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16. Abstract An investigation of the effect of axial load on the natural frequencies and mode shapes of uniform beams and of a cantilevered beam with a concentrated mass at the tip is presented. Characteristic equations which yield the frequencies and mode shape functions for the various cases are given. The solutions to these equations are presented by a series of graphs so that frequency as a function of axial load can readily be determined. The effect of axial load on the mode shapes are also depicted by another series of graphs.			
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EFFECT OF AXIAL LOAD ON MODE SHAPES AND FREQUENCIES OF BEAMS

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SUMMARY

An investigation of the effect of axial load on the natural frequencies and mode shapes of uniform beams with various types of boundary conditions and of a cantilevered beam with a concentrated mass at the tip is presented. This investigation yielded expressions for the mode shapes and characteristic equations for the various cases considered. For the uniform beams the characteristic equations were solved either numerically or in closed form, and the results are presented by a series of graphs showing the effect of preload on the different types of beams. The effect of axial load on the mode shapes is also shown in graphical form for several different loading conditions. For the cantilevered beam with a tip mass, two types of axial loads were considered: In the first case the axial load vector remained constant, and in the second case the load was directed through the root of the beam at all times. The results of this portion of the investigation are presented in graphs that show the effects of both tip-mass variation and axial load on the fundamental frequency of the system.

INTRODUCTION

In the design of certain spacecraft structural components it sometimes becomes necessary to determine the normal modes and frequencies of beam type components which are in a state of preload or prestress. This preload may result from inertial effects, as in the case of a spinning spacecraft, or it may be induced by mechanical means. For example, current designs for large, flexible solar arrays (refs. 1 to 4) are such that the boom that supports the array is in a state of prestress due to the tension that must be maintained in the solar cell substrate. Also, certain types of attachment and ejection mechanisms can cause preload in parts of the primary spacecraft structure. In most of these cases an exact solution for the normal modes and frequencies of the system would be intractable and recourse would be made to one of the many approximate solutions such as the Rayleigh-Ritz method, Galerkin's method, finite

elements, etc. (refs. 5 to 8). There exists, however, a class of problems for which an exact solution can be obtained. This class consists of uniform beams under constant axial load with various types of simple boundary conditions. This type of structure may represent some structural components reasonably well for preliminary design information. Even in those cases where it does not adequately model the structure, the modes obtained for these cases can be used to obtain Rayleigh-Ritz or Galerkin type of solutions.

For these reasons an investigation was made to determine the effect of an axial load on the modes and frequencies of a uniform beam with various boundary conditions and of a cantilevered beam with a tip mass. The boundary conditions considered for the uniform beam include all possible simple boundary conditions at both ends of the beam (i.e., free, hinged, clamped, and guided). For the cantilevered beam with a tip mass, two types of axial loads were considered. The first type was a constant axial load applied at the tip, and the second was an axial load whose direction is always through the fixed support. The second is the limiting case for some current large solar array designs if the mass of the blanket is negligible compared with the mass of the support beam. When the mass of the blanket is not negligible, which is usually the situation, these modes could still be used in a Rayleigh-Ritz analysis for this type of array.

SYMBOLS

A, B, C, D, E	arbitrary constants of integration
A_n, B_n	constants of integration corresponding to n^{th} mode of vibration
EI	uniform beam bending stiffness
k	axial load parameter, $\sqrt{P/EI}$
\bar{k}	nondimensional axial load parameter, $\sqrt{Pl/EI}$
\bar{k}_{cr}	nondimensional axial load parameter corresponding to critical load, $\sqrt{P_{cr} l^2/EI}$
l	length of beam
$M(x, t), m(x),$ $m(\bar{x})$	moment distribution along length of beam
\mathcal{M}_B	total mass of beam, ρl
\mathcal{M}_T	tip mass for cantilevered beam
P	axial load
P_{cr}	critical or Euler's buckling load for beam

$Q(x, t), q(x)$	shearing force distribution
$q(x)$	
t	time
$V(x, t), v(x)$	beam displacement
$\bar{v}(\bar{x})$	nondimensional beam displacement, v/l
$\bar{v}_n(\bar{x})$	mode shape of n^{th} mode
x	lengthwise coordinate
\bar{x}	nondimensional lengthwise coordinate, x/l
α_1, α_2	characteristic values, eq. (9)
α_{1n}, α_{2n}	characteristic values for n^{th} mode
β	frequency parameter, $\sqrt[4]{\rho\omega/EI}$
$\bar{\beta}$	nondimensional frequency parameter, βl
β_n	nondimensional frequency parameter corresponding to n^{th} mode of vibration, $\sqrt[4]{\rho\omega_n^2/EI}$
η_T	ratio of tip mass to total beam mass, M_T/M_B
ρ	mass per unit length of beam
ω	circular frequency of vibration
ω_n	circular natural frequency of n^{th} mode
Subscripts:	
B	beam
n	integral number designating natural bending mode and frequency of beam
T	tip of cantilevered beam
Superscript:	
'	differentiation with respect to \bar{x}

THEORETICAL ANALYSIS

Uniform Beam Under Axial Load

Equations of motion. - The development of the equations governing the bending vibrations of uniform beams are presented in numerous texts on vibration theory (refs. 5, 6, or 8). In the development that follows the effect of an axial load is included.

Consider a uniform beam vibrating freely under the action of a constant axial load P as shown in figure 1(a). Let $V(x, t)$ be the displacement of any point x along the neutral axis of the beam. The internal shear, moment, and inertial forces acting on an element dx in length will be as shown in figure 1(b). Summing forces and moments on this element yields the following two equilibrium equations:

$$\frac{\partial Q(x, t)}{\partial x} - \rho \frac{\partial^2 V(x, t)}{\partial t^2} = 0 \quad (1a)$$

$$\frac{\partial M(x, t)}{\partial x} + P \frac{\partial V(x, t)}{\partial x} + Q(x, t) = 0 \quad (1b)$$

From elementary beam theory the moment-curvature relationship for the sign convention depicted in figures 1 is

$$M(x, t) = EI \frac{\partial^2 V(x, t)}{\partial x^2} \quad (1c)$$

The shearing force $Q(x, t)$ can be expressed in terms of the displacement variable by using equations (1b) and (1c). Thus

$$Q(x, t) = - \left[EI \frac{\partial^3 V(x, t)}{\partial x^3} + P \frac{\partial V(x, t)}{\partial x} \right] \quad (2)$$

Finally, from equations (1a) and (2) the equation of motion in terms of the displacement variable can be written as

$$\frac{\partial^4 V}{\partial x^4} + \frac{P}{EI} \frac{\partial^2 V(x, t)}{\partial x^2} + \frac{\rho}{EI} \frac{\partial^2 V(x, t)}{\partial t^2} = 0 \quad (3)$$

Equation (3) represents the governing partial differential equation describing the motion of the beam, and equations (1c) and (2) represent the moment and shear distribution along the beam.

Solution of equation and boundary conditions. - The time variable in equations (1) to (3) can be eliminated by assuming a solution of the form

$$V(x, t) = v(x) \sin \omega t \quad (4a)$$

$$M(x, t) = m(x) \sin \omega t \quad (4b)$$

$$Q(x, t) = q(x) \sin \omega t \quad (4c)$$

Substituting equations (4) into equations (1) to (3) yields

$$\frac{d^4 v(x)}{dx^4} + k^2 \frac{d^2 v(x)}{dx^2} - \beta^4 v(x) = 0 \quad (5)$$

$$m(x) = EI \frac{d^2 v(x)}{dx^2} \quad (6a)$$

$$q(x) = -EI \left[\frac{d^3 v(x)}{dx^3} + k^2 \frac{dv(x)}{dx} \right] \quad (6b)$$

where $k^2 = P/EI$ and $\beta^4 = \rho\omega^2/EI$. It will be convenient at this point to put equation (5) into nondimensional form by letting $\bar{x} = x/l$, $\bar{v} = v/l$, $\bar{k} = kl$, and $\bar{\beta} = \beta l$. Using these expressions allows equations (5) and (6) to become

$$\bar{v}''''(\bar{x}) + \bar{k}^2 \bar{v}''(\bar{x}) - \bar{\beta}^4 \bar{v}(\bar{x}) = 0 \quad (7)$$

$$\left. \begin{aligned} m(\bar{x}) &= \frac{EI}{l} \bar{v}'' \\ q(\bar{x}) &= -\frac{EI}{l^3} \left\{ \bar{v}''''(\bar{x}) + \bar{k}^2 \bar{v}''(\bar{x}) \right\} \end{aligned} \right\} \quad (8)$$

The solution to equation (7), which can be readily verified by direct substitution, is given by

$$\bar{v}(\bar{x}) = A \cosh \alpha_1 \bar{x} + B \sinh \alpha_1 \bar{x} + D \cos \alpha_2 \bar{x} + E \sin \alpha_2 \bar{x} \quad (9)$$

where A, B, C, and D are constants and

$$\left. \begin{aligned} \alpha_1 &= \left(-\frac{\bar{k}^2}{2} + \sqrt{\frac{\bar{k}^4}{4} + \bar{\beta}^4} \right)^{1/2} \\ \alpha_2 &= \left(\frac{\bar{k}^2}{2} + \sqrt{\frac{\bar{k}^4}{4} + \bar{\beta}^4} \right)^{1/2} \end{aligned} \right\} \quad (10)$$

The frequency equation can now be determined from equation (9) once the boundary conditions are stipulated. The boundary conditions being considered at either end of the beam are as follows:

Pinned -

$$\bar{v} = 0 \text{ and } m = 0 \text{ or } \bar{v}'' = 0 \quad (11a)$$

Clamped -

$$\bar{v} = 0 \text{ and } \bar{v}' = 0 \quad (11b)$$

Free -

$$m = 0 \text{ or } \bar{v}'' = 0 \text{ and } q = 0 \text{ or } \bar{v}''' + \bar{k}^2 \bar{v}' = 0 \quad (11c)$$

Guided -

$$\bar{v}' = 0 \text{ and } q = 0 \text{ or } \bar{v}'''' = 0 \quad (11d)$$

The various combinations of these boundary conditions at both ends of the beam are listed in table I. If the boundary conditions given by equations (11) are used at $\bar{x} = 0$, two of the unknown constants in equation (9) can be evaluated, and the resulting solutions are for the beam free at $\bar{x} = 0$

$$\bar{v}(\bar{x}) = A \left(\cosh \alpha_1 \bar{x} + \frac{\alpha_1^2}{\alpha_2^2} \cos \alpha_2 \bar{x} \right) + B \left(\sinh \alpha_1 \bar{x} + \frac{\alpha_2}{\alpha_1} \sin \alpha_2 \bar{x} \right) \quad (12a)$$

for the beam pinned at $\bar{x} = 0$

$$\bar{v}(\bar{x}) = A \sinh \alpha_1 \bar{x} + E \sin \alpha_2 \bar{x} \quad (12b)$$

for the beam clamped at $\bar{x} = 0$

$$\bar{v}(\bar{x}) = A (\cosh \alpha_1 \bar{x} - \cos \alpha_2 \bar{x}) + B \left(\sinh \alpha_1 \bar{x} - \frac{\alpha_1}{\alpha_2} \sin \alpha_2 \bar{x} \right) \quad (12c)$$

and for the beam guided at $\bar{x} = 0$

$$\bar{v}(\bar{x}) = A \cosh \alpha_1 \bar{x} + D \cos \alpha_2 \bar{x} \quad (12d)$$

Note that each of equations (12) contains two unknowns. When the boundary conditions at $\bar{x} = 1$ are specified, these equations will yield two homogeneous, algebraic equations in these two unknowns. The determinant of the coefficients of the unknowns must be equal to zero for a nontrivial solution to exist. Equating this determinant to zero then yields the characteristic equation from which the frequency parameter $\bar{\beta}$ can be determined as a function of axial load P or axial load parameter \bar{k} . To illustrate, consider the case where the beam is free at $\bar{x} = 1$. From equation (11c) the boundary conditions are

$$\bar{v}''(1) = 0 \quad \bar{v}'''(1) + \bar{k}^2 \bar{v}'(1) = 0 \quad (13)$$

From equations (12c) and (13) the following two homogeneous, algebraic equations in A and B are obtained after some manipulation:

$$A(\alpha_1^3 \cosh \alpha_1 - \alpha_1^3 \cos \alpha_1) + B(\alpha_1^3 \sinh \alpha_1 - \alpha_2^3 \sin \alpha_2) = 0 \quad (14a)$$

$$A(\alpha_2^3 \sinh \alpha_1 + \alpha_1^3 \sin \alpha_2) + B(\alpha_2^3 \cosh \alpha_1 - \alpha_2^3 \cos \alpha_1) = 0 \quad (14b)$$

Equating the determinant of the coefficients of A and B in equations (14) to zero, expanding, and simplifying will then yield the following characteristic equation for the free-free beam:

$$2\bar{\beta}^6(1 - \cosh \alpha_1 \cos \alpha_2) + \bar{k}^2(\bar{k}^4 + 3\bar{\beta}^4) \sinh \alpha_1 \sin \alpha_2 = 0 \quad (15a)$$

Similarly, for the other boundary conditions at $\bar{x} = 1$ the following characteristic equations are obtained: for the free-guided beam

$$\alpha_2^3 \sinh \alpha_1 \cos \alpha_2 + \alpha_1^3 \cosh \alpha_1 \sin \alpha_2 = 0 \quad (15b)$$

for the free-pinned beam

$$-\alpha_2^3 \cosh \alpha_1 \sin \alpha_2 + \alpha_1^3 \sinh \alpha_1 \cos \alpha_2 = 0 \quad (15c)$$

for the clamped-free beam

$$2\bar{\beta}^4 + (2\bar{\beta}^4 + \bar{k}^4) \cosh \alpha_1 \cos \alpha_2 - \bar{\beta}^2 \bar{k}^2 \sinh \alpha_1 \sin \alpha_2 = 0 \quad (15d)$$

for the clamped-pinned beam

$$\alpha_1 \cosh \alpha_1 \sin \alpha_2 - \alpha_2 \sinh \alpha_1 \cos \alpha_2 = 0 \quad (15e)$$

for the clamped-guided beam

$$\alpha_1 \sinh \alpha_1 \cos \alpha_2 + \alpha_2 \cosh \alpha_1 \sin \alpha_2 = 0 \quad (15f)$$

for the clamped-clamped beam

$$2\bar{\beta}^2(1 - \cosh \alpha_1 \cos \alpha_2) - \bar{k}^2 \sinh \alpha_1 \sin \alpha_2 = 0 \quad (15g)$$

for the guided-guided and pinned-pinned beams

$$\sin \alpha_2 = 0 \quad (15h)$$

and for the guided-pinned beam

$$\cos \alpha_2 = 0 \quad (15i)$$

Equations (15g) and (15i) yield the following expressions for the frequency parameter:
for the pinned-pinned or guided-guided beam -

$$\bar{\beta}_n^2 = n\pi \sqrt{(n\pi)^2 - \bar{k}^2} \quad (16)$$

and for the pinned-guided beam

$$\bar{\beta}_n^2 = \frac{n\pi}{2} \sqrt{\left(\frac{n\pi}{2}\right)^2 - \bar{k}^2} \quad (17)$$

Equations (16) and (17) exhibit the main characteristics of all the frequency equations given by equations (15). Namely, as P (or \bar{k}) increase, the frequencies decrease until P equals the Euler buckling load P_{cr} of the beam. At this point the lowest elastic frequency becomes zero. For P greater than P_{cr} , the fundamental frequency becomes complex, and the corresponding mode shape is unstable (ref. 6, p. 451 and ref. 8, p. 302). Thus, the maximum value for P of practical interest is P_{cr} . The values of P_{cr} and \bar{k}_{cr} for beams with the different boundary conditions are tabulated in table II. To determine the frequency parameter for the beams with other boundary conditions, equations (15) can be solved numerically for $\bar{\beta}_n$ for any value of \bar{k} less than the corresponding \bar{k}_{cr} .

Cantilevered Beam Under Axial Load With Tip Mass

For the case of a cantilevered beam with a concentrated mass at the free end, two types of axial loads will be considered. In the first case (fig. 2(a)) the axial load will be taken as a constant in both magnitude and direction so that this case corresponds to the uniform clamped-free beam case except that a concentrated mass is added to the free end. In the second case (fig. 3(a)) the axial load is assumed to be directed through the root or fixed end of the beam at all times. For this case the axial load is not constant since it changes direction during the motion. This type of loading will be referred to as a directed axial load. For both cases the deflection is given by (eq. (12c) since the boundary condition at the fixed end is the same as for a uniform cantilevered beam. Also, the moment boundary condition at the free end is the same for both cases and is given by

$$\bar{v}''(1) = 0 \quad (18)$$

The difference between the two cases will be in the shear boundary condition at the free end. For the case of constant axial load, the shear boundary conditions can be found by satisfying the equilibrium condition at the free end which is depicted in figure 2(b).

Thus

$$q(1) + \mathcal{M}_T \omega^2 \bar{v}(1) = 0 \quad (19)$$

From equations (6) and (9) this boundary condition becomes

$$\bar{v}'''(1) + \bar{k}^2 \bar{v}'(1) + \eta_T \bar{\beta}^4 \bar{v}(1) = 0 \quad (20)$$

where η_T is the ratio of the tip mass \mathcal{M}_T to total beam mass, $\mathcal{M}_B = \rho l$. From equations (12c), (18), and (20) the following two homogeneous equations in A and B are obtained:

$$\left. \begin{aligned} & A \left(\alpha_1^2 \cosh \alpha_1 + \alpha_2^2 \cos \alpha_2 \right) + B \left(\alpha_1^2 \sinh \alpha_1 + \bar{\beta}^2 \sin \alpha_2 \right) = 0 \\ & A \left\{ \alpha_2^2 \sinh \alpha_1 - \bar{\beta}^2 \sin \alpha_2 + \eta_T \bar{\beta}^2 (\cosh \alpha_1 - \cos \alpha_2) \right\} \\ & \quad + B \left\{ \alpha_2^2 \cosh \alpha_1 + \alpha_1^2 \cos \alpha_2 + \eta_T \bar{\beta}^2 (\alpha_2 \sinh \alpha_1 - \alpha_1 \sin \alpha_2) \right\} = 0 \end{aligned} \right\} (21)$$

Equating the determinant of the coefficients of A and B to zero, expanding and simplifying yields the following characteristic equation for a cantilever beam with a tip mass under constant axial load.

$$\begin{aligned} & -2\bar{\beta}^4 + \bar{\beta}^2 \bar{k}^2 \sinh \alpha_1 \sin \alpha_2 - (2\bar{\beta}^4 + \bar{k}^4) \cosh \alpha_1 \cos \alpha_2 \\ & \quad + \eta_T \bar{\beta}^2 (\alpha_1^2 + \alpha_2^2) (\alpha_1 \cosh \alpha_1 \sin \alpha_2 - \alpha_2 \sinh \alpha_1 \cos \alpha_2) = 0 \end{aligned} \quad (22)$$

For the case of a cantilevered beam with a tip mass and a directed axial load, the condition at the free end is shown in figure 3(b). From this figure the equilibrium condition can be written as

$$\bar{v}'''(1) + \bar{k}^2 \bar{v}'(1) + (\eta_T \bar{\beta}^4 - \bar{k}^2) \bar{v}(1) = 0 \quad (23)$$

Comparing this equation with equation (20) shows that the term $\eta_T \bar{\beta}^4 - \bar{k}^2$ replaces $\eta_T \bar{\beta}^4$. Thus multiplying equation (22) by $\bar{\beta}^2$ and replacing $\eta_T \bar{\beta}^4$ with $(\eta_T \bar{\beta}^4 - \bar{k}^2)$ yields the characteristic equation for the beam with a directed axial load.

$$\begin{aligned} & -2\bar{\beta}^6 + \bar{\beta}^4 \bar{k}^2 \sinh \alpha_1 \sin \alpha_2 - (2\bar{\beta}^4 + \bar{k}^4) \cosh \alpha_1 \cos \alpha_2 \\ & \quad + (\eta_T \bar{\beta}^4 - \bar{k}^2) (\alpha_1^2 + \alpha_2^2) (\alpha_1 \cosh \alpha_1 \sin \alpha_2 - \alpha_2 \sinh \alpha_1 \cos \alpha_2) = 0 \end{aligned} \quad (24)$$

Mode Shapes For Beams Under Preload

The mode shapes for a beam under preload can be determined from equations (12) and one other boundary condition. For example, assume that the n^{th} root of the frequency equation is known and is given by $\bar{\beta}_n$. Corresponding to $\bar{\beta}_n$ there will be an

α_{1n} and α_{2n} which are obtained by substituting β_n into equations (10). For the case of a free-free beam, the displacement in the n^{th} mode is given by equation (12a) as

$$\bar{v}_n(\bar{x}) = A_n \left(\cosh \alpha_{1n} \bar{x} - \frac{\alpha_{1n}^2}{\alpha_{2n}^2} \cos \alpha_{2n} \bar{x} \right) + B_n \left(\sinh \alpha_{1n} \bar{x} + \frac{\alpha_{2n}}{\alpha_{1n}} \sin \alpha_{2n} \bar{x} \right) \quad (25)$$

For the free-free beam case either of equations (14) can be used to determine a relation between A_n and B_n . Using equation (14a) yields

$$B_n = -A_n \frac{(\cosh \alpha_{1n} - \cos \alpha_{2n})}{\left(\sinh \alpha_{1n} - \frac{\alpha_{2n}^3}{\alpha_{1n}^3} \sin \alpha_{2n} \right)} \quad (26)$$

Substituting equations (26) into (25) and arbitrarily setting $A_n = 1$ will then yield the desired mode shape or eigenfunction for the free-free beam. In a similar manner the mode shapes for the other boundary conditions can be determined. These functions are summarized in the following equations: for free-free and free-pinned beams

$$\bar{v}_n(\bar{x}) = \left(\cosh \alpha_{1n} \bar{x} + \frac{\alpha_{1n}^2}{\alpha_{2n}^2} \cos \alpha_{2n} \bar{x} \right) - \left(\frac{\cosh \alpha_{1n} - \cos \alpha_{2n}}{\sinh \alpha_{1n} - \frac{\alpha_{2n}^3}{\alpha_{1n}^3} \sin \alpha_{2n}} \right) \left(\sinh \alpha_{1n} \bar{x} + \frac{\alpha_{2n}}{\alpha_{1n}} \sin \alpha_{2n} \bar{x} \right) \quad (27a)$$

for a free-guided beam

$$\bar{v}_n(\bar{x}) = \cosh \alpha_{1n} \bar{x} + \frac{\alpha_{1n}^2}{\alpha_{2n}^2} \cos \alpha_{2n} \bar{x} - \left(\frac{\sinh \alpha_{1n} + \frac{\alpha_{1n}^3}{\alpha_{2n}^3} \sin \alpha_{2n}}{\cosh \alpha_{1n} - \cos \alpha_{2n}} \right) \left(\sinh \alpha_{1n} \bar{x} + \frac{\alpha_{2n}}{\alpha_{1n}} \sin \alpha_{2n} \bar{x} \right) \quad (27b)$$

for a clamped-free beam

$$\bar{v}_n(\bar{x}) = (\cosh \alpha_{1n} \bar{x} - \cos \alpha_{2n} \bar{x}) - \left(\frac{\cosh \alpha_{1n} + \frac{\alpha_{2n}^2}{\alpha_{1n}^2} \cos \alpha_{2n}}{\sinh \alpha_{1n} + \frac{\alpha_{2n}}{\alpha_{1n}} \sin \alpha_{2n}} \right) \left(\sinh \alpha_{1n} \bar{x} - \frac{\alpha_{1n}}{\alpha_{2n}} \sin \alpha_{2n} \bar{x} \right) \quad (27c)$$

for clamped-pin and clamped-clamped beams

$$\bar{v}_n(\bar{x}) = \cosh \alpha_{1n} \bar{x} - \cos \alpha_{2n} \bar{x} - \left(\frac{\cosh \alpha_{1n} - \cos \alpha_{2n}}{\sinh \alpha_{1n} - \frac{\alpha_{1n}}{\alpha_{2n}} \sin \alpha_{2n}} \right) \left(\sinh \alpha_{1n} \bar{x} - \frac{\alpha_{1n}}{\alpha_{2n}} \sin \alpha_{2n} \bar{x} \right) \quad (27d)$$

for a clamped-guide beam

$$\bar{v}_n(\bar{x}) = \cosh \alpha_{1n} \bar{x} - \cos \alpha_{2n} \bar{x} - \left(\frac{\sinh \alpha_{1n} + \frac{\alpha_{2n}}{\alpha_{1n}} \sin \alpha_{2n}}{\cosh \alpha_{1n} - \cos \alpha_{2n}} \right) \left(\sinh \alpha_{1n} \bar{x} - \frac{\alpha_{1n}}{\alpha_{2n}} \sin \alpha_{2n} \bar{x} \right) \quad (27e)$$

for a guided-guided beam

$$\bar{v}_n(\bar{x}) = \cos(n\pi\bar{x}) \quad (27f)$$

for a guided-pinned beam

$$v_n(\bar{x}) = \cos \left\{ \left(\frac{2n-1}{2} \right) \pi \bar{x} \right\} \quad (27g)$$

and for a pinned-pinned beam

$$\bar{v}_n(\bar{x}) = \sin(n\pi\bar{x}) \quad (27h)$$

It can readily be shown that the mode shapes for the cantilevered beams with a tip mass are the same as those given by equation (27c).

RESULTS AND DISCUSSION

Uniform Beam Under Axial Load

To determine the effect of axial load on the natural frequencies of a uniform beam, the characteristic equations (given by eqs. (15a) to (15g)) were solved numerically by the method of bisection (ref. 9). In this investigation the axial load ratio P/P_{cr} was varied between -1 and 1, and the first three lowest frequencies were determined. The results for the fundamental elastic body frequencies are presented in figure 4. The pinned-pinned, guided-guided, and pinned-guided curves in this figure were developed from equations (16) and (17). In this figure the curves for the free-free and clamped-clamped beams appear as a single curve. Although the frequencies for these two cases were not identically equal over the entire range of P/P_{cr} , the differences could not be shown for the scale used in the figure. The same is true for the free-guided and clamped-guided cases. The variation in frequency as a function of axial load for the first few lower frequencies for each uniform beam case is given in figure 5. It can be noted that the variation in the second and third frequency for all cases shown is almost linear for the range of P/P_{cr} considered. These higher mode frequencies will go to zero when the second and third buckling loads are reached. However, these portions of the curves are of no practical interest unless the beam is prevented from buckling in its fundamental mode.

To show the effect of axial loading on the mode shapes of the beam, equations (27) were used to determine the modes for $P/P_{cr} = -1.0, 0, \text{ and } 0.8$. These calculations were normalized, and the results are shown in figures 6 to 12 for the first three modes. These figures indicate that the effect of axial load is greatest on the fundamental mode and that the effect decreases rapidly as the mode number increases. The most

pronounced effect occurs for the free-pinned beam in its fundamental mode (fig. 8(a)). Note, however, that all the cases exhibit this effect as P/P_{cr} approaches 1.

Cantilevered Beam With Tip Mass

For a cantilevered beam with a tip mass equation (22) was solved for the case of a constant axial load and equation (24) for an axial load directed through the root of the beam. The method of solution of these equations was the same as that used for the uniform beams. The effects of both tip mass ratio, $\eta_T = \mathcal{M}_T/\mathcal{M}_B$, and axial load ratio P/P_{cr} on the fundamental frequencies were determined, and the results are shown in figures 13 to 16. Figure 13 shows the variation in the frequency parameters $\bar{\beta}_1^2 \sqrt{1 + \eta_T}$ as a function of tip mass ratio, η_T , for values of P/P_{cr} between -1 and 0.8. Note that $\bar{\beta}_1^2 \sqrt{1 + \eta_T}$ equals $\omega_1 \sqrt{(\mathcal{M}_T + \mathcal{M}_B)l^3/EI}$, so that the frequency parameter for this case is related to the total mass of the system and not just the beam mass. Figure 14 shows the same data plotted as a function of axial load ratio with the tip mass ratio as a parameter. Figures 15 and 16 are a similar set of graphs for the cantilever with a directed axial load.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, August 5, 1975,
506-22.

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TABLE I. - BOUNDARY CONDITIONS FOR UNIFORM BEAM UNDER CONSTANT AXIAL LOAD










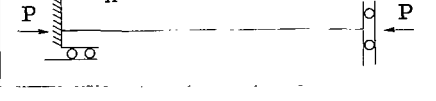
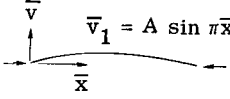
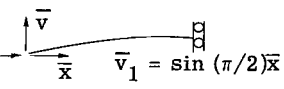
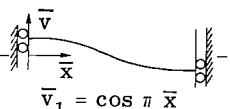
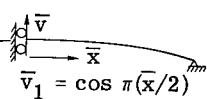
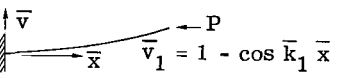
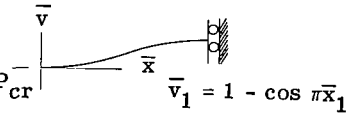
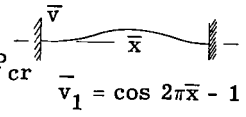
Free-free		$\bar{v}'''(0) = 0$ $\bar{v}''''(0) + \bar{k}^2 \bar{v}'(0) = 0$	$\bar{v}''(1) = 0$ $\bar{v}''''(1) + \bar{k}^2 \bar{v}'(1) = 0$
Free-guided		$\bar{v}''(0) = 0$ $\bar{v}''''(0) + \bar{k}^2 \bar{v}'(0) = 0$	$\bar{v}'(1) = 0$ $\bar{v}''''(1) + \bar{k}^2 \bar{v}'(1) = 0$
Free-pinned		$\bar{v}''(0) = 0$ $\bar{v}''''(0) + \bar{k}^2 \bar{v}'(0) = 0$	$v(1) = 0$ $v''(1) = 0$
Guided-guided		$\bar{v}'(0) = 0$ $\bar{v}''''(0) + \bar{k}^2 \bar{v}'(0) = 0$	$\bar{v}'(1) = 0$ $\bar{v}''''(1) + \bar{k}^2 \bar{v}'(1) = 0$
Guided-pinned		$\bar{v}'(0) = 0$ $\bar{v}''''(0) + \bar{k}^2 \bar{v}'(0) = 0$	$\bar{v}(1) = 0$ $\bar{v}''(1) = 0$
Clamped-free		$\bar{v}(0) = 0$ $\bar{v}'(0) = 0$	$\bar{v}''(1) = 0$ $\bar{v}''''(1) + \bar{k}^2 \bar{v}'(1) = 0$
Pinned-pinned		$\bar{v}(0) = 0$ $\bar{v}''(0) = 0$	$\bar{v}(1) = 0$ $\bar{v}''(1) = 0$
Clamped-pinned		$\bar{v}(0) = 0$ $\bar{v}'(0) = 0$	$v(1) = 0$ $v''(1) = 0$
Clamped-guided		$\bar{v}(0) = 0$ $\bar{v}'(0) = 0$	$\bar{v}'(1) = 0$ $\bar{v}''''(1) + \bar{k}^2 \bar{v}'(1) = 0$
Clamped-clamped		$\bar{v}(0) = 0$ $\bar{v}'(0) = 0$	$\bar{v}(1) = 0$ $\bar{v}'(1) = 0$

TABLE II. - BUCKLING CHARACTERISTICS OF SIMPLE BEAMS

Boundary conditions	Characteristic equation	Eigenvalue, \bar{k}_n	Critical load, P_{cr}	Buckling mode
Free-free	$\sin \bar{k}_n = 0$	$n\pi$	$\frac{\pi^2 EI}{l^2}$	 $\bar{v}_1 = A \sin \pi \bar{x}$
Free-guided	$\cos \bar{k}_n = 0$	$\frac{(2n-1)\pi}{2}$	$\frac{\pi^2 EI}{4 l^2}$	 $\bar{v}_1 = \sin(\pi/2)\bar{x}$
Free-pinned	$\sin \bar{k}_n = 0$	$n\pi$	$\frac{\pi^2 EI}{l^2}$	(Same as free-free)
Guided-guided	$\sin \bar{k}_n = 0$	$n\pi$	$\frac{\pi^2 EI}{l^2}$	 $\bar{v}_1 = \cos \pi \bar{x}$
Guided-pinned	$\cos \bar{k}_n = 0$	$\frac{(2n-1)\pi}{2}$	$\frac{\pi^2 EI}{4 l^2}$	 $\bar{v}_1 = \cos \pi(\bar{x}/2)$
Clamped-free	$\cos \bar{k}_n = 0$	$\frac{(2n-1)\pi}{2}$	$\frac{\pi^2 EI}{4 l^2}$	 $\bar{v}_1 = 1 - \cos \bar{k}_1 \bar{x}$
Pinned-pinned	$\sin \bar{k}_n = 0$	$n\pi$	$\frac{\pi^2 EI}{l^2}$	(Same as free-free)
Clamped-pinned	$\tan \bar{k}_n = \bar{k}_n$	4.493	$\frac{\pi^2 EI}{(0.699l)^2}$	
Clamped-guided	$\sin \bar{k}_n = 0$	$n\pi$	$\frac{\pi^2 EI}{l^2}$	 $\bar{v}_1 = 1 - \cos \pi \bar{x}_1$
Clamped-clamped	$\sin \frac{\bar{k}_n}{2} = 0$	$2n\pi$	$\frac{4\pi^2 EI}{l^2}$	 $\bar{v}_1 = \cos 2\pi \bar{x} - 1$

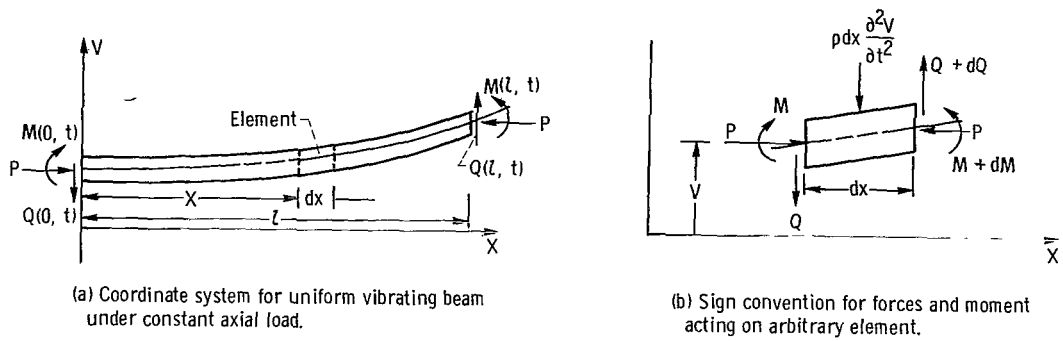


Figure 1. - Vibrating beam with arbitrary boundary conditions.

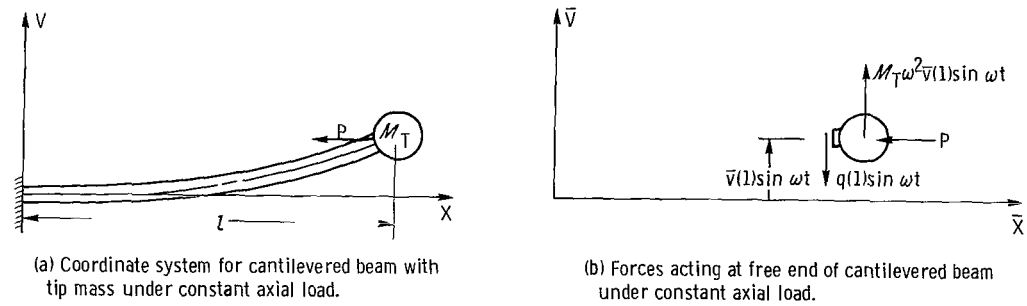


Figure 2. - Cantilevered beam with tip mass under constant axial load.

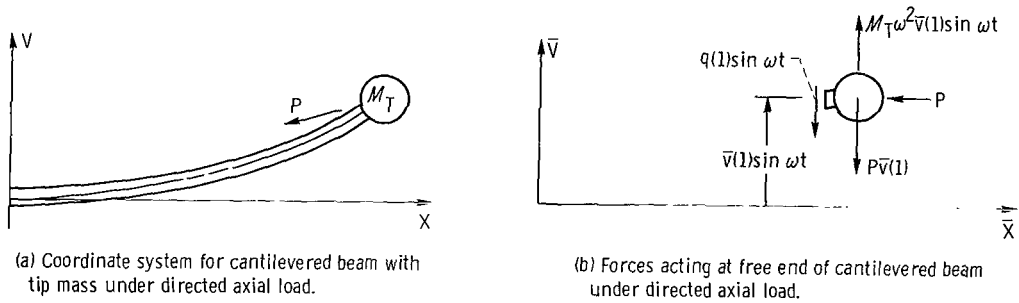


Figure 3. - Cantilevered beam with tip mass under directed axial load.

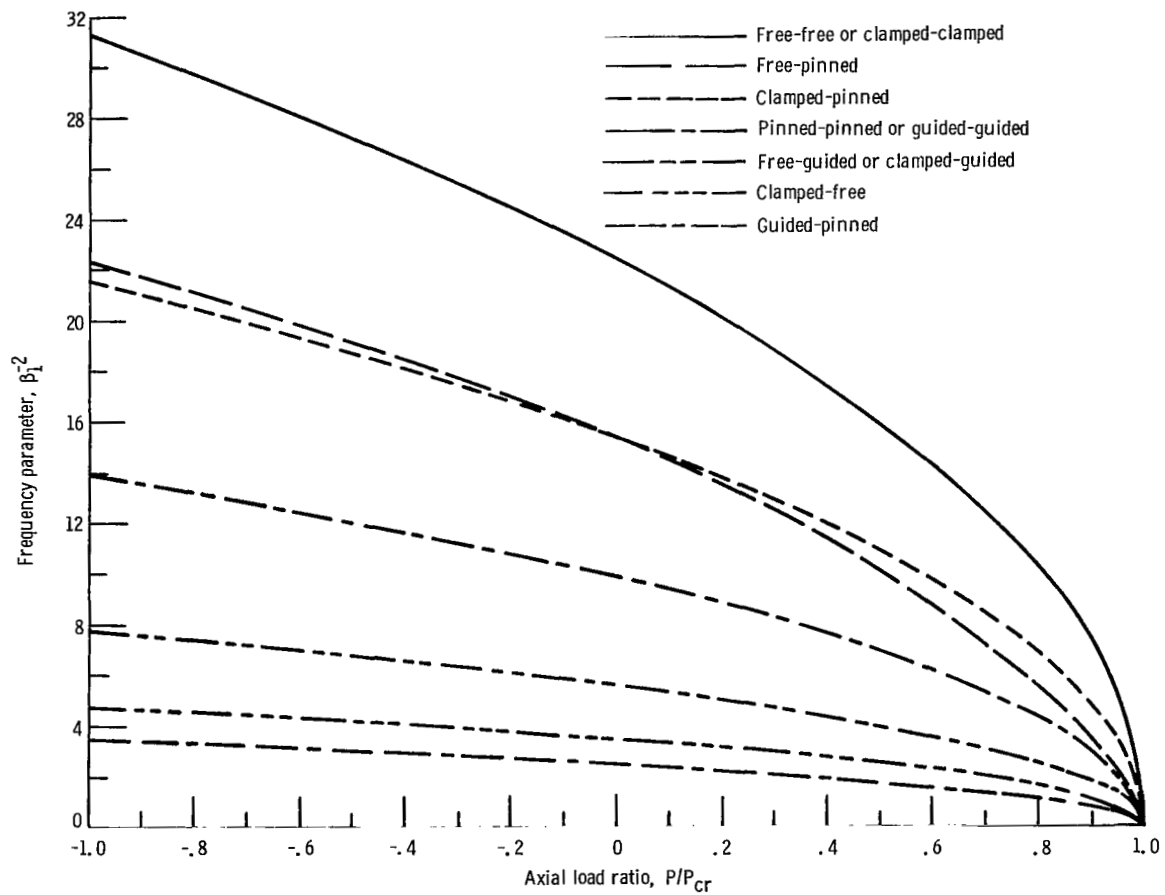


Figure 4. - Fundamental frequency as function of axial load for uniform beam with various boundary conditions.

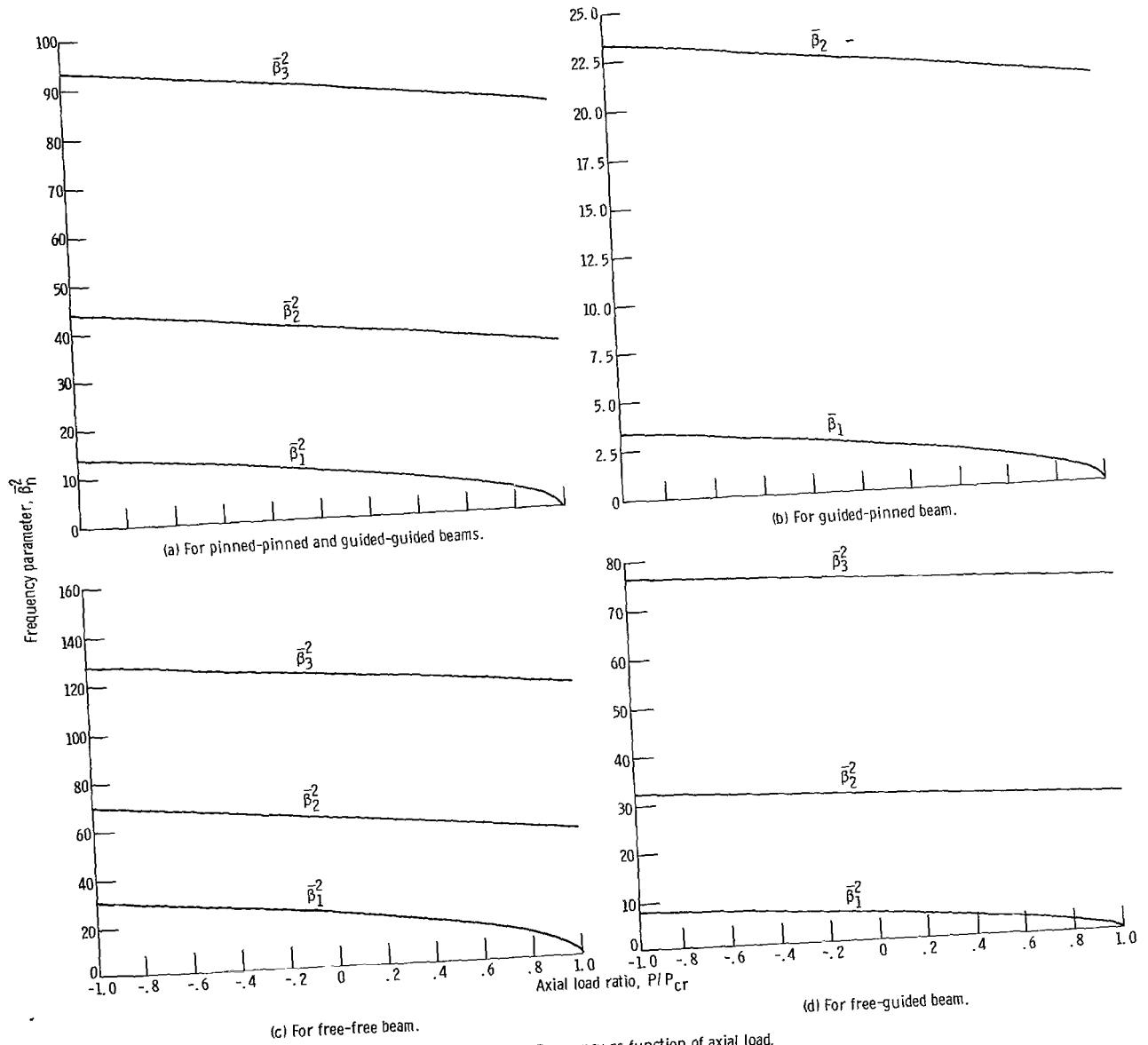


Figure 5. - Frequency as function of axial load.

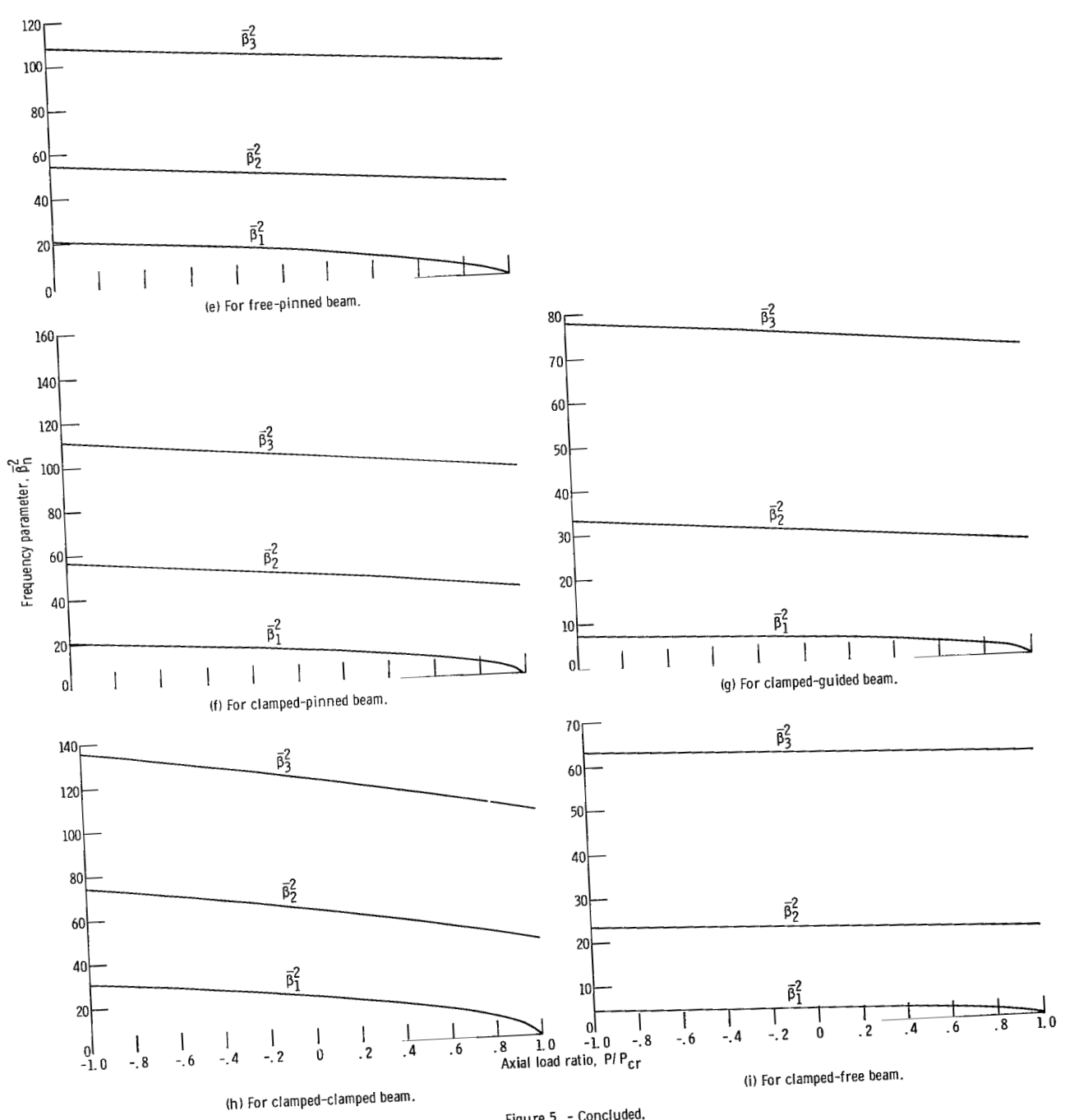


Figure 5. - Concluded.

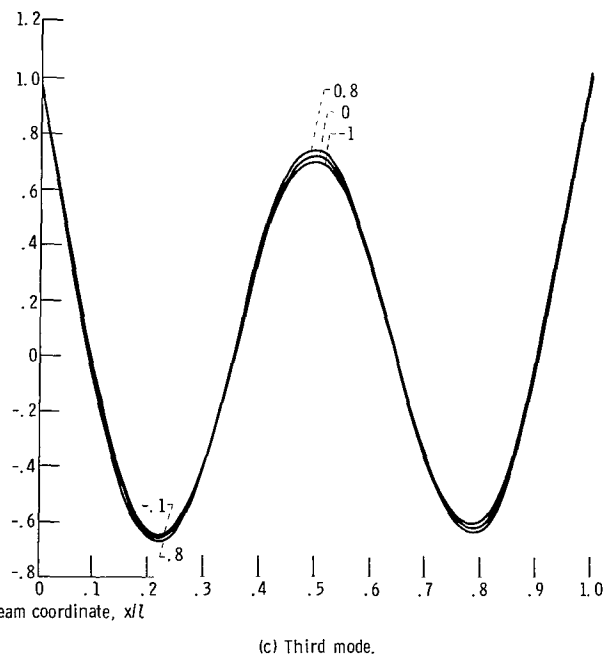
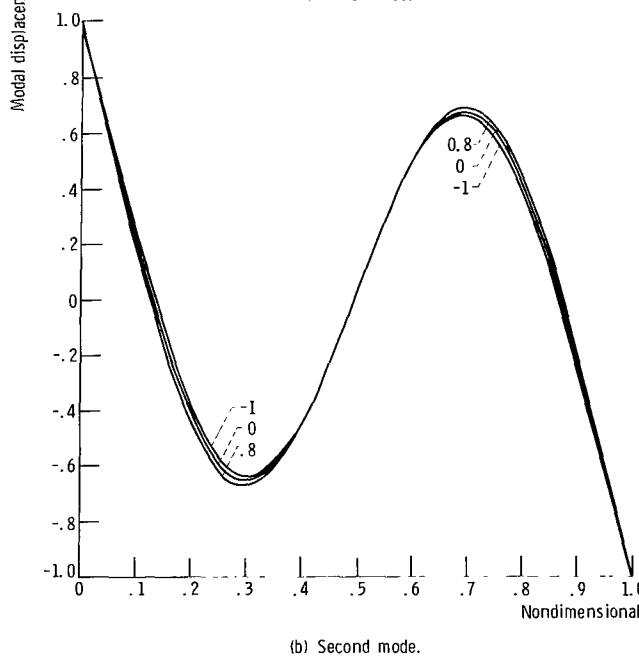
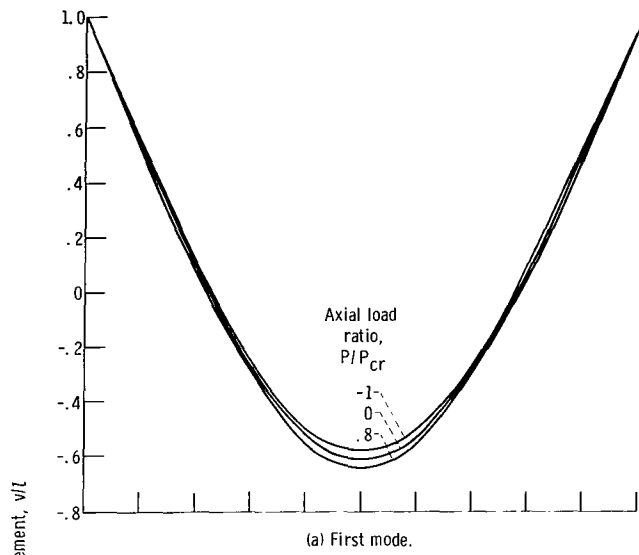


Figure 6. - Effect of axial load on mode shape of free-free beam.

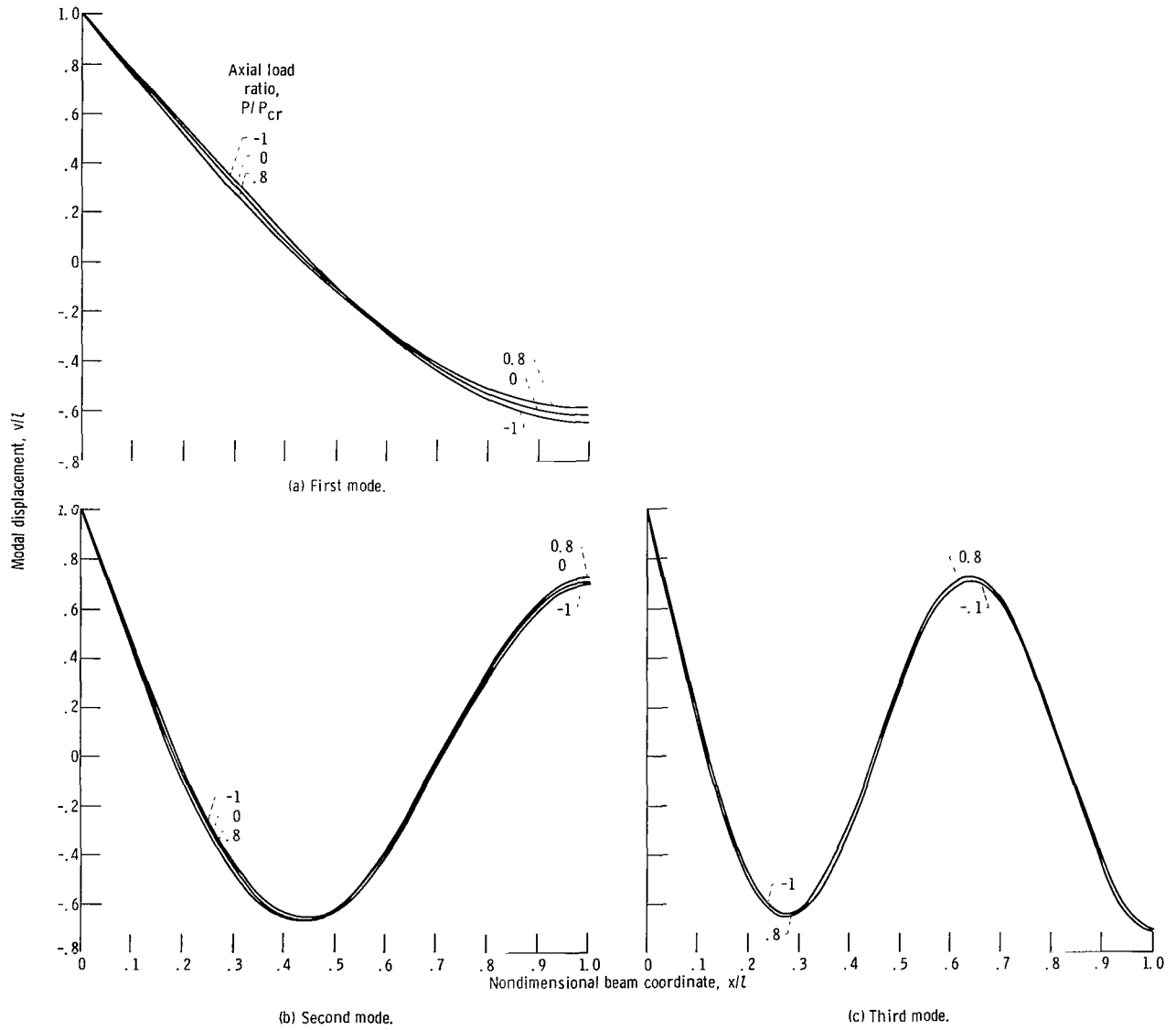


Figure 7. - Effect of axial load on mode shape of free-guided beam.

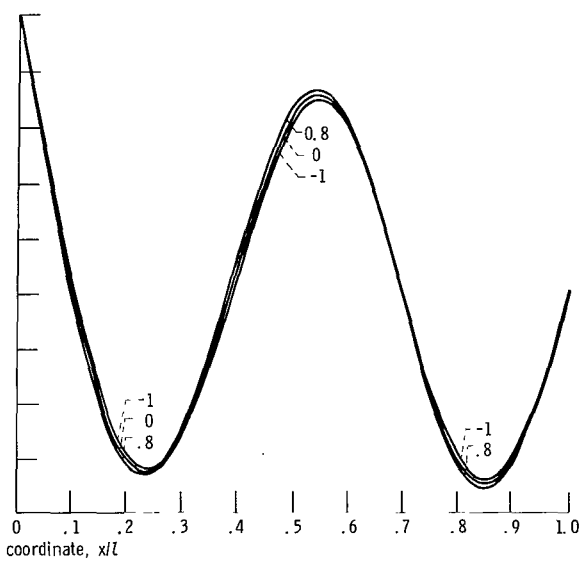
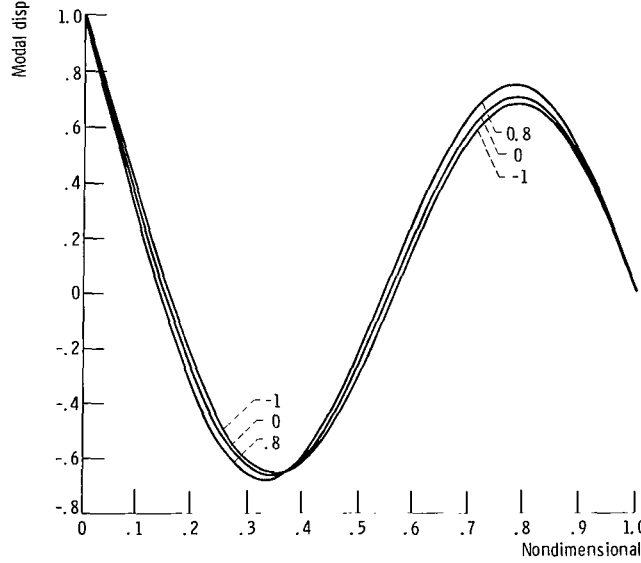
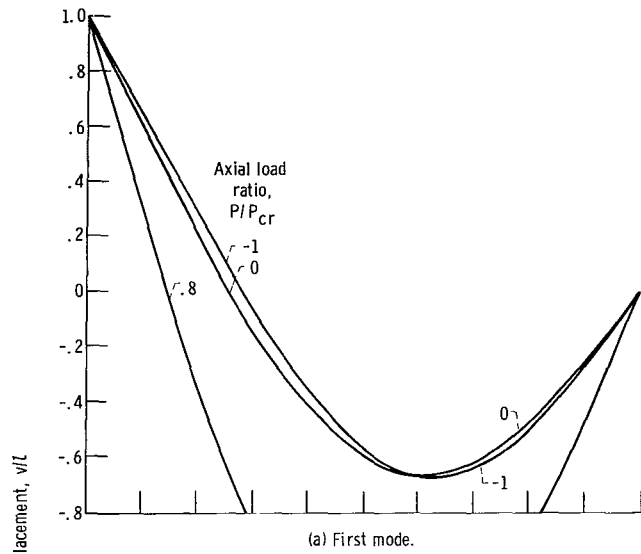


Figure 8. - Effect of axial load on mode shape of free-pinned beam.

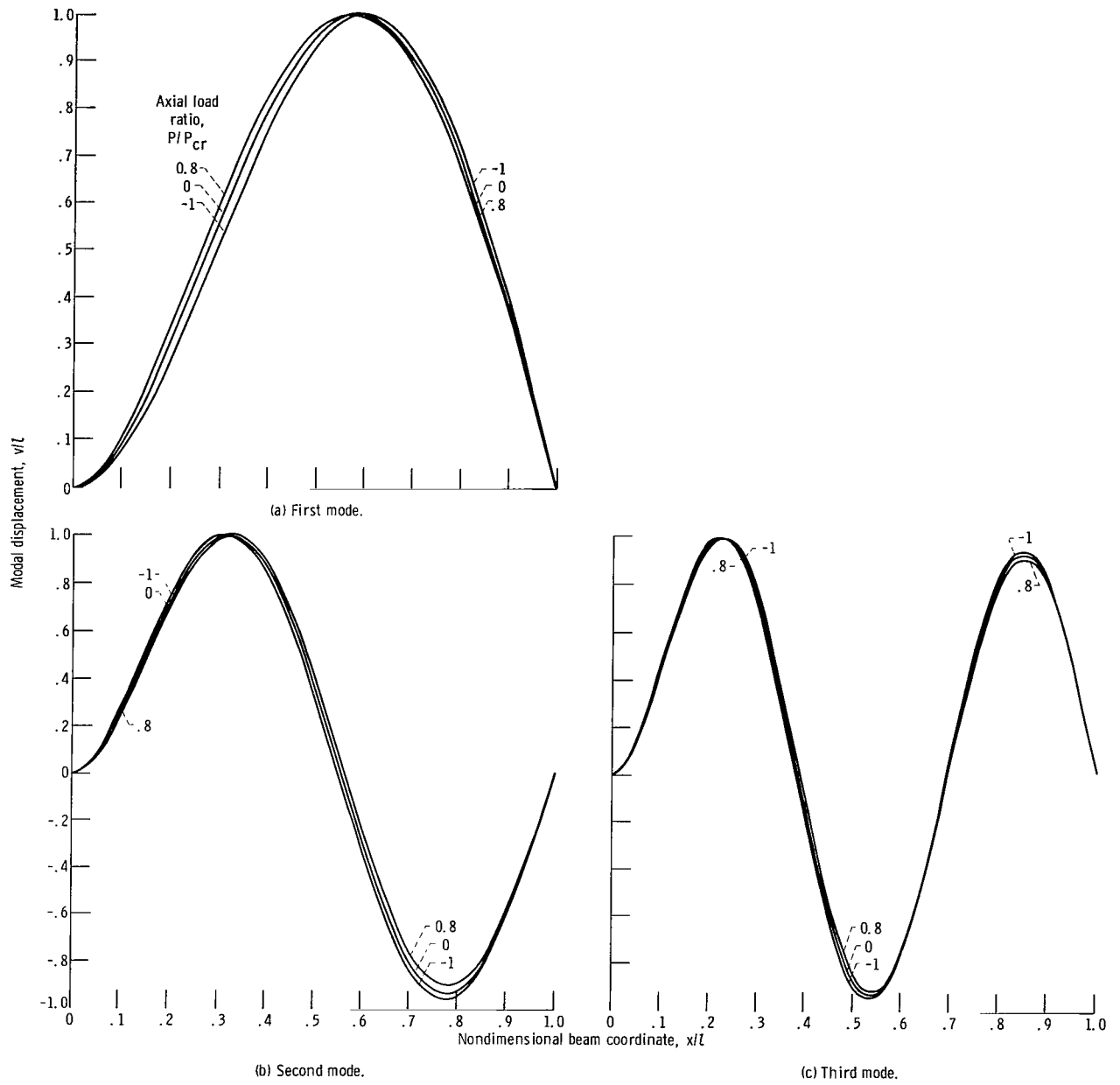


Figure 9. - Effect of axial load on mode shape of clamped-pinned beam.

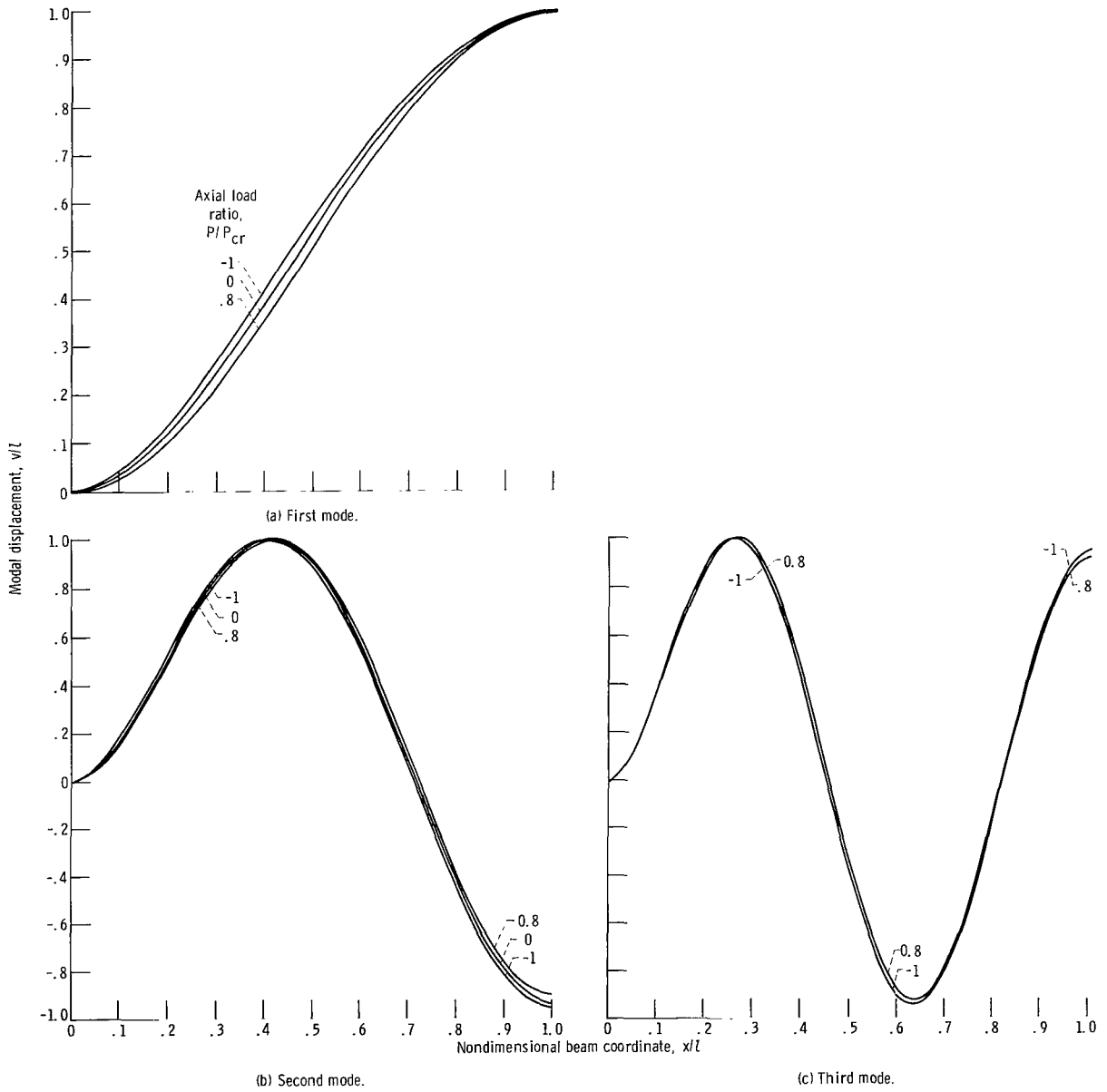


Figure 10. - Effect of axial load on mode shape of clamped-guided beam.

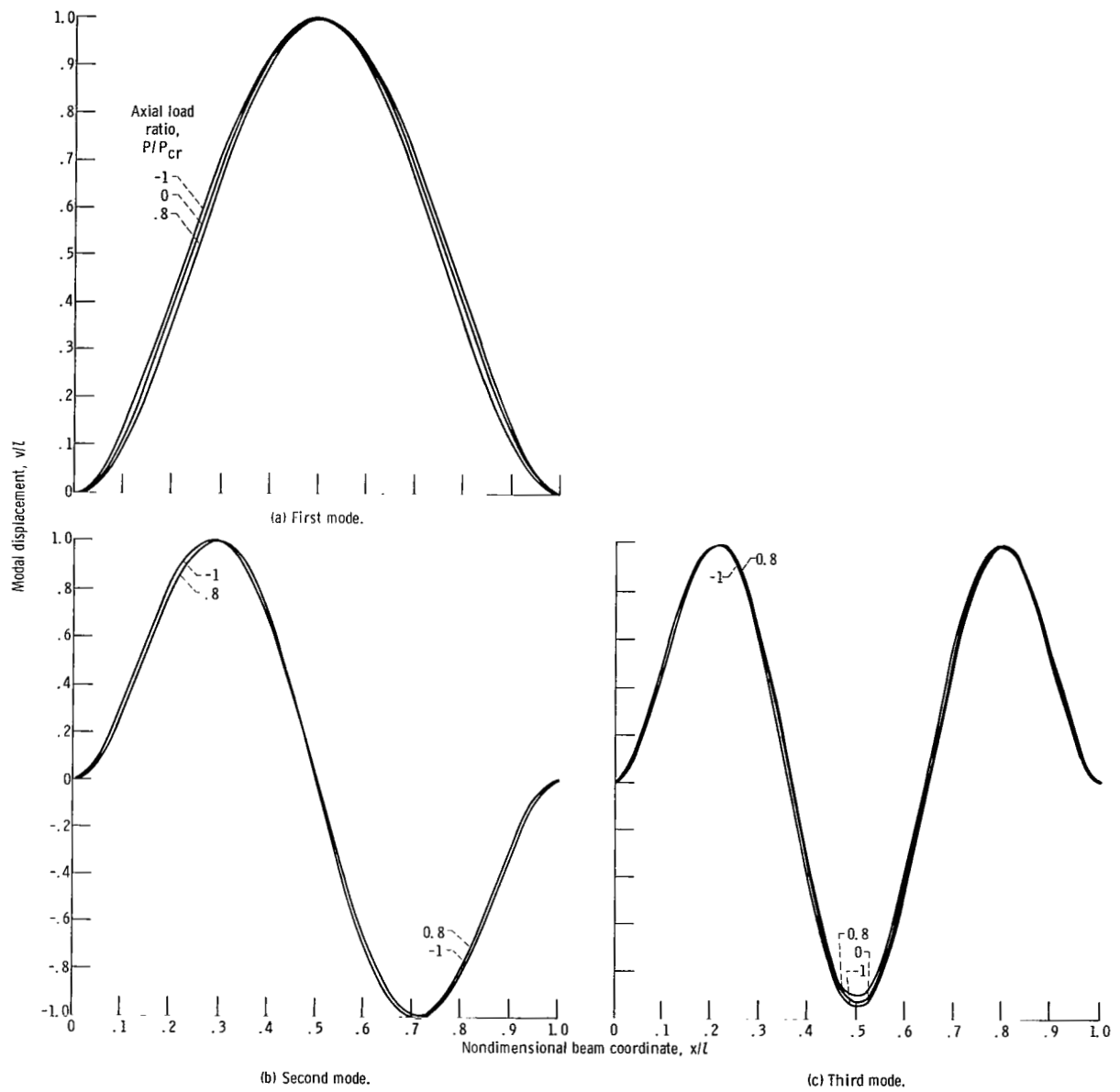
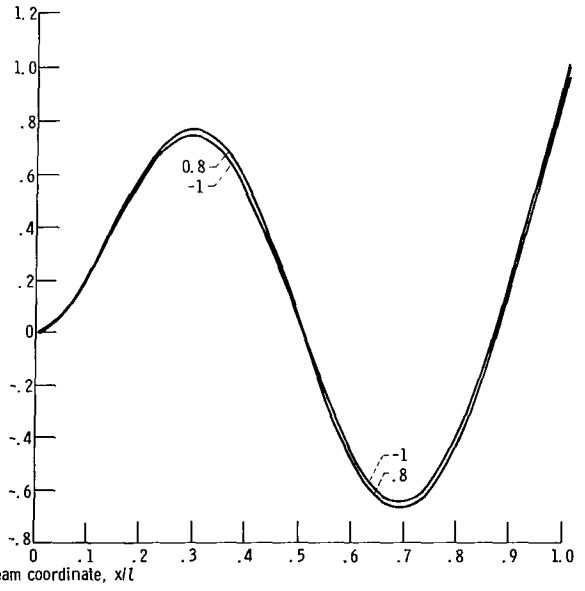
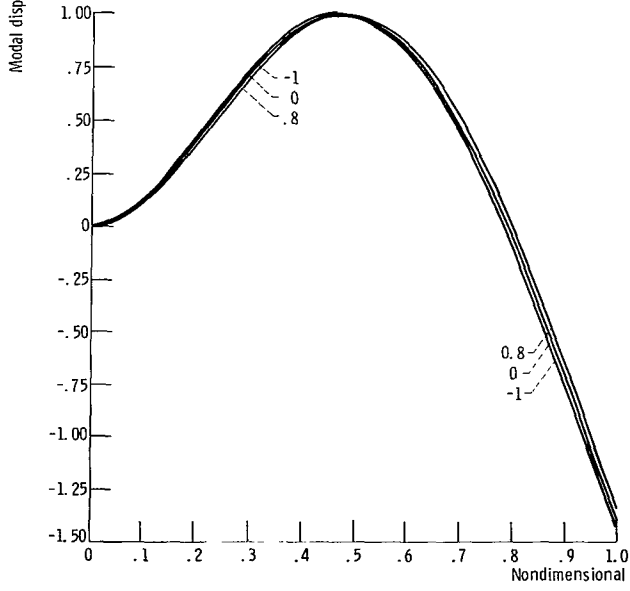
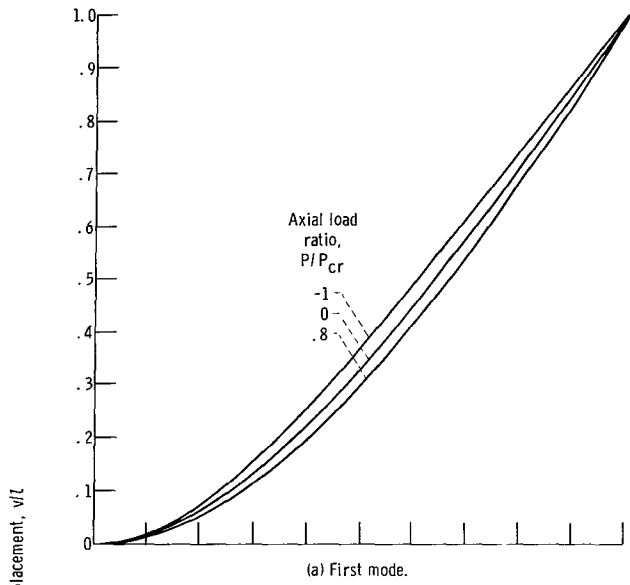


Figure 11. - Effect of axial load on mode shape of clamped-clamped beam.



(a) First mode. (b) Second mode. (c) Third mode.

Figure 12. - Effect of axial load on mode shape of clamped-free beam.

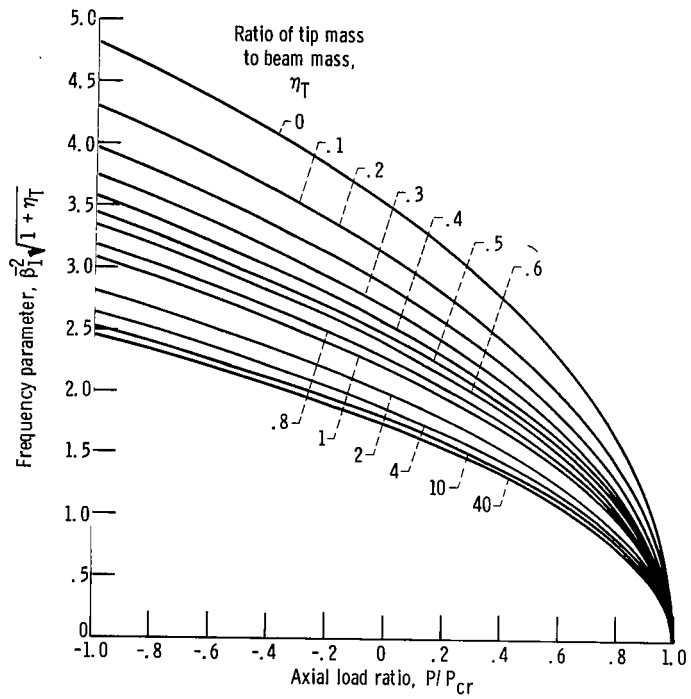


Figure 14. - Frequency as function of cantilever beam with tip mass.

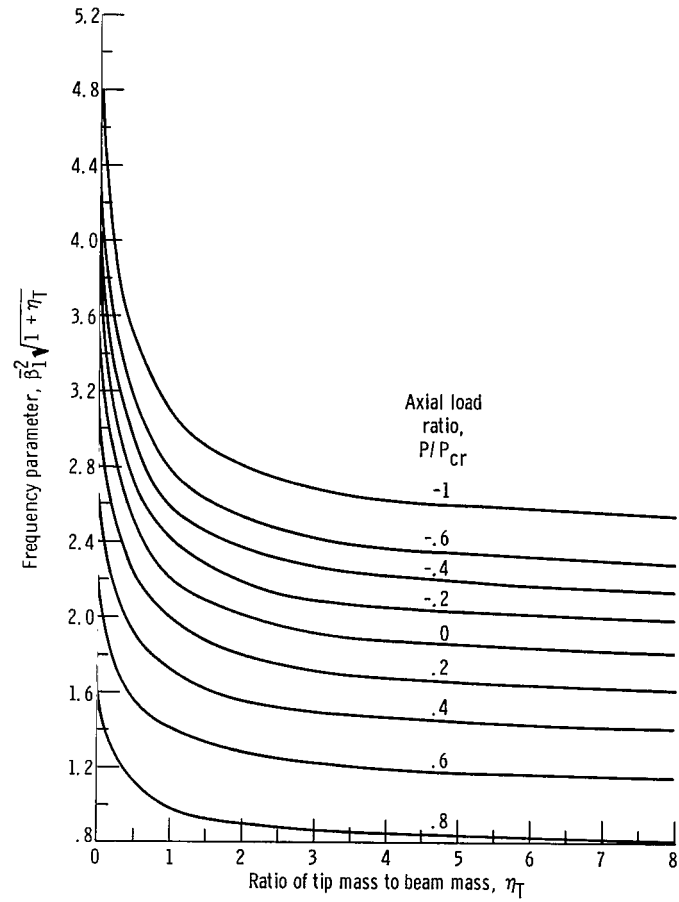


Figure 13. - Fundamental frequency of cantilevered beam with tip mass under constant axial load.

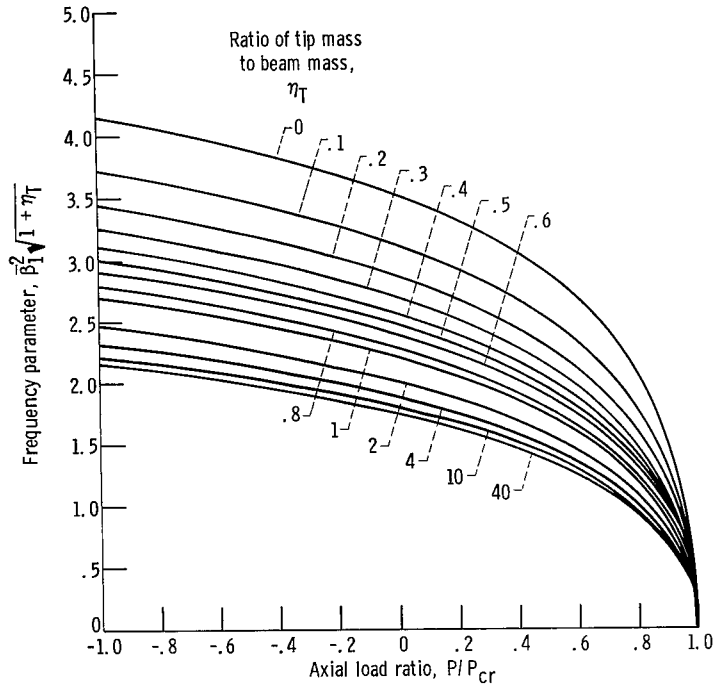


Figure 16. - Frequency as function of axial load for cantilevered beam with tip mass.

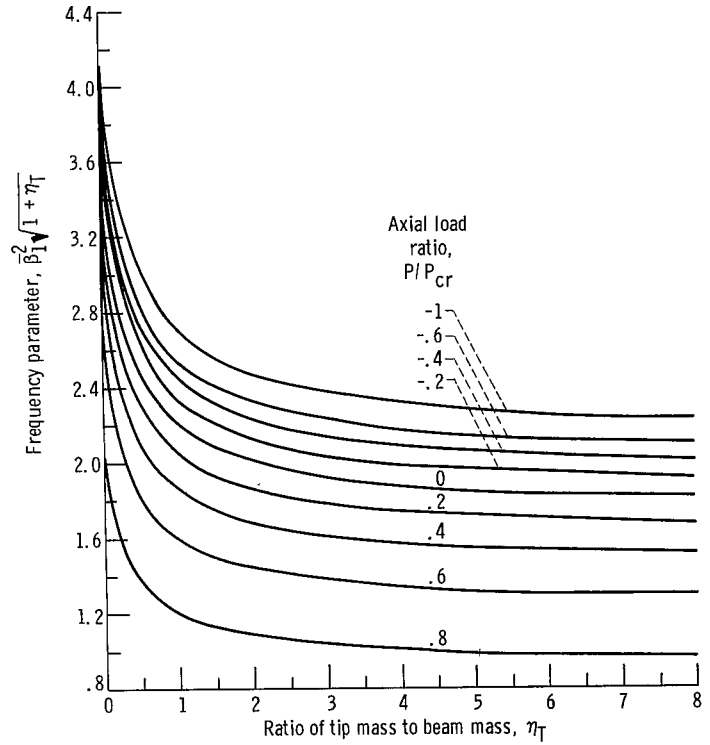


Figure 15. - Fundamental frequency of cantilevered beam with tip mass and axial load directed through root.

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