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**AN EFFECTIVE SOLUTION TO THE NONLINEAR, NON-  
STATIONARY NAVIER-STOKES EQUATIONS FOR  
TWO DIMENSIONS**

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# An Effective Solution to the Nonlinear, Nonstationary Navier-Stokes Equations for Two Dimensions

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## 1. Introduction

Classical fluid flow problems of aerodynamics stem from the nonlinear, nonstationary Navier-Stokes equations. When solving these problems for cases in which simplification is unacceptable, the practice has been to proceed in one of two ways: (1) the problem is considered within the framework of the Reynolds equations, or (2) the Navier-Stokes equations are dealt with numerically. There are, however, significant disadvantages to both the Reynolds equations approach and to existing numerical solutions. Relative to (1), recent experimental evidence indicates essential information about the flow dynamics is lost in the process of generating the Reynolds equations [1, p. 325], and relative to (2), existing numerical solutions of the Navier-Stokes equations completely lack explicit error estimates.

Whereas the development of more effective techniques for solving such fluid flow problems by means of the Reynolds equations is highly unlikely, future progress seems to lie in the direction of improved methods of handling such problems numerically.

Towards this end, a numerical solution with explicit error estimates is presented for the Navier-Stokes equations in two dimensions with suitable boundary conditions.

The solution, based on Newton's method as generalized by Kantorovich and Akilov [2],[3], is the limit of a sequence of functions that are solutions to a sequence of related linear partial differential equations.

Background material, preliminary results, principal development of the solution, and conclusions are presented in the sections that follow.

## 2. Background

The nonlinear, nonstationary Navier-Stokes problem is mathematically described by

$$\left. \begin{aligned} \Delta\psi_t - \nu\Delta\Delta\psi &= \psi_y\Delta\psi_x - \psi_x\Delta\psi_y + f_{1y} - f_{2x} \\ \psi(\partial S) &= b_1, \quad \Delta\psi(\partial S) = b_2, \quad \psi(S)|_{t=0} = \phi(S), \end{aligned} \right\} \quad (1)$$

where  $S$  is the interior of a compact two-dimensional region, and  $\partial S$  its boundary.

It will be assumed that  $f_1 \in C_0^1$ ,  $f_2 \in C_0^1$ , where  $C_T^N(S) \equiv \{f(x,y,t) | N \text{ times continuously differentiable in } (x,y) \in S, T \text{ times continuously differentiable in } t, 0 \leq t < \infty, \text{ inclusive of spatial derivatives up through order } [N/2]\}$ . Note that  $[ ]$  denotes greatest integer part.  $C_T^N(S)$  will be considered under the norm

$$\begin{aligned} \|\psi\|_{C_T^N} &= \sum_{i=0}^T \sum_{n=0}^{[N/2]} \sum_{m=0}^n \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} \left| \frac{\partial^{nti} \psi}{\partial t^i \partial x^m \partial y^{n-m}} \right| \\ &+ \sum_{n=[N/2]+1}^N \sum_{m=0}^n \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} \left| \frac{\partial^n \psi}{\partial x^m \partial y^{n-m}} \right|. \end{aligned}$$

Let

$$P(\psi) \equiv \Delta\psi_t - \nu\Delta\Delta\psi - \psi_y\Delta\psi_x + \psi_x\Delta\psi_y + f_{2x} - f_{1y}.$$

Equation (1) can be expressed in the equivalent form:

$$P(\psi) = 0, \quad \psi(\partial S) = b_1, \quad \Delta\psi(\partial S) = b_2, \quad \psi(S)|_{t=0} = \phi(S). \quad (2)$$

$P$  will be considered a mapping from  $C_1^4(S)$  into  $C_0^0(S)$ .

### 3. Preliminary Results

Lemma 1. The Fréchet derivative of  $P$  exists, and is given by

$$P'(\psi_0) = \Delta \frac{\partial}{\partial t} - \nu \Delta \Delta - \psi_{0y} \Delta \frac{\partial}{\partial x} - \Delta \psi_{0x} \frac{\partial}{\partial y} + \psi_{0x} \Delta \frac{\partial}{\partial y} + \Delta \psi_{0y} \frac{\partial}{\partial x}.$$

Proof. The Fréchet derivative is defined as follows:

$$\text{if } \lim_{\|a\|_{C_1^4} \rightarrow 0} \frac{\|P(\psi + a) - P(\psi) - La\|_{C_0^0}}{\|a\|_{C_1^4}} = 0 \text{ for some linear operator } L, \text{ then}$$

$P'(\psi_0) = L$ . Hence it can be directly shown that for  $\psi_0 \in C_1^4(S)$ ,  $P'(\psi_0)$  exists, and

$$P'(\psi_0) = \Delta \frac{\partial}{\partial t} - \nu \Delta \Delta - \psi_{0y} \Delta \frac{\partial}{\partial x} - \psi_{0x} \frac{\partial}{\partial y} + \psi_{0x} \Delta \frac{\partial}{\partial y} + \Delta \psi_{0y} \frac{\partial}{\partial x}.$$

(See ref. [4].)

Lemma 2.  $P''(\psi)$  exists for all  $\psi \in C_1^4(S)$ , and

$$P''(\psi_0)\phi\theta = \phi_y \Delta \theta_x + \theta_y \Delta \phi_x - \theta_x \Delta \phi_y - \phi_x \Delta \theta_y.$$

In addition,

$$\|P''(\psi_0)\| \leq 1.$$

Proof. By definition, if there exists a bilinear operator  $B$  such that

$$\lim_{\|\phi\|_{C_1^4} \rightarrow 0} \frac{\|P'(\psi_0 + \phi) - P'(\psi_0) - B\phi\|_{C_0^0}}{\|\phi\|_{C_1^4}} = 0,$$

then  $P''(\psi_0)$  exists and  $P''(\psi_0) = B$ . Hence it follows directly that

$$P''(\psi_0)\phi\theta = \phi_y \Delta \theta_x + \theta_y \Delta \phi_x - \theta_x \Delta \phi_y - \phi_x \Delta \theta_y$$

and that  $\|P''\| \leq 1$  [4].

In order to show that the "Newton-Kantorovich Algorithm"

$$\left. \begin{aligned} 0 &= P(\psi_m) + P'(\psi_m)(\psi_{m+1} - \psi_m) \\ \psi_m(\partial S) &= b_1, \quad \Delta\psi_m(\partial S) = b_2 \\ \psi_m|_{t=0} &= \phi(S), \quad m = 0, 1, 2, \dots, \end{aligned} \right\} \quad (3)$$

converges to a solution of Eq. (2), it is sufficient to show [2],[3] that

$$||P(\psi_0)|| \quad ||P'(\psi_0)^{-1}||^2 \quad ||P''(\psi_0)|| < \frac{1}{2}.$$

Since for a given  $\psi_0 \in C_1^4$ ,  $||P(\psi_0)||$  is readily calculated and

$||P''(\psi_0)|| \leq 1$ , it remains to calculate an upper estimate for  $||P'(\psi_0)^{-1}||$ .

#### 4. Main Development

Since, for a given  $\psi_0 \in C_1^4(S)$ , the sequence of  $\psi_n$ 's arising from Eq. (3) converges to  $\psi^*$ , the solution of Eq. (1), providing

$$||P(\psi_0)|| \quad ||P'(\psi_0)^{-1}||^2 \quad ||P''(\psi_0)|| \leq \frac{1}{2},$$

it remains to determine if and when  $P'(\psi_0)^{-1}$  exists, and estimate

$$||P'(\psi_0)^{-1}||.$$

To this end, let  $m = 0$  in Eq. (3). Therefore,

$$0 = P(\psi_0) + P'(\psi_0)(\psi_1 - \psi_0) \text{ in } C_0^0;$$

$$\psi_0(\partial S) = b_1, \quad \psi_1(\partial S) = b_1, \quad \Delta\psi_0(\partial S) = b_2, \quad \Delta\psi_1(\partial S) = b_2,$$

$$\psi_0|_{t=0} = \phi(S), \text{ and } \psi_1|_{t=0} = \phi(S).$$

Let  $\bar{\psi} = \psi_1 - \psi_0$ ; therefore,  $-P(\psi_0) = P'(\psi_0)\bar{\psi}$ . If  $\bar{\psi}$  exists, then

$$\bar{\psi} = -P'^{-1}(\psi_0)P(\psi_0).$$

Hence, we seek a solution  $\bar{\psi}$  of the problem

$$0 = P(\psi_0) + P'(\psi_0)\bar{\psi}, \quad \bar{\psi}(\partial S) = 0, \quad \Delta\bar{\psi}(\partial S) = 0, \text{ and } \bar{\psi}|_{t=0} = 0. \quad (4)$$

Let

$$P'(\psi_0) = \Delta \frac{\partial}{\partial t} - \nu \Delta \Delta + A,$$

where  $A$  is a linear operator mapping  $C_0^3$  into  $C_0^0(S)$ .

$$A(\psi_0)\phi = -\psi_{0y}\Delta\phi_x - \Delta\psi_{0x}\phi_y + \psi_{0x}\Delta\phi_y + \Delta\psi_{0y}\phi_x$$

Equation (4) can therefore be expressed by

$$\Delta\bar{\psi}_t - \nu\Delta\bar{\psi} = -P(\psi_0) - A(\psi_0)\bar{\psi}, \quad \bar{\psi}(\partial S) = 0, \quad \Delta\bar{\psi}(\partial S) = 0, \quad \bar{\psi}|_{t=0} = 0. \quad (5)$$

Let  $\bar{\bar{\psi}}$  be defined as follows:

$$\begin{aligned} \bar{\bar{\psi}}(x,y,t) = & -\int_S \int_0^t \int_S \left\{ [A(\psi_0)\bar{\bar{\psi}}](x'',y'',\tau) \right. \\ & \left. + P(\psi_0)(x'',y'',\tau) \right\} G^\nu(x,y,x',y',x'',y'',t-\tau) dx'' dy'' d\tau dx' dy', \end{aligned}$$

where

$$G^\nu(x,y,x',y',x'',y'',t) = G(x,y,x',y')H^\nu(x',y',x'',y'',t).$$

Lemma 3. If  $\bar{\bar{\psi}}$  of Eq. (6) exists, then  $\bar{\bar{\psi}}$  satisfies Eq. (4).

Proof. Since

$$G^\nu(x,y,x',y',x'',y'',t) = G(x,y,x',y')H^\nu(x',y',x'',y'',t)$$

where  $G$  is the Green's Function for Laplace's equation, and  $H^\nu$  the kernel function for the heat equation

$$\phi_t - \nu\Delta\phi = 0, \quad \phi(\partial S) = 0, \quad \phi|_{t=0} = 0,$$

it readily follows that

$$\bar{\bar{\psi}}(\partial S) = 0, \quad \Delta\bar{\bar{\psi}}(\partial S) = 0, \quad \text{and} \quad \bar{\bar{\psi}}|_{t=0} = 0.$$

Also,

$$\Delta\bar{\bar{\psi}}(x,y,t) = -\int_0^t \int_S [A(\psi_0)\bar{\bar{\psi}}(x'',y'',\tau) + P(\psi_0)(x'',y'',\tau)] H^\nu(x,y,x'',y'',t-\tau) dx'' dy'' d\tau.$$

$$\Delta\bar{\bar{\psi}}_t(x,y,t) = -\int_0^t \int_S [A(\psi_0)\bar{\bar{\psi}}(x'',y'',\tau) + P(\psi_0)] H_t^\nu(x,y,x'',y'',t-\tau) dx'' dy'' d\tau$$

$$- \int_S [A(\psi_0)\bar{\bar{\psi}}(x'',y'',t) + P(\psi_0)(x'',y'',t)] H^\nu(x,y,x'',y'',0) dx'' dy''.$$

$$\Delta\Delta\bar{\bar{\psi}}(x,y,\tau) = -\int_0^t \int_S [A(\psi_0)\bar{\bar{\psi}}(x'',y'',\tau) + P(\psi_0)(x'',y'',\tau)] \Delta H^\nu(x,y,x'',y'',t-\tau) dx'' dy'' d\tau.$$



Therefore,

$$\begin{aligned} \Delta \bar{\psi}_t - \nu \Delta \Delta \bar{\psi} &= - \int_0^t \int_S [A(\psi_0) + P(\psi_0)] [H_t^\nu(x, y, x'', y'', t - \tau) - \nu \Delta H^\nu] dx'' dy'' d\tau \\ &\quad - \int_S [A(\psi_0) \bar{\psi} + P(\psi_0)] H^\nu(x, y, x'', y'', 0) dx'' dy \\ &= -A(\psi_0) \bar{\psi} - P(\psi_0) . \end{aligned}$$

Sufficient conditions under which  $\bar{\psi}$  of Eq. (6) exists will now be developed. Define the linear mapping B:

$$Bq = - \int_S \int_0^t \int_S q(x'', y'', \tau) G^\nu(x, y, x', y', x'', y'', t - \tau) dx' dy' d\tau dx'' dy'' .$$

Therefore, Eq. (6) can be expressed in the form

$$\begin{aligned} \bar{\psi} - BA\bar{\psi} &= BP(\psi_0) , \\ (I - BA)\bar{\psi} &= BP(\psi_0) . \end{aligned}$$

Let  $\tilde{C}_0^3(S) \equiv$  the completion of  $C_0^3(S)$  under the norm

$$\begin{aligned} \|\psi\| &= \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} |\psi| + \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} |\psi_x| + \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} |\psi_y| \\ &\quad + \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} |\psi_{xx}| + \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} |\psi_{xy}| + \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} |\psi_{yy}| \\ &\quad + \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} |\Delta \psi_x| + \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} |\Delta \psi_y| . \end{aligned}$$

If (BA) is considered a mapping from  $\tilde{C}_0^3(S)$  into  $\tilde{C}_0^3(S)$ , and

$\|BA\| < 1$ , then  $(I - BA)^{-1}$  exists, and  $\bar{\psi} = (I - BA)^{-1}BP(\psi_0)$ . Therefore,

in order to show that  $\bar{\psi}$  exists, it is sufficient to show that

$\|B\| \|A\| < 1$ . To this end consider A as a mapping from  $\tilde{C}_0^3$  into  $C_0^0$ .

For  $\psi \in C_0^0$ ,  $\|\psi\|_{C_0^0} = \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} |\psi|$  ;

$$\begin{aligned}
\text{for } \psi \in \tilde{C}_0^3, \|\psi\|_{\tilde{C}_0^3} &= \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |\psi| + \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |\psi_x| + \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |\psi_y| \\
&+ \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |\psi_{yy}| + \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |\psi_{xx}| + \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |\psi_{xy}| \\
&+ \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |\Delta\psi_x| + \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |\Delta\psi_y|.
\end{aligned}$$

Since  $\|A\| = \sup_{\|\phi\|_{\tilde{C}_0^3} = 1} \|A\phi\|$  by definition,

$$A(\psi_0)\phi = -\psi_{0y} \Delta\phi_x - \Delta\psi_{0x} \phi_y + \psi_{0x} \Delta\phi_y + \Delta\psi_{0y} \phi_x,$$

and

$$\|A\phi\|_{C_0^0} = \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |A\phi| = \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} |-\psi_{0y} \Delta\phi_x - \Delta\psi_{0x} \phi_y + \psi_{0x} \Delta\phi_y + \Delta\psi_{0y} \phi_x|,$$

it follows that  $\|A\| \leq M_{\psi_0}$ , where

$$M_{\psi_0} = \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} (|\psi_{0y}|, |\psi_{0x}|, |\Delta\psi_{0x}|, |\Delta\psi_{0y}|).$$

Consider  $B$  as a mapping from  $C_0^0$  into  $\tilde{C}_0^3$ .  $\|B\| = \sup_{\|z\|_{C_0^0} = 1} (\|Bz\|_{\tilde{C}_0^3})$ ,

by definition. Therefore, since

$$\begin{aligned}
\|Bz\|_{\tilde{C}_0^3} &= \max |Bz| + \max |(Bz)_x| + \max |(Bz)_y| + \max |(Bz)_{xx}| + \max |(Bz)_{xy}| \\
&+ \max |(Bz)_{yy}| + \max |(\Delta Bz)_x| + \max |(\Delta Bz)_y|, \\
\|B\| &\leq \max_{\substack{x,y \in S \\ 0 \leq t < \infty}} \int_S \int_0^t \int_S H^\nu(x', y', x'', y'', t - \tau) [G(x, y, x', y') + |G_y| + |G_x| \\
&+ |G_{xx}| + |G_{yy}| + |G_{xy}|] dx'' dy'' d\tau dx' dy' \\
&+ \max \int_0^t \int_S [ |H_x^\nu(x, y, x'', y'', t - \tau)| + |H_y^\nu(x, y, x'', y'', t - \tau)| ] dx'' dy'' d\tau.
\end{aligned}$$

Let

$$N_B = \max_{\substack{x, y \in S \\ 0 \leq t < \infty}} \int_S \int_0^t \int_S H^v(x', y', x'', y'', t - \tau) (G + |G_x| + |G_y| + |G_{xx}| + |G_{yy}| + |G_{xy}|) dx'' dy'' d\tau,$$

and

$$M_{\psi_0} = \max(|\psi_0|, |\psi_{0x}|, |\Delta\psi_{0x}|, |\Delta\psi_{0y}|).$$

Therefore, for  $M_{\psi_0} N_B < 1$ ,  $\bar{\psi}$  exists, and  $\bar{\psi} = (I - BA)^{-1} BP(\psi_0)$ . Hence, from Lemma 3, it follows directly that  $-P'(\psi_0)^{-1} = (I - BA)^{-1} B$  and that

$$\|P'(\psi_0)^{-1}\| \leq \frac{\|B\|}{1 - \|B\| \|A\|} \leq \frac{N_B}{1 - N_B M_{\psi_0}}.$$

## 5. Conclusion

From the preceding sections it directly follows that if  $\psi_0$ , the initial educated guess, is judiciously chosen, a sequence of functions  $\psi_n$  ( $n = 1, 2, \dots$ ) can be directly constructed that converge to the exact solution of the nonlinear, nonstationary Navier-Stokes problem as expressed by Eq. (1). In addition, an explicit error estimate can be directly determined for each approximate solution  $\psi_n$ .

Explicitly then, for

$$\|P(\psi_0)\| < \frac{(1 - N_B M_{\psi_0})^2}{2N_B^2}$$

with  $M_{\psi_0} N_B < 1$ , the sequence of  $\psi_n$ 's determined by Eq. (3) exists and converges to the exact solution  $\psi^*$  of Eq. (1).

Moreover, under these conditions, the method yields the explicit error estimate

$$\|\psi^* - \psi_n\| \leq 2^{2^n - n} \left( \frac{N_B}{1 - M_{\psi_0} N_B} \right)^{2^{n+1} - 1} \|\mathbf{P}(\psi_0)\|^{2^n}.$$

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