NASA TECHNICAL MEMORANDUM

NASA TM X- 62,497

NASA TM X- 62,497

(NASA-TM-X-62497) A FAMILY OF APPROXIMATE SOLUTIONS AND EXPLICIT ERROR ESTIMATES FOR THE NONLINEAR STATIONARY NAVIER-STOKES PROBLEM (NASA) 11 p HC $3.50 CSCL 20D

N76-13428

Unclas
G3/34 05618

A FAMILY OF APPROXIMATE SOLUTIONS AND EXPLICIT ERROR ESTIMATES FOR THE NONLINEAR STATIONARY NAVIER-STOKES PROBLEM

Ralph E. Gabrielsen
Ames Research Center
and
U. S. Army Air Mobility R&D Laboratory
Moffett Field, Calif. 94035

and

Steven Karel
U. S. Army Element, NASA
Washington, D. C. 20546
and
Ames Research Center
Moffett Field, Calif. 94035

January 1975

1. Report No. NASA TM X-62,497	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle A FAMILY OF APPROXIMATE SOLUTIONS AND EXPLICIT ERROR ESTIMATES FOR THE NONLINEAR STATIONARY NAVIER-STOKES PROBLEM		5. Report Date
		6. Performing Organization Code
7. Author(s) Ralph E. Gabrielsen† and Steven Karel††		8. Performing Organization Report No. A-6319
		10. Work Unit No. 505-06-11
9. Performing Organization Name and Address *Ames Research Center and U.S. Army Air Mobility R&D Laboratory, Moffett Field, Calif. 94035, and †Ames Research Center, Moffett Field, Calif. 94035 and U.S. Army Air Element, NASA, Washington, D. C. 20546		11. Contract or Grant No. Technical Memorandum
		13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D. C. 20546		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

An algorithm for solving the nonlinear stationary Navier-Stokes problem is developed. Explicit error estimates are given.

17. Key Words (Suggested by Author(s))		18. Distribution Statement	
Navier-Stokes equations (stationary) Newton-Kantorovich method		Unlimited STAR Category — 34	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 10	22. Price* $3.25

*For sale by the National Technical Information Service, Springfield, Virginia 22151

A FAMILY OF APPROXIMATE SOLUTIONS AND EXPLICIT ERROR ESTIMATES FOR THE NONLINEAR STATIONARY NAVIER-STOKES PROBLEM

RALPH E. GABRIELSEN* AND STEVEN KAREL†

Abstract. An algorithm for solving the nonlinear stationary Navier-Stokes problem is developed. Explicit error estimates are given.

1. Introduction. Since the separation problem of aerodynamics is at present intractable, it has been decided to undertake a closely related problem via a mathematical technique that is potentially adaptable to the separation problem. Specifically, the problem under consideration is the "nonlinear stationary Navier-Stokes problem" of fluid dynamics. The generalized Newton's method, as developed by Kantorovich [5, 7] is used. Its application to this problem is of definite value for those seeking practical solutions of related fluid flow problems. The following questions are considered:

(i) Under what conditions does the sequence of functions obtained by Newton's method converge to the solution?

(ii) How should the initial guess be made, as a function of ν, so as to guarantee convergence?

(iii) At what rate does the sequence of approximate solutions converge?

Given S, a two-dimensional Green's domain, and $f_1(x,y) \in C^1(S)$, $f_2(x,y) \in C^1(S)$, the nonlinear stationary Navier-Stokes problem is:

*Ames Research Center and U.S. Army Air Mobility R&D Laboratory, Moffett Field, Calif. 94035.

†U.S. Army Element, NASA, Washington, D.C. 20546 and Ames Research Center, Moffett Field, Calif. 94035.

$$
(1) \quad \begin{cases} uu_x + vu_y + q_x - \nu \, \Delta u + f_1(x,y) = 0 \, , \\ uv_x + vv_y + q_y - \nu \, \Delta v + f_2(x,y) = 0 \, , \\ \qquad\qquad\qquad\qquad u_x + v_y = 0 \, , \\ \text{with boundary conditions} \\ \quad u(\partial S) = -b_2(\partial S) \, , \quad v(\partial S) = b_1(\partial S) \, . \end{cases}
$$

An equivalent expression is

$$
(2) \qquad P(\psi) \equiv \nu\Delta\Delta\psi + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y + f_{1_y} - f_{2_x} = 0
$$

$$
\psi_x(\partial S) = b_1(\partial S) \, , \quad \psi_y(\partial S) = b_2(\partial S) \, ,
$$

where

$$
\psi_x = v \, , \quad \psi_y = -u \, ,
$$

and P is a mapping from $C^4(S)$ into $C^0(S)$ with norm

$$
\|\psi\|_{C^N} = \sum_{n=0}^{N} \sum_{m=0}^{n} \max \left| \frac{\partial^n \psi}{\partial x^m \partial y^{n-m}} \right| \, .
$$

For clarity, the equivalence of (1) and (2) is now shown.

LEMMA (1) $\leftrightarrow$ (2).

Proof.

(1) $\rightarrow$ (2) directly follows from [6, Theorem 6, p. 131].

(2) $\rightarrow$ (1): (2) can be readily rearranged into the form

$$
(-\nu\Delta u + uu_x + vu_y + f_1)_y = (-\nu\Delta v + uv_x + vv_y + f_2)_x \, ,
$$

with

$$
u = \psi_y \, , \quad v = -\psi_x \, .
$$

Let

$$
\vec{Z} = (-\nu\Delta u + uu_x + vu_y + f_1)\vec{i} + (-\nu\Delta v + uv_x + vv_y + f_2)\vec{j} \, .
$$

Note the fact: if $\vec{V} = a\vec{i} + b\vec{j} \in C^1(S)$, then

$$
\exists F \ni \vec{V} = \nabla F \leftrightarrow a_y = b_x \, .
$$

Hence, $\exists\, q \ni \vec{Z} = \nabla q$.

Therefore, (2) $\rightarrow$ (1).

2. Main Results. We seek a solution to (2) by the generalized Newton's method. Consider the equation

$$P(\psi_0) + P'(\psi_0)(\psi - \psi_0) = 0 \; .$$

If ψ_1 is a solution of this equation, we can write a new equation

$$P(\psi_1) + P'(\psi_1)(\psi - \psi_1) = 0 \; .$$

Assume that for each $n \geq 0 \;\exists\; \psi_{n+1} \ni$

$$(3) \qquad \begin{aligned} &P(\psi_n) + P'(\psi_n)(\psi_{n+1} - \psi_n) = 0 \; , \\ &\psi_{n_x}(\partial S) = b_1 \; , \quad \psi_{n_y}(\partial S) = b_2 \; . \end{aligned}$$

(See LEMMA 1 for explicit expression for P'.)

If $\lim\limits_{n\to\infty} \psi_n$ exists, let ψ^* be the limit. Then

$$P(\psi^*) + P'(\psi^*)(\psi^* - \psi^*) = 0$$

$$P(\psi^*) = 0 \; .$$

Thus ψ^* is the desired solution. This is Newton's method.

Define

$$H_N = \max_{x',y' \,\epsilon\, S} \int \sum_{n=0}^{N} \sum_{m=0}^{n} \left| \frac{\partial^n G(x',y',x,y)}{\partial x'^n \partial y'^{n-m}} \right| dx\, dy$$

where G is the Green's function of the problem

$$\Delta\Delta\tilde{\psi} = 0 \text{ in } S \; , \quad \tilde{\psi}(\partial S) = 0 \; , \quad \tilde{\psi}_n(\partial S) = 0 \; .$$

[1], [2], [3], [4], [8].

Let $M_{\psi_0} = \max(\max|\Delta\psi_{0_y}|, \max|\Delta\psi_{0_x}|, \max|\psi_{0_x}|, \max|\psi_{0_y}|)$.

Therefore, based on the remarkable theory developed by Kantorovich, we obtain the following result:

THEOREM 1. *If the initial* ψ_0 *is* $\ni$

$$\psi_{0_x}\Big|_{\partial S} = b_1, \quad \psi_{0_y}\Big|_{\partial S} = b_2, \quad M_{\psi_0}H_3 < \nu ,$$

and

$$\|P(\psi_0)\|_{C^0} \le \frac{\nu^2\left(1 - \frac{1}{\nu} M_{\psi_0}H_3\right)^2}{2H_4^2} ,$$

then the Newton-Kantorovich sequence $\{\psi_m\}_{m=0}^{\infty}$ *does in fact converge to the unique solution of (2).*

Proof. By the theorem of Kantorovich [5, p. 708], it is sufficient to show that

$$\|P(\psi_0)\| \le \frac{1}{2\|P'(\psi_0)^{-1}\|^2\|P''\|} ;$$

this is shown by the following lemmas.

LEMMA 1. $P'(\psi)$, *the Frechet derivative exists at all points* ψ_0 *in the domain, and*

$$P'(\psi_0)g = \nu\Delta\Delta g + \psi_{0_y}\Delta g_x + \Delta\psi_{0_x}g_y - \Delta\psi_{0_y}g_x - \psi_{0_x}\Delta g_y .$$

Proof:

$$P(\psi) = \nu\Delta\Delta\psi + \psi_y\Delta\psi_x - \psi_x\Delta\psi_y + f_{1_y} - f_{2_x} .$$

If

$$\lim_{\|a\|\to 0} \frac{\|P(\psi+a) - P(\psi) - La\|}{\|a\|} = 0 ,$$

for some linear operator L, then define $P'(\psi_0) = L$.

$$P(\psi_0+a) - P(\psi_0) = \nu\Delta\Delta a + \psi_{0_y}\Delta a_x + a_y\Delta\psi_{0_x} + a_y\Delta a_x - a_x\Delta\psi_{0_y} - \psi_x\Delta a_y - a_x\Delta a_y .$$

Let

$$L(\psi_0)(a) = \nu\Delta\Delta a + \psi_{0_y}\Delta a_x + a_y\Delta\psi_{0_x} - a_x\Delta\psi_{0_y} - \psi_{0_x}\Delta a_y .$$

Then $L(\psi_0)$ is a linear operator. Therefore, $L(\psi) = P'(\psi)$ if

$$\lim_{\|a\|\to 0} \frac{\|P(\psi_0+a) - P(\psi_0) - La\|}{\|a\|} = \lim_{\|a\|\to 0} \frac{\|a_y\Delta a_x - a_x\Delta a_y\|}{\|a\|} = 0 .$$

$$a_y\Delta a_x - a_x\Delta a_y \in C^0 .$$

Therefore,

$$\|a_y \Delta a_x - a_x \Delta a_y\| = \max_S |a_y \Delta a_x - a_x \Delta a_y| .$$

$$a \in C^4 .$$

Therefore,

$$\|a\| = \sum_{n=0}^{4} \sum_{m=0}^{n} \max \left| \frac{\partial^n a}{\partial x^m \partial y^{n-m}} \right| .$$

$$|a_y \Delta a_x - a_x \Delta a_y| \le |a_y|(|a_{xxx}| + |a_{xyy}|) + |a_x|(|a_{xxy}| + |a_{yyy}|) .$$

Therefore,

$$\max_S |a_y \Delta a_x - a_x \Delta a_y| \le \max_S |a_y| \left(\max_S |a_{xxx}| + \max_S |a_{xyy}| \right) + \max_S |a_x| \left(\max_S |a_{xxy}| + \max_S |a_{yyy}| \right) .$$

So:

$$0 \le \lim_{\|a\| \to 0} \frac{\|P(\psi_0 + a) - P(\psi_0) - La\|}{\|a\|}$$

$$\le \lim_{\|a\| \to 0} \frac{\max_S |a_y| \left(\max_S |a_{xxx}| + \max_S |a_{xyy}| \right) + \max_S |a_x| \left(\max_S |a_{xxy}| + \max_S |a_{yyy}| \right)}{a}$$

$$\le \lim_{\|a\| \to 0} \frac{\|a\|(\|a\| + \|a\|) + \|a\|(\|a\| + \|a\|)}{\|a\|} .$$

Therefore,

$$\lim_{\|a\| \to 0} \frac{\|P(\psi_0 + a) - P(\psi_0) - La\|}{\|a\|} = 0 . \quad \text{Q.E.D.}$$

LEMMA 2. $P''(\psi)$ *exists at all points* ψ_0 *in* $C^4(S)$ *and*

$$P''(\psi_0)\psi\phi = \psi_y \Delta\phi_x + \phi_y \Delta\psi_x - \phi_x \Delta\psi_y - \psi_x \Delta\phi_y .$$

Proof. By definition, $P''(\psi_0)$ exists if there is a bilinear operator $B \ni$

$$\lim_{\|\phi\| \to 0} \frac{\|P'(\psi_0 + \phi) - P'(\psi_0) - B\phi\|}{\|\phi\|} = 0 .$$

If so, $P''(\psi_0)$ is defined to be B.

$$P'(\psi_0+\phi)g = \nu\Delta\Delta g + (\psi_0+\phi)_y\Delta g_x + g_y\Delta(\psi_0+\phi)_x - g_x\Delta(\psi_0+\phi)_y - (\psi_0+\phi)_x\Delta g_y .$$

$$[P'(\psi_0+\phi) - P'(\psi_0)]g = \phi_y\Delta g_x + g_y\Delta\phi_x - g_x\Delta\phi_y - \phi_x\Delta g_y .$$

Now,

$$[P'(\psi_0+\phi) - P'(\psi_0)](g_1+g_2) = [P'(\psi_0+\phi) - P'(\psi_0)]g_1 + [P'(\psi_0+\phi) - P'(\psi_0)]g_2 ,$$

and

$$[P'(\psi_0+\phi_1+\phi_2) - P'(\psi_0)]g = [P'(\psi_0+\phi_1) - P'(\psi_0)]g + [P'(\psi_0+\phi_2) - P'(\psi_0)]g .$$

Therefore, $[P'(\psi_0+\phi) - P'(\psi_0)]g$ is a bilinear operator of ϕ and g. Let $B\phi = P'(\psi_0+\phi) - P'(\psi_0)$. Then it follows that

$$P''(\psi_0)\phi\theta = \phi_y\Delta\theta_x + \theta_y\Delta\phi_x - \theta_x\Delta\phi_y - \phi_x\Delta\theta_y .$$

LEMMA 3. $P'(\psi_0)^{-1}$ *exists.*

Proof. Given $P(\psi_0) + P'(\psi_0)(\psi-\psi_o) = 0$, let $\tilde{\psi} = \psi - \psi_o$, then

$$P'(\psi_0)\tilde{\psi} = -P(\psi_0) .$$

Equivalently, $\nu\Delta\Delta\tilde{\psi} + \Delta\psi_{0_x}\tilde{\psi}_y + \psi_{0_y}\Delta\tilde{\psi}_x - \psi_{0_x}\Delta\tilde{\psi}_y - \Delta\psi_{0_y}\tilde{\psi}_x = -P(\psi_0)$. This equation can be abbreviated as $\nu\Delta\Delta\tilde{\psi} = \tilde{f}(\tilde{\psi}) + F$. Let G be the Green's function for $\tilde{\psi}$ (see Theorem 1), then

$$\tilde{\psi} = \frac{1}{\nu}\int G\tilde{f} + \frac{1}{\nu}\int GF .$$

Define the linear operators

$$A\tilde{\psi} = \frac{1}{\nu}\int G\tilde{f}(\tilde{\psi}) .$$

$$B[-P(\psi_0)] = \frac{1}{\nu}\int GF .$$

Then

$$\left(I - \frac{1}{\nu}A\right)\tilde{\psi} = B[-P(\psi_0)] .$$

Under proper conditions, as shown later in the proof, $\left(I - \frac{1}{\nu}A\right)^{-1}$ exists. Then

$$\tilde{\psi} = \left(I - \frac{1}{\nu} A\right)^{-1} B[-P(\psi_0)] .$$

Therefore,

$$P'(\psi_0)\left(I - \frac{1}{\nu} A\right)^{-1} B[-P(\psi_0)] = -P(\psi_0) ;$$

$$P'(\psi_0)\left(I - \frac{1}{\nu} A\right)^{-1} B = I ,$$

and $\left(I - \frac{1}{\nu} A\right)^{-1} B$ is a right inverse. Also, $\left(I - \frac{1}{\nu} A\right)^{-1} BP'(\psi_0)\tilde{\psi} = \tilde{\psi}$; $\left(I - \frac{1}{\nu} A\right)^{-1} BP'(\psi_0) = I$, and $\left(I - \frac{1}{\nu} A\right)^{-1} B$ is a left inverse. Therefore,

$$\left(I - \frac{1}{\nu} A\right)^{-1} B = [P'(\psi_0)]^{-1} .$$

We now show the conditions under which $\left(I - \frac{1}{\nu} A\right)^{-1}$ exists.

$$A\psi = \int_S G(\psi_x \Delta\psi_{0_y} - \psi_y \Delta\psi_{0_x} + \psi_{0_x}\Delta\psi_y - \psi_{0_y}\Delta\psi_x) dS$$

$$\|A\| = \sup_{\|\psi\| \le 1} \|A\psi\|$$

so

$$\|A\|_C = \sup_{\|\psi\| \le 1} \left[\max_{x',y' \varepsilon S} \left| \int_S G(\psi_x\Delta\psi_{0_y} - \psi_y\Delta\psi_{0_x} + \psi_{0_x}\Delta\psi_y - \psi_{0_y}\Delta\psi_x) dS \right| + \max (\ )_{x'} + \ldots \right.$$

$$\|A\|_C \le \sup_{\|\psi\| \le 1} \left| \max_{x',y' \varepsilon S} \int [|G| + |G_{x'}| + |G_{y'}| + \ldots] \cdot [|\psi_x||\Delta\psi_{0_y}| + \ldots] \right.$$

$$\|A\|_C \le \max_{\|\psi\| \le 1} \left(\max_{x',y' \varepsilon S} M_{\psi_0} \int [|G| + |G_{x'}| + |G_{y'}| + \ldots] \cdot [|\psi_x| + |\psi_y| + \ldots] \right) .$$

Therefore,

$$\|A\|_{C^4} \le M_{\psi_0}\left(\max \int [|G| + |G_{x'}| + |G_{y'}| + \ldots] dx\, dv\right) .$$

$$\|A\|_{C^4} \le M_{\psi_0} H_3 .$$

Therefore, $\left(I - \frac{1}{\nu} A\right)^{-1}$ exists if $M_{\psi_0} H_3 < \nu$. $\left(I - \frac{1}{\nu} A\right)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\nu^n} A^n$ exists.

LEMMA 4. $\|P''(\psi_0)\| \leq 1$.

Proof.

$$\|P''(\psi_0)\| = \max_{\substack{\|\phi\| \leq 1 \\ \|\theta\| \leq 1}} |\phi_y \Delta\theta_x + \theta_y \Delta\phi_x - \phi_x \Delta\theta_y - \theta_x \Delta\phi_y|$$

$$\leq \max(|\phi_y| + |\phi_{xxx}| + |\phi_{xyy}| + |\phi_x| + |\phi_{xxy}| + |\phi_{yyy}|) \leq 1 .$$

This completes the proof of Theorem 1.

THEOREM 2. *Under the hypothesis of Theorem 1, the error estimate for the mth approximate solution is expressed by*

$$\|\psi - \psi_m\| \leq \left(2^{2^m - m}\right) \left[\frac{H_4}{\nu\left(1 - \frac{1}{\nu} M_{\psi_0} H_3\right)}\right]^{2^{m+1}-1} \|P(\psi_0)\|^{2m} .$$

Proof. By the Kantorovich theory, this result follows from the hypothesis of Theorem 1.

COROLLARY 1. *As a function of* ν, *for fixed* m,

$$\|\psi - \psi_m\| = O\left(\nu^{-2^{m+1}+1}\right) .$$

COROLLARY 2. *If the hypothesis of Theorem 1 is satisfied, then for a given* $\epsilon > 0$, *there exists a denumerably infinite number of linear equations and solutions* ψ_m *as specified by (3) such that the entire family of* ψ_m*'s are within the* ϵ*-neighborhood of the exact solution* ψ *of (2), i.e.,*

$$\|\psi - \psi_n\|_{C^4} < \epsilon .$$

Proof. Follows directly from Theorem 2.

REFERENCES

[1] I. BABUŠKA, K. REKTORYS, and F. VYČICHLO, *Mathematische Elastizitatstheorie der Ebenen Probleme*, Akademi-Verlag, Berlin, 1960.

[2] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, Interscience, New York, N.Y., 1953.

[3] P. R. GARABEDIAN, *Partial Differential Equations*, John Wiley, New York, N.Y., 1964.

[4] L. V. KANTOROVICH and V. I. KRYLOV, *Approximate Methods of Higher Analysis*, Interscience, The Netherlands, 1958.

[5] L. V. KANTOROVICH and G. P. AKILOV, *Functional Analysis in Normed Spaces*, Pergamon Press, New York, N.Y., 1964.

[6] O. A. LADYZHENSKAIǍ, *The Mathematical Theory of Viscous Incompressible Flows*, 2nd English ed., rev. and Engl. translated from the Russian by Richard A. Silverman and John Chu, Gordon and Breach, New York, 1969.

[7] L. B. RALL, *Computational Solution of Nonlinear Operator Equations*, John Wiley, New York, N.Y., 1969.

[8] S. TIMOSHENKO and S. WOINOWSKY-KRIEGER, *Theory of Plates and Shells*, McGraw-Hill, New York, N.Y., 1959.