

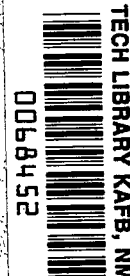
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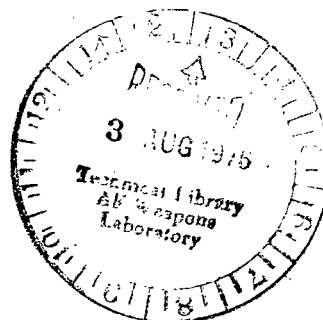
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ON THE FORMULATION OF THE AERODYNAMIC CHARACTERISTICS IN AIRCRAFT DYNAMICS

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16. Abstract The theory of functionals is used to reformulate the notions of aerodynamic indicial functions and superposition. Integral forms for the aerodynamic response to arbitrary motions are derived that are free of dependence on a linearity assumption. Simplifications of the integral forms lead to practicable nonlinear generalizations of the linear superposition and stability derivative formulations. Applied to arbitrary nonplanar motions, the generalization yields a form for the aerodynamic response that can be compounded of the contributions from a limited number of well-defined characteristic motions, in principle reproducible in the wind tunnel. Further generalizations that would enable the consideration of random fluctuations and multivalued aerodynamic responses are indicated.				13. Type of Report and Period Covered Technical Report	
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CONTENTS

	<u>Page</u>
FIGURE TITLES	v
NOTATION.	vi
1. INTRODUCTION.	1
2. COORDINATE SYSTEMS.	4
3. DEVELOPMENT OF INTEGRAL FORMS	8
3.1 Linear Aerodynamic Pitching-Moment Response.	8
3.2 Definition of a Functional	11
3.3 Nonlinear Aerodynamic Pitching-Moment Response	12
4. LIMITATIONS AND SIMPLIFICATION OF THE NONLINEAR INTEGRAL FORM.	16
4.1 Fundamental Assumptions.	17
4.2 Simplifying Assumptions.	18
5. APPLICATION OF THE SIMPLIFIED NONLINEAR INTEGRAL FORM TO PLANAR DYNAMIC STABILITY STUDIES	22
5.1 Stability Coefficient Formulation.	22
5.2 Interpretation of $C_{Z\delta}$	26
5.3 Theoretical Evaluation of Stability Coefficients	28
5.4 Experimental Evaluation of Stability Coefficients.	34
6. NONPLANAR MOTIONS	36
6.1 Approximate Formulation in the Aerodynamic Axis System .	37
6.2 Simplification of the Formulation in the Aerodynamic Axis System.	38

	<u>Page</u>
6.3 Approximate Formulation in the Body Axis System.	40
6.4 Simplification of the Formulation in the Body Axis System	44
6.5 Body of Revolution	47
6.6 Aircraft Spin Motions.	53
6.6.1 Aerodynamic Axis System	54
6.6.2 Body Axis System.	56
6.6.3 Spin Radius	57
6.7 Requirements for Experiments	58
7. CONCLUDING REMARKS.	60
REFERENCES.	62

FIGURES

<u>Figure</u>	<u>Title</u>	<u>Page</u>
1	Axes, angles, and velocity components in the crossflow and resultant angle-of-attack planes.	4
2	Summation of incremental responses.	9
3	Formation of indicial responses	14
4	Slowly varying motions.	20
5	Interpretation of $C_{Z\dot{\delta}}$	26
6	Definitive motions for calculation of stability coefficients.	28
7	Boundary conditions for indicial loading.	30
8	Indicial loading on two-dimensional wing at subsonic speed	31
9	Breakdown of indicial response.	32
10	Characteristic motions in the aerodynamic axis system. Linear dependence on angular rates.	40
11	Schematic representation of the equality between $(C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}})/\delta$ and the three damping coefficients	44
12	Characteristic motions in the body axis system. Linear dependence on angular rates	46
13	Variation of $\delta C_{m\dot{\sigma}}$ and $C_{n\dot{\phi}}$ with δ ; Mach number $= 2.0, l_{cg}/l = 0.61$	51
14	Maneuver showing origin of cross-coupling term $\hat{C}_{Y\dot{\alpha}}$	52
15	Characteristic motions in the aerodynamic axis system. Nonlinear dependence on coning rate	55
16	Characteristic motions in the body axis system. Nonlinear dependence on coning rate	57

NOTATION

\hat{C}_R	resultant force coefficient in the crossflow plane, $\hat{C}_R = -(\hat{C}_Y^2 + \hat{C}_Z^2)^{1/2}$
C_X	axial-force coefficient (along x_B), $2(\text{axial force})/\rho V^2 S$
C_Y	side-force coefficient in the aerodynamic axis system, (along y), $2(\text{side force})/\rho V^2 S$
C_Z	normal-force coefficient in the aerodynamic axis system (along z), $2(\text{normal force})/\rho V^2 S$
\hat{C}_Y, \hat{C}_Z	side-force and normal-force coefficients in the body axis system; along y_B, z_B , respectively
C_L	rolling-moment coefficient in the aerodynamic axis system (along x_B), $2\bar{L}/\rho V^2 S L$
C_m	pitching-moment coefficient in the aerodynamic axis system (along y), $2\bar{M}/\rho V^2 S L$
C_n	side-moment coefficient in the aerodynamic axis system (along z), $2\bar{N}/\rho V^2 S L$
$\hat{C}_L, \hat{C}_m, \hat{C}_n$	rolling, pitching, and yawing-moment coefficients in the body axis system; along x_B, y_B, z_B , respectively
ΔC_p	local loading coefficient (pressure coefficient on lower surface minus pressure coefficient on upper surface), Fig. 8
$G[\delta(\xi), \psi(\xi), \dot{\lambda}(\xi), q(\xi), r(\xi)]$	functional notation: value at $\xi = t$ of a time- dependent function which depends on all values taken by the five argument functions $\delta(\xi), \psi(\xi), \dot{\lambda}(\xi), q(\xi), r(\xi)$ over the time interval $0 \leq \xi \leq t$

\bar{L}	moment along longitudinal axis of body, Fig. 1
l	reference length
l_{cg}	distance along x_B from center of rotation to nose, Fig. 13
\bar{M}	moment along an axis normal to the plane of the resultant angle of attack (along y), Fig. 1
\bar{N}	moment along an axis in the plane of the resultant angle of attack (along z), Fig. 1
p_B, q_B, r_B	components along the x_B, y_B, z_B axes, respectively, of the total angular velocity of the body axes relative to inertial space
\tilde{q}, \tilde{r}	components of angular velocity along the \tilde{y}, \tilde{z} axes, respectively, Eq. (5)
q, r	components of angular velocity along the y, z axes, respectively, Eq. (6)
S	reference area
t	time
u_B, v_B, w_B	components of flight velocity along x_B, y_B, z_B axes, respectively, Fig. 1
V	magnitude of flight velocity vector
x_B, y_B, z_B	body-fixed axes, origin at mass-center, x_B coincident with a longitudinal axis of the body, Fig. 1
x_B, y, z	aerodynamic axes, origin at mass-center, x_B, z in the plane of the resultant angle of attack, y, z in the crossflow plane normal to the resultant angle-of- attack plane, Fig. 1

$x_B, \tilde{y}, \tilde{z}$	nonrolling axes (with respect to inertial space), origin at mass-center, \tilde{y}, \tilde{z} in the crossflow plane, Fig. 1
α	angle of attack in body axes, Eq. (7)
$\hat{\alpha}$	angle-of-attack parameter in body axis system, w_B/V
β	angle of sideslip in body axes, Eq. (7)
$\hat{\beta}$	angle-of-sideslip parameter in body axis system, v_B/V
γ	dimensionless axial component of velocity, Fig. 1 and Eq. (2)
δ	magnitude of the dimensionless crossflow velocity in the aerodynamic axis system, Fig. 1 and Eq. (2)
ϵ	$\tan \sigma$, Fig. 1 and Eq. (2)
λ	angular inclination from the \tilde{y} axis of the crossflow velocity vector, Fig. 1
ρ	atmospheric mass density
σ	resultant angle of attack defined by x_B axis and velocity vector, Fig. 1
$\dot{\phi}$	coning rate of x_B axis about the velocity vector of a body in level flight
$\tilde{\phi}$	angular inclination from the \tilde{y} axis of the y_B axis, Fig. 1 and Eq. (3)
ψ	angular inclination from the crossflow velocity vector of the z_B axis, Fig. 1
$(\dot{\quad})$	$d/dt (\quad)$

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1. INTRODUCTION

As aircraft design has evolved toward more slender planforms in response to the demand for higher speeds, aerodynamic phenomena (e.g., shock waves and concentrated vortex flows) have appeared that earlier were associated more with the flight of projectiles than with that of aircraft. The appearance of these flows has altered the aircraft's aerodynamic behavior, and consequently its dynamical behavior, in ways that often are not predictable on the basis of *linearized* formulations of the aerodynamic force and moment system. It is generally agreed that consistently successful predictions of dynamical behavior under the new conditions hinge on incorporating an adequate *nonlinear* depiction of the aerodynamic force and moment system within the equations of motion.

A number of fundamental questions are raised by the necessity of replacing the linear aerodynamic formulation with a nonlinear one. Concepts such as stability derivatives, indicial functions, and superposition, which were the main working tools in the analysis and prediction of dynamical motions, are all based implicitly on a linearity assumption. Are these concepts at all viable when the linearity

*Presented as a contribution to a course of lectures on "Aircraft Stability and Control" at the Von Karman Institute for Fluid Dynamics, Brussels, Belgium, May 12-16, 1975.

assumption is removed, or must they be abandoned? How are time-history effects to be treated? For nonplanar motions, how are coupling effects, which could be neglected under the linearity assumption, to be treated? What about experiments? How are their requirements and their interpretation changed with the adoption of a nonlinear aerodynamic formulation?

About ten years ago, the idea was introduced of using functional analysis as the vehicle for extending the linear formulation into the nonlinear domain [1,2]. The adoption of Volterra's original conception of a functional [3] enabled the construction (without invoking a linearity assumption) of an integral form for the aerodynamic response that included the most general linear formulation as a special case. A framework was thereby established of sufficient breadth to enable answering some of the questions just posed. Initially, only planar motions were studied with a view toward answering the questions about the adaptability of the linear concepts and the treatment of time-history effects [1,2]. Later efforts focused on nonplanar motions, first of bodies of revolution [4-6], then of more general bodies [7,8], aimed at answering the question about coupling effects. Within the past few years, effort has been directed toward making the formulation more applicable to the treatment of dynamical motions of aircraft within the stall and post-stall regimes [9].

The occasion of this course provides the authors with a welcome opportunity to present a connected account of the theory's main lines of development since its inception ten years ago. The paper begins with a brief review of the most general linear integral representation of the aerodynamic response for a planar motion involving two variables. An

examination of the principal failing of the linear integral forms leads naturally to the idea of replacing the indicial functions within the integrals by functionals, themselves dependent on the past motion. The formal (and rigorous) nature of this replacement is stressed to make clear that the treatment of more complicated motions involving any number of variables follows merely by an application of the same formalism. Then, simplifying approximations are introduced, aimed at reasonably specifying the degree to which the indicial responses can be cognizant of the past motion. The degree of cognizance allowed is shown largely to determine the generality of aerodynamic phenomena whose effects can be acknowledged within the scope of the resulting formulation. A simple specification leads to nonlinear forms of the aerodynamic response that can be recognized as practicable generalizations of the linear superposition and stability derivative formulations. Next, nonplanar motions are taken up and the question of coupling addressed. Simplification of the general formulation (allowing the indicial responses to have the same degree of cognizance of the past motion as before) yields a form for the total moment due to an arbitrary motion that can be compounded of the moment contributions from a limited number of well-defined characteristic motions. It is here that the particular importance of coning motion emerges as a motion characterizing the coupling problem. The characteristic motions in principle being reproducible in the wind tunnel, the result also provides the setting for a discussion of the kinds of wind-tunnel tests that, according to the formulation, may be required to enable consistently successful predictions of dynamical behavior.

2. COORDINATE SYSTEMS

Three coordinate systems will be used. They have a common origin at the body's mass-center and a common axis x_B aligned with a longitudinal axis of the body.

Axes x_B, y_B, z_B are body-fixed axes (Fig. 1). The flight velocity vector of magnitude V has components u_B, v_B, w_B resolved along x_B, y_B, z_B , respectively. Thus,

$$V = \sqrt{u_B^2 + v_B^2 + w_B^2} \quad (1)$$

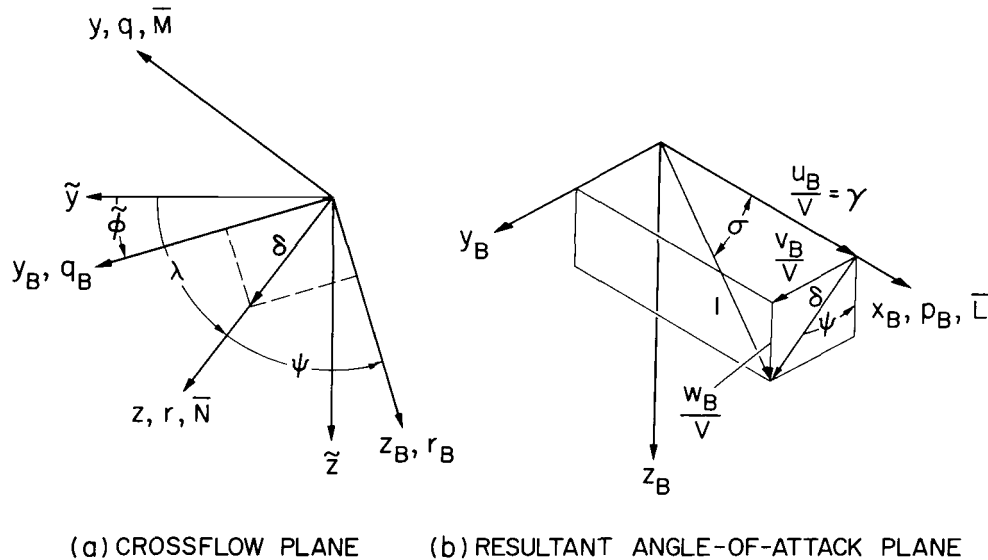


Fig. 1. Axes, angles, and velocity components in the crossflow and resultant angle-of-attack planes.

Resultant angle of attack σ is defined by the flight velocity vector and the x_B axis. The plane formed by y_B, z_B is called the crossflow

plane, illustrated in Fig. 1(a). The projection of a unit vector in the flight velocity direction onto the crossflow plane is a vector with magnitude δ ; it will be called the (dimensionless) crossflow velocity vector. Reference to Fig. 1(b) gives

$$\left. \begin{aligned} \delta &= \sqrt{\left(\frac{v_B}{V}\right)^2 + \left(\frac{w_B}{V}\right)^2} &= \sin \sigma \\ \gamma &= \frac{u_B}{V} &= \cos \sigma \\ \epsilon &= \frac{\delta}{\gamma} &= \tan \sigma \end{aligned} \right\} (2)$$

The components of the body's angular velocity relative to inertial space, resolved along x_B, y_B, z_B , are p_B, q_B, r_B , respectively.

A second axis system $x_B, \tilde{y}, \tilde{z}$ is nonrolling with respect to inertial space. Specifically, the $x_B, \tilde{y}, \tilde{z}$ axes have an angular velocity with respect to inertial space whose component resolved along the x_B axis is zero, while the components resolved along \tilde{y}, \tilde{z} are \tilde{q}, \tilde{r} , respectively. The angle $\tilde{\phi}$ through which the body axes have rolled at any time t can be defined relative to the nonrolling axis system as

$$\tilde{\phi} = \int_0^t p_B d\tau \quad (3)$$

The angular inclination λ of the crossflow velocity vector δ is measured relative to the nonrolling axis system, while ψ is the

angular inclination of the body axes from the crossflow velocity vector.

With the aid of Fig. 1(a), the body roll rate is seen to be

$$p_B = \dot{\lambda} + \dot{\psi} \quad (4)$$

The components p_B , \tilde{q} , \tilde{r} of the body's angular velocity vector resolved in the nonrolling axis system are related to those in the body axis system p_B , q_B , r_B through

$$\tilde{q} + i\tilde{r} = e^{i\phi}(q_B + ir_B) \quad (5)$$

Finally, an axis system x_B , y , z will be called the aerodynamic axis system. Axis z lies in the crossflow plane and is aligned with the direction of δ ; axis y lies in the crossflow plane aligned with a direction normal to the direction of δ . The components of the body's angular velocity resolved in the aerodynamic axis system p_B , q , r are related to those resolved in the body axis system through

$$q + ir = e^{i\psi}(q_B + ir_B) \quad (6)$$

It will be noted (cf. Fig. 1(b)) that δ and ψ are no more than the polar coordinates of the dimensionless velocities w_B/V and v_B/V in the body axis system. Let w_B/V be called the angle-of-attack parameter $\hat{\alpha}$ and v_B/V the angle-of-sideslip parameter $\hat{\beta}$; $\hat{\alpha}$ and $\hat{\beta}$ are related to the standard NASA definitions of angle of attack α and angle of sideslip β through

$$\left. \begin{aligned} \tan \alpha &= w_B/u_B = \hat{\alpha}/\gamma \\ \sin \beta &= v_B/V = \hat{\beta} \end{aligned} \right\} (7)$$

and to δ and ψ through

$$\hat{\alpha} + i\hat{\beta} = \delta e^{i\psi} \quad (8)$$

The components of the aerodynamic force coefficient resolved along the body axes x_B, y_B, z_B are $\hat{C}_X, \hat{C}_Y, \hat{C}_Z$, respectively, while the corresponding components of the aerodynamic moment coefficient (about the mass-center) are $\hat{C}_l, \hat{C}_m, \hat{C}_n$, respectively. Analogously, the components of the force and moment coefficients resolved along the aerodynamic axes x, y, z are C_X, C_Y, C_Z and C_l, C_m, C_n , respectively. Components of the aerodynamic moment coefficient resolved in the aerodynamic axis system are related to those resolved in the body axis system through

$$C_l = \hat{C}_l \quad (9)$$

$$C_m + iC_n = e^{i\psi}(\hat{C}_m + i\hat{C}_n)$$

Corresponding relations between the components of the aerodynamic force coefficient are obtained by replacing l, m, n by X, Y, Z , respectively in Eq. (9).

To completely describe the state of a six-degree-of-freedom motion of a rigid body, it is necessary to specify the velocity and angular velocity vectors of the body. These may be expressed in terms of their scalar components resolved in the body-fixed axes $u_B, v_B, w_B, p_B, q_B, r_B$, or equivalently by $\hat{\alpha}, \hat{\beta}, V, p_B, q_B, r_B$. Analogously, in the aerodynamic axis system the motion is specified by the scalar variables $\delta, \psi, V, p_B, q, r$, or by $\delta, \psi, V, \dot{\lambda}, q, r$, since p_B is related to $\dot{\psi}$ and $\dot{\lambda}$

through Eq. (4). These are the principal variables on which the aerodynamic force and moment formulation must depend in general, although, as will be seen, certain simplifications are possible in special cases.

3. DEVELOPMENT OF INTEGRAL FORMS

For simplicity, the following (removable) conditions are imposed in all of the work to follow: (1) the aircraft is considered to be a rigid body which, prior to time zero, has been in steady flight for a long time; (2) at time zero, the aircraft begins a maneuver in which altitude changes are sufficiently small that atmospheric density and kinematic viscosity along the flight path remain essentially constant; (3) the aircraft's velocity along the flight path also is specified to remain constant. Hence, dynamic pressure, Mach number, and Reynolds number as measured along the flight path remain fixed throughout the motion. Under these restrictions, a general motion referred to the body axis system is described by the variables $\hat{\alpha}$, $\hat{\beta}$, p_B , q_B , r_B ; referred to the aerodynamic axis system, the motion is described by the variables δ , ψ , $\dot{\lambda}$, q , r . The development of an integral form for the linear aerodynamic pitching-moment response to two of the variables will serve as a review of aerodynamic indicial functions and superposition.

3.1 Linear Aerodynamic Pitching-Moment Response

Let the aircraft begin a maneuver at time zero involving only two of the five flight variables on which the aerodynamic response depends (the others being fixed at zero). To fix ideas, the angle-of-attack parameter δ and the pitching velocity q in the aerodynamic axis

system are chosen, so that the motion is planar. The extension to a more general motion involving all five motion variables will be obvious. For brevity, attention is focused solely on the aerodynamic pitching moment. All that is said, however, will hold as well for any other of the force or moment components that may exist merely on the substitution of that component for C_m .

Consider the aerodynamic pitching-moment response to the variations in δ and q . It is usually admissible to break the variations into a large number of small step changes (cf. Fig. 2). In response to typical

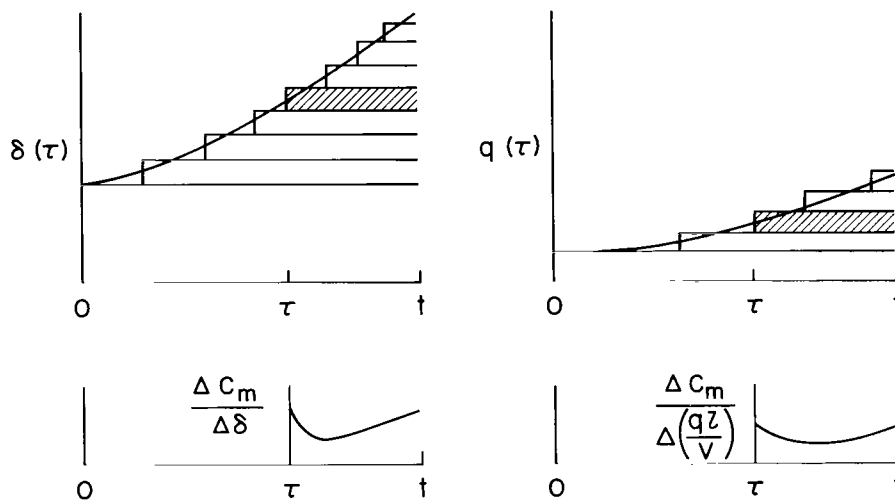


Fig. 2. Summation of incremental responses.

step changes $\Delta\delta$ and $\Delta(q\bar{z}/V)$ at time τ , there is an incremental change in pitching moment ΔC_m ; it is measured at a fixed time t subsequent to τ . The assumption of linearity is now invoked, having, in the context of this report, the following meaning: ΔC_m is said to

be independent of $\delta(\tau)$, $q(\tau)$, and the past values of these variables.

This enables writing ΔC_m in the form

$$\Delta C_m(t) = \frac{\Delta C_m(t - \tau)}{\Delta \delta} \Delta \delta + \frac{\Delta C_m(t - \tau)}{\Delta(q\ell/V)} \Delta(q\ell/V) \quad (10)$$

The form implies that $\Delta C_m/\Delta \delta$ and $\Delta C_m/\Delta(q\ell/V)$ are derivable from linear differential equations. That they depend on elapsed time $t - \tau$, rather than on t and τ separately implies that the coefficients of the differential equations are independent of time. It is important to note the following distinction: the significance of the linearity assumption does not rest on the assertion that ΔC_m is linearly dependent on $\Delta \delta$ and $\Delta(q\ell/V)$, or that the contributions to ΔC_m from the two increments are linearly additive. Both of these assertions are always justified whenever $\Delta \delta$ and $\Delta(q\ell/V)$ are so small that terms of $O((\Delta \delta)^2, (\Delta(q\ell/V))^2, (\Delta \delta \Delta(q\ell/V)))$ can be neglected. The significance of the assumption lies in saying that the two ratios are independent of δ and q . Thus, no matter how large the values of δ and q at the origins of the steps, the response functions $\Delta C_m/\Delta \delta$ and $\Delta C_m/\Delta(q\ell/V)$ are said to be the *same* functions, dependent only on $t - \tau$. The limits of these functions as $\Delta \delta$ and $\Delta(q\ell/V)$ approach zero

$$\left. \begin{aligned} \lim_{\Delta \delta \rightarrow 0} \frac{\Delta C_m(t - \tau)}{\Delta \delta} &= C_{m\delta}(t - \tau) \\ \lim_{\Delta(q\ell/V) \rightarrow 0} \frac{\Delta C_m(t - \tau)}{\Delta(q\ell/V)} &= C_{mq}(t - \tau) \end{aligned} \right\} (11)$$

are called the indicial pitching-moment responses per unit step changes in δ and $q\ell/V$, respectively. Every pair of steps in δ and $q\ell/V$

beginning at a value of τ less than t has a corresponding incremental pitching-moment response of the form Eq. (10) that contributes to the pitching moment at time t . The summation of these incremental responses to the steps that occur over the range of τ from zero to t then gives the aerodynamic pitching-moment response at time t to the variations in δ and q . As the indicial responses depend only on the time difference $t - \tau$, in the limit the summations take the form of the familiar convolution integral. The sum of the two contributions and the initial value of C_m then give the total pitching moment at time t

$$C_m(t) = C_m(0) + \int_0^t C_{m_\delta}(t - \tau) \frac{d}{d\tau} \delta(\tau) d\tau + \frac{z}{V} \int_0^t C_{m_q}(t - \tau) \frac{d}{d\tau} q(\tau) d\tau \quad (12)$$

For constant V and within the assumption of linearity, Eq. (12) is exact. Since all values of δ and q figure within the limits of the integrals, C_m at time t depends on the whole past of δ and q .

3.2 Definition of a Functional

The description of $C_m(t)$ as a function that depends on all of the past values of the argument functions δ and q corresponds mathematically to Volterra's description of a *functional* [3]. If Volterra's square-bracket notation is adopted, the assertion that $C_m(t)$ is a functional is indicated thus,

$$C_m(t) = G[\delta(\xi), q(\xi)] \quad (13)$$

where it is understood that ξ is a running variable in time, ranging over the interval zero to t .

In brief, just as an ordinary function $f(x)$ assigns a number to each x for which it is defined, a functional $F[y(\xi)]$ assigns a number to each function $y(\xi)$ of the set of functions (all of which are defined in some interval $a \leq \xi \leq b$) for which the functional is defined. Thus, Eq. (13) may be interpreted as follows: given any pair of functions $\delta(\xi), q(\xi)$ out of the collection of all such pairs defined in the interval $0 \leq \xi \leq t$, the functional G assigns a number to $C_m(t)$. It is the idea of a functional, with its formal recognition of the influence of past events on present behavior, that provides the key to generalizing the linear integral form.

3.3 Nonlinear Aerodynamic Pitching-Moment Response

An heuristic argument will show how the most general linear form for the aerodynamic response $C_m(t)$ to two of the flight variables (Eq. 12) can be freed of the linearity assumption. A more rigorous mathematical development yielding the same result is available in Ref. 1.

Attention is directed to the incremental form for $\Delta C_m(t)$, Eq. (10), where the principal limitation imposed by the linearity assumption first appears. As already noted, this limitation does not hinge on the assertion that $\Delta C_m(t)$ is linearly dependent on $\Delta\delta$ and $\Delta(q\bar{l}/V)$, or that the two incremental contributions are linearly additive. Both of these assertions remain in force in the more general development. The limitation is simply that the indicial responses are said to be independent of the past values of δ and q . Now, as described more precisely below, an indicial response to a step change in a motion is formed by taking a difference between the responses to two motions whose histories differ only by the step imposed on the second motion. Hence, the motions

prior to the step are identical. When the difference is taken in the linear case, the influence of this identical past cancels identically. In the absence of linearity, exact cancellation of the past influence is not to be expected, so that some remnant of that influence must be evidenced in the behavior of the indicial response. That is to say, the indicial response must be a functional. The way to eliminate the limitation of the linearity assumption in Eq. (10) thus becomes clear: replace the indicial functions by functionals, themselves dependent on values of δ and q prior to the origin of the steps. That this replacement can be rigorously justified is, in fact, the principal result of the mathematical development in Ref. 1.

If the notation already introduced for a functional is followed, the designation of the indicial responses as functionals is indicated thus,

$$\left. \begin{aligned} \lim_{\Delta\delta \rightarrow 0} \frac{\Delta C_m(t, \tau)}{\Delta\delta} &= C_{m\delta}[\delta(\xi), q(\xi); t, \tau] \\ \lim_{\Delta(qL/V) \rightarrow 0} \frac{\Delta C_m(t, \tau)}{\Delta(qL/V)} &= C_{mq}[\delta(\xi), q(\xi); t, \tau] \end{aligned} \right\} (14)$$

where, as before, t is the time at which the increment ΔC_m is measured, τ is the time at which the steps originate, and ξ is a running variable in time over the interval zero to τ ; that is, over the past for steps beginning at $\xi = \tau$.

Although the replacement of the indicial functions by functionals appears to be largely a matter of notation, the change is far-reaching and, for example, requires a more precise description of the formation of indicial responses than was necessary in the linear case. Two motions

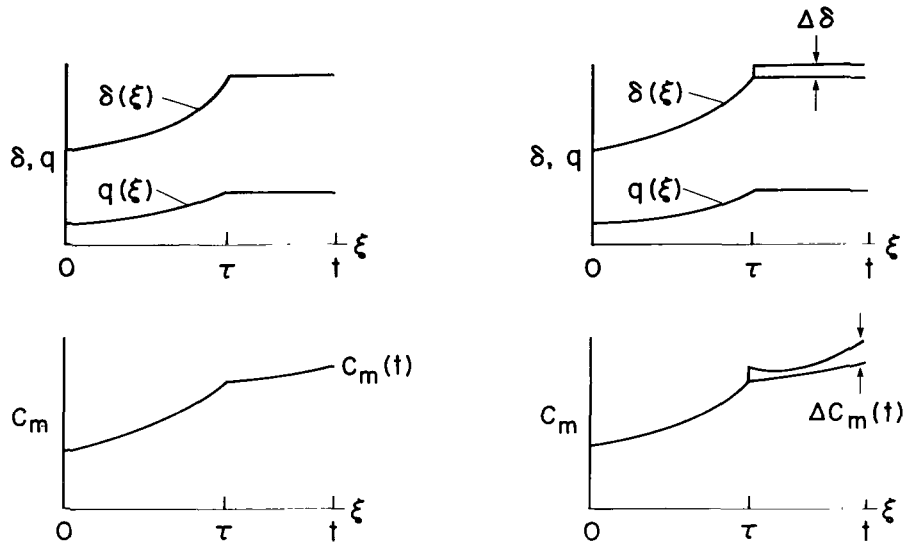


Fig. 3. Formation of indicial responses.

have to be considered (cf. Fig. 3): first, beginning at $\xi = 0$, the aircraft is made to execute the motion under study $\delta(\xi), \frac{q(\xi)l}{V}$. At a certain time τ , the motion is constrained such that the values of the flight variables existent at time τ , that is, $\delta(\tau), \frac{q(\tau)l}{V}$ remain constant thereafter. The pitching moment corresponding to this maneuver is measured at a time t , subsequent to τ . Second, the aircraft is made to execute precisely the same motion, beginning at $\xi = 0$ and constrained in the same way at $\xi = \tau$, except that at the latter time, one of the variables δ or $q l/V$ is given an incremental step $\Delta\delta$ or $\Delta(q l/V)$ over its value at $\xi = \tau$. Hence, if it is δ that is given an

increment $\Delta\delta$, the values of the flight variables for all times subsequent to τ are $\delta(\tau) + \Delta\delta, \frac{q(\tau)l}{V}$. The pitching moment corresponding to this maneuver is again measured at time t . The difference between the two measurements is divided by the incremental step $\Delta\delta$ or $\Delta(ql/V)$; the limit of this ratio as the magnitude of the step approaches zero is called the indicial pitching-moment response at time t per unit step at time τ of one of the two flight variables δ or ql/V . As indicated in Fig. 3, since the two motions prior to $\xi = \tau$ are identical, the ratio must be identically zero for all $\xi < \tau$. At $\xi = \tau$, a discontinuity in the ratio is permissible, reflecting the discontinuous change in one of the motion variables. For all values of $\xi > \tau$, the ratio must be continuous.

Formed as described above and as defined in Eq. (14), the indicial responses are suitably generalized to be free of dependence on a linearity assumption. With Eq. (14) replacing Eq. (11) in Eq. (10), the summation of incremental responses to yield an integral form for $C_m(t)$ follows as before. The resulting generalization of Eq. (12) is

$$C_m(t) = C_m(0) + \int_0^t C_{m\delta}[\delta(\xi), q(\xi); t, \tau] \frac{d}{d\tau} \delta(\tau) d\tau + \frac{l}{V} \int_0^t C_{mq}[\delta(\xi), q(\xi); t, \tau] \frac{d}{d\tau} q(\tau) d\tau \quad (15)$$

Equation (15), applicable to a planar motion involving arbitrary variations in δ and q , is believed to be of sufficient generality to

provide a framework for the study of a wide variety of nonlinear aerodynamic problems. Derivation of the integral forms corresponding to motions involving more than two variables follows simply by an application of the same formalism. In the next section, simplifying approximations are introduced into Eq. (15) which make the form more practicable at the price of narrowing the classes of aerodynamic behavior that can be acknowledged within its scope.

4. LIMITATIONS AND SIMPLIFICATION OF THE NONLINEAR INTEGRAL FORM

Conditions that flight-path properties remain constant during the motion and that the aircraft be a rigid body have been imposed at the outset. Imposing constant flight-path properties excludes from consideration the influence on motions of very large accelerations or variations in atmospheric density, such as might occur, for example, during atmospheric reentry. Omitting structural variables rules out the possibility of treating the buffeting problem, which involves interactions between the elastic airframe and random aerodynamic fluctuations [10]. However, the presence of fluctuations themselves can be acknowledged within the framework already established by adopting the notion of ensemble averaging (cf. Ref. 9). The remaining assumptions are of two main classes: first, fundamental assumptions associated with the use of functional analysis to develop the general integral form for the aerodynamic response; second, simplifying assumptions associated with the reduction of the general integral form to more practicable forms.

4.1 Fundamental Assumptions

Despite their generality, the nonlinear indicial responses defined by Eq. (14), and hence the corresponding integral form for the aerodynamic pitching moment, Eq. (15), already contain implicit assumptions that limit their applicability. These are principally that the indicial responses must exist for all values of their arguments (for $\xi > \tau$) and must be unique. Evidence for the existence of indicial responses is necessarily inferential, since it is doubtful whether they will ever be measured directly. However, the nonexistence of indicial responses follows from the nonexistence of their steady-state values, which are, of course, more amenable to measurement. In this respect, assuming the existence of the indicial responses for all values of their arguments (with $\xi > \tau$) strictly speaking implies the exclusion of cases where the variation of steady-state pitching moment with δ (e.g., at $q = 0$) becomes discontinuous either in its magnitude or slope at certain isolated values of δ , since $\lim_{\Delta\delta \rightarrow 0} \frac{\Delta C_m^{(\infty)}}{\Delta\delta}$ will not exist at these points. Such cases are known to characterize certain types of stall behavior (cf., for example, Ref. 11). Although these cases can be treated by an appropriate addition of jump conditions at the isolated points, for simplicity they will be excluded from further consideration here. Assuming uniqueness implies the exclusion of cases where more than one steady-state response to the same maneuver is possible; more precisely, cases must be excluded where the steady-state responses to repetitions of the same maneuver can be multivalued and probabilistic.

An example of aerodynamic behavior that must be excluded on this basis is the steady asymmetric vortex flow above a body of revolution at large δ when it is a matter of chance whether the vortices form a left-hand or a right-hand pattern. Given these fundamental limitations on the applicability of the general integral form for $C_m(t)$, Eq. (15), simplifying assumptions still need to be attached in order to reduce it to more usable forms.

4.2 Simplifying Assumptions

In the form Eq. (15) the indicial responses within the integrals are themselves functionals, depending in general on the whole past history of the motion $\delta(\xi), q(\xi)$. This makes the further use of the form exceedingly difficult, since the history of the motion normally is not known in advance but rather is desired as the solution of the equations of motion. Thus, when the past history is unspecified, the functionals also are unknown beforehand. Simplification of Eq. (15) hinges on replacing the functionals by appropriate functions whose dependence on the past is denoted by a limited number of parameters rather than by continuous functions. If $\delta(\xi), q(\xi)$ can be considered analytic functions in a neighborhood of $\xi = \tau$ (corresponding to the most recent past for an indicial response with origin at $\xi = \tau$), in principle their histories can be reconstructed from a knowledge of all of the coefficients of their Taylor series expansions about $\xi = \tau$. Thus, since $\delta(\xi), q(\xi)$ are equally represented by the coefficients of their expansions, the functional, with its dependence on $\delta(\xi), q(\xi)$, can be replaced without approximation by a function with a dependence

on all of the coefficients of the expansions of $\delta(\xi), q(\xi)$ about $\xi = \tau$.

The indicial response $C_{m\delta}$, for example, can be expressed as

$$C_{m\delta}[\delta(\xi), q(\xi); t, \tau] = C_{m\delta}(t, \tau; \delta(\tau), \dot{\delta}(\tau), \dots, q(\tau), \dot{q}(\tau), \dots) \quad (16)$$

(The reversal in the order of the dependencies anticipates the diminished role played by δ and q in the succeeding analysis.) Now physical reasoning suggests that the indicial response should have "forgotten" long-past events and thus should depend mainly on events in the most recent past. If this is assumed to be true, then so far as the effect of the past on the indicial response is concerned, the form of the past motion just prior to the origin of the step might just as well have existed for all earlier times. Hence, at most only the first few coefficients of the expansions of $\delta(\xi), q(\xi)$ need be retained to characterize correctly the most recent past, which is all the indicial response remembers. Retaining the first two coefficients of $\delta(\xi)$, for example, implies matching the true past history of δ in magnitude and slope at the origin of the step, thereby approximating $\delta(\xi)$ by a linear function of time $\delta(\xi) \approx \delta(\tau) - \dot{\delta}(\tau)(\tau - \xi)$. With an approximation of this order in force for both $\delta(\xi)$ and $q(\xi)$ in the indicial responses, the integral form replacing Eq. (15) becomes

$$C_m(t) = C_m(0) + \int_0^t C_{m\delta}(t, \tau; \delta(\tau), \dot{\delta}(\tau), q(\tau), \dot{q}(\tau)) \frac{d}{d\tau} \delta(\tau) d\tau \\ + \frac{\lambda}{V} \int_0^t C_{mq}(t, \tau; \delta(\tau), \dot{\delta}(\tau), q(\tau), \dot{q}(\tau)) \frac{d}{d\tau} q(\tau) d\tau \quad (17)$$

This form, while considerably more tractable than Eq. (15), is still sufficiently general to allow the treatment of motions involving

hysteresis effects. Retaining a dependence on $\dot{\delta}(\tau)$, for example, allows assigning different indicial responses to a step at a single pair of values $\delta(\tau), q(\tau)$, depending on the magnitude and sign of $\dot{\delta}(\tau)$. It is permissible, for example, to distinguish between indicial responses where δ was increasing or decreasing prior to the step. This will be valid when physical reasoning suggests that the particular response attached to a step is a deterministic result of the past history rather than the probabilistic result of an interaction with a random fluctuation. Although implications of the use of Eq. (17) will not be explored further in this study, the equation is believed to be both tractable enough and of sufficient scope to provide a framework for the study of rapidly varying maneuvers (for example, the rapid pull-up), where hysteresis effects governed by rate-dependent flow phenomena are known to be present in the aerodynamic response.

When deterministic hysteresis effects are absent, and if, additionally, the assumption of a slowly varying motion is introduced, then the dependence of the indicial responses on $\dot{\delta}(\tau), \dot{q}(\tau)$ will not be significant (cf. Fig. 4). So far as the indicial responses are concerned, omitting these dependencies in Eq. (17) implies that the motion prior to the origin of the steps is being approximated by the time-invariant

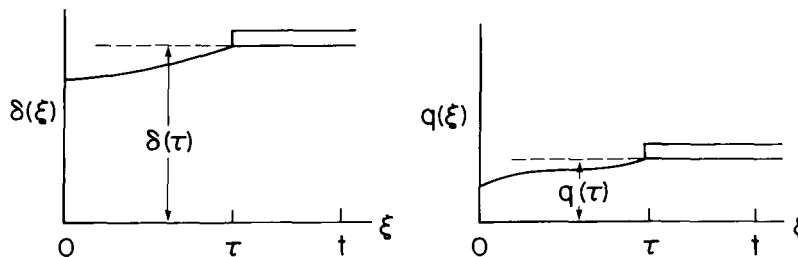


Fig. 4. Slowly varying motions.

motion $\delta(\xi) \approx \delta(\tau), q(\xi) \approx q(\tau)$. The indicial responses at any value of elapsed time, now dependent only on the magnitudes of δ and q just prior to the steps, must not only be continuous functions of $\delta(\tau), q(\tau)$, but henceforward also *single-valued* functions of $\delta(\tau), q(\tau)$. Further, with a given time-invariant past motion and with the already assumed constant flight-path properties, clearly an indicial response must have the same value after a given time has elapsed subsequent to the origin of a step no matter when the step occurs. That is, just as in the linear case, the indicial response must be a function of *elapsed* time $t - \tau$ rather than of t and τ separately. Finally, then, for slowly varying motions and within the additional restriction on the indicial responses of single-valuedness with respect to $\delta(\tau), q(\tau)$, and $t - \tau > 0$, a much more specific form of Eq. (15) may be written, still capable of embracing a fairly broad range of nonlinear aerodynamic problems. It is

$$\begin{aligned}
 C_m(t) = C_m(0) + \int_0^t C_{m\delta}(t - \tau; \delta(\tau), q(\tau)) \frac{d}{d\tau} \delta(\tau) d\tau \\
 + \frac{z}{V} \int_0^t C_{mq}(t - \tau; \delta(\tau), q(\tau)) \frac{d}{d\tau} q(\tau) d\tau
 \end{aligned} \tag{18}$$

Although the form of Eq. (18) represents a great simplification over that of Eq. (15), the equation still includes the full linear form (Eq. (12)) as a special case. Equation (18) is the basic integral form underlying the subsequent simplified formulations.

5. APPLICATION OF THE SIMPLIFIED NONLINEAR INTEGRAL FORM TO PLANAR DYNAMIC STABILITY STUDIES

Equation (18) is now applied to the study of aircraft dynamic stability. The rigid-body motions of aircraft are normally oscillatory, and moreover, the oscillations are generally of very low frequency. Several analytical benefits accrue from the latter fact. First, since the motions are slowly varying, the assumptions underlying Eq. (18) are particularly well grounded in this application. Second, Eq. (18) can be further simplified. The simplification, which in effect reduces Eq. (18) to an equation correct to the first order in frequency, parallels that realized in the linear case in the application of Eq. (12) to stability studies [12].

5.1 Stability Coefficient Formulation

The reduction of Eq. (18) to first order in frequency is carried out in detail so that, later, the analogous reduction of the more lengthy equations for nonplanar motions will need only to be indicated in passing. Equation (18) is first rearranged to give a more convenient form. From physical considerations, the indicial responses must approach steady-state values with increasing values of the argument $t - \tau$. To indicate this, the following notation is introduced (the notation parallels that of Ref. 12):

$$\left. \begin{aligned} C_{m\delta}(t - \tau; \delta(\tau), q(\tau)) &= C_{m\delta}(\infty; \delta(\tau), q(\tau)) - F_3(t - \tau; \delta(\tau), q(\tau)) \\ C_{mq}(t - \tau; \delta(\tau), q(\tau)) &= C_{mq}(\infty; \delta(\tau), q(\tau)) - F_4(t - \tau; \delta(\tau), q(\tau)) \end{aligned} \right\} (19)$$

where

$C_{m\delta}(\infty; \delta(\tau), q(\tau))$ rate of change with δ of the pitching-moment coefficient that would be measured in a steady flow, evaluated at the instantaneous value of $\delta(\tau)$ with q fixed at the instantaneous value $q(\tau)$,

$C_{mq}(\infty; \delta(\tau), q(\tau))$ rate of change with q of the pitching-moment coefficient that would be measured in a steady flow, evaluated at the instantaneous value $q(\tau)$ with δ fixed at the instantaneous value $\delta(\tau)$.

The functions F_3 and F_4 are termed deficiency functions; they tend to vanish with increasing values of the argument $t - \tau$. When Eqs. (19) are inserted in Eq. (18), the terms involving the steady-state parameters form a perfect differential which can be immediately integrated.

Equation (18) becomes

$$C_m(t) = C_m(\infty; \delta(t), q(t)) - \int_0^t F_3(t - \tau; \delta(\tau), q(\tau)) \frac{d}{d\tau} \delta(\tau) d\tau - \frac{z}{V} \int_0^t F_4(t - \tau; \delta(\tau), q(\tau)) \frac{d}{d\tau} q(\tau) d\tau \quad (20)$$

where

$C_m(\infty; \delta(t), q(t))$ total pitching-moment coefficient that would be measured in a steady flow with δ fixed at the instantaneous value $\delta(t)$ and q fixed at the instantaneous value $q(t)$.

Equation (20) is a form of Eq. (18) particularly amenable to approximation. Let it be assumed for illustration that the angle-of-attack parameter δ is essentially a harmonic function about a constant mean while the angle of pitch θ ($\dot{\theta} \equiv q$) is a purely harmonic function. That is

$$\left. \begin{aligned} \delta &\approx \delta_m + \delta_0 e^{i\omega t} \\ \theta &\approx \theta_0 e^{i\omega t} \end{aligned} \right\} (21)$$

Then clearly, since $q = \dot{\theta}$, q itself will be of first order in frequency ω . Hence q will be small for all values of time, and powers of q higher than the first will be of second and higher orders in frequency. Therefore, for any given values of t or τ , it is permissible to expand the terms in Eq. (20) in a Taylor series about $q = 0$ and to discard terms containing powers of q higher than the first. Terms in \dot{q} and $\dot{\delta}q$ likewise may be discarded as they will be of second order in frequency. The result of the expansion is

$$\begin{aligned} C_m(t) &= C_m(\infty; \delta(t), 0) + q(t) \frac{L}{V} C_{mq}(\infty; \delta(t), 0) \\ &\quad - \int_0^t F_3(t - \tau; \delta(\tau), 0) \frac{d}{d\tau} \delta(\tau) d\tau \end{aligned} \quad (22)$$

Definitions of $C_m(\infty; \delta(t), 0)$ and $C_{mq}(\infty; \delta(t), 0)$ follow from those given earlier with the substitution of $q(t) = 0$. The first two terms are clearly the nonlinear counterparts of the terms $\delta C_{m\delta}(\infty)$ and $(qL/V)C_{mq}(\infty)$ that appear in linear analyses based on the stability derivative concept. Therefore, the integral, when also reduced to the first order in frequency,

is anticipated to be the nonlinear counterpart of the term $(\dot{\delta}Z/V)C_{m\dot{\delta}}$.

This reduction is taken up next.

With the change in variable $t - \tau = \tau_1$, the integral becomes

$$I = \int_0^t F_3(\tau_1; \delta(t - \tau_1), 0) i\omega\delta_0 e^{i\omega(t-\tau_1)} d\tau_1 \quad (23)$$

which may be rewritten

$$I = \dot{\delta}(t) \int_0^t F_3(\tau_1; \delta(t - \tau_1), 0) e^{-i\omega\tau_1} d\tau_1 \quad (24)$$

Practically speaking, the deficiency function essentially vanishes after a relatively short period of time has elapsed. Let the value of τ_1 at which F_3 essentially vanishes be t_a , and consider events at a time t sufficiently removed from the start of the motion that $t > t_a$. Then the upper limit in Eq. (24) may be replaced by t_a , whereupon, with τ_1 bounded and ω small, the harmonic function may be expanded in powers of ω . Since $\dot{\delta}$ is itself of first order in ω , however, only the first term in the expansion, unity, contributes within the order of the approximation. Moreover, with respect to the parameter $\delta(t - \tau_1)$, a further simplification can be realized when the condition $t > t_a$ is invoked, for then $\delta(t - \tau_1) \approx \delta(t)$. The integral reduces to

$$I = \frac{\dot{\delta}(t)Z}{V} \left(\frac{V}{Z} \int_0^{t_a} F_3(\tau_1; \delta(t), 0) d\tau_1 \right) \quad (25)$$

where, as anticipated, the integral term within parentheses may be identified with $-C_{m\dot{\delta}}$.

Hence, just as in the linear case [12], to the first order in frequency $C_{m\dot{\delta}}$ is proportional to the area of the deficiency function, now, however, evaluated at and dependent on the particular value of the angle-of-attack parameter $\delta(t)$ under consideration. The nonlinear counterpart of the linear stability derivative formulation thus becomes

$$C_{m\dot{\delta}}(t) = C_{m\dot{\delta}}(\infty; \delta(t), 0) + q(t) \frac{z}{V} C_{m\dot{q}}(\infty; \delta(t), 0) + \dot{\delta}(t) \frac{z}{V} C_{m\dot{\delta}}(\delta(t)) \quad (26)$$

where

$$C_{m\dot{\delta}}(\delta(t)) = - \frac{V}{z} \int_0^{t_a} F_3(\tau_1; \delta(t), 0) d\tau_1 \quad (27)$$

5.2 Interpretation of $C_{Z\dot{\delta}}$

In Eqs. (26) and (27), just as in the linear theory, $C_{m\dot{\delta}}$, the term that accounts for the past, appears as an integral of the deficiency function. A physical argument will show why the term, or more directly, the analogous normal-force coefficient $C_{Z\dot{\delta}}$, continues to appear in this way. In Fig. 5, the aircraft on the left has been sinking, without pitching, for a long time at a constant rate. A force P must be

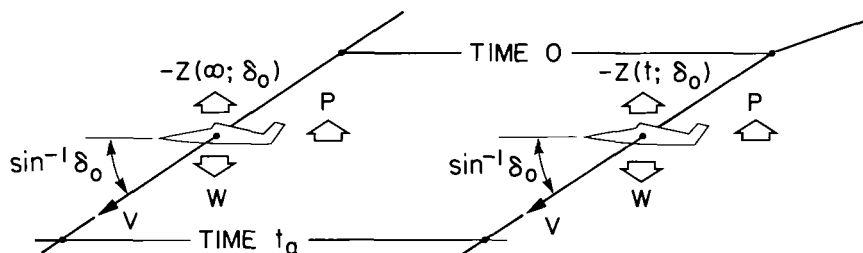


Fig. 5. Interpretation of $C_{Z\dot{\delta}}$.

applied to maintain the constant rate. The work done by the applied force over an arbitrarily large time interval zero to t_a is

$$Wk_1 = V\delta_0 \int_0^{t_a} P d\tau = V\delta_0 \int_0^{t_a} [W + Z(\infty; \delta_0)] d\tau \quad (28)$$

where W is the weight of the aircraft and $-Z(\infty; \delta_0)$ is the steady-state normal force due to the constant angle of attack. Now, as shown on the right side of Fig. 5, let the same aircraft experience a step change in δ at time zero and then undergo the same motion as in the first case. The work done over the same time interval by the force applied to maintain a constant rate is

$$Wk_2 = V\delta_0 \int_0^{t_a} [W + Z(\tau; \delta_0)] d\tau \quad (29)$$

The difference in work done is

$$Wk_1 - Wk_2 = V\delta_0 \int_0^{t_a} [Z(\infty; \delta_0) - Z(\tau; \delta_0)] d\tau \quad (30)$$

After identifying t_a with the time required for $Z(\tau; \delta_0)$ to reach steady state, one sees that the integral is the area enclosed by the indicial normal-force response curve and its steady-state value. That is to say, it is the area of the normal-force deficiency function, and it is therefore proportional to $C_{Z\delta}$. The energy of the aircraft is the same in both cases, since it undergoes the same motion. The energy expended by the applied force is different in the two cases. The balance of energy, which is Eq. (30), therefore must have been given to

or taken from the fluid. The term $C_{Z\dot{\delta}}$ is a measure of the energy given to or taken from the fluid whenever the angle of attack changes from one level to another. Because this assertion holds regardless of the magnitude of the angle of attack, it is reasonable that $C_{Z\dot{\delta}}$ (and $C_{m\dot{\delta}}$) should continue to appear as an integral of a deficiency function, even in the nonlinear analysis.

5.3 Theoretical Evaluation of Stability Coefficients

Of the three coefficients in Eq. (26), the first is the familiar static pitching-moment coefficient due to angle of attack. Since it is of fundamental importance in aircraft design, it has been studied extensively; a large body of both theoretical and experimental results may be called on to define it in particular cases. No such body of information exists for the remaining two coefficients. The specific problems which must be solved for these two coefficients are characterized by the motions illustrated in Fig. 6. Just as for the first term, solutions for C_{mq} may be derived from a steady-state equation.

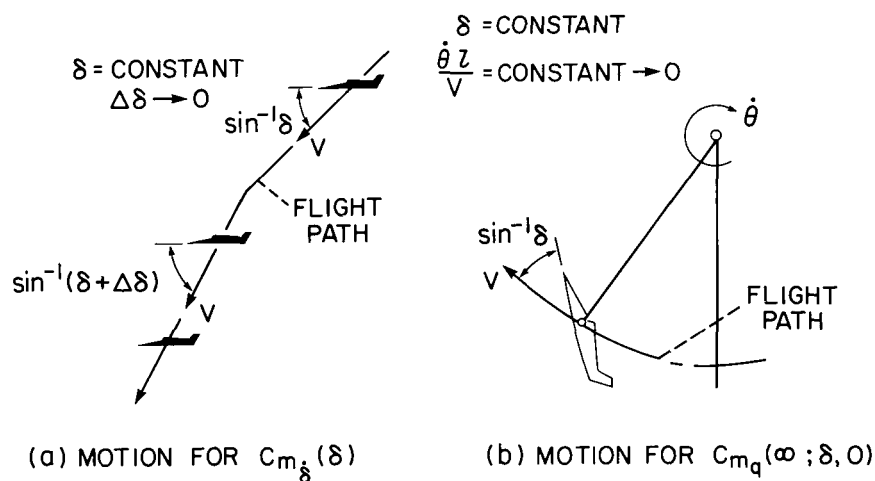


Fig. 6. Definitive motions for calculation of stability coefficients.

In fact, the theoretical problems for $C_{m\dot{q}}$ and for the static pitching-moment coefficient due to angle of attack differ little in principle; the computational methods that have been devised for the numerical solution of the latter problem should be largely transferable to the solution of the former. Solutions for $C_{m\dot{\delta}}$, on the other hand, must be derived from a time-dependent equation. Except for those cases where one of the three space coordinates can be omitted or where a similarity condition holds (e.g., two-dimensional flow, conical flow), the necessity of treating an essentially four-dimensional problem apparently places it beyond the capacities of even present-day automatic computing machines. At any rate, the authors are unaware of any attempts to treat the problem for $C_{m\dot{\delta}}$ with an allowance for the nonlinear dependence on δ . In view of this, some of the physical features of the indicial response and of the integral form for $C_{m\dot{\delta}}$ will be brought out that might assist in making at least order-of-magnitude estimates of $C_{m\dot{\delta}}$.

The two-dimensional wing is chosen for demonstration purposes since the important features of sound-wave propagation are most easily illustrated in this case. A subsonic flight speed is specified because here many of the results from the linear theory do not require modification in principle and thus can be adapted to guide the drawing of a physical picture of the indicial response. By means of these adaptations, the results may have more general bearing. Consider the boundary conditions corresponding to the motion illustrated in Fig. 6(a), where the flight speed V is subsonic. Let the wing move away from a coordinate system that is fixed in space at the position of the mass-center at time $\tau_1 = 0$ where the step change in the boundary conditions occurs (cf. Fig. 7).

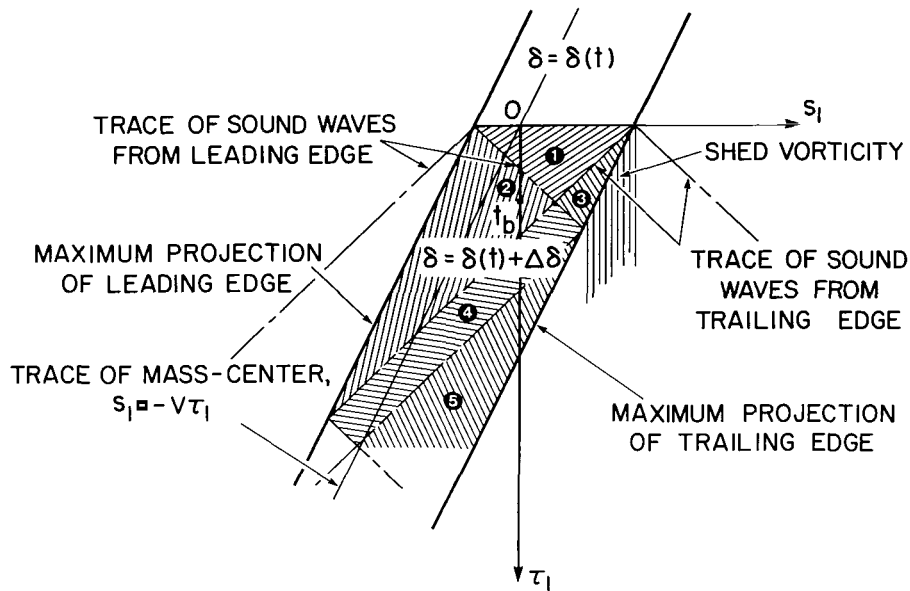


Fig. 7. Boundary conditions for indicial loading.

The distance traveled by the mass-center along the flight path is measured by a coordinate s_1 . Since the mass-center moves at constant speed, the trace of its path plotted against time τ_1 is a straight line. This is shown in Fig. 7, where the maximum projections of the leading and trailing edges are also shown, parallel to the trace of the mass-center. For $\tau_1 < 0$, the angle-of-attack parameter δ is constant at $\delta(t)$. A step change in δ occurs at $\tau_1 = 0$, so that for $\tau_1 \geq 0$, the angle-of-attack parameter is $\delta(t) + \Delta\delta$. Due to the impulsive change in δ , the loading on the wing at $\tau_1 = 0$ also undergoes a sudden change. The physical situation at this instant corresponds essentially to that described by piston theory (cf., for example, Ref. 13), which should give a reasonably accurate estimate of the initial change in loading. Reflecting the *local* character of the instantaneous response, the initial change in loading is essentially constant across the chord, so that the corresponding center of pressure is located very

near the mid-chord. Also at $\tau_1 = 0$, the sudden change in flow conditions causes disturbances to be propagated from every point on the chord line. Of particular importance are those from the leading and trailing edges. Each disturbance is propagated at the local speed of sound, so that, on a plot such as shown in Fig. 7, the zone of its influence is bounded by projections of an approximately conic surface whose origin is the point of the disturbance. As shown on Fig. 7, traces of the waves from the edges divide the wing into a number of distinct regions. Points in region (1) have not yet been made aware of the changed conditions at the leading and trailing edges by the arrival of the sound waves, and hence the loading in this region remains essentially unchanged from that existent at $\tau_1 = 0$ (cf. Fig. 8). This loading gradually disappears as the propagation of the two sound waves announces the new conditions to increasing portions of the wing, and it disappears completely at $\tau_1 = t_b$. Points in region (2) have been made aware of new conditions

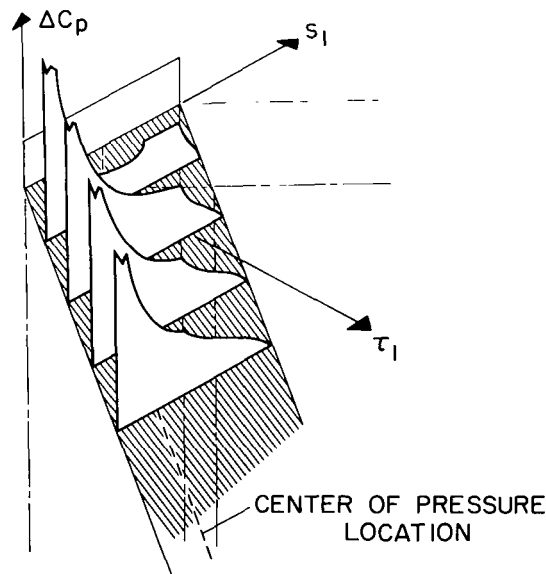


Fig. 8. Indicial loading on two-dimensional wing at subsonic speed.

at the leading edge but are as yet unaware of changed conditions at the trailing edge. The reverse is true for points in region (3). Points in region (4) have become aware of changes at both the leading and trailing edges, whereas points in region (5) in addition have become aware of the reflection from the trailing edge of the initial wave from the leading edge. Not long after the first reflection, the loading on the wing has essentially assumed the *form* of its ultimate steady-state loading (Fig. 8) and only its amplitude increases thereafter with increasing τ_1 , asymptotically approaching the steady-state amplitude. Therefore, as shown in Fig. 8, the center of pressure has essentially reached its steady-state position while the loading is still increasing in amplitude. The above behavior suggests breaking the indicial pitching-moment function into two separate contributions as shown in Fig. 9. The first variation represents the pitching-moment contribution of the integrated loading in region (1). Accordingly, it begins with the value $C_{m_\delta}(0; \delta(t), 0)$ and vanishes at $\tau_1 = t_b$. The second variation reflects the lumped contributions of the integrated loadings in the remaining regions; in conformity with the loading, its initial value

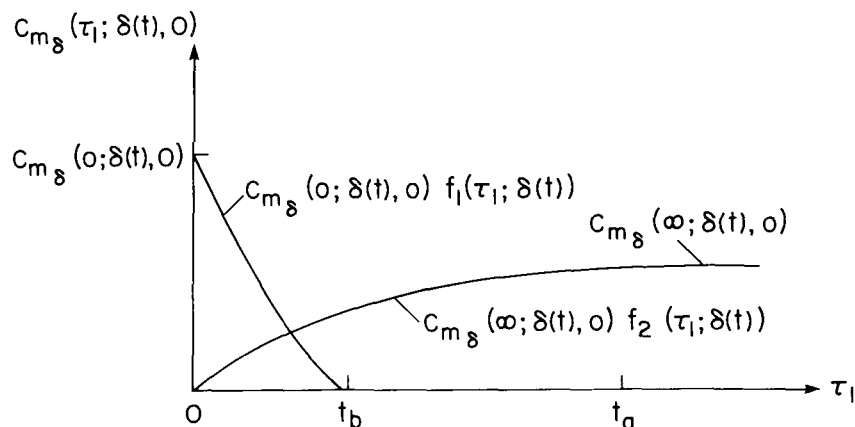


Fig. 9. Breakdown of indicial response.

is zero while its end value, essentially attained at $\tau_1 = t_a$, is $C_{m\delta}(\infty; \delta(t), 0)$. The sum of the two contributions is the indicial pitching-moment function $C_{m\delta}(\tau_1; \delta(t), 0)$. The end values of the two contributions are evidenced explicitly by introducing the normalized functions f_1 and f_2 as shown in the figure. This enables writing the indicial response in the form

$$C_{m\delta}(\tau_1; \delta(t), 0) = C_{m\delta}(0; \delta(t), 0)f_1(\tau_1; \delta(t)) + C_{m\delta}(\infty; \delta(t), 0)f_2(\tau_1; \delta(t)) \quad (31)$$

where f_1 and f_2 vary within the limits zero and unity. Then the deficiency function F_3 is

$$F_3(\tau_1; \delta(t), 0) = C_{m\delta}(\infty; \delta(t), 0)[1 - f_2(\tau_1; \delta(t))] - C_{m\delta}(0; \delta(t), 0)f_1(\tau_1; \delta(t)) \quad (32)$$

so that the integral for $C_{m\delta}^{\cdot}$ takes the form

$$\begin{aligned} \frac{L}{V} C_{m\delta}^{\cdot}(\delta(t)) = & C_{m\delta}(0; \delta(t), 0) \int_0^{t_b} f_1(\tau_1; \delta(t)) d\tau_1 \\ & - C_{m\delta}(\infty; \delta(t), 0) \int_0^{t_a} [1 - f_2(\tau_1; \delta(t))] d\tau_1 \end{aligned} \quad (33)$$

Equation (33) clearly shows the strong dependence of $C_{m\delta}^{\cdot}$ on the *steady-state* parameter $C_{m\delta}(\infty; \delta(t), 0)$. Further, the sign relation evidenced in Eq. (33) should be noted. Since the integral involving f_2

normally will be positive, the equation indicates that as $C_{m\delta}(\infty; \delta(t), 0)$ becomes more negative (i.e., statically more stable), $C_{m\delta}^*$ becomes more positive (i.e., dynamically more unstable).

Since f_1 and f_2 are normalized functions, cases are anticipated where their dependence on δ will not be significant. For example, the process of arriving at the form Eq. (33) suggests that it might be reasonably accurate to use the linear theory as a basis for estimating f_1 and f_2 in which case they would not show a dependence on δ . In such cases, the integrals in Eq. (33) become constants. When it can be further assumed that $C_{m\delta}(0; \delta(t), 0)$ is also independent of δ , the expression for $C_{m\delta}^*$ takes the simple form

$$C_{m\delta}^*(\delta(t)) = A + BC_{m\delta}(\infty; \delta(t), 0) \quad (34)$$

Despite differences in detail in the formation of the individual indicial responses, the representations of $C_{m\delta}^*$ given in Eqs. (33) and (34) should find applications to other more general categories of wings and bodies.

5.4 Experimental Evaluation of Stability Coefficients

Experimental evaluation of C_{mq} requires a true simulation of the pitching motion illustrated in Fig. 6(b); attempts to simulate this motion have proven so difficult as to be impracticable. On the other hand, an experimental determination of the indicial pitching-moment response, and thus $C_{m\delta}^*$, from the motion illustrated in Fig. 6(a) requires instrumentation having such rapid response characteristics as to be impracticable. The interpretation of $C_{Z\delta}^*$ given previously might provide a basis for an experimental determination of that coefficient

by means of appropriate energy measurements. However, difficulties remain in simulating the required motion, and in any case the interpretation does not suggest a basis for an analogous measurement of the more important coefficient $C_{m\dot{\delta}}$.

Rather than undertake these very difficult separate experiments, it can be argued that only one need be undertaken involving a combination of the terms C_{mq} and $C_{m\dot{\delta}}$, since they usually appear together in the equations of motion. While this need not be true in general, it is certainly the case when the aircraft is assumed to follow an essentially rectilinear flight path, i.e., when $q \approx \dot{\sigma}$. This is easily seen from Eq. (26), which can be rewritten in the form

$$C_m(t) = C_m(\infty; \delta(t), 0) + \dot{\sigma}(t) \frac{z}{V} \left[C_{mq}(\infty; \delta(t), 0) + \gamma C_{m\dot{\delta}}(\delta(t)) \right] + [q(t) - \dot{\sigma}(t)] \frac{z}{V} C_{mq}(\infty; \delta(t), 0) \quad (35)$$

The last term vanishes identically when the flight path is precisely rectilinear, and so may be neglected when departures from a rectilinear flight path are small. Hence, only a single measurement is required of the term $C_{mq} + \gamma C_{m\dot{\delta}}$, which is recognized as the damping-in-pitch coefficient. Clearly, a measurement of the coefficient that is in conformity with the way the term was derived requires a rectilinear flight path ($q = \dot{\sigma}$) and a motion involving small pitching oscillations in σ about a fixed σ equal to the instantaneous value of σ under consideration.

6. NONPLANAR MOTIONS

The treatment of nonplanar motions raises the question of the role played by *coupling* in a nonlinear formulation. By coupling is meant the following: in the analysis of, for example, a combined pitching and yawing motion, the linearity assumption (i.e., indicial responses independent of the past motion) allows the vector decomposition of the nonplanar motion into two orthogonal planar motions, the treatment of each planar motion as though the other were absent, and finally superposition of the separate results. In the nonlinear regime, where the indicial responses depend on all the variables of the past motion, the aerodynamic response to motion in one plane clearly will be influenced by the presence of the other motion. That influence is what is meant by coupling; the role assigned to it by the nonlinear formulation is studied in the sections to follow.

For simplicity, flight-path properties, including flight velocity, are again assumed to remain constant throughout the motion, so that a general motion involves five independent variables. These are taken to be the components of the aircraft's flight velocity and angular velocity resolved either in the aerodynamic axis system $(\delta, \psi, \dot{\lambda}, q, r)$ or in the body axis system $(\hat{\alpha}, \hat{\beta}, p_B, q_B, r_B)$. Having established the general integral form for the aerodynamic response to a motion involving two variables (Eq. 15), extending it to five variables is formal and requires no additional explanation. The general integral forms are assumed to have been derived and simplified in accordance with the same simplification invoked in the analysis of the planar motions; namely, that the indicial responses within the integrals, initially functionals, have been replaced by functions dependent only on the magnitude of the

past motion just prior to the origin of the steps. The analysis begins, then, with simplified integral forms analogous to the form applicable to planar motion, Eq. (18).

6.1 Approximate Formulation in the Aerodynamic Axis System

As noted above, the aerodynamic force and moment components resolved in the aerodynamic axis system are assumed to depend on the five variables δ , ψ , $\dot{\lambda}$, q , r . For example, the pitching-moment coefficient $C_m(t)$ is specified as a functional of the form

$$C_m(t) = G[\delta(\xi), \psi(\xi), \dot{\lambda}(\xi), q(\xi), r(\xi)] \quad (36)$$

The simplified integral form analogous to that of Eq. (18) thus contains five integrals, one for each variable. The indicial functions within the integrals depend on $t - \tau$ and the *magnitudes* of the five variables just prior to the origin of the steps. Just as in the planar case, the integrals can be further simplified by reducing them to forms correct to the first order in frequency, assuming that δ and ψ may be large but that the angular rates $\dot{\lambda}$, q , r remain small. If the procedure illustrated for the planar case (Eqs. 19-27) is followed, an expansion of the integrals about $\dot{\lambda} = 0$, $q = 0$, $r = 0$ yields, to first order in the rates, a sum of stability coefficients. The result is

$$\begin{aligned} C_m(t) = & C_m(\infty; \delta(t), \psi(t)) + \frac{\dot{\lambda}z}{V} C_{m\dot{\lambda}}(\infty; \delta(t), \psi(t)) \\ & + \frac{qz}{V} C_{mq}(\infty; \delta(t), \psi(t)) + \frac{rz}{V} C_{mr}(\infty; \delta(t), \psi(t)) \\ & + \frac{\dot{\delta}z}{V} C_{m\dot{\delta}}(\delta(t), \psi(t)) + \frac{\dot{\psi}z}{V} C_{m\dot{\psi}}(\delta(t), \psi(t)) \end{aligned} \quad (37)$$

where, as before, the infinity symbol indicates steady flow. For brevity the zeros belonging to $\dot{\lambda}$, q , r have been omitted. Analogous expressions for C_L and C_N and the axial, side and normal-force coefficients C_X , C_Y , C_Z are obtained by substituting these coefficients wherever C_m appears in Eq. (37).

6.2 Simplification of the Formulation in the Aerodynamic Axis System

Just as in the planar case (Eq. (35)), an additional simplification of Eq. (37) can be achieved by invoking the conditions of an almost rectilinear flight path. The conditions require, as before, $q \approx \dot{\sigma}$, and, in addition, $r \approx \varepsilon \dot{\lambda}$. Adding and subtracting the terms $(\dot{\sigma}l/V)C_{m_q}$ and $\varepsilon(\dot{\lambda}l/V)C_{m_r}$ and rewriting Eq. (37) yields

$$\begin{aligned}
 C_m(t) = & C_m(\infty; \delta(t), \psi(t)) + \frac{\dot{\psi}l}{V} C_{m_{\dot{\psi}}}(\delta(t), \psi(t)) \\
 & + \frac{\dot{\sigma}l}{V} \left[C_{m_q}(\infty; \delta(t), \psi(t)) + \gamma C_{m_{\dot{\sigma}}}(\delta(t), \psi(t)) \right] \\
 & + \frac{1}{\gamma} \frac{\dot{\lambda}l}{V} \left[\gamma C_{m_{\dot{\lambda}}}(\infty; \delta(t), \psi(t)) + \delta C_{m_r}(\infty; \delta(t), \psi(t)) \right] \\
 & + (q - \dot{\sigma}) \frac{l}{V} C_{m_q}(\infty; \delta(t), \psi(t)) \\
 & + (r - \varepsilon \dot{\lambda}) \frac{l}{V} C_{m_r}(\infty; \delta(t), \psi(t)) \tag{38}
 \end{aligned}$$

The last two terms vanish identically when the flight path is precisely rectilinear, and so may be neglected when departures from a rectilinear flight path are small. The remaining terms are identified by comparing them with those obtained when the flight path is precisely rectilinear, where exactly, $q = \dot{\sigma}$, $r = \delta \dot{\phi}$, $\dot{\lambda} = \gamma \dot{\phi}$, and $\dot{\phi}$ is the *coning* rate of the longitudinal axis around the flight velocity vector. The result is

$$\left. \begin{aligned} C_{m\sigma}^{\dot{}} &= C_{m_q} + \gamma C_{m\delta}^{\dot{}} \\ C_{m\phi}^{\dot{}} &= \gamma C_{m\lambda}^{\dot{}} + \delta C_{m_r} \end{aligned} \right\} (39)$$

Each of the terms in Eq. (38) is associated with a particular motion from which it may be evaluated. The term $C_m(\infty; \delta(t), \psi(t))$ is the pitching-moment coefficient that would be measured in a steady planar motion with δ and ψ at the fixed inclinations $\delta = \text{const}$, $\psi = \text{const}$. The term $C_{m_\psi}^{\dot{}}$ is a damping-in-roll coefficient that would be measured from small oscillations in ψ about $\psi = \text{const}$ with δ fixed at $\delta = \text{const}$ and $\dot{\phi}$ fixed at zero. The term $C_{m_\sigma}^{\dot{}}$ is, as before, the damping-in-pitch coefficient for planar pitching oscillations about an axis normal to the plane of σ , measured now, however, with *both* δ and ψ at the fixed inclinations $\delta = \text{const}$, $\psi = \text{const}$. The term $C_{m_\phi}^{\dot{}}$ is the rate of change with $\dot{\phi}Z/V$, evaluated at $\dot{\phi} = 0$, of the pitching-moment coefficient that would be measured in a steady coning motion, $\delta = \text{const}$, $\psi = \text{const}$, $\dot{\phi} = \text{const}$. Thus, coning motion emerges as one of the characteristic motions required in constructing the response to an arbitrary nonplanar motion.

In summary, with terms multiplied by $(q - \dot{\sigma})$ and $(r - \epsilon\dot{\lambda})$ neglected, the aerodynamic force and moment system takes the form

$$\begin{aligned} C_k(t) &= C_k(\infty; \delta(t), \psi(t)) + \frac{\dot{\psi}Z}{V} C_{k_\psi}^{\dot{}}(\delta(t), \psi(t)) \\ &+ \frac{\dot{\sigma}Z}{V} C_{k_\sigma}^{\dot{}}(\delta(t), \psi(t)) + \frac{\dot{\lambda}Z}{V} \frac{1}{\gamma} C_{k_\phi}^{\dot{}}(\infty; \delta(t), \psi(t)); \quad k = \begin{cases} X, Y, Z \\ Z, m, n \end{cases} \quad (40) \end{aligned}$$

Equation (40) suggests that, for nearly rectilinear flight paths, the forces and moments due to an arbitrary motion may be compounded of the

contributions from four simple motions: steady resultant angle of attack, oscillations in roll and pitch at constant resultant angle of attack, and coning at constant resultant angle of attack, all at a constant inclination of the body axes from the crossflow velocity vector. The motions are illustrated schematically in Fig. 10.

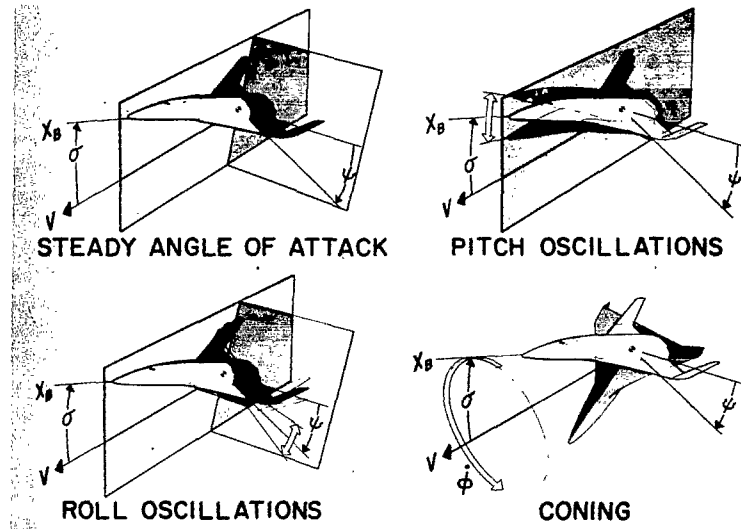


Fig. 10. Characteristic motions in the aerodynamic axis system.
Linear dependence on angular rates.

6.3 Approximate Formulation in the Body Axis System

The aerodynamic force and moment components resolved in the body axis system are assumed to depend on the five variables $\hat{\alpha}$, $\hat{\beta}$, p_B , q_B , r_B . For example, the pitching-moment coefficient $\hat{C}_m(t)$ is said to be a functional of the form

$$\hat{C}_m(t) = H[\hat{\alpha}(\xi), \hat{\beta}(\xi), p_B(\xi), q_B(\xi), r_B(\xi)] \quad (41)$$

The reductions of the integral form based on Eq. (41) parallel those of the preceding sections. Expanded about $p_B = 0$, $q_B = 0$, $r_B = 0$, the integral form for $\hat{C}_m(t)$ yields, to first order in the rates

$$\begin{aligned}
\hat{C}_m(t) = & \hat{C}_m(\infty; \hat{\alpha}(t), \hat{\beta}(t)) + \frac{p_B \dot{z}}{V} \hat{C}_{m_{p_B}}(\infty; \hat{\alpha}(t), \hat{\beta}(t)) \\
& + \frac{q_B \dot{z}}{V} \hat{C}_{m_{q_B}}(\infty; \hat{\alpha}(t), \hat{\beta}(t)) + \frac{r_B \dot{z}}{V} \hat{C}_{m_{r_B}}(\infty; \hat{\alpha}(t), \hat{\beta}(t)) \\
& + \frac{\dot{\alpha} z}{V} \hat{C}_{m_{\dot{\alpha}}}(\hat{\alpha}(t), \hat{\beta}(t)) + \frac{\dot{\beta} z}{V} \hat{C}_{m_{\dot{\beta}}}(\hat{\alpha}(t), \hat{\beta}(t))
\end{aligned} \tag{42}$$

where the zeros belonging to p_B , q_B , r_B have been omitted. Analogous expressions for \hat{C}_z and \hat{C}_n and the axial, side, and normal-force coefficients \hat{C}_x , \hat{C}_y , \hat{C}_z are obtained by substituting these coefficients wherever \hat{C}_m appears in Eq. (42).

That Eqs. (37) and (42) yield compatible forms may be verified by transferring \hat{C}_m and \hat{C}_n to the aerodynamic axis system by the use of Eq. (9) and then replacing the variables $\dot{\alpha}$, $\dot{\beta}$, p_B , q_B , r_B by variables in the aerodynamic axis system through the use of Eqs. (4), (6), and (8). It will be found that each coefficient in $C_m(t)$ and $C_n(t)$ can be matched with a combination of coefficients in $\hat{C}_m(t)$ and $\hat{C}_n(t)$ having the same multiplying variable. The matches for the coefficients in $C_m(t)$ yield:

$$\left. \begin{aligned}
C_m(\infty; \delta, \psi) &= \hat{C}_m(\infty; \hat{\alpha}, \hat{\beta}) \cos \psi - \hat{C}_n(\infty; \hat{\alpha}, \hat{\beta}) \sin \psi \\
C_{m_\lambda}(\infty; \delta, \psi) &= \hat{C}_{m_{p_B}}(\infty; \hat{\alpha}, \hat{\beta}) \cos \psi - \hat{C}_{n_{p_B}}(\infty; \hat{\alpha}, \hat{\beta}) \sin \psi \\
C_{m_q}(\infty; \delta, \psi) &= \hat{C}_{m_{q_B}}(\infty; \hat{\alpha}, \hat{\beta}) \cos^2 \psi + \hat{C}_{n_{r_B}}(\infty; \hat{\alpha}, \hat{\beta}) \sin^2 \psi \\
&\quad - \left[\hat{C}_{n_{q_B}}(\infty; \hat{\alpha}, \hat{\beta}) + \hat{C}_{m_{r_B}}(\infty; \hat{\alpha}, \hat{\beta}) \right] \cos \psi \sin \psi
\end{aligned} \right\} \tag{43}^*$$

*Eq. (43) continued on following page

$$\begin{aligned}
C_{m_r}(\infty; \delta, \psi) &= \hat{C}_{m_{rB}}(\infty; \hat{\alpha}, \hat{\beta}) \cos^2 \psi - \hat{C}_{n_{qB}}(\infty; \hat{\alpha}, \hat{\beta}) \sin^2 \psi \\
&\quad + \left[\hat{C}_{m_{qB}}(\infty; \hat{\alpha}, \hat{\beta}) - \hat{C}_{n_{rB}}(\infty; \hat{\alpha}, \hat{\beta}) \right] \cos \psi \sin \psi \\
C_{m_{\delta}}(\delta, \psi) &= \hat{C}_{m_{\alpha}}(\hat{\alpha}, \hat{\beta}) \cos^2 \psi - \hat{C}_{n_{\beta}}(\hat{\alpha}, \hat{\beta}) \sin^2 \psi \\
&\quad + \left[\hat{C}_{m_{\beta}}(\hat{\alpha}, \hat{\beta}) - \hat{C}_{n_{\alpha}}(\hat{\alpha}, \hat{\beta}) \right] \cos \psi \sin \psi \\
C_{m_{\psi}}(\delta, \psi) &= \hat{C}_{m_{pB}}(\infty; \hat{\alpha}, \hat{\beta}) \cos \psi - \hat{C}_{n_{pB}}(\infty; \hat{\alpha}, \hat{\beta}) \sin \psi \\
&\quad + \delta \hat{C}_{m_{\beta}}(\hat{\alpha}, \hat{\beta}) \cos^2 \psi + \delta \hat{C}_{n_{\alpha}}(\hat{\alpha}, \hat{\beta}) \sin^2 \psi \\
&\quad - \delta \left[\hat{C}_{m_{\alpha}}(\hat{\alpha}, \hat{\beta}) + \hat{C}_{n_{\beta}}(\hat{\alpha}, \hat{\beta}) \right] \cos \psi \sin \psi
\end{aligned} \tag{43}$$

The analogous matches for the coefficients in $C_n(t)$ may be obtained from Eq. (43) by replacing \hat{C}_{m_i} with \hat{C}_{n_i} and \hat{C}_{n_i} with $-\hat{C}_{m_i}$. The matches for the damping-in-pitch coefficient $C_{m_{\sigma}}$ and the side-moment coefficients $C_{n_{\phi}}$ and $(C_{n_{\phi}} - \gamma C_{n_{\psi}})$ are of particular interest. Equations (39) and (43) yield

$$\begin{aligned}
C_{m_{\sigma}}(\delta, \psi) &= C_{m_q} + \gamma C_{m_{\delta}} \\
&= (\hat{C}_{m_{qB}} + \gamma \hat{C}_{m_{\alpha}}) \cos^2 \psi + (\hat{C}_{n_{rB}} - \gamma \hat{C}_{n_{\beta}}) \sin^2 \psi \\
&\quad - \left[(\hat{C}_{n_{qB}} + \gamma \hat{C}_{n_{\alpha}}) + (\hat{C}_{m_{rB}} - \gamma \hat{C}_{m_{\beta}}) \right] \cos \psi \sin \psi
\end{aligned} \tag{44}$$

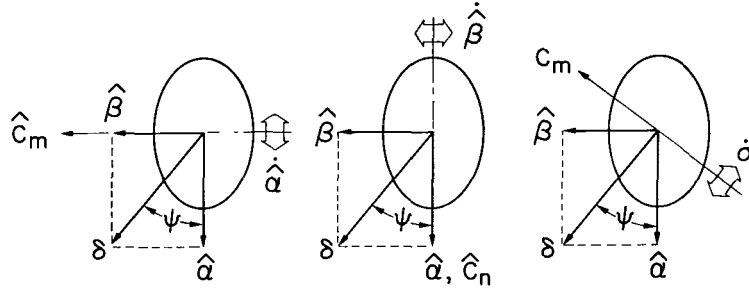
$$\begin{aligned}
C_{n\dot{\phi}}(\infty; \delta, \psi) &= \gamma C_{n\dot{\lambda}} + \delta C_{n\dot{r}} \\
&= \gamma (\hat{C}_{n_{p_B}} \cos \psi + \hat{C}_{m_{p_B}} \sin \psi) \\
&\quad + \delta \left[\hat{C}_{n_{r_B}} \cos^2 \psi + \hat{C}_{m_{q_B}} \sin^2 \psi \right. \\
&\quad \left. + (\hat{C}_{n_{q_B}} + \hat{C}_{m_{r_B}}) \cos \psi \sin \psi \right] \tag{45}
\end{aligned}$$

$$\begin{aligned}
C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}} &= \delta \left\{ (\hat{C}_{m_{q_B}} + \gamma \hat{C}_{m_{\dot{\alpha}}}) \sin^2 \psi \right. \\
&\quad + (\hat{C}_{n_{r_B}} - \gamma \hat{C}_{n_{\dot{\beta}}}) \cos^2 \psi \\
&\quad \left. + \left[(\hat{C}_{n_{q_B}} + \gamma \hat{C}_{n_{\dot{\alpha}}}) + (\hat{C}_{m_{r_B}} - \gamma \hat{C}_{m_{\dot{\beta}}}) \right] \cos \psi \sin \psi \right\} \tag{46}
\end{aligned}$$

Combining Eqs. (44) and (46) yields the following interesting equality:

$$C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}} + \delta C_{m_{\dot{\sigma}}} = \delta \left[(\hat{C}_{m_{q_B}} + \gamma \hat{C}_{m_{\dot{\alpha}}}) + (\hat{C}_{n_{r_B}} - \gamma \hat{C}_{n_{\dot{\beta}}}) \right] \tag{47}$$

The term $(\hat{C}_{m_{q_B}} + \gamma \hat{C}_{m_{\dot{\alpha}}})$ is recognized as being the planar damping-in-pitch coefficient measured along y_B for small oscillations in $\hat{\alpha}$ about $\hat{\alpha} = \text{const}$ with $\hat{\beta}$ held fixed at $\hat{\beta} = \text{const}$. Similarly, $(\hat{C}_{n_{r_B}} - \gamma \hat{C}_{n_{\dot{\beta}}})$ is the damping-in-yaw coefficient measured along z_B for small oscillations in $\hat{\beta}$ about $\hat{\beta} = \text{const}$, with $\hat{\alpha}$ held fixed at $\hat{\alpha} = \text{const}$. Thus, a measurement of $(C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}})$ would be equivalent to a measure of a combination of the three damping coefficients. The identity is shown schematically in Fig. 11. Equation (47) generalizes to the nonlinear case and to arbitrary bodies the relationship between $(C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}})$



$$\frac{C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}}}{\delta} = (\hat{C}_{m_{q_B}} + \gamma \hat{C}_{m_{\dot{\alpha}}}) + (\hat{C}_{n_{r_B}} - \gamma \hat{C}_{n_{\dot{\beta}}}) - C_{m_{\dot{\sigma}}}$$

Fig. 11. Schematic representation of the equality between

$$(C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}})/\delta \quad \text{and the three damping coefficients.}$$

and the damping coefficients that was pointed out and verified for bodies of revolution in the linear case in Ref. 5. It is noted that $(C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}})$ equals $\delta(\hat{C}_{m_{q_B}} + \gamma \hat{C}_{m_{\dot{\alpha}}})$ when $\hat{\alpha} = 0$ and equals $\delta(\hat{C}_{n_{r_B}} - \gamma \hat{C}_{n_{\dot{\beta}}})$ when $\hat{\beta} = 0$. Under conditions where a linear formulation of the moment system can be assumed to hold (e.g., when $\hat{\alpha} \rightarrow 0$, $\hat{\beta} \rightarrow 0$) it is consistent to assume that the couplings between motions in $\hat{\alpha}$ and $\hat{\beta}$ will be negligibly small. Under these conditions, the measurement of $(C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}})$ at $\hat{\alpha} = 0$ and again at $\hat{\beta} = 0$ is all that is required to yield measures of the damping coefficients characteristic of the two uncoupled modes.

6.4 Simplification of the Formulation in the Body Axis System

When the assumption of a nearly rectilinear flight path is justified, Eq. (42) can be simplified to yield a form analogous to that of

Eq. (38) in the aerodynamic axis system. The approximate expressions for q_B and r_B , consistent with $q \approx \dot{\sigma}$, $r \approx \epsilon \dot{\lambda}$, are

$$\left. \begin{aligned} q_B &\approx \frac{\dot{\hat{\alpha}}}{\gamma} + p_B \frac{\hat{\beta}}{\gamma} \\ r_B &\approx -\frac{\dot{\hat{\beta}}}{\gamma} + p_B \frac{\hat{\alpha}}{\gamma} \end{aligned} \right\} (48)$$

Substituting in Eq. (42) gives

$$\begin{aligned} \hat{C}_m(t) = & \hat{C}_m(\infty; \hat{\alpha}, \hat{\beta}) + \frac{1}{\gamma} \frac{p_B \dot{\lambda}}{V} \left[\gamma \hat{C}_{mp_B}(\infty; \hat{\alpha}, \hat{\beta}) + \hat{\beta} \hat{C}_{mq_B}(\infty; \hat{\alpha}, \hat{\beta}) \right. \\ & \left. + \hat{\alpha} \hat{C}_{mr_B}(\infty; \hat{\alpha}, \hat{\beta}) \right] + \frac{1}{\gamma} \frac{\dot{\hat{\alpha}} \dot{\lambda}}{V} \left[\hat{C}_{mq_B}(\infty; \hat{\alpha}, \hat{\beta}) \right. \\ & \left. + \gamma \hat{C}_{m\dot{\alpha}}(\hat{\alpha}, \hat{\beta}) \right] - \frac{1}{\gamma} \frac{\dot{\hat{\beta}} \dot{\lambda}}{V} \left[\hat{C}_{mr_B}(\infty; \hat{\alpha}, \hat{\beta}) - \gamma \hat{C}_{m\dot{\beta}}(\hat{\alpha}, \hat{\beta}) \right] \end{aligned} \quad (49)$$

Again, each of the terms in Eq. (49) is associated with a particular motion from which it can be evaluated. The first term is the pitching-moment coefficient along y_B that would be measured in steady planar motion with $\hat{\alpha}$ and $\hat{\beta}$ at the fixed inclinations $\hat{\alpha} = \text{const}$, $\hat{\beta} = \text{const}$. The combination of terms multiplied by $p_B \dot{\lambda}/V\gamma$ can be shown (from Eq. (45)) to be the rate of change with $\dot{\phi}\dot{\lambda}/V$, evaluated at $\dot{\phi} = 0$, of the pitching-moment coefficient along y_B that would be measured in a steady coning motion $\hat{\alpha} = \text{const}$, $\hat{\beta} = \text{const}$, $\dot{\phi} = \text{const}$. The third term has already been defined as the damping-in-pitch coefficient measured along y_B for small oscillations in $\hat{\alpha}$ about $\hat{\alpha} = \text{const}$ with $\hat{\beta}$ held fixed at $\hat{\beta} = \text{const}$. The term $(\hat{C}_{mr_B} - \gamma \hat{C}_{m\dot{\beta}})$ is a cross-coupling term resulting from the oscillatory motion in the $\hat{\beta}$ plane required for the

evaluation of the damping-in-yaw coefficient. This term and the analogous term in $\hat{C}_n(t)$, $(\hat{C}_{nq_B} + \gamma\hat{C}_{n\dot{\alpha}})$ are the nonlinear interaction terms that are normally excluded in the classical treatment and are missed by attempts to generalize from linear formulations based on the principle of superposition.

In summary, for nearly rectilinear flight paths, the aerodynamic force and moment system in body axes take the form

$$\hat{C}_k(t) = \hat{C}_k(\infty; \hat{\alpha}, \hat{\beta}) + \frac{1}{\gamma} \frac{P_B \bar{z}}{V} \hat{C}_{k\dot{\phi}}(\infty; \hat{\alpha}, \hat{\beta}) + \frac{1}{\gamma} \frac{\dot{\hat{\alpha}} \bar{z}}{V} \left[\hat{C}_{kq_B}(\infty; \hat{\alpha}, \hat{\beta}) + \gamma \hat{C}_{k\dot{\alpha}}(\hat{\alpha}, \hat{\beta}) \right] - \frac{1}{\gamma} \frac{\dot{\hat{\beta}} \bar{z}}{V} \left[\hat{C}_{kr_B}(\infty; \hat{\alpha}, \hat{\beta}) - \gamma \hat{C}_{k\dot{\beta}}(\hat{\alpha}, \hat{\beta}) \right]; \quad k = \begin{cases} X, Y, Z \\ Z, m, n \end{cases} \quad (50)$$

In the body axis system, the four characteristic motions are steady angle of attack and sideslip, coning at constant angle of attack and sideslip, and the oscillations in pitch and in yaw at constant angles of attack and sideslip. The oscillations-in-roll motion that was required in the aerodynamic axis system is, in effect, incorporated in the oscillations in pitch and yaw motions in the body axis system. An oscillation in $\hat{\alpha}$ with $\hat{\beta}$ held fixed, for example, will be seen to involve an oscillation in ψ . The four motions are illustrated schematically in Fig. 12.

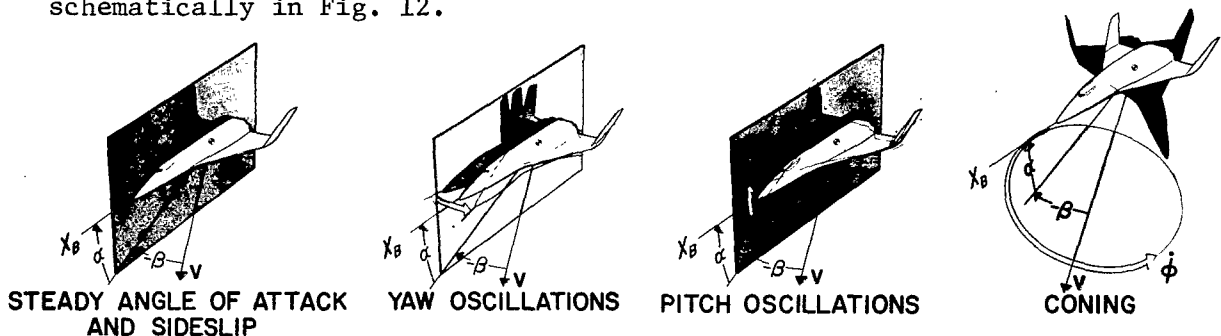


Fig. 12. Characteristic motions in the body axis system.
Linear dependence on angular rates.

6.5 Body of Revolution

Symmetry considerations in the body axis system can be used to further specify the form of the aerodynamic coefficients. The arguments will be carried out for the body of revolution, since they can be carried farthest in this case and will indicate the form of the interaction terms.

Consider an arbitrary motion in $\hat{\alpha}$ and $\hat{\beta}$ with $p_B = 0$. The most general form that can be written for the pitching moment that is of third order in the angles and first order in the rates is

$$\begin{aligned} \hat{C}_m(t) = & a_0 + a_1\hat{\alpha} + a_2\hat{\alpha}^2 + a_3\hat{\alpha}^3 + a_4\hat{\beta} + a_5\hat{\beta}^2 + a_6\hat{\beta}^3 + a_7\hat{\alpha}\hat{\beta} + a_8\hat{\alpha}^2\hat{\beta} + a_9\hat{\alpha}\hat{\beta}^2 \\ & + \frac{1}{\gamma} \frac{\dot{\hat{\alpha}}\hat{z}}{V} (b_0 + b_1\hat{\alpha} + b_2\hat{\alpha}^2 + b_3\hat{\beta} + b_4\hat{\beta}^2 + b_5\hat{\alpha}\hat{\beta}) \\ & + \frac{1}{\gamma} \frac{\dot{\hat{\beta}}\hat{z}}{V} (c_0 + c_1\hat{\alpha} + c_2\hat{\alpha}^2 + c_3\hat{\beta} + c_4\hat{\beta}^2 + c_5\hat{\alpha}\hat{\beta}) \end{aligned} \quad (51)$$

The requirement that \hat{C}_m be an odd function of $\hat{\alpha}$ and an even function of $\hat{\beta}$ reduces Eq. (51) to

$$\hat{C}_m(t) = \hat{\alpha}(a_1 + a_3\hat{\alpha}^2 + a_9\hat{\beta}^2) + \frac{1}{\gamma} \frac{\dot{\hat{\alpha}}\hat{z}}{V} (b_0 + b_2\hat{\alpha}^2 + b_4\hat{\beta}^2) + \frac{1}{\gamma} \frac{\dot{\hat{\beta}}\hat{z}}{V} (c_5\hat{\alpha}\hat{\beta}) \quad (52)$$

The requirement that \hat{C}_n be equal to $-\hat{C}_m$ when $\hat{\alpha}$ and $\hat{\beta}$ are interchanged gives for $\hat{C}_n(t)$

$$-\hat{C}_n(t) = \hat{\beta}(a_1 + a_3\hat{\beta}^2 + a_9\hat{\alpha}^2) + \frac{1}{\gamma} \frac{\dot{\hat{\beta}}\hat{z}}{V} (b_0 + b_2\hat{\beta}^2 + b_4\hat{\alpha}^2) + \frac{1}{\gamma} \frac{\dot{\hat{\alpha}}\hat{z}}{V} (c_5\hat{\alpha}\hat{\beta}) \quad (53)$$

Finally, the requirement that the total moment in the crossflow plane be normal to the direction of δ when the motion is planar ($\dot{\psi} = 0$) gives

$$\left. \begin{aligned} a_3 &= a_9 \\ b_4 &= b_2 - c_5 \end{aligned} \right\} (54)$$

so that

$$\begin{aligned} \hat{C}_m(t) &= \hat{\alpha}(a_1 + a_3\delta^2) + \frac{1}{\gamma} \frac{\dot{\hat{\alpha}}\dot{z}}{V} (b_0 + b_2\delta^2 - c_5\hat{\beta}^2) + \frac{1}{\gamma} \frac{\dot{\hat{\beta}}\dot{z}}{V} (c_5\hat{\alpha}\hat{\beta}) \\ -\hat{C}_n(t) &= \hat{\beta}(a_1 + a_3\delta^2) + \frac{1}{\gamma} \frac{\dot{\hat{\beta}}\dot{z}}{V} (b_0 + b_2\delta^2 - c_5\hat{\alpha}^2) + \frac{1}{\gamma} \frac{\dot{\hat{\alpha}}\dot{z}}{V} (c_5\hat{\alpha}\hat{\beta}) \end{aligned} \quad (55)$$

Matching coefficients in Eq. (50) yields

$$\left. \begin{aligned} \hat{C}_m(\infty; \hat{\alpha}, \hat{\beta}) &= \hat{\alpha}(a_1 + a_3\delta^2) \\ \hat{C}_{mq_B} + \gamma \hat{C}_{m\dot{\alpha}} &= (b_0 + b_2\delta^2 - c_5\hat{\beta}^2) \\ \hat{C}_{mr_B} - \gamma \hat{C}_{m\dot{\beta}} &= -c_5\hat{\alpha}\hat{\beta} \end{aligned} \right\} (56a)$$

$$\left. \begin{aligned} \hat{C}_n(\infty; \hat{\alpha}, \hat{\beta}) &= -\hat{\beta}(a_1 + a_3\delta^2) \\ \hat{C}_{nr_B} - \gamma \hat{C}_{n\dot{\beta}} &= (b_0 + b_2\delta^2 - c_5\hat{\alpha}^2) \\ \hat{C}_{nq_B} + \gamma \hat{C}_{n\dot{\alpha}} &= -c_5\hat{\alpha}\hat{\beta} \end{aligned} \right\} (56b)$$

For the body of revolution, the interaction terms are equal and their form to second order is a product of $\hat{\alpha}$ and $\hat{\beta}$. Equation (56) can be

used to find the analogous terms in the aerodynamic axis system.

Equations (43) - (47) yield

$$\left. \begin{aligned} C_{\underline{m}}(\infty; \delta, \psi) &= \delta(a_1 + a_3\delta^2) \\ C_{\underline{m}\dot{\sigma}}(\delta, \psi) &= (b_0 + b_2\delta^2) \\ C_{\underline{m}\dot{\phi}} - \gamma C_{\underline{m}\dot{\psi}} &= 0 \end{aligned} \right\} (57a)$$

$$\left. \begin{aligned} C_{\underline{n}}(\infty; \delta, \psi) &= 0 \\ C_{\underline{n}\dot{\sigma}}(\delta, \psi) &= 0 \\ C_{\underline{n}\dot{\phi}} - \gamma C_{\underline{n}\dot{\psi}} &= \delta(b_0 + b_2\delta^2 - c_5\delta^2) \end{aligned} \right\} (57b)$$

As required, the coefficients in the aerodynamic axis system are functions only of δ . Note that $C_{\underline{m}\dot{\sigma}}$ equals $(\hat{C}_{\underline{m}q_B} + \gamma\hat{C}_{\underline{m}\dot{\alpha}})$ only when $\hat{\beta} = 0$ and equals $(\hat{C}_{\underline{n}r_B} - \gamma\hat{C}_{\underline{n}\dot{\beta}})$ only when $\hat{\alpha} = 0$. That this must be true may be verified by inspection of Fig. 11. The relationship between $C_{\underline{m}\dot{\sigma}}$ and $(C_{\underline{n}\dot{\phi}} - \gamma C_{\underline{n}\dot{\psi}})/\delta$ is of particular interest. Equation (57) indicates that to first order in δ (i.e., small δ where a linearized theory can be expected to hold), the two terms must be equal; but this requirement does *not* hold for larger values of δ where terms of $O(\delta^2)$ must be retained. The consequences of the breakdown of the equality have been discussed several times (cf., for example, Refs. 6 and 14). Briefly, breakdown of the equality is proof that the linear stability derivative formulation cannot be extrapolated into the nonlinear regime simply by allowing the coefficients that appear as

identical constants in the linear formulation to become identical functions of δ . A formulation incorporating this erroneous assumption can be shown to yield misleading results when used in a program for extracting nonlinear aerodynamic coefficients from free-flight data [6].

The significant aspects of the relationship between $C_{m\dot{\sigma}}$ and $(C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}})/\delta$ have been confirmed by the results of two recent experiments. The first involved spinning and coning motions of a slender cone at a supersonic speed [5]. The results, obtained with an apparatus that allows investigation of separate or combined spinning and coning motions, are important in two respects: first, experiments with a purely spinning cone at constant δ (i.e., the classical Magnus experiment $\delta = \text{const}$, $\dot{\psi} = \text{const}$, $\dot{\phi} = 0$, which, for bodies of revolution, can be used to replace the oscillations-in-roll experiment required to measure $C_{n\dot{\psi}}$ for nonaxisymmetric bodies) failed to reveal a measurable Magnus moment coefficient $C_{n\dot{\psi}}$; and second, the results confirmed that, with $C_{n\dot{\psi}}$ negligibly small, the equality that should exist for small δ is an equality between the side-moment coefficient due to coning $C_{n\dot{\phi}}/\delta$ and $C_{m\dot{\sigma}}$. The main results are shown in Fig. 13. Measured values of $C_{n\dot{\phi}}$ are seen to be in excellent agreement at small δ with the straight-line variation $\delta C_{m\dot{\sigma}}$ obtained from a linear potential theory [15]. The breakdown of the equality at larger δ has been confirmed by the results of a second experiment carried out by Iyengar [16]. This was a careful oscillations-in-pitch experiment for $C_{m\dot{\sigma}}$ over a large range of δ with both model and wind-tunnel conditions the same as those used in the coning experiment. The results, also shown in Fig. 13, clearly confirm that the equality does not hold at

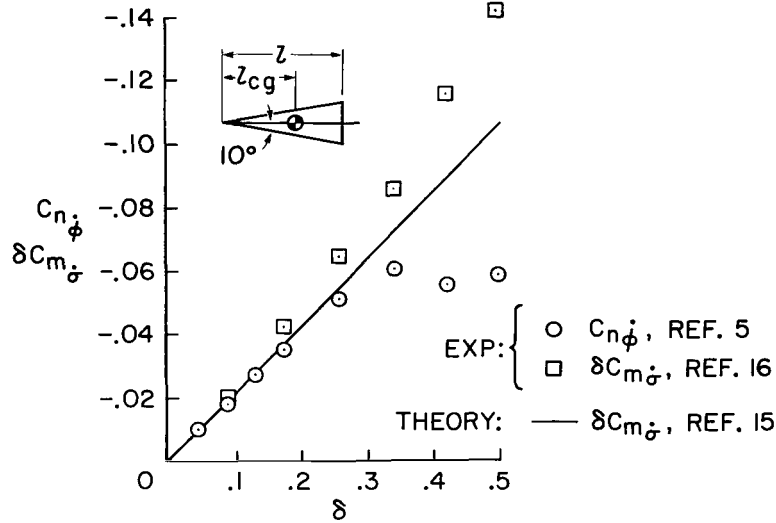


Fig. 13. Variation of $\delta C_{m\dot{\sigma}}$ and $C_{n\dot{\phi}}$ with δ ;
Mach number = 2.0, $l_{cg}/L = 0.61$.

large δ . Equation (57) shows that the source of the inequality between $C_{m\dot{\sigma}}$ and $(C_{n\dot{\phi}} - \gamma C_{n\dot{\psi}})/\delta$ lies in the term c_5 , the coefficient of the cross-coupling terms $(\hat{C}_{m_{r_B}} - \gamma \hat{C}_{m_{\dot{\beta}}})$ and $(\hat{C}_{n_{q_B}} + \gamma \hat{C}_{n_{\dot{\alpha}}})$. The experimental results indicate that $|c_5| > |b_2|$, where b_2 is the coefficient of the nonlinear contribution to $C_{m\dot{\sigma}}$. That is, the cross-coupling terms are at least of the same order of magnitude as the nonlinear part of the principal damping coefficient, and so must be retained for consistency whenever nonlinearities are admitted into the principal terms of the aerodynamic formulation. Why the cross-coupling terms must exist when $\hat{\alpha}$ and $\hat{\beta}$ become sufficiently large is indicated below.

Consider a body of revolution in uniform sinking and sideslipping motion $\hat{\alpha} = \text{const}$, $\hat{\beta} = \text{const}$, with $p_B, q_B, r_B = 0$. The motion is illustrated in Fig. 14. The resultant normal-force coefficient $\hat{C}_R(\infty; \hat{\alpha}, \hat{\beta})$ initially is aligned with the resultant angle-of-attack vector. At time zero, the body is given a step change in $\hat{\alpha}$ with $\hat{\beta}$ held fixed.

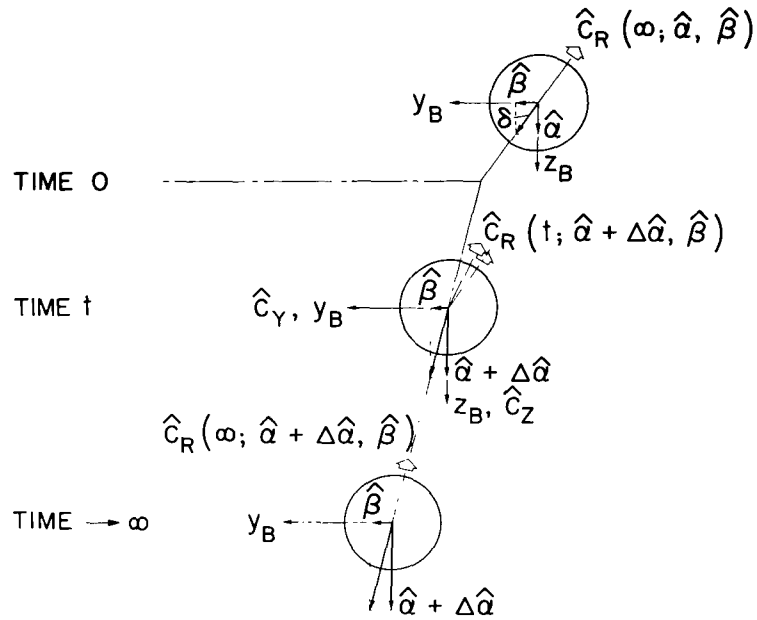


Fig. 14. Maneuver showing origin of cross-coupling term $\hat{C}_{Y\dot{\alpha}}$.

This is the experiment that would yield the indicial normal-force response $\hat{C}_{Z\dot{\alpha}}(t; \hat{\alpha}, \hat{\beta})$ to which the term $\hat{C}_{Z\dot{\alpha}}$ is related by an integral. As shown in Fig. 14, subsequent to the change in $\hat{\alpha}$ the resultant normal-force coefficient $\hat{C}_R(t; \hat{\alpha} + \Delta\hat{\alpha}, \hat{\beta})$, in addition to growing, must rotate in order that it be aligned with the resultant angle of attack again after a steady state has been established. Clearly, a step change in $\hat{\alpha}$ will induce a time-dependent side force in the interval during which the resultant normal force rotates to its new position. The induced time-dependent side force is related by an integral to the cross-coupling term $\hat{C}_{Y\dot{\alpha}}$ that appears in the expression for $\hat{C}_Y(t)$. Such a term must exist whenever a maneuver in $\hat{\alpha}$ occurs in the presence of a nonzero $\hat{\beta}$.

6.6 Aircraft Spin Motions

The emergence of coning motion as a characteristic motion in *both* the aerodynamic and body axis systems emphasizes its importance in the nonlinear formulation. With $p_B = 0$, coning motion is equivalently the simultaneous periodic variation of two orthogonal planar motions; in thus enabling two planar motions to interact, coning motion can be said to characterize the coupling problem.

Further evidence of its important is the obvious similarity between coning motion and the steady spin of an aircraft, suggesting that a moment formulation based on either Eq. (40) or Eq. (50) will properly describe the aerodynamic responses of spinning aircraft. This view is encouraged by the success achieved in two attempts to reproduce aircraft or model spin motions by calculations based on aerodynamic formulations bearing a similarity to those proposed here. In the first [17], the actual spins of an F-100 aircraft were reproduced by calculations based on an aerodynamic formulation that called principally for wind-tunnel measurements of the conventional static forces and moments. In the second [18], the spins of a delta-wing model in a spin tunnel were reproduced by calculations based on an aerodynamic formulation that called principally for wind-tunnel measurements of the forces and moments on a model in coning motion. The formulations based on Eqs. (40) and (50) in effect include these motions within the four characteristic motions whose force and moment contributions are required to build up the response to an arbitrary motion. It is known, however, that in the establishment of a spin the large asymmetric regions of separated flow on the wings of the aircraft may cause the aerodynamic responses to be

nonlinear functions of the spin rate, even at low spin rates. This contradicts the assumption underlying the development of Eqs. (40) and (50) that the aerodynamic responses be linear functions of the rates. The authors have shown recently [9] how the formulations can be generalized to allow a nonlinear dependence on the coning rate, which should make them more fully applicable to the analysis of spin motions. The restriction imposed previously on the degree of cognizance of the past motion allowed the indicial response remains in force, however, so that aerodynamic hysteresis effects still cannot be acknowledged within the scope of the otherwise extended formulations. The main results are indicated below.

6.6.1 *Aerodynamic Axis System*: The generalized formulation paralleling that of Eq. (40) (again assuming a nearly rectilinear flight path) takes the form

$$\begin{aligned}
 C_k(t) = & C_k \left(\infty; \delta(t), \psi(t), \frac{\dot{\lambda}}{\gamma}(t) \right) + \frac{\dot{\sigma}Z}{V} C_{k\sigma} \left(\delta(t), \psi(t), \frac{\dot{\lambda}}{\gamma}(t) \right) \\
 & + \frac{\dot{\psi}Z}{V} C_{k\psi} \left(\delta(t), \psi(t), \frac{\dot{\lambda}}{\gamma}(t) \right); \quad k = \begin{array}{l} X, Y, Z \\ Z, m, n \end{array} \quad (58)
 \end{aligned}$$

That Eq. (58) reverts to Eq. (40) when the linearity condition on coning rate is reimposed can be seen by expanding the terms in Eq. (58) to first order about $\dot{\lambda}/\gamma = 0$. The first term in Eq. (58) is the coefficient that would be measured in a steady coning motion $\delta = \text{const}$, $\psi = \text{const}$, $\dot{\phi} = \dot{\lambda}/\gamma = \text{const}$. As before, the term $C_{k\sigma}$ is the damping-in-pitch coefficient that would be measured from small oscillations in σ about $\sigma = \text{const}$ with ψ fixed at $\psi = \text{const}$ but now, in addition, *in the*

presence of a steady coning motion $\dot{\phi} = \dot{\lambda}/\gamma = \text{const.}$ Similarly, $C_{k,\psi}$ is the damping-in-roll coefficient that would be measured from small oscillations in ψ about $\psi = \text{const}$ with δ fixed at $\delta = \text{const}$ and in the presence of a steady coning motion $\dot{\phi} = \dot{\lambda}/\gamma = \text{const.}$ The indicated functional dependence on $\delta, \psi, \dot{\lambda}/\gamma$ must be interpreted as follows: for flight with given values of $\delta, \psi, \dot{\lambda}, q, r$ at a particular instant, the aerodynamic coefficients that are to be associated with that instant are those evaluated around a coning motion having constant values of δ and ψ equal to the instantaneous flight values and a constant value of coning rate equal to the instantaneous value of $\dot{\lambda}/\gamma$.

Thus, the four contributions required in Eq. (40) to build up the response to an arbitrary motion reduce to three when a nonlinear dependence on coning rate is admitted. This is because the first term in Eq. (58) $C_k(\infty; \delta, \psi, \dot{\lambda}/\gamma)$ is the general term which, it now appears, replaces two terms in Eq. (40) representing the expansion of $C_k(\infty; \delta, \psi, \dot{\lambda}/\gamma)$ around $\dot{\lambda}/\gamma = 0$ to first order in $\dot{\lambda}/\gamma$. The more important change, however, at least from the experimental standpoint, is that retaining a nonlinear dependence on coning rate requires for consistency that the oscillatory experiments be carried out in the presence of coning motion. The three motions are illustrated schematically in Fig. 15.

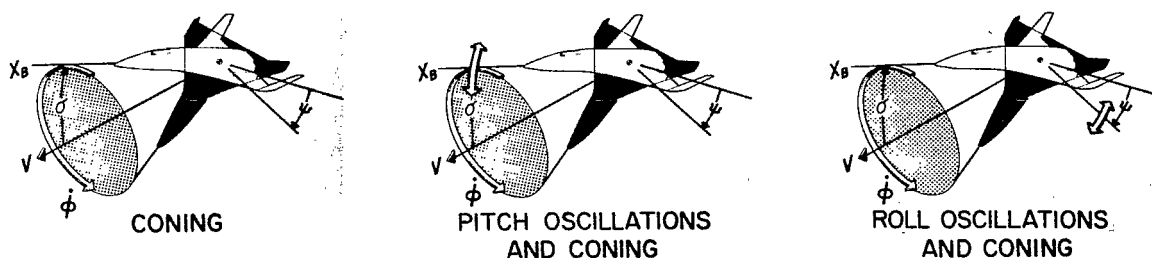


Fig. 15. Characteristic motions in the aerodynamic axis system. Nonlinear dependence on coning rate.

6.6.2 *Body Axis System:* The generalized formulation paralleling that of Eq. (50) takes the form

$$\hat{C}_k(t) = \hat{C}_k(\infty; \hat{\alpha}, \hat{\beta}, p_B/\gamma) + \frac{1}{\gamma} \frac{\dot{\hat{\alpha}}z}{V} \left[\hat{C}_{kq_B}(\infty; \hat{\alpha}, \hat{\beta}, p_B/\gamma) + \gamma C_{k\dot{\alpha}}(\hat{\alpha}, \hat{\beta}, p_B/\gamma) \right] - \frac{1}{\gamma} \frac{\dot{\hat{\beta}}z}{V} \left[\hat{C}_{kr_B}(\infty; \hat{\alpha}, \hat{\beta}, p_B/\gamma) - \gamma \hat{C}_{k\dot{\beta}}(\hat{\alpha}, \hat{\beta}, p_B/\gamma) \right]; \quad k = \begin{cases} X, Y, Z \\ z, m, n \end{cases} \quad (59)$$

Equation (59) reverts to Eq. (50) upon expanding the terms in Eq. (59)

to first order about $p_B/\gamma = 0$. The first term in Eq. (59) is the coefficient that would be measured in a steady coning motion

$\dot{\phi} = p_B/\gamma = \text{const}$ with $\hat{\alpha}$ and $\hat{\beta}$ at the fixed inclinations $\hat{\alpha} = \text{const}$,

$\hat{\beta} = \text{const}$. The second term is the damping-in-pitch coefficient that

would be measured from small oscillations in $\hat{\alpha}$ about $\hat{\alpha} = \text{const}$ with

$\hat{\beta}$ fixed at $\hat{\beta} = \text{const}$ and in the presence of a steady coning motion

$\dot{\phi} = p_B/\gamma = \text{const}$. The third term results from small oscillations in

$\hat{\beta}$ about $\hat{\beta} = \text{const}$ with $\hat{\alpha}$ fixed at $\hat{\alpha} = \text{const}$ and in the presence

of a steady coning motion $\dot{\phi} = p_B/\gamma$. It should be noted that the coning

rate $\dot{\phi} = p_B/\gamma$ on which the terms depend in Eq. (59) is not equal in

magnitude to the coning rate $\dot{\phi} = \dot{\lambda}/\gamma$ in Eq. (58). The rates differ

by $\dot{\psi}/\gamma$. Thus, in the body axis system, the appropriate constant value

of the coning rate for the aerodynamic coefficients that are to be

associated with an instantaneous flight condition is that formed from

the instantaneous value of p_B/γ . The three motions required in the

body axis system are illustrated schematically in Fig. 16.

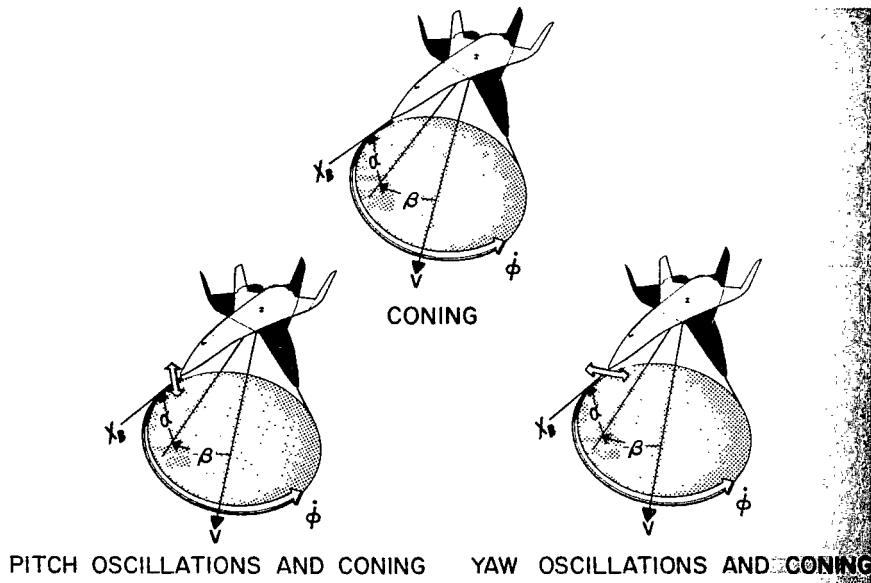


Fig. 16. Characteristic motions in the body axis system.
Nonlinear dependence on coning rate.

6.6.3 *Spin Radius*: Equations (58) and (59) were derived on the assumption of a nearly rectilinear flight path, which would apparently restrict their application to spin motions having essentially zero spin radius. It can be shown, however, that the results will apply as well to spin motions having constant spin radius. Motions having a constant spin radius can be characterized by the existence of a point other than the mass-center about which the body rotates. This point, which lies on the body x_B axis, is itself in essentially rectilinear motion. The existence of such a point usually will guarantee fulfillment of the conditions under which Eqs. (58) and (59) apply, namely, $q - \dot{\sigma} \approx 0$, $r - \epsilon \dot{\lambda} \approx 0$. The principal restriction is that $|qx/V|, |rx/V| \ll 1$, where x is the distance along x_B between the mass-center and the point in nearly rectilinear motion. Variations in spin radius $x \sin \sigma$ also can be tolerated under the additional restriction $|\dot{x}/V| \ll 1$.

6.7 Requirements for Experiments

Within the assumption of a linear dependence of the moment on coning rate, the analysis suggests that the moment contributions resulting from four characteristic motions are required to completely specify the nonlinear moment system for arbitrary motions about nearly rectilinear flight paths (Figs. 10 and 12). For wind-tunnel tests in the aerodynamic axis system, two kinds of apparatus would appear to be necessary:

- (1) A coning and spinning apparatus similar to the one described in Ref. 5. (For nonaxisymmetric bodies, the spin motor that reproduced the constant spin rate $\dot{\psi}$ would have to be replaced by a device reproducing small oscillations in ψ about $\psi = \text{const.}$) Such an apparatus should be capable of measuring the moment contributions due to steady resultant angle of attack, coning at constant resultant angle of attack, and oscillations in roll at constant resultant angle of attack.
- (2) An oscillations-in-pitch apparatus for measuring the moments due to small oscillations in σ about a fixed σ with the axis of rotation oriented normal to the σ plane. It is emphasized that for nonaxisymmetric bodies, the oscillations-in-pitch apparatus must be capable of measuring not only the pitching moment but also the induced side and rolling moments.

Wind-tunnel tests in the body axis system require the same coning experiment as in the aerodynamic axis system and, in addition, separate oscillations-in-pitch and oscillations-in-yaw experiments. The oscillations-in-pitch device must be capable of simultaneously measuring all three moment components due to small oscillations in $\hat{\alpha}$ about a

fixed $\hat{\alpha}$ with $\hat{\beta}$ held fixed. The same device can be used in the oscillations-in-yaw experiment, in which the roles of $\hat{\alpha}$ and $\hat{\beta}$ are reversed. Hence, experiments carried out in the body axis system, requiring only a single oscillatory device, would appear to have an advantage over those in the aerodynamic axis system from the standpoint of economics. It is essential, however, that the oscillatory experiments in the body axis system enable the measurement of the nonlinear interaction terms *even for bodies of revolution*. Recently, the successful development of a device capable of measuring these interaction terms has been reported [19].

In programs designed to extract the nonlinear aerodynamic coefficients from free-flight data, of course it is recommended that a form for the aerodynamic moment system based on Eq. (40) or Eq. (50) be incorporated. It has already been noted [6] that procedures not allowing for the presence of interaction terms in the representation of the moment system can assign erroneous weights to the remaining terms.

Eliminating the assumption of a linear dependence of the moment on coning rate reduces the number of characteristic motions required from four to three: in either axis system, a coning motion and two oscillatory motions in the presence of coning (Figs. 15 and 16). Experiments designed to reproduce the motions in the wind tunnel again require the coning apparatus and the types of oscillatory devices just described. The significant additional requirement that each of the oscillatory experiments be carried out in the presence of coning means, of course, that now the oscillatory devices must be incorporated in the coning apparatus. These obviously difficult experiments, involving oscillatory

and coning motions in combination, are required only where the moment contribution due to steady coning shows a significant nonlinear dependence on coning rate. Otherwise, the experiments may be conducted separately as described above. Useful surveys of the needs and capabilities for carrying out experiments in the wind tunnel involving oscillatory, rotary, and combined motions at high angles of attack are available in Refs. 20 and 21.

7. CONCLUDING REMARKS

A study aimed at extending the notions of aerodynamic indicial functions and superposition integrals into the nonlinear regime has shown that replacing the indicial functions within the integrals by functionals, themselves dependent on the past motion, achieves the desired objective. A simple specification of the degree to which the reformulated indicial responses are cognizant of the past motion led to practicable nonlinear generalizations of the linear superposition and stability derivative formulations of the aerodynamic response to arbitrary motions. Applied to arbitrary nonplanar motions, the generalization yielded a form for the aerodynamic response built up of the contributions from a limited number of well-defined characteristic motions, in principle reproducible in the wind tunnel. Further generalizations, enabling the acknowledgment of more general categories of aerodynamic phenomena within the scope of the formulation, are possible. The characteristic feature of hysteresis, multivalued aerodynamic responses, could be acknowledged,

for example, by extending the degree to which the indicial responses are cognizant of the past motion one step beyond that specified in this study.

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Moffett Field, Calif., 94035, May 1975

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