NASA TECHNICAL REPORT

NASA TR R-451

NASA TR R-451

THEORY OF NOISE GENERATION FROM MOVING BODIES WITH AN APPLICATION TO HELICOPTER ROTORS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • DECEMBER 1975

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TR R-451		

4. Title and Subtitle	5. Report Date
THEORY OF NOISE GENERATION FROM MOVING BODIES WITH AN APPLICATION TO HELICOPTER ROTORS	December 1975
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
F. Farassat, The George Washington University, Joint Institute for Acoustics and Flight Sciences	L-10379
9. Performing Organization Name and Address	10. Work Unit No.
NASA Langley Research Center Hampton, Va. 23665	505-10-26-02
	11. Contract or Grant No.
	13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address	Technical Report
National Aeronautics and Space Administration Washington, D.C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

Several expressions for the determination of the acoustic field of moving bodies are presented in this report. The analysis is based on the Ffowcs Williams-Hawkings equation. Applying some criteria proposed herein, one of these expressions is singled out for numerical computation of acoustic pressure signature. The compactness of sources is not assumed and the main results are not restricted by the observer position. The distinction between compact and noncompact sources on moving surfaces is discussed. Some thickness noise calculations of helicopter rotors and comparison with experiments are included which suggest this mechanism as the source of high-speed blade slap of rotors.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Aeroacoustics Aerodynamic noise Helicopter rotor noise	Unclassified – Unlimited Subject Category 71

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	61	$4.25

*For sale by the National Technical Information Service, Springfield, Virginia 22161

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THEORY OF NOISE GENERATION FROM MOVING BODIES WITH AN APPLICATION TO HELICOPTER ROTORS

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SUMMARY

In this report, several formulations for the determination of the acoustic field of bodies of arbitrary shape and motion are presented. The analysis is based on the Ffowcs Williams-Hawkings equation which is rederived by an embedding procedure and is given in the appendix. The required parameters are the body geometry, time history of motion and its surface pressure distribution. Based on criteria proposed in the report, one of these formulations which consists of an observer time derivative and integrals over the curve of intersection of a collapsing sphere and the body, is singled out for numerical calculation of acoustic pressure signature. Compact or noncompact source distributions on a moving body are defined in terms of the time scale of fluctuations of the sources and the length scale of a virtual radiating surface whose geometry depends on the body shape and motion. A new expression is derived for the computation of the acoustic pressure signature of rotating blades. This expression is not restricted by compactness or far-field assumptions. A careful study of singularities of the various forms of the solution is undertaken and the nature of these singularities is explained. Some examples of thickness noise of helicopter rotors in hover and forward flight are worked out. These examples suggest strongly that high-speed blade slap of rotors is mainly the thickness noise phenomenon.

INTRODUCTION

The generation of sound from moving bodies, such as a rotating propeller or helicopter rotor blades, is an undesirable byproduct of many useful machines. The estimation of the sound field produced by the motion of such bodies is, therefore, essential in designing quiet machines. The theory for estimation of this sound has been developing gradually in recent years. Generally, one is concerned with the motion of a slender body such as a fan or a propeller blade with a rather complicated flow pattern around it. The pressure disturbance of the fluid medium caused by the motion of the body is felt as sound when the disturbance passes over any observer in the medium. However, without proper approximations, one is faced with a difficult theoretical problem in calculating the acoustic pressure.

An important approximation is introduced by assuming that the body does not disturb the medium appreciably and thus the nonlinearities may have negligible effects. The governing equation for the acoustic pressure is then a wave equation with some inhomogeneous source terms depending on the net local force of the body on the fluid and the normal velocity of the surface of the body. Historically, the effects of these two terms were studied separately.

Gutin (ref. 1) was first to calculate the sound from static propellers. He employed a result derived by Lamb (ref. 2) for the acoustic field of a stationary concentrated force. Because of the periodicity of the propeller forces, one is able to replace them by stationary and harmonically oscillating forces which are distributed on the propeller plane. Gutin applied Lamb's result to these forces. For harmonic analysis the governing wave equation was converted into the Helmholtz equation whose solution gave the acoustic pressure spectrum.

Garrick and Watkins (ref. 3) removed Gutin's restriction of static propellers by first studying the acoustic field of a concentrated force in uniform rectilinear motion. The observer is in the frame fixed to the source. In the nonrotating frame moving with the propeller, the analysis of Gutin was repeated and the acoustic pressure spectrum was obtained.

Deming (ref. 4) studied the effect of the source term involving the normal velocity distribution of the body surface for static propellers. He utilized Rayleigh's relation for the piston radiation into a semi-infinite space and obtained the acoustic pressure spectrum. This noise is referred to as the thickness noise. Arnoldi (ref. 5) extended Deming's work to propellers in uniform axial motion. Lyon (ref. 6) has studied the thickness noise of helicopter rotors in forward flight.

One of the common assumptions in treating the problem of acoustic radiation is the compactness of the sources. A stationary source is compact if its dimension is much smaller than the wave length of radiation. This definition needs to be modified for moving sources. Effectively a compact source may be treated as a point source and considerable simplification is introduced in the acoustic analysis (ref. 7). Even the forward flight effects of helicopter rotors may be studied numerically rather easily in this case (ref. 8).

It is the main purpose of this report to develop a theory for the calculation of the acoustic pressure signature for bodies in arbitrary motion in the ground-fixed coordinate system where the observer is not limited to the far field and no compactness assumption is made. The starting point of the present analysis is the equation developed by Ffowcs Williams and Hawkings, denoted the FW-H equation, using the acoustic analogy (ref. 9).

With a view toward application to helicopter rotors and propellers, the second-order source term which depends on Lighthill's stress tensor is neglected. The resulting equation is valid in the entire unbounded space and contains source terms which involve Dirac delta functions. These terms indeed come from the boundary conditions of the problem. One may then solve the FW-H equation by using the Green's function for the unbounded space.

An interesting fact about the FW-H equation is that it is possible to obtain many different forms of the solution (ref. 9). All these forms have some type of singularities for bodies moving at high speeds. In this report several forms of the solution of the FW-H equation are presented. The criteria which are proposed later will be applied to single out one form for the computation of the acoustic pressure signature. The distinction between the compact and noncompact source models for moving bodies is discussed in some detail. In many situations of interest, one cannot assume that the sources that generate the sound are compact.

A new expression for the determination of the acoustic field of high-speed helicopter rotors and propellers is derived and presented here. The part of this expression called the thickness noise is used for some trend calculations in the case of helicopter rotors. These calculations indicate that the thickness noise may be the mechanism of high-speed blade slap of helicopters. Some comparison with experiments is given to support this proposition.

The mathematical derivation and notes presented in the appendix are supplementary to the main body of the report. The technique of embedding a problem with a restricted domain into a problem with unbounded domain is discussed first. This technique is employed to derive the FW-H equation. The different forms of the solution of the FW-H equation are obtained and their relation to each other is explained in the appendix. A special emphasis is put on the problem of singularities in the solution of the FW-H equation. The nature of these singularities is also discussed in the appendix.

SYMBOLS

c	speed of sound
D	see equation (30)
DP	disk plane of helicopter rotor on propeller
E_{ij}	viscous stress tensor

$F(\vec{y}; \vec{x},t) = f(\vec{y},t-r/c) = 0$ equation of Σ-surface

$f(\ )$ arbitrary function

$f(\vec{y},\tau) = 0$ equation of surface of body

$G(x,\xi),\tilde{G}(x,\xi)$ Green's functions

g $= \tau - t + r/c$

$H(\ )$ Heaviside function

$h(\ )$ arbitrary function

$h(\eta_1',\eta_2')$ camber surface of blade in $\vec{\eta}\,'$-frame

$L(\vec{x},t)$ length scale of Σ-surface for a fixed observer position $\vec{x}$ and time t

l length scale of the body

M_n Mach number based on velocity normal to body, v_n/c

M_r Mach number based on velocity in radiation direction, v_r/c

M_t tip Mach number

m-curves system of curves on Σ-surface orthogonal to Γ-curves

N_i unit outward normal to Σ-surface

n_i unit outward normal to body

P_{ij} compressive stress tensor

p,p_1,p_2 pressure, acoustic pressure

p_T pressure distribution on body due to thickness

$Q(\vec{y},\tau), Q_i(\vec{y},\tau)$	Source distribution functions
q_i	net force of body on fluid
r, r_1	distance between source point and observer
ret	subscript denoting retarded time
$\hat{r}_i$	unit vector in radiation direction
$\hat{r}_i'$	components of unit vector $\vec{\hat{r}}$ in $\vec{\eta}\,'$-frame
r_{min}	minimum distance from observer to Σ-surface
r_o	see figure A2
S	surface of body $f(\vec{y},\tau) = 0$
S*	part of body surface whose sources contribute to acoustic pressure at position $\vec{x}$ and observer time t
T_{ij}	Lighthill stress tensor
$\tilde{T}(\vec{y})$	arbitrary bounded function
$T(\eta_1', \eta_2')$	thickness distribution function of blade in $\vec{\eta}\,'$-frame
t	observer time
t_p, t_v	time scales of pressure fluctuations and variation of speed of body
u_i	fluid velocity
u_n	fluid velocity normal to the body
V_i	vehicle velocity

V_i'	component of vehicle velocity $\vec{V}$ in $\vec{\eta}\,'$-frame
$\tilde{V}_1'$	$= -V_1' + \eta_2'\Omega$
$\tilde{V}_2'$	$= -V_2' - \eta_1'\Omega$
v_i	velocity of body
$v_i(\vec{\zeta},\tau)$	velocity of a point of the body $f(\vec{y},\tau) = 0$ in Lagrangian coordinates
v_n	component of velocity normal to body surface, $\vec{v} \cdot \vec{n}$
v_r	component of velocity of body surface in radiation direction, $\vec{v} \cdot \vec{r}$
x_i	observer position vector
$\vec{y}$-frame	Cartesian coordinate system fixed to the undisturbed medium
y_i	source position vector
Γ	curve of intersection of body and sphere $g = 0$
Δp	pressure distribution on rotor blade due to lift
$\delta(\ )$	Dirac delta function
δ_{ij}	Kronecker delta
$\vec{\zeta}$	Lagrangian coordinate of a point on the body
$\vec{\eta},\vec{\eta}\,'$	Cartesian coordinate frames
θ	angle between n_i and $\hat{r}_i$
$\tilde{\theta}$	angle between N_i and the radiation direction $\hat{r}_i$
θ_h	angle between normal to camber surface and $\hat{r}_i$

Λ	$= \left(1 + M_n^2 - 2M_n \cos\theta\right)^{1/2}$
ρ	density
ρ_o	density of undisturbed medium
$\tilde{\rho}$	$= \rho - \rho_o$
Σ	surface generated by $F(\vec{y}; \vec{x}, t) = 0$
τ, τ_1, τ_2	source times
$\tau^*, \bar{\tau}^*$	emission time of source
$\tau_c(\vec{x}, t)$	time spent for sphere $g = 0$ to cross body for fixed observer position $\vec{x}$ and observer time t
τ_o	see figure A2
ϕ	angle between η_1'- and η_1-axes
$\phi(\vec{x},t), \phi_1(\vec{x},t), \phi_2(\vec{x},t)$	unknown functions in wave equations (see appendix)
Ω	angular velocity of rotor
$\square^2$	wave operator, $\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} - \nabla^2$
∇^2	Laplacian operator, $\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2}$

Subscripts 1, 2, and 3 are used to denote components of a vector or partial derivatives along the corresponding axis.

Vectors are also denoted by an arrow over the symbol.

Summation convention on subscripts i,j is used throughout this report.

The element of the volume in each reference frame is written in terms of the differential of the symbol used to designate the frame (i.e., $d\vec{y}$ is the element of volume in the $\vec{y}$-frame).

THEORETICAL FORMULATION

General Discussion

For a body in motion, whose surface is described as $f(\vec{y},\tau) = 0$, the governing equation for the determination of the acoustic pressure p is the Ffowcs Williams-Hawkings (FW-H) equation (ref. 9):

$$\Box^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial \tau^2} - \nabla^2 p = \frac{\partial}{\partial \tau}\left[\rho_o v_n |\nabla f| \ \delta(f)\right] - \frac{\partial}{\partial y_i}\left[P_{ij}\frac{\partial f}{\partial y_j}\ \delta(f)\right] + \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \tag{1}$$

The summation convention on subscripts i and j is used here. The derivation of this equation is based on the acoustic analogy and it is, therefore, valid in the entire unbounded space. The $\vec{y}$-frame in which equation (1) holds is fixed with respect to the undisturbed medium. The above equation is derived in the appendix.

Equation (1) is the generalization of the equation derived by Lighthill (ref. 10) in connection with his jet noise theory. The first two inhomogeneous source terms in equation (1) are absent in Lighthill's theory and the contribution from the remaining term $\partial^2 T_{ij}/\partial y_i \partial y_j$ is the so-called quadrupole noise due to turbulence. The first term in the above equation arises as a result of the motion of the surface in the normal direction. Each surface element can be viewed as a small piston acting on the fluid with speed v_n. The second term comes from the local surface stress P_{ij} which consists of viscous stress E_{ij} and the thermodynamic pressure p. Physically, this term is the contribution of the net force acting on the fluid due to the viscous stress and the pressure distribution on the body surface.

It is assumed that the flow parameters on the moving surface are known. These parameters are obtained experimentally or by aerodynamic calculations. Obviously, this assumption simplifies the acoustic analysis considerably. As in the case of jet noise theory, not all the features of the acoustic field are expected to be exhibited by the present formulation. This is because of the fact that in the situations of interest, the flow around the body is quite complex and the required flow parameters could be specified to a limited degree. Despite this shortcoming, the acoustic analogy appears as one of the most useful methods for the study of the noise from bodies in motion.

Formulation

Since in the cases of interest, one is concerned with the motion of a slender body such as a rotor or fan blade, the region of turbulent flow is small and of relatively low intensity. Therefore, the term involving T_{ij} in equation (1) will be neglected. It is well known that turbulence is a very inefficient noise producing mechanism (ref. 10). This is yet another reason to neglect the last term of equation (1) which is a volume source. Turbulence has another effect on the acoustic field by producing fluctuating pressure on the body surface which is taken care of analytically through the second term in equation (1). The viscosity effect will also be neglected. Under the above assumptions, the governing equation for the generation of the sound for moving bodies is:

$$\Box^2 p = \frac{\partial}{\partial \tau}\left[\rho_o v_n |\nabla f|\ \delta(f)\right] - \frac{\partial}{\partial y_i}\left[p n_i |\nabla f|\ \delta(f)\right] \tag{2}$$

Here, $\vec{n}$ is the unit outward normal to the body. The pressure on the right side of the equality is the surface pressure. In the solution of this equation, the geometry and the time history of the motion of the body will be assumed known. By the nature of the acoustic analogy, the surface pressure distribution is a given parameter of the problem.

Solution of Governing Equation

In view of the fact that it is possible to get numerous forms of the solution of the FW-H equation, some criteria for the selection of a suitable form for the computational work are needed. The following criteria are therefore proposed:

(a) The solution should require as little information as possible about the source distribution on the surface of the body.

(b) The singularities in the solution should be integrable.

(c) The computation time should be as low as feasible.

The solution of equation (2) is $p = p_1 + p_2$ where p_1 and p_2 are the solutions of the following equations:

$$\Box^2 p_1 = \frac{\partial}{\partial \tau}\left[\rho_o v_n |\nabla f|\ \delta(f)\right] \tag{3}$$

$$\Box^2 p_2 = -\frac{\partial}{\partial y_i}\left[p n_i |\nabla f|\ \delta(f)\right] \tag{4}$$

These equations are of the types discussed in the appendix (eqs. (A31) and (A32), respectively). In the following paragraphs the solutions of equations (3) and (4) are written with the above criteria in mind.

Consider the surface $F(\vec{y}; \vec{x}, t) = f(\vec{y}, t - r/c) = 0$. Here, $\vec{x}$ and t are the observer position and time, respectively. These two parameters are kept fixed in the following discussion. It is shown in the appendix that the surface $F = 0$, denoted by Σ, serves as a virtual radiating surface for the body $f(\vec{y}, \tau) = 0$ which is in motion. The Σ-surface is the locus of the points on the body whose signals arrive simultaneously to the observer at time t. This surface is generated by the curves of the intersection of the body and the collapsing sphere $g = \tau - t + r/c = 0$ for $-\infty < \tau \leqq t$. These curves of intersection are called the Γ-curves here. The collapsing sphere in the process of crossing a two-bladed rotor system is shown in figure 1. (See also fig. (A1).) In terms of the Σ-surface, the solution of equation (3), using equation (A34) with $Q(\vec{y}, \tau) = \rho_o v_n$, is

$$4\pi\, p_1(\vec{x}, t) = \frac{\partial}{\partial t} \int_{F=0} \frac{1}{r} \left(\frac{\rho_o v_n}{\Lambda}\right)_{ret} d\Sigma \tag{5}$$

The symbol Λ is used for $\left(1 + M_n^2 - 2M_n \cos \theta\right)^{1/2}$ where $M_n = v_n/c$ and θ is the angle between $\vec{n}$ and $\vec{r}$ $(= \vec{x} - \vec{y},\ \vec{y}$ is the source position). The subscript ret stands for retarded time. The singularities in the integrand of equation (5), which appear when $\Lambda = 0$, are shown to be integrable in the appendix. The reason for keeping the time derivative outside the integral rather than using the result similar to equation (A44) is twofold. First, the order of singularity in the integrand is increased when the derivative is taken inside the integral. Second, as seen from equation (A44), one is required to calculate more parameters on the Σ-surface than those needed in equation (5). Also, when equation (A44) is expanded by performing the differentiations in the integrand, the result is a very long and complicated expression which does not seem to be useful for numerical computation.

For equation (4), the solution equivalent to equation (A46) with $Q_i(\vec{y}, \tau) = p n_i$, is

$$4\pi\, p_2(\vec{x}, t) = \frac{1}{c} \frac{\partial}{\partial t} \int_{F=0} \frac{1}{r} \left(\frac{p \cos \theta}{\Lambda}\right)_{ret} d\Sigma + \int_{F=0} \frac{1}{r^2} \left(\frac{p \cos \theta}{\Lambda}\right)_{ret} d\Sigma \tag{6}$$

In the discussion following equation (A46), the reason for preferring this form of solution to those involving space derivatives, such as equation (A36), is stated. The remarks concerning equation (5) apply equally here.

The solution of equation (2), from equations (5) and (6), is

$$4\pi\, p(\vec{x},t) = 4\pi(p_1 + p_2) = \frac{1}{c}\frac{\partial}{\partial t}\int_{F=0} \frac{1}{r}\left(\frac{\rho_o c v_n + p\cos\theta}{\Lambda}\right)_{ret} d\Sigma + \int_{F=0} \frac{1}{r^2}\left(\frac{p\cos\theta}{\Lambda}\right)_{ret} d\Sigma \tag{7}$$

Although the Σ-surface is defined analytically by $F(\vec{y};\vec{x},t) = f(\vec{y},t-r/c) = 0$, in general the above equation cannot be integrated in closed form even if the surface pressure p is known analytically. This is because of the implicit representation of the Σ-surface which may have a complicated geometrical shape in the situations of interest. (See, for instance, figs. A3 and A5.) Because of this difficulty one needs to introduce a change of variables of integration to get other forms of the solution. However, one must be aware of the additional singularities introduced in the solution by such transformations. When the relation of equation (A68) is employed, the following equivalent forms of equation (7) are obtained:

$$4\pi\, p(\vec{x},t) = \frac{\partial}{\partial t}\int_{\tau_1}^{\tau_2}\int_{\Gamma} \frac{\rho_o c v_n + p\cos\theta}{r\sin\theta}\, d\Gamma\, d\tau + \int_{\tau_1}^{\tau_2}\int_{\Gamma} \frac{c p \cot\theta}{r^2}\, d\Gamma\, d\tau \tag{8}$$

$$4\pi\, p(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\int_{S*}\left(\frac{\rho_o c v_n + p\cos\theta}{r\left|1 - M_r\right|}\right)_{\tau*} dS + \int_{S*}\left(\frac{p\cos\theta}{r^2\left|1 - M_r\right|}\right)_{\tau*} dS \tag{9}$$

As described earlier, Γ in equation (8) is the curve of intersection of the collapsing sphere $g = \tau - t + r/c = 0$ and the body $f(\vec{y},\tau) = 0$. This sphere enters and leaves the body at the source times τ_1 and τ_2. If, for a fixed $\vec{x}$ and t, the collapsing sphere enters and leaves the body more than once, then the integration in equation (8) is carried out over all the periods for which the intersection occurs.

Equation (9) applies to rigid bodies only. The time $\tau*$ is the emission time of the sources on the element of surface area dS of the body. The region of the body surface which contributes to the acoustic pressure at $\vec{x}$ and at the time t is denoted by $S*$. If all the points on the body surface move at speeds less than the speed of sound c, then $S*$ is the entire surface of the body. Otherwise, one has $S* = S*(\vec{x},t)$. In the preceding equation, M_r is defined as v_r/c where $v_r = \vec{v}\cdot\vec{\hat{r}}$ and $\vec{v}$ is the local velocity of the body surface. The time derivatives in equations (7), (8), and (9) should be taken numerically in the applications.

For the application of equation (8), one needs to construct the Γ-curves. This is, however, relatively simple. At any source time τ_1, one starts with the largest sphere with its center at the observer position $\vec{x}$ which touches the body. Let the radius of this sphere shrink at the speed of sound. The body should continue its motion. The intersection of the sphere and the body forms the Γ-curves. The sphere is allowed to collapse until it leaves the body completely at the time τ_2. The observer time corresponding to this sweep is $t = \tau_1 + r_1/c$ where r_1 is the radius of the sphere at the source time τ_1. One must make sure that the sphere does not intersect the body prior to τ_1 and after τ_2. This may be figured by knowing the time history of the motion of the body. Otherwise, the contribution of further intersections to the integrals in equation (8) must be included in the calculation of the acoustic pressure. The singularities of equation (8) are discussed in the appendix. They are integrable and could be dealt with in numerical work.

If a Lagrangian coordinate system $\vec{\xi}$ is introduced for the points on the body, the following transformation rule exists between the $\vec{y}$- and $\vec{\xi}$-frames:

$$\vec{y} = \vec{\xi} + \int_0^\tau \vec{v}(\vec{\xi},\tau)\, d\tau = \vec{y}(\vec{\xi},\tau) \tag{10}$$

Here it is assumed that the two frames coincide at $\tau = 0$. The velocity of a point on the body with coordinate $\vec{\xi}$ and at the time τ is $\vec{v}(\vec{\xi},\tau)$. The emission time τ^* is the solution or the solutions of the equation

$$g = \tau^* - t + \frac{\left|\vec{x} - \vec{y}(\vec{\xi},\tau^*)\right|}{c} = 0 \tag{11}$$

for fixed $\vec{x}$, t, and $\vec{\xi}$. In practice, this equation is transcendental and rarely can a simple analytic solution be obtained. However, the above equation can usually be solved numerically by an iteration technique. It must be mentioned that, formally, one may write

$$\tau^* = t - \frac{\left|\vec{x} - \vec{y}(\vec{\xi},\tau^*)\right|}{c} = t - \frac{r(\vec{\xi},\tau^*;\vec{x})}{c}$$

Because of this relation, it is customary to use the retarded time notation for formulas with integrals over the body surface. Such a notation could lead to confusion when taking a derivative inside an integral. For this reason, to convert the space derivatives in equation (A36) to a time derivative, the Σ-surface solution is used in the appendix even if the solution consisting of an integral over the body surface could also be employed. In the latter case the conversion is not as straight forward as that based on the Σ-surface solution.

For numerical calculations based on equation (9), the body surface S is first divided into a number of small elements. For each element, equation (11) should be employed to evaluate the corresponding source time τ^* (keeping $\vec{x}$ and t fixed) at an appropriate point on this element. However, because τ^* needs to be calculated a large number of times for each t, the use of equation (9) may require an excessive amount of computation time. The problem of the singularities of this equation is not easy to handle in numerical work.

Going back to the criteria proposed earlier in this section, it is seen that the most suitable form of the solution of the FW-H equation, equation (2), for computational purposes is given by equation (8). This solution satisfies criterion (a) by requiring only the surface pressure and not the pressure gradient. The choice of Γ and τ as the variables of integration is to satisfy the criteria (b) and (c). Also only the observer time derivative and not the derivatives with respect to x_i (such as in eqs. (A35) and (A36)) appears in equation (8). This reduces the computation time and thus favorably affects criterion (c).

Compact and Noncompact Sources

The terms compact and noncompact sources are used very frequently in acoustics. The purpose of this section is to bring out more clearly the distinction between compact and noncompact sources on moving bodies. Using the Σ-surface solution, one can define these terms precisely.

To be specific, the following equation will be considered:

$$\Box^2 p = -\frac{\partial}{\partial y_i}\left[p n_i |\nabla f| \, \delta(f)\right] \tag{12}$$

This equation is similar to equation (A32) with $Q_i(\vec{y},\tau) = -pn_i|\nabla f| \, \delta(f)$. The solution given by equation (A36) is

$$4\pi \, p(\vec{x},t) = -\frac{\partial}{\partial x_i}\int_{F=0} \frac{1}{r}\left(\frac{pn_i}{\Lambda}\right)_{ret} d\Sigma \tag{13}$$

After using the relation of equation (A68), one obtains

$$4\pi \, p(\vec{x},t) = -\frac{\partial}{\partial x_i}\int_{Body} \left(\frac{pn_i}{r|1 - M_r|}\right)_{\tau^*} dS \tag{14}$$

where τ^* is the solution of equation (11) for points on the surface S of the body.

For a fixed observer position $\vec{x}$ and time t, let the Σ-surface have the largest dimension $L(\vec{x},t)$ and let the time that the sphere $g = 0$ takes to cross the entire body be $\tau_c(\vec{x},t)$. If the time scale t_p of the pressure variation on the surface S and the time scale t_v of the variation of speed of the body are much greater than $\tau_c(\vec{x},t)$ and if the shortest distance from the observer to the Σ-surface r_{min} is much greater than $L(\vec{x},t)$, then equation (14) can be approximated as follows:

$$4\pi\, p(\vec{x},t) = -\frac{\partial}{\partial x_i}\frac{1}{\left(r\left|1 - M_r\right|\right)_{\bar{\tau}*}}\int_S (pn_i)_{\tau*}\, dS = -\frac{\partial}{\partial x_i}\left(\frac{q_i}{r\left|1 - M_r\right|}\right)_{\bar{\tau}*} \tag{15}$$

where $\vec{q}$ is the net force on the medium due to the pressure on the surface S and $\bar{\tau}*$ is the mean source time which is obtained by solving

$$\bar{\tau}* - t + \frac{\left|\vec{x} - \vec{y}_B(\bar{\tau}*)\right|}{c} = 0 \tag{16}$$

Here the mean location of the body is given by $\vec{y}_B(\bar{\tau}*)$. It is obvious that, effectively, the body is treated as a point source and in fact equation (15) is the solution of

$$\square^2 p = -\frac{\partial}{\partial y_i}\left\{q_i(\tau)\,\delta\left[\vec{y} - \vec{y}_B(\tau)\right]\right\} \tag{17}$$

which is what Lowson solved and published in 1965 (ref. 7).

It is seen that the crucial assumptions in treating the body as a point source are

$$\left.\begin{aligned} &\tau_p \gg \tau_c(\vec{x},t) \\ &\tau_v \gg \tau_c(\vec{x},t) \\ &r_{min} \gg L(\vec{x},t) \end{aligned}\right\} \tag{18}$$

Under these conditions, the source distribution is called compact. Otherwise, it is called noncompact. These conditions are dependent on observer time and position. Therefore, if an observer time interval is considered, all the above conditions should hold for this interval. Because of the dependence on observer position, the term "compact source distribution" is misleading since compactness is not a characteristic of the source distribution per se. For low subsonic speeds of the body, the length scale $L(\vec{x},t)$ and the typical length of the body l are of the same order of magnitude so that the third condition of equation (18) can be replaced by $r_{min} \gg l$.

The most important fact about compact source representation is that one is able to relate the acoustic field to global parameters of the motion of the body such as the net force $\vec{q}$ of the body on the fluid or in the case of thickness noise to net rate of mass injection. However, in the case where the source distribution is noncompact, the local pressure on the Σ-surface at the source time τ $(=t - r/c)$ and not the net force on the body at the time τ contributes to the acoustic pressure at the observer.

For rotating bodies, even at moderate tip speeds, some of the conditions of equations (18) could not be satisfied for the observer in or close to the plane of rotation. This may explain the poor agreement of the compact source theories with experiments so far.

APPLICATION TO HIGH-SPEED PROPELLERS AND HELICOPTER ROTORS

Specialization of General Equation

The solution of equation (2) is used here to derive a useful expression for high-speed propellers and helicopter rotors. The particular solution employed is equation (8). The main assumption here is that the intersection of the collapsing sphere $g = 0$ with the blade can be approximated by the curve of intersection of $g = 0$ with the disk plane. Because of this approximation, it is suggested that the observer be located in the region between 45° above or below the plane of rotation during the period when $g = 0$ crosses the system. This is the region where the noncompact source formulation should be used so that this requirement does not affect the usefulness of the results. The observer position is otherwise arbitrary. It can be in either the near or the far field.

A single blade is considered first. Three coordinate systems are used as shown in figure 2. The first which is called the $\vec{\eta}\,'$-frame is fixed to the blade such that the $\eta_1'\eta_2'$-plane coincides with the plane of rotation. The η_2'-axis is along the span and the η_1'-axis is parallel to the chord of the blade. The origin of this frame is at the rotation center. The second system is the $\vec{\eta}$-frame with its origin at the same location as the $\vec{\eta}\,'$-frame but is nonrotating. The $\eta_1\eta_2$-plane coincides with the disk plane. The axes of this frame are parallel to those of the third system, the $\vec{y}$-frame, which is fixed to the undisturbed medium. The position vector of the origin of the $\vec{\eta}$-frame and the $\vec{\eta}\,'$-frame is $\vec{\xi}(\tau)$ and the angle between the η_1'-axis and the η-axis is ϕ. In the following discussion, the components of any vector in the $\vec{\eta}\,'$-frame are primed for distinction. The vehicle speed with respect to the $\vec{y}$-frame which is $d\vec{\xi}/d\tau$ is denoted by $\vec{V}$.

The coordinate transformations between the preceding frames are

$$\vec{\eta} = \vec{y} - \vec{\xi} \tag{19}$$

$$\left.\begin{aligned}\eta_1' &= \eta_1 \cos\phi + \eta_2 \sin\phi \\ \eta_2' &= -\eta_1 \sin\phi + \eta_2 \cos\phi \\ \eta_3' &= \eta_3\end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned}\eta_1' &= (y_1 - \xi_1)\cos\phi + (y_2 - \xi_2)\sin\phi \\ \eta_2' &= -(y_1 - \xi_1)\sin\phi + (y_2 - \xi_2)\cos\phi \\ \eta_3' &= y_3 - \xi_3\end{aligned}\right\} \tag{21}$$

Let $\eta_3' = h(\eta_1',\eta_2')$ and $\eta_3' = T(\eta_1',\eta_2')$ be the equations of the camber surface and the thickness distribution, respectively. The equation of the surface of the blade in the $\vec{\eta}\,'$-frame is $\tilde{f}(\vec{\eta}\,') = 0$ where

$$\tilde{f}(\vec{\eta}\,') = \left\{\begin{array}{ll}\eta_3' - T(\eta_1',\eta_2') - h(\eta_1',\eta_2') & \text{(US)} \\ -\eta_3' - T(\eta_1',\eta_2') + h(\eta_1',\eta_2') & \text{(LS)}\end{array}\right\} \tag{22}$$

Here the abbreviations US and LS stand for the upper surface and the lower surface of the blade, respectively. The preceding equations are written in such a way that $\tilde{f}(\vec{\eta}\,') < 0$ inside the blade and $\tilde{f}(\vec{\eta}\,') > 0$ outside the blade. The equation of the blade surface in the $\vec{y}$-frame is

$$f(\vec{y},\tau) = \tilde{f}\left[\vec{\eta}\,'(\vec{y},\tau)\right] = 0 \tag{23}$$

where $\vec{\eta}\,'(\vec{y},\tau)$ is given by equations (21).

The normal velocity v_n of the blade surface is found from the relation

$$v_n = \frac{-\partial f/\partial \tau}{|\nabla f|} \tag{24}$$

From equation (23) one can obtain the following results:

$$\frac{\partial f}{\partial \tau} = \begin{cases} (T_1 + h_1)(V'_1 - \eta'_2\Omega) + (T_2 + h_2)(V'_2 + \eta'_1\Omega) - V'_3 & \text{(US)} \\ (T_1 - h_1)(V'_1 - \eta'_2\Omega) + (T_2 - h_2)(V'_2 + \eta'_1\Omega) + V'_3 & \text{(LS)} \end{cases} \tag{25}$$

$$|\nabla f|^2 = \begin{cases} 1 + (T_1 + h_1)^2 + (T_2 + h_2)^2 & \text{(US)} \\ 1 + (T_1 - h_1)^2 + (T_2 - h_2)^2 & \text{(LS)} \end{cases} \tag{26}$$

where (V'_1, V'_2, V'_3) are the components of the vehicle velocity $\vec{V}$ in the $\vec{\eta}\,'$-frame, Ω is the angular velocity of the blade system, and the subscripts 1 and 2 on the functions T and h designate partial derivatives with respect to η'_1 and η'_2, respectively. Since the blade may be considered a slender body, one can assume that

$$\left.\begin{aligned} &|h_1| << 1 \qquad |h_2| << 1 \qquad |T_2| << 1 \\ &|T_1| << 1 \qquad \text{(Except near leading edge)} \end{aligned}\right\} \tag{27}$$

For this reason, one obtains

$$|\nabla f|^2 \approx 1 + T_1^{\,2} \tag{28}$$

over the upper and lower surfaces of the blade.

Equation (23) also may be employed to obtain the relation

$$\nabla f \cdot \vec{\hat{r}} = |\nabla f| \cos\theta = \begin{cases} -(T_1 + h_1)\hat{r}'_1 - (T_2 + h_2)\hat{r}'_2 + \hat{r}'_3 & \text{(US)} \\ -(T_1 - h_1)\hat{r}'_1 - (T_2 - h_2)\hat{r}'_2 - \hat{r}'_3 & \text{(LS)} \end{cases} \tag{29}$$

where $(\hat{r}'_1, \hat{r}'_2, \hat{r}'_3)$ are the components of the unit radiation vector $\vec{\hat{r}}$ in the $\vec{\eta}\,'$-frame. Combining the preceding equation and equations (27) and (28) yields

$$|\nabla f|^2 \sin^2\theta = 1 - \hat{r}'^2_3 + T_1^{\,2}\left(1 - \hat{r}'^2_1\right) = D^2 \tag{30}$$

Equation (8) will be rewritten as follows:

$$4\pi\, p(\vec{x},t) = \frac{\partial}{\partial t}\int_{\tau_1}^{\tau_2}\int_{\Gamma} \frac{\rho_o c v_n}{r \sin\theta}\, d\Gamma\, d\tau + \frac{\partial}{\partial t}\int_{\tau_1}^{\tau_2}\int_{\Gamma} \frac{p \cot\theta}{r}\, d\Gamma\, d\tau + \int_{\tau_1}^{\tau_2}\int_{\Gamma} \frac{cp \cot\theta}{r^2}\, d\Gamma\, d\tau \tag{31}$$

For the term involving v_n, which is in fact the solution of equation (3), equations (24), (25), and (30) give

$$p_1(\vec{x},t) = \frac{\rho_o c}{4\pi}\frac{\partial}{\partial t}\int_{\tau_1}^{\tau_2}\int_{\Gamma} \frac{v_n}{r \sin\theta}\, d\Gamma\, d\tau = \frac{\rho_o c}{2\pi}\frac{\partial}{\partial t}\int_{\tau_1}^{\tau_2}\int_{\Gamma(DP)} \frac{T_1\tilde{V}_1' + T_2\tilde{V}_2'}{rD}\, d\Gamma\, d\tau \tag{32}$$

where $\tilde{V}_1'$ and $\tilde{V}_2'$ are given by the following relations:

$$\left.\begin{aligned} \tilde{V}_1' &= -V_1' + \eta_2'\Omega \\ \tilde{V}_2' &= -V_2' - \eta_1'\Omega \end{aligned}\right\} \tag{33}$$

The abbreviation DP stands for the disk plane of the helicopter rotor or propeller.

For the terms which involve p on the right side of equation (31), consider the following integral:

$$I = \int_{\tau_1}^{\tau_2}\int_{\Gamma} \frac{p \cot\theta}{r}\, d\Gamma\, d\tau \tag{34}$$

When equations (29) and (30) are used, one obtains

$$\begin{aligned} I = {} & \int_{\tau_1}^{\tau_2}\int_{\Gamma(DP)} \frac{(p_U - p_L)\left(-h_1\hat{r}_1' - h_2\hat{r}_2' + \hat{r}_3'\right)}{rD}\, d\Gamma\, d\tau \\ & - \int_{\tau_1}^{\tau_2}\int_{\Gamma(DP)} \frac{(p_U + p_L)\left(T_1\hat{r}_1' + T_2\hat{r}_2'\right)}{rD}\, d\Gamma\, d\tau \end{aligned} \tag{35}$$

where the subscripts U and L stand for the upper and lower surfaces of the blade, respectively. Using the notation $\Delta p \equiv p_L - p_U$ and $p_T \equiv (p_U + p_L)/2$, equation (35) is written as

$$I = -\int_{\tau_1}^{\tau_2}\int_{\Gamma(DP)} \frac{\Delta p \cos \theta_h}{rD}\, d\Gamma\, d\tau - 2\int_{\tau_1}^{\tau_2}\int_{\Gamma(DP)} \frac{p_T\left(T_1\hat{r}_1' + T_2\hat{r}_2'\right)}{rD}\, d\Gamma\, d\tau \tag{36}$$

where θ_h is the angle between the normal to the camber surface (pointing into the region $\eta_3' > 0$) and the radiation direction. In equation (36), the parameters Δp and p_T are pressure distribution due to lift and thickness, respectively. Since, in the situation of interest, these two parameters are of the same order of magnitude, the two integrals in equation (36) are of the same order of magnitude if the observer is in or very near the $\eta_1\eta_2$-plane when the sphere $g = 0$ crosses the blade. Directly above and below this plane, the first integral in equation (36) dominates.

The contribution from the terms containing pressure on the right side of equation (31) is, therefore,

$$p_2(\vec{x},t) = -\frac{\partial}{\partial t}\int_{\tau_1}^{\tau_2}\int_{\Gamma(DP)} \frac{\Delta p \cos \theta_h + 2p_T\left(T_1\hat{r}_1' + T_2\hat{r}_2'\right)}{4\pi rD}\, d\Gamma\, d\tau$$

$$- c\int_{\tau_1}^{\tau_2}\int_{\Gamma(DP)} \frac{\Delta p \cos \theta_h + 2p_T\left(T_1\hat{r}_1' + T_2\hat{r}_2'\right)}{4\pi r^2 D}\, d\Gamma\, d\tau \tag{37}$$

This is in fact the solution of equation (4). For several blades the acoustic pressure from each blade must be added linearly in equations (32) and (37). Essentially a similar result consisting of partial derivatives with respect to observer position rather than time t was obtained and reported in reference 11.

The contribution to the acoustic pressure denoted by $p_1(\vec{x},t)$ will be called the thickness noise although for a blade with zero lift, $\Delta p = 0$, the acoustic pressure has a nonzero contribution from p_2 since $p_T \neq 0$ on the blade. This definition is consistent with what has been called the thickness noise in the literature. (See refs. 4, 5, and 12.)

There is some evidence (ref. 6) that in or close to the $\eta_1\eta_2$-plane, the plane of rotation, the contribution of thickness noise p_1 is greater than p_2 at high tip speeds. This fact and the fact that the calculation of p_1 only requires knowledge of blade geometry and kinematics led to the development of a computer program to calculate the thickness noise (ref. 14). At present, the lack of sufficient information about the surface pressure distribution on rotating blade systems prevents a reliable calculation of p_2. However, the output of the thickness noise program and a comparison with experimental results may be employed to find under what conditions the acoustic pressure p can be approximated by p_1.

Thickness Noise of Helicopter Rotors

To study the thickness noise of high-speed helicopter rotors, equation (32) was used to develop a computer program for a rotor system with rectangular blades and a uniform airfoil section along the span. The program and the user's guide are presented in reference 14.

In this program, the Γ-curves are approximated by the arcs of intersection of the collapsing sphere $g = 0$ and the disk plane which is in the $\eta_1\eta_2$-plane. The double integral of equation (32) is approximated by a finite difference technique with a smaller mesh size near the leading edge of the blade. The derivative with respect to the observer time is taken numerically after the result of the integration is smoothed to reduce enlargement of errors. As expected, due to the construction of many Γ-curves for each observer time, the computation time is very long. A special effort should be made to design an efficient algorithm for the computation of noise from noncompact source distributions.

The computer program was used to investigate the relation between thickness noise and high-speed blade slap of helicopter rotors. The particular features of blade slap of concern here are

(a) Impulsive acoustic pressure signature

(b) Strong dependence of sound intensity on tip Mach number

(c) Directionality in the plane of rotation

The following numerical examples are presented to point out that thickness noise indeed exhibits all the above characteristics of high-speed blade slap. The rotor system for figures 3 to 5 has two blades with biconvex parabolic arc airfoil sections. The rotor diameter is 10 m and the blade chord is 0.4 m. The thickness ratio of each blade is 10 percent. The observer is in the rotation plane 50 m from the center of rotation. For figures 6 and 7 the rotor parameters are those of a test helicopter. The two-bladed rotor diameter is 13.42 m and the blade chord is 0.68 m. The thickness ratio of the blades is 9.3 percent. The rotor speed is 324 rpm. The helicopter is at an altitude of 15 m and the observer is 80 m ahead of the helicopter.

Figure 3 shows the theoretical variation of the acoustic pressure signature of a hovering helicopter as the tip speed is increased. The observer is in the plane of rotation. As the tip Mach number is increased, one sees a steepening of the signature, becoming impulsive above $M_t = 0.6$. The shape of the acoustic pressure signature changes at and above sonic tip speed with the new shape having two positive peaks and a negative peak of approximately equal magnitude. Between $M_t = 0.6$ and near sonic speed, the negative peak is dominant. The absolute value of the negative peak varies as a function of tip Mach number according to $\exp\left(8.94M_t^2\right)$ for this particular configuration in the range

$M_t = 0.4$ to 1.1. Therefore, both features (a) and (b) of the blade slap are exhibited by the thickness noise.

Figure 4 shows the theoretical variation of the acoustic pressure signature with the observer elevation. The observer distance is 50 m from the center of rotation of the hovering rotor system. The tip Mach number is kept fixed at $M_t = 0.9$. The directionality of the signal in the plane of rotation and the reduction in the impulsiveness of the signature at higher elevation is observed in this figure. Feature (c) of high-speed blade slap is, therefore, displayed by the thickness noise.

Figure 5 shows that it is the region near the tip of the rotor that generates most of the noise. The cut-out of the blade system of figures 3 and 4 is increased from 14 percent to 84 percent. Comparing this figure with figure 3(f), one can see that the pressure signature has not changed significantly either in shape or level.

Figure 6 shows the theoretical effect of change in the thickness distribution on the acoustic pressure signature. Note that in this figure, the rotor system is different from previous figures and a steady forward speed is also included. It is seen that the changes in the pressure signatures are significant and should be considered in designing a low-noise profile for the region near the blade tips. From equation (32), it may be shown that the acoustic pressure for the same section type (e.g., NACA four-digit airfoil or supercritical airfoil) is proportional to the thickness ratio. Although the supercritical airfoil section under the assumed conditions generates a higher noise level, one may use blades of smaller thickness ratio, as compared with other types, and obtain good aerodynamic characteristics as well as low noise level.

Figure 7 shows comparison of theoretical and experimental pressure signatures for the rotor system of figure 6 with an NACA four-digit airfoil section. The theoretical pressure signature is due to thickness alone but the agreement, particularly at higher forward speed, is very good. The most probable source of discrepancy at lower speeds is the neglect of the pressure sources on the blade surfaces of the rotor.

Since thickness noise does indeed display all the observed features of high-speed blade slap of helicopters, including the correct level of acoustic pressure, the thickness noise is proposed as the mechanism of the slap phenomenon. The theory and the numerical examples suggest several methods of controlling this noise:

(1) Reduce the tip speed of the rotor as much as possible (see fig. 3)

(2) Modify the tip region by

- (a) Reducing the thickness ratio; the acoustic pressure is directly proportional to thickness ratio (see eq. (32))
- (b) Changing the thickness distribution of the airfoil section (see fig. 6)
- (c) Altering the planform

The alteration of the planform is not included in the computer program presented in reference 14. However, its effect on the thickness noise can be suspected by the changes in the Σ-surface it produces for different observer times. This effect may be studied numerically.

CONCLUDING REMARKS

Several expressions for the determination of the acoustic pressure signature of bodies in motion are presented herein. Although all these expressions are of theoretical value, in practical situations, one form, consisting of an observer time derivative and integrals over the curve of intersection of a collapsing sphere and the body, was found to be the most useful for numerical work. No assumptions regarding the compactness of the sources or the location of the observer were made. An expression for the calculation of the acoustic pressure signature of rotating blades was derived. Using this expression, some thickness noise calculations were presented which, after comparison with experimental results, indicated that the thickness noise should be suspected as the cause of the high-speed blade slap of helicopter rotors. Further experiments are needed to verify this result.

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October 31, 1975

APPENDIX

MATHEMATICAL DERIVATIONS AND NOTES

Embedding Procedure

As mentioned in the Introduction, to use the Green's function for the unbounded space, the original problem, restricted to the domain outside of the moving body, is embedded into a class of problems defined in the entire unbounded space. The merits of this technique will be discussed shortly. The procedure will be illustrated by an example using an ordinary differential equation.

Consider the differential equation

$$\left.\begin{array}{ll} y'' - y = f(x) & \left(\begin{array}{l} x \text{ in } [a,\infty) \\ f(x) \to 0 \text{ as } x \to \infty \end{array}\right) \\ \text{with boundary conditions} & \\ \quad y(a) = y_a & \\ \quad y \to 0 \text{ as } x \to \infty & \end{array}\right\} \quad \text{(A1)}$$

This problem can be solved by finding the solutions of the following equations:

$$\left.\begin{array}{ll} y_1'' - y_1 = f(x) & \left(\begin{array}{l} x \text{ in } [a,\infty) \\ f(x) \to 0 \text{ as } x \to \infty \end{array}\right) \\ \text{with boundary conditions} & \\ \quad y_1(a) = 0 & \\ \quad y_1 \to 0 \text{ as } x \to \infty & \end{array}\right\} \quad \text{(A2)}$$

$$\left.\begin{array}{ll} y_2'' - y_2 = 0 & \left(x \text{ in } [a,\infty)\right) \\ \text{with boundary conditions} & \\ \quad y_2(a) = y_a & \\ \quad y_2 \to 0 \text{ as } x \to \infty & \end{array}\right\} \quad \text{(A3)}$$

The solution to the original problem is then $y = y_1 + y_2$ due to linearity of equations (A1).

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The Green's function for equations (A2) is (ref. 15)

$$G(x,\xi) = e^{a-\xi} \sinh (a - x) H(\xi - x) + \sinh (a - \xi) e^{a-x} H(x-\xi)$$

where $H(x)$ is the Heaviside function defined by

$$H(x) = \begin{Bmatrix} 1 & (x > 1) \\ 0 & (x < 0) \end{Bmatrix} \tag{A4}$$

The solution for y is then the following:

$$y(x) = \langle G(x,\xi), f(\xi) \rangle + y_2(x)$$

$$\equiv \int_a^\infty G(x,\xi) f(\xi) \, d\xi + y_2(x)$$

$$= e^{a-x} \int_a^x \sinh (a - \xi) f(\xi) \, d\xi + \sinh (a - x) \int_x^\infty e^{a-\xi} f(\xi) \, d\xi + y_a e^{a-x} \tag{A5}$$

The solution of equations (A3) which is

$$y_2 = y_a e^{a-x} \tag{A6}$$

has been utilized in equation (A5).

For use in the upcoming discussion the following relation is obtained by evaluating the derivative of the solution of $y(x)$ in equation (A5) at $x = a$:

$$y_a + y_a' = -\int_a^\infty e^{a-\xi} f(\xi) \, d\xi \tag{A7}$$

where $y_a' \equiv y'(a)$. Now consider the following differential equation:

$$\tilde{y}'' - \tilde{y} = h(x) \qquad \begin{pmatrix} x \text{ in } (-\infty,\infty) \\ h(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty \end{pmatrix}$$

with boundary conditions

$$\tilde{y} \rightarrow 0 \text{ as } |x| \rightarrow \infty \tag{A8}$$

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The Green's function for this problem is

$$\tilde{G}(x,\xi) = -\frac{1}{2}\left[e^{x-\xi}\, H(\xi - x) + e^{-x+\xi}\, H(x-\xi)\right]$$

The solution of equation (A8) is, therefore,

$$\tilde{y}(x) = \langle \tilde{G}(x,\xi), h(\xi) \rangle$$

$$= -\frac{1}{2}\, e^{-x} \int_{-\infty}^{x} e^{\xi}\, h(\xi)\, d\xi - \frac{1}{2}\, e^{x} \int_{x}^{\infty} e^{-\xi}\, h(\xi)\, d\xi \tag{A9}$$

Let $y(x)$ be the solution of equations (A1) corresponding to a given function $f(x)$ satisfying the condition $f(x) \to 0$ as $x \to \infty$. The function $\tilde{y}(x) = H(x-a)\, y(x)$ certainly satisfies the boundary conditions of the equations (A8). What is the corresponding source function $h(x)$? This function turns out to be a generalized function. It is obtained as follows:

$$\tilde{y}'(x) = H(x-a)\, y'(x) + H'(x-a)\, y(x)$$

$$= H(x-a)\, y'(x) + \delta(x-a)\, y(x)$$

$$= H(x-a)\, y'(x) + \delta(x-a)\, y_a$$

$$\tilde{y}''(x) = H(x-a)\, y''(x) + \delta(x-a)\, y'(x) + \delta'(x-a)\, y_a$$

$$= H(x-a)\, y''(x) + \delta(x-a)\, y'_a + \delta'(x-a)\, y_a$$

$$\tilde{y}''(x) - \tilde{y}(x) = H(x-a)\left[y''(x) - y(x)\right] + \delta(x-a)\, y'_a + \delta'(x-a)\, y_a$$

$$= H(x-a)\, f(x) + \delta(x-a)\, y'_a + \delta'(x-a)\, y_a$$

where $\delta(x-a)$ is the Dirac delta function. And, thus, the function $h(x)$ is given by

$$h(x) = H(x-a)\, f(x) + \delta(x-a)\, y'_a + \delta'(x-a)\, y_a \tag{A10}$$

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The function $h(x)$ is not determined completely since y_a is known but y'_a is not given as seen from equations (A1). Assume that y'_a is either measured by an experiment or is otherwise known. It will be calculated shortly. Note that the domain of the definition of $h(x)$ is the entire unbounded real line. What has been achieved is that the problem described by equations (A1) may be viewed as a special case of the class of problems described by equations (A8). This technique which extends the domain of the definition of a differential equation will be called the embedding procedure. It is clear that if the Green's function for the problem with the larger domain is known, then the new problem may be solved assuming that the appropriate additional boundary conditions, such as y'_a in equation (A10), are known. Since in many problems, particularly those involving partial differential equations, it is hard to find the Green's function, the embedding procedure extends the usefulness of the available Green's functions.

To convince the reader that one obtains the same solution of equation (A5) for the problem of equations (A1), the source function $h(x)$ in equation (A10) is substituted in equation (A5). Some elementary properties of the generalized function theory (ref. 16, pp. 1-25) will be used here. One obtains

$$
\begin{aligned}
H(x-a)\, y(x) &= \tilde{y}(x) \\
&= \left\langle \tilde{G}(x,\xi), h(\xi) \right\rangle \\
&= \left\langle \tilde{G}(x,\xi), H(\xi-a)\, f(\xi) + \delta(\xi-a)\, y'_a + \delta'(\xi-a)\, y_a \right\rangle \\
&= \left\langle \tilde{G}(x,\xi), H(\xi-a)\, f(\xi) \right\rangle + y'_a\, \tilde{G}(x,a) - y_a \frac{\partial \tilde{G}}{\partial \xi}(x,a) \\
&= -\frac{1}{2}\left[e^{-x} \int_a^x e^{\xi} f(\xi)\, d\xi + e^{x} \int_x^{\infty} e^{-\xi} f(\xi)\, d\xi \right] \\
&\quad - \frac{y_a + y'_a}{2} e^{x-a} H(x-a) + \frac{y_a - y'_a}{2} e^{-x+a} H(x-a)
\end{aligned}
\tag{A11}
$$

Since one is interested in the value of y when x is in $[a,\infty)$, from equations (A1) the following is obtained:

$$
y(x) = -\frac{1}{2}\left[e^{-x} \int_a^x e^{\xi} f(\xi)\, d\xi + e^{x} \int_x^{\infty} e^{-\xi} f(\xi)\, d\xi \right] + \frac{y_a - y'_a}{2} e^{-x+a} \tag{A12}
$$

Letting $x = a$, precisely the same relation as equation (A7) is obtained. This indicates that given y_a, one cannot choose y'_a arbitrarily, but it is obtained uniquely in the process of solving the problem. The relation between the boundary conditions, equation (A7), will be called the compatibility condition. Substituting y'_a in terms of y_a from equation (A7) in equation (A12), one obtains the same solution as equation (A5).

Looking back, it is seen that the solution of the problem described by equations (A1) with restricted domain was obtained by first embedding the problem in the class of problems described by equations (A8) with unbounded domain. This provided the inhomogeneous source term which is the generalized function $h(x)$ involving conditions at the point $x = a$ not entirely supplied by the original problem, equations (A1). However, the method eventually led to the compatibility condition (A7) which yielded the unique solution to equations (A1).

For the Ffowcs Williams-Hawkings (FW-H) equation, the compatibility condition is in fact an integral equation in terms of the surface pressure of the body which is usually difficult to solve. One may then use experimentally obtained surface pressure distributions. The usefulness of the method presented here should not be overlooked as it provides a systematic and neat method of attack which may work.

Derivation of Ffowcs Williams-Hawkings Equation by Embedding Procedure

Let the surface of the moving body be given by the function $f(\vec{y},\tau) = 0$. The function $f(\vec{y},\tau)$ is defined to be positive outside and negative inside the body. Following the procedure of the preceding section, all the flow parameters will be multiplied by the function $H[f(\vec{y},\tau)]$, where $H(f)$ is the Heaviside function, in order to embed the problem of the external flow and acoustics of this body into the class of problems valid in the entire unbounded space.

It is required on physical grounds that all flow parameters vanish at infinity. For this reason, the quantity $\rho - \rho_o$ where ρ_o is the density of the undisturbed medium, will be used as the density parameter, rather than ρ itself. However, if ρ is multiplied by a quantity that vanishes at infinity, such as the flow velocity, it is kept unchanged. Outside the body, the continuity and momentum equations hold true. Using summation convention on subscripts i and j, these equations are

Continuity equation:

$$\frac{\partial}{\partial \tau}(\rho - \rho_o) + \frac{\partial}{\partial y_i}(\rho u_i) = 0 \qquad \text{(A13)}$$

Momentum equation:

$$\frac{\partial}{\partial \tau}(\rho u_i) + \frac{\partial}{\partial y_j}\left(P_{ij} + \rho u_i u_j\right) = 0 \tag{A14}$$

where u_i is the fluid velocity, τ is the time, and P_{ij} is the compressive stress tensor. The parameter $\rho - \rho_o$ will be designated $\tilde{\rho}$. The frame of reference $\vec{y}$ is fixed with respect to the undisturbed medium.

The continuity equation, valid in the entire unbounded space, is

$$\frac{\partial}{\partial \tau}\left[\tilde{\rho}\ H(f)\right] + \frac{\partial}{\partial y_i}\left[\rho u_i\ H(f)\right] = H(f)\left[\frac{\partial \tilde{\rho}}{\partial \tau} + \frac{\partial}{\partial y_i}(\rho u_i)\right] + \tilde{\rho}\left(\frac{\partial H}{\partial \tau} + u_i\ \frac{\partial H}{\partial y_i}\right) + \rho_o u_i\ \frac{\partial H}{\partial y_i}$$

$$= \tilde{\rho}\left(\frac{\partial H}{\partial \tau} + u_i\ \frac{\partial H}{\partial y_i}\right) + \rho_o u_i\ \frac{\partial H}{\partial y_i} \tag{A15}$$

Note that

$$\frac{\partial H}{\partial \tau} + u_i\ \frac{\partial H}{\partial y_i} = \delta(f)\left(\frac{\partial f}{\partial \tau} + u_i\ \frac{\partial f}{\partial y_i}\right)$$

$$= \delta(f)\ \frac{\partial f}{\partial y_i}(u_i - v_i)$$

$$= \delta(f)\ |\nabla f|(u_n - v_n) = 0 \tag{A16}$$

where v_i is the velocity of the body and the no-penetration condition $u_n = v_n$ is used on the body. The subscript n denotes the component normal to the body. Using the preceding result, equation (A15) will be written in the following form:

$$\frac{\partial}{\partial \tau}\left[\tilde{\rho}\ H(f)\right] + \frac{\partial}{\partial y_i}\left[\rho u_i\ H(f)\right] = \rho_o u_i\ \delta(f)\ \frac{\partial f}{\partial y_i}$$

$$= \rho_o u_n |\nabla f|\ \delta(f)$$

$$= \rho_o v_n |\nabla f|\ \delta(f) \tag{A17}$$

Using the above technique, the momentum equation can be written as:

$$\frac{\partial}{\partial\tau}\left[\rho u_i\ H(f)\right] + \frac{\partial}{\partial y_j}\left[\left(P_{ij} + \rho u_i u_j\right)\ H(f)\right] = P_{ij}\ \frac{\partial H}{\partial y_j}$$

$$= P_{ij}\ \frac{\partial f}{\partial y_j}\ \delta(f) \tag{A18}$$

In this relation, the result of equation (A16) has been applied. Taking the time derivative of both sides of equation (A17) and the space derivative with respect to y_i of both sides of equation (A18) and then subtracting the latter from the former, one obtains

$$\frac{\partial^2}{\partial\tau^2}\left[\tilde{\rho}\ H(f)\right] - \frac{\partial^2}{\partial y_i \partial y_j}\left[\left(P_{ij} + \rho u_i u_j\right)\ H(f)\right] = \frac{\partial}{\partial\tau}\left[\rho_o v_n |\nabla f|\ \delta(f)\right] - \frac{\partial}{\partial y_i}\left[P_{ij}\ \frac{\partial f}{\partial y_j}\ \delta(f)\right] \tag{A19}$$

Now add and subtract the term $c^2\ \partial^2\left[\tilde{\rho}\ H(f)\right]\Big/\partial y_i^2$ on the left side of equation (A19) and rearrange the term involving P_{ij} to get

$$\frac{\partial^2}{\partial\tau^2}\left[\tilde{\rho}\ H(f)\right] - c^2\ \frac{\partial^2}{\partial y_i^2}\left[\tilde{\rho}\ H(f)\right] = \frac{\partial}{\partial\tau}\left[\rho_o v_n |\nabla f|\ \delta(f)\right] - \frac{\partial}{\partial y_i}\left[P_{ij}\ \frac{\partial f}{\partial y_j}\ \delta(f)\right]$$

$$+ \frac{\partial^2}{\partial y_i \partial y_j}\left[\left(P_{ij} + \rho u_i u_j - c^2\tilde{\rho}\ \delta_{ij}\right)\ H(f)\right] \tag{A20}$$

Here c is the speed of sound in the undisturbed medium. The quantity

$$T_{ij} = P_{ij} + \rho u_i u_j - c^2\tilde{\rho}\ \delta_{ij} \tag{A21}$$

is called the Lighthill stress tensor, well known in jet noise theory. Let $p = c^2\tilde{\rho}$. This quantity is the acoustic pressure in the region not too close to the body. It is the perturbation pressure in linear theory even close to the body itself. Using this notation, equation (A20) will be written as

$$\Box^2\left[p\ H(f)\right] \equiv \left(\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2} - \nabla^2\right)\left[p\ H(f)\right]$$

$$= \frac{\partial}{\partial\tau}\left[\rho_o v_n |\nabla f|\ \delta(f)\right] - \frac{\partial}{\partial y_i}\left[P_{ij}\ \frac{\partial f}{\partial y_j}\ \delta(f)\right] + \frac{\partial^2}{\partial y_i \partial y_j}\left[T_{ij}\ H(f)\right] \tag{A22}$$

The above equation was derived first by Ffowcs Williams and Hawkings (ref. 9). They based their derivation of the generalized mass continuity equation (eq. (A17)) and momentum equation (eq. (A18)) on the study of mass and momentum flow through a control volume, respectively. The body is represented by a surface of discontinuity in the unbounded medium. Inside the body, the fluid is at rest with the conditions of the undisturbed medium. The method presented in this appendix was known to Ffowcs Williams and Hawkings and the technique, without using the terminology of embedding, was sketched in their original work (ref. 9). Note that the FW-H equation is valid even if shock discontinuities are present in the flow field. This is because of the fact that this equation is based on conservation of mass and momentum which should hold in the presence of shock waves.

It should be emphasized that as in the case of ordinary differential equations, the source terms in equation (A22) can not all be specified arbitrarily. For example, the surface pressure on the body should be specified in such a way that if the observer is located on the surface of the body at the time t, the calculated acoustic pressure p should be the same as the local surface pressure at the time. This is the compatibility condition.

When obtaining the solution of equation (A22), the function which is solved for will be written as $p(\vec{x},t)$ rather than $p(\vec{x},t)\,H(f)$. The latter function is actually what the solution represents. Mathematically, the solution is valid anywhere including inside the body for which $p(\vec{x},t)\,H(f) = 0$. This fact can be used to advantage in some problems. When the formal solution of equation (A22) is regarded as an integral equation in terms of the body surface pressure, one can show that $H(f) = 1/2$ on the surface of the body.

Solution of Wave Equation With Generalized Functions as Source Terms

Let $f(\vec{y},\tau) = 0$ be the equation of the surface of a moving body. The function $f(\vec{y},\tau)$ is defined such that its gradient ∇f points outside the body. The solution of the inhomogeneous wave equation

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial\tau^2} - \nabla^2\phi = Q(\vec{y},\tau)\,|\nabla f|\,\delta(f) \tag{A23}$$

will be considered first. Next the solutions of the wave equation with the following source terms

$$\frac{\partial}{\partial\tau}\Big[Q(\vec{y},\tau)\,|\nabla f|\,\delta(f)\Big], \frac{\partial}{\partial y_i}\Big[Q_i(\vec{y},\tau)\,|\nabla f|\,\delta(f)\Big]$$

will be given. In the preceding equation, the functions $Q(\vec{y},\tau)$ and $Q_i(\vec{y},\tau)$ are given piecewise smooth functions on the body surface. Note that the choice of the preceding source terms corresponds to those of FW-H equation (A22).

The Green's function for the solution of the wave equation in the unbounded space is $\delta(g)/4\pi r$ where $g = \tau - t + r/c$, τ is the source time, t is the observer time, $r = |\vec{x} - \vec{y}|$ and $\vec{x}$ and $\vec{y}$ are the observer and source locations respectively. When $\vec{x}$ and t are fixed and the source time τ varies from $-\infty$ to t the surface $g = 0$ is a sphere with center at $\vec{x}$ whose radius collapses at the speed of sound. The solution of the equation (A23) is thus

$$4\pi\,\phi(\vec{x},t) = \int \frac{Q(\vec{y},\tau)}{r} |\nabla f|\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau \tag{A24}$$

The volume integral extends over the unbounded three-dimensional space and the time integral is over the interval $-\infty < \tau \leqq t$.

The integrals involving a product of two delta functions are studied in the literature of generalized functions (ref. 16, pp. 209-247). The result is an integral over the subspace where $f = 0$ and $g = 0$. To get this integral, the variables of integration in the preceding equation should be transformed to a coordinate system which has f and g as new variables. The details are outlined in reference 17. One writes:

$$\begin{aligned} d\vec{y} &= \frac{df\,dg\,dy_1}{\partial(f,g)/\partial(y_2,y_3)} \\ &= \frac{df\,dg\,dy_1}{(\nabla f \times \nabla g)\cdot\vec{e}_1} \\ &= \frac{c\,df\,dg\,d\Gamma}{|\nabla f|\sin\theta} \end{aligned} \tag{A25}$$

where $\vec{e}_1$ is the unit vector along y_1-axis and Γ is the curve of intersection of the surfaces f = Constant and g = Constant. The angle θ is between ∇f and $\vec{r}$ $(=\vec{x} - \vec{y})$. In the preceding equation one may use other combinations of axes to get the same final result. Substituting equation (A25) in equation (A24) and integrating with respect to f and g, one obtains

$$4\pi\,\phi(\vec{x},t) = \int_{\tau_1}^{\tau_2} \int_{\Gamma} \frac{cQ(\vec{y},\tau)}{r\sin\theta}\,d\Gamma\,d\tau \tag{A26}$$

Here, because of the products of delta functions in the integrand of equation (A24), Γ is the curve of intersection of the surfaces $f = 0$ and $g = 0$. (See figs. 1 and A1.) The source times τ_1 and τ_2 correspond to the times when the sphere $g = 0$ enters and leaves the body, respectively. The condition $\theta = 0$ will be discussed separately later on.

Another form of the solution of equation (A23) may be written by noting that the subspace for which $f = g = 0$, keeping $\vec{x}$ and t fixed, is the surface

$$F(\vec{y};\ \vec{x},t) = f\left(\vec{y},t-\frac{r}{c}\right) = 0$$

This equation is obtained by solving for τ from $g = 0$ and then substituting in $f(\vec{y},\tau) = 0$. This surface, designated as Σ, is the locus of all the points on the surface $f(\vec{y},\tau) = 0$ whose signals arrive simultaneously at the observer location $\vec{x}$ and at the time t. To visualize this surface, consider a large sphere enclosing the body whose radius collapses at the speed of sound to radius zero at observer time t with the center at observer position $\vec{x}$. As the sphere contracts, let the body trace its history in time. The surface generated by the curves of intersection of the sphere and the body is in fact the Σ-surface. Incidentally, these curves of intersection are the Γ-curves employed in the equation (A26). Note that for points on a Γ-curve, or a point lying on the Σ-surface, with coordinate $\vec{y}$, the source time τ is related to t by the relation $\tau = t - \frac{r}{c}$ where $r = \left|\vec{x} - \vec{y}\right|$. In figure A1 the situation is illustrated for the moving body $f(\vec{y},\tau) = 0$. The sphere and the body at the source time τ are shown with dotted lines. Their intersection, the Γ-curve corresponding to the time τ, lies on the Σ-surface. This surface is formed by all the Γ-curves for $-\infty < \tau \leqq t$. The Σ-surface is not necessarily simply connected.

Now that the Σ-surface has been defined, use the relationship $\delta(F)\ \delta(g) = \delta(f)\ \delta(g)$ in the integrand of equation (A24). This follows from the fact that the subspace defined by $f = 0$ and $g = 0$ is the same as one defined by $F = 0$ and $g = 0$, once again keeping $\vec{x}$ and t fixed. Equation (A24) may be written as

$$4\pi\ \phi(\vec{x},t) = \int \frac{Q(\vec{y},\tau)}{r} \left|\nabla f\right|\ \delta(F)\ \delta(g)\ d\vec{y}\ d\tau \tag{A27}$$

The domain of the volume integral is the unbounded three-dimensional space and that of the time integral is $-\infty < \tau \leqq t$. The following result is employed to integrate the above:

$$d\vec{y}\ d\tau = \frac{dy_1\ dy_2\ dF\ dg}{\partial(F,g)/\partial(y_3,\tau)}$$

$$= \frac{dy_1\ dy_2\ dF\ dg}{\partial F/\partial y_3}$$

$$= \frac{dy_1\ dy_2\ dF\ dg}{N_3|\nabla F|}$$

$$= \frac{d\Sigma\ dF\ dg}{|\nabla F|} \tag{A28}$$

Here $d\Sigma$ is the element of the surface of $F(\vec{y};\vec{x},t) = 0$ and $\vec{N}$ is the outward unit normal to this surface. The symbol $\partial(F,g)/\partial(y_3,\tau)$ stands for Jacobian.

The following expression is obtained from the relation

$$F(\vec{y};\ \vec{x},t) = f\left(\vec{y},t-\frac{r}{c}\right) = \left[f(\vec{y},\tau)\right]_{ret} = 0$$

where the subscript ret stands for retarded time:

$$|\nabla F| = \left[|\nabla f|\ \left(1 + M_n^2 - 2M_n \cos\theta\right)^{1/2}\right]_{ret}$$

In this expression, $M_n = v_n/c$, $v_n = -(\partial f/\partial\tau)/|\nabla f|$ is the normal velocity of the body, and θ is the angle between the outward normal to the body and the vector in the radiation direction $\vec{r}$. The symbol Λ will be used in this report as

$$\Lambda = \left(1 + M_n^2 - 2M_n \cos\theta\right)^{1/2} \tag{A29}$$

When equation (A28) is substituted in equation (A27) and the result integrated with respect to F and g, one obtains a new form of the solution for equation (A23). This form is

$$4\pi\ \phi(\vec{x},t) = \int_{F=0} \frac{1}{r}\left[\frac{Q(\vec{y},\tau)}{\Lambda}\right]_{ret} d\Sigma \tag{A30}$$

where the integral is over the Σ-surface ($\vec{x}$ and t are kept fixed). This result was originally published in reference 18. It was derived from equation (A26) using geometric arguments.

From equation (A30), it is seen that the Σ-surface serves as a virtual radiating body whose surface structure depends on the body shape and motion as well as the observer position and time. For very small values of M_n, the Σ-surface may be approximated by the surface of the body itself. From the definition of Λ, one can show that the condition $\Lambda = 0$ occurs only when $M_n = 1$ and $\theta = 0$, simultaneously. This is equivalent to the body having sonic speed at the point of tangency with the sphere $g = 0$. It will be shown later that the singularity $\Lambda = 0$ is integrable in the situations of interest.

Next the solutions of the following equations will be studied:

$$\square^2 \phi_1 = \frac{\partial}{\partial \tau}\left[Q(\vec{y},\tau)\ |\nabla f|\ \delta(f)\right] \tag{A31}$$

$$\square^2 \phi_2 = \frac{\partial}{\partial y_i}\left[Q_i(\vec{y},\tau)\ |\nabla f|\ \delta(f)\right] \tag{A32}$$

Since the operator $\square^2$ commutes with operators $\partial/\partial\tau$ and $\partial/\partial y_i$, from equations (A26) and (A30), it follows that

$$4\pi\ \phi_1(\vec{x},t) = \frac{\partial}{\partial t}\int_{\tau_1}^{\tau_2}\int_\Gamma \frac{cQ(\vec{y},\tau)}{r \sin\theta}\, d\Gamma\, d\tau \tag{A33}$$

$$4\pi\ \phi_1(\vec{x},t) = \frac{\partial}{\partial t}\int_{F=0} \frac{1}{r}\left[\frac{Q(\vec{y},\tau)}{\Lambda}\right]_{ret} d\Sigma \tag{A34}$$

$$4\pi\ \phi_2(\vec{x},t) = \frac{\partial}{\partial x_i}\int_{\tau_1}^{\tau_2}\int_\Gamma \frac{cQ_i(\vec{y},\tau)}{r \sin\theta}\, d\Gamma\, d\tau \tag{A35}$$

$$4\pi\ \phi_2(\vec{x},t) = \frac{\partial}{\partial x_i}\int_{F=0} \frac{1}{r}\left[\frac{Q_i(\vec{y},\tau)}{\Lambda}\right]_{ret} d\Sigma \tag{A36}$$

Equations (A33) and (A35) may be written in other forms by taking the derivatives inside the integrals. The method is presented in reference 17. However, the resulting equations may not be suitable for numerical integration in the case of complex geometries. The derivatives in equations (A34) and (A36) may also be taken inside the integrals. The technique will only be worked out in detail for equation (A34) as follows:

$$\frac{\partial}{\partial t}\int_{F=0}\frac{1}{r}\left[\frac{Q(\vec{y},\tau)}{\Lambda}\right]_{ret} d\Sigma = \frac{\partial}{\partial t}\int_{F=0}\frac{\tilde{Q}(\vec{y};\vec{x},t)}{r}\frac{d\Sigma}{|\nabla F|}$$

$$= \frac{\partial}{\partial t}\int_{-\infty}^{+\infty}\frac{\tilde{Q}}{r}\,\delta(F)\,d\vec{y}$$

$$= \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}\left(\frac{\tilde{Q}}{r}\right)\delta(F)\,d\vec{y} + \int_{-\infty}^{+\infty}\frac{\tilde{Q}}{r}\frac{\partial F}{\partial t}\,\delta'(F)\,d\vec{y} \qquad \text{(A37)}$$

where $\tilde{Q}(\vec{y};\vec{x},t) = \left[Q(\vec{y},\tau)\,|\nabla f|\right]_{ret}$. In the preceding equation the following identity, in which $\tilde{T}(\vec{y})$ is an arbitrary function, is used:

$$\int_{F=0}\tilde{T}(\vec{y})\,d\Sigma = \int_{-\infty}^{+\infty}|\nabla F|\,\tilde{T}(\vec{y})\,\delta(F)\,d\vec{y} \qquad \text{(A38)}$$

To get this useful identity, note that

$$d\vec{y} = \frac{dF\,dy_1\,dy_2}{\partial F/\partial y_3}$$

$$= \frac{dF\,d\Sigma}{|\nabla F|} \qquad \text{(A39)}$$

Substitute this relation on the right side of equation (A38) and then integrate with respect to variable F.

Applying equation (A38), one integral in equation (A37) may be written as

$$\int_{-\infty}^{+\infty}\frac{\partial}{\partial t}\left(\frac{\tilde{Q}}{r}\right)\delta(F)\,d\vec{y} = \int_{F=0}\frac{1}{r}\frac{\partial\tilde{Q}}{\partial t}\frac{d\Sigma}{|\nabla F|} \qquad \text{(A40)}$$

The following identity, closely related to one used by Ffowcs Williams and Hawkings (ref. 9, eq. 6.2), is required for the second integral on the right side of equation (A37):

$$\int_{-\infty}^{+\infty} T(\vec{y})\,\delta'(F)\,d\vec{y} = -\int_{F=0} \frac{\partial}{\partial y_i}\left[\frac{\partial F}{\partial y_i}\,\frac{T(\vec{y})}{|\nabla F|^2}\right]\frac{d\Sigma}{|\nabla F|} \tag{A41}$$

The summation convention on i is used in equation (A41). Derivation of this identity is not trivial. It is obtained by writing $d\Sigma = \sum_{i=1}^{3} N_i\, d\vec{y}_i$ where $\vec{N}$ is the unit normal vector to the Σ-surface. The symbol $d\vec{y}_i$ is defined as the element of the area of the coordinate plane whose normal is parallel to y_i-axis, that is, $d\vec{y}_1 = dy_2\,dy_3$, etc. Substituting this relation in equation (A39) and then using the result on the left side of equation (A41), one obtains

$$\int_{-\infty}^{-\infty} T(\vec{y})\,\delta'(F)\,d\vec{y} = \int_{-\infty}^{+\infty} \sum_{i=1}^{3} \frac{T(\vec{y})\,N_i}{|\nabla F|}\,\delta'(F)\,dF\,d\vec{y}_i$$

$$= -\int_{-\infty}^{+\infty} \sum_{i=1}^{3} \left.\frac{\partial}{\partial F}\right|_{\vec{y}_i}\left[\frac{T(\vec{y})\,N_i}{|\nabla F|}\right]\delta(F)\,dF\,d\vec{y}_i \tag{A42}$$

where $\left.\frac{\partial}{\partial F}\right|_{\vec{y}_i}$ indicates that while taking the derivative $\partial/\partial F$, the coordinates in the plane in which $d\vec{y}_i$ is defined must be kept fixed. For example, $\left.\frac{\partial}{\partial F}\right|_{\vec{y}_1}$ means that the coordinates y_2 and y_3 are kept fixed. It may be shown that

$$\left.\frac{\partial}{\partial F}\right|_{\vec{y}_i} = \frac{1}{\partial F/\partial y_i}\,\frac{\partial}{\partial y_i} = \frac{1}{|\nabla F|\,N_i}\,\frac{\partial}{\partial y_i} \qquad (i = 1,\ 2,\ 3;\ \text{no sum on } i)$$

and also $d\vec{y}_i/N_i = d\Sigma$ (i = 1, 2, 3; no sum on i). When these two results are used in equation (A42) and the latter integrated with respect to F, the result is equation (A41).

The last integral of equation (A37), by the use of the identity in equation (A41), yields

$$\int_{-\infty}^{+\infty} \frac{\tilde{Q}}{r}\,\frac{\partial F}{\partial t}\,\delta'(F)\,d\vec{y}_i = -\int_{F=0} \frac{\partial}{\partial y_i}\left(\frac{\partial F}{\partial y_i}\,\frac{\partial F}{\partial t}\,\frac{\tilde{Q}}{r|\nabla F|^2}\right)\frac{d\Sigma}{|\nabla F|} \tag{A43}$$

Equation (A34) can thus be written as

$$4\pi\, \phi_1(\vec{x},t) = \frac{\partial}{\partial t}\int_{F=0} \frac{1}{r}\left(\frac{Q}{\Lambda}\right)_{ret} d\Sigma$$

$$= \int_{F=0} \left[\frac{1}{r}\frac{\partial \tilde{Q}}{\partial t} - \frac{\partial}{\partial y_i}\left(\frac{\partial F}{\partial t}\frac{\tilde{Q}}{r|\nabla F|} N_i\right)\right]\frac{d\Sigma}{|\nabla F|} \tag{A44}$$

Here the relation $N_i = (\partial F/\partial y_i)/|\nabla F|$ is used.

When precisely the same technique is applied to equation (A36), one obtains

$$4\pi\, \phi_2(\vec{x},t) = \int_{F=0} \left[\frac{\partial}{\partial x_i}\left(\frac{\tilde{Q}_i}{r}\right) - \frac{\partial}{\partial y_i}\left(\frac{\partial F}{\partial x_j}\frac{\tilde{Q}_j}{r|\nabla F|} N_i\right)\right]\frac{d\Sigma}{|\nabla F|} \tag{A45}$$

where $\tilde{Q}_j = \left(|\nabla f|Q_j\right)_{ret}$.

The preceding equation will be written in yet another form which is very useful in applications. First, note that

$$\frac{\partial}{\partial x_i}\left(\frac{\tilde{Q}_i}{r}\right) = -\frac{\tilde{Q}_r}{r^2} - \frac{1}{cr}\frac{\partial \tilde{Q}_r}{\partial t}$$

and

$$\frac{\partial F}{\partial x_i} = -\frac{\hat{r}_i}{c}\frac{\partial F}{\partial t}$$

where $\tilde{Q}_r = \tilde{Q}_i\hat{r}_i$ When these are substituted in equation (A45), the following is obtained:

$$4\pi\, \phi_2(\vec{x},t) = -\int_{F=0} \frac{\tilde{Q}_r}{r^2}\frac{d\Sigma}{|\nabla F|} - \frac{1}{c}\int_{F=0} \left[\frac{1}{r}\frac{\partial \tilde{Q}_r}{\partial t} - \frac{\partial}{\partial y_i}\left(\frac{\partial F}{\partial t}\frac{\tilde{Q}_r}{r|\nabla F|} N_i\right)\right]\frac{d\Sigma}{|\nabla F|}$$

Now compare the second integral on the right side of this equation with the integral in equation (A44). Since Q is arbitrary in equation (A44), one may write

$$\int_{F=0} \left[\frac{1}{r}\frac{\partial \tilde{Q}_r}{\partial t} - \frac{\partial}{\partial y_i}\left(\frac{\partial F}{\partial t}\frac{\tilde{Q}_r}{r|\nabla F|} N_i\right)\right]\frac{d\Sigma}{|\nabla F|} = \frac{\partial}{\partial t}\int_{F=0} \frac{\tilde{Q}_r}{r}\frac{d\Sigma}{|\nabla F|}$$

Note that $|\nabla F| = \left(|\nabla f|\Lambda\right)_{\mathrm{ret}}$. When the above relation is substituted in the equation preceding it, the new form of solution for equation (A36) becomes

$$4\pi\,\phi_2(\vec{x},t) = -\int_{F=0} \frac{\tilde{Q}_r}{r^2}\frac{d\Sigma}{|\nabla F|} - \frac{1}{c}\frac{\partial}{\partial t}\int_{F=0}\frac{\tilde{Q}_r}{r}\frac{d\Sigma}{|\nabla F|}$$

$$= -\int_{F=0}\frac{1}{r^2}\left(\frac{Q_r}{\Lambda}\right)_{\mathrm{ret}} d\Sigma - \frac{1}{c}\frac{\partial}{\partial t}\int_{F=0}\frac{1}{r}\left(\frac{Q_r}{\Lambda}\right) d\Sigma \tag{A46}$$

where $Q_r = Q_i\hat{r}_i$.

The importance of the last result lies in the fact that now only one derivative, that is, the time derivative, appears in the solution which may be performed numerically for a fixed observer position. This is considerably simpler than using equation (A36) in which the space derivatives must be calculated numerically at a fixed observer time.

Equation (A46) may also be written as

$$4\pi\,\phi_2(\vec{x},t) = -\left(\int_{\tau_1}^{\tau_2}\int_{\Gamma}\frac{cQ_r}{r^2\sin\theta}\,d\Gamma\,d\tau + \frac{\partial}{\partial t}\int_{\tau_1}^{\tau_2}\int_{\Gamma}\frac{Q_r}{r\sin\theta}\,d\Gamma\,d\tau\right) \tag{A47}$$

To get this from equation (A46), compare equation (A26) with equation (A30). It can also be obtained by using equation (A68).

It is seen that one may write the solution of the wave equations considered in this section of the appendix in many different forms. The solution of the FW-H equation satisfying the criteria proposed in this report is based on equations (A33) and (A47).

Discussion of Singularities of the Solution of Wave Equation

One disturbing feature of the different forms of the solution of the wave equation is the appearance of singularities such as $\Lambda = 0$ in equation (A34). These singularities limit the applicability of the solution. It may appear that such singularities are genuine. In this section, it will be shown that, provided that the right variables are chosen in the solution of equation (A23), the singularities are integrable. In the case of the following expressions or other equivalent forms:

$$I_1 = \frac{\partial}{\partial t}\int_{F=0}\frac{1}{r}\left(\frac{Q}{\Lambda}\right)_{\mathrm{ret}} d\Sigma \tag{A48a}$$

$$I_2 = \frac{\partial}{\partial x_i} \int_{F=0} \frac{1}{r}\left(\frac{Q_i}{\Lambda}\right)_{ret} d\Sigma \tag{A48b}$$

it is not permissible to take the derivatives inside the integral without proper interpretation. The order of the singularity in the integrand increases when the derivatives are taken inside. One should then use Hadamard's method of selecting the finite part of a divergent integral or its modern equivalent form in generalized function theory. Due to multidimensionality of the integrals, the use of such a technique is an unnecessary complication which could be avoided by applying numerical differentiation with respect to the observer time. The ensuing discussion will, therefore, be confined to the study of the singularities of the following integral:

$$I = \int_{F=0} \frac{1}{r}\left(\frac{Q}{\Lambda}\right)_{ret} d\Sigma \tag{A49}$$

In this section, it will be shown that the singularities of equation (A49) are integrable.

Since in the following discussion $\vec{x}$ and t are kept fixed, the dependence on these variables is suppressed in the notation and the integrand will be written as $\tilde{T}(\vec{y}) = \tilde{Q}(\vec{y}; \vec{x},t)/r$. Assume that $\tilde{T}(\vec{y})$ is bounded on the Σ-surface. Thus the following equivalent form of equation (A49) will be considered for the study of the singularities:

$$I = \int_{F=0} \tilde{T}(\vec{y}) \frac{d\Sigma}{|\nabla F|} \tag{A50}$$

since $\nabla F = |\nabla f| \Lambda_{ret}$ and $\tilde{Q} = |\nabla f| Q_{ret}$. The set of points on the Σ-surface where $|\nabla F| = 0$ is called the critical points of the surface. At these points, provided that $f = 0$ has no critical points, $M_n = 1$ and $\theta = 0$. This means that the surfaces of $f = 0$ and $g = 0$ are tangent, and normal body velocity is equal to the speed of sound c. In the following discussion, it is assumed that not all the second partial derivatives with respect to variables y_i are zero at critical points of $F(\vec{y}; \vec{x},t) = 0$. The following three cases are distinguished.

Case (i). - If the sphere $g = 0$ is tangent to the body for an extended period of time, when the body is on one side of the sphere, the Σ-surface corresponding to this period degenerates into a curve in space. Since this curve has no surface content, the integral of equation (A50) is zero. This curve itself can become an isolated point in space if $g = 0$ and $f = 0$ become tangent and immediately separate. Again one obtains $I = 0$ in equation (A50).

Case (ii).- There may be isolated points on the Σ-surface where $|\nabla F| = 0$. This situation can occur, for example, at points where the sphere $g = 0$ enters or leaves the body at the speed of sound. Such points may also appear when $g = 0$ is in the process of crossing the body. To visualize this latter condition, consider a finite wing with a rectangular planform and a symmetric section moving in a straight path with variable speed. (See fig. A2(a).) Let the observer be in the plane of symmetry of the wing as shown in figure A2(b). If the body accelerates to sonic speed, and subsequently decelerates without spending some time at sonic speed, the Σ-surface corresponding to $t = \tau_o + r_o/c$ has a singularity at point B. This surface is shown in figure A3.

This type of singularity appears most often in calculations for rotating bodies. In such a case, the tangent plane to the Σ-surface at the point B is not defined. In fact, in the vicinity of B, the Σ-surface looks like a cone since it can be represented by the function $\tilde{F}(\vec{y}) = F_{ij}y_iy_j = 0$ in a coordinate system with origin at the point B where $F_{ij} = \partial^2 F / \partial y_i \partial y_j \big|_B$. For a small patch of area near B on Σ-surface, designated as D, one has

$$I_D = \int_D T(\vec{y}) \frac{d\Sigma}{|\nabla F|} = T(\vec{\xi}) \int_D \frac{d\Sigma}{|\nabla F|}$$

for some $\vec{\xi}$ in region D. Thus, consider the following integral near point B:

$$I' = \int_D \frac{d\Sigma}{|\nabla F|} \tag{A51}$$

Let the y_1y_2-plane coincide with the tangent plane to the sphere $g = 0$ when it is at the point B. In this plane, set a polar coordinate (R,φ) such that

$$y_1 = R \cos \varphi \tag{A52a}$$

$$y_2 = R \sin \varphi \tag{A52b}$$

Then equation (A51) becomes

$$I' = \int_D \frac{R dR d\varphi}{|\partial\tilde{F}/\partial y_3|} \tag{A53}$$

and from $\tilde{F}(\vec{y}) = F_{ij}y_i y_j = 0$, one obtains

$$\frac{\partial \tilde{F}}{\partial y_3} = F_{13}y_1 + F_{23}y_2 + F_{33}y_3 \tag{A54}$$

Solving for y_3 from the equation of the surface near B, and substituting in equation (A54), and finally using the relations of equations (A52), the following result is obtained:

$$I' = \int_0^{R_o} \int_{D_\varphi} \frac{d\varphi \, dR}{\sqrt{2A \cos^2 \varphi + B \sin 2\varphi + 2C \sin^2 \varphi}}$$

$$= R_o \int_{D_\varphi} \frac{d\varphi}{\sqrt{(A + C) + (A - C) \cos 2\varphi + B \sin 2\varphi}} \tag{A55}$$

where

$$A = \left(F_{13}^2 - F_{11}F_{33}\right)/2$$

$$B = \left(F_{13}F_{23} - F_{12}F_{33}\right)/2$$

$$C = \left(F_{23}^2 - F_{22}F_{33}\right)/2$$

The domain of φ, denoted by D_φ, is the set of values of φ which makes the denominator in the integrand of equation (A55) real. This domain is not necessarily 2π. Even when the denominator in equation (A55) is zero, there is not a genuine singularity since it behaves as $\sqrt{|\varphi - \varphi_0|}$ where φ_0 is the value of φ which makes the denominator zero. If the region D is taken in such a way that its projection in the y_1y_2-plane lies in a circle of radius R_o, one obtains $I' \propto R_o$. Therefore, by taking R_o small enough, one can neglect a patch of area near the type of singularity discussed here without introducing large errors in calculations.

Case (iii). - If in the processes of crossing the body, the sphere $g = 0$ becomes tangent to the body at a point whose normal velocity is sonic and remains sonic for extended periods of time, the Σ-surface intersects itself along a curve. This situation is uncommon for rotating bodies. Let the speed vary as shown in figure A4, and using the wing of figure A2, let the sphere be tangent to the wing at the point B when $\tau = \tau_o$. The Σ-

surface is shown in figure A5. The structure of this surface near point B is presented in figure A6.

Again, a new $\vec{y}$-frame with origin at point B will be introduced with the y_1y_2-plane coinciding with the tangent plane to the sphere $g = 0$ when it is at point B. This plane is tangent to the Γ-curves which cross each other passing through point B. The slopes $\tilde{m}_1$ and $\tilde{m}_2$ of the tangents to Γ-curves in y_1y_2-plane are given by the solution of the equation

$$F_{22}\tilde{m}^2 + 2F_{12}\tilde{m} + F_{11} = 0 \tag{A56}$$

where

$$F_{ij} = \left.\frac{\partial^2 F}{\partial y_i \partial y_j}\right|_B$$

The analysis of case (ii) can be applied on that part of the Σ-surface where $y_3 < 0$. (See fig. A6.) This shows that, for this region of the surface, I' exists and can be made arbitrarily small if the region D is taken as small.

For the intersecting part of the Σ-surface in region $y_3 > 0$, one can set up a new $\vec{y}$-frame at any point P on curve BC with the y_1y_2-plane coinciding with the tangent plane to the sphere $g = 0$ at P. The normals to the two intersecting parts of the Σ-surface are defined and could be obtained by using equation (A56) with F_{ij} evaluated at point P. One obtains the following relations on each intersecting part of Σ-surface, say, in A'BC near point P

$$|\nabla F| = \vec{y} \cdot \nabla |\nabla F|\big|_P + O\left(y_i^2\right) \tag{A57a}$$

$$\nabla |\nabla F|\big|_P = N_i \, \nabla F_i\big|_P \tag{A57b}$$

where $F_i = \partial F/\partial y_i$ and N_i is the unit normal to A'BC. Using equations (A57), one can write

$$I' = \lim_{\epsilon \to 0} \int_0^b \int_\epsilon^\delta \frac{d\Gamma \, dy_3}{\alpha\Gamma + \beta y_3} \tag{A58}$$

where α and β are related to the second partial derivatives of $F(\vec{y}; \vec{x},t)$ at point P and $\delta > \epsilon$ is a small fixed parameter. The above integral exists and can be made as small as one wishes by taking the value of the integration limit, δ small. This means that one may neglect a strip of surface along the intersecting part of the Σ-surface which includes the curve of intersection BC without producing large errors in calculation.

Relation of Σ-Surface Solution to Other Forms of the Solution of Wave Equation

The Γ-curves and the system of curves on the Σ-surface orthogonal to them, denoted by m-curves, form a grid on the Σ-surface. Except at points on this surface where $\theta = 0$, one has the relation

$$d\Sigma = d\Gamma \, dm \tag{A59}$$

At points where $\theta = 0$, the Γ-curves or m-curves intersect themselves as shown in figure A7. Figure A7(a) illustrates the case when $g = 0$ enters or leaves the body. Figure A7(b) is for the condition $\theta = 0$ when the sphere $g = 0$ is in the process of crossing the body. The third case (fig. A7(c)) corresponds to figure A6, when the Σ-surface intersects itself. In any case, one may neglect a small patch of area near the points for which $\theta = 0$, even when $M_n = 1$.

In figure A8 a region of the Σ-surface, where $\theta \neq 0$ and $M_n \neq 1$, is shown. It is seen that

$$dm = \frac{c\,d\tau}{\sin \tilde{\theta}} \tag{A60}$$

where $\tilde{\theta}$ is the angle between N_i and the radiation direction $\hat{r}_i$. But

$$\sin \tilde{\theta} = \sqrt{1 - (\vec{N} \cdot \hat{\vec{r}})^2}$$

and

$$\vec{N} = \frac{\nabla F}{|\nabla F|}$$

$$= \frac{\vec{n} - M_n \hat{\vec{r}}}{\Lambda} \tag{A61}$$

thus,

$$\sin\tilde{\theta} = \frac{\sin\theta}{\Lambda} \tag{A62}$$

Therefore, from equations (A59) to (A62), one gets

$$dm = \frac{c\Lambda\ d\tau}{\sin\theta} \tag{A63}$$

and

$$d\Sigma = \frac{c\Lambda\ d\Gamma\ d\tau}{\sin\theta} \tag{A64}$$

To get a relation between $d\Sigma$ and the element of area of the body surface dS, consider a portion of the Σ-surface corresponding to the observer time t. (See fig. A8.) It is assumed that the body is rigid. Point A which lies on the Γ-curve at source time τ, denoted as $\Gamma(\tau)$, would move to point B on the body at source time $\tau + d\tau$. Note that $\vec{v}$ is the local velocity of the body at the point A when $\Gamma(\tau)$ was formed. Let C be the point of intersection of the m-curve from A and the curve $\Gamma(\tau + d\tau)$. Let $\vec{\Gamma}$ and $\vec{m}$ be the unit tangents to Γ-curve and m-curve at C, respectively. Then dS is given by

$$dS = \left|\overrightarrow{CB} \times d\vec{\Gamma}\right| \tag{A65}$$

where $d\vec{\Gamma} = \vec{\Gamma}\ d\Gamma$. It can be seen that

$$\overrightarrow{CB} = -d\vec{m} + \vec{v}\ d\tau$$

Here $d\vec{m} = \vec{m}\ dm$. Substituting for $\overrightarrow{CB}$ into equation (A65), one gets

$$dS = \left|-\vec{m} \times \vec{\Gamma}(dm\ d\Gamma) + \vec{v} \times \vec{\Gamma}(d\Gamma\ d\tau)\right| \tag{A66}$$

From figure A9 and the definition of $d\Sigma$, the following may be shown

$$\begin{aligned} -\vec{m} \times \vec{\Gamma}(dm\ d\Gamma) &= \vec{N}\ d\Sigma \\ &= \frac{\vec{n} - M_n\vec{\hat{r}}}{\Lambda}\ d\Sigma \end{aligned} \tag{A67}$$

In the last step of equation (A67), $\vec{N}$ is substituted from equation (A61). Also since $\vec{\Gamma} = \vec{\hat{r}} \times \vec{n}/\sin\theta$, it may be shown that

$$\vec{V} \times \vec{\Gamma}(d\Gamma\, d\tau) = \vec{v} \times (\vec{\hat{r}} \times \vec{n})\frac{d\Gamma\, d\tau}{\sin\theta}$$

$$= (v_n\vec{\hat{r}} - v_r\vec{n})\frac{d\Sigma}{C\Lambda}$$

$$= (M_n\vec{\hat{r}} - M_r\vec{n})\frac{d\Sigma}{\Lambda} \tag{A68}$$

where $v_n = \vec{v} \cdot \vec{n}$, $v_r = \vec{v} \cdot \vec{\hat{r}}$, and $M_r = v_r/c$. Here equation (A64) is used to give

$$\frac{d\Gamma\, d\tau}{\sin\theta} = \frac{d\Sigma}{c\Lambda}$$

From equations (A65), (A67), and (A68), the relation between $d\Sigma$ and dS is obtained which is

$$dS = \left|1 - M_r\right| \frac{d\Sigma}{\Lambda} \tag{A69}$$

For use in different forms of the solution of the wave equation, equations (A64) and (A69) give

$$\frac{d\Sigma}{\Lambda} = \frac{c\, d\Gamma\, d\tau}{\sin\theta} = \frac{dS}{\left|1 - M_r\right|} \tag{A70}$$

It is seen that when $M_r = 1$, one gets a singularity in the solution in the form of an integral over the body surface S. This condition occurs when $|\vec{v}| \geqq c$. When this happens, the Γ-curves corresponding to the same observer time t intersect one another or become tangent. These curves may cover a portion of the surface S more than once. For example, if the body is moving at steady rectilinear supersonic speed, then only that portion of the body which is within the forward Mach cone from the observer will be covered twice by the Γ-curves. Note that only for the solution involving an integral over the body surface, it is assumed that the body be rigid. The other forms of the solution apply for general surfaces which deform while in motion.

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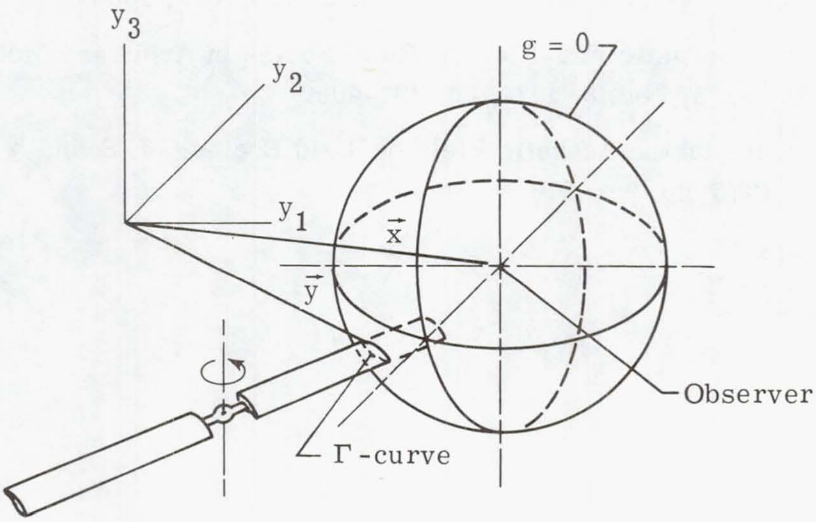


Figure 1.- Formation of a Γ-curve by the intersection of the collapsing sphere $g = 0$ and the body (a helicopter rotor).

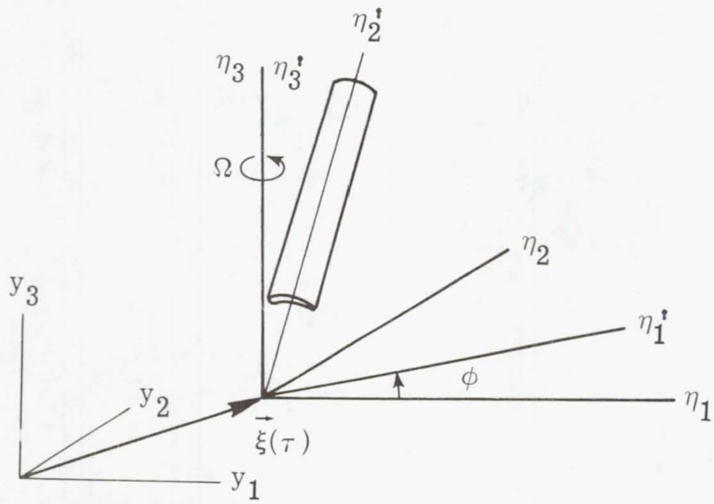
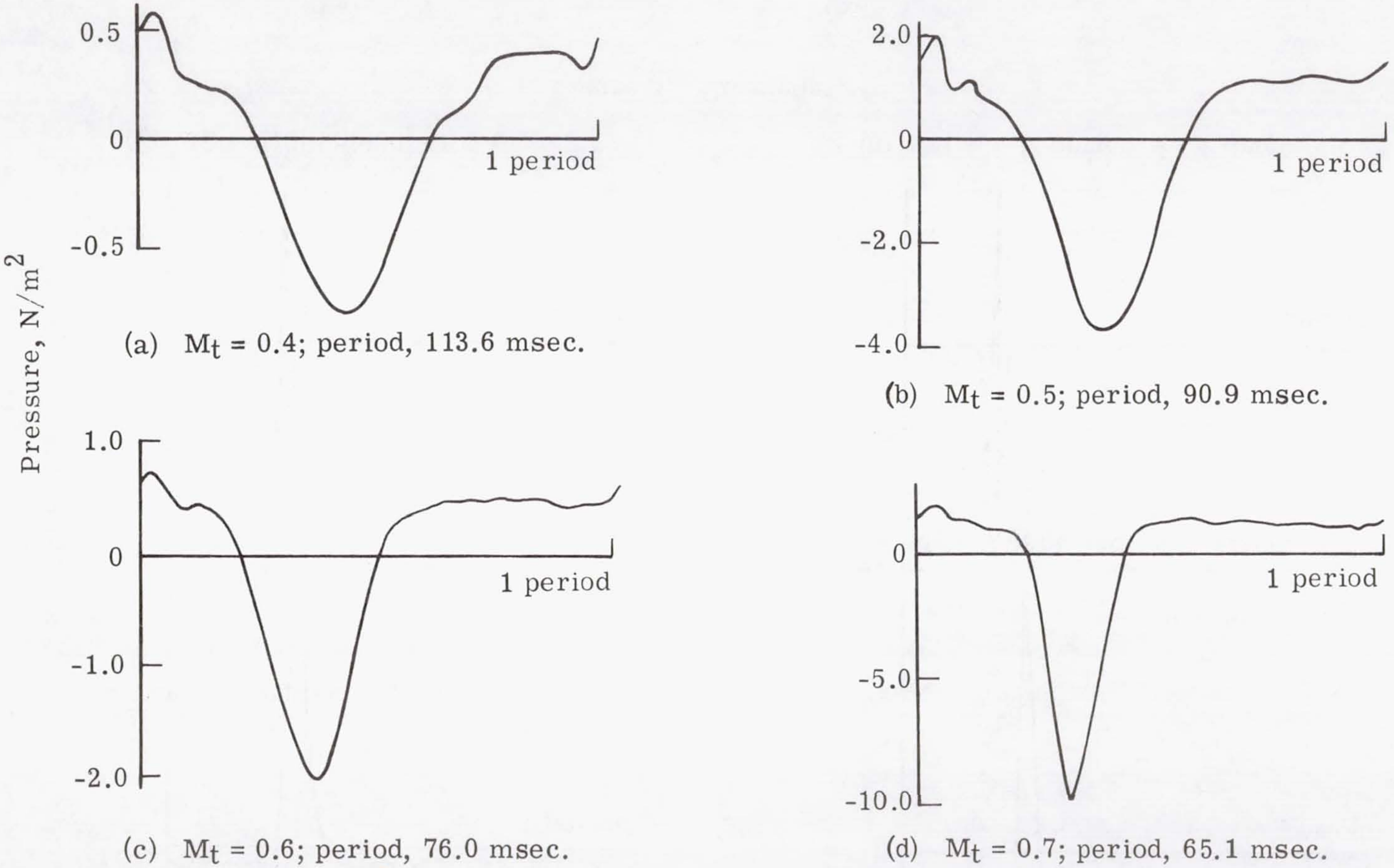


Figure 2.- Coordinate frames used in determination of the acoustic field of rotors and propellers.



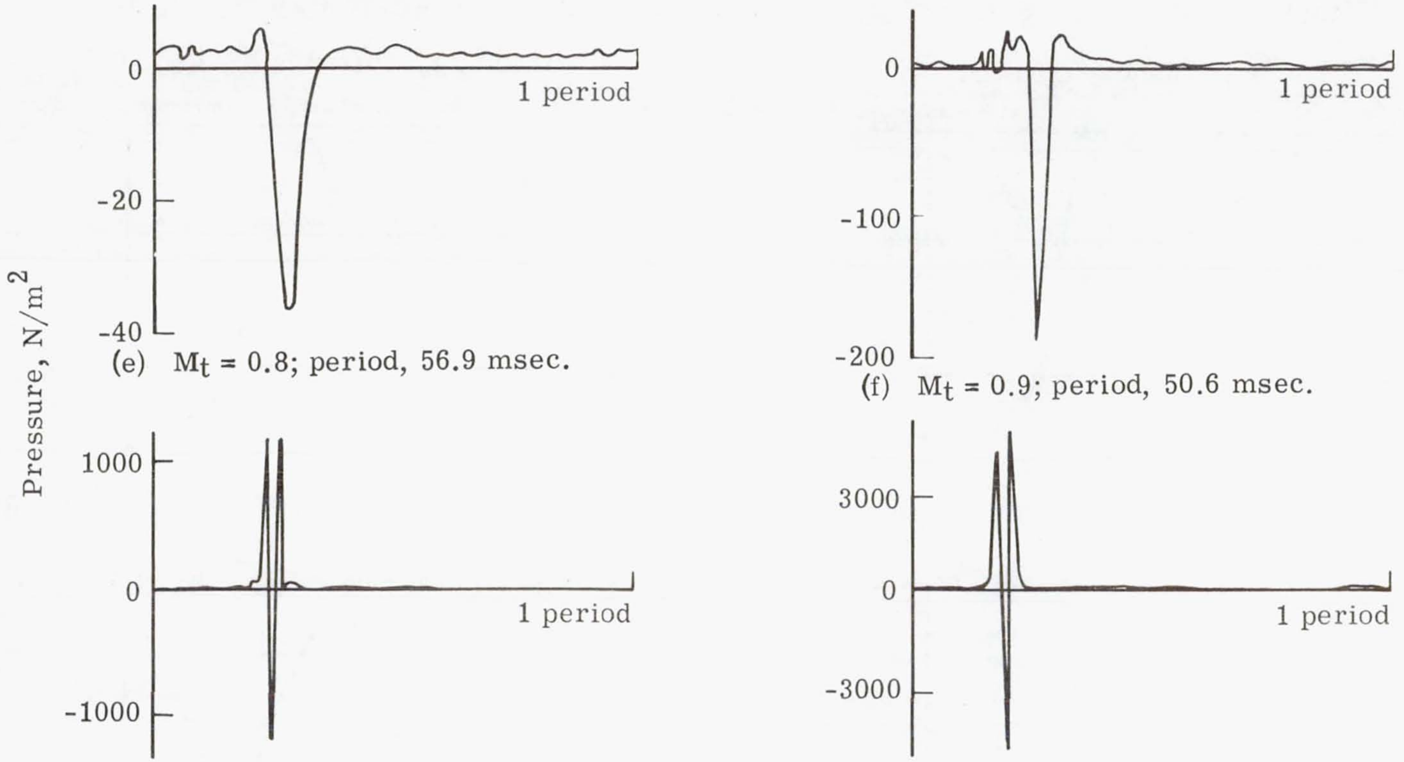
(a) M_t = 0.4; period, 113.6 msec.

(b) M_t = 0.5; period, 90.9 msec.

(c) M_t = 0.6; period, 76.0 msec.

(d) M_t = 0.7; period, 65.1 msec.

Figure 3. - Theoretical variation of the acoustic pressure signature of a hovering helicopter. Thickness noise only. (Note change of scale.)



(e) M_t = 0.8; period, 56.9 msec.

(f) M_t = 0.9; period, 50.6 msec.

(g) M_t = 1.0; period, 45.5 msec.

(h) M_t = 1.1; period, 41.4 msec.

Figure 3.- Concluded.

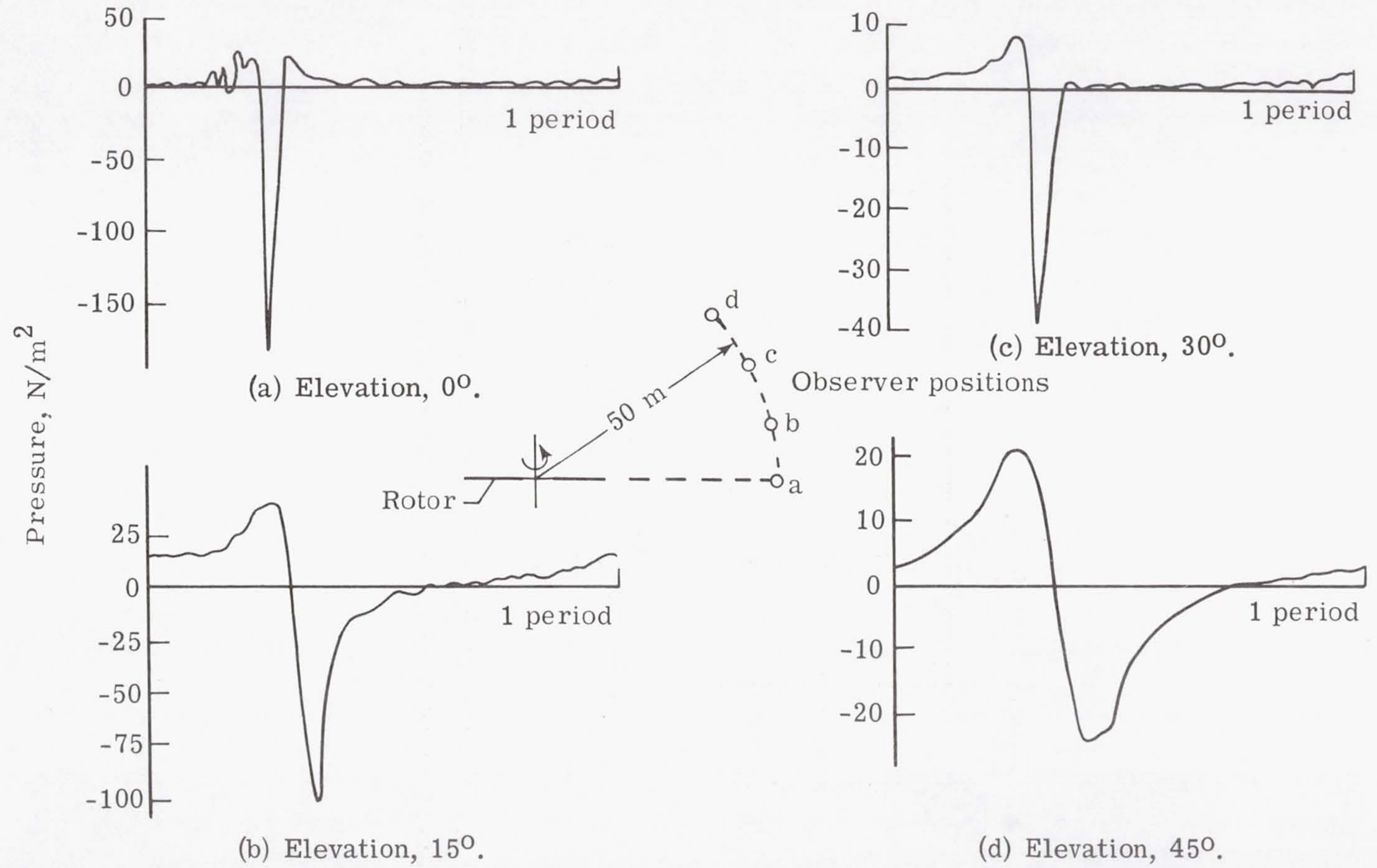


Figure 4.- Theoretical variation of the acoustic pressure signature with the observer elevation for a hovering helicopter. Period, 50.6 msec. Thickness noise only. (Note the change of scale.)

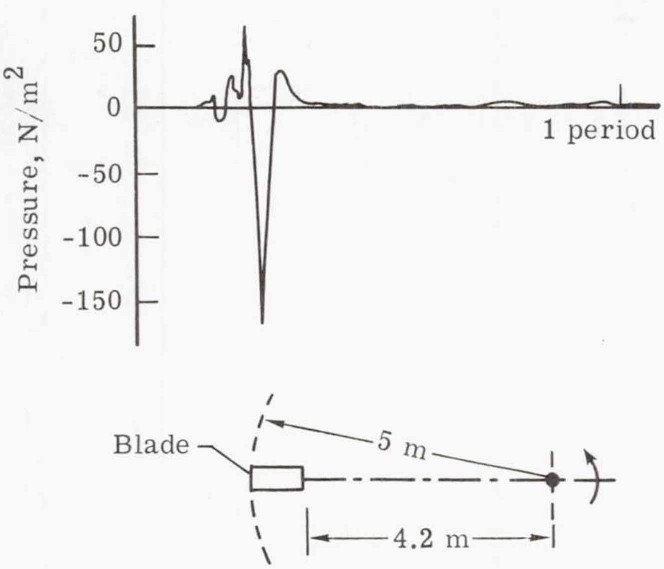
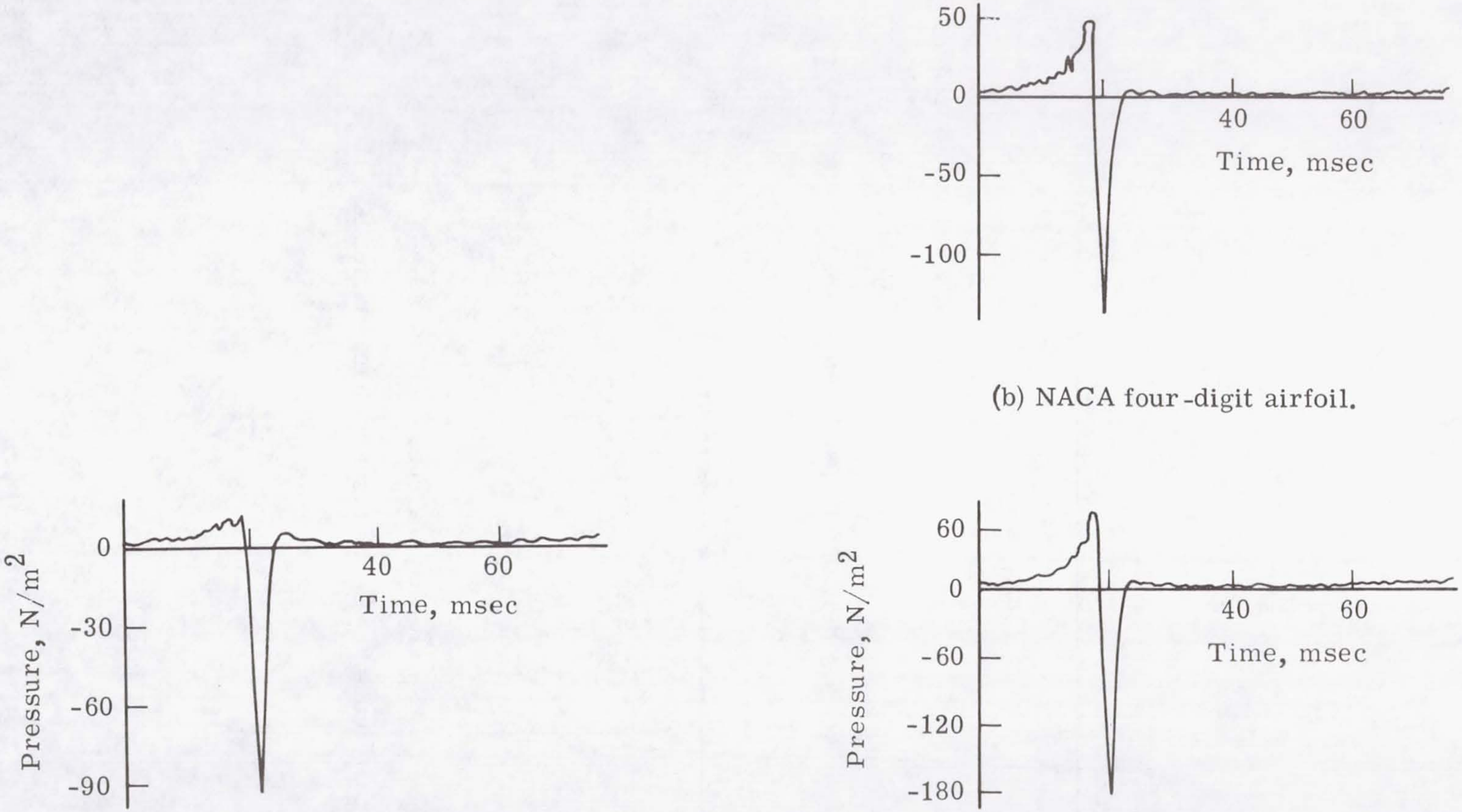


Figure 5.- Influence of the region near the blade tip on the generation of thickness noise of a hovering helicopter. Period, 50.6 msec.

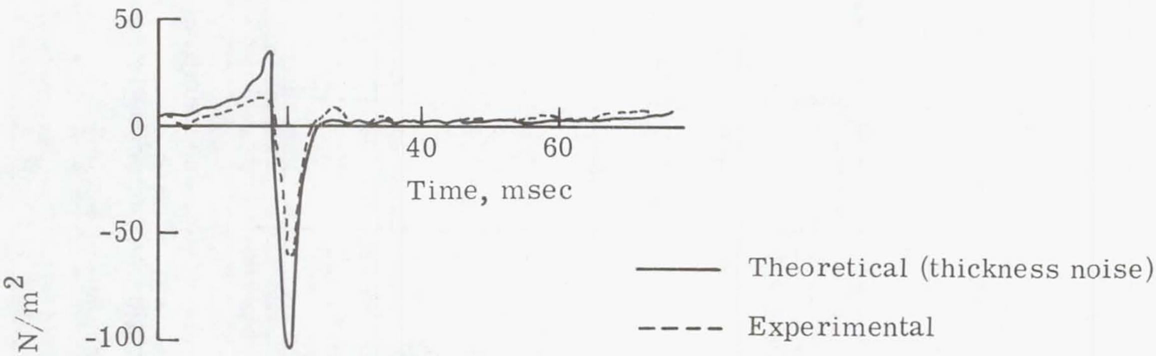


(b) NACA four-digit airfoil.

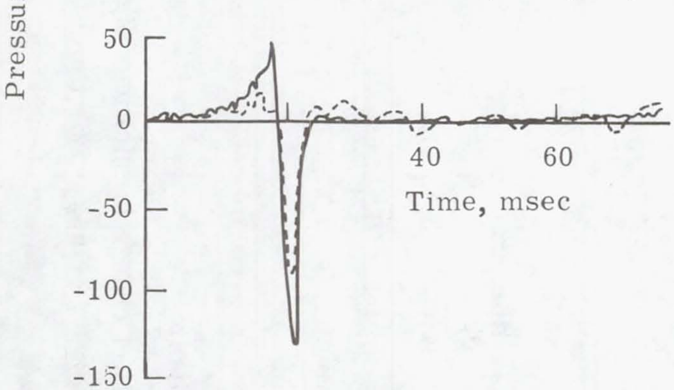
(a) Biconvex airfoil (parabolic arc).

(c) Supercritical airfoil.

Figure 6.- Theoretical effect of change in thickness distribution on the acoustic pressure signature. Thickness noise only. (Note the change of scale.)



(a) Helicopter speed, 140 knots.



(b) Helicopter speed, 170 knots.

Figure 7. - Comparison with experiments.

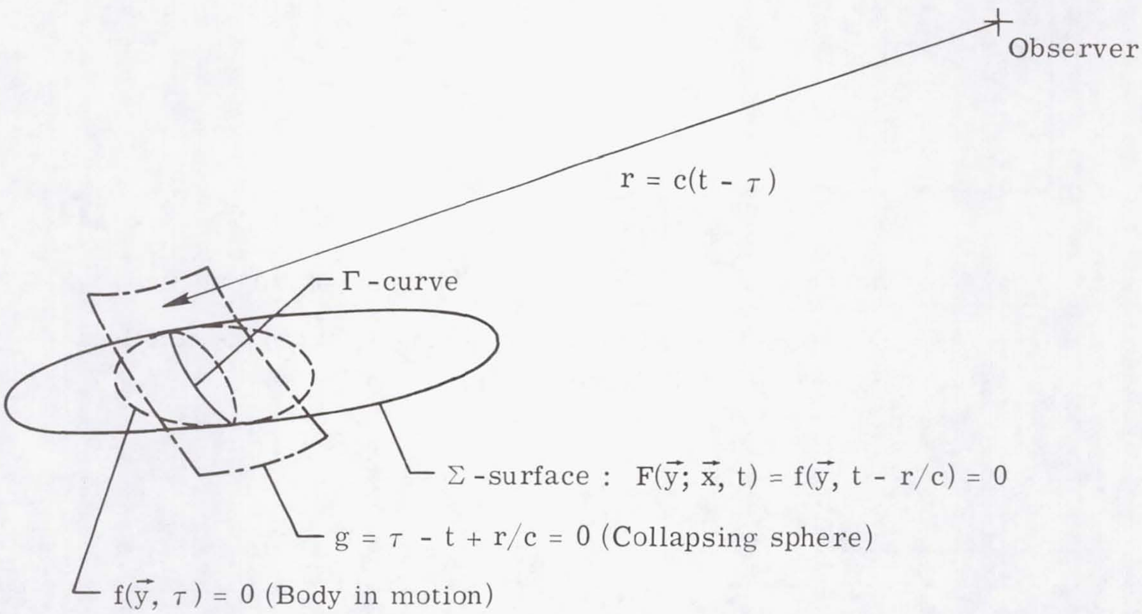
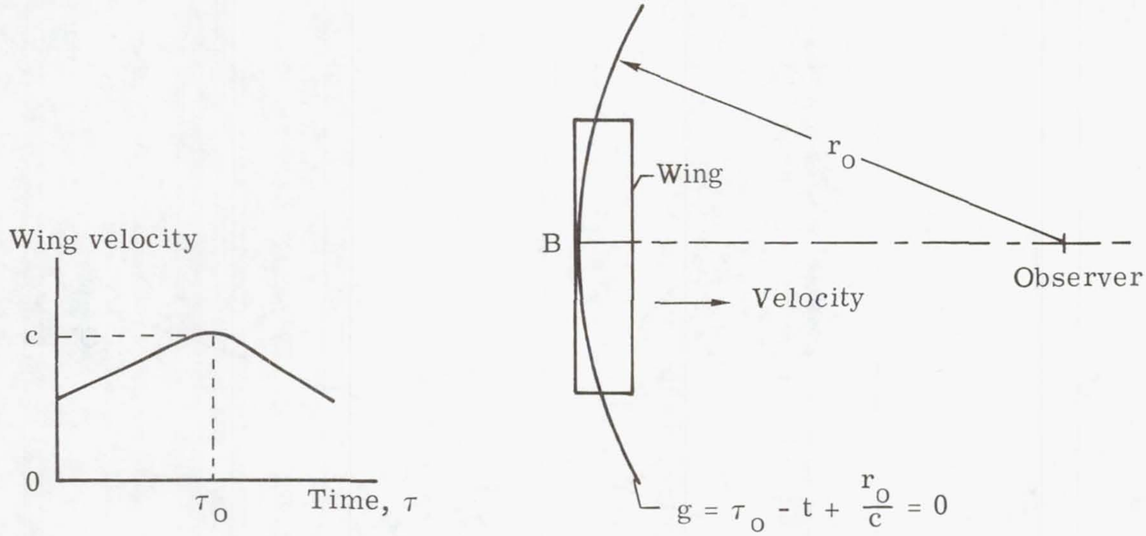


Figure A1.- The collapsing sphere $g = 0$ in the process of intersecting the moving body $f(\vec{y}, \tau) = 0$ and forming the Γ-curve. The Σ-surface is the surface generated by all Γ-curves as the source time τ varies from $-\infty$ to t for fixed observer position $\vec{x}$ and time t.



(a) Wing velocity. (b) Wing position at time τ_o.

Figure A2.- Example of a condition causing singularity in the solution of the wave equation.

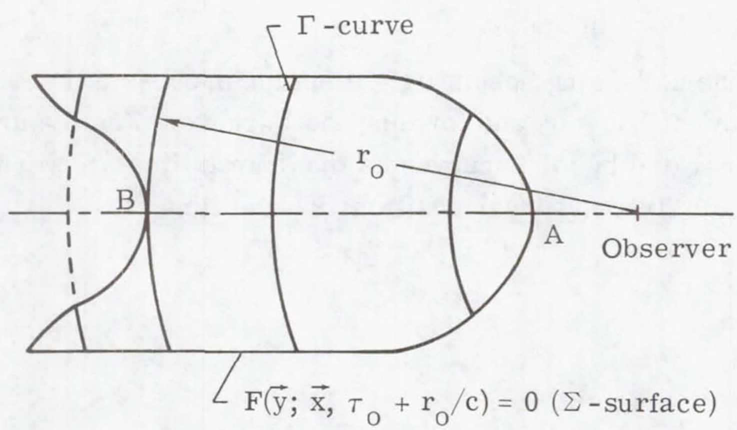


Figure A3.- The Σ-surface corresponding to the observer time $\tau_o + r_o/c$ of the wing of figure A2 some of the Γ-curves are also shown.

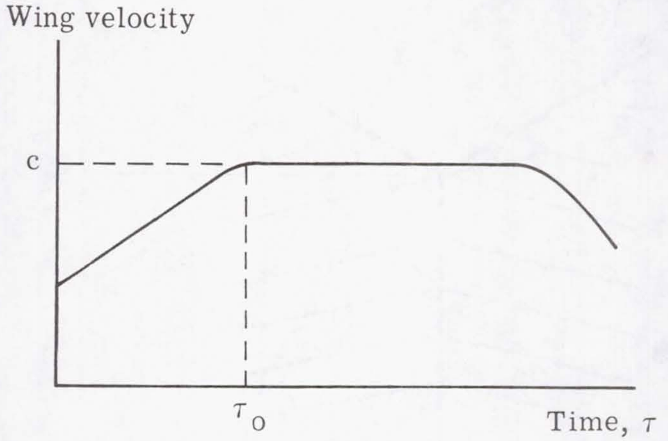


Figure A4.- Time history of the motion of the wing of figure A2 causing the Σ-surface to intersect itself.

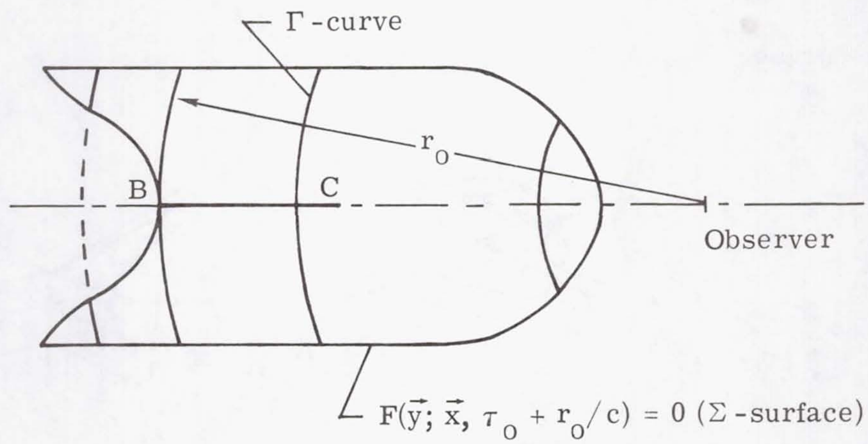


Figure A5.- Σ-surface of the wing of figure A2 corresponding to the observer time $r_o + \tau_o/c$. Some of the Γ-curves are also shown.

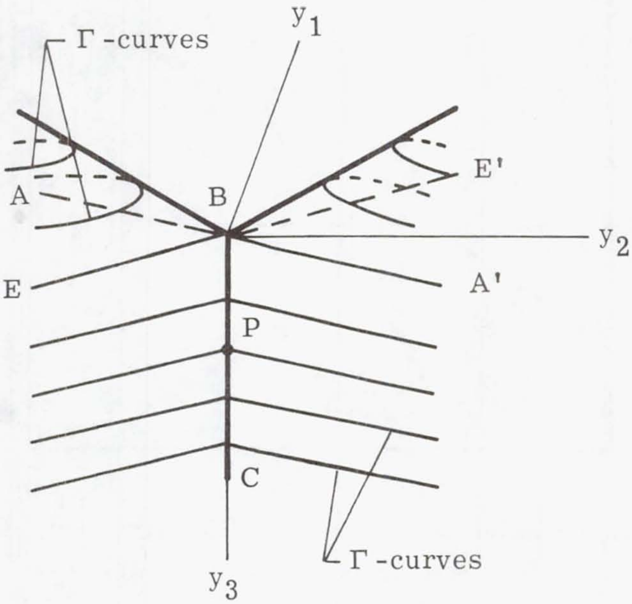
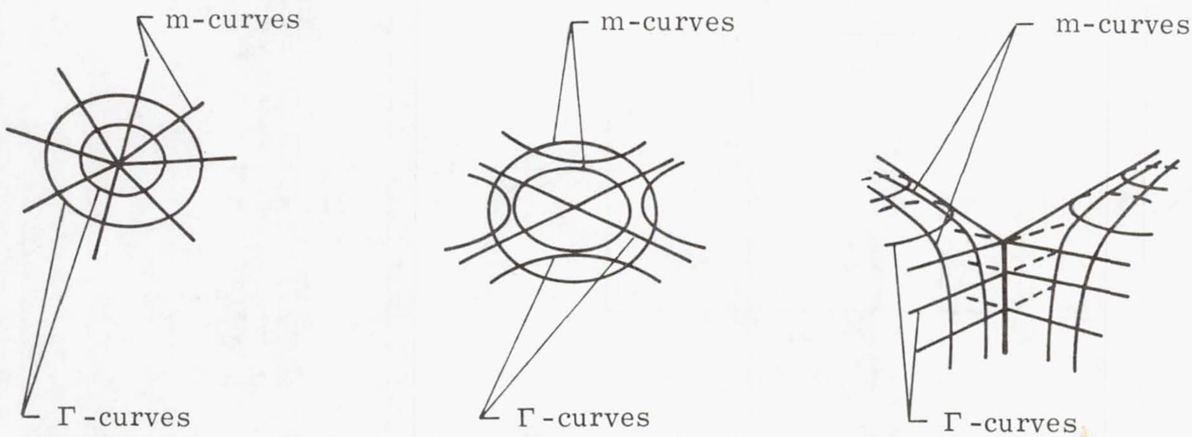


Figure A6.- Structure of the Σ-surface near point B of figure A5.



(a) Point A of figure A3. (b) Point B of figure A3. (c) Point B of figure A5.

Figure A7.- The Γ- and m-curves in various situations causing singularities in (Γ, τ)-formulation of the solution of the wave equation.

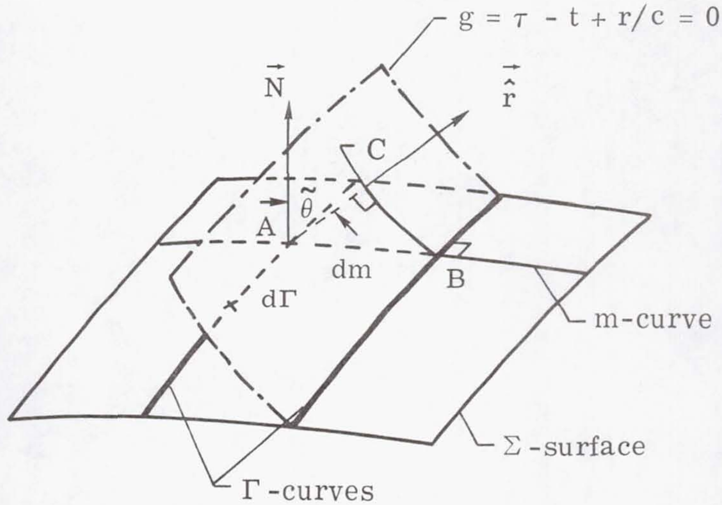


Figure A8.- Diagram for determination of the relation between the Σ-surface and (Γ, τ)-formulations. AC = c dτ; AB = dm.

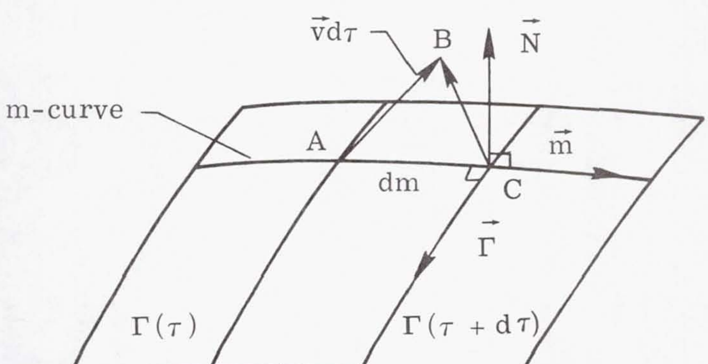


Figure A9.- Diagram for determination of the relation between the elements of the areas of the Σ-surface and the body surface S. (The unit vectors $\vec{\Gamma}$ and $\vec{m}$ are tangent to Γ-curve and m-curve, respectively.)