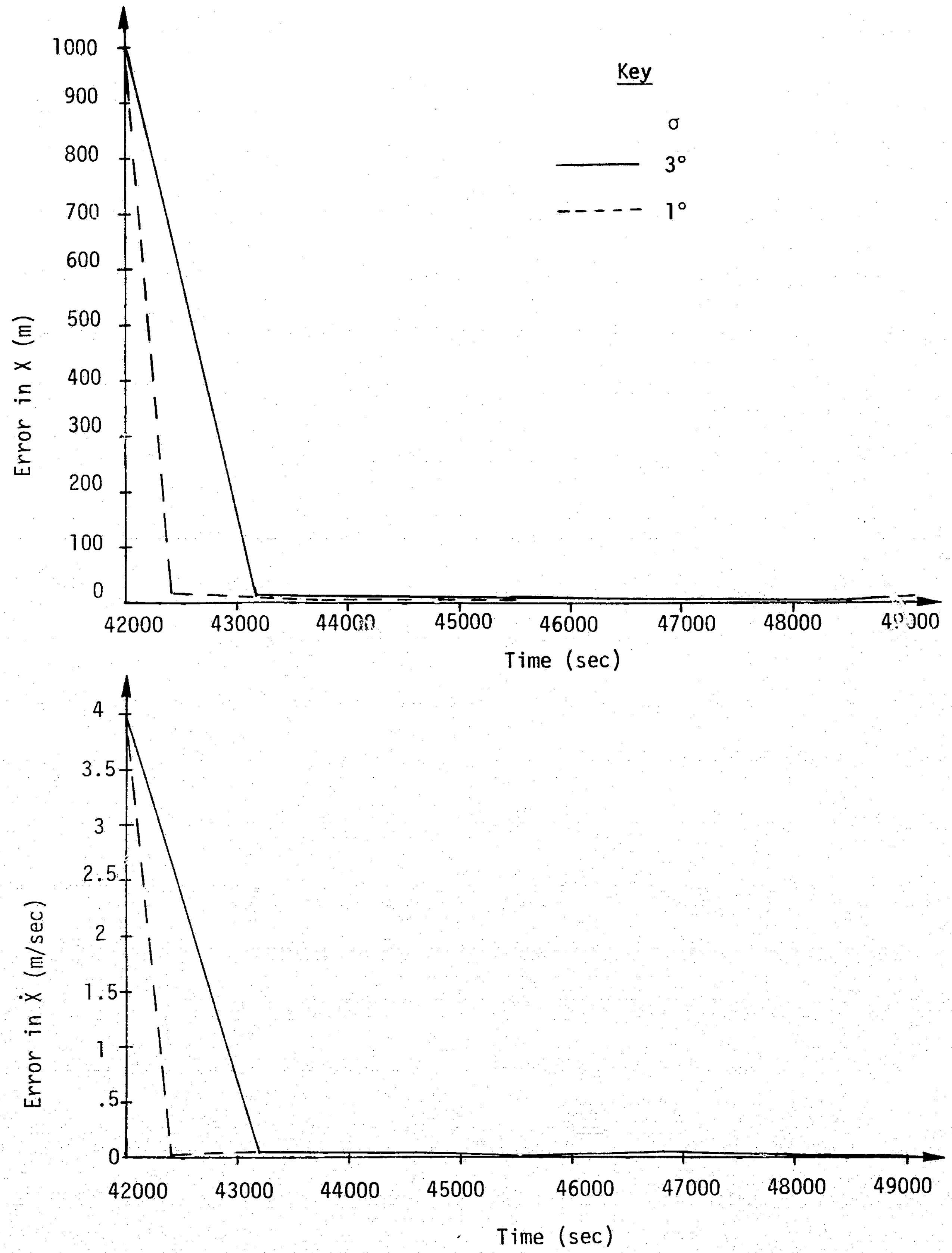


Figure V-8. PLOT OF ERROR IN X AND $\dot{X}$ DURING FIRST COAST

F. Study of Unfiltered Noisy Accelerometer Data vs. Filtered Noisy Data

It was suspected that, due to the indirect measurements used during a burn, the filter might prove ineffectual. Accordingly, a series of tests were made to compare the filtered and unfiltered responses. This was done by bypassing the filter and using the unfiltered state estimate as the system input. The state estimate was obtained by integrating the accelerometer measurements. All of the pertinent data used to carry out these tests was the same as that used previously.

The results shown in Figure V-9 and V-10 bore out the prediction. The curves for various noises were plotted, but they were found to be very similar to one another. Therefore only tests using different offsets are shown.

The small differences between the two curves demonstrates dramatically the poor nature of the (relative) burn measurements. In essence these tests show that little "cleaning up" of the state estimate is possible during a burn due to the lack of true state measurements available. This shows that the six vector filter of positions and velocities with accelerometer data during the burn is of limited value. One possible remedy for this would be to incorporate a nine vector of positions, velocities, and accelerations.

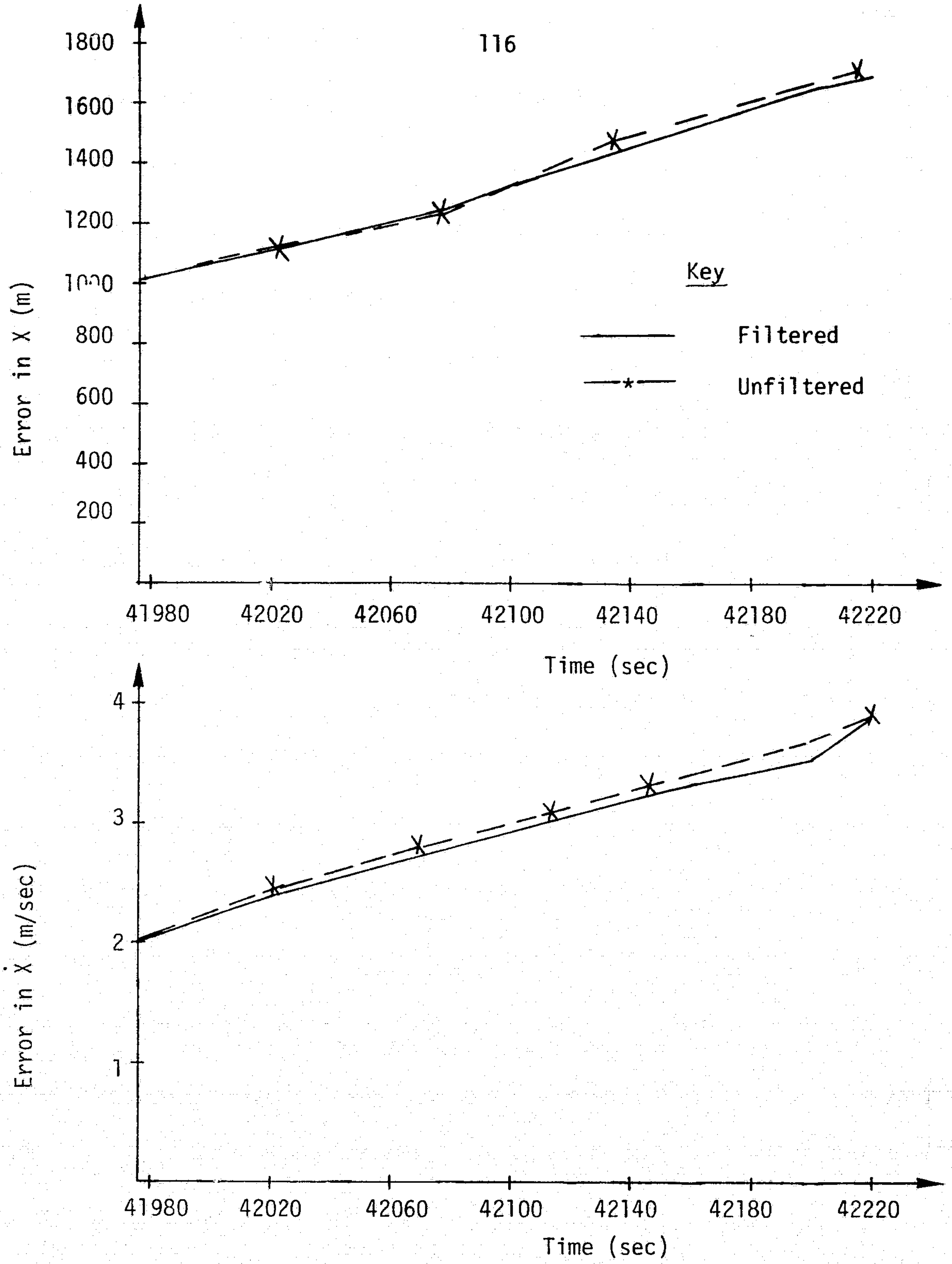


Figure V-9. PLOT OF ERROR IN X AND $\dot{X}$ FOR FILTERED AND UNFILTERED RESPONSES

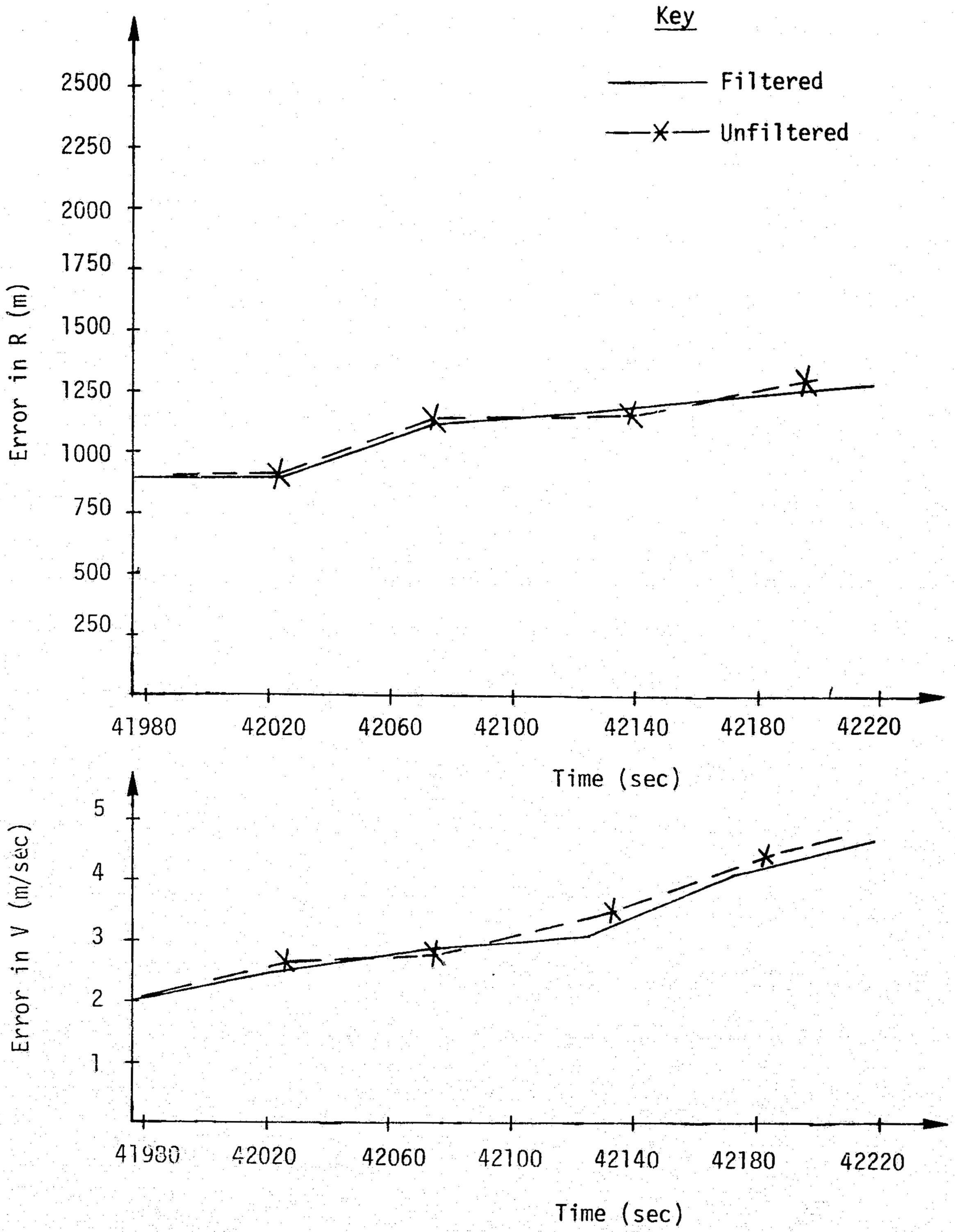


Figure V-10. PLOT OF ERROR IN R AND V FOR FILTERED AND UNFILTERED RESPONSES

G. Propagation Time Study

Filter accuracy depends on many parameters associated with the filter. One of the basic parameters is the propagation time for the filter, Δt. This is the time interval in which the filter updates or propagates the state vector. In this study various values are used in the filter to determine a range of propagation times in which the filter can still function properly, as well as to determine a best or optimal propagation time so that errors in the state vector will meet the desired tracking specifications. From previous knowledge and literature on Kalman Filtering it was known that the propagation time for the filter should probably be between .5 and 2 seconds [25]. It was believed that the smaller the propagation time the better the filtered states should track the actual states and that for larger propagation times, the filter would be less accurate. These ideas seem reasonable when viewed from the standpoint of taking measurements. If measurements are taken at smaller time intervals, the filtered states should be closer to the actual states.

The simulation runs made for this study were for three different propagation times, .5, 2 and 5 seconds, with .5 and 2 representing lower and upper bounds for the propagation time. The 5 second run was used to determine the rate at which the filter would diverge for an unreasonably large propagation time. In all cases the initial offset in the state vector was selected as $\Delta \underline{X}^T$ = [1000 500 1000 2 1 2]. This offset was used since it was specified as a normal uncertainty in the state vector prior to the first burn. The measurement system used for all simulations was that of accelerometer data being used to construct measurement states. The measurement noise was set at .01% of the vehicle acceleration.

The results of these studies are presented graphically in Figure V-11 and V-12. Figure V-11 shows curves of the position error norm and velocity error norm vs time for the first burn. These results are as expected. In all cases the rate in which the normed error increases is reduced for a smaller propagation time. The most significant difference in the three simulations appears in the normed velocity error curves. The normed position curves show a smaller amount of difference. This is due to the method of constructing the states in the measurement system. The filter is not able to reduce these errors below the value of the initial offset due to the lack of true measurements. The filter is forced to use the initial state vector as accurate. Therefore, if the filter tracks closely to the initial value then it is assumed that if a better initial value of the state vector were known the errors could be reduced. The error in the X and $\dot{X}$ states is shown in Figure V-12 as an illustration of a typical single axis error response. Again, results were as expected.

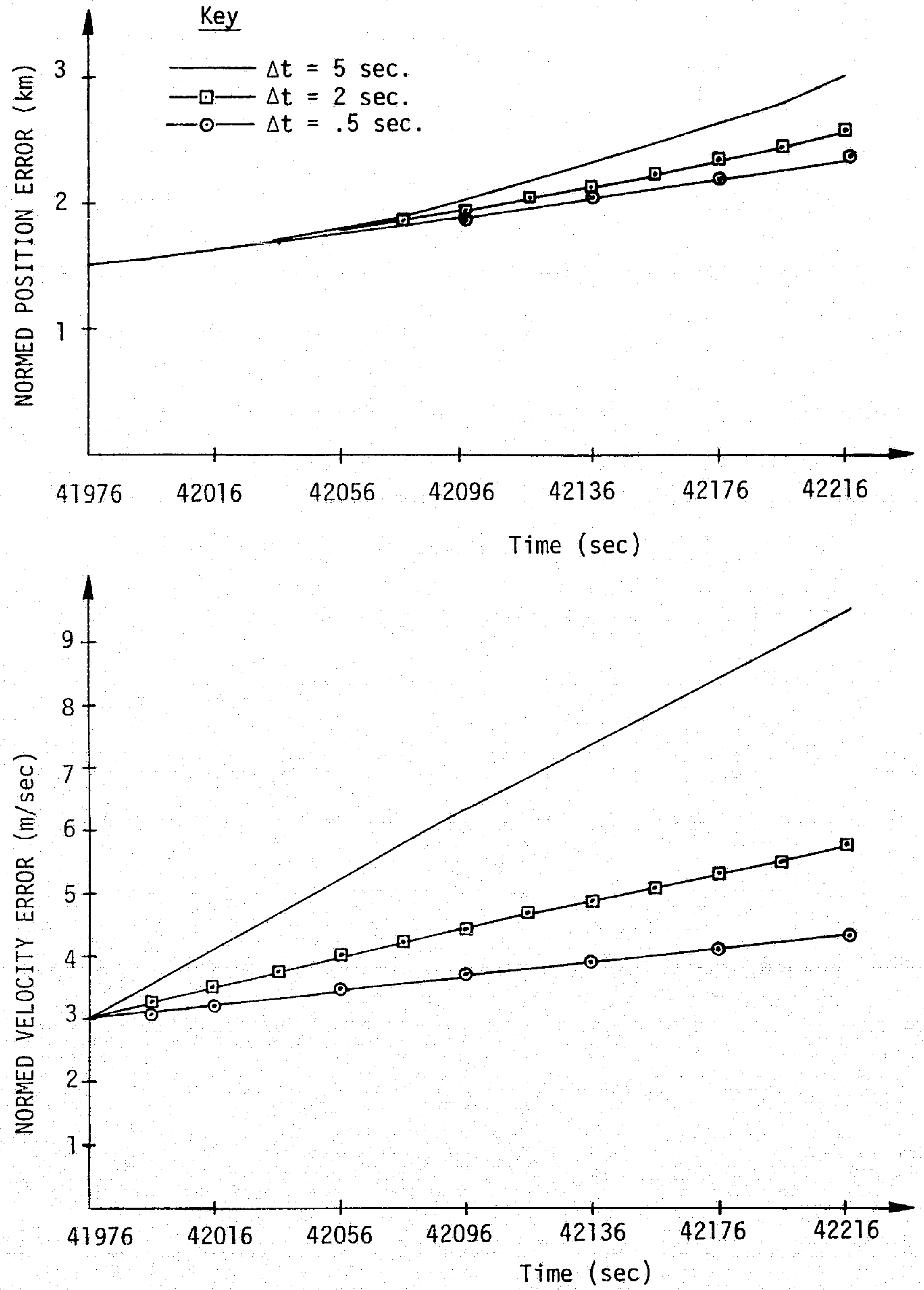


Figure V-11. NORMED POSITION AND VELOCITY ERROR vs. TIME FOR DIFFERENT PROPAGATION TIMES

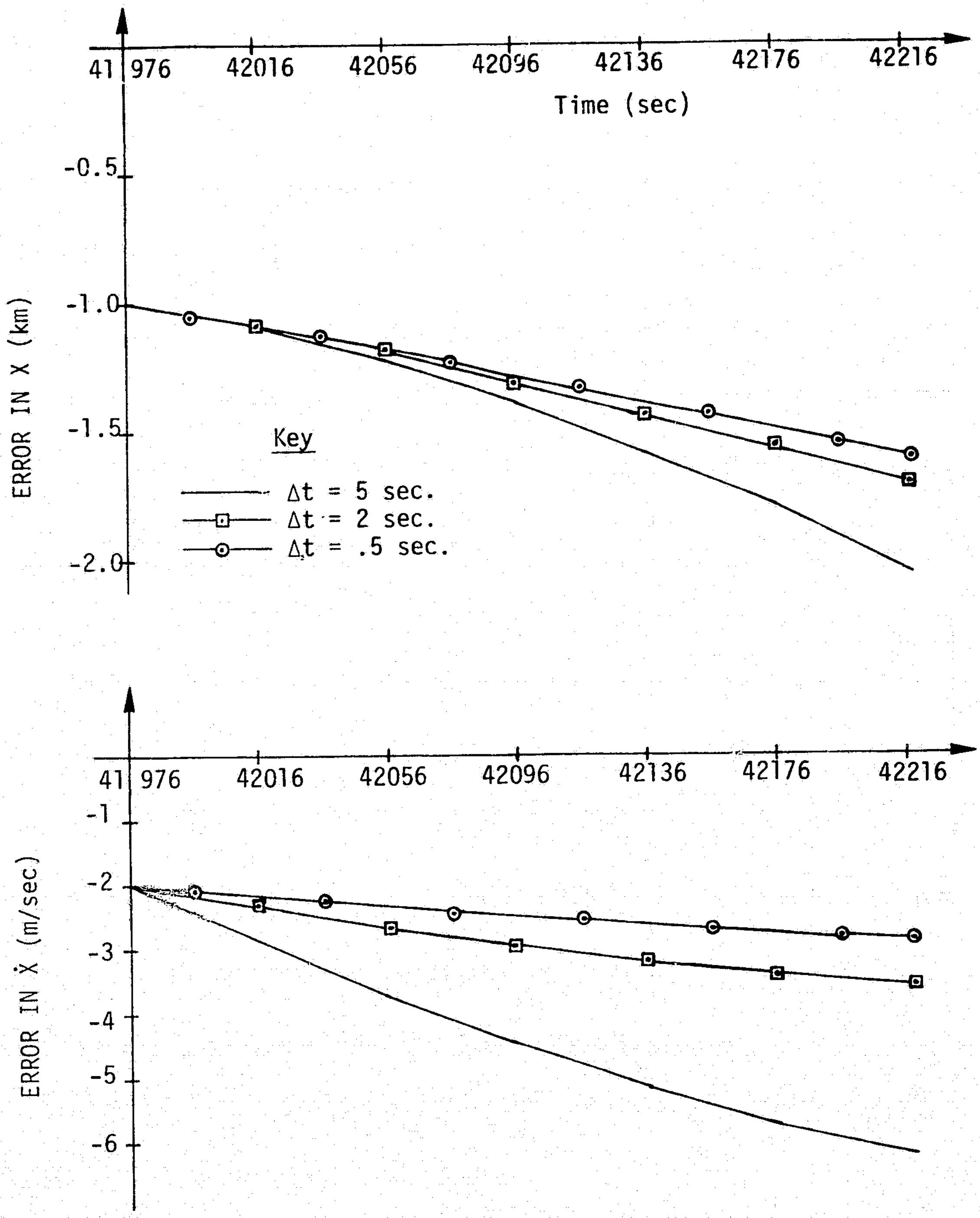


Figure V-12. ERROR IN X AND Ẋ vs. TIME FOR DIFFERENT PROPAGATION TIMES

H. Variation of Initial State Vector Offset

As stated previously, the initial state vector prior to a burn will always contain an error. In GSP this initial error was modeled as an additive offset to the actual state vector. That is

$$\underline{X}_i(0) = \underline{X}_a(0) + \Delta\underline{X}_{offset} \tag{V-9}$$

where i and a signify initial and actual, respectively. Therefore from (V-1) a positive number in the $\Delta\underline{X}_{offset}$ vector means the initial state is larger than the actual state. However, when the error between the actual and estimated state is computed a negative quantity would be found. The error in the state vector is computed by

$$\underline{X}_a - \hat{\underline{X}} = \underline{X}_e \tag{V-10}$$

The purpose of this study was to observe the filter error response to variations in the initial state offset vector. It was decided to keep the initial normed position and velocity errors constant and only change the sign of certain values in the initial state offset vector. The magnitude in the initial offset vector were set at reasonable maximums that were specified. In the previous studies the initial state vector was greater than the actual state vector in each individual state. By constructing the initial state vector in this manner the normed position and velocity errors were never reduced below the initial value. However, in this study, where the initial state vector is greater in some states and less in others than the actual state vector, it is not clear whether the normed position and velocity errors will follow a similar trend. Since the measurement system employed in the filter does not use exact state information it is expected that the error trend should be similar.

In the simulations made for this study the measurement system was that of the constructed measurement states and using $\hat{\underline{X}}(k+1/k)$ in the gravity computation. The measurement noise was again set to .01% of the vehicle's acceleration. The propagation time was set to .5 seconds. The results of the simulations are shown in Figures V-13 and V-14.

From these results it can be seen that for some offset variations, these particular errors follow the same trends as the results from previous studies. However, for other offsets the normed position and velocity errors are actually reduced below their initial value. This can only be attributed to the particular offset value used and no general statement can be made regarding the reduction or increase in the error. Again, these point to the fact that the filter response, either good or bad, depends highly on the accuracy of the initial offset vector.

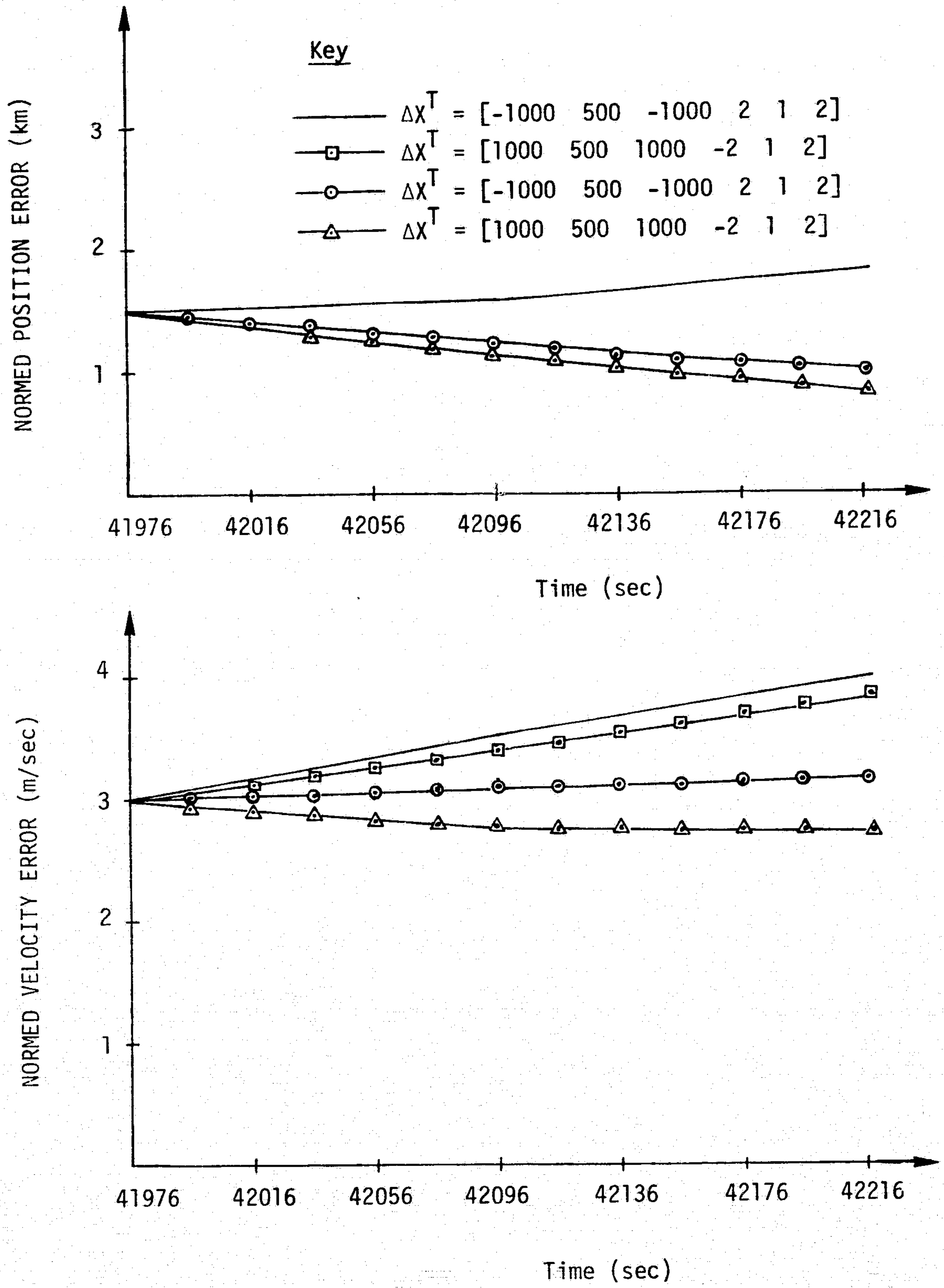


Figure V-13. NORMED POSITION AND VELOCITY ERROR vs. TIME FOR DIFFERENT INITIAL OFFSETS

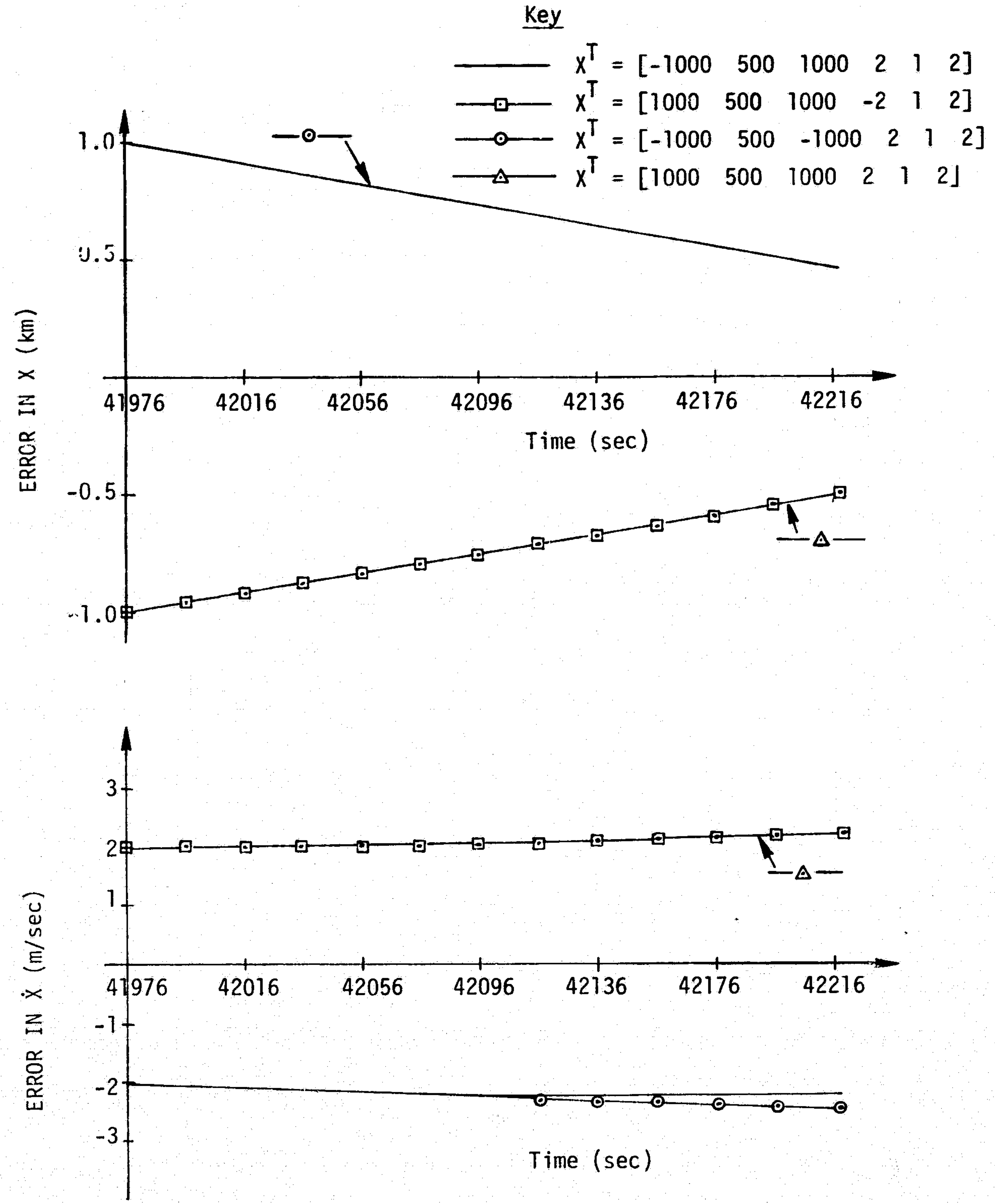


Figure V-14. ERROR IN X AND $\dot{X}$ vs. TIME
FOR DIFFERENT INITIAL OFFSETS

I. Gravity Computation Studies

In the filtering process, a gravity model is used to obtain the acceleration of the Space Tug due to gravity. The acceleration due to gravity is then added to the sensed acceleration received from the accelerometers to give the total vehicle acceleration during the burn phase of flight. This is then used in the measurement system to obtain constructed measurement states. The purpose of the study made in this section centers around the computation of the gravity model. The gravity model needs as one of its inputs a value of the state vector. In previous simulations the value of the state vector used was the value $\hat{\underline{X}}(k/k)$. It was realized, however, that at the time the gravity model was called for in the measurement system, the one step propagated state vector $\hat{\underline{X}}(k+1/k)$ was available. The one step propagated state vector is a predicted value for the state vector at time t_{k+1} given measurement information at time t_k. By using $\hat{\underline{X}}(k+1/k)$ instead of $\hat{\underline{X}}(k/k)$ in the gravity model computation, it was believed that the acceleration due to gravity could be improved. This implied that improvement in the filtered state vector at time t_{k+1}, $\hat{\underline{X}}(k+1/k+1)$, could be obtained.

The simulation runs made to verify the above assumptions used an initial offset in the state vector of $\Delta\underline{X}^T = [1000\ 500\ 1000\ 2\ 1\ 2]$ and a measurement noise of .01% of the vehicle acceleration. The results of the simulation runs are shown in Figures V-15 and V-16. The normed position and velocity curves both show a reduction in error as expected. The larger amount of difference again appears in the normed velocity error curves for the reasons discussed in the previous section. Figure V-16 shows the errors in the X and $\dot{X}$ states. There is a small improvement in the X state error for the two cases; however the improvement in the $\dot{X}$ state shows a reduction below the error of the initial $\dot{X}$ state. The increase in the normed error shows that the other states are increasing at a rate greater than the rate by which the X state decreases.

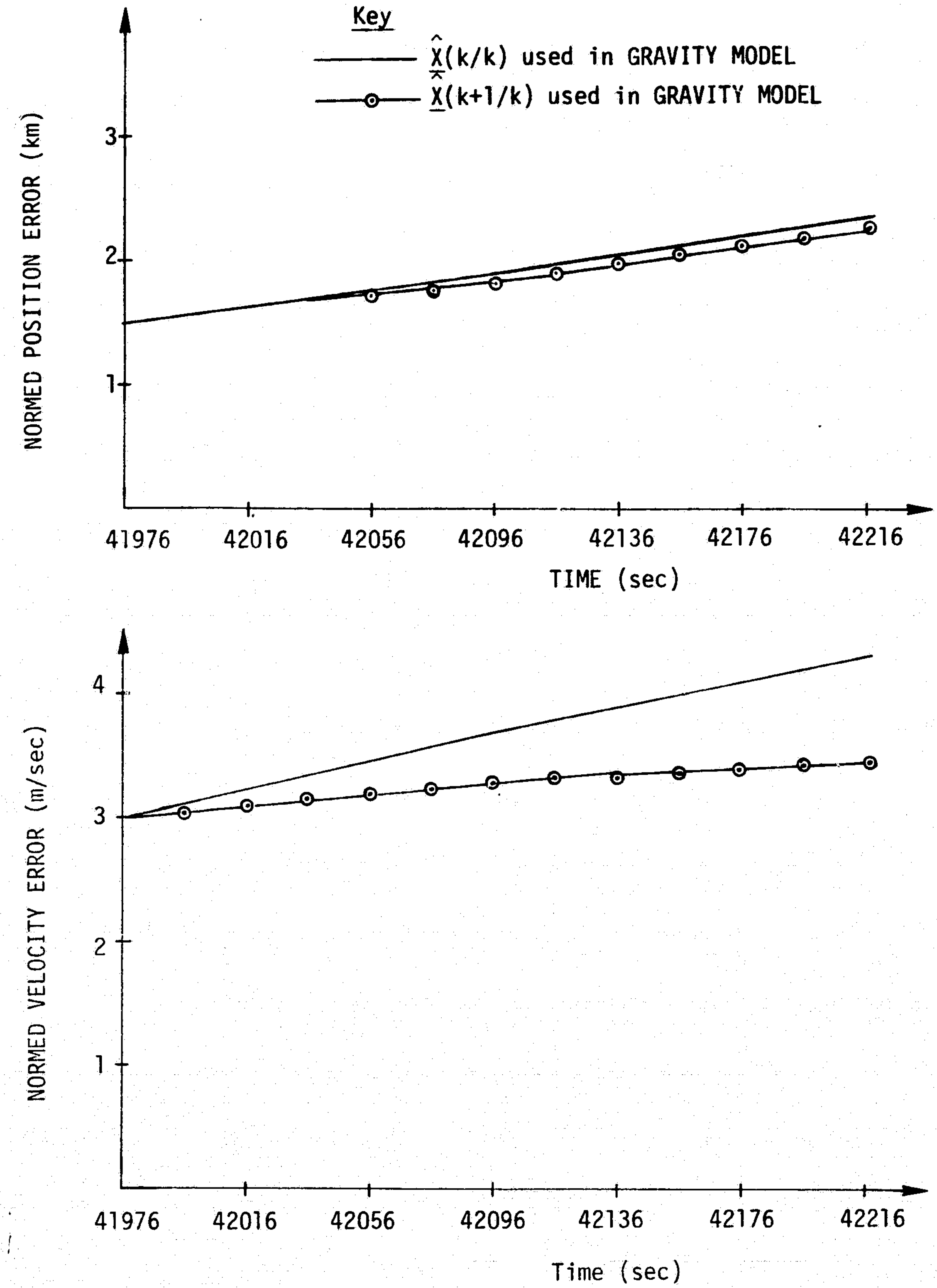


Figure V-15. NORMED POSITION AND VELOCITY ERROR VS. TIME FOR DIFFERENT COMPUTATION TIMES

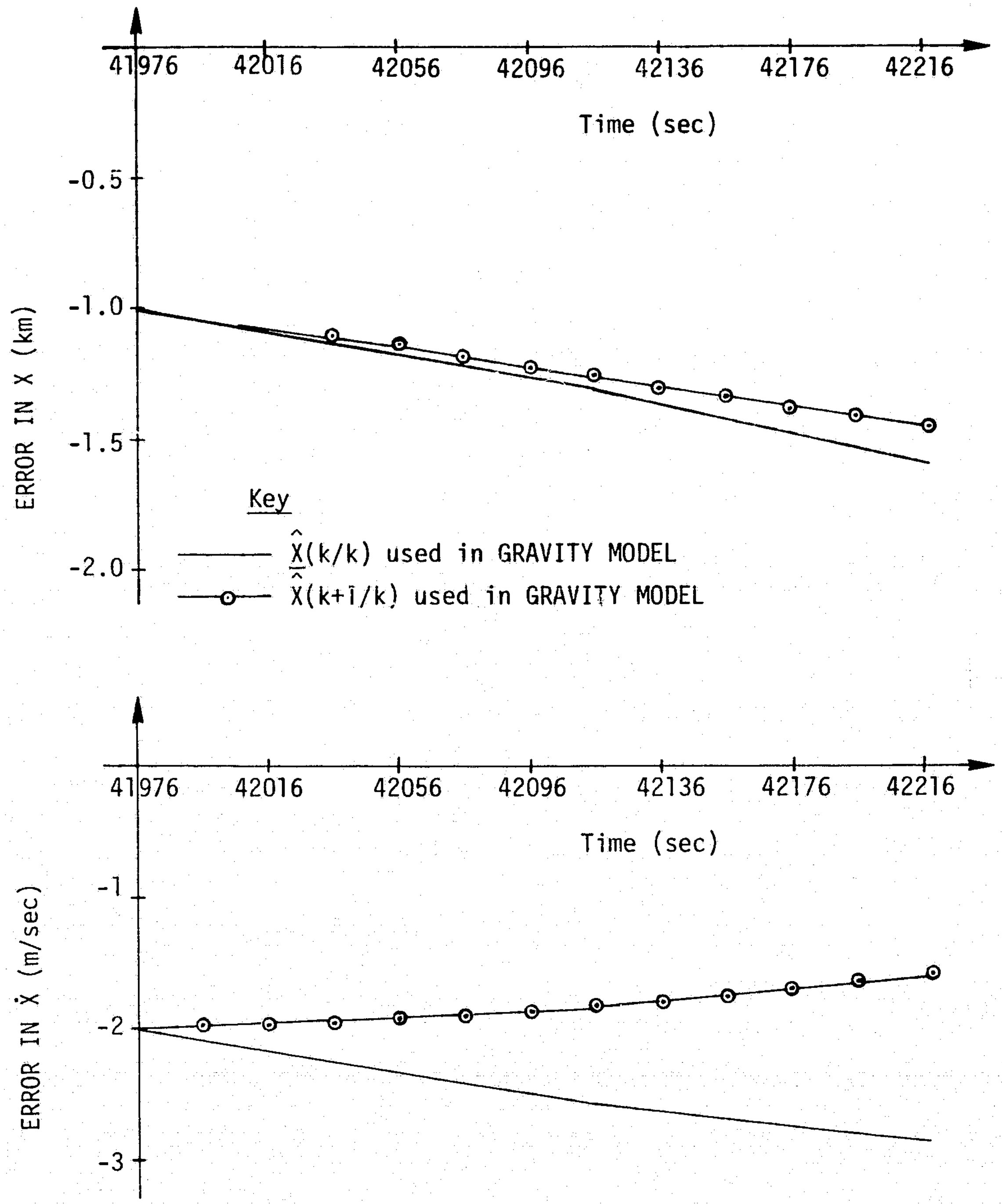


Figure V-16. ERROR IN X AND $\dot{X}$ VS. TIME FOR DIFFERENT COMPUTATION TIMES

VI. FINDINGS, CONCLUSIONS, AND RECOMMENDATIONS

In this chapter are outlined the overall results of the Space Tug Navigation study. In addition are included recommendations based on an extension of the work reported herein using as a basis the computer simulation package developed under the contract.

Based on the study of the simulation results presented in Chapter V and previous theoretical analysis the following findings are presented:

1. Overall tracking accuracy is independent of the choice of the initial covariance matrix, P(0/0), for a reasonable range of values (See Chapter V-Section D).

2. Position and velocity error norms are highly dependent on initial state indeterminacy. In general, the rate of growth of the error is a function of the individual state component inderminacies. (X_e, Y_e, Z_e, $\dot{X}_e$, $\dot{Y}_e$, $\dot{Z}_e$) (Refer to Chapter V-Section H)

3. It appears as if the position and velocity error norms are related to the state estimation filter propagation time, Δt, by

   $$|| \underline{X}_e(t) || \cong k\,\Delta t + || \underline{X}_e(0) ||$$

   where, k is an unknown constant and $\underline{X}_e(0)$ is the initial state indeterminacy.

4. The overall tracking accuracy is directly related to the accuracy of the state vector estimate used in the gravity model during the burn phase acceleration computation (See Chapter V-Section I).

5. It was found in the computation of the partial derivatives of DCX, DCY, and DCZ that these partial derivatives were insensitive to a large number of intermediate partial derivative computations. From simulation studies of various mission profiles it was observed that the

   $\frac{\partial K_1}{\partial \zeta_i}$, $\frac{\partial K_2}{\partial \zeta_i}$, $\frac{\partial K_3}{\partial \zeta_i}$, and $\frac{\partial K_4}{\partial \zeta_i}$, where ζ_i is the i^{th} element of the state

   vector, could be set to zero with minimal degradation of overall tracking accuracy (See IGM Guidance Law in Chapter III). Because of the interrelationship of the aforementioned partial derivatives with others, 84 partial derivative computations were therefore eliminated, thus greatly reducing the computational burden.

6. As shown in the results of Chapter V-Section F, the unfiltered noisy accelerometer results are only slightly worse than the filtered measurements during the burn. This was as expected since only relative state vector measurements are available at this time. It is felt that a ninth order filter formulation, including vehicle accelerations as states themselves, would dramatically improve the filtered results at the expense of a larger computational burden.

7. It appears that the tracking limits specified can be met by the filtering scheme developed. However, the tracking accuracy is close to the upper limits specified and further work is needed to improve them.

From the foregoing findings and the overall results from Chapter V, the following general conclusions can be stated:

(1) The overall tracking accuracy can probably meet specifications. During the burn, however, the initial state indeterminacy principally determines the tracking error. The propagation time also affects accuracy.

(2) The general simulator package GSP is a flexible, efficient simulator that is capable of handling a variety of nonlinear state estimation jobs. Due to its design flexibility, it is also useful with aircraft and rocket navigation problems in addition to that of the Space Tug.

Many ideas suggest themselves as possible areas of future work. Although these ideas are quite varied and interrelated, they can be lumped under the following six major recommendations for future work:

1) Investigation of trade-offs between navigational accuracy and computational requirements. The low-cost nature of the mission dictates the need for a low-cost, simple on-board computer. By reducing the volume of calculations, the computation time and hence the computational burden can correspondingly be reduced. To implement computational simplifications, the sensitivity of the overall tracking accuracy to each on-board calculation must be determined. Since the bulk of the computations are partial derivatives used in the one-step propagation, a good starting

point would be to try to further reduce these by elimination as described in Chapter III. In addition, a simplification may be possible in the gravity and drag models.

2) Study the use of different measurement systems. As explained in Chapter IV, the indirect measurements used during the burn make reduction of the initial error difficult. Thus, it would be desirous to obtain direct state information from the measurements. One possible suggestion is to use the Interferometer Landmark Tracking (ILT) System during both the burn and the coast phases.

3) Find a way to reduce the initial state error. This ties in closely with (2) in that one possible remedy might be found by using different measurement configurations. Another area to be investigated is the frequency, type, and accuracy requirements of ground based updates.

4) Augment the present six-state vector to include the accelerations. By yielding better data on the accelerations, this nine-state vector might improve the position and velocity estimates. Such an approach would involve modeling the dynamics of the third derivative of position (jerk), however.

5) Determine the relationship between the propagation time and tracking accuracy, the purpose being to determine the longest propagation time possible and still meet mission tracking specifications.

6) Investigate the use of a non-linear filtering scheme to replace the present linearized one. Although it would increase the analytical complexity, it is expected that this new scheme would reduce the computational burden and increase tracking accuracy.

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APPENDIX A

Φ MATRIX PARTIAL DERIVATIVE COMPUTATION

In the development of the state estimator, it was necessary to obtain discrete linearized equations based on the system dynamics in order to propagate the state vector.

The basic system dynamics considered for the 3-degree of freedom flight are

$$\begin{bmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{bmatrix} = \begin{bmatrix} \frac{T}{M} DCX + \ddot{X}_g \\ \frac{T}{M} DCY + \ddot{Y}_g \\ \frac{T}{M} DCZ + \ddot{Z}_g \end{bmatrix} \tag{A-1}$$

where T = thrust (fixed)

M = vehicle mass (time-varying)

$\ddot{X}_g$, $\ddot{Y}_g$, $\ddot{Z}_g$ = gravity terms

DCX, DCY, DCZ = direction cosines of engine gimbals

The state vector is defined as

$$\underline{X}^T = [X \; Y \; Z \; \dot{X} \; \dot{Y} \; \dot{Z}] \tag{A-2}$$

In terms of the state equations, (A-1) can be written as

$$\dot{\underline{X}} = \underline{f}(\underline{X},t) \qquad \text{(NON-LINEAR SYSTEM)} \tag{A-3}$$

Expanding this in a Taylor Series gives

$$\underline{X}(k + 1) = \underline{X}(k) + \dot{\underline{X}}\, \Delta t + \ddot{\underline{X}}\, \frac{\Delta t^2}{2} + \text{H.O.T.} \tag{A-4}$$

where Δt is the discrete propagation time and H.O.T. represents the higher order terms in the series expansion. Since $\dot{\underline{X}} = \underline{f}(\underline{X}, t)$ and $\ddot{\underline{X}} = \frac{\partial f}{\partial \underline{X}} \dot{\underline{X}}$, (A-4) can be written as

$$X(k+1) \approx \underline{X}(k) + \underline{f}(\underline{X},t)\Delta t + \frac{\partial f}{\partial \underline{X}}\, \underline{f}\, \frac{\Delta t^2}{2} + \underline{n}(k) \tag{A-5}$$

where $\underline{n}(k)$ is lumped noise due to series truncation.

After defining the terms

$$\text{JACOBIAN MATRIX} = A(\underline{X},t) = \left.\frac{\partial f}{\partial \underline{X}}\right|_{\underline{X}(k)} \tag{A-6}$$

and

$$\Phi(k + 1,\ k) = [A(\underline{X},t)\, \frac{\Delta t}{2} + I] \tag{A-7}$$

equation (A-5) can be written as

$$\underline{X}(k+1) = \underline{X}(k) + \Phi(k+1,\ k)\underline{f}(\underline{X},t)\Delta t + \underline{n}(k) \tag{A-8}$$

As evidenced from (A-6) and (A-7) $\Phi(k+1, k)$ is found from taking successive partial derivatives of the system dynamics with respect to the elements of the state vector. It was assumed that DCX, DCY, DCZ, and the gravity terms are the only terms in (A-1) that are functions of the states, therefore the partial derivatives of these first three terms are needed in terms of known quantities. This enables propagation of DCX, DCY, and DCZ from one discrete time instance to the next. However, generation of these partial derivatives proved to be quite laborious due to the number of terms involved in the I.G.M. law.

The complete derivation of these terms with respect to one element, X, follows. This derivation is shown in successive steps in order to show the precise procedure and all quantities are as defined in I.G.M.

$$\begin{bmatrix} DCX \\ DCY \\ DCZ \end{bmatrix} = [K]^{-1} \begin{bmatrix} \sin \chi_p' \cos \chi_y' \\ \sin \chi_y' \\ \cos \chi_p' \cos \chi_y' \end{bmatrix}$$

$$K = [\Phi_T]BA \ , \quad BA = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

$$[\Phi_T] = \begin{bmatrix} \cos \phi_T & 0 & \sin \phi_T \\ 0 & 1 & 0 \\ -\sin \phi_T & 0 & \cos \phi_T \end{bmatrix}$$

$$A = \begin{bmatrix} \cos \phi_L & \sin \phi_L \sin A_z & -\sin \phi_L \cos A_z \\ -\sin \phi_L & \cos \phi_L \sin A_z & -\cos \phi_L \cos A_z \\ 0 & \cos A_z & \sin A_z \end{bmatrix}$$

$$B = \begin{bmatrix} \cos \theta_N & 0 & \sin \theta_N \\ \sin \theta_n \sin i & \cos i & -\cos \theta_N \sin i \\ -\sin \theta_n \cos i & \sin i & \cos \theta_N \cos i \end{bmatrix}$$

For now:

$$[K]^{-1} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$DCX = C_{11} \sin \chi_p' \cos \chi_y' + C_{12} \sin \chi_y' + C_{13} \cos \chi_p' \cos \chi_y'$$

$$DCY = C_{21} \sin \chi_p' \cos \chi_y' + C_{22} \sin \chi_y' + C_{23} \cos \chi_p' \cos \chi_y'$$

$$DCZ = C_{31} \sin \chi_p' \cos \chi_y' + C_{32} \sin \chi_y' + C_{33} \cos \chi_p' \cos \chi_y'$$

$C_{11} \ldots C_{33}$ are functions of the states; however, through simulation they were shown to have little effect in the following partial derivatives. Hence, their derivative w.r.t. the states are considered zero.

STEP 1.

$$\frac{\partial DCX}{\partial X} = C_{11}\left(\cos \chi_y' \operatorname{cox} \chi_p' \frac{\partial \chi_p'}{\partial X} - \sin \chi_p' \sin \chi_y' \frac{\partial \chi_y'}{\partial X}\right)$$

$$+ C_{12}\left(\cos \chi_y' \frac{\partial \chi_y'}{\partial X}\right) - C_{13}\left(\cos \chi_p' \sin \chi_y' \frac{\partial \chi_y'}{X} +\right.$$

$$\left.\cos \chi_y' \sin \chi_p' \frac{\partial \chi_p'}{\partial X}\right)$$

STEP 2.

$$\frac{\partial \chi_p'}{\partial X} = \frac{-\partial K_1}{\partial X} + \frac{\partial K_2}{\partial X} \quad \delta t + \frac{\partial \tilde{\chi}_p}{\partial X}$$

$$\frac{\partial \chi_y'}{\partial X} = \frac{-\partial K_3}{\partial X} + \frac{\partial K_4}{\partial X} \quad \delta t + \frac{\partial \tilde{\chi}_y}{\partial X}$$

Now solve for $\frac{\partial \chi_p'}{\partial X}$ and $\frac{\partial \chi_y'}{\partial X}$ in known quantities.

STEP 3.

$$\frac{\partial K_1}{\partial X} = \frac{(A_pD_p-B_pC_p)\left(\frac{\partial}{\partial X}(B_pE_p)\right) - (B_pE_p)\left(\frac{\partial}{\partial X}(A_pB_p)-\frac{\partial}{\partial X}(B_pC_p)\right)}{(A_pD_p - B_pC_p)^2}$$

$$\frac{\partial K_2}{\partial X} = \frac{\partial}{\partial X}\left(K_1\frac{A_p}{B_p}\right) = \left(\frac{A_p}{B_p}\right)\frac{\partial K_1}{\partial X} + K_1\frac{\partial}{\partial X}\left(\frac{A_p}{B_p}\right)$$

$$\frac{\partial K_3}{\partial X} = \frac{(A_yD_y-B_yC_y)\frac{\partial}{\partial X}(B_yE_y) - (B_yE_y)\left(\frac{\partial}{\partial X}(A_yD_y)-\frac{\partial}{\partial X}(B_yC_y)\right)}{(A_yD_y - B_yC_y)^2}$$

$$\frac{\partial K_4}{\partial X} = \left(\frac{A_p}{B_p}\right)\frac{\partial K_3}{\partial X} + (K_3)\frac{\partial}{\partial X}\left(\frac{A_p}{B_p}\right)$$

$$\frac{\partial\tilde{\chi}_p}{\partial X} = \frac{\partial}{\partial X}\left\{\tan^{-1}\left(\frac{\Delta\dot{\xi}}{\Delta\dot{\zeta}}\right)\right\}$$

$$\frac{\partial\tilde{\chi}_y}{\partial X} = \frac{\partial}{\partial X}\left\{\tan^{-1}\left(\Delta\dot{\eta}/(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)^{\frac{1}{2}}\right)\right\}$$

STEP 4.

$$\frac{\partial}{\partial X}[B_pE_p] = \frac{\partial}{\partial X}\left((MB_y-NP)(\xi - \xi_T + \dot{\xi}T_g + \tfrac{1}{2}\ddot{\xi}_gT_g^2 - [MS-NQ]\sin\tilde{\chi}_p)\right)$$

$$= [\xi - \xi_T + \dot{\xi}T_g + \tfrac{1}{2}\ddot{\xi}_gT_g^2 - MS\sin\tilde{\chi}_p + NQ\sin\tilde{\chi}_p]\frac{\partial}{\partial X}(MB_y-NP)$$

$$+ [MBy-NP]\left\{\frac{\partial}{\partial X}\xi - \frac{\partial}{\partial X}\xi_T + \frac{\partial}{\partial X}(\dot{\xi}T_g) + \tfrac{1}{2}\frac{\partial}{\partial X}(\ddot{\xi}_gT_g^2)\right.$$

$$\left.- \frac{\partial}{\partial X}(MS\sin\tilde{\chi}_p) + \frac{\partial}{\partial X}(NQ\sin\tilde{\chi}_p)\right\}$$

$$\frac{\partial}{\partial X}(A_pB_p) = A_p \frac{\partial}{\partial X}(B_p) + B_p \frac{\partial}{\partial X}(A_p)$$

$$= (MA_y - NB_y)\left(\frac{\partial}{\partial X}[MB_y] - \frac{\partial}{\partial X}[NP]\right) +$$

$$(MB_y - NP)\left(\frac{\partial}{\partial X}[MA_y] - \frac{\partial}{\partial X}[NB_y]\right)$$

$$\frac{\partial}{\partial X}(B_pC_p) = B_p \frac{\partial}{\partial X}[C_p] + C_p \frac{\partial}{\partial X}[B_p]$$

$$= (MB_y\text{-}NP)\left\{\cos \tilde{\chi}_p \frac{\partial}{\partial X}(MS\text{-}NQ) + (MS\text{-}NQ)\frac{\partial}{\partial X}(\cos \tilde{\chi}_p)\right\}$$

$$+ (MS\text{-}NQ)\cos \tilde{\chi}_p \left\{\frac{\partial}{\partial X}(MB_y) - \frac{\partial}{\partial X}(NP)\right\}$$

$$\frac{\partial}{\partial X}\left(\frac{A_p}{B_p}\right) = \frac{B_p\left(\frac{\partial}{\partial X}(MA_y) - \frac{\partial}{\partial X}(NB_y)\right) - A_p\left(\frac{\partial}{\partial X}(MB_y) - \frac{\partial}{\partial X}(NP)\right)}{(B_p)^2}$$

$$\frac{\partial}{\partial X}(B_yE_y) = B_y \frac{\partial}{\partial X}(E_y) + E_y \frac{\partial}{\partial X}(B_y)$$

$$= B_y\left\{\frac{\partial}{\partial X}(\eta) + \frac{\partial}{\partial X}(\dot{\eta}T_g) + (\tfrac{1}{2})\frac{\partial}{\partial X}(\ddot{\eta}_g T_g^{\,2})\right.$$

$$\left.- \frac{\partial}{\partial X}(S \sin \tilde{\chi}_y)\right\} + E_y\left\{\frac{\partial J}{\partial X} + \frac{\partial}{\partial X}(T_g\ V_{ex}\ L)\right\}$$

$$\frac{\partial}{\partial X}[A_yD_y] = A_y \frac{\partial}{\partial X}(D_y) + D_y \frac{\partial}{\partial X}(A_y)$$

$$= A_y\left\{Q\frac{\partial}{\partial X}(\cos \tilde{\chi}_y) + \text{cox}\ \tilde{\chi}_y \frac{\partial Q}{\partial X}\right\} +$$

$$D_y\left\{V_{ex}\frac{\partial L}{\partial X} + L\frac{\partial V_{ex}}{\partial X}\right\}$$

$$\frac{\partial}{\partial X}[B_y C_y] = B_y \frac{\partial}{\partial X}(C_y) + C_y \frac{\partial}{\partial X}(B_y)$$

$$= B_y \left\{ S \frac{\partial}{\partial X}(\cos \tilde{\chi}_y) + \cos \tilde{\chi}_y \frac{\partial}{\partial X}(S) \right\} +$$

$$C_y \left\{ \frac{\partial J}{\partial X} + \frac{\partial}{\partial X}(T_g V_{ex} L) \right\}$$

$$\frac{\partial}{\partial X}\left(\tan^{-1}\left(\frac{\Delta\dot{\xi}}{\Delta\dot{\zeta}}\right)\right) = \frac{(\Delta\dot{\zeta})\frac{\partial}{\partial X}(\Delta\dot{\xi}) - (\Delta\dot{\xi})\frac{\partial}{\partial x}(\Delta\dot{\zeta})}{(\Delta\dot{\xi})^2 + (\Delta\dot{\zeta})^2}$$

$$\frac{\partial}{\partial X}\left(\tan^{-1}(\Delta\dot{\eta}/(\Delta\dot{\xi}^2+\Delta\dot{\zeta}^2)^{\frac{1}{2}})\right) = \frac{(\Delta\dot{\xi}^2+\Delta\dot{\zeta}^2)^{\frac{1}{2}}\frac{\partial\Delta\dot{\eta}}{\partial X} - \Delta\dot{\eta}\frac{\partial}{\partial X}(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)^{\frac{1}{2}}}{(\Delta\dot{\xi})^2 + (\Delta\dot{\zeta})^2 + (\Delta\dot{\eta})^2}$$

STEP 5.

$$\frac{\partial}{\partial X}(MB_y - NP) = M\frac{\partial}{\partial X}(B_y) + B_y\frac{\partial}{\partial X}(M) - N\frac{\partial}{\partial X}(P) - P\frac{\partial}{\partial X}(N)$$

$$= M\left(\frac{\partial J}{\partial X} + \frac{\partial}{\partial X}(T_g V_{ex} L)\right) + B_y\left(\frac{\partial}{\partial X}(\cos \tilde{\chi}_y) + K_3\frac{\partial}{\partial X}(\sin \check{\chi}_y)\right.$$

$$\left. + (\sin \tilde{\chi}_y)\frac{\partial}{\partial X}(K_3)\right) - N(-\tfrac{1}{2}[V_{ex}\frac{\partial}{\partial X}(T_g{}^2) + T_g{}^2\frac{\partial}{\partial X}(V_{ex})] +$$

$$[\tau\frac{\partial}{\partial X}(J) + J\frac{\partial}{\partial X}(\tau)]) - P(K_4\frac{\partial}{\partial X}(\sin \tilde{\chi}_y) + \sin \tilde{\chi}_y \frac{\partial}{\partial X}(K_4))$$

$$\frac{\partial}{\partial X}(\xi) = \frac{\partial}{\partial X}[k_{11} X + k_{12} Y + k_{13} Z]$$

$$= [k_{11} + X\frac{\partial}{\partial X}(k_{11})] + Y\frac{\partial}{\partial X}(k_{12}) + Z\frac{\partial}{\partial X}(k_{13})$$

$$\frac{\partial}{\partial X}(\xi_T) = \frac{\frac{\partial}{\partial X}(P_T)[1+e_T \cos \theta_T] + P_T[e_T\frac{\partial}{\partial X}(\cos \theta_T) + \cos \theta_T \frac{\partial}{\partial X}(e_T)]}{(1 + e_T \cos \theta_T)^2}$$

$$\frac{\partial}{\partial X}(\dot{\xi}T_g) = \dot{\xi}\frac{\partial}{\partial X}(T_g) + T_g\frac{\partial}{\partial X}(\dot{\xi})$$

$$\frac{\partial}{\partial X}(\ddot{\xi}_g T_g^{\,2}) = \ddot{\xi}_g\frac{\partial}{\partial X}(T_g^{\,2}) + T_g^{\,2}\frac{\partial}{\partial X}(\ddot{\xi}_g)$$

$$\frac{\partial}{\partial X}(MS\sin\tilde{\chi}_p) = M\left\{S\frac{\partial(\sin\tilde{\chi}_p)}{\partial X} + \sin\tilde{\chi}_p\frac{\partial S}{\partial X}\right\} +$$

$$S\sin\tilde{\chi}_p\left\{\frac{\partial}{\partial X}(\cos\tilde{\chi}_y) + K_3\frac{\partial}{\partial X}(\sin\tilde{\chi}_y) + (\sin\tilde{\chi}_y)\frac{\partial}{\partial X}(K_3)\right\}$$

$$\frac{\partial}{\partial X}(NQ\sin\tilde{\chi}_p) = N\left\{Q\frac{\partial}{\partial X}(\sin\tilde{\chi}_p) + \sin\tilde{\chi}_p\frac{\partial}{\partial X}(Q)\right\}$$

$$+ (Q\sin\tilde{\chi}_p)\left\{K_4\frac{\partial}{\partial X}(\sin\tilde{\chi}_y) + \sin\tilde{\chi}_y\frac{\partial}{\partial X}(K_4)\right\}$$

$$\frac{\partial}{\partial X}(MB_y) = M\left\{\frac{\partial J}{\partial X} + \frac{\partial}{\partial X}(T_g V_{ex} L)\right\} + B_y\left\{\frac{\partial}{\partial X}(\cos\tilde{\chi}_y) + K_3\frac{\partial}{\partial X}(\sin\tilde{\chi}_y)\right.$$

$$\left. + (\sin\tilde{\chi}_y)\frac{\partial}{\partial X}(K_3)\right\}$$

$$\frac{\partial}{\partial X}(NP) = N\left\{-\tfrac{1}{2}[V_{ex}\frac{\partial}{\partial X}(T_g^{\,2}) + T_g^{\,2}\frac{\partial}{\partial X}(V_{ex})] + [\tau\frac{\partial}{\partial X}(J) + J\frac{\partial}{\partial X}(\tau)]\right\}$$

$$+ P\left\{K_4\frac{\partial}{\partial X}(\sin\tilde{\chi}_y) + \sin\tilde{\chi}_y\frac{\partial}{\partial X}(K_4)\right\}$$

$$\frac{\partial}{\partial X}(MA_y) = M\left\{V_{ex}\frac{\partial}{\partial X}(L) + L\frac{\partial}{\partial X}(V_{ex})\right\} +$$

$$A_y\left\{\frac{\partial}{\partial X}(\cos\tilde{\chi}_y) + K_3\frac{\partial}{\partial X}(\sin\tilde{\chi}_y) + (\sin\tilde{\chi}_y\frac{\partial}{\partial X}(K_3))\right\}$$

$$\frac{\partial}{\partial X}(NB_y) = N\left\{\frac{\partial J}{\partial X} + \frac{\partial}{\partial X}(T_g V_{ex} L)\right\} + B_y\left\{K_4\frac{\partial}{\partial X}(\sin\tilde{\chi}_y) + \sin\tilde{\chi}_y\frac{\partial}{\partial X}(K_4)\right\}$$

$$\frac{\partial}{\partial X}(MS-NQ) = \left(M\frac{\partial}{\partial X}(S) + S\frac{\partial}{\partial X}(M)\right) - \left(N\frac{\partial}{\partial X}(Q) + Q\frac{\partial}{\partial X}(N)\right)$$

$$\frac{\partial M}{\partial X} = \frac{\partial}{\partial X}(\cos\tilde{\chi}_y) + K_3\frac{\partial}{\partial X}(\sin\tilde{\chi}_y) + (\sin\tilde{\chi}_y)\frac{\partial}{\partial X}(K_3)$$

$$\frac{\partial N}{\partial X} = K_4\frac{\partial}{\partial X}(\sin\tilde{\chi}_y) + \sin\tilde{\chi}_y\frac{\partial}{\partial X}(K_4)$$

$$\frac{\partial}{\partial X}(\cos\tilde{\chi}_p) = \frac{\partial}{\partial X}\cos\left\{\tan^{-1}\left(\frac{\Delta\dot{\xi}}{\Delta\dot{\zeta}}\right)\right\}$$

$$= -\sin\left\{\tan^{-1}\left(\frac{\Delta\dot{\xi}}{\Delta\dot{\zeta}}\right)\right\}\left\{\frac{(\Delta\dot{\zeta})\frac{\partial}{\partial X}(\Delta\dot{\xi}) - (\Delta\dot{\xi})\frac{\partial}{\partial X}(\Delta\dot{\zeta})}{(\Delta\dot{\xi})^2 + (\Delta\dot{\zeta})^2}\right\}$$

$$\frac{\partial}{\partial X}(\eta) = \frac{\partial}{\partial X}[k_{21}X + k_{22}Y + k_{23}Z]$$

$$= k_{21} + X\frac{\partial}{\partial X}(k_{21}) + Y\frac{\partial}{\partial X}(k_{22}) + Z\frac{\partial}{\partial X}(k_{23})$$

$$\frac{\partial}{\partial X}(\dot{\eta}\,T_g) = \dot{\eta}\frac{\partial}{\partial X}(T_g) + T_g\frac{\partial}{\partial X}(\dot{\eta})$$

$$\frac{\partial}{\partial X}(\dot{\eta}) = \frac{\partial}{\partial X}[k_{21}\dot{X} + k_{22}\dot{Y} + k_{23}\dot{Z}]$$

$$= \dot{X}\frac{\partial}{\partial X}(k_{21})^{\nearrow 0} + \dot{Y}\frac{\partial}{\partial X}(k_{22})^{\nearrow 0} + \dot{Z}\frac{\partial}{\partial X}(k_{23})^{\nearrow 0}$$

$$\frac{\partial}{\partial X}(\ddot{\eta}\,T_g^{\,2}) = \ddot{\eta}_g\frac{\partial}{\partial X}(T_g^{\,2}) + T_g^{\,2}\frac{\partial}{\partial X}(\ddot{\eta}_g)$$

$$\frac{\partial}{\partial X}(\ddot{\eta}_g) = \tfrac{1}{2}\left\{\frac{\partial}{\partial X}(\ddot{\eta}_{GT})^{\nearrow 0} + k_{21}\frac{\partial}{\partial X}(\ddot{X}_g) + \ddot{X}_g\frac{\partial}{\partial X}(k_{21})^{\nearrow 0} + k_{22}\frac{\partial}{\partial X}(\ddot{Y}_g) + \ddot{Y}_g\frac{\partial}{\partial X}(k_{22})^{\nearrow 0} + k_{23}\frac{\partial}{\partial X}(\ddot{Z}_g) + \ddot{Z}_g\frac{\partial}{\partial X}(k_{23})^{\nearrow 0}\right\}$$

$$\frac{\partial}{\partial X}(S \sin \tilde{\chi}_y) = S\frac{\partial}{\partial X}\sin\tilde{\chi}_y + \sin\tilde{\chi}_y\frac{\partial}{\partial X}(S)$$

$$\frac{\partial J}{\partial X} = V_{E_X}\quad\frac{\tau}{\tau - T_g}\quad\frac{\partial T_g}{\partial X} - \frac{\partial T_g}{\partial X}$$

$$\frac{\partial}{\partial X}(T_g V_{ex} L) = T_g\quad V_{ex}\frac{\partial L}{\partial X} + \frac{\partial V_{ex}}{\partial X}\quad + V_{ex}L\frac{\partial}{\partial X}(T_g)$$

$$\frac{\partial}{\partial X}(\cos\tilde{\chi}_y) = \frac{\partial}{\partial X}\cos\left\{\tan^{-1}\left(\frac{\Delta\dot{\eta}}{(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)^{1/2}}\right)\right\}$$

$$= -\sin\left\{\tan^{-1}\left(\frac{\Delta\dot{\eta}}{(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)^{1/2}}\right)\right\}\cdot$$

$$\left\{\left(\frac{1}{1 + \frac{(\Delta\dot{\eta})^2}{(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)}}\right)\left(\frac{(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)^{1/2}\frac{\partial}{\partial X}(\Delta\dot{\eta})}{\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2}\right) - \left(\frac{\Delta\dot{\eta}\ \frac{\partial}{\partial X}(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)^{1/2}}{\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2}\right)\right\}$$

$$\frac{\partial}{\partial X}(Q) = V_{ex}T_g \frac{\partial T_g}{\partial X} + \tau V_{ex} \left\{\left(\frac{\tau}{\tau - T_g}\right)\frac{\partial T_g}{\partial X} - \frac{\partial T_g}{\partial X} - L\frac{\partial T_g}{\partial X} - T_g\left(\frac{1}{\tau - T_g}\right)\frac{\partial T_g}{\partial X}\right\}$$

$$\frac{\partial}{\partial X}(L) = \left(\frac{1}{\tau - T_g}\right)\frac{\partial T_g}{\partial X}$$

$$\frac{\partial V_{ex}}{\partial X} = 0$$

$$\frac{\partial S}{\partial X} = V_{ex}\left\{\left(\frac{\tau}{\tau - T_g}\right)\frac{\partial T_g}{\partial X} - \frac{\partial T_g}{\partial X}\right\} - V_{ex}L\frac{\partial T_g}{\partial X} - V_{ex}T_g\left(\frac{1}{\tau - T_g}\right)\frac{\partial T_g}{\partial X}$$

$$\frac{\partial \Delta\dot{\xi}}{\partial X} = \frac{\partial \Delta\dot{\xi}^*}{\partial X} - \ddot{\xi}_g\frac{\partial \Delta T}{\partial X} - \Delta T\frac{\partial \ddot{\xi}_g}{\partial X}$$

$$\frac{\partial}{\partial X}(\Delta\dot{\zeta}) = \frac{\partial \Delta\dot{\zeta}^*}{\partial X} - \ddot{\zeta}_g\frac{\partial \Delta T}{\partial X} - \Delta T\frac{\partial \ddot{\zeta}_g}{\partial X}$$

$$\frac{\partial}{\partial X}(\Delta\dot{\eta}) = \frac{\partial}{\partial X}(\Delta\dot{\eta}^*) - \ddot{\eta}_g\frac{\partial}{\partial X}(\Delta T) - \Delta T\frac{\partial}{\partial X}(\ddot{\eta}_g)$$

$$\frac{\partial}{\partial X}[\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2]^{\frac{1}{2}} = \frac{\Delta\dot{\xi}\frac{\partial \Delta\dot{\xi}}{\partial X} + \Delta\dot{\zeta}\frac{\partial \Delta\dot{\zeta}}{\partial X}}{[\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2]^{\frac{1}{2}}}$$

STEP 6.

$$\frac{\partial}{\partial X}(\sin \tilde{\chi}_y) = \cos \tilde{\chi}_y \left\{ \left[\frac{1}{1 + \frac{(\Delta\dot{\eta})^2}{(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)}} \right] \cdot \left(\frac{(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)^{\frac{1}{2}} \frac{\partial}{\partial X}(\Delta\dot{\eta})}{\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2} \right) - \left(\frac{\Delta\dot{\eta}(\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2)^{\frac{1}{2}}}{\Delta\dot{\xi}^2 + \Delta\dot{\zeta}^2} \right) \right\}$$

$$\frac{\partial}{\partial X}(T_g^{\,2}) = 2T_g \frac{\partial T_g}{\partial X}$$

$$\frac{\partial}{\partial X}(\tau) = 0$$

$$\frac{\partial}{\partial X}(k_{11}) = -q_{11} \sin(\phi_T) \frac{\partial \phi_T}{\partial X} + q_{31} \cos(\phi_T) \frac{\partial \phi_T}{\partial X}$$

$$\frac{\partial}{\partial X}(k_{12}) = -q_{12} \sin(\phi_T) \frac{\partial \phi_T}{\partial X} + q_{32} \cos(\phi_T) \frac{\partial \phi_T}{\partial X}$$

$$\frac{\partial}{\partial X}(k_{13}) = -q_{13} \sin(\phi_T) \frac{\partial \phi_T}{\partial X} + q_{33} \cos(\phi_T) \frac{\partial \phi_T}{\partial X}$$

$$\frac{\partial}{\partial X}(P_T) = 0$$

$$\frac{\partial}{\partial X}(\cos \theta_T) = \frac{\partial \phi_T}{\partial X} + \cancelto{0}{\frac{\partial \omega_T}{\partial X}}$$

$$\frac{\partial}{\partial X}(e_T) = 0$$

$$\frac{\partial}{\partial X}(\dot{\xi}) = \dot{X}(-q_{11}\ \sin(\phi_T)\ \frac{\partial \phi_T}{\partial X} + q_{31}\ \cos(\phi_T)\ \frac{\partial \phi_T}{\partial X}\ ) +$$

$$\dot{Y}(q_{12}\ \sin(\phi_T)\ \frac{\partial \phi_T}{\partial X} + q_{32}\ \cos(\phi_T)\ \frac{\partial \phi_T}{\partial X}\ ) +$$

$$\dot{Z}(-q_{13}\ \sin(\phi_T)\ \frac{\partial \phi_T}{\partial X} + q_{23}\ \cos(\phi_T)\ \frac{\partial \phi_T}{\partial X}\ )$$

$$\frac{\partial \ddot{\xi}_g}{\partial X} = \tfrac{1}{2} \left\{ \frac{\partial G_T}{\partial X} + (\cos\ \phi_T\ q_{11} + \sin\ \phi_T\ q_{31})\ \frac{\partial \ddot{X}_g}{\partial X} + \right.$$

$$(\cos\ \phi_T\ q_{12} + \sin\ \phi_T\ q_{32})\ \frac{\partial \ddot{Y}_g}{\partial X} + (\cos\ \phi_T\ q_{13} +$$

$$\sin\ \phi_T\ q_{33})\ \frac{\partial \ddot{Z}_g}{\partial X} + \ddot{X}_g(-q_{11}\ \sin\ \phi_T\ \frac{\partial \phi_T}{\partial X} + q_{31}\ \cos\ \phi_T\ \frac{\partial \phi_T}{\partial X}\ )$$

$$+ \ddot{Y}_g(-q_{12}\ \sin\ \phi_T\ \frac{\partial \phi_T}{\partial X} + q_{32}\ \cos\ \phi_T\ \frac{\partial \phi_T}{\partial X}\ ) +$$

$$\left. \ddot{Z}_g(-q_{13}\ \sin\ \phi_T\ \frac{\partial \phi_T}{\partial X} + q_{33}\ \cos(\phi_T)\ \frac{\partial \phi_T}{\partial X}\ ) \right\}$$

$$\frac{\partial}{\partial X}(\sin\ \tilde{\chi}_p) = \cos\left[\tan^{-1}\left(\frac{\Delta\dot{\xi}}{\Delta\dot{\zeta}}\right)\right] \left(\frac{(\Delta\dot{\xi})\ \frac{\partial}{\partial X}\ (\Delta\dot{\xi}) - (\Delta\dot{\xi})\ \frac{\partial}{\partial X}\ (\Delta\dot{\zeta})}{(\Delta\dot{\xi})^2 + (\Delta\dot{\zeta})^2} \right)$$

$$\frac{\partial}{\partial X}\ (k_{21}) = 0$$

$$\frac{\partial}{\partial X}\ (k_{22}) = 0$$

$$\frac{\partial}{\partial X}\ (k_{23}) = 0$$

$$\frac{\partial}{\partial X}\ (\ddot{\eta}_{GT}) = 0$$

$$\frac{\partial}{\partial X}(\ddot{X}_g) = \quad \frac{\partial}{\partial X}(\ddot{Y}_g) = \quad \frac{\partial}{\partial X}(\ddot{Z}_g) = \quad \Bigg\}$$

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$$\frac{\partial}{\partial X}(\Delta\dot{\xi}^*) = \frac{\partial}{\partial X}(\dot{\xi}_T) - \frac{\partial}{\partial X}(\dot{\xi}) - \ddot{\xi}_g \frac{\partial T_g}{\partial X} - T_g \frac{\partial \ddot{\xi}_g}{\partial X}$$

$$\frac{\partial}{\partial X}(\Delta T) = \frac{1}{V_{ex}}\left\{G\left(-\frac{\partial T_g}{\partial X}\right) + (\tau - T_g)\frac{\partial G}{\partial X}\right\}$$

$$\frac{\partial}{\partial X}(\Delta\dot{\zeta}^*) = \frac{\partial}{\partial X}(\dot{\zeta}_T) - \frac{\partial}{\partial X}(\dot{\zeta}) - T_g \frac{\partial \ddot{\zeta}_g}{\partial X} - \ddot{\zeta}_g \frac{\partial T_g}{\partial X}$$

$$\frac{\partial}{\partial X}(\ddot{\zeta}_g) = \tfrac{1}{2}\Bigg\{(-\sin\phi_T\, q_{11} + \cos\phi_T\, q_{31})\frac{\partial \ddot{X}_g}{\partial X} +$$

$$(-\sin\phi_T\, q_{12} + q_{32}\cos\phi_T)\frac{\partial \ddot{Y}_g}{\partial X} +$$

$$(-q_{13}\sin\phi_T + q_{33}\cos\phi_T)\frac{\partial \ddot{Z}_g}{\partial X} + \ddot{X}_g\Big(-q_{11}\cos\phi_T\frac{\partial\phi_T}{\partial X}$$

$$- q_{31}\sin\phi_T\frac{\partial\phi_T}{\partial X}\Big) + \ddot{Y}_g\Big(-q_{12}\cos\phi_T\frac{\partial\phi_T}{\partial X} - q_{32}\sin\phi_T\frac{\partial\phi_T}{\partial X}\Big)$$

$$+ \ddot{Z}_g\Big(-q_{13}\cos\phi_T\frac{\partial\phi_T}{\partial X} - q_{33}\sin\phi_T\frac{\partial\phi_T}{\partial X}\Big)\Bigg\}$$

$$\frac{\partial\Delta\dot{\eta}^*}{\partial X} = \frac{\partial\dot{\eta}_T}{\partial X} - \frac{\partial\dot{\eta}}{\partial X} - \frac{\partial\ddot{\eta}_g}{\partial X}T_g - \ddot{\eta}_g\frac{\partial T_g}{\partial X}$$

$$\frac{\partial}{\partial X}(\ddot{\eta}_g) = q_{21}\frac{\partial\ddot{X}_g}{\partial X} + q_{22}\frac{\partial\ddot{Y}_g}{\partial X} + q_{23}\frac{\partial\ddot{Z}_g}{\partial X}$$

STEP 7.

$$\frac{\partial \phi_T}{\partial X} = \frac{\partial \phi_{ET}}{\partial X} + ½(T_g + FK6\ T_g)\left\{\frac{V}{R}(-\sin\gamma)\ \frac{\partial \gamma}{\partial X} + \right.$$

$$\frac{\cos\gamma}{R^2}\left(R\ \frac{\partial V}{\partial X} - V\ \frac{\partial R}{\partial X}\right) + \frac{V_T}{R_T}\ (-\sin\gamma_T\ \frac{\partial \gamma_T}{\partial X}) +$$

$$\left.\frac{\cos\gamma_T}{R_T{}^2}\ (R_T\ \frac{\partial V_T}{\partial X} - V_T\ \frac{\partial R_T}{\partial X})\right\} \quad +$$

$$½\left\{\frac{V\ \cos\gamma}{R} + \frac{V_T}{R_T}\cos\gamma_T\right\}\left\{\frac{\partial T_g}{\partial X} + FK6\ \frac{\partial T_g}{\partial X} + T_g\ \frac{\partial FK6}{\partial X}\right\}$$

$$\frac{\partial G_T}{\partial X} = \frac{2\mu}{R_T{}^3}\ \ \frac{\partial R_T}{\partial X}$$

$$\frac{\partial \dot{\xi}_T}{\partial X} = V_T\ \cos\gamma_T\ \frac{\partial \gamma_T}{\partial X} + \sin\gamma_T\ \frac{\partial V_T}{\partial X}$$

$$\frac{\partial G}{\partial X} = ½\left\{V_{ex}L(2\Delta\dot{\xi}^*\ \frac{\partial \Delta\dot{\xi}^*}{\partial X} + 2\Delta\dot{\eta}^*\ \frac{\partial \Delta\dot{\eta}^*}{\partial X}) - (\Delta\dot{\xi}^{*2} + \Delta\dot{\eta}^{*2} + \Delta\dot{\zeta}^{*2})\right.$$

$$\left.\cdot\ (V_{ex}\ \frac{\partial L}{\partial X} + L\ \frac{\partial V_{ex}}{\partial X})\right\}\frac{1}{(V_{ex}L)^2} - ½(V_{ex}\ \frac{\partial L}{\partial X} + L\ \frac{\partial V_{ex}}{\partial X})$$

$$\frac{\partial \dot{\zeta}_T}{\partial X} = -V_T\ \sin\gamma_T\ \frac{\partial \gamma_T}{\partial X} + \cos\gamma_T\ \frac{\partial V_T}{\partial X}$$

$$\frac{\partial \dot{\zeta}}{\partial X} = \dot{X}(-q_{11} \cos \phi_T \frac{\partial \phi_T}{\partial X} - q_{31} \sin \phi_T \frac{\partial \phi_T}{\partial X})$$

$$+ \dot{Y}(-q_{12} \cos \phi_T \frac{\partial \phi_T}{\partial X} - q_{32} \sin \phi_T \frac{\partial \phi_T}{\partial X})$$

$$+ \dot{Z}(-q_{13} \cos \phi_T \frac{\partial \phi_T}{\partial X} - q_{33} \sin \phi_T \frac{\partial \phi_T}{\partial X}$$

$$\frac{\partial \dot{\eta}_T}{\partial X} = 0$$

$$\frac{\partial \dot{\eta}}{\partial X} = 0$$

STEP 8.

$$\frac{\partial \phi_{ET}}{\partial X} = \frac{X_F\, q_{31} - Z_F\, q_{11}}{X_F^{\,2} + Z_F^{\,2}}$$

$$\frac{\partial \gamma}{\partial X} = \left\{1 - \frac{X\dot{X} + Y\dot{Y} + Z\dot{Z}}{RV}\right\}^{-\frac{1}{2}} \left\{\frac{RV\dot{X} - (X\dot{X} + Y\dot{Y} + Z\dot{Z})(V\, \frac{\partial R}{\partial X})}{R^2V^2}\right\}$$

$$\frac{\partial V}{\partial X} = 0$$

$$\frac{\partial R}{\partial X} = \frac{X}{[X^2 + Y^2 + Z^2]^{\frac{1}{2}}}$$

$$\frac{\partial \gamma_T}{\partial X} = \left(1 + \frac{e_T \sin\theta_T}{1+e_T\cos\theta_T}\right)^{-1} \left(\frac{(e_T+e_T{}^2\cos\theta_T)\,\frac{\partial \sin\theta_T}{\partial X} - (e_T{}^2\sin\theta_T\,\frac{\partial \cos\theta_T}{\partial X})}{(1 + e_T\cos\theta_T)^2}\right)$$

$$\frac{\partial V_T}{\partial X} = \left(\frac{\mu}{P_T}\right)^{\frac{1}{2}} \left\{ \frac{-\sin\theta_T \frac{\partial \theta_T}{\partial X}}{[1 + e_T(e_T + 2\cos\theta_T)]^{\frac{1}{2}}} \right\}$$

$$\frac{\partial R_T}{\partial X} = \frac{P_T\, e_T \sin\theta_T \frac{\partial \theta_T}{\partial X}}{(1 + e_T\cos\theta_T)^2}$$

$$\frac{\partial FK6}{\partial X} = (2)(C1)(T_g)\frac{\partial T_g}{\partial X} + (C2)\frac{\partial T_g}{\partial X}$$

where

$$C1 = B_{22}\,\Delta a^2 + B_{21}\,\Delta a + B_{20}$$

$$C2 = B_{12}\,\Delta a^2 + B_{11}\,\Delta a + B_{10}$$

and B_{10}, B_{11}, B_{12}, B_{20}, B_{21}, and B_{22} are constants determined elsewhere and Δa is the difference between semi-major axes of the initial Parking Orbit and the Final orbit.

STEP 9.

$$\frac{\partial}{\partial X}\sin\theta_T = \cos(\theta_T)\frac{\partial \theta_T}{\partial X}$$

$$\frac{\partial}{\partial X}\cos\theta_T = -\sin(\theta_T)\frac{\partial \theta_T}{\partial X}$$

$$\frac{\partial \theta_T}{\partial X} = \frac{\partial \phi_T}{\partial X} = \frac{\partial \phi_{ET}}{\partial X} + \tfrac{1}{2}(T_g + FK6\ T_g) \cdot$$

$$\left\{ \frac{V}{R}(-\sin\gamma)\frac{\partial \gamma}{\partial X} - \frac{\cos(\gamma)\cdot V}{R^2}\frac{\partial R}{\partial X} - \frac{V_T}{R_T}\sin\gamma_T\frac{\partial \gamma_T}{\partial X}\right.$$

$$\left. + \frac{\cos\gamma_T}{R_T}\frac{\partial V_T}{\partial X} - \frac{V_T\cos\gamma_T}{R_T^{\ 2}}\frac{\partial R_T}{\partial X}\right\} +$$

$$\tfrac{1}{2}\left\{\frac{V\cos\gamma}{R} + \frac{V_T}{R_T}\cos\gamma_T\right\}\left\{\frac{\partial T_g}{\partial X} + FK6\frac{\partial T_g}{\partial X} + T_g\frac{\partial FK6}{\partial X}\right\}$$

All partial derivative terms appearing in the above expression have been previously determined. It was noted that the term $\frac{\partial T_g}{\partial X}$ is common in the derivative term expressions. Thus, if $\frac{\partial \theta_T}{\partial X}$ can be expressed as a function of $\frac{\partial T_g}{\partial X}$ and $\frac{\partial T_g}{\partial X}$ can be expressed in closed form, then all of the partial derivatives could be evaluated. After proper substitutions, $\frac{\partial \theta_T}{\partial X}$ can be shown to be,

$$\frac{\partial \theta_T}{\partial X} = \frac{E1 + A5\frac{\partial T_g}{\partial X}}{1 + A_2 + A_3 + A_4}$$

where,

$$E1 = \frac{X_F\ Q(3,1) - Z_F\ Q(1,1)}{X_F^{\ 2} + Z_F^{\ 2}} - W2\left\{\frac{RV\dot{X} - (X\dot{X} + Y\dot{Y} + Z\dot{Z})\frac{VX}{R}}{(RV)^2}\right\}$$

$$\cdot \left\{\frac{1}{\left\{1 - \left(\frac{X\dot{X} + Y\dot{Y} + Z\dot{Z}}{RV}\right)^2\right\}^{\frac{1}{2}}}\right\} - W4\left(\frac{X}{R}\right)$$

$$A2 = W5 \left\{ \frac{e_T^{\,2} + e_T \cos \theta_T}{1 + 2e_T \cos \theta_T + e_T^{\,2}} \right\}$$

$$A3 = W6 \left\{ \left(\frac{\mu}{P_T}\right)^{\frac{1}{2}} \frac{e_T \sin \theta_T}{(1 + e_T^{\,2} + 2e_T \cos \theta_T)^{\frac{1}{2}}} \right\}$$

$$A4 = W7 \left\{ \frac{P_T\, e_T \sin \theta_T}{(1 + e_T \cos \theta_T)^2} \right\}$$

$$A5 = W8 + W9[2(C1FK6)T_g + C2FK6]$$

and

$$W1 = \frac{V \cos \gamma}{R} + \frac{V_T \cos \gamma_T}{R_T}$$

$$W2 = \tfrac{1}{2}[T_g + FK6(T_g)]\, \frac{V}{R} \sin \gamma$$

$$W4 = \tfrac{1}{2}[T_g + FK6(T_g)]\, \frac{V}{R^2} \cos \gamma$$

$$W5 = \tfrac{1}{2}[T_g + FK6(T_g)]\, \frac{V_T}{R_T} \sin \gamma_T$$

$$W6 = \tfrac{1}{2}[T_g + FK6(T_g)]\, \frac{\cos \gamma_T}{R_T}$$

$$W7 = V_T(W6)$$

$$W8 = (\tfrac{1}{2})(W1)(1 + FK6)$$

$$W9 = (\tfrac{1}{2})(W1)\ T_g$$

As a final task, $\frac{\partial T_g}{\partial X}$ must be expressed in closed form.

At this point, it would be advantageous to discuss T_g as it appears in the I.G.M. law. The equation for T_g given in I.G.M. is,

$$T_g = T_g + \Delta T,$$

where it appears that this is merely an update of T_g. But, acutally, T_g has been updated prior to this step and ΔT is actually a correction factor for the error in the update.

The discrete equations are

$$\overset{\text{actual}}{T_g(k)} = \overset{\text{theor.}}{T_g(k)} + \overset{\text{computed}}{\Delta T(k)}$$

$$\overset{\text{theor.}}{T_g(k+1)} = \overset{\text{theor.}}{T_g(k)} + \overset{\text{computed}}{\Delta T(k)} + \delta t(k)$$

and taking the first partial derivative,

$$\overset{\text{actual}}{\frac{\partial T_g(k)}{\partial X}} = \overset{\overset{0}{\nearrow}}{\overset{\text{theor.}}{\cancel{\frac{\partial T_g(k)}{\partial X}}}} + \overset{\text{computed}}{\frac{\partial \Delta T(k)}{\partial X}} \tag{A-9}$$

Solving for the $\frac{\partial \Delta T}{\partial X}$ in terms of T_g

$$\frac{\partial \Delta T}{\partial X} = \left(\frac{\tau - T_g}{V_{ex}}\right)\frac{\partial G}{\partial X} - \left(\frac{G}{V_{ex}}\right)\frac{\partial T_g}{\partial X}$$

From Step 7,

$$\frac{\partial G}{\partial X} = \tfrac{1}{2}\left\{V_{ex}\ L\left(2\Delta\dot{\xi}^{*}\ \frac{\partial\Delta\dot{\xi}^{*}}{\partial X} + 2\Delta\dot{\eta}^{*}\ \frac{\partial\Delta\dot{\eta}^{*}}{\partial X}\right) - (\Delta\dot{\xi}^{*2} + \Delta\dot{\eta}^{*2} + \Delta\dot{\zeta}^{*2})\right.$$

$$\left.\cdot\left(V_{ex}\ \frac{\partial L}{\partial X} + L\ \frac{\partial V_{ex}}{\partial X}\right)\right\}\ \frac{1}{(V_{ex}L)^2} - \tfrac{1}{2}\left(V_{ex}\ \frac{\partial L}{\partial X} + L\ \frac{\partial V_{ex}}{\partial X}\right)$$

Substituting back into this equation using the partial derivatives obtained in terms of T_g, and solving for $\frac{\partial\Delta T}{\partial X}$ one obtains,

$$\frac{\partial\Delta T}{\partial X} = \left\{\left(\frac{\tau - T_g}{2\ V_{ex}}\right)A6 - \frac{G}{V_{ex}}\right\}\frac{\partial T_g}{\partial X} + E2\left(\frac{\tau - T_g}{2\ V_{ex}}\right)$$

where,

$$A6 = \left(\frac{A5(S1 - S2 - S3 - S4 - S5 - S6 + S7 + S8 + S9)}{1 + A2 + A3 + A4}\right) - M1$$

$$S1 = M3\left\{\frac{e_T^{\ 2} + e_T\cos\theta_T}{1 + 2e_T\cos\theta_T + e_T^{\ 2}}\right\}$$

$$S2 = M4\left\{\left(\frac{\mu}{P_T}\right)^{\frac{1}{2}}\ \frac{e_T\sin\theta_T}{[1 + e_T^{\ 2} + 2e_T\cos\theta_T]^{\frac{1}{2}}}\right\}$$

$$S3 = M5\left\{\left(\frac{2\mu}{R_T^{\ 3}}\right)\ \frac{P_T\ e_T\sin\theta_T}{[1 + e_T\cos\theta_T]^2}\right\}$$

$$S4 = M6[q_{31}\cos\phi_T - q_{11}\sin\phi_T]$$

$$S5 = M7[q_{32}\cos\phi_T - q_{12}\sin\phi_T]$$

$$S6 = M8[q_{33}\cos\phi_T - q_{13}\sin\phi_T]$$

$$S7 = N3[q_{11} \cos \phi_T + q_{31} \sin \phi_T]$$

$$S8 = N4[q_{12} \cos \phi_T + q_{32} \sin \phi_T]$$

$$S9 = N5[q_{13} \cos \phi_T + q_{33} \sin \phi_T]$$

$$M1 = \left\{ \left(\frac{2\Delta\dot{\xi}^* \ddot{\xi}_g + 2\Delta\dot{\eta}^* \ddot{\eta}_g + 2\Delta\dot{\zeta}^* \ddot{\zeta}_g}{V_{ex} L} \right) + \left(\frac{(\Delta\dot{\xi}^*)^2 + (\Delta\dot{\eta}^*)^2 + (\Delta\dot{\zeta}^*)^2}{V_{ex} L^2 (\tau - T_g)} \right) + \frac{V_{ex}}{\tau - T_g} \right\}$$

$$M2 = \left\{ \left(\frac{\Delta\dot{\xi}^* T_g}{V_{ex} L} \left(k_{11} \frac{\partial \ddot{X}_g}{\partial X} + k_{12} \frac{\partial \ddot{Y}_g}{\partial X} + k_{13} \frac{\partial \ddot{Z}_g}{\partial X} \right) \right) + \left(\frac{\Delta\dot{\eta}^* T_g}{V_{ex} L} \left(k_{21} \frac{\partial \ddot{X}_g}{\partial X} + k_{22} \frac{\partial \ddot{Y}_g}{\partial X} + k_{23} \frac{\partial \ddot{Z}_g}{\partial X} \right) \right) + \left(\frac{\Delta\dot{\zeta}^* T_g}{V_{ex} L} \left(k_{31} \frac{\partial \ddot{X}_g}{\partial X} + k_{32} \frac{\partial \ddot{Y}_g}{\partial X} + k_{33} \frac{\partial \ddot{Z}_g}{\partial X} \right) \right) \right\}$$

$$M3 = \left(\frac{2\Delta\dot{\xi}^*}{V_{ex} L} \right) V_T \cos \gamma_T - \left(\frac{2\Delta\dot{\zeta}^*}{V_{ex} L} \right) V_T \sin \gamma_T$$

$$M4 = \left(\frac{2\Delta\dot{\xi}^*}{V_{ex} L} \right) \sin \gamma_T + \left(\frac{2\Delta\dot{\zeta}^*}{V_{ex} L} \right) \cos \gamma_T$$

$$M5 = \frac{\Delta\dot{\xi}^* T_g}{V_{ex} L}$$

$$M6 = \left(\frac{2\Delta\dot{\xi}^*}{V_{ex} L} \right) \left(\dot{X} + \frac{T_g}{2} \ddot{X}_g \right)$$

$$M7 = \left(\frac{2\Delta\dot{\xi}^*}{V_{ex}L}\right)\left(\dot{Y} + \frac{T_g}{2}\ddot{Y}_g\right)$$

$$M8 = \left(\frac{2\Delta\dot{\xi}^*}{V_{ex}L}\right)\left(\dot{Z} + \frac{T_g}{2}\ddot{Z}_g\right)$$

$$M9 = \left(\frac{2\Delta\dot{\eta}^*}{V_{ex}L}\right)\left(\dot{X} + \frac{T_g}{2}\ddot{X}_g\right)$$

$$N1 = \left(\frac{2\Delta\dot{\eta}^*}{V_{ex}L}\right)\left(\dot{Y} + \frac{T_g}{2}\ddot{Y}_g\right)$$

$$N2 = \left(\frac{2\Delta\dot{\eta}^*}{V_{ex}L}\right)\left(\dot{Z} + \frac{T_g}{2}\ddot{Z}_g\right)$$

$$N3 = \left(\frac{2\Delta\dot{\zeta}^*}{V_{ex}L}\right)\left(\dot{X} + \frac{T_g}{2}\ddot{X}_g\right)$$

$$N4 = \left(\frac{2\Delta\dot{\zeta}^*}{V_{ex}L}\right)\left(\dot{Y} + \frac{T_g}{2}\ddot{Y}_g\right)$$

$$N5 = \left(\frac{2\Delta\dot{\zeta}^*}{V_{ex}L}\right)\left(\dot{Z} + \frac{T_g}{2}\ddot{Z}_g\right)$$

$$E2 = \left\{\frac{E1(S1 - S2 - S3 - S4 - S5 - S6 + S7 + S8 + S9)}{1 + A2 + A3 + A4}\right\} M2$$

and E1, A2, A3, A4, and A5 are as defined previously.

From (A-9) the final step in the derivation is

$$\frac{\partial \Delta T}{\partial X} = \frac{\partial T_g}{\partial X}$$

$$\frac{\partial T_g}{\partial X}\left\{1 - \left(\frac{\tau - T_g}{2\ V_{ex}}\right) A6 + \frac{G}{V_{ex}}\right\} = \left(\frac{\tau - T_g}{2\ V_{ex}}\right) E2$$

$$\frac{\partial T_g}{\partial X} = \frac{E2}{\left(\frac{2\ V_{ex}}{\tau - T_g}\right) - A6 + \left(\frac{2G}{\tau - T_g}\right)}$$

Similar expressions can be derived for the partials with respect to the five remaining elements in the state vector. Due to the extreme size of the listing it is omitted from this report, however, it is available upon request as an Addendum.

APPENDIX B

DATA INPUT/OUTPUT

In this Appendix, the form of data input needed for computer cards is shown and an example output given and discussed as to individual term meanings so there can be no confusion as to what the output yields. It should be noted that in all cases, double precision notation is required, although not all signifigant digits need be or can always be input or output.

A. Input From Data Cards

For different data, different formats are used. A listing of the groups of variables will be given, along with the format code called for.

1. Earth Orbit Data

ECC, ECCAML, ANM, TREF, AMAJOR

Format is 5D13.8 - up to 8 decimals per number, 13 spaces total (including sign and exponent), D format, up to 5 numbers on a card.

An example card field is shown below.

```
COL 1            14             27
      .016727260D00  8.30127160D-3  1.99083972D-7

      0.00000000D00  1.49467000D11
COL 40           53
```

2. IGM Initialization Integers

ISW, NCYL, NFK6

Format is 16I5 - up to 16 different five-digit integers (only 3 are used here) which are right-justified. An example card field is shown below

COL	5	10	15
	2	2	0

3. Tug Characteristic Data

THRUST, SISP, WTSTAG, TF, TIME

Format is 6D13.8 - up to 8 decimals per number, 13 spaces total, D format, up to 6 numbers on a card. An example is shown below

COL	1	14	27
	1.50000000D04	4.44000000D02	5.85000000D04
	1.00000000D01	4.19755900D04	
COL	40	53	

4. Exact Tug State Vector

X, DX

Format is 6D13.8 - same as is subsection 3. An example is shown below

COL.	1	14	27
	-8.7143370D04	-2.4140460D05	-6.6738760D06
	7.73020900D03	-1.7556700D01	-1.0550210D02
COL.	40	53	66

5. State Offset Vector

XOFFST

Format is 6D13.8 - same as in subsection 3. An example card is shown below

```
COL. 1        14       27        40     53     66
     1000.0D0 500.0D0  1000.0D0  2.0D0  1.0D0  2.0D0
```

6. Burn-Coast Initialization Times

DTC, DPRTC, DTB, DPRTB

Format is 6D13.8 - same as in subsection 3. An example card is shown below

```
COL. 1                14               27
     6.00000000D016   6.00000000D01    .500000000D00
     .500000000D00
COL. 40
```

7. Launch Coordinates and IGM Control Times

AZ, PHIL, TPHIT, TCHIT

Format is 6D13.8 - same as in subsection 3. An example card is shown below

```
COL. 1              14              27
     8.33911243D01  2.86080000D01   9.9999990D06
     1.50000000D01
COL. 40
```

8. Simulator Mode Selection Integer

ICHECK

Format is I1- integer value is 1, 2, or 3 in column 1.

9. End-of-Burn Orbital Parameters

RT, VT, THT, PO2, E2, ARGPER, DINCL, DNODE, TGB, RANT, WDROP

Format is 6D13.8 - same as in subsection 3. There are two cards required for this data.

COL.	1	14	27
	6.69584200D06	8.36041600D03	1.25450100D00
	7.85809400D06	1.7547030D-01	1.80689300D02
COL.	40	53	66

COL.	1	14	27
	2.82827000D01	9.94007300D01	2.42300500D02
	1.71028100D02	0.00000000D00	
COL.	40	53	

10. End-of-Coast Time

TCOAST

Format is D13.8 - one number with 13 total digits or less, only 8 of which may be decimals. An example card is shown below

COL. 1

4.90717848D04

B. GSP Output

This section discusses and displays representation data output so as to clarify the purpose of the data presented.

On the next page is shown the first output sheet containing an explanation of the mode selection, i.e. deterministic, full filter, or passenger mode, along with the IGM data input from cards as discussed in the previous section of this Appendix. This is for two purposes: (1) to insure that the data has been read properly, and (2) it provides a concise listing of the simulation run values to aid in proper analysis of the flight data at a later time.

On the pages following the previous sheet are representative output data from GSP, as commanded by subroutines NPRINT and OUTPUT. The first block set of data represents initial conditions data existing before the simulation begins. Subheadings "Begin Burn", "Begin Coast", "End Burn", "End Coast" appear as necessary to clearly dileneate the flight phase the printed data is compatable with. Note that TIME, listed at the beginning of each block of data, represents the reference time at which the data is available. This time is referenced to the initial Earth launch of the Space Shuttle - Space Tug combination. For instance, TIME = 41980.0 means 41,980 seconds (or about 11 1/2 hours) have elapsed from launch initiation. A defining list of symbols for the terms in these data blocks is given in Table B-1.

The sections of output data not in blocks, i.e. "FILTERED STATES", represents output data from the state estimator navigation system.

```
ICHECK = 2

IF ICHECK=1,TRUE DATA USED STATE ESTIMATOR IS BYPASSED

IF ICHECK=2, THE STATE ESTIMATOR IS USED

IF ICHECK=3,THE STATE ESTIMATOR IS CARRIEDALONG AS     A PASSENGER AND TRUE DATA USED AT THEBEGINNING OF EACH CYCLE

 ECC=   0.16727D-01      AML=     .83013D-02     ANM=    0.19908D-06     TREF=   0.0             AMAJOR=   0.14947D 12


                                            IGM INPUT DATA

ISW =  2      NCYL=  2      NFK6=  0

THRUST=  0.15000000D 05        SISP=  0.44400000D 03     WTSTAGE=  0.58500000D 05           TF=  0.10000000D 02
  TIME=  0.41975590D 05           X= -0.87143370D 05           Y= -0.24140460D 06           Z= -0.66738760D 07
    XD=  0.77302090D 04          YD= -0.17556700D 02          ZD= -0.10550210D 03          DTC=  0.60000000D 02
 DPRTC=  0.30000000D 03         DTB=  0.20000000D 01      DPRTB=  0.20000000D 02           AZ=  0.83391124D 02
  PHIL=  0.28608000D 02       TPHIT=  0.99999990D 07      TCHIT=  0.15000000D 02
    RT=  0.66958420D 07          VT=  0.83604160D 04        THT=  0.12545010D 01       PO2 =  0.78580940D 07
    E2=  0.17547030D 00      ARGPER=  0.18068930D 03       INCL=  0.28282700D 02        NODE=  0.99400730D 02
   TGO=  0.24230050D 03       RANGE=  0.17102810D 03      WDROP=  0.0
                                   THE ERROR IN THE X VECTOR INITIALLY IS
       0.50000000D 03       0.25000000D 03       0.50000000D 03       0.10000000D 01       0.50000000D 00       0.10000000D 01
TIME AT END OF COAST (IF ANY) SHOULD BE =.0
```

```
BEGIN BURN

TIME  0.42080000D 05
X-13  0.73237933D 06 Y-13 -0.23911664D 06 Z-13 -0.66366439D 07 DX-13  0.79515273D 04 DY-13  0.62213715D 02 DZ-13  0.82794971D 03
R     0.66812124D 07 V     0.79947582D 04 PATH  0.33665682D 00   INCL  0.28704890D 02 NODE  0.99443666D 02   RAN   0.17801119D 03
GX -0.98008876D 00   FMXM  0.26189763D 01 DDX  0.16388875D 01 WT  0.24935171D 05 WTLBS  0.54972635D 05 FLOW -0.15324069D 02
GY  0.31969474D 00   FMYM  0.46330699D 00 DDY  0.78300172D 00 F  0.66723333D 05 FLBS  0.15000000D 05 PHIT  0.18773460D 03
GZ  0.88831707D 01   FMZM  0.29429255D 00 DDZ  0.91774532D 01 ISP  0.44400000D 03 F/M  0.26758723D 01 TGO  0.13828462D 03
CHIP  0.83588824D 02 CHIDP-0.10066420D 00 CHIY -0.99705762D 01 CHIDY -0.52290033D-03 CHIPC  0.83387496D 02 CHIYC -0.99716220D 01
FK1 -0.11874721D 00 FK2 -0.16919643D-02 FK3 -0.49535328D-02 FK4 -0.70597704D-04 CHIPP  0.90062152D 01 CHIPY  0.90740807D 01
 CHI(1)  0.97834179D 00 CHI(2)  0.17316039D 00 CHI(3)  0.11341439D 00 CHITP  0.23963859D 01 CHITY  0.87983541D 01
P     0.71575674D 07 E     0.71575030D-01 ARG   0.18703429D 03TRU      0.50454800D 01A      0.71944243D 07PSEC    0.60730039D 04
RP    0.66794832D 07 RA    0.77093655D 07 VP   0.79966897D 04  VA    0.69284243D 04 AP    0.16269827D 03  AA    0.71879021D 03
```



```
TIME =  0.42080000D 05

                                FILTERED STATES =

   0.7323793319D 06   -0.2391166361D 06   -0.6636643932D 07    0.7951527340D 04    0.6221371795D 02    0.8279497615D 03

STATE ERROR = ACTUAL-ESTIMATED

  -0.5040105316D-04    0.1241467544D-05    0.1851096749D-04    0.4117787284D-05   -0.3354812662D-05   -0.5445883659D-04

POSITION ERROR NORM =  0.5370715998D-04

 VELOCITY ERROR NORM =  0.54717235D-04

                                EXACT MEASUREMENTS =

   0.7323793319D 06   -0.2391166361D 06   -0.6636643932D 07    0.7951527344D 04    0.6221371460D.02    0.8279497070D 03

                                AVAILABLE MEASUREMENTS =

   0.7323793319D 06   -0.2391166361D 06   -0.6636643932D 07    0.7951527344D 04    0.6221371460D 02    0.8279497070D 03

                                ESTIMATED MEASUREMENTS =

   0.7323793446D 06   -0.2391166364D 06   -0.6636643937D 07    0.7951526357D 04    0.6221455859D 02    0.8279634101D 03

                         MEASUREMENTS RESIDUAL = AVAILABLE - ESTIMATED

  -0.1274732335D-01    0.3147879023D-03    0.4630323499D-02    0.9870320828D-03   -0.8439874834D-03   -0.1370302960D-01
```

```
COVARIANCE MATRIX IS
  0.996032340 01-0.98611533D-11-0.11640129D-10 0.32155185D-04-0.54685336D-08 0.26681144D-07
 -0.98611533D-11 0.99603234D 01-0.19376949D-11 0.18847466D-07 0.32043393D-04 0.39255761D-08
 -0.11640129D-10-0.19376949D-11 0.99603234D 01-0.10857068D-07-0.12531808D-08 0.32048907D-04
  0.32155185D-04 0.18847466D-07-0.10857068D-07 0.99603168D 01 0.62482393D-07 0.76991317D-07
 -0.54685336D-08 0.32043393D-04-0.12531808D-08 0.62482393D-07 0.99603158D 01 0.12770947D-07
  0.26681144D-07 0.39255761D-08 0.32048907D-04 0.76991317D-07 0.12770947D-07 0.99603158D 01
```

```
THE STATE TRANSITION MATRIX IS
  0.10000000D 01 0.0            0.0            0.10000000D 01 0.0            0.0
  0.0            0.10000000D 01 0.0            0.0            0.10000000D 01 0.0
  0.0            0.0            0.10000000D 01 0.0            0.0            0.10000000D 01
 -0.94453921D-05 0.48118226D-04 0.29159796D-04 0.99953321D 00 0.45813094D-02 0.28155053D-02
  0.47853490D-05-0.27952899D-03 0.70022364D-06-0.13490143D-02 0.97329360D 00-0.36755696D-03
  0.36452925D-04 0.98620389D-05-0.27024906D-03 0.64838620D-02 0.11482428D-02 0.97471380D 00
```

```
THE ROOTS OF THE STATE TRANSITION MATRIX
          REAL                   IMAGINARY
   0.99999237E 00           -0.11186264E-02
   0.99999237E 00            0.11186264E-02
   0.98659629E 00           -0.10322601E-01
   0.98659629E 00            0.10322601E-01
   0.98718184E 00           -0.10299139E-01
   0.98718184E 00            0.10299139E-01
```

```
DELCHK =  0.20000000D 01
                         GRAVITY TERMS

XCDG=    -0.12875D-05   -0.13900D-07   -0.42223D-06   0.0    0.0    0.0

YCDG=    -0.13908D-07   -0.13286D-05    0.13603D-06   0.0    0.0    0.0

ZCDG=    -0.42218D-06    0.14346D-06    0.26162D-05   0.0    0.0    0.0


NOT CONTROLLABLE  DET =     0.0
                           NY =   6            NP =   6               IER =   1

OBSERVABLE        DET =     1.00000D 00
```

THE OBSERVATION MATRIX H IS

0.100000000D 01	0.0	0.0	0.0	0.0	0.0
0.0	0.100000000D 01	0.0	0.0	0.0	0.0
0.0	0.0	0.100000000D 01	0.0	0.0	0.0
0.0	0.0	0.0	0.100000000D 01	0.0	0.0
0.0	0.0	0.0	0.0	0.100000000D 01	0.0
0.0	0.0	0.0	0.0	0.0	0.100000000D 01

Definitions follow:

Name	Definition
Filtered States	State vector information actually available to guidance law (is in error).
State Error	$\underline{e} = \underline{X}(k)_{\text{actual}} - \hat{\underline{X}}(k/k)$
Position Error Norm	$\sqrt{e_1^2 + e_2^2 + e_3^2}$
Velocity Error Norm	$\sqrt{e_4^2 + e_5^2 + e_6^2}$
Exact Measurements	actual measurements without noise
Available Measurements	actual measurements plus noise
Estimated Measurements	results which come from filter one step propagation, $\hat{\underline{Z}}(k+1/k) = H(k+1)\hat{\underline{X}}(k+1/k)$
Measurement Residual	(available-estimated) measurement indicates how well filter is working.
Covariance Matrix	State covariance matrix
State Transition Matrix	$\Phi(k+1,k)$ entries
DELCHK	Integration step size
Gravity Terms XDDG YDDG ZDDG	partial derivatives of gravity terms; going across page, $\frac{\partial \ddot{x}_g}{\partial x}, \frac{\partial \ddot{x}_g}{\partial y}, \ldots, \frac{\partial \ddot{x}_g}{\partial \dot{y}}, \frac{\partial \ddot{x}_g}{\partial \dot{z}}$
Not Controllable - Controllable Not Observable - Observable	Headings from CONOBS referring to observability/controllability
DET	determinant of a square matrix used in CONOBS analysis
H matrix	Elements of H matrix; 6x6 during burn; 3x6 during coast

Table B-1. LIST OF SYMBOLS IN OUTPUT DATA BLOCK

Symbol	Definition
X-13	Exact X coordinate of the Tug in Apollo 13 ECLC. (meter)
Y-13	Exact Y coordinate of the Tug in Apollo 13 ECLC. (meter)
Z-13	Exact Z coordinate of the Tug in Apollo 13 ECLC. (meter)
DX-13	Exact $\dot{X}$ coordinate of the Tug in Apollo 13 ECLC. (m/s)
DY-13	Exact $\dot{Y}$ coordinate of the Tug in Apollo 13 ECLC. (m/s)
DZ-13	Exact $\dot{Z}$ coordinate of the Tug in Apollo 13 ECLC. (m/s)
R	Instantaneous radius of the Tug from earth (meter)
V	Instantaneous velocity of the Tug (m/s)
PATH	Flight path angle γ in IGM (in degrees)
INCL	Present inclination angle of orbit (in degrees)
NODE	Present flight orbit node (in degrees)
RAN	Range angle (degrees)
GX	Instantaneous gravity in the X coordinate (m/s^2)
GY	Instantaneous gravity in the Y coordinate (m/s^2)
FMXM	Acceleration due to engine thruster in the X coordinate (m/s^2)
FMYM	Acceleration due to engine thruster in the Y coordinate (m/s^2)
GZ	Instantaneous gravity in the Z coordinate (m/s^2)

Symbol	Definition
FMZM	Acceleration due to engine thruster in the Z coordinate (m/s^2)
DDX	Instantaneous acceleration of the Tug in the X coordinate (m/s^2)
DDZ	Instantaneous acceleration of the Tug in the Z coordinate. (m/s^2)
WT	Vehicle mass (kilograms)
WTLBS	Weight of vehicle (pounds referenced to sea level)
FLOW	Mass flow rate (kg/sec)
F	Vehicle Thrust (in Newtons)
FLBS	Vehicle Thrust (in pounds referenced to sea level)
PHIT	ϕ_T - Terminal Range angle
ISP	Specific impulse (in seconds)
F/M	Instantaneous Tug acceleration due to thrusting (m/s^2)
TGO	Time-to-go (seconds)
CHIP	χ_p - Pitch burn angle (degrees)
CHIDP	$\dot{\chi}_p$ - Pitch burn angle rate (degrees/seconds)
CHIY	χ_y - Yaw burn angle (degrees)
CHIDY	$\dot{\chi}_y$ - Yaw burn angle rate (degrees/seconds)
CHIPC	One stage computation of χ_p (degrees)
CHIYC	One stage computation of χ_y (degrees)
P	Semi-latus rectum of instantaneous orbit (meters)
DDY	Instantaneous acceleration of the Tug in the Y coordinate m/s^2

Symbol	Definition
E	Instantaneous eccentricity of the present orbit (dimensionless)
ARG	Argument of the perigee (degrees)
TRU	True anomaly (degrees)
A	Orbital semi-major axis (meters)
PSEC	Orbital Period (seconds)
RP	Perigee of instantaneous orbit (meters)(low point)
RA	Apogee of instantaneous orbit (high point)(meters)
VP	Velocity at perigee (meter/sec)
AP	Orbit perigee (nautical miles) 1 n.m. = 6080.27 ft.
AA	Orbit apogee (nautical miles)
VA	Velocity at apogee (meter/sec)

APPENDIX C

SPACE TUG NAVIGATION

SPECIFICATIONS

The state vector errors allowable in the initial state vector prior to a burn, measured in a local vertical coordinate system are:

X - position	1.0 Km
Y - position	0.1 Km
Z - position	1.0 Km
X - velocity	2.0 m/sec
Y - velocity	0.5 m/sec
Z - velocity	2.0 m/sec

The errors in the platform attitude angles should initially be less than:

θ_x	3.0 min
θ_y	3.0
θ_z	3.0

The navigation system position and velocity errors during powered flight should be maintained less than:

(excluding initialization errors)

Position (all axes)	2 Km
Velocity (all axes)	5 m/sec

Guidance placement accuracy is bound by the following limitations:

	Position	Velocity
Geosynchronous Orbit	50 Km	10 m/s
Low Earth Orbit	10 Km	10 m/s

The operation of the navigation system should not place any constraints on the orientation of the vehicle.

APPENDIX D

SYMBOLS AND DEFINITIONS OF COMPUTER VARIABLES IN THE GSP LISTING

On the following pages are a listing of the computer variable names used in GSP and the corresponding variable names implied in the IGM guidance law. It is included to help in the understanding and use of the simulator.

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DLDX	$\partial L/\partial X$	DQDXD	$\partial Q/\partial \dot{X}$
DLDY	$\partial L/\partial Y$	DQDYD	$\partial Q/\partial \dot{Y}$
DLDZ	$\partial L/\partial Z$	DQDZD	$\partial Q/\partial \dot{Z}$
DLDXD	$\partial L/\partial \dot{X}$	DUDX	$\partial U/\partial X$
DLDYD	$\partial L/\partial \dot{Y}$	DUDY	$\partial U/\partial Y$
DLDZD	$\partial L/\partial \dot{Z}$	DUDZ	$\partial U/\partial Z$
DJDX	$\partial J/\partial X$	DUDXD	$\partial U/\partial \dot{X}$
DJDY	$\partial J/\partial Y$	DUDYD	$\partial U/\partial \dot{Y}$
DJDZ	$\partial J/\partial Z$	DUDZD	$\partial U/\partial \dot{Z}$
DJDXD	$\partial J/\partial \dot{X}$	DAYDX	$\partial A_Y/\partial X$
DJDYD	$\partial J/\partial \dot{Y}$	DAYDY	$\partial A_Y/\partial Y$
DJDZD	$\partial J/\partial \dot{Z}$	DAYDZ	$\partial A_Y/\partial Z$
DPDX	$\partial P/\partial X$	DAYDXD	$\partial A_Y/\partial \dot{X}$
DPDY	$\partial P/\partial Y$	DAYDYD	$\partial A_Y/\partial \dot{Y}$
DPDZ	$\partial P/\partial Z$	DAYDZD	$\partial A_Y/\partial \dot{Z}$
DPDXD	$\partial P/\partial \dot{X}$	DBYDX	$\partial B_Y/\partial X$
DPDYD	$\partial P/\partial \dot{Y}$	DBYDY	$\partial B_Y/\partial Y$
DPDZD	$\partial P/\partial \dot{Z}$	DBYDZ	$\partial B_Y/\partial Z$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DSDX	$\partial S/\partial X$	DBYDXD	$\partial B_Y/\partial \dot{X}$
DSDY	$\partial S/\partial Y$	DBYDYD	$\partial B_Y/\partial \dot{Y}$
DSDZ	$\partial S/\partial Z$	DBYDZD	$\partial B_Y/\partial \dot{Z}$
DSDXD	$\partial S/\partial \dot{X}$	DETAX	$\partial \eta/\partial X$
DSDYD	$\partial S/\partial \dot{Y}$	DETAY	$\partial \eta/\partial Y$
DSDZD	$\partial S/\partial \dot{Z}$	DETAZ	$\partial \eta/\partial Z$
DQDX	$\partial Q/\partial X$	DETAXD	$\partial \eta/\partial \dot{X}$
DQDY	$\partial Q/\partial Y$	DETAYD	$\partial \eta/\partial \dot{Y}$
DQDZ	$\partial Q/\partial Z$	DETAZD	$\partial \eta/\partial \dot{Z}$
ETADX	$\partial \dot{\eta}/\partial X$	ZTADXD	$\partial \dot{\zeta}/\partial \dot{X}$
ETADY	$\partial \dot{\eta}/\partial Y$	ZTADYD	$\partial \dot{\zeta}/\partial \dot{Y}$
ETADZ	$\partial \dot{\eta}/\partial Z$	ZTADZD	$\partial \dot{\zeta}/\partial \dot{Z}$
ETADXD	$\partial \dot{\eta}/\partial \dot{X}$	XDDGX	$\partial \ddot{X}_G/\partial X$
ETADYD	$\partial \dot{\eta}/\partial \dot{Y}$	XDDGY	$\partial \ddot{X}_G/\partial Y$
ETADZD	$\partial \dot{\eta}/\partial \dot{Z}$	XDDGZ	$\partial \ddot{X}_G/\partial Z$
DXIDX	$\partial \xi/\partial X$	XDDGXD	$\partial \ddot{X}_G/\partial \dot{X}$
DXIDY	$\partial \xi/\partial Y$	XDDGYD	$\partial \ddot{X}_G/\partial \dot{Y}$
DXIDZ	$\partial \xi/\partial Z$	XDDGZD	$\partial \ddot{X}_G/\partial \dot{Z}$
DXIDXD	$\partial \xi/\partial \dot{X}$	YDDGX	$\partial \ddot{Y}_G/\partial X$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DXIDYD	$\partial \xi / \partial \dot{Y}$	YDDGY	$\partial \ddot{Y}_G / \partial Y$
DXIDZD	$\partial \xi / \partial \dot{Z}$	YDDGZ	$\partial \ddot{Y}_G / \partial Z$
XIDDX	$\partial \dot{\xi} / \partial X$	YDDGXD	$\partial \ddot{Y}_G / \partial \dot{X}$
XIDDY	$\partial \dot{\xi} / \partial Y$	YDDGYD	$\partial \ddot{Y}_G / \partial \dot{Y}$
XIDDZ	$\partial \dot{\xi} / \partial Z$	YDDGZD	$\partial \ddot{Y}_G / \partial \dot{Z}$
XIDDXD	$\partial \dot{\xi} / \partial \dot{X}$	ZDDGX	$\partial \ddot{Z}_G / \partial X$
XIDDYD	$\partial \dot{\xi} / \partial \dot{Y}$	ZDDGY	$\partial \ddot{Z}_G / \partial Y$
XIDDZD	$\partial \dot{\xi} / \partial \dot{Z}$	ZDDGZ	$\partial \ddot{Z}_G / \partial Z$
ZTAX	$\partial \zeta / \partial X$	ZDDGXD	$\partial \ddot{Z}_G / \partial \dot{X}$
ZTAY	$\partial \zeta / \partial Y$	ZDDGYD	$\partial \ddot{Z}_G / \partial \dot{Y}$
ZTAZ	$\partial \zeta / \partial Z$	ZDDGZD	$\partial \ddot{Z}_G / \partial \dot{Z}$
ZTAXD	$\partial \zeta / \partial \dot{X}$	ETDGX	$\partial \ddot{\eta}_g / \partial X$
ZTAYD	$\partial \zeta / \partial \dot{Y}$	ETDGY	$\partial \ddot{\eta}_g / \partial Y$
ZTAZD	$\partial \zeta / \partial \dot{Z}$	ETDGZ	$\partial \ddot{\eta}_g / \partial Z$
ZTADX	$\partial \dot{\zeta} / \partial X$	ETDGXD	$\partial \ddot{\eta}_g / \partial \dot{X}$
ZTADY	$\partial \dot{\zeta} / \partial Y$	ETDGYD	$\partial \ddot{\eta}_g / \partial \dot{Y}$
ZTADZ	$\partial \dot{\zeta} / \partial Z$	ETDGZD	$\partial \ddot{\eta}_g / \partial \dot{Z}$
XIDGX	$\partial \ddot{\xi}_g / \partial X$	DRDXD	$\partial R / \partial \dot{X}$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
XIDGY	$\partial\ddot{\xi}_g/\partial Y$	DRDYD	$\partial R/\partial\dot{Y}$
XIDGZ	$\partial\ddot{\xi}_g/\partial Z$	DRDZD	$\partial R/\partial\dot{Z}$
XIDGXD	$\partial\ddot{\xi}_g/\partial\dot{X}$	XLRVX	$\partial(\frac{X\cdot\dot{X}}{RV})/\partial X$
XIDGYD	$\partial\ddot{\xi}_g/\partial\dot{Y}$	XLRVY	$\partial(\frac{X\cdot\dot{X}}{RV})/\partial Y$
XIDGZD	$\partial\ddot{\xi}_g/\partial\dot{Z}$	XLRVZ	$\partial(\frac{X\cdot\dot{X}}{RV})/\partial Z$
ZTDGX	$\partial\ddot{\zeta}_g/\partial X$	XLRVXD	$\partial(\frac{X\cdot X}{RV})/\partial\dot{X}$
ZTDGY	$\partial\ddot{\zeta}_g/\partial Y$	XLRVYD	$\partial(\frac{X\cdot\dot{X}}{RV})/\partial\dot{Y}$
ZTDGZ	$\partial\ddot{\zeta}_g/\partial Z$	XLRVZD	$\partial(\frac{X\cdot\dot{X}}{RV})/\partial\dot{Z}$
ZTDGXD	$\partial\ddot{\zeta}_g/\partial\dot{X}$	DGAMX	$\partial\gamma/\partial X$
ZTDGYD	$\partial\ddot{\zeta}_g/\partial\dot{Y}$	DGAMY	$\partial\gamma/\partial Y$
ZTDGZD	$\partial\ddot{\zeta}_g/\partial\dot{Z}$	DGAMZ	$\partial\gamma/\partial Z$
ETDTX	$\partial\dot{\eta}_T/\partial X$	DGAMXD	$\partial\gamma/\partial\dot{X}$
ETDTY	$\partial\dot{\eta}_T/\partial Y$	DGAMYD	$\partial\gamma/\partial\dot{Y}$
ETDTZ	$\partial\dot{\eta}_T/\partial Z$	DGAMZD	$\partial\gamma/\partial\dot{Z}$
ETDTXD	$\partial\dot{\eta}_T/\partial\dot{X}$	DXFDX	$\partial X_F/\partial X$
ETDTYD	$\partial\dot{\eta}_T/\partial\dot{Y}$	DXFDY	$\partial X_F/\partial Y$
ETDTZD	$\partial\dot{\eta}_T/\partial\dot{Z}$	DXFDZ	$\partial X_F/\partial Z$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DVDX	$\partial V/\partial X$	DXFDXD	$\partial X_F/\partial \dot{X}$
DVDY	$\partial V/\partial Y$	DXFDYD	$\partial X_F/\partial \dot{Y}$
DVDZ	$\partial V/\partial Z$	DXFDZD	$\partial X_F/\partial \dot{Z}$
DVDXD	$\partial V/\partial \dot{X}$	DZLXX	$\partial(\frac{Z_F}{X_F})/\partial X$
DVDYD	$\partial V/\partial \dot{Y}$	DZLXY	$\partial(\frac{Z_F}{X_F})/\partial Y$
DVDZD	$\partial V/\partial \dot{Z}$	DZLXZ	$\partial(\frac{Z_F}{X_F})/\partial Z$
DRDX	$\partial R/\partial X$	DZLXXD	$\partial(\frac{Z_F}{X_F})/\partial \dot{X}$
DRDY	$\partial R/\partial Y$	DZLXYD	$\partial(\frac{Z_F}{X_F})/\partial \dot{Y}$
DRDZ	$\partial R/\partial Z$	DZDXZD	$\partial(\frac{Z_F}{X_F})/\partial \dot{Z}$
DFETX	$\partial \phi_{ET}/\partial X$	DRTXD	$\partial R_T/\partial \dot{X}$
DFETY	$\partial \phi_{ET}/\partial Y$	DRTYD	$\partial R_T/\partial \dot{Y}$
DFETZ	$\partial \phi_{ET}/\partial Z$	DRTZD	$\partial R_T/\partial \dot{Z}$
DFETXD	$\partial \phi_{ET}/\partial \dot{X}$	DGMTX	$\partial \gamma_T/\partial X$
DFETYD	$\partial \phi_{ET}/\partial \dot{Y}$	DGMTY	$\partial \gamma_T/\partial Y$
DFETZD	$\partial \phi_{ET}/\partial \dot{Z}$	DGMTZ	$\partial \gamma_T/\partial Z$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DETX	$\partial E_T/\partial X$	DGMTXD	$\partial \gamma_T/\partial \dot{X}$
DETY	$\partial E_T/\partial Y$	DGMTYD	$\partial \gamma_T/\partial \dot{Y}$
DETZ	$\partial E_T/\partial Z$	DGMTZD	$\partial \gamma_T/\partial \dot{Z}$
DETXD	$\partial E_T/\partial \dot{X}$	DVTX	$\partial V_T/\partial X$
DETYD	$\partial E_T/\partial \dot{Y}$	DVTY	$\partial V_T/\partial Y$
DETZD	$\partial E_T/\partial \dot{Z}$	DVTZ	$\partial V_T/\partial Z$
DFK6X	$\partial FK6/\partial X$	DVTXD	$\partial V_T/\partial \dot{X}$
DFK6Y	$\partial FK6/\partial Y$	DVTYD	$\partial V_T/\partial \dot{Y}$
DFK6Z	$\partial FK6/\partial Z$	DVTZD	$\partial V_T/\partial \dot{Z}$
DFK6XD	$\partial FK6/\partial \dot{X}$	DGTX	$\partial G_T/\partial X$
DFK6YD	$\partial FK6/\partial \dot{Y}$	DGTY	$\partial G_T/\partial Y$
DFK6ZD	$\partial FK6/\partial \dot{Z}$	DGTZ	$\partial G_T/\partial Z$
DFTX	$\partial \phi_T/\partial X$	DGTXD	$\partial G_T/\partial \dot{X}$
DFTY	$\partial \phi_T/\partial Y$	DGTYD	$\partial G_T/\partial \dot{Y}$
DFTZ	$\partial \phi_T/\partial Z$	DGTZD	$\partial G_T/\partial \dot{Z}$
DFTXD	$\partial \phi_T/\partial \dot{X}$	XIDTX	$\partial \dot{\xi}_T/\partial X$
DFTYD	$\partial \phi_T/\partial \dot{Y}$	XIDTY	$\partial \dot{\xi}_T/\partial Y$
DFTZD	$\partial \phi_T/\partial \dot{Z}$	XIDTZ	$\partial \dot{\xi}_T/\partial Z$

SYMBOL	DEFINITION
DRTX	$\partial R_T/\partial X$
DRTY	$\partial R_T/\partial Y$
DRTZ	$\partial R_T/\partial Z$
ZTDTX	$\partial \dot{\zeta}_T/\partial X$
ZTDTY	$\partial \dot{\zeta}_T/\partial Y$
ZTDTZ	$\partial \dot{\zeta}_T/\partial Z$
ZTDTXD	$\partial \dot{\zeta}_T/\partial \dot{X}$
ZTDTYD	$\partial \dot{\zeta}_T/\partial \dot{Y}$
ZTDTZD	$\partial \dot{\zeta}_T/\partial \dot{Z}$
XIGTX	$\partial \ddot{\xi}_{GT}/\partial X$
XIGTY	$\partial \ddot{\xi}_{GT}/\partial Y$
XIGTZ	$\partial \ddot{\xi}_{GT}/\partial Z$
XIGTXD	$\partial \ddot{\xi}_{GT}/\partial \dot{X}$
XIGTYD	$\partial \ddot{\xi}_{GT}/\partial \dot{Y}$
XIGTZD	$\partial \ddot{\xi}_{GT}/\partial \dot{Z}$
XIDSX	$\partial \Delta\dot{\xi}^*/\partial X$
XIDSY	$\partial \Delta\dot{\xi}^*/\partial Y$
XIDSZ	$\partial \Delta\dot{\xi}^*/\partial Z$
XIDTXD	$\partial \dot{\xi}_T/\partial \dot{X}$
XIDTYD	$\partial \dot{\xi}_T/\partial \dot{Y}$
XIDTZD	$\partial \dot{\xi}_T/\partial \dot{Z}$
ZTDSXD	$\partial \Delta\dot{\zeta}^*/\partial \dot{X}$
ZTDSYD	$\partial \Delta\dot{\zeta}^*/\partial \dot{Y}$
ZTDSZD	$\partial \Delta\dot{\zeta}^*/\partial \dot{Z}$
DGDX	$\partial G/\partial X$
DGDY	$\partial G/\partial Y$
DGDZ	$\partial G/\partial Z$
DGDXD	$\partial G/\partial \dot{X}$
DGDYD	$\partial G/\partial \dot{Y}$
DGDZD	$\partial G/\partial \dot{Z}$
DDTDX	$\partial \Delta T/\partial X$
DDTDY	$\partial \Delta T/\partial Y$
DDTDZ	$\partial \Delta T/\partial Z$
DDTDXD	$\partial \Delta T/\partial \dot{X}$
DDTDYD	$\partial \Delta T/\partial \dot{Y}$
DDTDZD	$\partial \Delta T/\partial \dot{Z}$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
XIDSXD	$\partial\Delta\dot{\xi}^*/\partial\dot{X}$	LXIDX	$\partial\dot{\xi}/\partial X$
XIDSYD	$\partial\Delta\dot{\xi}^*/\partial\dot{Y}$	LXIDY	$\partial\dot{\xi}/\partial Y$
XIDSZD	$\partial\Delta\dot{\xi}^*/\partial\dot{Z}$	LXIDZ	$\partial\dot{\xi}/\partial Z$
ETDSX	$\partial\Delta\dot{\eta}^*/\partial X$	LXIDXD	$\partial\dot{\xi}/\partial\dot{X}$
ETDSY	$\partial\Delta\dot{\eta}^*/\partial Y$	LXIDYD	$\partial\dot{\xi}/\partial\dot{Y}$
ETDSZ	$\partial\Delta\dot{\eta}^*/\partial Z$	LXIDZD	$\partial\dot{\xi}/\partial\dot{Z}$
ETDSXD	$\partial\Delta\dot{\eta}^*/\partial\dot{X}$	LETDX	$\partial\Delta\dot{\eta}/\partial X$
ETDSYD	$\partial\Delta\dot{\eta}^*/\partial\dot{Y}$	LETDY	$\partial\Delta\dot{\eta}/\partial Y$
ETDSZD	$\partial\Delta\dot{\eta}^*/\partial\dot{Z}$	LETDZ	$\partial\Delta\dot{\eta}/\partial Z$
ZTDSX	$\partial\Delta\dot{\zeta}^*/\partial X$	LETDXD	$\partial\Delta\dot{\eta}/\partial\dot{X}$
ZTDSY	$\partial\Delta\dot{\zeta}^*/\partial Y$	LETDYD	$\partial\Delta\dot{\eta}/\partial\dot{Y}$
ZTDSZ	$\partial\Delta\dot{\zeta}^*/\partial Z$	LETDZD	$\partial\Delta\dot{\eta}/\partial\dot{Z}$
LZTDX	$\partial\Delta\dot{\zeta}/\partial X$	DDYDXD	$\partial D_Y/\partial\dot{X}$
LZTDY	$\partial\Delta\dot{\zeta}/\partial Y$	DDYDYD	$\partial D_Y/\partial\dot{Y}$
LZTDZ	$\partial\Delta\dot{\zeta}/\partial Z$	DDYDZD	$\partial D_Y/\partial\dot{Z}$
LZTDXD	$\partial\Delta\dot{\zeta}/\partial\dot{X}$	DEYDX	$\partial E_Y/\partial X$
LZTDYD	$\partial\Delta\dot{\zeta}/\partial\dot{Y}$	DEYDY	$\partial E_Y/\partial Y$
LZTDZD	$\partial\Delta\dot{\zeta}/\partial\dot{Z}$	DEYDZ	$\partial E_Y/\partial Z$

SYMBOL	DEFINITION
KIPTX	$\partial \tilde{x}_P / \partial X$
KIPTY	$\partial \tilde{x}_P / \partial Y$
KIPTZ	$\partial \tilde{x}_P / \partial Z$
KIPTXD	$\partial \tilde{x}_P / \partial \dot{X}$
KIPTYD	$\partial \tilde{x}_P / \partial \dot{Y}$
KIPTZD	$\partial \tilde{x}_P / \partial \dot{Z}$
KIYTX	$\partial \tilde{x}_Y / \partial X$
KIYTY	$\partial \tilde{x}_Y / \partial Y$
KIYTZ	$\partial \tilde{x}_Y / \partial Z$
KIYTXD	$\partial \tilde{x}_Y / \partial \dot{X}$
KIYTYD	$\partial \tilde{x}_Y / \partial \dot{Y}$
KIYTZD	$\partial \tilde{x}_Y / \partial \dot{Z}$
DCYDX	$\partial C_Y / \partial X$
DCYDY	$\partial C_Y / \partial Y$
DCYDZ	$\partial C_Y / \partial Z$
DCYDXD	$\partial C_Y / \partial \dot{X}$
DCYDYD	$\partial C_Y / \partial \dot{Y}$
DCYDZD	$\partial C_Y / \partial \dot{Z}$

SYMBOL	DEFINITION
DEYDXD	$\partial E_Y / \partial \dot{X}$
DEYDYD	$\partial E_Y / \partial \dot{Y}$
DEYDZD	$\partial E_Y / \partial \dot{Z}$
DK3DX	$\partial K_3 / \partial X$
DK3DY	$\partial K_3 / \partial Y$
DK3DZ	$\partial K_3 / \partial Z$
DK3DXD	$\partial K_3 / \partial \dot{X}$
DK3DYD	$\partial K_3 / \partial \dot{Y}$
DK3DZD	$\partial K_3 / \partial \dot{Z}$
DK4DX	$\partial K_4 / \partial X$
DK4DY	$\partial K_4 / \partial Y$
DK4DZ	$\partial K_4 / \partial Z$
DK4DXD	$\partial K_4 / \partial \dot{X}$
DK4DYD	$\partial K_4 / \partial \dot{Y}$
DK4DZD	$\partial K_4 / \partial \dot{Z}$
DMDX	$\partial M / \partial X$
DMDY	$\partial M / \partial Y$
DMDZ	$\partial M / \partial Z$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DDYDX	$\partial D_Y/\partial X$	DMDXD	$\partial M/\partial \dot{X}$
DDYDY	$\partial D_Y/\partial Y$	DMDYD	$\partial M/\partial \dot{Y}$
DDYDZ	$\partial D_Y/\partial Z$	DMDZD	$\partial M/\partial \dot{Z}$
DNDX	$\partial N/\partial X$	DDPDXD	$\partial D_P/\partial \dot{X}$
DNDY	$\partial N/\partial Y$	DDPDYD	$\partial D_P/\partial \dot{Y}$
DNDZ	$\partial N/\partial Z$	DDPDZD	$\partial D_P/\partial \dot{Z}$
DNDXD	$\partial N/\partial \dot{X}$	DEPDX	$\partial E_P/\partial X$
DNDYD	$\partial N/\partial \dot{Y}$	DEPDY	$\partial E_P/\partial Y$
DNDZD	$\partial N/\partial \dot{Z}$	DEPDZ	$\partial E_P/\partial Z$
DAPDX	$\partial A_P/\partial X$	DEPDXD	$\partial E_P/\partial \dot{X}$
DAPDY	$\partial A_P/\partial Y$	DEPDYD	$\partial E_P/\partial \dot{Y}$
DAPDZ	$\partial A_P/\partial Z$	DEPDZD	$\partial E_P/\partial \dot{Z}$
DAPDXD	$\partial A_P/\partial \dot{X}$	DK1DX	$\partial K_1/\partial X$
DAPDYD	$\partial A_P/\partial \dot{Y}$	DK1DY	$\partial K_1/\partial Y$
DAPDZD	$\partial A_P/\partial \dot{Z}$	DK1DZ	$\partial K_1/\partial Z$
DBPDX	$\partial B_P/\partial X$	DK1DXD	$\partial K_1/\partial \dot{X}$
DBPDY	$\partial B_P/\partial Y$	DK1DYD	$\partial K_1/\partial \dot{Y}$
DBPDZ	$\partial B_P/\partial Z$	DK1DZD	$\partial K_1/\partial \dot{Z}$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DBPDXD	$\partial B_p/\partial \dot{X}$	DK2DX	$\partial K_2/\partial X$
DBPDYD	$\partial B_p/\partial \dot{Y}$	DK2DY	$\partial K_2/\partial Y$
DBPDZD	$\partial B_p/\partial \dot{Z}$	DK2DZ	$\partial K_2/\partial Z$
DCPDX	$\partial C_p/\partial X$	DK2DXD	$\partial K_2/\partial \dot{X}$
DCPDY	$\partial C_p/\partial Y$	DK2DYD	$\partial K_2/\partial \dot{Y}$
DCPDZ	$\partial C_p/\partial Z$	DK2DZD	$\partial K_2/\partial \dot{Z}$
DCPDXD	$\partial C_p/\partial \dot{X}$	KIPPX	$\partial \chi'_p/\partial X$
DCPDYD	$\partial C_p/\partial \dot{Y}$	KIPPY	$\partial \chi'_p/\partial Y$
DCPDZD	$\partial C_p/\partial \dot{Z}$	KIPPZ	$\partial \chi'_p/\partial Z$
DDPDX	$\partial D_p/\partial X$	KIPPXD	$\partial \chi'_p/\partial \dot{X}$
DDPDY	$\partial D_p/\partial Y$	KIPPYD	$\partial \chi'_p/\partial \dot{Y}$
DDPDZ	$\partial D_p/\partial Z$	KIPPZD	$\partial \chi'_p/\partial \dot{Z}$
KIYPX	$\partial \chi'_Y/\partial X$	DYFDXD	$\partial Y_F/\partial \dot{X}$
KIYPY	$\partial \chi'_Y/\partial Y$	DYFDYD	$\partial Y_F/\partial \dot{Y}$
KIYPZ	$\partial \chi'_Y/\partial Z$	DYFDZD	$\partial Y_F/\partial \dot{Z}$
KIYPXD	$\partial \chi'_Y/\partial \dot{X}$	DTHTX	$\partial \theta_T/\partial X$
KIYPYD	$\partial \chi'_Y/\partial \dot{Y}$	DTHTY	$\partial \theta_T/\partial Y$
KIYPZD	$\partial \chi'_Y/\partial \dot{Z}$	DTHTZ	$\partial \theta_T/\partial Z$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
ETGTX	$\partial \ddot{\eta}_{GT}/\partial X$	DTHTXD	$\partial \theta_T/\partial \dot{X}$
ETGTY	$\partial \ddot{\eta}_{GT}/\partial Y$	DTHTYD	$\partial \theta_T/\partial \dot{Y}$
ETGTZ	$\partial \ddot{\eta}_{GT}/\partial Z$	DTHTZD	$\partial \theta_T/\partial \dot{Z}$
ETGTXD	$\partial \ddot{\eta}_{GT}/\partial \dot{X}$	TAU	τ
ETGTYD	$\partial \ddot{\eta}_{GT}/\partial \dot{Y}$	TG	T_g
ETGTZD	$\partial \ddot{\eta}_{GT}/\partial \dot{Z}$	J	J
TGDX	$\partial T_g/\partial X$	L(FL)	L
TGDY	$\partial T_g/\partial Y$	P	P
TGDZ	$\partial T_g/\partial Z$	S	S
TGDXD	$\partial T_g/\partial \dot{X}$	QC(Q)	Q
TGDYD	$\partial T_g/\partial \dot{Y}$	U(UU)	U
TGDZD	$\partial T_g/\partial \dot{Z}$	AY	A_Y
DZFDX	$\partial Z_F/\partial X$	BY	B_Y
DZFDY	$\partial Z_F/\partial Y$	ETA(XI(21))	η
DZFDZ	$\partial Z_F/\partial Z$	ETAD(XID(2))	$\dot{\eta}$
DZFDXD	$\partial Z_F/\partial \dot{X}$	XI(XI(1))	ξ
DZFDYD	$\partial Z_F/\partial \dot{Y}$	XID(XID(1))	$\dot{\xi}$
DZFDZD	$\partial Z_F/\partial \dot{Z}$	ZTA(XI(3))	ζ

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DYFDX	$\partial Y_F/\partial X$	ZTAD(XID(3))	$\dot{\zeta}$
DYFDY	$\partial Y_F/\partial Y$	* XDDG(GX(1))	$\ddot{X}_G$
DYFDZ	$\partial Y_F/\partial Z$	* YDDG(GX(2))	$\ddot{Y}_G$
* ZDDG(GX(3))	$\ddot{Z}_G$	LXID(DXID(1))	$\Delta\dot{\xi}$
ETDDG(XIDD(2))	$\ddot{\eta}_G$	LETD(DXID(2))	$\Delta\dot{\eta}$
XIDDG(XIDD(1))	$\ddot{\xi}_G$	LZTD(DXID(3))	$\Delta\dot{\zeta}$
ZTDDG(XIDD(3))	$\ddot{\zeta}_G$	KIPT(CHITP)	$\tilde{\chi}_P$
ETDT(XIDT(2))	$\dot{\eta}_T$	VEX	V_{EX}
V	V	KIYT(CHITY)	$\tilde{\chi}_Y$
R	R	CY	C_Y
XDLRV	$(X \cdot \dot{X}/RV)$	DY	D_Y
GAM	γ	EY	E_Y
ZFLXF	(Z_F/X_F)	K4(FK4)	K4
FET	ϕ_{ET}	M(AM)	M
ET(E2)	E_T	N(AN)	N
FK6	FK6	AP	A_P
FT	ϕ_T	BP	B_P
RT	R_T	CP	C_P
		K3(FK3)	K_3

SYMBOL	DEFINITION	SYMBOL	DEFINITION
GAMT(THT)	γ_T	DP	D_P
VT	V_T	EP	E_P
GT	G_T	K1(FK1)	K_1
XIDT(XIDT(1))	$\dot{\xi}_T$	K2(FK2)	K_2
ZTDT(XIDT(3))	$\dot{\zeta}_T$	KIPP(CHIPP)	χ'_P
XIDDGT(XIDDT(1))	$\ddot{\xi}_{GT}$	KIYP(CHIPY)	χ'_Y
LXIDS(DXIDS(1))	$\Delta\dot{\xi}^*$	ETDDGT(XIDDT(2))	$\ddot{\eta}_{GT}$
LETDS(DXIDS(2))	$\Delta\dot{\eta}^*$	XF(XF(1))	X_F
LZTDS(DXIDS(3))	$\Delta\dot{\zeta}^*$	YF(XF(2))	Y_F
G(GG)	G	ZF(XF(3))	Z_F
DLT(DTN)	ΔT	THET	Θ_T
MU(U)(GM)	μ	DK22X	$\partial K_{22}/\partial X$
PT(PO2)	P_T	DK22Y	$\partial K_{22}/\partial Y$
XIT(XIT(1))	ξ_T	DK22Z	$\partial K_{22}/\partial Z$
DK11X	$\partial K_{11}/\partial X$	DK22XD	$\partial K_{22}/\partial \dot{X}$
DK11Y	$\partial K_{11}/\partial Y$	DK22YD	$\partial K_{22}/\partial \dot{Y}$
DK11Z	$\partial K_{11}/\partial Z$	DK22ZD	$\partial K_{22}/\partial \dot{Z}$
DK11XD	$\partial K_{11}/\partial \dot{X}$	DK23X	$\partial K_{23}/\partial X$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DK21XD	$\partial K_{21}/\partial \dot{X}$	DK33X	$\partial K_{33}/\partial X$
DK21YD	$\partial K_{21}/\partial \dot{Y}$	DK33Y	$\partial K_{33}/\partial Y$
DK21ZD	$\partial K_{21}/\partial \dot{Z}$	DK33Z	$\partial K_{33}/\partial Z$
DK33XD	$\partial K_{33}/\partial \dot{X}$	DEQDX	$\partial EQ/\partial X$
DK33YD	$\partial K_{33}/\partial \dot{Y}$	DEQDY	$\partial EQ/\partial Y$
DK33ZD	$\partial K_{33}/\partial \dot{Z}$	DEQDZ	$\partial EQ/\partial Z$
CKINV	$K^{-1} = C$	DEQDXD	$\partial EQ/\partial \dot{X}$
ERAD	EARTH'S RADIUS	DEQDYD	$\partial EQ/\partial \dot{Y}$
DCPHX	$\partial(\text{CSC } \phi)/\partial X$	DEQDZD	$\partial EQ/\partial \dot{Z}$
DCPHY	$\partial(\text{CSC } \phi)/\partial Y$	EQS	$(EQ)^2$
DCPHZ	$\partial(\text{CSC } \phi)/\partial Z$	CINT	DUMMY VARIABLE = $(K^{-1})(EQ)$
DCPHXD	$\partial(\text{CSC } \phi)/\partial \dot{X}$	DC11X	$\partial C_{11}/\partial X$
DCPHYD	$\partial(\text{CSC } \phi)/\partial \dot{Y}$	DC11Y	$\partial C_{11}/\partial Y$
DCPHZD	$\partial(\text{CSC } \phi)/\partial \dot{Z}$	DC11Z	$\partial C_{11}/\partial Z$
DAL2X	$\partial(\text{COS } \alpha_2)/\partial X$	DC11XD	$\partial C_{11}/\partial \dot{X}$
DAL2Y	$\partial(\text{COS } \alpha_2)/\partial Y$	DC11YD	$\partial C_{11}/\partial \dot{Y}$
DAL2Z	$\partial(\text{COS } \alpha_2)/\partial Z$	DC11ZD	$\partial C_{11}/\partial \dot{Z}$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DAL2XD	$\partial(\cos \alpha_2)/\partial\dot{X}$	DC12X	$\partial C_{12}/\partial X$
DAL2YD	$\partial(\cos \alpha_2)/\partial\dot{Y}$	DC12Y	$\partial C_{12}/\partial Y$
DAL2ZD	$\partial(\cos \alpha_2)/\partial\dot{Z}$	DC12Z	$\partial C_{12}/\partial Z$
DBT2X	$\partial(\cos \beta_2)/\partial X$	DC12XD	$\partial C_{12}/\partial\dot{X}$
DBT2Y	$\partial(\cos \beta_2)/\partial Y$	DC12YD	$\partial C_{12}/\partial\dot{Y}$
DBT2Z	$\partial(\cos \beta_2)/\partial Z$	DC12ZD	$\partial C_{12}/\partial\dot{Z}$
DBT2XD	$\partial(\cos \beta_2)/\partial\dot{X}$	DC13X	$\partial C_{13}/\partial X$
DBT2YD	$\partial(\cos \beta_2)/\partial\dot{Y}$	DC13Y	$\partial C_{13}/\partial Y$
DBT2ZD	$\partial(\cos \beta_2)/\partial\dot{Z}$	DC13Z	$\partial C_{13}/\partial Z$
XSUN	SUN	DC13XD	$\partial C_{13}/\partial\dot{X}$
YSUN	COORDINATES	DC13YD	$\partial C_{13}/\partial\dot{Y}$
ZSUN	IN ECLC	DC13ZD	$\partial C_{13}/\partial\dot{Z}$
EQ		DC21X	$\partial C_{21}/\partial X$
DC21Y	$\partial C_{21}/\partial Y$	DC32YD	$\partial C_{32}/\partial\dot{Y}$
DC21Z	$\partial C_{21}/\partial Z$	DC32ZD	$\partial C_{32}/\partial\dot{Z}$
DC21XD	$\partial C_{21}/\partial\dot{X}$	DC33X	$\partial C_{33}/\partial X$
DC21YD	$\partial C_{21}/\partial\dot{Y}$	DC33Y	$\partial C_{33}/\partial Y$
DC21ZD	$\partial C_{21}/\partial\dot{Z}$	DC33Z	$\partial C_{33}/\partial Z$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DC22X	$\partial C_{22}/\partial X$	DC33XD	$\partial C_{33}/\partial \dot{X}$
DC22Y	$\partial C_{22}/\partial Y$	DC33YD	$\partial C_{33}/\partial \dot{Y}$
DC22Z	$\partial C_{22}/\partial Z$	DC33ZD	$\partial C_{33}/\partial \dot{Z}$
DC22XD	$\partial C_{22}/\partial \dot{X}$	DDCXX	$\partial DCX/\partial X$
DC22YD	$\partial C_{22}/\partial \dot{Y}$	DDCXY	$\partial DCX/\partial Y$
DC22ZD	$\partial C_{22}/\partial \dot{Z}$	DDCXZ	$\partial DCX/\partial Z$
DC23X	$\partial C_{23}/\partial X$	DDCXXD	$\partial DCX/\partial \dot{X}$
DC23Y	$\partial C_{23}/\partial Y$	DDCXYD	$\partial DCX/\partial \dot{Y}$
DC23Z	$\partial C_{23}/\partial Z$	DDCXZD	$\partial DCX/\partial \dot{Z}$
DC23XD	$\partial C_{23}/\partial \dot{X}$	DDCYX	$\partial DCY/\partial X$
DC23YD	$\partial C_{23}/\partial \dot{Y}$	DDCYY	$\partial DCY/\partial Y$
DC23ZD	$\partial C_{23}/\partial \dot{Z}$	DDCYZ	$\partial DCY/\partial Z$
DC31X	$\partial C_{31}/\partial X$	DDCYXD	$\partial DCY/\partial \dot{X}$
DC31Y	$\partial C_{31}/\partial Y$	DDCYYD	$\partial DCY/\partial \dot{Y}$
DC31Z	$\partial C_{31}/\partial Z$	DDCYZD	$\partial DCY/\partial \dot{Z}$
DC31XD	$\partial C_{31}/\partial \dot{X}$	DDCZX	$\partial DCZ/\partial X$
DC31YD	$\partial C_{31}/\partial \dot{Y}$	DDCZY	$\partial DCZ/\partial Y$
DC31ZD	$\partial C_{31}/\partial \dot{Z}$	DDCZZ	$\partial DCZ/\partial Z$

SYMBOL	DEFINITION	SYMBOL	DEFINITION
DC32X	$\partial C_{32}/\partial X$	DDCZXD	$\partial DCZ/\partial \dot{X}$
DC32Y	$\partial C_{32}/\partial Y$	DDCZYD	$\partial DCZ/\partial \dot{Y}$
DC32Z	$\partial C_{32}/\partial Z$	DDCZZD	$\partial DCZ/\partial \dot{Z}$
DC32XD	$\partial C_{32}/\partial \dot{X}$		
ESTM	$T^{(1)}$	POLO	P(k/k)
XBT		P1L1	P(k+1/k+1)
YBT		P1LO	P(k+1/k)
ZBT		XHOLO	$\hat{X}$(k/k)
DNM1	LUMPED PARAMETER	XH1LO	$\hat{X}$(k+1/k)
DNM2	LUMPED PARAMETER	XH1L1	$\hat{X}$(k+1/k+1)
YEA	LUMPED PARAMETER	QK	Q(MATRIX)
YEB	LUMPED PARAMETER	HK1	H(MEASUREMENT MATRIX)
YEC	LUMPED PARAMETER	RK1	R(MATRIX)
YED	LUMPED PARAMETER	KK1	K(KALMAN GAIN MATRIX)
YEE	LUMPED PARAMETER	ZLHXH	$\tilde{Z}$
YEF	LUMPED PARAMETER	PHI1LO	
AZ	A_Z	FT(PHIT)	ϕ_T
PHIL	ϕ_L	X	X

SYMBOL	DEFINITIONS	SYMBOL	DEFINITIONS
PHIX		Y	Y
PHIY		Z	Z
PHIZ			
NUM	LUMPED PARAMETER	XD(DX(1))	$\dot{X}$
DEM	LUMPED PARAMETER	YD(DX(2))	$\dot{Y}$
CSCPHI	CSC(ϕ)	ZD(DX(3))	$\dot{Z}$
CALPH2	COS(α_2)	K(FK)	K(MATRIX)
CBET2	COS(β_2)	CHIPC	χ_P
ZK1	NOISY OBSERVATIONS	CHIYC	χ_Y
ZK2	PERFECT OBSERVATIONS	ENGACC	ENGINE ACCELERATION
WT		* XDDG(PARTL)	XGRV(1) (GRAV)
THRUST		* YDDG(PARTL)	XGRV(2) (GRAV)
		* ZDDG(PARTL)	XGRV(3) (GRAV)

ADDENDUM TO SPACE TUG REPORT CONTRACT NAS8-29852 PARTIAL DERIVATIVES FOR THE STATE TRANSITION MATRIX $\Phi(k+1, k)$ AND THE OBSERVATION MATRIX $H(k+1)$

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This Addendum is to supplement the report on the Space Tug under NASA contract #NAS8-29852. The purpose is not to present a complete derivation of the Generalized Simulation Package, (GSP), but to give insight to the approach in which the one-step propogation method was utilized in the IGM package to estimate the desired burn angles for the guidance routine.

A complete listing of all partial derivatives and lumped parameters used in the partial derivative routines of the GSP program are given. These equations are listed in their order of computation and should serve as a reference for translation of the program variable names to the notation of the IGM package.

This listing goes along with the GSP program listing, available on request from the Department of Electrical Engineering, Auburn University. Together they give an overall picture of the analytical approach taken in the Tug navigation study. In addition, a brief discussion and derivation showing a step-by-step process of the partial derivative computation for one of the terms needed in the burn angle estimations is given in the Appendix of the Space Tug report.

$$\frac{\partial T_g}{\partial x} = \left[\frac{T - T_g}{2 V_{ex}}\right]\left[\frac{E2}{1 - A6\left[\frac{T - T_g}{2 V_{ex}}\right] + \frac{G}{V_{ex}}}\right]$$

$$\frac{\partial T_g}{\partial y} = \frac{A7}{\left[\frac{2 V_{ex}}{T - T_g}\right] - A6 + \left[\frac{2G}{T - T_g}\right]}$$

$$\frac{\partial T_g}{\partial z} = \frac{B2}{\left[\frac{2 V_{ex}}{T - T_g}\right] - A6 + \left[\frac{2G}{T - T_g}\right]}$$

$$\frac{\partial T_g}{\partial \dot{x}} = \frac{NUM}{DEM}$$

$$\frac{\partial T_g}{\partial \dot{y}} = \left[\frac{T - T_g}{V_{ex}}\right]\left[\frac{-C1 - C2 + D1(DNUM1)}{DNUM2}\right]$$

$$\frac{\partial T_g}{\partial \dot{z}} = \left[\frac{T - T_g}{V_{ex}}\right]\left[\frac{-C3 - C4 + D2(DNUM1)}{DNUM2}\right]$$

$$\frac{\partial \Lambda T}{\partial x} = \frac{\partial T_g}{\partial x}$$

$$\frac{\partial \Lambda T}{\partial y} = \frac{\partial T_g}{\partial y}$$

$$\frac{\partial \Delta T}{\partial z} = \frac{\partial T_g}{\partial z}$$

$$\frac{\partial \Delta T}{\partial \dot{x}} = \frac{\partial T_g}{\partial \dot{x}}$$

$$\frac{\partial \Delta T}{\partial \dot{y}} = \frac{\partial T_g}{\partial \dot{y}}$$

$$\frac{\partial \Delta T}{\partial \dot{z}} = \frac{\partial T_g}{\partial \dot{z}}$$

$$\frac{\partial \phi_T}{\partial x} = \left[\frac{E1 + A5 \frac{\partial T_g}{\partial x}}{1 + A2 + A3 + A4} \right]$$

$$\frac{\partial \phi_T}{\partial y} = \left[\frac{A1 + A5 \frac{\partial T_g}{\partial y}}{1 + A2 + A3 + A4} \right]$$

$$\frac{\partial \phi_T}{\partial z} = \left[\frac{B1 + A5 \frac{\partial T_g}{\partial z}}{1 + A2 + A3 + A4} \right]$$

$$\frac{\partial \phi_T}{\partial \dot{x}} = \frac{T_g (1/2)(1 + FK6)(L2) + (L1) \frac{\partial T_g}{\partial \dot{x}}}{\left[1 + (\frac{1}{2}) T_g (1 + FK6)(L3) \right]}$$

$$\frac{\partial \phi_T}{\partial \dot{y}} = \frac{T_g (1/2)(1 + FK6)(D1) + (L1) \frac{\partial T_g}{\partial \dot{y}}}{\left[1 + (1/2) T_g (1 + FK6)(L3) \right]}$$

$$\frac{\partial \phi_T}{\partial \dot{z}} = \frac{T_g (1/2)(1 + FK6)(D2) + (L1) \frac{\partial T_g}{\partial \dot{z}}}{\left[1 + (1/2) T_g (1 + FK6)(L3) \right]}$$

$$\frac{\partial \theta_T}{\partial x} = \frac{\partial \phi_T}{\partial x}$$

$$\frac{\partial \theta_T}{\partial y} = \frac{\partial \phi_T}{\partial y}$$

$$\frac{\partial \theta_T}{\partial z} = \frac{\partial \phi_T}{\partial z}$$

$$\frac{\partial \theta_T}{\partial \dot{x}} = \frac{\partial \phi_T}{\partial \dot{x}}$$

$$\frac{\partial \theta_T}{\partial \dot{y}} = \frac{\partial \phi_T}{\partial \dot{y}}$$

$$\frac{\partial \theta_T}{\partial \dot{z}} = \frac{\partial \phi_T}{\partial \dot{z}}$$

$$\frac{\partial L}{\partial x} = \left[\frac{1}{\tau - T_g}\right] \frac{\partial T_g}{\partial x}$$

$$\frac{\partial L}{\partial y} = \left[\frac{1}{\tau - T_g}\right] \frac{\partial T_g}{\partial y}$$

$$\frac{\partial L}{\partial z} = \left[\frac{1}{\tau - T_g}\right] \frac{\partial T_g}{\partial z}$$

$$\frac{\partial L}{\partial \dot{x}} = \left[\frac{1}{\tau - T_g}\right] \frac{\partial T_g}{\partial \dot{x}}$$

$$\frac{\partial L}{\partial \dot{y}} = \left[\frac{1}{\tau - T_g}\right] \frac{\partial T_g}{\partial \dot{y}}$$

$$\frac{\partial L}{\partial \dot{z}} = \left[\frac{1}{\tau - T_g}\right] \frac{\partial T_g}{\partial \dot{z}}$$

$$\frac{\partial J}{\partial x} = V_{ex}\left\{\frac{\partial T_g}{\partial x}\left[\frac{\tau}{\tau - T_g}\right] - \frac{\partial T_g}{\partial x}\right\}$$

$$\frac{\partial J}{\partial y} = V_{ex}\left[\tau \frac{\partial L}{\partial y} - \frac{\partial T_g}{\partial y}\right]$$

$$\frac{\partial J}{\partial z} = V_{ex}\left[\tau \frac{\partial L}{\partial z} - \frac{\partial T_g}{\partial z}\right]$$

$$\frac{\partial J}{\partial \dot{x}} = V_{ex}\left[\Upsilon \frac{\partial L}{\partial \dot{x}} - \frac{\partial T_g}{\partial \dot{x}} \right]$$

$$\frac{\partial J}{\partial \dot{y}} = V_{ex}\left[\Upsilon \frac{\partial L}{\partial \dot{y}} - \frac{\partial T_g}{\partial \dot{y}} \right]$$

$$\frac{\partial J}{\partial \dot{z}} = V_{ex}\left[\Upsilon \frac{\partial L}{\partial \dot{z}} - \frac{\partial T_g}{\partial \dot{z}} \right]$$

$$\frac{\partial P}{\partial x} = -V_{ex} T_g \frac{\partial T_g}{\partial x} + \Upsilon \frac{\partial J}{\partial x}$$

$$\frac{\partial P}{\partial y} = -V_{ex} T_g \frac{\partial T_g}{\partial y} + \Upsilon \frac{\partial J}{\partial y}$$

$$\frac{\partial P}{\partial z} = -V_{ex} T_g \frac{\partial T_g}{\partial z} + \Upsilon \frac{\partial J}{\partial z}$$

$$\frac{\partial P}{\partial \dot{x}} = -V_{ex} T_g \frac{\partial T_g}{\partial \dot{x}} + \Upsilon \frac{\partial J}{\partial \dot{x}}$$

$$\frac{\partial P}{\partial \dot{y}} = -V_{ex} T_g \frac{\partial T_g}{\partial \dot{y}} + \Upsilon \frac{\partial J}{\partial \dot{y}}$$

$$\frac{\partial P}{\partial \dot{z}} = -V_{ex} T_g \frac{\partial T_g}{\partial \dot{z}} + \Upsilon \frac{\partial J}{\partial \dot{z}}$$

$$\frac{\partial S}{\partial x} = \frac{\partial J}{\partial x} - V_{ex} L \frac{\partial T_g}{\partial x} - V_{ex} T_g \frac{\partial L}{\partial x}$$

$$\frac{\partial S}{\partial y} = \frac{\partial J}{\partial y} - V_{ex} L \frac{\partial T_g}{\partial y} - V_{ex} T_g \frac{\partial L}{\partial y}$$

$$\frac{\partial S}{\partial z} = \frac{\partial J}{\partial z} - V_{ex} L \frac{\partial T_g}{\partial z} - V_{ex} T_g \frac{\partial L}{\partial z}$$

$$\frac{\partial S}{\partial \dot{x}} = \frac{\partial J}{\partial \dot{x}} - V_{ex} L \frac{\partial T_g}{\partial \dot{x}} - V_{ex} T_g \frac{\partial L}{\partial \dot{x}}$$

$$\frac{\partial S}{\partial \dot{y}} = \frac{\partial J}{\partial \dot{y}} - V_{ex} L \frac{\partial T_g}{\partial \dot{y}} - V_{ex} T_g \frac{\partial L}{\partial \dot{y}}$$

$$\frac{\partial S}{\partial \dot{z}} = \frac{\partial J}{\partial \dot{z}} - V_{ex} L \frac{\partial T_g}{\partial \dot{z}} - V_{ex} T_g \frac{\partial L}{\partial \dot{z}}$$

$$\frac{\partial Q}{\partial x} = T_g V_{ex} \frac{\partial T_g}{\partial x} + \tau \frac{\partial S}{\partial x}$$

$$\frac{\partial Q}{\partial y} = T_g V_{ex} \frac{\partial T_g}{\partial y} + \tau \frac{\partial S}{\partial y}$$

$$\frac{\partial Q}{\partial z} = T_g V_{ex} \frac{\partial T_g}{\partial z} + \tau \frac{\partial S}{\partial z}$$

$$\frac{\partial Q}{\partial \dot{x}} = T_g V_{ex} \frac{\partial T_g}{\partial \dot{x}} + \tau \frac{\partial S}{\partial \dot{x}}$$

$$\frac{\partial Q}{\partial \dot{y}} = T_g V_{ex} \frac{\partial T_g}{\partial \dot{y}} + \tau \frac{\partial S}{\partial \dot{y}}$$

$$\frac{\partial Q}{\partial \dot{z}} = T_g V_{ex} \frac{\partial T_g}{\partial \dot{z}} + \tau \frac{\partial S}{\partial \dot{z}}$$

$$\frac{\partial \bar{U}}{\partial x} = (1/2) V_{ex} T_g^2 \frac{\partial T_g}{\partial x} + \tau \frac{\partial Q}{\partial x}$$

$$\frac{\partial \bar{U}}{\partial y} = (1/2) V_{ex} T_g^2 \frac{\partial T_g}{\partial y} + \tau \frac{\partial Q}{\partial y}$$

$$\frac{\partial \bar{U}}{\partial z} = (1/2) V_{ex} T_g^2 \frac{\partial T_g}{\partial z} + \tau \frac{\partial Q}{\partial z}$$

$$\frac{\partial \bar{U}}{\partial \dot{x}} = (1/2) V_{ex} T_g^2 \frac{\partial T_g}{\partial \dot{x}} + \tau \frac{\partial Q}{\partial \dot{x}}$$

$$\frac{\partial \bar{U}}{\partial \dot{y}} = (1/2) V_{ex} T_g^2 \frac{\partial T_g}{\partial \dot{y}} + \tau \frac{\partial Q}{\partial \dot{y}}$$

$$\frac{\partial \bar{U}}{\partial \dot{z}} = (1/2) V_{ex} T_g^2 \frac{\partial T_g}{\partial \dot{z}} + \tau \frac{\partial Q}{\partial \dot{z}}$$

$$\frac{\partial A_y}{\partial \dot{x}} = V_{ex} \frac{\partial L}{\partial \dot{x}}$$

$$\frac{\partial A_y}{\partial \dot{y}} = V_{ex} \frac{\partial L}{\partial \dot{y}}$$

$$\frac{\partial A_y}{\partial \dot{z}} = V_{ex} \frac{\partial L}{\partial \dot{z}}$$

$$\frac{\partial A_y}{\partial x} = V_{ex} \frac{\partial L}{\partial x}$$

$$\frac{\partial A_y}{\partial y} = V_{ex} \frac{\partial L}{\partial y}$$

$$\frac{\partial A_y}{\partial z} = V_{ex} \frac{\partial L}{\partial z}$$

$$\frac{\partial B_y}{\partial x} = \frac{\partial J}{\partial x} + V_{ex} L \frac{\partial T_g}{\partial x} + V_{ex} T_g \frac{\partial L}{\partial x}$$

$$\frac{\partial B_y}{\partial y} = \frac{\partial J}{\partial y} + V_{ex} L \frac{\partial T_g}{\partial y} + V_{ex} T_g \frac{\partial L}{\partial y}$$

$$\frac{\partial B_y}{\partial z} = \frac{\partial J}{\partial z} + V_{ex} L \frac{\partial T_g}{\partial z} + V_{ex} T_g \frac{\partial L}{\partial z}$$

$$\frac{\partial B_y}{\partial \dot{x}} = \frac{\partial J}{\partial \dot{x}} + V_{ex} L \frac{\partial T_g}{\partial \dot{x}} + V_{ex} T_g \frac{\partial L}{\partial \dot{x}}$$

$$\frac{\partial B_y}{\partial \dot{y}} = \frac{\partial J}{\partial \dot{y}} + V_{ex} L \frac{\partial T_g}{\partial \dot{y}} + V_{ex} T_g \frac{\partial L}{\partial \dot{y}}$$

$$\frac{\partial B_y}{\partial \dot{z}} = \frac{\partial J}{\partial \dot{z}} + V_{ex} L \frac{\partial T_g}{\partial \dot{z}} + V_{ex} T_g \frac{\partial L}{\partial \dot{z}}$$

$$\frac{\partial \eta}{\partial x} = g_{z1}$$

$$\frac{\partial \eta}{\partial y} = g_{z2}$$

$$\frac{\partial \eta}{\partial z} = g_{z3}$$

$$\frac{\partial \eta}{\partial \dot{x}} = \frac{\partial \eta}{\partial \dot{y}} = \frac{\partial \eta}{\partial \dot{z}} = 0$$

$$\frac{\partial \xi}{\partial x} = \left[g_{11}\cos\phi_T + g_{31}\sin\phi_T\right] + \left[x\left(-g_{11}\sin\phi_T \cdot \frac{\partial \phi_T}{\partial x} + g_{31}\cos\phi_T\frac{\partial \phi_T}{\partial x}\right) + y\left(-g_{12}\sin\phi_T\frac{\partial \phi_T}{\partial x} + g_{32}\cos\phi_T\frac{\partial \phi_T}{\partial x}\right) + z\left(-g_{13}\sin\phi_T\frac{\partial \phi_T}{\partial x} + g_{33}\cos\phi_T\frac{\partial \phi_T}{\partial x}\right)\right]$$

$$\frac{\partial \xi}{\partial y} = \left[g_{12}\cos\phi_T + \sin\phi_T\, g_{32}\right] + \left[x\left(-g_{11}\sin\phi_T \cdot \frac{\partial \phi_T}{\partial y} + g_{31}\cos\phi_T\frac{\partial \phi_T}{\partial y}\right) + y\left(-g_{12}\sin\phi_T\frac{\partial \phi_T}{\partial y} + g_{32}\cos\phi_T\frac{\partial \phi_T}{\partial y}\right) + z\left(-g_{13}\sin\phi_T\frac{\partial \phi_T}{\partial y} + g_{33}\cos\phi_T\frac{\partial \phi_T}{\partial y}\right)\right]$$

$$\frac{\partial \xi}{\partial z} = \left[g_{13}\cos\phi_T + g_{33}\sin\phi_T\right] + \left[x\left(-g_{11}\sin\phi_T\frac{\partial \phi_T}{\partial z} + g_{31}\cos\phi_T\frac{\partial \phi_T}{\partial z}\right) + y\left(-g_{12}\sin\phi_T\frac{\partial \phi_T}{\partial z} + g_{32}\cos\phi_T\frac{\partial \phi_T}{\partial z}\right) + z\left(-g_{13}\sin\phi_T\frac{\partial \phi_T}{\partial z} + g_{33}\cos\phi_T\frac{\partial \phi_T}{\partial z}\right)\right]$$

$$\frac{\partial \dot{\xi}}{\partial \dot{x}} = x\left[-g_{11}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{x}} + g_{31}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{x}}\right] + y\left[-g_{12}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{x}} + g_{32}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{x}}\right] + z\left[-g_{13}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{x}} + g_{33}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{x}}\right]$$

$$\frac{\partial \dot{\xi}}{\partial \dot{y}} = x\left[-g_{11}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{y}} + g_{31}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{y}}\right] + y\left[-g_{12}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{y}} + g_{32}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{y}}\right] + z\left[-g_{13}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{y}} + g_{33}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{y}}\right]$$

$$\frac{\partial \dot{\xi}}{\partial \dot{z}} = x\left[-g_{11}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{z}} + g_{31}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{z}}\right] + y\left[-g_{12}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{z}} + g_{32}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{z}}\right] + z\left[-g_{13}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{z}} + g_{33}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{z}}\right]$$

$$\frac{\partial \dot{\xi}}{\partial x} = \left[-g_{11}\sin\phi_T + g_{31}\cos\phi_T\right] + x\left[-g_{11}\cos\phi_T \frac{\partial \phi_T}{\partial x} - g_{31}\sin\phi_T \frac{\partial \phi_T}{\partial x}\right] + y\left[-g_{12}\cos\phi_T \frac{\partial \phi_T}{\partial x} - g_{32}\sin\phi_T \frac{\partial \phi_T}{\partial x}\right] + z\left[-g_{13}\cos\phi_T \frac{\partial \phi_T}{\partial x} - g_{33}\sin\phi_T \frac{\partial \phi_T}{\partial x}\right]$$

$$\frac{\partial \xi}{\partial y} = \left[-g_{12} \sin \phi_T + g_{32} \cos \phi_T\right]$$

$$+ x\left[-g_{11} \cos \phi_T \frac{\partial \phi_T}{\partial y} - g_{31} \sin \phi_T \frac{\partial \phi_T}{\partial y}\right]$$

$$+ y\left[-g_{12} \cos \phi_T \frac{\partial \phi_T}{\partial y} - g_{32} \sin \phi_T \frac{\partial \phi_T}{\partial y}\right]$$

$$+ z\left[-g_{13} \cos \phi_T \frac{\partial \phi_T}{\partial y} - g_{33} \sin \phi_T \frac{\partial \phi_T}{\partial y}\right]$$

$$\frac{\partial \xi}{\partial z} = \left[-g_{13} \sin \phi_T + g_{33} \cos \phi_T\right]$$

$$+ x\left[-g_{11} \cos \phi_T \frac{\partial \phi_T}{\partial z} - g_{31} \sin \phi_T \frac{\partial \phi_T}{\partial z}\right]$$

$$+ y\left[-g_{12} \cos \phi_T \frac{\partial \phi_T}{\partial z} - g_{32} \sin \phi_T \frac{\partial \phi_T}{\partial z}\right]$$

$$+ z\left[-g_{13} \cos \phi_T \frac{\partial \phi_T}{\partial z} - g_{33} \sin \phi_T \frac{\partial \phi_T}{\partial z}\right]$$

$$\frac{\partial \xi}{\partial \dot{x}} = x\left[-g_{11} \cos \phi_T \frac{\partial \phi_T}{\partial \dot{x}} - g_{31} \sin \phi_T \frac{\partial \phi_T}{\partial \dot{x}}\right]$$

$$+ y\left[-g_{12} \cos \phi_T \frac{\partial \phi_T}{\partial \dot{x}} - g_{32} \sin \phi_T \frac{\partial \phi_T}{\partial \dot{x}}\right]$$

$$+ z\left[-g_{13} \cos \phi_T \frac{\partial \phi_T}{\partial \dot{x}} - g_{33} \sin \phi_T \frac{\partial \phi_T}{\partial \dot{x}}\right]$$

$$\frac{\partial \xi}{\partial \dot{y}} = x\left[-g_{11} \cos \phi_T \frac{\partial \phi_T}{\partial \dot{y}} - g_{31} \sin \phi_T \frac{\partial \phi_T}{\partial \dot{y}}\right]$$

$$+ y\left[-g_{12} \cos \phi_T \frac{\partial \phi_T}{\partial \dot{y}} - g_{32} \sin \phi_T \frac{\partial \phi_T}{\partial \dot{y}}\right]$$

$$+ z\left[-g_{13} \cos \phi_T \frac{\partial \phi_T}{\partial \dot{y}} - g_{33} \sin \phi_T \frac{\partial \phi_T}{\partial \dot{y}}\right]$$

$$\frac{\partial \dot{\xi}}{\partial \dot{z}} = x\left[-g_{11}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{z}} - g_{31}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{z}}\right]$$

$$+ y\left[-g_{12}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{z}} - g_{32}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{z}}\right]$$

$$+ z\left[-g_{13}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{z}} - g_{33}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{z}}\right]$$

$$\frac{\partial \dot{\eta}}{\partial x} = \dot{x}\frac{\partial}{\partial x}(k_{21})^{\nearrow 0} + \dot{y}\frac{\partial}{\partial x}(k_{22})^{\nearrow 0} + \dot{z}\frac{\partial}{\partial x}(k_{23})^{\nearrow 0}$$

$$\frac{\partial \dot{\eta}}{\partial y} = \dot{x}\frac{\partial}{\partial y}(k_{21})^{\nearrow 0} + \dot{y}\frac{\partial}{\partial y}(k_{22})^{\nearrow 0} + \dot{z}\frac{\partial}{\partial y}(k_{23})^{\nearrow 0}$$

$$\frac{\partial \dot{\eta}}{\partial z} = \dot{x}\frac{\partial}{\partial z}(k_{21})^{\nearrow 0} + \dot{y}\frac{\partial}{\partial z}(k_{22})^{\nearrow 0} + \dot{z}\frac{\partial}{\partial z}(k_{23})^{\nearrow 0}$$

$$\frac{\partial \dot{\eta}}{\partial \dot{x}} = k_{21} = g_{21}$$

$$\frac{\partial \dot{\eta}}{\partial \dot{y}} = k_{22} = g_{22}$$

$$\frac{\partial \dot{\eta}}{\partial \dot{z}} = k_{23} = g_{23}$$

$$\frac{\partial \ddot{\eta}_3}{\partial x} = \left(\frac{1}{z}\right)\left[\frac{\partial \ddot{\eta}}{\partial x}^{\nearrow 0} + k_{21}\frac{\partial}{\partial x}(\ddot{x}_g) + \ddot{x}_g\frac{\partial}{\partial x}(k_{21})^{\nearrow 0}\right.$$

$$+ k_{22}\frac{\partial}{\partial x}(\ddot{y}_g) + \ddot{y}_g\frac{\partial}{\partial x}(k_{22})^{\nearrow 0} + k_{23}\frac{\partial}{\partial x}(\ddot{z}_g)$$

$$\left. + \ddot{z}_g\frac{\partial}{\partial x}(k_{23})^{\nearrow 0}\right]$$

$$= \left(\frac{1}{z}\right)\left[g_{21}\frac{\partial}{\partial x}(\ddot{x}_g) + g_{22}\frac{\partial}{\partial x}(\ddot{y}_g) + g_{23}\frac{\partial}{\partial x}(\ddot{z}_g)\right]$$

likewise

$$\frac{\partial \ddot{\eta}_3}{\partial y} = \left(\frac{1}{z}\right)\left[g_{21}\frac{\partial}{\partial y}(\ddot{x}_g) + g_{22}\frac{\partial}{\partial y}(\ddot{y}_g) + g_{23}\frac{\partial}{\partial y}(\ddot{z}_g)\right]$$

$$\frac{\partial \ddot{\eta}_g}{\partial \ddot{z}} = (1/2)\left[g_{21}\frac{\partial}{\partial \ddot{z}}(\ddot{x}_g) + g_{22}\frac{\partial}{\partial \ddot{z}}(\ddot{y}_g) + g_{23}\frac{\partial}{\partial \ddot{z}}(\ddot{z}_g)\right]$$

$$\frac{\partial \ddot{\eta}_g}{\partial \dot{x}} = (1/2)\left[\frac{\partial}{\partial \dot{x}}(\eta_{GT})^{0} + k_{21}\frac{\partial}{\partial \dot{x}}(\ddot{x}_g)^{0} + \ddot{x}_g\frac{\partial}{\partial \dot{x}}(k_{21})^{0}\right.$$

$$+ k_{22}\frac{\partial}{\partial \dot{x}}(\ddot{y}_g)^{0} + \ddot{y}_g\frac{\partial}{\partial \dot{x}}(k_{22})^{0} + k_{23}\frac{\partial}{\partial \dot{x}}(\ddot{z}_g)^{0}$$

$$\left. + \ddot{z}_g\frac{\partial}{\partial \dot{x}}(k_{23})^{0}\right]$$

likewise;

$$\frac{\partial \ddot{\eta}_g}{\partial \dot{y}} = \frac{\partial \ddot{\eta}_g}{\partial \dot{z}} = 0$$

$$\frac{\partial}{\partial x}(\ddot{\eta}_g) = (1/2)\left[(-g_{11}\sin\phi_T + g_{31}\cos\phi_T)\frac{\partial \ddot{x}_g}{\partial x}\right.$$

$$+ (-g_{12}\sin\phi_T + g_{32}\cos\phi_T)\frac{\partial}{\partial x}(\ddot{y}_g)$$

$$+ (-g_{13}\sin\phi_T + g_{33}\cos\phi_T)\frac{\partial}{\partial x}(\ddot{z}_g)$$

$$+ \ddot{x}_g\left(-g_{11}\cos\phi_T\frac{\partial \phi_T}{\partial x} - g_{31}\sin\phi_T\frac{\partial \phi_T}{\partial x}\right)$$

$$+ \ddot{y}_g\left(-g_{12}\cos\phi_T\frac{\partial \phi_T}{\partial x} - g_{32}\sin\phi_T\frac{\partial \phi_T}{\partial x}\right)$$

$$\left. + \ddot{z}_g\left(-g_{13}\cos\phi_T\frac{\partial \phi_T}{\partial x} - g_{33}\sin\phi_T\frac{\partial \phi_T}{\partial x}\right)\right]$$

$$\frac{\partial}{\partial y}(\ddot{z}_g) = (1/2)\Big[(-g_{11}\sin\phi_T + g_{31}\cos\phi_T)\frac{\partial}{\partial y}(\ddot{x}_g)$$

$$+ (-g_{12}\sin\phi_T + g_{32}\cos\phi_T)\frac{\partial}{\partial y}(\ddot{y}_g)$$

$$+ (-g_{13}\sin\phi_T + g_{33}\cos\phi_T)\frac{\partial}{\partial y}(\ddot{z}_g)$$

$$+ \ddot{x}_g\left(-g_{11}\cos\phi_T\frac{\partial\phi_T}{\partial y} - g_{31}\sin\phi_T\frac{\partial\phi_T}{\partial y}\right)$$

$$+ \ddot{y}_g\left(-g_{12}\cos\phi_T\frac{\partial\phi_T}{\partial y} - g_{32}\sin\phi_T\frac{\partial\phi_T}{\partial y}\right)$$

$$+ \ddot{z}_g\left(-g_{13}\cos\phi_T\frac{\partial\phi_T}{\partial y} - g_{33}\sin\phi_T\frac{\partial\phi_T}{\partial y}\right)\Big]$$

$$\frac{\partial}{\partial z}(\ddot{z}_g) = (1/2)\Big[(-g_{11}\sin\phi_T + g_{31}\cos\phi_T)\frac{\partial}{\partial z}(\ddot{x}_g)$$

$$+ (-g_{12}\sin\phi_T + g_{32}\cos\phi_T)\frac{\partial}{\partial z}(\ddot{y}_g)$$

$$+ (-g_{13}\sin\phi_T + g_{33}\cos\phi_T)\frac{\partial}{\partial z}(\ddot{z}_g)$$

$$+ \ddot{x}_g\left(-g_{11}\cos\phi_T\frac{\partial\phi_T}{\partial z} - g_{31}\sin\phi_T\frac{\partial\phi_T}{\partial z}\right)$$

$$+ \ddot{y}_g\left(-g_{12}\cos\phi_T\frac{\partial\phi_T}{\partial z} - g_{32}\sin\phi_T\frac{\partial\phi_T}{\partial z}\right)$$

$$+ \ddot{z}_g\left(-g_{13}\cos\phi_T\frac{\partial\phi_T}{\partial z} - g_{33}\sin\phi_T\frac{\partial\phi_T}{\partial z}\right)\Big]$$

$$\frac{\partial}{\partial \dot{x}}(\ddot{z}_g) = (1/2)\Big[\ddot{x}_g\left(-g_{11}\cos\phi_T\frac{\partial\phi_T}{\partial \dot{x}} - g_{31}\sin\phi_T\frac{\partial\phi_T}{\partial \dot{x}}\right)$$

$$+ \ddot{y}_g\left(-g_{12}\cos\phi_T\frac{\partial\phi_T}{\partial \dot{x}} - g_{32}\sin\phi_T\frac{\partial\phi_T}{\partial \dot{x}}\right)$$

$$+ \ddot{z}_g\left(-g_{13}\cos\phi_T\frac{\partial\phi_T}{\partial \dot{x}} - g_{33}\sin\phi_T\frac{\partial\phi_T}{\partial \dot{x}}\right)\Big]$$

$$\frac{\partial}{\partial \dot{y}}(\ddot{\zeta}_g) = (1/2)\Big[\ddot{x}_g\Big(-g_{11}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{y}} - g_{31}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{y}}\Big)$$

$$+ \ddot{y}_g\Big(-g_{12}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{y}} - g_{32}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{y}}\Big)$$

$$+ \ddot{z}_g\Big(-g_{13}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{y}} - g_{33}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{y}}\Big)\Big]$$

$$\frac{\partial}{\partial \dot{z}}(\ddot{\zeta}_g) = (1/2)\Big[\ddot{x}_g\Big(-g_{11}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{z}} - g_{31}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{z}}\Big)$$

$$+ \ddot{y}_g\Big(-g_{12}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{z}} - g_{32}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{z}}\Big)$$

$$+ \ddot{z}_g\Big(-g_{13}\cos\phi_T \frac{\partial \phi_T}{\partial \dot{z}} - g_{33}\sin\phi_T \frac{\partial \phi_T}{\partial \dot{z}}\Big)\Big]$$

$$\frac{\partial}{\partial x}(\dot{\eta}_T) = \frac{\partial}{\partial y}(\dot{\eta}_T) = \frac{\partial}{\partial z}(\dot{\eta}_T) = 0$$

$$\frac{\partial}{\partial \dot{x}}(\dot{\eta}_T) = \frac{\partial}{\partial \dot{y}}(\dot{\eta}_T) = \frac{\partial}{\partial \dot{z}}(\dot{\eta}_T) = 0$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} = 0$$

$$\frac{\partial V}{\partial \dot{x}} = \frac{\dot{x}}{V}$$

$$\frac{\partial V}{\partial \dot{y}} = \frac{\dot{y}}{V}$$

$$\frac{\partial V}{\partial \dot{z}} = \frac{\dot{z}}{V}$$

$$\frac{\partial R}{\partial x} = \frac{x}{R}$$

$$\frac{\partial R}{\partial y} = \frac{y}{R}$$

$$\frac{\partial R}{\partial z} = \frac{z}{R}$$

$$\frac{\partial R}{\partial \dot{x}} = \frac{\partial R}{\partial \dot{y}} = \frac{\partial R}{\partial \dot{z}} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{x \cdot \dot{x}}{RV}\right) = \frac{R \cdot V \cdot \dot{x} - (x \cdot \dot{x} + y \cdot \dot{y} + z \cdot \dot{z}) \frac{\partial R}{\partial x} V}{R^2 V^2}$$

$$\frac{\partial}{\partial y}\left(\frac{x \cdot \dot{x}}{RV}\right) = \frac{R \cdot V \cdot \dot{y} - (x \cdot \dot{x} + y \cdot \dot{y} + z \cdot \dot{z}) V \frac{\partial R}{\partial y}}{R^2 V^2}$$

$$\frac{\partial}{\partial z}\left(\frac{x \cdot \dot{x}}{RV}\right) = \frac{R \cdot V \cdot \dot{z} - (x \cdot \dot{x} + y \cdot \dot{y} + z \cdot \dot{z}) V \frac{\partial R}{\partial z}}{R^2 V^2}$$

$$\frac{\partial}{\partial \dot{x}}\left(\frac{x \cdot \dot{x}}{RV}\right) = \frac{R \cdot V \cdot x - (x \cdot \dot{x} + y \cdot \dot{y} + z \cdot \dot{z}) R \frac{\partial V}{\partial \dot{x}}}{R^2 V^2}$$

$$\frac{\partial}{\partial \dot{y}}\left(\frac{x \cdot \dot{x}}{RV}\right) = \frac{R \cdot V \cdot y - (x \cdot \dot{x} + y \cdot \dot{y} + z \cdot \dot{z}) R \frac{\partial V}{\partial \dot{y}}}{R^2 V^2}$$

$$\frac{\partial}{\partial \dot{z}}\left(\frac{x \cdot \dot{x}}{RV}\right) = \frac{R \cdot V \cdot z - (x \cdot \dot{x} + y \cdot \dot{y} + z \cdot \dot{z}) R \frac{\partial V}{\partial \dot{z}}}{R^2 V^2}$$

$$\frac{\partial \gamma}{\partial x} = \frac{\frac{\partial}{\partial x}\left[\frac{x \cdot \dot{x}}{RV}\right]}{\sqrt{1 - \left[\frac{x \cdot \dot{x}}{RV}\right]^2}}$$

$$\frac{\partial \gamma}{\partial y} = \frac{\frac{\partial}{\partial y}\left[\frac{x \cdot \dot{x}}{RV}\right]}{\sqrt{1 - \left[\frac{x \cdot \dot{x}}{RV}\right]^2}}$$

$$\frac{\partial \gamma}{\partial z} = \frac{\frac{\partial}{\partial z}\left[\frac{X \cdot \dot{X}}{RV}\right]}{\sqrt{1 - \left[\frac{X \cdot \dot{X}}{RV}\right]^2}}$$

$$\frac{\partial \gamma}{\partial \dot{x}} = \frac{\frac{\partial}{\partial \dot{x}}\left[\frac{X \cdot \dot{X}}{RV}\right]}{\sqrt{1 - \left[\frac{X \cdot \dot{X}}{RV}\right]^2}}$$

$$\frac{\partial \gamma}{\partial \dot{y}} = \frac{\frac{\partial}{\partial \dot{y}}\left[\frac{X \cdot \dot{X}}{RV}\right]}{\sqrt{1 - \left[\frac{X \cdot \dot{X}}{RV}\right]^2}}$$

$$\frac{\partial \gamma}{\partial \dot{z}} = \frac{\frac{\partial}{\partial \dot{z}}\left[\frac{X \cdot \dot{X}}{RV}\right]}{\sqrt{1 - \left[\frac{X \cdot \dot{X}}{RV}\right]^2}}$$

$$\frac{\partial X_F}{\partial x} = g_{11}$$

$$\frac{\partial X_F}{\partial y} = g_{12}$$

$$\frac{\partial X_F}{\partial z} = g_{13}$$

$$\frac{\partial X_F}{\partial \dot{x}} = \frac{\partial X_F}{\partial \dot{y}} = \frac{\partial X_F}{\partial \dot{z}} = 0$$

$$\frac{\partial Y_F}{\partial x} = g_{21}$$

$$\frac{\partial Y_F}{\partial y} = g_{22}$$

$$\frac{\partial Y_F}{\partial z} = g_{23}$$

$$\frac{\partial Y_F}{\partial \dot{x}} = \frac{\partial Y_F}{\partial \dot{y}} = \frac{\partial Z_F}{\partial \dot{y}} = 0$$

$$\frac{\partial Z_F}{\partial x} = \gamma_{31}$$

$$\frac{\partial Z_F}{\partial y} = \gamma_{32}$$

$$\frac{\partial Z_F}{\partial z} = \gamma_{33}$$

$$\frac{\partial Z_F}{\partial \dot{x}} = \frac{\partial Z_F}{\partial \dot{y}} = \frac{\partial Z_F}{\partial \dot{z}} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{Z_F}{X_F}\right) = \frac{X_F\left(\frac{\partial Z_F}{\partial x}\right) - Z_F\left(\frac{\partial X_F}{\partial y}\right)}{X_F^{2}}$$

$$\frac{\partial}{\partial y}\left(\frac{Z_F}{X_F}\right) = \frac{X_F\left(\frac{\partial Z_F}{\partial y}\right) - Z_F\left(\frac{\partial X_F}{\partial y}\right)}{X_F^{2}}$$

$$\frac{\partial}{\partial z}\left(\frac{Z_F}{X_F}\right) = \frac{X_F\left(\frac{\partial Z_F}{\partial z}\right) - Z_F\left(\frac{\partial X_F}{\partial z}\right)}{X_F^{2}}$$

$$\frac{\partial}{\partial \dot{x}}\left(\frac{Z_F}{X_F}\right) = \frac{\partial}{\partial \dot{y}}\left(\frac{Z_F}{X_F}\right) = \frac{\partial}{\partial \dot{z}}\left(\frac{Z_F}{X_F}\right) = 0$$

$$\frac{\partial}{\partial x}(\phi_{ET}) = \left[1 + \left(\frac{Z_F}{X_F}\right)^{2}\right]^{-1} \frac{\partial}{\partial x}\left(\frac{Z_F}{X_F}\right)$$

$$\frac{\partial}{\partial y}(\phi_{ET}) = \left[1 + \left(\frac{Z_F}{X_F}\right)^{2}\right]^{-1} \frac{\partial}{\partial y}\left(\frac{Z_F}{X_F}\right)$$

$$\frac{\partial}{\partial z}(\phi_{ET}) = \left[1 + \left(\frac{Z_F}{X_F}\right)^{2}\right]^{-1} \frac{\partial}{\partial z}\left(\frac{Z_F}{X_F}\right)$$

$$\frac{\partial}{\partial \dot{x}}(\phi_{ET}) = \frac{\partial}{\partial \dot{y}}(\phi_{ET}) = \frac{\partial}{\partial \dot{z}}(\phi_{ET}) = 0$$

$$\frac{\partial E_T}{\partial x} = \frac{\partial E_T}{\partial y} = \frac{\partial E_T}{\partial z} = 0$$

$$\frac{\partial E_T}{\partial \dot{x}} = \frac{\partial E_T}{\partial \dot{y}} = \frac{\partial E_T}{\partial \dot{z}} = 0$$

$$\frac{\partial FK6}{\partial x} = (2)(C1FK6)T_g \frac{\partial T_g}{\partial x} + (C2FK6)\frac{\partial T_g}{\partial x}$$

$$\frac{\partial FK6}{\partial y} = (2)(C1FK6)T_g \frac{\partial T_g}{\partial y} + (C2FK6)\frac{\partial T_g}{\partial y}$$

$$\frac{\partial FK6}{\partial z} = (2)(C1FK6)T_g \frac{\partial T_g}{\partial z} + (C2FK6)\frac{\partial T_g}{\partial z}$$

$$\frac{\partial FK6}{\partial \dot{x}} = (2)(C1FK6)T_g \frac{\partial T_g}{\partial \dot{x}} + (C2FK6)\frac{\partial T_g}{\partial \dot{x}}$$

$$\frac{\partial FK6}{\partial \dot{y}} = (2)(C1FK6)T_g \frac{\partial T_g}{\partial \dot{y}} + (C2FK6)\frac{\partial T_g}{\partial \dot{y}}$$

$$\frac{\partial FK6}{\partial \dot{z}} = (2)(C1FK6)T_g \frac{\partial T_g}{\partial \dot{z}} + (C2FK6)\frac{\partial T_g}{\partial \dot{z}}$$

$$\frac{\partial R_T}{\partial x} = \frac{P_T\, e_T \sin\theta_T \frac{\partial \theta_T}{\partial x}}{(1 + e_T \cos\theta_T)^2}$$

$$\frac{\partial R_T}{\partial y} = \frac{P_T\, e_T \sin\theta_T \frac{\partial \theta_T}{\partial y}}{(1 + e_T \cos\theta_T)^2}$$

$$\frac{\partial R_T}{\partial z} = \frac{P_T\, e_T \sin\theta_T \frac{\partial \theta_T}{\partial z}}{(1 + e_T \cos\theta_T)^2}$$

$$\frac{\partial R_T}{\partial \dot{x}} = \frac{P_T\, e_T \sin\theta_T \frac{\partial \theta_T}{\partial \dot{x}}}{(1 + e_T \cos\theta_T)^2}$$

$$\frac{\partial R_T}{\partial \dot{y}} = \frac{p_T \, e_T \, \sin\theta_T \, \frac{\partial \theta_T}{\partial \dot{y}}}{(1 + e_T \cos\theta_T)^2}$$

$$\frac{\partial R_T}{\partial \dot{z}} = \frac{p_T \, e_T \, \sin\theta_T \, \frac{\partial \theta_T}{\partial \dot{z}}}{(1 + e_T \cos\theta_T)^2}$$

$$\frac{\partial \gamma_T}{\partial x} = (SMPLFY) \frac{\partial \theta_T}{\partial x}$$

$$\frac{\partial \gamma_T}{\partial y} = (SMPLFY) \frac{\partial \theta_T}{\partial y}$$

$$\frac{\partial \gamma_T}{\partial z} = (SMPLFY) \frac{\partial \theta_T}{\partial z}$$

$$\frac{\partial \gamma_T}{\partial \dot{x}} = (SMPLFY) \frac{\partial \theta_T}{\partial \dot{x}}$$

$$\frac{\partial \gamma_T}{\partial \dot{y}} = (SMPLFY) \frac{\partial \theta_T}{\partial \dot{y}}$$

$$\frac{\partial \gamma_T}{\partial \dot{z}} = (SMPLFY) \frac{\partial \theta_T}{\partial \dot{z}}$$

$$\frac{\partial V_T}{\partial x} = -\left(-\sqrt{\frac{\mu}{p_T}}\right)\left(\sin\theta_T \frac{\partial \theta_T}{\partial x}\right)\left[1 + e_T(e_T + 2\cos\theta_T)\right]^{-1/2}$$

$$\frac{\partial V_T}{\partial y} = -\left(\sqrt{\frac{\mu}{p_T}}\right)\left(\sin\theta_T \frac{\partial \theta_T}{\partial y}\right)\left[1 + e_T(e_T + 2\cos\theta_T)\right]^{-1/2}$$

$$\frac{\partial V_T}{\partial z} = -\left(-\sqrt{\frac{\mu}{p_T}}\right)\left(\sin\theta_T \frac{\partial \theta_T}{\partial z}\right)\left[1 + e_T(e_T + 2\cos\theta_T)\right]^{-1/2}$$

$$\frac{\partial V_T}{\partial \dot{x}} = -\left(-\sqrt{\frac{\mu}{p_T}}\right)\left(\sin\theta_T \frac{\partial \theta_T}{\partial \dot{x}}\right)\left[1 + e_T(e_T + 2\cos\theta_T)\right]^{-1/2}$$

$$\frac{\partial V_T}{\partial \dot{y}} = -\left(-\sqrt{\frac{\mu}{p_T}}\right)\left(\sin\theta_T \frac{\partial \theta_T}{\partial \dot{y}}\right)\left[1 + e_T(e_T + 2\cos\theta_T)\right]^{-1/2}$$

$$\frac{\partial V_T}{\partial \dot{z}} = -\left(\sqrt{\frac{\mu}{P_T}}\right)\left(\sin\theta_T \frac{\partial \theta_T}{\partial \dot{z}}\right)\left[1+e_T\left(e_T+2\cos\theta_T\right)\right]^{-1/2}$$

$$\frac{\partial G_T}{\partial x} = \left(\frac{2\mu}{R_T^3}\right) \frac{\partial R_T}{\partial x}$$

$$\frac{\partial G_T}{\partial y} = (2\mu)\left(R_T^{-3}\right) \frac{\partial R_T}{\partial y}$$

$$\frac{\partial G_T}{\partial z} = (2\mu)\left(R_T^{-3}\right) \frac{\partial R_T}{\partial z}$$

$$\frac{\partial G_T}{\partial \dot{x}} = (2\mu)\left(R_T^{-3}\right) \frac{\partial R_T}{\partial \dot{x}}$$

$$\frac{\partial G_T}{\partial \dot{y}} = (2\mu)\left(R_T^{-3}\right) \frac{\partial R_T}{\partial \dot{y}}$$

$$\frac{\partial G_T}{\partial \dot{z}} = (2\mu)\left(R_T^{-3}\right) \frac{\partial R_T}{\partial \dot{z}}$$

$$\frac{\partial}{\partial x}\left(\dot{\xi}_T\right) = \sin\gamma_T \frac{\partial V_T}{\partial x} + V_T \cos\gamma_T \frac{\partial \gamma_T}{\partial x}$$

$$\frac{\partial}{\partial y}\left(\dot{\xi}_T\right) = \sin\gamma_T \frac{\partial V_T}{\partial y} + V_T \cos\gamma_T \frac{\partial \gamma_T}{\partial y}$$

$$\frac{\partial}{\partial z}\left(\dot{\xi}_T\right) = \sin\gamma_T \frac{\partial V_T}{\partial z} + V_T \cos\gamma_T \frac{\partial \gamma_T}{\partial z}$$

$$\frac{\partial}{\partial \dot{x}}\left(\dot{\xi}_T\right) = \sin\gamma_T \frac{\partial V_T}{\partial \dot{x}} + V_T \cos\gamma_T \frac{\partial \gamma_T}{\partial \dot{x}}$$

$$\frac{\partial}{\partial \dot{y}}\left(\dot{\xi}_T\right) = \sin\gamma_T \frac{\partial V_T}{\partial \dot{y}} + V_T \cos\gamma_T \frac{\partial \gamma_T}{\partial \dot{y}}$$

$$\frac{\partial}{\partial \dot{z}}\left(\dot{\xi}_T\right) = \sin\gamma_T \frac{\partial V_T}{\partial \dot{z}} + V_T \cos\gamma_T \frac{\partial \gamma_T}{\partial \dot{z}}$$

$$\frac{\partial}{\partial x}\left(\dot{\zeta}_T\right) = \cos\gamma_T \frac{\partial V_T}{\partial x} - V_T \sin\gamma_T \frac{\partial \gamma_T}{\partial x}$$

$$\frac{\partial}{\partial y}(\dot{\xi}_T) = \cos\gamma_T \frac{\partial V_T}{\partial y} - V_T \sin\gamma_T \frac{\partial \gamma_T}{\partial y}$$

$$\frac{\partial}{\partial z}(\dot{\xi}_T) = \cos\gamma_T \frac{\partial V_T}{\partial z} - V_T \sin\gamma_T \frac{\partial \gamma_T}{\partial z}$$

$$\frac{\partial}{\partial \dot{x}}(\dot{\xi}_T) = \cos\gamma_T \frac{\partial V_T}{\partial \dot{x}} - V_T \sin\gamma_T \frac{\partial \gamma_T}{\partial \dot{x}}$$

$$\frac{\partial}{\partial \dot{y}}(\dot{\xi}_T) = \cos\gamma_T \frac{\partial V_T}{\partial \dot{y}} - V_T \sin\gamma_T \frac{\partial \gamma_T}{\partial \dot{y}}$$

$$\frac{\partial}{\partial \dot{z}}(\dot{\xi}_T) = \cos\gamma_T \frac{\partial V_T}{\partial \dot{z}} - V_T \sin\gamma_T \frac{\partial \gamma_T}{\partial \dot{z}}$$

$$\frac{\partial}{\partial x}(\dot{\xi}) = \dot{x}\left[\cancelto{0}{\frac{\partial}{\partial x}(k_{11})}\right] + \dot{y}\left[\cancelto{0}{\frac{\partial}{\partial x}(k_{12})}\right] + \dot{z}\left[\cancelto{0}{\frac{\partial}{\partial x}(k_{13})}\right]$$

$$\frac{\partial}{\partial y}(\dot{\xi}) = \dot{x}\left[\cancelto{0}{\frac{\partial}{\partial y}(k_{11})}\right] + \dot{y}\left[\cancelto{0}{\frac{\partial}{\partial y}(k_{12})}\right] + \dot{z}\left[\cancelto{0}{\frac{\partial}{\partial y}(k_{13})}\right]$$

$$\frac{\partial}{\partial z}(\dot{\xi}) = \dot{x}\left[\cancelto{0}{\frac{\partial}{\partial y}(k_{11})}\right] + \dot{y}\left[\cancelto{0}{\frac{\partial}{\partial y}(k_{12})}\right] + \dot{z}\left[\cancelto{0}{\frac{\partial}{\partial y}(k_{13})}\right]$$

$$\frac{\partial}{\partial \dot{x}}(\dot{\xi}) = k_{11} + \dot{x}\left[\cancelto{0}{\frac{\partial}{\partial \dot{x}}(k_{11})}\right] + \dot{y}\left[\cancelto{0}{\frac{\partial}{\partial \dot{x}}(k_{12})}\right] + \dot{z}\left[\cancelto{0}{\frac{\partial}{\partial \dot{x}}(k_{13})}\right]$$

$$\frac{\partial}{\partial \dot{y}}(\dot{\xi}) = k_{12} + \dot{x}\left[\cancelto{0}{\frac{\partial}{\partial \dot{y}}(k_{11})}\right] + \dot{y}\left[\cancelto{0}{\frac{\partial}{\partial \dot{y}}(k_{12})}\right] + \dot{z}\left[\cancelto{0}{\frac{\partial}{\partial \dot{y}}(k_{13})}\right]$$

$$\frac{\partial}{\partial \dot{z}}(\dot{\xi}) = k_{13} + \dot{x}\left[\cancelto{0}{\frac{\partial}{\partial \dot{z}}(k_{11})}\right] + \dot{y}\left[\cancelto{0}{\frac{\partial}{\partial \dot{z}}(k_{12})}\right] + \dot{z}\left[\cancelto{0}{\frac{\partial}{\partial \dot{z}}(k_{13})}\right]$$

$$\frac{\partial}{\partial x}(\ddot{\xi}_{GT}) = \frac{\partial G_T}{\partial x}$$

$$\frac{\partial}{\partial y}(\ddot{\xi}_{GT}) = \frac{\partial G_T}{\partial y}$$

$$\frac{\partial}{\partial z}(\ddot{\xi}_{GT}) = \frac{\partial G_T}{\partial z}$$

$$\frac{\partial}{\partial \dot{x}}(\ddot{\xi}_{GT}) = \frac{\partial G_T}{\partial \dot{x}}$$

$$\frac{\partial}{\partial \dot{y}}\left(\ddot{\xi}_{GT}\right) = \frac{\partial G_T}{\partial \dot{y}}$$

$$\frac{\partial}{\partial \dot{z}}\left(\ddot{\xi}_{GT}\right) = \frac{\partial G_T}{\partial \dot{z}}$$

$$\frac{\partial}{\partial x}\left(\ddot{\xi}_{G}\right) = \left(\tfrac{1}{2}\right)\left[\frac{\partial G_T}{\partial x} + k_{11}\frac{\partial \ddot{x}_G}{\partial x} + k_{12}\frac{\partial \ddot{y}_G}{\partial x} + k_{13}\frac{\partial \ddot{z}_G}{\partial x} + \ddot{x}_G \frac{\partial}{\partial x}(k_{11}) + \ddot{y}_G \frac{\partial}{\partial x}(k_{12}) + \ddot{z}_G \frac{\partial}{\partial x}(k_{13})\right]$$

$$\frac{\partial}{\partial y}\left(\ddot{\xi}_{G}\right) = \left(\tfrac{1}{2}\right)\left[\frac{\partial G_T}{\partial y} + k_{11}\frac{\partial \ddot{x}_G}{\partial y} + k_{12}\frac{\partial \ddot{y}_G}{\partial y} + k_{13}\frac{\partial \ddot{z}_G}{\partial y} + \ddot{x}_G \frac{\partial}{\partial y}(k_{11}) + \ddot{y}_G \frac{\partial}{\partial y}(k_{12}) + \ddot{z}_G \frac{\partial}{\partial y}(k_{13})\right]$$

$$\frac{\partial}{\partial z}\left(\ddot{\xi}_{G}\right) = \left(\tfrac{1}{2}\right)\left[\frac{\partial G_T}{\partial z} + k_{11}\frac{\partial \ddot{x}_G}{\partial z} + k_{12}\frac{\partial \ddot{y}_G}{\partial z} + k_{13}\frac{\partial \ddot{z}_G}{\partial z} + \ddot{x}_G \frac{\partial}{\partial z}(k_{11}) + \ddot{y}_G \frac{\partial}{\partial z}(k_{12}) + \ddot{z}_G \frac{\partial}{\partial z}(k_{13})\right]$$

$$\frac{\partial}{\partial \dot{x}}\left(\ddot{\xi}_{G}\right) = \left(\tfrac{1}{2}\right)\left[\frac{\partial G_T}{\partial \dot{x}} + \ddot{x}_G \frac{\partial}{\partial \dot{x}}(k_{11}) + \ddot{y}_G \frac{\partial}{\partial \dot{x}}(k_{12}) + \ddot{z}_G \frac{\partial}{\partial \dot{x}}(k_{13})\right]$$

$$\frac{\partial}{\partial \dot{y}}\left(\ddot{\xi}_{G}\right) = \left(\tfrac{1}{2}\right)\left[\frac{\partial G_T}{\partial \dot{y}} + \ddot{x}_G \frac{\partial}{\partial \dot{y}}(k_{11}) + \ddot{y}_G \frac{\partial}{\partial \dot{y}}(k_{12}) + \ddot{z}_G \frac{\partial}{\partial \dot{y}}(k_{13})\right]$$

$$\frac{\partial}{\partial \dot{z}}\left(\ddot{\xi}_{G}\right) = \left(\tfrac{1}{2}\right)\left[\frac{\partial G_T}{\partial \dot{z}} + \ddot{x}_G \frac{\partial}{\partial \dot{z}}(k_{11}) + \ddot{y}_G \frac{\partial}{\partial \dot{z}}(k_{12}) + \ddot{z}_G \frac{\partial}{\partial \dot{z}}(k_{13})\right]$$

$$\frac{\partial}{\partial x}\left(\Delta \ddot{\xi}^{*}\right) = \frac{\partial}{\partial x}\left(\ddot{\xi}_T\right) - \frac{\partial}{\partial x}\left(\ddot{\xi}\right) - \ddot{\xi}_G \frac{\partial T_g}{\partial x} - T_g \frac{\partial}{\partial x}\left(\ddot{\xi}_G\right)$$

$$\frac{\partial}{\partial y}\left(\Delta \ddot{\xi}^{*}\right) = \frac{\partial}{\partial y}\left(\ddot{\xi}_T\right) - \frac{\partial}{\partial y}\left(\ddot{\xi}\right) - \ddot{\xi}_G \frac{\partial T_g}{\partial y} - T_g \frac{\partial}{\partial y}\left(\ddot{\xi}_G\right)$$

$$\frac{\partial}{\partial z}\left(\Delta \ddot{\xi}^{*}\right) = \frac{\partial}{\partial z}\left(\ddot{\xi}_T\right) - \frac{\partial}{\partial z}\left(\ddot{\xi}\right) - \ddot{\xi}_G \frac{\partial T_g}{\partial z} - T_g \frac{\partial}{\partial z}\left(\ddot{\xi}_G\right)$$

$$\frac{\partial}{\partial x}(\Delta \dot{\xi}^*) = \frac{\partial}{\partial x}(\dot{\xi}_T) - \frac{\partial}{\partial x}(\dot{\xi}) - \ddot{\xi}_G \frac{\partial T_g}{\partial x} - T_g \frac{\partial}{\partial x}(\ddot{\xi}_G)$$

$$\frac{\partial}{\partial y}(\Delta \dot{\xi}^*) = \frac{\partial}{\partial y}(\dot{\xi}_T) - \frac{\partial}{\partial y}(\dot{\xi}) - \ddot{\xi}_G \frac{\partial T_g}{\partial y} - T_g \frac{\partial}{\partial y}(\ddot{\xi}_G)$$

$$\frac{\partial}{\partial z}(\Delta \dot{\xi}^*) = \frac{\partial}{\partial z}(\dot{\xi}_T) - \frac{\partial}{\partial z}(\dot{\xi}) - \ddot{\xi}_G \frac{\partial T_g}{\partial z} - T_g \frac{\partial}{\partial z}(\ddot{\xi}_G)$$

$$\frac{\partial}{\partial x}(\Delta \dot{\eta}^*) = \frac{\partial}{\partial x}(\dot{\eta}_T) - \frac{\partial}{\partial x}(\dot{\eta}) - \ddot{\eta}_G \frac{\partial T_g}{\partial x} - T_g \frac{\partial}{\partial x}(\ddot{\eta}_G)$$

$$\frac{\partial}{\partial y}(\Delta \dot{\eta}^*) = \frac{\partial}{\partial y}(\dot{\eta}_T) - \frac{\partial}{\partial y}(\dot{\eta}) - \ddot{\eta}_G \frac{\partial T_g}{\partial y} - T_g \frac{\partial}{\partial y}(\ddot{\eta}_G)$$

$$\frac{\partial}{\partial z}(\Delta \dot{\eta}^*) = \frac{\partial}{\partial z}(\dot{\eta}_T) - \frac{\partial}{\partial z}(\dot{\eta}) - \ddot{\eta}_G \frac{\partial T_g}{\partial z} - T_g \frac{\partial}{\partial z}(\ddot{\eta}_G)$$

$$\frac{\partial}{\partial \dot{x}}(\Delta \dot{\eta}^*) = \frac{\partial}{\partial \dot{x}}(\dot{\eta}_T) - \frac{\partial}{\partial \dot{x}}(\dot{\eta}) - \ddot{\eta}_G \frac{\partial T_g}{\partial \dot{x}} - T_g \frac{\partial}{\partial \dot{x}}(\ddot{\eta}_G)$$

$$\frac{\partial}{\partial \dot{y}}(\Delta \dot{\eta}^*) = \frac{\partial}{\partial \dot{y}}(\dot{\eta}_T) - \frac{\partial}{\partial \dot{y}}(\dot{\eta}) - \ddot{\eta}_G \frac{\partial T_g}{\partial \dot{y}} - T_g \frac{\partial}{\partial \dot{y}}(\ddot{\eta}_G)$$

$$\frac{\partial}{\partial \dot{z}}(\Delta \dot{\eta}^*) = \frac{\partial}{\partial \dot{z}}(\dot{\eta}_T) - \frac{\partial}{\partial \dot{z}}(\dot{\eta}) - \ddot{\eta}_G \frac{\partial T_g}{\partial \dot{z}} - T_g \frac{\partial}{\partial \dot{z}}(\ddot{\eta}_G)$$

$$\frac{\partial}{\partial x}(\dot{\xi}) = \ldots + \dot{x}\,\cancelto{0}{\frac{\partial}{\partial x}(k_{31})} + \dot{y}\,\cancelto{0}{\frac{\partial}{\partial x}(k_{32})} + \dot{z}\,\cancelto{0}{\frac{\partial}{\partial x}(k_{33})}$$

$$\frac{\partial}{\partial y}(\dot{\xi}) = \dot{x}\,\cancelto{0}{\frac{\partial}{\partial y}(k_{31})} + \dot{y}\,\cancelto{0}{\frac{\partial}{\partial y}(k_{32})} + \dot{z}\,\cancelto{0}{\frac{\partial}{\partial y}(k_{33})}$$

$$\frac{\partial}{\partial z}(\dot{\xi}) = \dot{x}\,\cancelto{0}{\frac{\partial}{\partial z}(k_{31})} + \dot{y}\,\cancelto{0}{\frac{\partial}{\partial z}(k_{32})} + \dot{z}\,\cancelto{0}{\frac{\partial}{\partial z}(k_{33})}$$

$$\frac{\partial}{\partial \dot{x}}(\dot{\xi}) = k_{31} + \dot{x}\frac{\partial}{\partial \dot{x}}(k_{31}) + \dot{y}\frac{\partial}{\partial \dot{x}}(k_{32}) + \dot{z}\frac{\partial}{\partial \dot{x}}(k_{33})$$

$$\frac{\partial}{\partial \dot{y}}(\dot{\xi}) = k_{32} + \dot{x}\frac{\partial}{\partial \dot{y}}(k_{31}) + \dot{y}\frac{\partial}{\partial \dot{y}}(k_{32}) + \dot{z}\frac{\partial}{\partial \dot{y}}(k_{33})$$

$$\frac{\partial}{\partial \dot{z}}(\dot{\xi}) = k_{33} + \dot{x}\frac{\partial}{\partial \dot{z}}(k_{31}) + \dot{y}\frac{\partial}{\partial \dot{z}}(k_{32}) + \dot{z}\frac{\partial}{\partial \dot{z}}(k_{33})$$

$$\frac{\partial}{\partial x}(\Delta \dot{\xi}^*) = \frac{\partial}{\partial x}(\dot{\xi}_T) - \cancelto{0}{\frac{\partial}{\partial x}(\dot{\xi})} - T_g \frac{\partial \ddot{\xi}_G}{\partial x} - \ddot{\xi}_G \frac{\partial T_g}{\partial x}$$

$$\frac{\partial}{\partial y}(\Delta \dot{\xi}^*) = \frac{\partial}{\partial y}(\dot{\xi}_T) - \cancelto{0}{\frac{\partial}{\partial y}(\dot{\xi})} - T_g \frac{\partial \ddot{\xi}_G}{\partial y} - \ddot{\xi}_G \frac{\partial T_g}{\partial y}$$

$$\frac{\partial}{\partial z}(\Delta \dot{\xi}^*) = \frac{\partial}{\partial z}(\dot{\xi}_T) - \cancelto{0}{\frac{\partial}{\partial z}(\dot{\xi})} - T_g \frac{\partial \ddot{\xi}_G}{\partial z} - \ddot{\xi}_G \frac{\partial T_g}{\partial z}$$

$$\frac{\partial}{\partial \dot{x}}(\Delta \dot{\xi}^*) = \frac{\partial}{\partial \dot{x}}(\dot{\xi}_T) - \frac{\partial}{\partial \dot{x}}(\dot{\xi}) - T_g \frac{\partial \ddot{\xi}_G}{\partial \dot{x}} - \ddot{\xi}_G \frac{\partial T_g}{\partial \dot{x}}$$

$$\frac{\partial}{\partial \dot{y}}(\Delta \dot{\xi}^*) = \frac{\partial}{\partial \dot{y}}(\dot{\xi}_T) - \frac{\partial}{\partial \dot{y}}(\dot{\xi}) - T_g \frac{\partial \ddot{\xi}_G}{\partial \dot{y}} - \ddot{\xi}_G \frac{\partial T_g}{\partial \dot{y}}$$

$$\frac{\partial}{\partial \dot{z}}(\Delta \dot{\xi}^*) = \frac{\partial}{\partial \dot{z}}(\dot{\xi}_T) - \frac{\partial}{\partial \dot{z}}(\dot{\xi}) - T_g \frac{\partial \ddot{\xi}_G}{\partial \dot{z}} - \ddot{\xi}_G \frac{\partial T_g}{\partial \dot{z}}$$

$$\frac{\partial G}{\partial x} = (1/2)\left(\frac{1}{V_{ex} L}\right)^2 \Big\{ \Big[\Big(2\Delta \dot{\xi}^* \frac{\partial \Delta \dot{\xi}^*}{\partial x} + 2\Delta \dot{\eta}^* \frac{\partial \Delta \dot{\eta}^*}{\partial x}$$
$$+ 2\Delta \dot{\zeta}^* \frac{\partial \Delta \dot{\zeta}^*}{\partial x} \Big) V_{ex} L \Big] - \Big[(\Delta \dot{\xi}^*)^2 + (\Delta \dot{\eta}^*)^2$$
$$+ (\Delta \dot{\zeta}^*)^2 \Big] \Big[V_{ex} \frac{\partial L}{\partial x} + L \cancelto{0}{\frac{\partial V_{ex}}{\partial x}} \Big] \Big\} - (1/2) \Big[L \cancelto{0}{\frac{\partial V_{ex}}{\partial x}} + V_{ex} \frac{\partial L}{\partial x}$$

$$\frac{\partial G}{\partial y} = (1/2)\left(\frac{1}{V_{ex} L}\right)^2 \Big\{ \Big[\Big(2\Delta \dot{\xi}^* \frac{\partial \Delta \dot{\xi}^*}{\partial y} + 2\Delta \dot{\eta}^* \frac{\partial \Delta \dot{\eta}^*}{\partial y}$$
$$+ 2\Delta \dot{\zeta}^* \frac{\partial \Delta \dot{\zeta}^*}{\partial y} \Big) V_{ex} L \Big] - \Big[(\Delta \dot{\xi}^*)^2 + (\Delta \dot{\eta}^*)^2$$
$$+ (\Delta \dot{\zeta}^*)^2 \Big] \Big[V_{ex} \frac{\partial L}{\partial y} + L \cancelto{0}{\frac{\partial V_{ex}}{\partial y}} \Big] \Big\} - (1/2) \Big[L \cancelto{0}{\frac{\partial V_{ex}}{\partial y}} + V_{ex} \frac{\partial L}{\partial y} \Big]$$

$$\frac{\partial G}{\partial z} = (1/2)\left(\frac{1}{V_{ex} L}\right)^2 \Big\{ \Big[\Big(2\Delta \dot{\xi}^* \frac{\partial \Delta \dot{\xi}^*}{\partial z} + 2\Delta \dot{\eta}^* \frac{\partial \Delta \dot{\eta}^*}{\partial z}$$
$$+ 2\Delta \dot{\zeta}^* \frac{\partial \Delta \dot{\zeta}^*}{\partial z} \Big) V_{ex} L \Big] - \Big[(\Delta \dot{\xi}^*)^2 + (\Delta \dot{\eta}^*)^2 + (\Delta \dot{\zeta}^*)^2 \Big]$$
$$\cdot \Big[V_{ex} \frac{\partial L}{\partial z} + L \cancelto{0}{\frac{\partial V_{ex}}{\partial z}} \Big] \Big\} - (1/2) \Big[L \cancelto{0}{\frac{\partial V_{ex}}{\partial z}} + V_{ex} \frac{\partial L}{\partial z} \Big]$$

$$\frac{\partial G}{\partial \dot{x}} = \left(\tfrac{1}{2}\right)\left(\frac{1}{V_{ex} L}\right)^2 \Big\{ \Big[\Big(2\Delta \dot{\xi}^* \frac{\partial \Delta \dot{\xi}^*}{\partial \dot{x}} + 2\Delta \dot{\eta}^* \frac{\partial \Delta \dot{\eta}^*}{\partial \dot{x}}$$

$$+ 2\Delta \dot{\zeta}^* \frac{\partial \Delta \dot{\zeta}^*}{\partial \dot{x}} \Big) V_{ex} L \Big] - \Big[(\Delta \dot{\xi}^*)^2 + (\Delta \dot{\eta}^*)^2$$

$$+ (\Delta \dot{\zeta}^*)^2 \Big] \Big[V_{ex} \frac{\partial L}{\partial \dot{x}} + L \cancelto{0}{\frac{\partial V_{ex}}{\partial \dot{x}}} \Big] \Big\} - \left(\tfrac{1}{2}\right) \Big[L \cancelto{0}{\frac{\partial V_{ex}}{\partial \dot{x}}} + V_{ex} \frac{\partial L}{\partial \dot{x}} \Big]$$

$$\frac{\partial G}{\partial \dot{y}} = \left(\tfrac{1}{2}\right)\left(\frac{1}{V_{ex} L}\right)^2 \Big\{ \Big[\Big(2\Delta \dot{\xi}^* \frac{\partial \Delta \dot{\xi}^*}{\partial \dot{y}} + 2\Delta \dot{\eta}^* \frac{\partial \Delta \dot{\eta}^*}{\partial \dot{y}}$$

$$+ 2\Delta \dot{\zeta}^* \frac{\partial \Delta \dot{\zeta}^*}{\partial \dot{y}} \Big) V_{ex} L \Big] - \Big[(\Delta \dot{\xi}^*)^2 + (\Delta \dot{\eta}^*)^2 + (\Delta \dot{\zeta}^*)^2 \Big]$$

$$\cdot \Big[V_{ex} \frac{\partial L}{\partial \dot{y}} + L \cancelto{0}{\frac{\partial V_{ex}}{\partial \dot{y}}} \Big] \Big\} - \left(\tfrac{1}{2}\right) \Big[L \cancelto{0}{\frac{\partial V_{ex}}{\partial \dot{y}}} + V_{ex} \frac{\partial L}{\partial \dot{y}} \Big]$$

$$\frac{\partial G}{\partial \dot{z}} = \left(\tfrac{1}{2}\right)\left(\frac{1}{V_{ex} L}\right)^2 \Big\{ \Big[\Big(2\Delta \dot{\xi}^* \frac{\partial \Delta \dot{\xi}^*}{\partial \dot{z}} + 2\Delta \dot{\eta}^* \frac{\partial \Delta \dot{\eta}^*}{\partial \dot{z}}$$

$$+ 2\Delta \dot{\zeta}^* \frac{\partial \Delta \dot{\zeta}^*}{\partial \dot{z}} \Big) V_{ex} L \Big] - \Big[(\Delta \dot{\xi}^*)^2 + (\Delta \dot{\eta}^*)^2 + (\Delta \dot{\zeta}^*)^2 \Big]$$

$$\cdot \Big[V_{ex} \frac{\partial L}{\partial \dot{z}} + L \cancelto{0}{\frac{\partial V_{ex}}{\partial \dot{z}}} \Big] \Big\} - \left(\tfrac{1}{2}\right) \Big[L \cancelto{0}{\frac{\partial V_{ex}}{\partial \dot{z}}} + V_{ex} \frac{\partial L}{\partial \dot{z}} \Big]$$

$$\frac{\partial}{\partial x}(\Delta \dot{\xi}) = \frac{\partial}{\partial x}(\Delta \dot{\xi}^*) - \ddot{\xi}_G \frac{\partial \Delta T}{\partial x} - \Delta T \frac{\partial}{\partial x}(\ddot{\xi}_G)$$

$$\frac{\partial}{\partial y}(\Delta \dot{\xi}) = \frac{\partial}{\partial y}(\Delta \dot{\xi}^*) - \ddot{\xi}_G \frac{\partial \Delta T}{\partial y} - \Delta T \frac{\partial}{\partial y}(\ddot{\xi}_G)$$

$$\frac{\partial}{\partial z}(\Delta \dot{\xi}) = \frac{\partial}{\partial z}(\Delta \dot{\xi}^*) - \ddot{\xi}_G \frac{\partial \Delta T}{\partial z} - \Delta T \frac{\partial}{\partial z}(\ddot{\xi}_G)$$

$$\frac{\partial}{\partial \dot{x}}(\Delta \dot{\xi}) = \frac{\partial}{\partial \dot{x}}(\Delta \dot{\xi}^*) - \ddot{\xi}_G \frac{\partial \Delta T}{\partial \dot{x}} - \Delta T \frac{\partial}{\partial \dot{x}}(\ddot{\xi}_G)$$

$$\frac{\partial}{\partial \dot{y}}(\Delta \dot{\xi}) = \frac{\partial}{\partial \dot{y}}(\Delta \dot{\xi}^*) - \ddot{\xi}_G \frac{\partial \Delta T}{\partial \dot{y}} - \Delta T \frac{\partial}{\partial \dot{y}}(\ddot{\xi}_G)$$

$$\frac{\partial}{\partial \dot{z}}(\Delta \dot{\xi}) = \frac{\partial}{\partial \dot{z}}(\Delta \dot{\xi}^*) - \ddot{\xi}_G \frac{\partial \Delta T}{\partial \dot{z}} - \Delta T \frac{\partial}{\partial \dot{z}}(\ddot{\xi}_G)$$

$$\frac{\partial \Delta \ddot{\eta}}{\partial x} = \frac{\partial}{\partial x}(\Delta \ddot{\eta}^{*}) - \ddot{\eta}_G \frac{\partial \Delta T}{\partial x} - \Delta T \frac{\partial}{\partial x}(\ddot{\eta}_G)$$

$$\frac{\partial \Delta \ddot{\eta}}{\partial y} = \frac{\partial}{\partial y}(\Delta \ddot{\eta}^{*}) - \ddot{\eta}_G \frac{\partial \Delta T}{\partial y} - \Delta T \frac{\partial}{\partial y}(\ddot{\eta}_G)$$

$$\frac{\partial \Delta \ddot{\eta}}{\partial z} = \frac{\partial}{\partial z}(\Delta \ddot{\eta}^{*}) - \ddot{\eta}_G \frac{\partial \Delta T}{\partial z} - \Delta T \frac{\partial}{\partial z}(\ddot{\eta}_G)$$

$$\frac{\partial \Delta \ddot{\eta}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}(\Delta \ddot{\eta}^{*}) - \ddot{\eta}_G \frac{\partial \Delta T}{\partial \dot{x}} - \Delta T \frac{\partial}{\partial \dot{x}}(\ddot{\eta}_G)$$

$$\frac{\partial \Delta \ddot{\eta}}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}(\Delta \ddot{\eta}^{*}) - \ddot{\eta}_G \frac{\partial \Delta T}{\partial \dot{y}} - \Delta T \frac{\partial}{\partial \dot{y}}(\ddot{\eta}_G)$$

$$\frac{\partial \Delta \ddot{\eta}}{\partial \dot{z}} = \frac{\partial}{\partial \dot{z}}(\Delta \ddot{\eta}^{*}) - \ddot{\eta}_G \frac{\partial \Delta T}{\partial \dot{z}} - \Delta T \frac{\partial}{\partial \dot{z}}(\ddot{\eta}_G)$$

$$\frac{\partial \Delta \ddot{\zeta}}{\partial x} = \frac{\partial}{\partial x}(\Delta \ddot{\zeta}^{*}) - \ddot{\zeta}_G \frac{\partial \Delta T}{\partial x} - \Delta T \frac{\partial}{\partial x}(\ddot{\zeta}_G)$$

$$\frac{\partial \Delta \ddot{\zeta}}{\partial y} = \frac{\partial}{\partial y}(\Delta \ddot{\zeta}^{*}) - \ddot{\zeta}_G \frac{\partial \Delta T}{\partial y} - \Delta T \frac{\partial}{\partial y}(\ddot{\zeta}_G)$$

$$\frac{\partial \Delta \ddot{\zeta}}{\partial z} = \frac{\partial}{\partial z}(\Delta \ddot{\zeta}^{*}) - \ddot{\zeta}_G \frac{\partial \Delta T}{\partial z} - \Delta T \frac{\partial}{\partial z}(\ddot{\zeta}_G)$$

$$\frac{\partial \Delta \ddot{\zeta}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}(\Delta \ddot{\zeta}^{*}) - \ddot{\zeta}_G \frac{\partial \Delta T}{\partial \dot{x}} - \Delta T \frac{\partial}{\partial \dot{x}}(\ddot{\zeta}_G)$$

$$\frac{\partial \Delta \ddot{\zeta}}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}(\Delta \ddot{\zeta}^{*}) - \ddot{\zeta}_G \frac{\partial \Delta T}{\partial \dot{y}} - \Delta T \frac{\partial}{\partial \dot{y}}(\ddot{\zeta}_G)$$

$$\frac{\partial \Delta \ddot{\zeta}}{\partial \dot{z}} = \frac{\partial}{\partial \dot{z}}(\Delta \ddot{\zeta}^{*}) - \ddot{\zeta}_G \frac{\partial \Delta T}{\partial \dot{z}} - \Delta T \frac{\partial}{\partial \dot{z}}(\ddot{\zeta}_G)$$

$$\frac{\partial \tilde{x}_P}{\partial x} = \left[(\Delta \ddot{\xi})^2 + (\Delta \ddot{\zeta})^2\right]^{-1}\left[\Delta \ddot{\xi} \frac{\partial \Delta \ddot{\zeta}}{\partial x} - \Delta \ddot{\zeta} \frac{\partial \Delta \ddot{\xi}}{\partial x}\right]$$

$$\frac{\partial Z_P}{\partial y} = \left[(\Delta \ddot{\xi})^2 + (\Delta \ddot{\zeta})^2\right]^{-1}\left[\Delta \ddot{\xi} \frac{\partial \Delta \ddot{\zeta}}{\partial y} - \Delta \ddot{\zeta} \frac{\partial \Delta \ddot{\xi}}{\partial y}\right]$$

$$\frac{\partial \tilde{\chi}_p}{\partial z} = \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1} \left[\Delta \dot{\zeta} \frac{\partial \Delta \dot{\xi}}{\partial z} - \Delta \dot{\xi} \frac{\partial \Delta \dot{\zeta}}{\partial z}\right]$$

$$\frac{\partial \tilde{\chi}_p}{\partial \dot{x}} = \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1} \left[\Delta \dot{\zeta} \frac{\partial \Delta \dot{\xi}}{\partial \dot{x}} - \Delta \dot{\xi} \frac{\partial \Delta \dot{\zeta}}{\partial \dot{x}}\right]$$

$$\frac{\partial \tilde{\chi}_p}{\partial \dot{y}} = \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1} \left[\Delta \dot{\zeta} \frac{\partial \Delta \dot{\xi}}{\partial \dot{y}} - \Delta \dot{\xi} \frac{\partial \Delta \dot{\zeta}}{\partial \dot{y}}\right]$$

$$\frac{\partial \tilde{\chi}_p}{\partial \dot{z}} = \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1} \left[\Delta \dot{\zeta} \frac{\partial \Delta \dot{\xi}}{\partial \dot{z}} - \Delta \dot{\xi} \frac{\partial \Delta \dot{\zeta}}{\partial \dot{z}}\right]$$

$$\frac{\partial \tilde{\chi}_y}{\partial x} = \left[1 + \frac{(\Delta \dot{\eta})^2}{(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2}\right]^{-1} \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1}$$

$$\cdot \left\{ \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{1/2} \frac{\partial \Delta \dot{\eta}}{\partial x} - (1/2)\left[\Delta \dot{\eta} \left[(\Delta \dot{\xi})^2\right.\right.\right.$$

$$\left.\left.\left. + (\Delta \dot{\zeta})^2\right]^{-1/2}\right]\left[2\Delta \dot{\xi} \frac{\partial \Delta \dot{\xi}}{\partial x} + 2\Delta \dot{\zeta} \frac{\partial \Delta \dot{\zeta}}{\partial x}\right] \right\}$$

$$\frac{\partial \tilde{\chi}_y}{\partial y} = \left[1 + \frac{(\Delta \dot{\eta})^2}{(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2}\right]^{-1} \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1}$$

$$\cdot \left\{ \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{1/2} \frac{\partial \Delta \dot{\eta}}{\partial y} - (1/2)\left[\Delta \dot{\eta} \left[(\Delta \dot{\xi})^2\right.\right.\right.$$

$$\left.\left.\left. + (\Delta \dot{\zeta})^2\right]^{-1/2}\right]\left[2\Delta \dot{\xi} \frac{\partial \Delta \dot{\xi}}{\partial y} + 2\Delta \dot{\zeta} \frac{\partial \Delta \dot{\zeta}}{\partial y}\right] \right\}$$

$$\frac{\partial \tilde{\chi}_y}{\partial z} = \left[1 + \frac{(\Delta \dot{\eta})^2}{(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2}\right]^{-1} \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1}$$

$$\cdot \left\{ \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{1/2} \frac{\partial \Delta \dot{\eta}}{\partial z} - (1/2)\left[\Delta \dot{\eta}\left[(\Delta \dot{\xi})^2\right.\right.\right.$$

$$\left.\left.\left. + (\Delta \dot{\zeta})^2\right]^{-1/2}\right]\left[2\Delta \dot{\xi} \frac{\partial \Delta \dot{\xi}}{\partial z} + 2\Delta \dot{\zeta} \frac{\partial \Delta \dot{\zeta}}{\partial z}\right]\right\}$$

$$\frac{\partial \tilde{\chi}_y}{\partial \dot{x}} = \left[1 + \frac{(\Delta \dot{\eta})^2}{(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2}\right]^{-1} \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1}$$

$$\cdot \left\{ \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{1/2} \frac{\partial \Delta \dot{\eta}}{\partial \dot{x}} - (1/2)\left[\Delta \dot{\eta}\left[(\Delta \dot{\xi})^2\right.\right.\right.$$

$$\left.\left.\left. + (\Delta \dot{\zeta})^2\right]^{-1/2}\right]\left[2\Delta \dot{\xi} \frac{\partial \Delta \dot{\xi}}{\partial \dot{x}} + 2\Delta \dot{\zeta} \frac{\partial \Delta \dot{\zeta}}{\partial \dot{x}}\right]\right\}$$

$$\frac{\partial \tilde{\chi}_y}{\partial \dot{y}} = \left[1 + \frac{(\Delta \dot{\eta})^2}{(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2}\right]^{-1} \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1}$$

$$\cdot \left\{ \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{1/2} \frac{\partial \Delta \dot{\eta}}{\partial \dot{y}} - (1/2)\left[\Delta \dot{\eta}\left[(\Delta \dot{\xi})^2\right.\right.\right.$$

$$\left.\left.\left. + (\Delta \dot{\zeta})^2\right]^{-1/2}\right]\left[2\Delta \dot{\xi} \frac{\partial \Delta \dot{\xi}}{\partial \dot{y}} + 2\Delta \dot{\zeta} \frac{\partial \Delta \dot{\zeta}}{\partial \dot{y}}\right]\right\}$$

$$\frac{\partial \tilde{\chi}_y}{\partial \dot{z}} = \left[1 + \frac{(\Delta \dot{\eta})^2}{(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2}\right]^{-1} \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{-1}$$

$$\cdot \left\{ \left[(\Delta \dot{\xi})^2 + (\Delta \dot{\zeta})^2\right]^{1/2} \frac{\partial \Delta \dot{\eta}}{\partial \dot{z}} - (1/2)\left[\Delta \dot{\eta}\left[(\Delta \dot{\xi})^2\right.\right.\right.$$

$$\left.\left.\left. + (\Delta \dot{\zeta})^2\right]^{-1/2}\right]\left[2\Delta \dot{\xi} \frac{\partial \Delta \dot{\xi}}{\partial \dot{z}} + 2\Delta \dot{\zeta} \frac{\partial \Delta \dot{\zeta}}{\partial \dot{z}}\right]\right\}$$

$$\frac{\partial C_y}{\partial x} = -S \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial x} + \cos\tilde{\chi}_y \frac{\partial S}{\partial x}$$

$$\frac{\partial C_y}{\partial y} = -S \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial y} + \cos\tilde{\chi}_y \frac{\partial S}{\partial y}$$

$$\frac{\partial C_y}{\partial z} = -S \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial z} + \cos\tilde{\chi}_y \frac{\partial S}{\partial z}$$

$$\frac{\partial C_y}{\partial \dot{x}} = -S \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{x}} + \cos\tilde{\chi}_y \frac{\partial S}{\partial \dot{x}}$$

$$\frac{\partial C_y}{\partial \dot{y}} = -S \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{y}} + \cos\tilde{\chi}_y \frac{\partial S}{\partial \dot{y}}$$

$$\frac{\partial C_y}{\partial \dot{z}} = -S \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{z}} + \cos\tilde{\chi}_y \frac{\partial S}{\partial \dot{z}}$$

$$\frac{\partial D_y}{\partial x} = -Q \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial x} + \cos\tilde{\chi}_y \frac{\partial Q}{\partial x}$$

$$\frac{\partial D_y}{\partial y} = -Q \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial y} + \cos\tilde{\chi}_y \frac{\partial Q}{\partial y}$$

$$\frac{\partial D_y}{\partial z} = -Q \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial z} + \cos\tilde{\chi}_y \frac{\partial Q}{\partial z}$$

$$\frac{\partial D_y}{\partial \dot{x}} = -Q \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{x}} + \cos\tilde{\chi}_y \frac{\partial Q}{\partial \dot{x}}$$

$$\frac{\partial D_y}{\partial \dot{y}} = -Q \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{y}} + \cos\tilde{\chi}_y \frac{\partial Q}{\partial \dot{y}}$$

$$\frac{\partial D_y}{\partial \dot{z}} = -Q \sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{z}} + \cos\tilde{\chi}_y \frac{\partial Q}{\partial \dot{z}}$$

$$\frac{\partial E_y}{\partial x} = \frac{\partial \eta}{\partial x} + \dot{\eta}\frac{\partial T_g}{\partial x} + T_g\frac{\partial \dot{\eta}}{\partial x} + (1/2)\,T_g^2\,\frac{\partial \ddot{\eta}_G}{\partial x}$$

$$+ \ddot{\eta}_G T_g \frac{\partial T_g}{\partial x} - S\cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial x} - \sin\tilde{\chi}_y \frac{\partial S}{\partial x}$$

$$\frac{\partial E_y}{\partial y} = \frac{\partial \eta}{\partial y} + \dot{\eta}\frac{\partial T_g}{\partial y} + T_g\frac{\partial \dot{\eta}}{\partial y} + (1/2)\,T_g^2\,\frac{\partial \ddot{\eta}_G}{\partial y}$$

$$+ \ddot{\eta}_G T_g \frac{\partial T_g}{\partial y} - S\cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial y} - \sin\tilde{\chi}_y \frac{\partial S}{\partial y}$$

$$\frac{\partial E_y}{\partial z} = \frac{\partial \eta}{\partial z} + \dot{\eta}\frac{\partial T_g}{\partial z} + T_g\frac{\partial \dot{\eta}}{\partial z} + (1/2)\,T_g^2\,\frac{\partial \ddot{\eta}_G}{\partial z}$$

$$+ \ddot{\eta}_G T_g \frac{\partial T_g}{\partial z} - S\cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial z} - \sin\tilde{\chi}_y \frac{\partial S}{\partial z}$$

$$\frac{\partial E_y}{\partial \dot{x}} = \frac{\partial \eta}{\partial \dot{x}} + \dot{\eta}\frac{\partial T_g}{\partial \dot{x}} + T_g\frac{\partial \dot{\eta}}{\partial \dot{x}} + (1/2)\,T_g^2\,\frac{\partial \ddot{\eta}_G}{\partial \dot{x}}$$

$$+ \ddot{\eta}_G T_g \frac{\partial T_g}{\partial \dot{x}} - S\cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{x}} - \sin\tilde{\chi}_y \frac{\partial S}{\partial \dot{x}}$$

$$\frac{\partial E_y}{\partial \dot{y}} = \frac{\partial \eta}{\partial \dot{y}} + \dot{\eta}\frac{\partial T_g}{\partial \dot{y}} + T_g\frac{\partial \dot{\eta}}{\partial \dot{y}} + (1/2)\,T_g^2\,\frac{\partial \ddot{\eta}_G}{\partial \dot{y}}$$

$$+ \ddot{\eta}_G T_g \frac{\partial T_g}{\partial \dot{y}} - S\cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{y}} - \sin\tilde{\chi}_y \frac{\partial S}{\partial \dot{y}}$$

$$\frac{\partial E_y}{\partial \dot{z}} = \frac{\partial \eta}{\partial \dot{z}} + \dot{\eta}\frac{\partial T_g}{\partial \dot{z}} + T_g\frac{\partial \dot{\eta}}{\partial \dot{z}} + (1/2)\,T_g^2\,\frac{\partial \ddot{\eta}_G}{\partial \dot{z}}$$

$$+ \ddot{\eta}_G T_g \frac{\partial T_g}{\partial \dot{z}} - S\cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{z}} - \sin\tilde{\chi}_y \frac{\partial S}{\partial \dot{z}}$$

$$\frac{\partial K_3}{\partial x} = \left[A_y D_y - B_y C_y\right]^{-2} \left\{ \left[E_y \frac{\partial B_y}{\partial x} + B_y \frac{\partial E_y}{\partial x}\right] \right.$$

$$\circ \left[A_y D_y - B_y C_y\right] - B_y E_y \left[A_y \frac{\partial D_y}{\partial x} + D_y \frac{\partial A_y}{\partial x}\right.$$

$$\left.\left. - B_y \frac{\partial C_y}{\partial x} - C_y \frac{\partial B_y}{\partial x}\right]\right\}$$

$$\frac{\partial K_3}{\partial y} = \left[A_y D_y - B_y C_y\right]^{-2} \left\{ \left[E_y \frac{\partial B_y}{\partial y} + B_y \frac{\partial E_y}{\partial y}\right] \right.$$

$$\circ \left[A_y D_y - B_y C_y\right] - B_y E_y \left[A_y \frac{\partial D_y}{\partial y} + D_y \frac{\partial A_y}{\partial y}\right.$$

$$\left.\left. - B_y \frac{\partial C_y}{\partial y} - C_y \frac{\partial B_y}{\partial y}\right]\right\}$$

$$\frac{\partial K_3}{\partial z} = \left[A_y D_y - B_y C_y\right]^{-2} \left\{ \left[E_y \frac{\partial B_y}{\partial z} + B_y \frac{\partial E_y}{\partial z}\right] \right.$$

$$\circ \left[A_y D_y - B_y C_y\right] - B_y E_y \left[A_y \frac{\partial D_y}{\partial z} + D_y \frac{\partial A_y}{\partial z}\right.$$

$$\left.\left. - B_y \frac{\partial C_y}{\partial z} - C_y \frac{\partial B_y}{\partial z}\right]\right\}$$

$$\frac{\partial K_3}{\partial \dot{x}} = \left[A_y D_y - B_y C_y\right]^{-2} \left\{ \left[E_y \frac{\partial B_y}{\partial \dot{x}} + B_y \frac{\partial E_y}{\partial \dot{x}}\right] \right.$$

$$\circ \left[A_y D_y - B_y C_y\right] - B_y E_y \left[A_y \frac{\partial D_y}{\partial \dot{x}} + D_y \frac{\partial A_y}{\partial \dot{x}}\right.$$

$$\left.\left. - B_y \frac{\partial C_y}{\partial \dot{x}} - C_y \frac{\partial B_y}{\partial \dot{x}}\right]\right\}$$

$$\frac{\partial K_3}{\partial \dot{y}} = \left[A_y D_y - B_y C_y\right]^{-2} \left\{ \left[E_y \frac{\partial B_y}{\partial \dot{y}} + B_y \frac{\partial E_y}{\partial \dot{y}}\right] \right.$$

$$\cdot \left[A_y D_y - B_y C_y\right] - B_y E_y \left[A_y \frac{\partial D_y}{\partial \dot{y}} + D_y \frac{\partial A_y}{\partial \dot{y}}\right.$$

$$\left.\left. - B_y \frac{\partial C_y}{\partial \dot{y}} - C_y \frac{\partial B_y}{\partial \dot{y}} \right] \right\}$$

$$\frac{\partial K_3}{\partial \dot{z}} = \left[A_y D_y - B_y C_y\right]^{-2} \left\{ \left[E_y \frac{\partial B_y}{\partial \dot{z}} + B_y \frac{\partial E_y}{\partial \dot{z}}\right] \right.$$

$$\cdot \left[A_y D_y - B_y C_y\right] - B_y E_y \left[A_y \frac{\partial D_y}{\partial \dot{z}} + D_y \frac{\partial A_y}{\partial \dot{z}}\right.$$

$$\left.\left. - B_y \frac{\partial C_y}{\partial \dot{z}} - C_y \frac{\partial B_y}{\partial \dot{z}} \right] \right\}$$

$$\frac{\partial K_4}{\partial x} = \left[B_y\right]^{-1} \left[A_y \frac{\partial K_3}{\partial x} + K_3 \frac{\partial A_y}{\partial x}\right] - K_3 A_y \frac{\partial B_y}{\partial x} (B_y)^{-2}$$

$$\frac{\partial K_4}{\partial y} = \left[B_y\right]^{-1} \left[A_y \frac{\partial K_3}{\partial y} + K_3 \frac{\partial A_y}{\partial y}\right] - K_3 A_y \frac{\partial B_y}{\partial y} (B_y)^{-2}$$

$$\frac{\partial K_4}{\partial z} = \left[B_y\right]^{-1} \left[A_y \frac{\partial K_3}{\partial z} + K_3 \frac{\partial A_y}{\partial z}\right] - K_3 A_y \frac{\partial B_y}{\partial z} (B_y)^{-2}$$

$$\frac{\partial K_4}{\partial \dot{x}} = \left[B_y\right]^{-1} \left[A_y \frac{\partial K_3}{\partial \dot{x}} + K_3 \frac{\partial A_y}{\partial \dot{x}}\right] - K_3 A_y \frac{\partial B_y}{\partial \dot{x}} (B_y)^{-2}$$

$$\frac{\partial K_4}{\partial \dot{y}} = \left[B_y\right]^{-1} \left[A_y \frac{\partial K_3}{\partial \dot{y}} + K_3 \frac{\partial A_y}{\partial \dot{y}}\right] - K_3 A_y \frac{\partial B_y}{\partial \dot{y}} (B_y)^{-2}$$

$$\frac{\partial K_4}{\partial \dot{z}} = \left[B_y\right]^{-1} \left[A_y \frac{\partial K_3}{\partial \dot{z}} + K_3 \frac{\partial A_y}{\partial \dot{z}}\right] - K_3 A_y \frac{\partial B_y}{\partial \dot{z}} (B_y)^{-2}$$

$$\frac{\partial M}{\partial x} = -\sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial x} + K_3 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial x} + \sin\tilde{\chi}_y \frac{\partial K_3}{\partial x}$$

$$\frac{\partial M}{\partial y} = -\sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial y} + K_3 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial y} + \sin\tilde{\chi}_y \frac{\partial K_3}{\partial y}$$

$$\frac{\partial M}{\partial z} = -\sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial z} + K_3 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial z} + \sin\tilde{\chi}_y \frac{\partial K_3}{\partial z}$$

$$\frac{\partial M}{\partial \dot{x}} = -\sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{x}} + K_3 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{x}} + \sin\tilde{\chi}_y \frac{\partial K_3}{\partial \dot{x}}$$

$$\frac{\partial M}{\partial \dot{y}} = -\sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{y}} + K_3 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{y}} + \sin\tilde{\chi}_y \frac{\partial K_3}{\partial \dot{y}}$$

$$\frac{\partial M}{\partial \dot{z}} = -\sin\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{z}} + K_3 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{z}} + \sin\tilde{\chi}_y \frac{\partial K_3}{\partial \dot{z}}$$

$$\frac{\partial N}{\partial x} = K_4 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial x} + \sin\tilde{\chi}_y \frac{\partial K_4}{\partial x}$$

$$\frac{\partial N}{\partial y} = K_4 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial y} + \sin\tilde{\chi}_y \frac{\partial K_4}{\partial y}$$

$$\frac{\partial N}{\partial z} = K_4 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial z} + \sin\tilde{\chi}_y \frac{\partial K_4}{\partial z}$$

$$\frac{\partial N}{\partial \dot{x}} = K_4 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{x}} + \sin\tilde{\chi}_y \frac{\partial K_4}{\partial \dot{x}}$$

$$\frac{\partial N}{\partial \dot{y}} = K_4 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{y}} + \sin\tilde{\chi}_y \frac{\partial K_4}{\partial \dot{y}}$$

$$\frac{\partial N}{\partial \dot{z}} = K_4 \cos\tilde{\chi}_y \frac{\partial \tilde{\chi}_y}{\partial \dot{z}} + \sin\tilde{\chi}_y \frac{\partial K_4}{\partial \dot{z}}$$

$$\frac{\partial A_p}{\partial x} = M \frac{\partial A_y}{\partial x} + A_y \frac{\partial M}{\partial x} - N \frac{\partial B_y}{\partial x} - B_y \frac{\partial N}{\partial x}$$

$$\frac{\partial A_p}{\partial y} = M \frac{\partial A_y}{\partial y} + A_y \frac{\partial M}{\partial y} - N \frac{\partial B_y}{\partial y} - B_y \frac{\partial N}{\partial y}$$

$$\frac{\partial A_p}{\partial z} = M \frac{\partial A_y}{\partial z} + A_y \frac{\partial M}{\partial z} - N \frac{\partial B_y}{\partial z} - B_y \frac{\partial N}{\partial z}$$

$$\frac{\partial A_p}{\partial \dot{x}} = M \frac{\partial A_y}{\partial \dot{x}} + A_y \frac{\partial M}{\partial \dot{x}} - N \frac{\partial B_y}{\partial \dot{x}} - B_y \frac{\partial N}{\partial \dot{x}}$$

$$\frac{\partial A_p}{\partial \dot{y}} = M \frac{\partial A_y}{\partial \dot{y}} + A_y \frac{\partial M}{\partial \dot{y}} - N \frac{\partial B_y}{\partial \dot{y}} - B_y \frac{\partial N}{\partial \dot{y}}$$

$$\frac{\partial A_p}{\partial \dot{z}} = M \frac{\partial A_y}{\partial \dot{z}} + A_y \frac{\partial M}{\partial \dot{z}} - N \frac{\partial B_y}{\partial \dot{z}} - B_y \frac{\partial N}{\partial \dot{z}}$$

$$\frac{\partial B_p}{\partial x} = M \frac{\partial B_y}{\partial x} + B_y \frac{\partial M}{\partial x} - N \frac{\partial P}{\partial x} - P \frac{\partial N}{\partial x}$$

$$\frac{\partial B_p}{\partial y} = M \frac{\partial B_y}{\partial y} + B_y \frac{\partial M}{\partial y} - N \frac{\partial P}{\partial y} - P \frac{\partial N}{\partial y}$$

$$\frac{\partial B_p}{\partial z} = M \frac{\partial B_y}{\partial z} + B_y \frac{\partial M}{\partial z} - N \frac{\partial P}{\partial z} - P \frac{\partial N}{\partial z}$$

$$\frac{\partial B_p}{\partial \dot{x}} = M \frac{\partial B_y}{\partial \dot{x}} + B_y \frac{\partial M}{\partial \dot{x}} - N \frac{\partial P}{\partial \dot{x}} - P \frac{\partial N}{\partial \dot{x}}$$

$$\frac{\partial B_p}{\partial \dot{y}} = M \frac{\partial B_y}{\partial \dot{y}} + B_y \frac{\partial M}{\partial \dot{y}} - N \frac{\partial P}{\partial \dot{y}} - P \frac{\partial N}{\partial \dot{y}}$$

$$\frac{\partial B_p}{\partial \dot{z}} = M \frac{\partial B_y}{\partial \dot{z}} + B_y \frac{\partial M}{\partial \dot{z}} - N \frac{\partial P}{\partial \dot{z}} - P \frac{\partial N}{\partial \dot{z}}$$

$$\frac{\partial C_p}{\partial x} = -\sin \tilde{\chi}_p \left[M \cdot S - N \cdot Q \right] \frac{\partial \tilde{\chi}_p}{\partial x}$$

$$+ \cos \tilde{\chi}_p \left[M \frac{\partial S}{\partial x} + S \frac{\partial M}{\partial x} - N \frac{\partial Q}{\partial x} - Q \frac{\partial N}{\partial x} \right]$$

$$\frac{\partial C_p}{\partial y} = -\sin \tilde{\chi}_p \left[M \cdot S - N \cdot Q \right] \frac{\partial \tilde{\chi}_p}{\partial y}$$

$$+ \cos \tilde{\chi}_p \left[M \frac{\partial S}{\partial y} + S \frac{\partial M}{\partial y} - N \frac{\partial Q}{\partial y} - Q \frac{\partial N}{\partial y} \right]$$

$$\frac{\partial C_p}{\partial z} = -\sin \tilde{\chi}_p \left[M \cdot S - N \cdot Q \right] \frac{\partial \tilde{\chi}_p}{\partial z}$$

$$+ \cos \tilde{\chi}_p \left[M \frac{\partial S}{\partial z} + S \frac{\partial M}{\partial z} - N \frac{\partial Q}{\partial z} - Q \frac{\partial N}{\partial z} \right]$$

$$\frac{\partial C_p}{\partial \dot{x}} = -\sin \tilde{\chi}_p \left[M \cdot S - N \cdot Q \right] \frac{\partial \tilde{\chi}_p}{\partial \dot{x}}$$

$$+ \cos \tilde{\chi}_p \left[M \frac{\partial S}{\partial \dot{x}} + S \frac{\partial M}{\partial \dot{x}} - N \frac{\partial Q}{\partial \dot{x}} - Q \frac{\partial N}{\partial x} \right]$$

$$\frac{\partial C_p}{\partial \dot{y}} = -\sin \tilde{\chi}_p \left[M \cdot S - N \cdot Q \right] \frac{\partial \tilde{\chi}_p}{\partial \dot{y}}$$

$$+ \cos \tilde{\chi}_p \left[M \frac{\partial S}{\partial \dot{y}} + S \frac{\partial M}{\partial \dot{y}} - N \frac{\partial Q}{\partial \dot{y}} - Q \frac{\partial N}{\partial \dot{y}} \right]$$

$$\frac{\partial C_p}{\partial \dot{z}} = -\sin \tilde{\chi}_p \left[M \cdot S - N \cdot Q \right] \frac{\partial \tilde{\chi}_p}{\partial \dot{z}}$$

$$+ \cos \tilde{\chi}_p \left[M \frac{\partial S}{\partial \dot{z}} + S \frac{\partial M}{\partial \dot{z}} - N \frac{\partial Q}{\partial \dot{z}} - Q \frac{\partial N}{\partial \dot{z}} \right]$$

$$\frac{\partial D_P}{\partial x} = -\sin\tilde{\chi}_p\left[M\cdot Q - N\cdot \bar{U}\right]\frac{\partial \tilde{\chi}_p}{\partial x}$$

$$+\cos\tilde{\chi}_p\left[M\frac{\partial Q}{\partial x} + Q\frac{\partial M}{\partial x} - N\frac{\partial \bar{U}}{\partial x} - \bar{U}\frac{\partial N}{\partial x}\right]$$

$$\frac{\partial D_P}{\partial y} = -\sin\tilde{\chi}_p\left[M\cdot Q - N\cdot \bar{U}\right]\frac{\partial \tilde{\chi}_p}{\partial y}$$

$$+\cos\tilde{\chi}_p\left[M\frac{\partial Q}{\partial y} + Q\frac{\partial M}{\partial y} - N\frac{\partial \bar{U}}{\partial y} - \bar{U}\frac{\partial N}{\partial y}\right]$$

$$\frac{\partial D_P}{\partial z} = -\sin\tilde{\chi}_p\left[M\cdot Q - N\cdot \bar{U}\right]\frac{\partial \tilde{\chi}_p}{\partial z}$$

$$+\cos\tilde{\chi}_p\left[M\frac{\partial Q}{\partial z} + Q\frac{\partial M}{\partial z} - N\frac{\partial \bar{U}}{\partial z} - \bar{U}\frac{\partial N}{\partial z}\right]$$

$$\frac{\partial D_P}{\partial \dot{x}} = -\sin\tilde{\chi}_p\left[M\cdot Q - N\cdot \bar{U}\right]\frac{\partial \tilde{\chi}_p}{\partial \dot{x}}$$

$$+\cos\tilde{\chi}_p\left[M\frac{\partial Q}{\partial \dot{x}} + Q\frac{\partial M}{\partial \dot{x}} - N\frac{\partial \bar{U}}{\partial \dot{x}} - \bar{U}\frac{\partial N}{\partial \dot{x}}\right]$$

$$\frac{\partial D_P}{\partial \dot{y}} = -\sin\tilde{\chi}_p\left[M\cdot Q - N\cdot \bar{U}\right]\frac{\partial \tilde{\chi}_p}{\partial \dot{y}}$$

$$+\cos\tilde{\chi}_p\left[M\frac{\partial Q}{\partial \dot{y}} + Q\frac{\partial M}{\partial \dot{y}} - N\frac{\partial \bar{U}}{\partial \dot{y}} - \bar{U}\frac{\partial N}{\partial \dot{y}}\right]$$

$$\frac{\partial D_P}{\partial \dot{z}} = -\sin\tilde{\chi}_p\left[M\cdot Q - N\cdot \bar{U}\right]\frac{\partial \tilde{\chi}_p}{\partial \dot{z}}$$

$$+\cos\tilde{\chi}_p\left[M\frac{\partial Q}{\partial \dot{z}} + Q\frac{\partial M}{\partial \dot{z}} - N\frac{\partial \bar{U}}{\partial \dot{z}} - \bar{U}\frac{\partial N}{\partial \dot{z}}\right]$$

$$\frac{\partial \xi_T}{\partial x} = \frac{\partial R_T}{\partial x}$$

$$\frac{\partial \xi_T}{\partial y} = \frac{\partial R_T}{\partial y}$$

$$\frac{\partial \xi_T}{\partial z} = \frac{\partial R_T}{\partial z}$$

$$\frac{\partial \xi_T}{\partial \dot{x}} = \frac{\partial R_T}{\partial \dot{x}}$$

$$\frac{\partial \xi_T}{\partial \dot{y}} = \frac{\partial R_T}{\partial \dot{y}}$$

$$\frac{\partial \xi_T}{\partial \dot{z}} = \frac{\partial R_T}{\partial \dot{z}}$$

$$\frac{\partial E_P}{\partial x} = \frac{\partial \xi}{\partial x} - \frac{\partial \xi_T}{\partial x} + \dot{\xi} \frac{\partial T_g}{\partial x} + T_g \frac{\partial \dot{\xi}}{\partial x} + \frac{1}{2} T_g^2 \frac{\partial \ddot{\xi}_G}{\partial x}$$

$$+ T_g \ddot{\xi}_G \frac{\partial T_g}{\partial x} - (MS-NQ) \cos \tilde{\chi}_P \frac{\partial \tilde{\chi}_P}{\partial x} - \sin(\tilde{\chi}_P) \frac{\partial [MS-}{\partial x}$$

$$\frac{\partial E_P}{\partial y} = \frac{\partial \xi}{\partial y} - \frac{\partial \xi_T}{\partial y} + \dot{\xi} \frac{\partial T_g}{\partial y} + T_g \frac{\partial \dot{\xi}}{\partial y} + \frac{1}{2} T_g^2 \frac{\partial \ddot{\xi}_G}{\partial y}$$

$$+ T_g \ddot{\xi}_G \frac{\partial T_g}{\partial y} - (MS-NQ) \cos \tilde{\chi}_P \frac{\partial \tilde{\chi}_P}{\partial y} - \sin(\tilde{\chi}_P) \frac{\partial [MS-NQ]}{\partial y}$$

$$\frac{\partial E_P}{\partial z} = \frac{\partial \xi}{\partial z} - \frac{\partial \xi_T}{\partial z} + \dot{\xi} \frac{\partial T_g}{\partial z} + T_g \frac{\partial \dot{\xi}}{\partial z} + \frac{1}{2} T_g^2 \frac{\partial \ddot{\xi}_G}{\partial z}$$

$$+ T_g \ddot{\xi}_G \frac{\partial T_g}{\partial z} - (MS-NQ) \cos \tilde{\chi}_P \frac{\partial \tilde{\chi}_P}{\partial z} - \sin(\tilde{\chi}_P) \frac{\partial [MS-NQ}{\partial z}$$

$$\frac{\partial E_p}{\partial \dot{x}} = \frac{\partial \xi}{\partial \dot{x}} - \frac{\partial \xi_T}{\partial \dot{x}} + \dot{\xi} \frac{\partial T_g}{\partial \dot{x}} + T_g \frac{\partial \dot{\xi}}{\partial \dot{x}} + \frac{1}{2} T_g^2 \frac{\partial \ddot{\xi}_G}{\partial \dot{x}}$$

$$+ T_g \ddot{\xi}_G \frac{\partial T_g}{\partial \dot{x}} - (MS-NQ) \cos \tilde{\chi}_p \frac{\partial \tilde{\chi}_p}{\partial \dot{x}} - \sin(\tilde{\chi}_p) \frac{\partial [MS-NQ]}{\partial \dot{x}}$$

$$\frac{\partial E_p}{\partial \dot{y}} = \frac{\partial \xi}{\partial \dot{y}} - \frac{\partial \xi_T}{\partial \dot{y}} + \dot{\xi} \frac{\partial T_g}{\partial \dot{y}} + T_g \frac{\partial \dot{\xi}}{\partial \dot{y}} + \frac{1}{2} T_g^2 \frac{\partial \ddot{\xi}_G}{\partial \dot{y}}$$

$$+ T_g \ddot{\xi}_G \frac{\partial T_g}{\partial \dot{y}} - (MS-NQ) \cos \tilde{\chi}_p \frac{\partial \tilde{\chi}_p}{\partial \dot{y}} - \sin(\tilde{\chi}_p) \frac{\partial [MS-NQ]}{\partial \dot{y}}$$

$$\frac{\partial E_p}{\partial \dot{z}} = \frac{\partial \xi}{\partial \dot{z}} - \frac{\partial \xi_T}{\partial \dot{z}} + \dot{\xi} \frac{\partial T_g}{\partial \dot{z}} + T_g \frac{\partial \dot{\xi}}{\partial \dot{z}} + \frac{1}{2} T_g^2 \frac{\partial \ddot{\xi}_G}{\partial \dot{z}}$$

$$+ T_g \ddot{\xi}_G \frac{\partial T_g}{\partial \dot{z}} - (MS-NQ) \cos \tilde{\chi}_p \frac{\partial \tilde{\chi}_p}{\partial \dot{z}} - \sin(\tilde{\chi}_p) \frac{\partial [MS-NQ]}{\partial \dot{z}}$$

$$\frac{\partial K_1}{\partial x} = \left\{ \left[B_p \frac{\partial E_p}{\partial x} + E_p \frac{\partial B_p}{\partial x} \right] \left[A_p D_p - B_p C_p \right] - \right.$$

$$\left. B_p E_p \left[A_p \frac{\partial D_p}{\partial x} + D_p \frac{\partial A_p}{\partial x} - B_p \frac{\partial C_p}{\partial x} - C_p \frac{\partial B_p}{\partial x} \right] \right\}$$

$$\cdot \left[A_p D_p - B_p C_p \right]^{-2}$$

$$\frac{\partial K_1}{\partial y} = \left\{ \left[B_p \frac{\partial E_p}{\partial y} + E_p \frac{\partial B_p}{\partial y} \right] \left[A_p D_p - B_p C_p \right] - \right.$$

$$\left. B_p E_p \left[A_p \frac{\partial D_p}{\partial y} + D_p \frac{\partial A_p}{\partial y} - B_p \frac{\partial C_p}{\partial y} - C_p \frac{\partial B_p}{\partial y} \right] \right\}$$

$$\cdot \left[A_p D_p - B_p C_p \right]^{-2}$$

$$\frac{\partial K_1}{\partial z} = \left\{ \left[B_p \frac{\partial E_p}{\partial z} + E_p \frac{\partial B_p}{\partial z} \right] \left[A_p D_p - B_p C_p \right] - B_p E_p \left[A_p \frac{\partial D_p}{\partial z} + D_p \frac{\partial A_p}{\partial z} - B_p \frac{\partial C_p}{\partial z} - C_p \frac{\partial B_p}{\partial z} \right] \right\} \left[A_p D_p - B_p C_p \right]^{-2}$$

$$\frac{\partial K_1}{\partial \dot{x}} = \left\{ \left[B_p \frac{\partial E_p}{\partial \dot{x}} + E_p \frac{\partial B_p}{\partial \dot{x}} \right] \left[A_p D_p - B_p C_p \right] - B_p E_p \left[A_p \frac{\partial D_p}{\partial \dot{x}} + D_p \frac{\partial A_p}{\partial \dot{x}} - B_p \frac{\partial C_p}{\partial \dot{x}} - C_p \frac{\partial B_p}{\partial \dot{x}} \right] \right\} \left[A_p D_p - B_p C_p \right]^{-2}$$

$$\frac{\partial K_1}{\partial \dot{y}} = \left\{ \left[B_p \frac{\partial E_p}{\partial \dot{y}} + E_p \frac{\partial B_p}{\partial \dot{y}} \right] \left[A_p D_p - B_p C_p \right] - B_p E_p \left[A_p \frac{\partial D_p}{\partial \dot{y}} + D_p \frac{\partial A_p}{\partial \dot{y}} - B_p \frac{\partial C_p}{\partial \dot{y}} - C_p \frac{\partial B_p}{\partial \dot{y}} \right] \right\} \left[A_p D_p - B_p C_p \right]^{-2}$$

$$\frac{\partial K_1}{\partial \dot{z}} = \left\{ \left[B_p \frac{\partial E_p}{\partial \dot{z}} + E_p \frac{\partial B_p}{\partial \dot{z}} \right] \left[A_p D_p - B_p C_p \right] - B_p E_p \left[A_p \frac{\partial D_p}{\partial \dot{z}} + D_p \frac{\partial A_p}{\partial \dot{z}} - B_p \frac{\partial C_p}{\partial \dot{z}} - C_p \frac{\partial B_p}{\partial \dot{z}} \right] \right\} \left[A_p D_p - B_p C_p \right]^{-2}$$

$$\frac{\partial K_2}{\partial x} = \left\{ \left[K_1 \frac{\partial A_p}{\partial x} + A_p \frac{\partial K_1}{\partial x} \right] B_p - \left[K_1 A_p \frac{\partial B_p}{\partial x} \right] \right\} \left[B_p \right]^{-2}$$

$$\frac{\partial K_2}{\partial y} = \left\{ \left[K_1 \frac{\partial A_p}{\partial y} + A_p \frac{\partial K_1}{\partial y} \right] B_p - \left[K_1 A_p \frac{\partial B_p}{\partial y} \right] \right\} \left[B_p \right]^{-2}$$

$$\frac{\partial K_2}{\partial z} = \left\{ \left[K_1 \frac{\partial A_p}{\partial z} + A_p \frac{\partial K_1}{\partial z} \right] B_p - \left[K_1 A_p \frac{\partial B_p}{\partial z} \right] \right\} \left[B_p \right]^{-2}$$

$$\frac{\partial K_2}{\partial \dot{x}} = \left\{ \left[K_1 \frac{\partial A_p}{\partial \dot{x}} + A_p \frac{\partial K_1}{\partial \dot{x}} \right] B_p - \left[K_1 A_p \frac{\partial B_p}{\partial \dot{x}} \right] \right\} \left[B_p \right]^{-2}$$

$$\frac{\partial K_2}{\partial \dot{y}} = \left\{ \left[K_1 \frac{\partial A_p}{\partial \dot{y}} + A_p \frac{\partial K_1}{\partial \dot{y}} \right] B_p - \left[K_1 A_p \frac{\partial B_p}{\partial \dot{y}} \right] \right\} \left[B_p \right]^{-2}$$

$$\frac{\partial K_2}{\partial \dot{z}} = \left\{ \left[K_1 \frac{\partial A_p}{\partial \dot{z}} + A_p \frac{\partial K_1}{\partial \dot{z}} \right] B_p - \left[K_1 A_p \frac{\partial B_p}{\partial \dot{z}} \right] \right\} \left[B_p \right]^{-2}$$

$$\frac{\partial \chi_p'}{\partial x} = \frac{\partial \tilde{\chi}_p}{\partial x} - \frac{\partial K_1}{\partial x} + \frac{\partial K_2}{\partial x} \delta_T + K_2 \cancelto{0}{\frac{\partial \delta_T}{\partial x}}$$

$$\frac{\partial \chi_p'}{\partial y} = \frac{\partial \tilde{\chi}_p}{\partial y} - \frac{\partial K_1}{\partial y} + \frac{\partial K_2}{\partial y} \delta_T + K_2 \cancelto{0}{\frac{\partial \delta_T}{\partial y}}$$

$$\frac{\partial \chi_p'}{\partial z} = \frac{\partial \tilde{\chi}_p}{\partial z} - \frac{\partial K_1}{\partial z} + \frac{\partial K_2}{\partial z} \delta_T + K_2 \cancelto{0}{\frac{\partial \delta_T}{\partial z}}$$

$$\frac{\partial \chi_p'}{\partial \dot{x}} = \frac{\partial \tilde{\chi}_p}{\partial \dot{x}} - \frac{\partial K_1}{\partial \dot{x}} + \frac{\partial K_2}{\partial \dot{x}} \delta_T + K_2 \cancelto{0}{\frac{\partial \delta_T}{\partial \dot{x}}}$$

$$\frac{\partial \chi_p'}{\partial \dot{y}} = \frac{\partial \tilde{\chi}_p}{\partial \dot{y}} - \frac{\partial K_1}{\partial \dot{y}} + \frac{\partial K_2}{\partial \dot{y}} \delta_T + K_2 \cancelto{0}{\frac{\partial \delta_T}{\partial \dot{y}}}$$

$$\frac{\partial \chi_p'}{\partial \dot{z}} = \frac{\partial \tilde{\chi}_p}{\partial \dot{z}} - \frac{\partial K_1}{\partial \dot{z}} + \frac{\partial K_2}{\partial \dot{z}} \delta_T + K_2 \cancelto{0}{\frac{\partial \delta_T}{\partial \dot{z}}}$$

$$\frac{\partial \chi'_y}{\partial x} = \frac{\partial \tilde{\chi}_y}{\partial x} - \frac{\partial K_3}{\partial x} + \delta_T \frac{\partial K_4}{\partial x} + K_4 \cancelto{0}{\frac{\partial \delta_T}{\partial x}}$$

$$\frac{\partial \chi'_y}{\partial y} = \frac{\partial \tilde{\chi}_y}{\partial y} - \frac{\partial K_3}{\partial y} + \delta_T \frac{\partial K_4}{\partial y} + K_4 \cancelto{0}{\frac{\partial \delta_T}{\partial y}}$$

$$\frac{\partial \chi'_y}{\partial z} = \frac{\partial \tilde{\chi}_y}{\partial z} - \frac{\partial K_3}{\partial z} + \delta_T \frac{\partial K_4}{\partial z} + K_4 \cancelto{0}{\frac{\partial \delta_T}{\partial z}}$$

$$\frac{\partial \chi'_y}{\partial \dot{x}} = \frac{\partial \tilde{\chi}_y}{\partial \dot{x}} - \frac{\partial K_3}{\partial \dot{x}} + \delta_T \frac{\partial K_4}{\partial \dot{x}} + K_4 \cancelto{0}{\frac{\partial \delta_T}{\partial \dot{x}}}$$

$$\frac{\partial \chi'_y}{\partial \dot{y}} = \frac{\partial \tilde{\chi}_y}{\partial \dot{y}} - \frac{\partial K_3}{\partial \dot{y}} + \delta_T \frac{\partial K_4}{\partial \dot{y}} + K_4 \cancelto{0}{\frac{\partial \delta_T}{\partial \dot{y}}}$$

$$\frac{\partial \chi'_y}{\partial \dot{z}} = \frac{\partial \tilde{\chi}_y}{\partial \dot{z}} - \frac{\partial K_3}{\partial \dot{z}} + \delta_T \frac{\partial K_4}{\partial \dot{z}} + K_4 \cancelto{0}{\frac{\partial \delta_T}{\partial \dot{z}}}$$

$$\frac{\partial \ddot{\eta}_{GT}}{\partial x} = \frac{\partial \ddot{\eta}_{GT}}{\partial y} = \frac{\partial \ddot{\eta}_{GT}}{\partial z} = 0$$

$$\frac{\partial \ddot{\eta}_{GT}}{\partial \dot{x}} = \frac{\partial \ddot{\eta}_{GT}}{\partial \dot{y}} = \frac{\partial \ddot{\eta}_{GT}}{\partial \dot{z}} = 0$$

$$\frac{\partial R_{11}}{\partial x} = (YEA)\ \frac{\partial \phi_T}{\partial x}$$

$$\frac{\partial R_{12}}{\partial x} = (YEB)\ \frac{\partial \phi_T}{\partial x}$$

$$\frac{\partial R_{13}}{\partial x} = (YEC)\ \frac{\partial \phi_T}{\partial x}$$

$$\frac{\partial R_{21}}{\partial x} = \frac{\partial R_{22}}{\partial x} = \frac{\partial R_{23}}{\partial x} = 0$$

$$\frac{\partial k_{31}}{\partial x} = (YED)\frac{\partial \phi_T}{\partial x}$$

$$\frac{\partial k_{32}}{\partial x} = (YEE)\frac{\partial \phi_T}{\partial x}$$

$$\frac{\partial k_{33}}{\partial x} = (YEF)\frac{\partial \phi_T}{\partial x}$$

$$\frac{\partial k_{11}}{\partial y} = (YEA)\frac{\partial \phi_T}{\partial y}$$

$$\frac{\partial k_{12}}{\partial y} = (YEB)\frac{\partial \phi_T}{\partial y}$$

$$\frac{\partial k_{13}}{\partial y} = (YEC)\frac{\partial \phi_T}{\partial y}$$

$$\frac{\partial k_{21}}{\partial y} = \frac{\partial k_{22}}{\partial y} = \frac{\partial k_{23}}{\partial y} = 0$$

$$\frac{\partial k_{31}}{\partial y} = (YED)\frac{\partial \phi_T}{\partial y}$$

$$\frac{\partial k_{32}}{\partial y} = (YEE)\frac{\partial \phi_T}{\partial y}$$

$$\frac{\partial k_{33}}{\partial y} = (YEF)\frac{\partial \phi_T}{\partial y}$$

$$\frac{\partial k_{11}}{\partial z} = (YEA)\frac{\partial \phi_T}{\partial z}$$

$$\frac{\partial k_{12}}{\partial z} = (YEB)\frac{\partial \phi_T}{\partial z}$$

$$\frac{\partial k_{13}}{\partial z} = (YEC)\frac{\partial \phi_T}{\partial z}$$

$$\frac{\partial R_{21}}{\partial z} = \frac{\partial R_{22}}{\partial z} = \frac{\partial R_{23}}{\partial z} = 0$$

$$\frac{\partial R_{31}}{\partial z} = (YED)\,\frac{\partial \phi_T}{\partial z}$$

$$\frac{\partial R_{32}}{\partial z} = (YEE)\,\frac{\partial \phi_T}{\partial z}$$

$$\frac{\partial R_{33}}{\partial z} = (YEF)\,\frac{\partial \phi_T}{\partial z}$$

$$\frac{\partial R_{11}}{\partial \dot{x}} = (YEA)\,\frac{\partial \phi_T}{\partial \dot{x}}$$

$$\frac{\partial R_{12}}{\partial \dot{x}} = (YEB)\,\frac{\partial \phi_T}{\partial \dot{x}}$$

$$\frac{\partial R_{13}}{\partial \dot{x}} = (YEC)\,\frac{\partial \phi_T}{\partial \dot{x}}$$

$$\frac{\partial R_{21}}{\partial \dot{x}} = \frac{\partial R_{22}}{\partial \dot{x}} = \frac{\partial R_{23}}{\partial \dot{x}} = 0$$

$$\frac{\partial R_{31}}{\partial \dot{x}} = (YED)\,\frac{\partial \phi_T}{\partial \dot{x}}$$

$$\frac{\partial R_{32}}{\partial \dot{x}} = (YEE)\,\frac{\partial \phi_T}{\partial \dot{x}}$$

$$\frac{\partial R_{33}}{\partial \dot{x}} = (YEF)\,\frac{\partial \phi_T}{\partial \dot{x}}$$

$$\frac{\partial R_{11}}{\partial \dot{y}} = (YEA)\,\frac{\partial \phi_T}{\partial \dot{y}}$$

$$\frac{\partial R_{12}}{\partial \dot{y}} = (YEB)\frac{\partial \phi_T}{\partial \dot{y}}$$

$$\frac{\partial R_{13}}{\partial \dot{y}} = (YEC)\frac{\partial \phi_T}{\partial y}$$

$$\frac{\partial R_{21}}{\partial \dot{y}} = \frac{\partial R_{22}}{\partial \dot{y}} = \frac{\partial R_{23}}{\partial \dot{y}} = 0$$

$$\frac{\partial R_{31}}{\partial \dot{y}} = (YED)\frac{\partial \phi_T}{\partial \dot{y}}$$

$$\frac{\partial R_{32}}{\partial \dot{y}} = (YEE)\frac{\partial \phi_T}{\partial \dot{y}}$$

$$\frac{\partial R_{33}}{\partial \dot{y}} = (YEF)\frac{\partial \phi_T}{\partial \dot{y}}$$

$$\frac{\partial R_{11}}{\partial \dot{z}} = (YEA)\frac{\partial \phi_T}{\partial \dot{z}}$$

$$\frac{\partial R_{12}}{\partial \dot{z}} = (YEB)\frac{\partial \phi_T}{\partial \dot{z}}$$

$$\frac{\partial R_{13}}{\partial \dot{z}} = (YEC)\frac{\partial \phi_T}{\partial \dot{z}}$$

$$\frac{\partial R_{21}}{\partial \dot{z}} = \frac{\partial R_{22}}{\partial \dot{z}} = \frac{\partial R_{23}}{\partial \dot{z}} = 0$$

$$\frac{\partial R_{31}}{\partial \dot{z}} = (YED)\frac{\partial \phi_T}{\partial \dot{z}}$$

$$\frac{\partial R_{32}}{\partial \dot{z}} = (YEE)\frac{\partial \phi_T}{\partial \dot{z}}$$

$$\frac{\partial R_{33}}{\partial \dot{z}} = (YEF)\frac{\partial \phi_T}{\partial \dot{z}}$$

$$\frac{\partial EQ}{\partial x} = CINT_{11}\frac{\partial R_{11}}{\partial x} + CINT_{21}\frac{\partial R_{12}}{\partial x} + CINT_{31}\frac{\partial R_{13}}{\partial x}$$

$$+ CINT_{12}\cancelto{0}{\frac{\partial R_{21}}{\partial x}} + CINT_{22}\cancelto{0}{\frac{\partial R_{22}}{\partial x}} + CINT_{32}\cancelto{0}{\frac{\partial R_{23}}{\partial x}}$$

$$+ CINT_{13}\frac{\partial R_{31}}{\partial x} + CINT_{23}\frac{\partial R_{32}}{\partial x} + CINT_{33}\frac{\partial R_{33}}{\partial x}$$

$$\frac{\partial EQ}{\partial y} = CINT_{11}\frac{\partial R_{11}}{\partial y} + CINT_{21}\frac{\partial R_{12}}{\partial y} + CINT_{31}\frac{\partial R_{13}}{\partial y}$$

$$+ CINT_{12}\cancelto{0}{\frac{\partial R_{21}}{\partial y}} + CINT_{22}\cancelto{0}{\frac{\partial R_{22}}{\partial y}} + CINT_{32}\cancelto{0}{\frac{\partial R_{23}}{\partial y}}$$

$$+ CINT_{13}\frac{\partial R_{31}}{\partial y} + CINT_{23}\frac{\partial R_{32}}{\partial y} + CINT_{33}\frac{\partial R_{33}}{\partial y}$$

$$\frac{\partial EQ}{\partial z} = CINT_{11}\frac{\partial R_{11}}{\partial z} + CINT_{21}\frac{\partial R_{12}}{\partial z} + CINT_{31}\frac{\partial R_{13}}{\partial z}$$

$$+ CINT_{13}\frac{\partial R_{31}}{\partial z} + CINT_{23}\frac{\partial R_{32}}{\partial z} + CINT_{33}\frac{\partial R_{33}}{\partial z}$$

$$\frac{\partial EQ}{\partial \dot{x}} = CINT_{11}\frac{\partial R_{11}}{\partial \dot{x}} + CINT_{21}\frac{\partial R_{12}}{\partial \dot{x}} + CINT_{31}\frac{\partial R_{13}}{\partial \dot{x}}$$

$$CINT_{13}\frac{\partial R_{31}}{\partial \dot{x}} + CINT_{23}\frac{\partial R_{32}}{\partial \dot{x}} + CINT_{33}\frac{\partial R_{33}}{\partial \dot{x}}$$

$$\frac{\partial EQ}{\partial \dot{y}} = CINT_{11}\frac{\partial R_{11}}{\partial \dot{y}} + CINT_{21}\frac{\partial R_{12}}{\partial \dot{y}} + CINT_{31}\frac{\partial R_{13}}{\partial \dot{y}}$$

$$+ CINT_{13}\frac{\partial R_{31}}{\partial \dot{y}} + CINT_{23}\frac{\partial R_{32}}{\partial \dot{y}} + CINT_{33}\frac{\partial R_{33}}{\partial \dot{y}}$$

$$\frac{\partial EQ}{\partial \dot{z}} = CINT_{11}\frac{\partial R_{11}}{\partial \dot{z}} + CINT_{21}\frac{\partial R_{12}}{\partial \dot{z}} + CINT_{31}\frac{\partial R_{13}}{\partial \dot{z}}$$

$$+ CINT_{13}\frac{\partial R_{31}}{\partial \dot{z}} + CINT_{23}\frac{\partial R_{32}}{\partial \dot{z}} + CINT_{33}\frac{\partial R_{33}}{\partial \dot{z}}$$

$$\frac{\partial C_{11}}{\partial x} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{33}}{\partial x} + k_{33}\frac{\partial k_{22}}{\partial x} - k_{23}\frac{\partial k_{32}}{\partial x}\right] - CINT_{11}\frac{\partial EQ}{\partial x}[EQ]^{-2}$$

$$\frac{\partial C_{11}}{\partial y} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{33}}{\partial y} + k_{33}\frac{\partial k_{22}}{\partial y} - k_{23}\frac{\partial k_{32}}{\partial y}\right] - CINT_{11}\frac{\partial EQ}{\partial y}[EQ]^{-2}$$

$$\frac{\partial C_{11}}{\partial z} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{33}}{\partial z} + k_{33}\frac{\partial k_{22}}{\partial z} - k_{23}\frac{\partial k_{32}}{\partial z}\right] - CINT_{11}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{11}}{\partial \dot{x}} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{33}}{\partial \dot{x}} + k_{33}\frac{\partial k_{22}}{\partial \dot{x}} - k_{23}\frac{\partial k_{32}}{\partial \dot{x}}\right] - CINT_{11}\frac{\partial EQ}{\partial \dot{x}}[EQ]^{-2}$$

$$\frac{\partial C_{11}}{\partial \dot{y}} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{33}}{\partial \dot{y}} + k_{33}\frac{\partial k_{22}}{\partial \dot{y}} - k_{23}\frac{\partial k_{32}}{\partial \dot{y}}\right] - CINT_{11}\frac{\partial EQ}{\partial \dot{y}}[EQ]^{-2}$$

$$\frac{\partial C_{11}}{\partial \dot{z}} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{33}}{\partial \dot{z}} + k_{33}\frac{\partial k_{22}}{\partial \dot{z}} - k_{23}\frac{\partial k_{32}}{\partial \dot{z}}\right] - CINT_{11}\frac{\partial EQ}{\partial \dot{z}}[EQ]^{-2}$$

$$\frac{\partial C_{12}}{\partial x} = (EQ)^{-1}\left[k_{13}\frac{\partial k_{32}}{\partial x} + k_{32}\frac{\partial k_{13}}{\partial x} - k_{12}\frac{\partial k_{33}}{\partial x} - k_{33}\frac{\partial k_{12}}{\partial x}\right] - CINT_{12}\frac{\partial EQ}{\partial x}[EQ]^{-2}$$

$$\frac{\partial C_{12}}{\partial y} = (EQ)^{-1}\left[k_{13}\frac{\partial k_{32}}{\partial y} + k_{32}\frac{\partial k_{13}}{\partial y} - k_{12}\frac{\partial k_{33}}{\partial y} - k_{33}\frac{\partial k_{12}}{\partial y}\right] - CINT_{12}\frac{\partial EQ}{\partial y}[EQ]^{-2}$$

$$\frac{\partial C_{12}}{\partial z} = (EQ)^{-1}\left[k_{13}\frac{\partial k_{32}}{\partial z} + k_{32}\frac{\partial k_{13}}{\partial z} - k_{12}\frac{\partial k_{33}}{\partial z} - k_{23}\frac{\partial k_{12}}{\partial z}\right] - CINT_{12}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{12}}{\partial \dot{x}} = (EQ)^{-1}\left[k_{13}\frac{\partial k_{32}}{\partial \dot{x}} + k_{32}\frac{\partial k_{13}}{\partial \dot{x}} - k_{12}\frac{\partial k_{33}}{\partial \dot{x}} - k_{33}\frac{\partial k_{12}}{\partial \dot{x}}\right] - CINT_{12}\frac{\partial EQ}{\partial \dot{x}}[EQ]^{-2}$$

$$\frac{\partial C_{12}}{\partial \dot{y}} = (EQ)^{-1}\left[k_{13}\frac{\partial k_{32}}{\partial \dot{y}} + k_{32}\frac{\partial k_{13}}{\partial \dot{y}} - k_{12}\frac{\partial k_{33}}{\partial \dot{y}} - k_{33}\frac{\partial k_{12}}{\partial \dot{y}}\right] - CINT_{12}\frac{\partial EQ}{\partial \dot{y}}[EQ]^{-2}$$

$$\frac{\partial C_{12}}{\partial \dot{z}} = (EQ)^{-1}\left[k_{13}\frac{\partial k_{32}}{\partial \dot{z}} + k_{32}\frac{\partial k_{13}}{\partial \dot{z}} - k_{12}\frac{\partial k_{33}}{\partial \dot{z}} - k_{33}\frac{\partial k_{12}}{\partial \dot{z}}\right] - CINT_{12}\frac{\partial EQ}{\partial \dot{z}}[EQ]^{-2}$$

$$\frac{\partial C_{13}}{\partial x} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{12}}{\partial x} - k_{22}\frac{\partial k_{13}}{\partial x}\right] - CINT_{13}\frac{\partial EQ}{\partial x}[EQ]^{-2}$$

$$\frac{\partial C_{13}}{\partial y} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{12}}{\partial y} - k_{22}\frac{\partial k_{13}}{\partial y}\right] - CINT_{13}\frac{\partial EQ}{\partial y}[EQ]^{-2}$$

$$\frac{\partial C_{13}}{\partial z} = (EQ^{-1})\left[k_{23}\frac{\partial k_{12}}{\partial z} - k_{22}\frac{\partial k_{13}}{\partial z}\right] - CINT_{13}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{13}}{\partial \dot{x}} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{12}}{\partial \dot{x}} - k_{22}\frac{\partial k_{13}}{\partial \dot{x}}\right] - CINT_{13}\frac{\partial EQ}{\partial \dot{x}}[EQ]^{-2}$$

$$\frac{\partial C_{13}}{\partial \dot{y}} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{12}}{\partial \dot{y}} - k_{22}\frac{\partial k_{13}}{\partial \dot{y}}\right] - CINT_{13}\frac{\partial EQ}{\partial \dot{y}}[EQ]^{-2}$$

$$\frac{\partial C_{13}}{\partial \dot{z}} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{12}}{\partial \dot{z}} - k_{22}\frac{\partial k_{13}}{\partial \dot{z}}\right] - CINT_{13}\frac{\partial EQ}{\partial \dot{z}}[EQ]^{-2}$$

$$\frac{\partial C_{21}}{\partial x} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{31}}{\partial x} - k_{21}\frac{\partial k_{33}}{\partial x}\right] - CINT_{21}\frac{\partial EQ}{\partial x}[EQ]^{-2}$$

$$\frac{\partial C_{21}}{\partial y} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{31}}{\partial y} - k_{21}\frac{\partial k_{33}}{\partial y}\right] - CINT_{21}\frac{\partial EQ}{\partial y}[EQ]^{-2}$$

$$\frac{\partial C_{21}}{\partial z} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{31}}{\partial z} - k_{21}\frac{\partial k_{33}}{\partial z}\right] - CINT_{21}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{21}}{\partial \dot{x}} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{31}}{\partial \dot{x}} - k_{21}\frac{\partial k_{33}}{\partial \dot{x}}\right] - CINT_{21}\frac{\partial EQ}{\partial \dot{x}}[EQ]^{-2}$$

$$\frac{\partial C_{21}}{\partial \dot{y}} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{31}}{\partial \dot{y}} - k_{21}\frac{\partial k_{33}}{\partial \dot{y}}\right] - CINT_{21}\frac{\partial EQ}{\partial \dot{y}}[EQ]^{-2}$$

$$\frac{\partial C_{21}}{\partial \dot{z}} = (EQ)^{-1}\left[k_{23}\frac{\partial k_{31}}{\partial \dot{z}} - k_{21}\frac{\partial k_{33}}{\partial \dot{z}}\right] - CINT_{21}\frac{\partial EQ}{\partial \dot{z}}[EQ]^{-2}$$

$$\frac{\partial C_{22}}{\partial x} = (EQ)^{-1}\left[k_{11}\frac{\partial k_{33}}{\partial x} + k_{33}\frac{\partial k_{11}}{\partial x} - k_{13}\frac{\partial k_{31}}{\partial x} - k_{31}\frac{\partial k_{13}}{\partial x}\right]$$

$$- CINT_{22}\frac{\partial EQ}{\partial x}[EQ]^{-2}$$

$$\frac{\partial C_{22}}{\partial y} = (EQ)^{-1}\left[R_{11}\frac{\partial R_{33}}{\partial y} + R_{33}\frac{\partial R_{11}}{\partial y} - R_{13}\frac{\partial R_{31}}{\partial y} - R_{31}\frac{\partial R_{13}}{\partial y}\right] - CINT_{22}\frac{\partial EQ}{\partial y}[EQ]^{-2}$$

$$\frac{\partial C_{22}}{\partial z} = (EQ)^{-1}\left[R_{11}\frac{\partial R_{33}}{\partial z} + R_{33}\frac{\partial R_{11}}{\partial z} - R_{13}\frac{\partial R_{31}}{\partial z} - R_{31}\frac{\partial R_{13}}{\partial z}\right] - CINT_{22}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{22}}{\partial \dot{x}} = (EQ)^{-1}\left[R_{11}\frac{\partial R_{33}}{\partial \dot{x}} + R_{33}\frac{\partial R_{11}}{\partial \dot{x}} - R_{13}\frac{\partial R_{31}}{\partial \dot{x}} - R_{31}\frac{\partial R_{13}}{\partial \dot{x}}\right] - CINT_{22}\frac{\partial EQ}{\partial \dot{x}}[EQ]^{-2}$$

$$\frac{\partial C_{22}}{\partial \dot{y}} = (EQ)^{-1}\left[R_{11}\frac{\partial R_{33}}{\partial \dot{y}} + R_{33}\frac{\partial R_{11}}{\partial \dot{y}} - R_{13}\frac{\partial R_{31}}{\partial \dot{y}} - R_{31}\frac{\partial R_{13}}{\partial \dot{y}}\right] - CINT_{22}\frac{\partial EQ}{\partial \dot{y}}[EQ]^{-2}$$

$$\frac{\partial C_{22}}{\partial \dot{z}} = (EQ)^{-1}\left[R_{11}\frac{\partial R_{33}}{\partial \dot{z}} + R_{33}\frac{\partial R_{11}}{\partial \dot{z}} - R_{13}\frac{\partial R_{31}}{\partial \dot{z}} - R_{31}\frac{\partial R_{13}}{\partial \dot{z}}\right] - CINT_{22}\frac{\partial EQ}{\partial \dot{z}}[EQ]^{-2}$$

$$\frac{\partial C_{23}}{\partial x} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{13}}{\partial x} - R_{23}\frac{\partial R_{11}}{\partial x}\right] - CINT_{23}\frac{\partial EQ}{\partial x}[EQ]^{-2}$$

$$\frac{\partial C_{23}}{\partial y} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{13}}{\partial y} - R_{23}\frac{\partial R_{11}}{\partial y}\right] - CINT_{23}\frac{\partial EQ}{\partial y}[EQ]^{-2}$$

$$\frac{\partial C_{23}}{\partial z} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{13}}{\partial z} - R_{23}\frac{\partial R_{11}}{\partial z}\right] - CINT_{23}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{23}}{\partial \dot{x}} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{13}}{\partial \dot{x}} - R_{23}\frac{\partial R_{11}}{\partial \dot{x}}\right] - CINT_{23}\frac{\partial EQ}{\partial \dot{x}}[EQ]^{-2}$$

$$\frac{\partial C_{23}}{\partial \dot{y}} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{13}}{\partial \dot{y}} - R_{23}\frac{\partial R_{11}}{\partial \dot{y}}\right] - CINT_{23}\frac{\partial EQ}{\partial \dot{y}}[EQ]^{-2}$$

$$\frac{\partial C_{23}}{\partial \dot{z}} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{13}}{\partial \dot{z}} - R_{23}\frac{\partial R_{11}}{\partial \dot{z}}\right] - CINT_{23}\frac{\partial EQ}{\partial \dot{z}}[EQ]^{-2}$$

$$\frac{\partial C_{31}}{\partial x} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{32}}{\partial x} - R_{22}\frac{\partial R_{31}}{\partial x}\right] - CINT_{31}\frac{\partial EQ}{\partial x}[EQ]^{-2}$$

$$\frac{\partial C_{31}}{\partial y} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{32}}{\partial y} - R_{22}\frac{\partial R_{31}}{\partial y}\right] - CINT_{31}\frac{\partial EQ}{\partial y}[EQ]^{-2}$$

$$\frac{\partial C_{31}}{\partial z} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{32}}{\partial z} - R_{22}\frac{\partial R_{31}}{\partial z}\right] - CINT_{31}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{31}}{\partial \dot{x}} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{32}}{\partial \dot{x}} - R_{22}\frac{\partial R_{31}}{\partial \dot{x}}\right] - CINT_{31}\frac{\partial EQ}{\partial \dot{x}}[EQ]^{-2}$$

$$\frac{\partial C_{31}}{\partial \dot{y}} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{32}}{\partial \dot{y}} - R_{22}\frac{\partial R_{31}}{\partial \dot{y}}\right] - CINT_{31}\frac{\partial EQ}{\partial \dot{y}}[EQ]^{-2}$$

$$\frac{\partial C_{31}}{\partial \dot{z}} = (EQ)^{-1}\left[R_{21}\frac{\partial R_{32}}{\partial \dot{z}} - R_{22}\frac{\partial R_{31}}{\partial \dot{z}}\right] - CINT_{31}\frac{\partial EQ}{\partial \dot{z}}[EQ]^{-2}$$

$$\frac{\partial C_{32}}{\partial x} = (EQ)^{-1}\left[R_{12}\frac{\partial R_{31}}{\partial x} + R_{31}\frac{\partial R_{12}}{\partial x} - R_{11}\frac{\partial R_{32}}{\partial x} - R_{32}\frac{\partial R_{11}}{\partial x}\right] - CINT_{32}\frac{\partial EQ}{\partial x}[EQ]^{-2}$$

$$\frac{\partial C_{32}}{\partial y} = (EQ)^{-1}\left[R_{12}\frac{\partial R_{31}}{\partial y} + R_{31}\frac{\partial R_{12}}{\partial y} - R_{11}\frac{\partial R_{32}}{\partial y} - R_{32}\frac{\partial R_{11}}{\partial y}\right] - CINT_{32}\frac{\partial EQ}{\partial y}[EQ]^{-2}$$

$$\frac{\partial C_{32}}{\partial z} = (EQ)^{-1}\left[R_{12}\frac{\partial R_{31}}{\partial z} + R_{31}\frac{\partial R_{12}}{\partial z} - R_{11}\frac{\partial R_{32}}{\partial z} - R_{32}\frac{\partial R_{11}}{\partial z}\right] - CINT_{32}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{32}}{\partial \dot{x}} = (EQ)^{-1}\left[R_{12}\frac{\partial R_{31}}{\partial \dot{x}} + R_{31}\frac{\partial R_{12}}{\partial \dot{x}} - R_{11}\frac{\partial R_{32}}{\partial \dot{x}} - R_{32}\frac{\partial R_{11}}{\partial \dot{x}}\right] - CINT_{32}\frac{\partial EQ}{\partial \dot{x}}[EQ]^{-2}$$

$$\frac{\partial C_{32}}{\partial \dot{y}} = (EQ)^{-1}\left[R_{12}\frac{\partial R_{31}}{\partial \dot{y}} + R_{31}\frac{\partial R_{12}}{\partial \dot{y}} - R_{11}\frac{\partial R_{32}}{\partial \dot{y}} - R_{32}\frac{\partial R_{11}}{\partial \dot{y}}\right] - CINT_{32}\frac{\partial EQ}{\partial \dot{y}}[EQ]^{-2}$$

$$\frac{\partial C_{32}}{\partial z} = (EQ)^{-1}\left[k_{12}\frac{\partial k_{31}}{\partial z} + k_{31}\frac{\partial k_{12}}{\partial z} - k_{11}\frac{\partial k_{32}}{\partial z} - k_{32}\frac{\partial k_{11}}{\partial z}\right] - CINT_{32}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{33}}{\partial x} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{11}}{\partial x} - k_{21}\frac{\partial k_{12}}{\partial x}\right] - CINT_{33}\frac{\partial EQ}{\partial x}[EQ]^{-2}$$

$$\frac{\partial C_{33}}{\partial y} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{11}}{\partial y} - k_{21}\frac{\partial k_{12}}{\partial y}\right] - CINT_{33}\frac{\partial EQ}{\partial y}[EQ]^{-2}$$

$$\frac{\partial C_{33}}{\partial z} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{11}}{\partial z} - k_{21}\frac{\partial k_{12}}{\partial z}\right] - CINT_{33}\frac{\partial EQ}{\partial z}[EQ]^{-2}$$

$$\frac{\partial C_{33}}{\partial \dot{x}} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{11}}{\partial \dot{x}} - k_{21}\frac{\partial k_{12}}{\partial \dot{x}}\right] - CINT_{33}\frac{\partial EQ}{\partial \dot{x}}[EQ]^{-2}$$

$$\frac{\partial C_{33}}{\partial \dot{y}} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{11}}{\partial \dot{y}} - k_{21}\frac{\partial k_{12}}{\partial \dot{y}}\right] - CINT_{33}\frac{\partial EQ}{\partial \dot{y}}[EQ]^{-2}$$

$$\frac{\partial C_{33}}{\partial \dot{z}} = (EQ)^{-1}\left[k_{22}\frac{\partial k_{11}}{\partial \dot{z}} - k_{21}\frac{\partial k_{12}}{\partial \dot{z}}\right] - CINT_{33}\frac{\partial EQ}{\partial \dot{z}}[EQ]^{-2}$$

$$\frac{\partial DCX}{\partial x} = C_{11}\left[\cos x'_y \cos x'_p \frac{\partial x'_p}{\partial x} - \sin x'_p \sin x'_y \frac{\partial x'_y}{\partial x}\right]$$
$$+ \frac{\partial C_{11}}{\partial x}\left[\sin x'_p \cos x'_y\right] + C_{12}\left[\cos x'_y \frac{\partial x'_y}{\partial x}\right]$$
$$+ \frac{\partial C_{12}}{\partial x}\sin x'_y - C_{13}\left[\cos x'_p \sin x'_y \frac{\partial x'_y}{\partial x} + \cos x'_y \sin x'_p \cdot \frac{\partial x'_p}{\partial x}\right] + \frac{\partial C_{13}}{\partial x}\left[\cos x'_p \cos x'_y\right]$$

$$\frac{\partial DCX}{\partial y} = C_{11}\left[\cos x'_y \cos x'_p \frac{\partial x'_p}{\partial y} - \sin x'_p \sin x'_y \frac{\partial x'_y}{\partial y}\right]$$
$$+ \frac{\partial C_{11}}{\partial y}\left[\sin x'_p \cos x'_y\right] + C_{12}\left[\cos x'_y \frac{\partial x'_y}{\partial y}\right] + \frac{\partial C_{12}}{\partial y}\sin x'_y$$
$$- C_{13}\left[\cos x'_p \sin x'_y \frac{\partial x'_y}{\partial y} + \cos x'_y \sin x'_p \frac{\partial x'_p}{\partial y}\right] + \frac{\partial C_{13}}{\partial y} \cdot \left[\cos x'_p \cos x'_y\right]$$

$$\frac{\partial DCX}{\partial z} = C_{11}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial z} - \sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial z}\right]$$

$$+ \frac{\partial C_{11}}{\partial z}\left[\sin\chi'_p \cos\chi'_y\right] + C_{12}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial z}\right] + \frac{\partial C_{12}}{\partial z}\sin\chi'_y$$

$$- C_{13}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial z} + \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial z}\right]$$

$$+ \frac{\partial C_{13}}{\partial z}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCX}{\partial \dot{x}} = C_{11}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial \dot{x}} - \sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{x}}\right]$$

$$+ \frac{\partial C_{11}}{\partial \dot{x}}\left[\sin\chi'_p \cos\chi'_y\right] + C_{12}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial \dot{x}}\right] + \frac{\partial C_{12}}{\partial \dot{x}}\sin\chi'_y$$

$$- C_{13}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{x}} + \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial \dot{x}}\right]$$

$$+ \frac{\partial C_{13}}{\partial \dot{x}}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCX}{\partial \dot{y}} = C_{11}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial \dot{y}} - \sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{y}}\right]$$

$$+ \frac{\partial C_{11}}{\partial \dot{y}}\left[\sin\chi'_p \cos\chi'_y\right] + C_{12}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial \dot{y}}\right] + \frac{\partial C_{12}}{\partial \dot{y}}\sin\chi'_y$$

$$- C_{13}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{y}} + \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial \dot{y}}\right]$$

$$+ \frac{\partial C_{13}}{\partial \dot{y}}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCX}{\partial \dot{z}} = C_{11}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial \dot{z}} - \sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{z}}\right]$$

$$+ \frac{\partial C_{11}}{\partial \dot{z}}\left[\sin\chi'_p \cos\chi'_y\right] + C_{12}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial \dot{z}}\right] + \frac{\partial C_{12}}{\partial \dot{z}}\sin\chi'_y$$

$$- C_{13}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{z}} + \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial \dot{z}}\right]$$

$$+ \frac{\partial C_{13}}{\partial \dot{z}}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCY}{\partial x} = C_{z1}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial x} - \sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial x}\right]$$

$$+ \frac{\partial C_{z1}}{\partial x}\left[\sin\chi'_p \cos\chi'_y\right] + C_{z2}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial x}\right]$$

$$+ \frac{\partial C_{z2}}{\partial x} \sin\chi'_y - C_{z3}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial x} + \right.$$

$$\left. \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial x}\right] + \frac{\partial C_{z3}}{\partial x}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCY}{\partial y} = C_{z1}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial y}\right] - C_{z1}\left[\sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial y}\right]$$

$$+ \frac{\partial C_{z1}}{\partial y}\left[\sin\chi'_p \cos\chi'_y\right] + C_{z2}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial y}\right]$$

$$+ \frac{\partial C_{z2}}{\partial y} \sin\chi'_y - C_{z3}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial y} + \right.$$

$$\left. \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial y}\right] + \frac{\partial C_{z3}}{\partial y}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCY}{\partial z} = C_{z1}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial z}\right] - C_{z1}\left[\sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial z}\right]$$

$$+ \frac{\partial C_{z1}}{\partial z}\left[\sin\chi'_p \cos\chi'_y\right] + C_{z2}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial z}\right]$$

$$+ \frac{\partial C_{z2}}{\partial z} \sin\chi'_y - C_{z3}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial z} + \right.$$

$$\left. \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial z}\right] + \frac{\partial C_{z3}}{\partial z}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCY}{\partial \dot{x}} = C_{z1}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial \dot{x}}\right] - C_{z1}\left[\sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{x}}\right]$$

$$+ \frac{\partial C_{z1}}{\partial \dot{x}}\left[\sin\chi'_p \cos\chi'_y\right] + C_{z2}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial \dot{x}}\right] + \frac{\partial C_{z2}}{\partial \dot{x}} \sin\chi'_y$$

$$- C_{z3}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{x}} + \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial \dot{x}}\right]$$

$$+ \frac{\partial C_{z3}}{\partial \dot{x}}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCY}{\partial \dot{y}} = C_{21}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial \dot{y}}\right] - C_{21}\left[\sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{y}}\right]$$

$$+ \frac{\partial C_{21}}{\partial \dot{y}}\left[\sin\chi'_p \cos\chi'_y\right] + C_{22}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial \dot{y}}\right] + \frac{\partial C_{22}}{\partial \dot{y}} \sin\chi'_y$$

$$- C_{23}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{y}} + \cos\chi'_y \sin\chi'_p\right] + \frac{\partial C_{23}}{\partial \dot{y}}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCY}{\partial \dot{z}} = C_{21}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial \dot{y}} - \sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{y}}\right]$$

$$+ \frac{\partial C_{21}}{\partial \dot{z}}\left[\sin\chi'_p \cos\chi'_y\right] + C_{22}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial \dot{z}}\right] + \frac{\partial C_{22}}{\partial \dot{z}} \sin\chi'_y$$

$$- C_{23}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{z}} + \cos\chi'_y \sin\chi'_p\right] + \frac{\partial C_{23}}{\partial \dot{z}}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCZ}{\partial x} = C_{31}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial x} - \sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial x}\right]$$

$$+ \frac{\partial C_{31}}{\partial x}\left[\sin\chi'_p \cos\chi'_y\right] + C_{32}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial x}\right] + \frac{\partial C_{32}}{\partial x} \sin\chi'_y$$

$$- C_{33}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial x} + \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial x}\right] + \frac{\partial C_{33}}{\partial x}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCZ}{\partial y} = C_{31}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial y} - \sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial y}\right]$$

$$+ \frac{\partial C_{31}}{\partial y}\left[\sin\chi'_p \cos\chi'_y\right] + C_{32}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial y}\right] + \frac{\partial C_{32}}{\partial y} \sin\chi'_y$$

$$- C_{33}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial y} + \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial y}\right] +$$

$$\frac{\partial C_{33}}{\partial y}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$\frac{\partial DCZ}{\partial Z} = C_{31}\left[\cos \chi'_y \cos \chi'_p \frac{\partial \chi'_p}{\partial Z}\right] - C_{31}\left[\sin \chi'_p \sin \chi'_y \frac{\partial \chi'_y}{\partial Z}\right]$$

$$+ \frac{\partial C_{31}}{\partial Z}\left[\sin \chi'_p \cos \chi_y\right] + C_{32}\left[\cos \chi'_y \frac{\partial \chi'_y}{\partial Z}\right]$$

$$+ \frac{\partial C_{32}}{\partial Z} \sin \chi'_y - C_{33}\left[\cos \chi'_p \sin \chi'_y \frac{\partial \chi'_y}{\partial Z} + \cos \chi'_y\right.$$

$$\left.\cdot \sin \chi'_p \frac{\partial \chi'_p}{\partial Z}\right] + \frac{\partial C_{33}}{\partial Z}\left[\cos \chi'_p \cos \chi'_y\right]$$

$$\frac{\partial DCZ}{\partial \dot{X}} = C_{31}\left[\cos \chi'_y \cos \chi'_p \frac{\partial \chi'_p}{\partial \dot{X}} - \sin \chi'_p \sin \chi'_y \frac{\partial \chi'_y}{\partial \dot{X}}\right]$$

$$+ \frac{\partial C_{31}}{\partial \dot{X}}\left[\sin \chi'_p \cos \chi'_y\right] + C_{32}\left[\cos \chi'_y \frac{\partial \chi'_y}{\partial \dot{X}}\right]$$

$$+ \frac{\partial C_{32}}{\partial \dot{X}} \sin \chi'_y - C_{33}\left[\cos \chi'_p \sin \chi'_y \frac{\partial \chi'_y}{\partial \dot{X}} + \cos \chi'_y\right.$$

$$\left.\cdot \sin \chi'_p \frac{\partial \chi'_p}{\partial \dot{X}}\right] + \frac{\partial C_{33}}{\partial \dot{X}}\left[\cos \chi'_p \cos \chi'_y\right]$$

$$\frac{\partial DCZ}{\partial \dot{y}} = C_{31}\left[\cos \chi'_y \cos \chi'_p \frac{\partial \chi'_p}{\partial \dot{y}} - \sin \chi'_p \sin \chi'_y \frac{\partial \chi'_y}{\partial \dot{y}}\right]$$

$$+ \frac{\partial C_{31}}{\partial \dot{y}}\left[\sin \chi'_p \cos \chi'_y\right] + C_{32}\left[\cos \chi'_y \frac{\partial \chi'_y}{\partial \dot{y}}\right]$$

$$+ \frac{\partial C_{32}}{\partial \dot{y}} \sin \chi'_y - C_{33}\left[\cos \chi'_p \sin \chi'_y \frac{\partial \chi'_y}{\partial \dot{y}} + \cos \chi'_y\right.$$

$$\left.\cdot \sin \chi'_p \frac{\partial \chi'_p}{\partial \dot{y}}\right] + \frac{\partial C_{33}}{\partial \dot{y}}\left[\cos \chi'_p \cos \chi'_y\right]$$

$$\frac{\partial DCZ}{\partial \dot{Z}} = C_{21}\left[\cos\chi'_y \cos\chi'_p \frac{\partial \chi'_p}{\partial \dot{Z}} - \sin\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{Z}}\right]$$

$$+ \frac{\partial C_{21}}{\partial \dot{Z}}\left[\sin\chi'_p \cos\chi'_y\right] + C_{32}\left[\cos\chi'_y \frac{\partial \chi'_y}{\partial \dot{Z}}\right]$$

$$+ \frac{\partial C_{32}}{\partial \dot{Z}} \sin\chi'_y - C_{33}\left[\cos\chi'_p \sin\chi'_y \frac{\partial \chi'_y}{\partial \dot{Z}}\right.$$

$$\left. + \cos\chi'_y \sin\chi'_p \frac{\partial \chi'_p}{\partial \dot{Z}}\right] + \frac{\partial C_{33}}{\partial \dot{Z}}\left[\cos\chi'_p \cos\chi'_y\right]$$

$$R = \left[x^2 + y^2 + z^2\right]^{1/2}$$

$$V = \left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right]^{1/2}$$

$$T1 = -GM$$

$$T2 = -(GM)(AJBESL)(AEG)^2$$

$$T3 = -3(GM)(DGRAV)(AEG)^4$$

$$T4 = -3(GM)(HGRAV)(AEG)^3$$

$$T5 = 5(GM)(AJBESL)(AEG)^2$$

$$T6 = 63$$

$$T7 = 42(GM)(DGRAV)(AEG)^4$$

$$T8 = 7(GM)(HGRAV)(AEG)^3$$

$$T9 = .6(GM)(HGRAV)(AEG)^2$$

$$T10 = -2(AJBESL)(GM)(AEG)^2$$

$$T11 = -.2(GM)(HGRAV)(AEG)^3$$

$$T12 = -\left(\frac{12}{7}\right)(GM)(DGRAV)(AEG)^4$$

$$T13 = -\left(\frac{23}{7}\right)(GM)(DGRAV)(AEG)^4$$

$$VG1 = (.87663975)\cos(1.8324607)$$

$$VG2 = -.47748893$$

$$VG3 = (.87663975)\sin(1.8324607)$$

$$\alpha_1 = ALPH1 = (VG1)(x) + (VG2)(y) + (VG3)(z)$$

$$ZFLXF = (z_F/x_F)$$

$$XDLRV = \frac{x\cdot\dot{x} + y\cdot\dot{y} + z\dot{z}}{R\cdot V}$$

$$H1 = \frac{(\Delta\dot{\xi}^*)g_{11} + (\Delta\dot{\eta}^*)g_{21} + (\Delta\dot{\zeta}^*)g_{31}}{(V_{ex})(L)}$$

$$H2 = 0$$

$$I1 = \left[\frac{\ddot{\xi}_G(\Delta\dot{\xi}^*) + \ddot{\eta}_G(\Delta\dot{\eta}^*) + \ddot{\zeta}_G(\Delta\dot{\zeta}^*)}{(V_{ex})(L)}\right] + \left[\frac{(\Delta\dot{\xi}^*)^2 + (\Delta\dot{\eta}^*)^2 + (\Delta\dot{\zeta}^*)^2}{2(V_{ex})(L)^2(\tau - T_g)}\right] + \left[\frac{V_{ex}}{2(\tau - T_g)}\right]$$

$$I2 = \frac{\left[(\Delta\dot{\xi}^*)\cos\gamma_T - (\Delta\dot{\zeta}^*)\sin\gamma_T\right]V_T}{(V_{ex})(L)}$$

$$I3 = \frac{(\Delta\dot{\xi}^*)\sin\gamma_T + (\Delta\dot{\zeta}^*)\cos\gamma_T}{(V_{ex})(L)}$$

$$J1 = (\Delta\dot{\xi}^*)\left(\dot{x} + \frac{\ddot{x}_G\cdot T_g}{2}\right)(g_{31}\cos\phi_T - g_{11}\sin\phi_T)$$

$$J2 = (\Delta\dot{\xi}^*)\left(\dot{y} + \frac{\ddot{y}_G\cdot T_g}{2}\right)(g_{32}\cos\phi_T - g_{12}\sin\phi_T)$$

$$J3 = (\Delta \dot{\xi}^*)\left(\dot{z} + \frac{\ddot{z}_G \cdot T_g}{2}\right)(g_{33} \cos \phi_T - g_{13} \sin \phi_T)$$

$$J4 = (\Delta \dot{\xi}^*)\left(\dot{x} + \frac{\ddot{x}_G \cdot T_g}{2}\right)(g_{11} \cos \phi_T + g_{31} \sin \phi_T)$$

$$J5 = (\Delta \dot{\xi}^*)\left(\dot{y} + \frac{\ddot{y}_G \cdot T_g}{2}\right)(g_{12} \cos \phi_T + g_{32} \sin \phi_T)$$

$$J6 = (\Delta \dot{\xi}^*)\left(\ddot{z} + \frac{\dddot{z}_G \cdot T_g}{2}\right)(g_{13} \cos \phi_T + g_{33} \sin \phi_T)$$

$$C1 = \frac{(\Delta \dot{\xi}^*) k_{12} + (\Delta \dot{\eta}^*) k_{22} + (\Delta \dot{\zeta}^*) k_{32}}{(V_{ex})(L)}$$

$$C2 = 0$$

$$C3 = \frac{(\Delta \dot{\xi}^*) k_{13} + (\Delta \dot{\eta}^*) k_{23} + (\Delta \dot{\zeta}^*) k_{33}}{(V_{ex})(L)}$$

$$C4 = 0$$

$$L1 = \left[\frac{V \cos \gamma}{R} + \frac{V_T \cos \gamma_T}{R_T}\right]\left[\frac{1 + FK6 + 2(C1FK6)T_g^2 + (C2FK6)T_g \ldots}{2}\right.$$

$$L2 = -\left[\frac{V \sin \gamma \left[V X - \frac{\dot{x}(x\dot{x} + y\dot{y} + z\dot{z})}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right]}{\sqrt{1 - \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{RV}\right)^2}\,(RV)^2}\right]$$

$$+ \left[\frac{\dot{x} \cos \gamma}{R(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}}\right]$$

$$L3 = \left[\frac{V_T(\sin\gamma_T)(e_T^2 + e_T\cos\theta_T)}{R_T(1+e_T^2+2e_T\cos\theta_T)}\right]$$

$$+\left[\frac{\left(\frac{\mu}{R_T}\right)^{1/2}(\cos\gamma_T)\, e_T \sin\theta_T}{R_T(1+e_T^2+2e_T\cos\theta_T)^{1/2}}\right]$$

$$+\left[\frac{(\cos\gamma_T)\quad V_T\, e_T \sin\theta_T}{(R_T)^2 (1+e_T\cos\theta_T)^2}\right]$$

$$L4 = \left[\frac{(I2)(e_T^2 + e_T\cos\theta_T)}{(1+e_T^2+2e_T\cos\theta_T)}\right] - \left[\frac{(I3)\,\mu^{1/2}\, e_T \sin\theta_T}{\left[P_T(1+e_T^2+2e_T\cos\theta_T)\right]^{1/2}}\right]$$

$$-\left[\frac{(\Delta\dot{\xi}^*)\,T_g\,(\mu)\,P_T\,(e_T)\sin\theta_T}{V_{ex}\,(L)\,R_T^3\,(1+e_T\cos\theta_T)^2}\right]$$

$$L6 = \left[\frac{1}{V_{ex}L}\right]\left[J1 + J2 + J3 - J4 - J5 - J6\right]$$

$$L7 = L4 - L6$$

$$NUM = \left[\frac{\gamma - T_g}{V_{ex}}\right](L7)\left(\frac{T_g}{2}\right)\left[1 + FK6\right] L2 \left[1 + \frac{T_g}{2}(1+FK6)L \ldots\right.$$

$$-\left[\frac{\gamma - T_g}{V_{ex}}\right]\left[H1 + H2\right]$$

$$DEM = 1 + \left[\frac{\gamma - T_g}{V_{ex}}\right](I1) - \left[(\gamma - T_g)(L7)(L1)\right]$$

$$\circ\left[V_{ex}\left(1 + \frac{T_g}{2}\left[1+FK6\right]L3\right)\right]$$

$$ALF3 = -(9)\alpha_1^2 (T13)\left(\frac{1}{r''}\right)$$

$$CKXX = X(ALF3)$$

$$CKXY = ALF3$$

$$CKXZ = Z(ALF3)$$

$$CKYX = CKXX$$

$$CKYY = ALF3$$

$$CKYZ = CKXZ$$

$$CKZX = (YG3)(CKXX)$$

$$CKZY = Y(ALF3)$$

$$CKZZ = CKXZ$$

$$DKXX = \left[-9x^2(T7)\alpha_1 - 9x^2(T8)\alpha_1^3\right]\left[\frac{1}{r''}\right]$$

$$DKXY = \left[-9XY\alpha_1(T7 + T8\alpha_1^2)\right]\left[\frac{1}{r''}\right]$$

$$DKXZ = \left[-9XZ(T7\alpha_1 + T8\alpha_1^3)\right]\left[\frac{1}{r''}\right]$$

$$DKYX = DKXY$$

$$DKYY = -9y^2(\alpha_1)\left[T7 + T8\alpha_1^2\right]\left[\frac{1}{r''}\right]$$

$$DKYZ = -9ZY(\alpha_1)\left[T7 + T8\alpha_1^2\right]\left[\frac{1}{r''}\right]$$

$$DKZX = DKXZ$$

$$DKZY = -DKYZ$$

$$DKZZ = 9Z^2(\alpha_1)\left[T7 + T8\alpha_1^2\right]\left[\frac{1}{r''}\right]$$

$$M1 = \left[\frac{2\Delta\dot{\xi}^{*}\ddot{\xi}_g + 2\Delta\dot{\eta}^{*}\ddot{\eta}_g + 2\Delta\dot{\zeta}^{*}\ddot{\zeta}_g}{V_{ex}\, L}\right]$$

$$+ \left[\frac{(\Delta\dot{\xi}^{*})^2 + (\Delta\dot{\eta}^{*})^2 + (\Delta\dot{\zeta}^{*})^2}{V_{ex}\, L^2\,(\tau - T_g)}\right] + \left[\frac{V_{ex}}{\tau - T_g}\right]$$

$$M2 = \left[\frac{\Delta\dot{\xi}^{*} T_g}{V_{ex} L}\left(k_{11}\frac{\partial \ddot{x}_G}{\partial X} + k_{12}\frac{\partial \ddot{y}_G}{\partial X} + k_{13}\frac{\partial \ddot{z}_G}{\partial X}\right)\right]$$

$$+ \left[\frac{\Delta\dot{\eta}^{*} T_g}{V_{ex} L}\left(k_{21}\frac{\partial \ddot{x}_G}{\partial X} + k_{22}\frac{\partial \ddot{y}_G}{\partial X} + k_{23}\frac{\partial \ddot{z}_G}{\partial X}\right)\right]$$

$$+ \left[\frac{\Delta\dot{\zeta}^{*} T_g}{V_{ex} L}\left(k_{31}\frac{\partial \ddot{x}_G}{\partial X} + k_{32}\frac{\partial \ddot{y}_G}{\partial X} + k_{33}\frac{\partial \ddot{z}_G}{\partial X}\right)\right]$$

$$M3 = \left[\left(\frac{2\Delta\dot{\xi}^{*}}{V_{ex} L}\right) V_T \cos\gamma_T - \left(\frac{2\Delta\dot{\zeta}^{*}}{V_{ex} L}\right) V_T \sin\gamma_T\right]$$

$$M4 = \left[\left(\frac{2\Delta\dot{\xi}^{*}}{V_{ex} L}\right)\sin\gamma_T + \left(\frac{2\Delta\dot{\zeta}^{*}}{V_{ex} L}\right)\cos\gamma_T\right]$$

$$M5 = \left[\frac{\Delta\dot{\xi}^{*}\, T_g}{V_{ex} L}\right]$$

$$M6 = \left[\frac{2\Delta\dot{\xi}^{*}}{V_{ex} L}\right]\left[\dot{x} + \frac{T_g}{2}\ddot{x}_G\right]$$

$$M7 = \left[\frac{2\Delta\dot{\xi}^{*}}{V_{ex} L}\right]\left[\dot{y} + \frac{T_g}{2}\ddot{y}_G\right]$$

$$M3 = \left[\frac{2\Delta \dot{\zeta}^*}{V_{ex} L}\right]\left[\dot{z} + \frac{T_g}{2}\ddot{z}_G\right]$$

$$M9 = \left[\frac{2\Delta \dot{\eta}^*}{V_{ex} L}\right]\left[\dot{x} + \frac{T_g}{2}\ddot{x}_G\right]$$

$$M10 = \left[\frac{T_g \Delta \dot{\zeta}^*}{V_{ex} L}\left(k_{11}\frac{\partial \ddot{x}_G}{\partial y} + k_{12}\frac{\partial \ddot{y}_G}{\partial y} + k_{13}\frac{\partial \ddot{z}_G}{\partial y}\right)\right]$$

$$+ \left[\frac{T_g \Delta \dot{\eta}^*}{V_{ex} L}\left(k_{21}\frac{\partial \ddot{x}_G}{\partial y} + k_{22}\frac{\partial \ddot{y}_G}{\partial y} + k_{23}\frac{\partial \ddot{z}_G}{\partial y}\right)\right]$$

$$+ \left[\frac{T_g \Delta \dot{\zeta}^*}{V_{ex} L}\left(k_{31}\frac{\partial \ddot{x}_G}{\partial y} + k_{32}\frac{\partial \ddot{y}_G}{\partial y} + k_{33}\frac{\partial \ddot{z}_G}{\partial y}\right)\right]$$

$$M11 = \left[\frac{T_g \Delta \dot{\zeta}^*}{V_{ex} L}\left(k_{11}\frac{\partial \ddot{x}_G}{\partial z} + k_{12}\frac{\partial \ddot{y}_G}{\partial z} + k_{13}\frac{\partial \ddot{z}_G}{\partial z}\right)\right]$$

$$+ \left[\frac{T_g \Delta \dot{\eta}^*}{V_{ex} L}\left(k_{21}\frac{\partial \ddot{x}_G}{\partial z} + k_{22}\frac{\partial \ddot{y}_G}{\partial z} + k_{23}\frac{\partial \ddot{z}_G}{\partial z}\right)\right]$$

$$+ \left[\frac{T_g \Delta \dot{\zeta}^*}{V_{ex} L}\right]\left(k_{31}\frac{\partial \ddot{x}_G}{\partial z} + k_{32}\frac{\partial \ddot{y}_G}{\partial z} + k_{33}\frac{\partial \ddot{z}_G}{\partial z}\right)$$

$$N1 = \left[\frac{2\Delta \dot{\eta}^*}{V_{ex} L}\right]\left[\dot{y} + \frac{T_g}{2}\ddot{y}_G\right]$$

$$N2 = \left[\frac{2\Delta \dot{\eta}^*}{V_{ex} L}\right]\left[\dot{z} + \frac{T_g}{2}\ddot{z}_G\right]$$

$$N3 = \left[\frac{2\Delta \dot{\xi}^*}{V_{ex} L}\right]\left[\dot{x} + \frac{T_g}{2}\ddot{x}_g\right]$$

$$N4 = \left[\frac{2\Delta \dot{\xi}^*}{V_{ex} L}\right]\left[\dot{y} + \frac{T_g}{2}\ddot{y}_g\right]$$

$$N5 = \left[\frac{2\Delta \dot{\xi}^*}{V_{ex} L}\right]\left[\dot{z} + \frac{T_g}{2}\ddot{z}_g\right]$$

$$S1 = M3\left[\frac{e_T^2 + e_T \cos\theta_T}{1 + 2 e_T \cos\theta_T + e_T^2}\right]$$

$$S2 = M4\left[\left(\frac{\mu}{P_T}\right)^{1/2}\left(\frac{e_T \sin\theta_T}{\left[1 + e_T^2 + 2 e_T \cos\theta_T\right]^{1/2}}\right)\right]$$

$$S3 = M5\left[\left(\frac{2\mu}{R_T^3}\right)\left(\frac{P_T e_T \sin\theta_T}{\left[1 + e_T \cos e_T\right]^2}\right)\right]$$

$$S4 = M6\left[g_{31}\cos\phi_T - g_{11}\sin\phi_T\right]$$

$$S5 = M7\left[g_{32}\cos\phi_T - g_{12}\sin\phi_T\right]$$

$$S6 = M8\left[g_{33}\cos\phi_T - g_{13}\sin\phi_T\right]$$

$$S7 = N3\left[g_{11}\cos\phi_T + g_{31}\sin\phi_T\right]$$

$$S8 = N4\left[q_{12}\cos\phi_T + q_{32}\sin\phi_T\right]$$

$$S9 = N5\left[q_{13}\cos\phi_T + q_{33}\sin\phi_T\right]$$

$$W1 = \left[\frac{V\cos\gamma}{R} + \frac{V_T\cos\gamma_T}{R_T}\right]$$

$$W2 = \left(\frac{1}{2}\right)\left[T_g(1+FK6)\right]\left(\frac{V}{R}\right)\sin\gamma$$

$$W4 = \left(\frac{1}{2}\right)\left[T_g(1+FK6)\right]\left(\frac{V}{R^2}\right)\cos\gamma$$

$$W5 = \left(\frac{1}{2}\right)\left[T_g(1+FK6)\right]\left(\frac{V_T}{R_T}\right)\sin\gamma_T$$

$$W6 = \left(\frac{1}{2}\right)\left[T_g(1+FK6)\right]\left(\frac{1}{R_T}\right)\cos\gamma_T$$

$$W7 = V_T(W6)$$

$$W8 = \left(\frac{1}{2}\right)(W1)(1+FK6)$$

$$W9 = \left(\frac{1}{2}\right)(W1)\,T_g$$

$$A1 = \left[\frac{X_F q_{32} - Z_F q_{12}}{X_F^2 + Z_F^2}\right] - \left[\frac{W2}{\left[1 - \left(\frac{x\dot{x} - y\dot{y} + z\dot{z}}{RV}\right)^2\right]^{1/2}}\right]$$

$$\cdot \left[\frac{RV\ddot{y} - (x\dot{x} + y\dot{y} + z\dot{z})\frac{VY}{R}}{(RV)^2}\right] - \left[\frac{(W4)\,Y}{R}\right]$$

$$A2 = W5\left[\frac{e_T^2 + e_T\cos\theta_T}{1 + 2e_T\cos\theta_T + e_T^2}\right]$$

$$A3 = W6\left[\left(\frac{\mu}{P_T}\right)^{1/2}\left(\frac{e_T\sin\theta_T}{(1 + e_T^2 + 2e_T\cos\theta_T)^{1/2}}\right)\right]$$

$$A4 = W7\left[\frac{P_T\, e_T\sin\theta_T}{(1 + e_T\cos\theta_T)^2}\right]$$

$$A5 = W8 + W9\left[\tfrac{1}{2}(C1FK6)T_g + C2FK6\right]$$

$$A6 = \left[\frac{A5(S1 - S2 - S3 - S4 - S5 - S6 + S7 + S8 + S9)}{1 + A2 + A3 + A4}\right] - M.$$

$$A7 = A1\left[\frac{(A6 + M1)}{A5}\right] - M10$$

$$B1 = \left[\frac{x_F g_{33} - z_F g_{13}}{x_F^2 + z_F^2}\right] - \left[\frac{W2}{\left[1 - \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{RV}\right)^2\right]^{1/2}}\right]$$

$$\cdot\left[\frac{RV\dot{z} - (x\dot{x} + y\dot{y} + z\dot{z})\frac{Vz}{R}}{(RV)^2}\right] - \left[\frac{(W4)\,z}{R}\right]$$

$$B2 = \left[\frac{B1(S1 - S2 - S3 - S4 - S5 - S6 + S7 + S8 + S9)}{1 + A2 + A3 + A4}\right] -$$

$$E1 = \left[\frac{x_F\, g_{31} - z_F\, g_{11}}{x_F^2 + z_F^2}\right] - W2\left[\frac{RV\dot{x} - (x\dot{x} + y\dot{y} + z\dot{z})\frac{VX}{R}}{(RV)^2}\right]$$

$$\cdot\left[\frac{1}{\left[1 - \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{RV}\right)^2\right]^{1/2}}\right] - \left[\frac{(W4)\ X}{R}\right]$$

$$E2 = \left[\frac{E1\,(S1 - S2 - S3 - S4 - S5 - S6 + S7 + S8 + S9)}{1 + A2 + A3 + A4}\right] M2$$

$$D1 = -\frac{V}{R}\sin\gamma\left[\frac{RVy - (x\dot{x} + y\dot{y} + z\dot{z})\,\frac{R\dot{y}}{V}}{\left[1 - \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{RV}\right)^2\right]^{1/2}(RV)^2}\right]$$

$$+\left[\frac{\cos\gamma}{R}\left(\frac{\dot{y}}{V}\right)\right]$$

$$D2 = -\frac{V}{R}\sin\gamma\left[\frac{RVz - (x\dot{x} + y\dot{y} + z\dot{z})\,R\,\frac{\dot{z}}{V}}{\left[1 - \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{RV}\right)^2\right]^{1/2}(RV)^2}\right]$$

$$+\left[\frac{\cos\gamma\ \dot{z}}{RV}\right]$$

$$YEA = -g_{11}\sin\phi_T + g_{31}\cos\phi_T$$

$$YEB = -g_{12}\sin\phi_T + g_{32}\cos\phi_T$$

$$YEC = -g_{13}\sin\phi_T + g_{33}\cos\phi_T$$

$$YED = -g_{11} \cos\phi_T - g_{31} \sin\phi_T$$

$$YEE = -g_{12} \cos\phi_T - g_{32} \sin\phi_T$$

$$YEF = -g_{13} \cos\phi_T - g_{33} \sin\phi_T$$

$$ZFLXF = z_F / x_F$$

$$SMPLFY = \left[\frac{e_T^2 + e_T \cos\theta_T}{1 + 2 e_T \cos\theta_T + e_T^2} \right]$$

$$CONTROL(1) = \left(\frac{1}{V_{ex}}\right) \left[(T - T_g) \frac{\partial G}{\partial x} - G \frac{\partial T_g}{\partial x} \right]$$

$$CONTROL(2) = \left(\frac{1}{V_{ex}}\right) \left[(T - T_g) \frac{\partial G}{\partial y} - G \frac{\partial T_g}{\partial y} \right]$$

$$CONTROL(3) = \left(\frac{1}{V_{ex}}\right) \left[(T - T_g) \frac{\partial G}{\partial z} - G \frac{\partial T_g}{\partial z} \right]$$

$$CONTROL(4) = \left(\frac{1}{V_{ex}}\right) \left[(T - T_g) \frac{\partial G}{\partial x} - G \frac{\partial T_g}{\partial x} \right]$$

$$CONTROL(5) = \left(\frac{1}{V_{ex}}\right) \left[(T - T_g) \frac{\partial G}{\partial y} - G \frac{\partial T_g}{\partial y} \right]$$

$$CONTROL(6) = \left(\frac{1}{V_{ex}}\right) \left[(T - T_g) \frac{\partial G}{\partial z} - G \frac{\partial T_g}{\partial z} \right]$$

$$CINT(1,1) = k_{22} k_{33} - k_{23} k_{32}$$
$$CINT(1,2) = k_{13} k_{32} - k_{12} k_{33}$$
$$CINT(1,3) = k_{12} k_{23} - k_{13} k_{22}$$
$$CINT(2,1) = k_{23} k_{31} - k_{21} k_{33}$$
$$CINT(2,2) = k_{11} k_{33} - k_{13} k_{31}$$
$$CINT(2,3) = k_{13} k_{21} - k_{11} k_{23}$$
$$CINT(3,1) = k_{21} k_{32} - k_{22} k_{31}$$
$$CINT(3,2) = k_{12} k_{31} - k_{11} k_{32}$$
$$CINT(3,3) = k_{11} k_{22} - k_{12} k_{21}$$

$$EQ = k_{11}\, CINT(1,1) + k_{21}\, CINT(2,1) + k_{31}\, CINT(1,3)$$

$$\frac{\partial \ddot{x}_G}{\partial x} = \frac{T1}{R^3} + \frac{-3x^2 T1 + T2}{R^5} + \frac{1}{R^7}\Big[-5x^2 T2 + T3$$

$$+ (T4)\alpha_1 + (T5)\alpha_1^2 + (T6)\alpha_2^4 + x(VG1)(T4)$$

$$+ 2x(T5)(VG1)\alpha_1 + 4x(T6)(VG1)\alpha_1^3\Big]$$

$$+ \frac{1}{R^9}\Big[-7x^2(T3) - 7x^2(T4)\alpha_1 - 7x^2(T5)\alpha_1^2$$

$$- 7x^2(T6)\alpha_1^4 + (T7)\alpha_1 + (T8)\alpha_1^3 + x(T7)(VG1)$$

$$+ 3x(T8)(VG1)\alpha_1^2\Big] + DKXX + (VG1)\Big[\frac{(T10)(VG}{R^5}$$

$$+ \frac{1}{R^7}\Big[-5(T9)x - 5(T10)(VG1)x^2 - 5(T10)xy(VG2)$$

$$- 5(T10)xz(VG3) + (T12)(VG1)\Big] +$$

$$\frac{1}{R^9}\Big[-7(T11)x - 7(T12)x^2(VG1) - 7xy(T12)(VG2)$$

$$-7(T12)(VG3)xz + 3(T13)(VG1)\alpha_1^2\Big] + CKXX\Big]$$

$$\frac{\partial \ddot{x}_G}{\partial y} = \left(\frac{1}{R^5}\right)\Big[-3xy(T1)\Big] + \left(\frac{1}{R^7}\right)\Big[-5xy(T2) + (T4)x(VG2) +$$

$$2(T5)x\alpha_1(VG2) + 4(T6)x\alpha_1^3(VG2)\Big] + \left(\frac{1}{R^9}\right)$$

$$\cdot\Big[-7xy(T3 + \alpha_1 T4 + \alpha_1^2 T5 + \alpha_1^4 T6) + (T7)x(VG2)$$

$$+ 3x(T8)(VG2)\alpha_1^2\Big] + DKXY + (VG1)\Big[\left(\frac{1}{R^5}\right)\Big[(T10)(VG2)\Big]$$

$$+ \left(\frac{1}{R^7}\right)\Big[-5(T9)y - 5y(T10)\alpha_1 + (T12)(VG2)\Big] + \left(\frac{1}{R^9}\right)\Big[-7(T11$$

$$- 7(T12)\alpha_1 y + 3(T13)(VG2)\alpha_1^2\Big] + (-CKXY)\Big]$$

$$\frac{\partial \ddot{x}_G}{\partial z} = \left(\frac{1}{R^5}\right)\left[-3xz(T1)\right] + \left(\frac{1}{R^7}\right)\Big[-5xz(T2) + (T4)x(VG3)$$

$$+ 2(T5)x(\alpha_1)(VG3) + 4(T6)x(\alpha_1^3)(VG3)\Big]$$

$$+ \left(\frac{1}{R^9}\right)\Big[-7xz(T3 + \alpha_1 T4 + \alpha_1^2 T5 + \alpha_1^4 T6)$$

$$+ (T7)x(VG3) + 3(T8)x(\alpha_1^2)(VG3)\Big] + DKxz$$

$$+ VG1\Big[\left(\frac{1}{R^5}\right)(T10)(VG3) + \left(\frac{1}{R^7}\right)\Big[(T12)(VG3) - 5(T9)z$$

$$- 5z(T10)\alpha_1\Big] + \left(\frac{1}{R^9}\right)\Big[3(VG3)(\alpha_1^2)(T13) - 7(T11)z$$

$$- 7(T12)\alpha_1 z\Big] - CKxz\Big]$$

$$\frac{\partial \ddot{x}_G}{\partial \dot{x}} = 0$$

$$\frac{\partial \ddot{x}_G}{\partial \dot{y}} = 0$$

$$\frac{\partial \ddot{x}_G}{\partial \dot{z}} = 0$$

$$\frac{\partial \ddot{y}_G}{\partial x} = \left(\frac{1}{R^5}\right)(-3XY)(T1) + \left(\frac{1}{R^7}\right)\Big[-5XY(T2) + (VG1)\,Y(T4)$$

$$+ 2(T5)\,Y(VG1)\alpha_1 + 4(T6)(VG1)(\alpha_1^{3})\,Y\Big]$$

$$+ \left(\frac{1}{R^9}\right)\Big[-7XY\big[T3 + \alpha_1 T4 + \alpha_1^{2}\,T5 + \alpha_1^{4}\,T6\big]$$

$$+ Y(T7)(VG1) + 3Y(T8)(VG1)\alpha_1^{2}\Big] + DKYX$$

$$+ (VG2)\Big[\left(\frac{1}{R^5}\right)(T10)(VG1) + \left(\frac{1}{R^7}\right)\Big[-5\big[(T9)X$$

$$+ (T10)(VG1)X^2 + (T10)(VG3)XZ + (T10)XY(VG2)\big]$$

$$+ (T12)(VG1)\Big] + \left(\frac{1}{R^9}\right)\Big[-7\big[(T11)X + (T12)X^2(VG1)$$

$$+ XY(VG2)(T12) + (T12)(VG3)XZ\big] + 3(T13)(VG1)\alpha_1^{2}$$

$$- CKYX\Big]$$

$$\frac{\partial \ddot{y}_G}{\partial y} = (T1)\left(\frac{1}{R^5}\right) + \left(\frac{1}{R^5}\right)\big[T2 - 3(T1)Y^2\big] + \left(\frac{1}{R^7}\right)\Big[T3 -$$

$$5Y^2(T2) + (T4)\alpha_1 + (T5)\alpha_1^{2} + (T6)\alpha_1^{4} + Y(VG2)(T4)$$

$$+ VG2\big[2Y(T5)\alpha_1 + 4Y(T6)\alpha_1^{3}\big]\Big] + \left(\frac{1}{R^9}\right)\Big[-7Y^2(T3$$

$$+ \alpha_1 T4 + \alpha_1^{2}\,T5 + \alpha_1^{4}\,T6) + (T7)\alpha_1 + (T8)\alpha_1^{3} + Y(T7)(VG2)$$

$$+ 3Y(T8)(VG2)(\alpha_1^{2})\Big] + DKYY + (VG2)\Big[(T10)(VG2)\left(\frac{1}{R^5}\right)$$

$$+ \left(\frac{1}{R^7}\right)\big[(T12)(VG2) - 5(T9)Y - 5(T10)Y(\alpha_1)\big] + \left(\frac{1}{R^9}\right)$$

$$\cdot\big[3(T13)(VG2)\alpha_1^{2} - 7(T12)\alpha_1 Y - 7(T11)Y\big] - CKYY\Big]$$

$$\frac{\partial \ddot{y}_6}{\partial z} = \left(\frac{1}{R^5}\right)(-3zy)(T1) + \left(\frac{1}{R^7}\right)\Big[y(VG3)(T4) - 5yz(T2)$$

$$+ 2y(T5)(VG3)\alpha_1 + 4y(T6)(VG3)\alpha_1^3\Big] + \left(\frac{1}{R^9}\right)$$

$$\cdot\Big[-7yz(T3 + \alpha_1 T4 + \alpha_1^2 T5 + \alpha_1^4 T6) + y(T7)(VG3)$$

$$+ 3y(T8)(VG3)\alpha_1^2\Big] + DKyz + (VG2)\Big[\left(\frac{1}{R^5}\right)(T10)(VG$$

$$+ \left(\frac{1}{R^7}\right)\Big[(T12)(VG3) - 5(T9)z - 5z(T10)\alpha_1\Big] + \left(\frac{1}{R^9}\right)$$

$$\cdot\Big[3(VG3)\alpha_1^2(T13) - 7(T11)z - 7(T12)\alpha_1 z\Big] - CKyz\Big]$$

$$\frac{\partial \ddot{y}_6}{\partial \dot{x}} = \frac{\partial \ddot{y}_6}{\partial \dot{y}} = \frac{\partial \ddot{y}_6}{\partial \dot{z}} = 0$$

$$\frac{\partial \ddot{z}_6}{\partial x} = \left(\frac{1}{R^5}\right)(-3xz)(T1) + \left(\frac{1}{R^7}\right)\Big[-5xz(T2) + z(VG1)(T4)$$

$$+ 2z(T5)(VG1)\alpha_1 + 4z(T6)(VG1)\alpha_1^3\Big] + \left(\frac{1}{R^9}\right)$$

$$\cdot\Big[-7xz(T3 + \alpha_1 T4 + \alpha_1^2 T5 + \alpha_1^4 T6) + z(T7)(VG1)$$

$$+ 3z(T8)(VG1)\alpha_1^2\Big] + DKzx + VG3\Big[\left(\frac{1}{R^5}\right)(T10)(VG1)$$

$$+ \left(\frac{1}{R^7}\right)\Big[-5\big[(T9)x + (T10)(VG1)x^2 + (T10)(VG3)xz +$$

$$(T10)xy(VG2)\big] + (T12)(VG1)\Big] + \left(\frac{1}{R^9}\right)\Big[-7(T11)x + (T12)x^2$$

$$(VG1) + xy(VG2)(T12) + (VG3)xz(T12)\Big] + 3(T13)(VG1)\alpha_1^2$$

$$- CKzx\left(\frac{1}{VG3}\right)\Big]$$

$$\frac{\partial \ddot{z}_G}{\partial y} = \left(\frac{1}{R^5}\right)\left[-3 z y (T1)\right] + \left(\frac{1}{R^7}\right)\Big[z(T4)(VG2) - 5 y z (T2)$$

$$+ 2(VG2)(T5)\alpha_1 + 4 z (T6)(VG2)\alpha_1^3\Big] + \left(\frac{1}{R^9}\right)\Big[-7 y z (T3$$

$$+ T4\alpha_1 + T5\alpha_1^2 + T6\alpha_1^4) + z(T7)(VG2) + 3 z (T8)(VG2)\alpha_1^2$$

$$+ DK z y + VG3\Big[\left(\frac{1}{R^5}\right)(T10)(VG2) + \left(\frac{1}{R^7}\right)\Big[-5(T9) y$$

$$- 5(T10) y \alpha_1 + (T12)(VG2)\Big] + \left(\frac{1}{R^7}\right)\Big[3(T13)(VG2)\alpha_1^2$$

$$- 7(T11) y - 7(T12)\alpha_1 y\Big] - CK z y\Big]$$

$$\frac{\partial \ddot{z}_G}{\partial z} = \left(\frac{1}{R^3}\right)(T1) + \left(\frac{1}{R^5}\right)\left[-3 z^2 (T1) + (T2)\right] + \left(\frac{1}{R^7}\right)\Big[T3$$

$$- 5 z^2 (T2) + (T4)\alpha_1 + (T5)\alpha_1^2 + (T6)\alpha_1^4 + z(VG3)(T4)$$

$$+ 2 z (T5)(VG3)\alpha_1 + 4 z (T6)(VG3)\alpha_1^3\Big] + \left(\frac{1}{R^9}\right)\Big[-7 z^2 (T3$$

$$+ T4\alpha_1 + T5\alpha_1^2 + T6\alpha_1^4) + (T7)\alpha_1 + (T8)\alpha_1^3 + z(T7)(V\ldots$$

$$+ 3 z (T8)(VG3)\alpha_1^2\Big] + DK z z + (VG3)\Big[\left(\frac{1}{R^5}\right)(T10)(VG3)$$

$$+ \left(\frac{1}{R^7}\right)\Big[(T12)(VG3) - 5(T9) z - 5 z (T10)\alpha_1\Big] + \left(\frac{1}{R^9}\right)$$

$$\cdot\Big[3(VG3)\alpha_1^2 (T13) - 7(T11) z - 7(T12)\alpha_1 z\Big] - CK z z\Big]$$

$$\frac{\partial \ddot{z}_G}{\partial \dot{x}} = \frac{\partial \ddot{z}_G}{\partial \dot{y}} = \frac{\partial \ddot{z}_G}{\partial \dot{z}} = 0$$