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PURDUE UNIVERSITY

Lafayette, Indiana 47907

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# PROPAGATION OF HIGH AMPLITUDE HIGHER ORDER SOUNDS IN SLIGHTLY SOFT RECTANGULAR DUCTS, CARRYING MEAN FLOW

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Submitted by:

K.S. Wang, Graduate Research Assistant
P.G. Vaidya, Principal Investigator and Project Director

Approved by:

Raymond Cohen, Director
Ray W. Herrick Laboratories

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# LIST OF SYMBOLS

| | |
|---|---|
| $a$ | Width of duct. |
| $a_j$ | Phase shift of $j^{th}$ component. Ref. eq. 25. |
| $A_j$ | Amplitude of $j^{th}$ component. Ref. eq. 25. |
| $c_o$ | Sound speed at infinitesimal amplitude |
| $f_j$ | jth dissipation, see eq. 51. |
| $j, j_1, j_2$ | Positive integers, see eqs. 41 and 42. |
| $\ell$ | Positive integer. |
| $M$ | Mach number of the mean flow = $\frac{Um}{c_o}$ |
| $\bar{m}$ | Wave number in traverse direction. |
| $m$ | $= \bar{m}\,\pi$ |
| $N_1, N_2$ | Nonlinear differential operators. See egs. 4 & 5. |
| $p$ | Positive integer. |
| $P$ | Pressure. |
| $\bar{R}_o$ | Resistance of the wall. |
| $\bar{t}$ | Time. |
| $t$ | Nondimensional time = $\frac{\bar{t}c_o}{a}$ . |
| $u$ | Acoustic velocity. |
| $U_m$ | Mean velocity. |
| $\bar{x}$ | Tranverse distance. |
| $x$ | Nondimensional transverse distance, $x = \frac{\bar{x}}{a}$ |
| $Y_1, Y_2, Y_m, Y_m$ | Parameters, see eqs. 39, 40, 67, and 68. |
| $\bar{z}$ | Propagating direction. |
| $z$ | Nondimensional propagating direction, $z = \frac{\bar{z}}{a}$ . |
| $z_o, z_1, z_2$ | Multiple scales in propagation direction, see eq. 32. |
| $\alpha$ | Wave number in z direction |

| | |
|---|---|
| $\gamma$ | Specific heat ratio. |
| $\varepsilon$ | Peak Mach number of acoustic wave, $\frac{u}{c_o}$ perturbation parameter. |
| $\eta$ | Constant, see eq. 43. |
| $\eta_1$ | Conductance (Inverse of resistance) of the wall. |
| $\theta$ | $\alpha z - \omega t$. |
| $\mu$ | Parameter, see eq. 64. |
| $\rho_o$ | Density |
| $\phi$ | Nondimensional acoustic velocity potential, $\phi = \frac{\bar{\phi}}{\varepsilon c_o a}$ . |
| $\phi_j$ | $j^{th}$ order velocity potential component, see eq. 34. |
| $\bar{\chi}_o$ | Reactance of the wall. |
| $\chi_o$ | Ratio of Reactance of Resistance $(\bar{\chi}_o/\bar{R}o)$. |
| $\Psi_j$ | Component of the second order solution eq. 37. |
| $\tilde{\Psi}_j$ | Component of the second order solution eq. 37. |
| $\omega$ | Nondimensional angular frequency (or reduced frequency). $\omega = \frac{\bar{\omega}a}{c_o}$ . |

## ABSTRACT

The resonance expansion method, developed earlier to study the propagation of sound in rigid rectanglular ducts, has been applied to the case of slightly soft ducts. Expressions for the generation and decay of various harmonics is obtained. The effect of wall admittance is seen through a dissipation function in the system of nonlinear differential equations, governing the generation of harmonics. As the wall admittance increases, the resonance is reduced. For a given wall admittance this phenomenon is stronger at higher input intensities. Both the first and second order solutions are obtained and the results are extended to the case of ducts having mean flow.

# I. INTRODUCTION

This report deals with the propagation of higher order mode, finite amplitude sound in slightly soft rectangular ducts, carrying mean flow. This report is a comparison of a report which deals with the rigid wall boundary condition (1). Conventional methods of analysis have failed to solve, satisfactorily both these problems.

It was shown in the case of the rigid rectangular duct that the sound wave undergoes a nonlinear resonance. This is due to the fact that the eigenvalues of the higher order modes are integral multiples of the eigenvalue of the first mode. In reference 1 it was shown that a new method could be used to analyze such a problem. This method was called resonance-expansion method. It will be seen that the resonance phenomenon persists for a soft duct as long as the admittance of the duct is not very large. Larger the amplitude of the sound waves, greater is the extent of the admittance to which this phenomenon will persist.

Previous researchers (2,3) have attempted a solution of the soft wall problem by a direct multiple-scale expansion method. Unfortunately, the solution obtained by this method is not valid as the linear attenuation coefficient approaches to the order of the perturbation parameter $\varepsilon$, (based on initial amplitude). This is due to the result that, in such a case, the direct method leads to secular terms in the second order solution. This alters the accuracy of the modal amplitudes in the first order solution.

There exists an intrisic need to have a unified theory which explains the curious behavior in the entire domain of wall admittance. It would be seen that the solution described in this report approaches the one for the rigid wall case as the admittance tends to zero and to that from a straight forward solution as the admittance increases. There is also a pragmatic need for such a solution as it promises a practical way to reduce resonance by means of a relatively small amount of sound absorbent.

It should be noted that in view of the equivalent impedance concept of Cremer (4), (extended to higher order modes by Vaidya (5)), effects of viscosity and heat conduction could be lumped on the wall of a duct as a small admittance. Thus this report could as well be viewed as one describing sound propagation in a rigid rectangular duct, carrying viscous, heat conducting fluid.

## II. FORMULATION

The problem formulated herein is idealized to the extent that an acoustic field is considered within an infinite duct, therefore only outgoing waves exist. (see Figure 1)

The following set of non-dimensional veriables are introduced, where ($\bar{\ }$) indicated the dimensional quantity:

$$x = x/a, \quad z = \bar{z}/a, \quad t = \bar{t}c_o/a, \quad p = \bar{p}/\rho_o c_o{}^2, \quad \phi = \bar{\phi}/\varepsilon c_o a \tag{1}$$

where $\phi$ is the nondimensional velocity potential. Therefore, we have

$$\bar{V} = \nabla\phi \tag{2}$$

In terms of the non-dimensional quantities, the governing non-linear equations for the inviscid polytropic gas are (1) ($\varepsilon$ is amplitude of input wave).

$$\nabla^2\phi - \phi_{tt} = \varepsilon N_1(\phi) + \varepsilon^2 N_2(\phi) \tag{3}$$

$$N_1\ (\phi) = (\gamma-1)\phi_t\nabla^2\phi + 2\nabla\phi\cdot(\nabla\phi_t) \tag{4}$$

$$N_2\ (\phi) = (\gamma-1)\ (\nabla\phi)^2\ \nabla^2\phi/2 + (\nabla\phi\cdot\nabla)\ (\nabla\phi)^2/2 \tag{5}$$

Where

$$\nabla^2\ (\ \ ) = \partial^2(\ \ )/\partial x^2 + \partial^2(\ \ )/\partial z^2 \tag{6}$$

$$\nabla(\ ) = [\partial(\ )/\ x]\ \bar{i} + [\partial(\ )/\ z]\ \bar{k} \tag{7}$$

$$[\nabla(\ \ )]^2 = [\partial(\ \ )/\partial x]^2 + [\partial(\ \ )/\partial z]^2 \tag{8}$$

The relation between the pressure and velocity is (1)

$$\gamma p = \{(1-\gamma)\ [\varepsilon\phi_t + \varepsilon^2(\nabla\phi)^2/2] + 1\}^{\gamma/(\gamma-1)} \tag{9}$$

The formulation of the problem is completed once the boundary conditions are descirbed. Here we use a modified form of the semi-empirical form proposed by Zorumski and Parrott(6), (It's also used by Nayfeh and Tsai (3)) that is

$$\bar{p}-p_o = (\bar{R}(\bar{u}) - \frac{\bar{\chi}(\bar{u})}{\bar{\omega}} \frac{\partial}{\partial \bar{t}}) \; \bar{u} \text{ at } x = 1 \tag{10}$$

where $\bar{u} = \phi_x$, is the velocity in the x direction. The form of this equation differs from that used in reference 3 in that the equation reduced to the familiar form

$$\bar{p}-p_o = (\bar{R} + i\bar{\chi}) \; \bar{u} \tag{10a}$$

in the linear monochromatic case.

In particular the following forms for $\bar{R}$ and $\bar{X}$ could be considered:

$$\bar{R}(u) = \bar{R}_0 + \bar{R}_2 \; (u/c_o)^2 \tag{11}$$

$$\bar{\chi}(u) = \bar{\chi}_0 + \bar{\chi}_2 \; (u/c_o)^2 \tag{12}$$

It will be seen later on in this analysis that $\bar{R}_2$ and $\bar{\chi}_2$ play no part up to the second order solution. Therefore ordinary linear boundary conditions would be adequate. However, since the nonlinear boundary conditions are found in the current literature, we have chosen to retain them and to demonstrate their relative lack of significance.

Substiduting equations (11) and (12) into (10) and non-dimensionalizing the resulting equation leads to

$$p-\gamma^{-1} = \varepsilon[\bar{R}_o/\rho_o c_o - \bar{\chi}_o/\omega\rho_o c_o \frac{\partial}{\partial t} + O(\varepsilon^2)]\,\phi_x$$

Expanding the right-hand side of equation (9) into Taylor's series, the polynomial relation between p and $(\phi_x, \phi_z)$ can be found. Therefore equation (13) becomes

$$\text{at } \mathbf{x} = 1,\ -\eta_1\ [\phi_t + \varepsilon(\phi_{\mathbf{x}}^2 + \phi_z^2 - \phi_t^2)/2 + O(\varepsilon^2)]$$

$$= [1 - \chi_o/\omega \frac{\partial}{\partial t} + O(\varepsilon^2)]\phi_{\mathbf{x}} \tag{14}$$

Where

$$\eta_1 = \frac{\rho_o c_o}{\bar{R}_o} \quad , \quad \chi_o = \frac{\bar{\chi}_o}{\bar{R}_o} \tag{14a}$$

For the wall of duct with small admittance, let $O(\eta_1) \sim \varepsilon$ and $O(\chi_o) \sim 1$. Using this relation equation (14) can be arranged as

$$\text{at } x = 1,\ (1 - \chi_o/\omega\ \partial/\partial t)\phi_x + \varepsilon(\eta_1/\varepsilon)\phi_t + O(\varepsilon^2) = 0 \tag{15}$$

The condition at $x = 0$ is

$$\phi_x = 0 \qquad \text{(for symmetric case)} \tag{16}$$

$$\phi = 0 \qquad \text{(for anti-symmetric case)} \tag{17}$$

For the report, we consider the solution based on (16). The initial condition (or the input condition) at $z = 0$ is

$$\phi = \cos m\,\mathbf{x} \cos \omega\, t + O(\varepsilon) \tag{18}$$

Where $m = \bar{m}\pi$ ($\bar{m} = 1, 2, 3, \ldots\ldots$). The correction, $O(\varepsilon)$, is found such that the slightly soft wall condition is satisfied. (see the second order solution, section 3.3).

# III PERTURBATION SOLUTIONS

To determine an approximate solution for this system, we use the perturbation expansion. Let

$$\phi = \phi_o + \varepsilon\phi_1 + O(\varepsilon^2) \tag{19}$$

where $\phi_1$ and $\phi_2$ are the functions of $z_0$, $z_1$, ....., t and x. The $z_i$'s are defined

$$z_i = \varepsilon^i z \tag{20}$$

$z_o$ is a fast scale characterizing the wave length while $z_i$ (i = 1, 2, 3,.....) is a slow scale characterizing the amplitude modulation (ref. 2).

## 3.1 The First Order System

Substituting equation (19) into the governing system, the first order system is (let $z_o$ be denoted as z)

$$\nabla^2\phi_o - \phi_{o,tt} = 0$$

$$\text{at } z_o = 0, \quad \phi_o = \text{Re}\,[\cos mx \cos \omega t] \tag{22}$$

$$\text{at } x = 0, \quad \phi_{o,x} = 0 \tag{23}$$

$$\text{at } x = 1, \quad \phi_{o,x} - \frac{\chi_o}{\omega}\phi_{o,xt} = 0 \tag{24}$$

Therefore the solution of $\phi_o$ is (1)

$$\phi_o = \sum_{j=1}^{N} A_j(z_1) \cos jmx \cos[j\theta + a_j(z_1)] \tag{25}$$

where

$$\alpha^2 = \omega^2 - m^2, \; \theta = \alpha x - \omega t$$

Note that the form of equation (25) is of the resonent expansion type. This form could be generalized for various types of inputs. As a simple example, consider

$$A_j(O) = \begin{cases} 1 & j = 1 \\ O & j \neq 1 \end{cases} \tag{26}$$

$$a_j(O) = O \qquad j = 1,2,3,\ldots\ldots \tag{27}$$

It would be assumed, at first that the order of the phases $a_j$ is less than or equal to $\varepsilon$. In Appendix B, this restriction is removed. It follows from equations (23) and (24) that

$$m = \bar{m}\,\pi \qquad \bar{m} = 1,\ 2,\ 3,\ldots\ldots\ldots \tag{28}$$

The behavior of $A_j$'s and $a_j$'s can be determined by the condition that the second order system should be without secular terms.

### 3.2 Solvability Condition for $A_j$

The governing equation for second order is

$$\nabla^2\phi_1 - \phi_{1,tt} - \sum_{j=1}^{N} 2j\alpha\, A_j' \cos jmx \sin(j\theta + a_j)$$

$$- \sum_{j=1}^{N} 2j\alpha\, A_j a_j' \cos jmx \cos(j\theta + a_j) = N_1(\phi_o) \tag{29}$$

In view of the assumption stated above, the phases shift can be neglected with the possible exception of its derivative, $a_j'$. Equation (29) could be rewritten as

$$\nabla^2\phi_1 - \phi_{1,tt} = \sum_{j=1}^{N} 2j\alpha\, A_j' \cos jmx \sin j\theta$$

$$+ \sum_{j=1}^{N} 2j\alpha\, a_j' A_j \cos jmx \cos j\theta + N_1(\phi_o) \qquad (30)$$

where

$$N_1(\phi_o) = \sum_{j=1}^{N} \sum_{\ell=1}^{N} \bar{F}_1(x) j\ell^2 A_j A_\ell \sin (j+\ell)\theta$$

$$+ \sum_{j=1}^{N} \sum_{\ell=1}^{N} \bar{F}_2(x)\, j\ell^2 A_j A_\ell \sin (j-\ell)\theta \qquad (31)$$

$$\bar{F}_1(x) = [-\tfrac{1}{2}(\gamma-1)\omega^3-\alpha^2\omega] \cos jmx \cos \ell mx + m^2\omega \sin jmx \sin \ell mx \qquad (32)$$

$$\bar{F}_2(x) = [-\tfrac{1}{2}(\gamma-1)\omega^3-\alpha^2\omega] \cos jmx \cos \ell mx - m^2\omega \sin jmx \sin \ell mx \qquad (33)$$

The boundary conditions are

$$\text{at } z = 0, \quad \phi_1 = 0$$

$$\text{at } x = 0, \quad \phi_{1,x} = 0$$

$$\text{at } x = 1, \quad \phi_{1,x} - \chi_o/\omega\, \phi_{1,xt} = -\eta_1/\varepsilon\phi_{o,t} \qquad (36)$$

Let the solution of $\phi_1$ be of the form

$$\phi_1 = \sum_{j=1}^{N} \tilde{\Psi}_j (x,z_1) \sin j\theta + \sum_{j=1}^{N} \Psi_j (x,z_1) \cos j\theta \qquad (37)$$

Substitution of equation (37) into (30),(34),(35), and (36) yields the differential systems for $\tilde{\Psi}_j$ and $\Psi_j$. For $\tilde{\Psi}_j$,

$$\tilde{\Psi}''_j + j^2 m^2 \tilde{\Psi}_j = 2j\alpha A'_j \cos jmx +$$

$$\sum_{\ell=1}^{j_1} j\ell(j-\ell) A_{j-\ell}A_\ell [-Y_1 \cos jmx + Y_2 \cos(j-2\ell)mx] +$$

$$\eta[-Y_1 \cos jmx + Y_2] (j/2)^3 A^2_{j/2} -$$

$$\sum_{j=1}^{j_2} j\ell(j+\ell) A_{j+\ell}A_\ell [-Y_1 \cos jmx + Y_2 \cos(j+2\ell)mx] \qquad (38)$$

where

$$Y_1 = (\gamma+1)\omega^3/4 \qquad (39)$$

$$Y_2 = \omega[m^2 - (\gamma-1)\omega^2 - \alpha^2]/4 \qquad (40)$$

$$j_1 = \begin{cases} j/2 - 1 & \text{if } j/2 \text{ is an integer} \\ (j-1)/2 & \text{if } j/2 \text{ isn't an integer} \end{cases} \qquad (41)$$

$$j_2 = N - j \qquad (42)$$

$$\eta = \begin{cases} 1 & \text{if } j/2 \text{ is an integer} \\ 0 & \text{if } j/2 \text{ isn't an integer} \end{cases} \qquad (43)$$

The boundary conditions are

$$\text{at } x = 0, \quad \tilde{\Psi}_j = 0 \qquad (44)$$

$$\text{at } x = 1, \quad \tilde{\Psi}'_j - j\chi_o \Psi'_j = -(\eta_1/\varepsilon)\cos jm\, A'_j \qquad (45)$$

For $\Psi_j$ the equation is

$$\Psi''_j + j^2m^2\Psi_j = 2j\alpha\, a'_j A_j \cos jmx \qquad (46)$$

The boundary conditions are

$$\text{at } x = 0, \qquad \Psi_j' = 0 \tag{47}$$

$$\text{at } x = 1, \qquad \Psi_j' + j\chi_o \tilde{\Psi}_j' = 0 \tag{48}$$

Eliminating $\Psi_j'$ (1) between equations (45) and (48) yields

$$\text{at } x = 1, \quad \tilde{\Psi}_j' = -(\eta_1/\varepsilon) j\omega \cos jm\, A_j/(1 + j^2\chi_o{}^2) \tag{49}$$

It is shown in the appendix A that the system for $\tilde{\Psi}_j$ (equations (38), (44), and (49)) has a solution, if and only if

$$A_j' + f_j A_j - \frac{\gamma_1}{2\alpha} \Big[ \sum_{\ell=1}^{j_1} \ell(j-\ell)\, A_{j-\ell} A_\ell +$$

$$\frac{\eta}{2} (j/2)^2\, A_{j/2}^2 + \sum_{\ell=1}^{j_1} \ell\,(j+\ell)\, A_{j+\ell} A_\ell \Big] = 0, \; j = 1,2,3,\dots. \tag{50}$$

with

$$f_j = (\eta_1/\varepsilon)\omega / [\alpha(1+j^2\chi_o{}^2)] \quad j = 1,2,3,\dots\dots \tag{51}$$

or

$$f_j = \frac{\omega}{\varepsilon\alpha} \; \frac{\bar{\rho}_o \bar{c}_o \bar{R}_o}{\bar{R}_o^2 + j^2\, \bar{\chi}_o^2} \tag{52}$$

or

$$\varepsilon\, f_j = \frac{\bar{k}\bar{a}}{\sqrt{\bar{k}^2\bar{a}^2 - \bar{m}^2\pi^2}} \; \frac{\bar{\rho}_o \bar{c}_o \bar{R}_o}{\bar{R}_o + j^2 \bar{\chi}_o^2} \tag{53}$$

where $\bar{k}$ is the dimensional wave number, (so that $\bar{\omega} = \bar{k}\bar{c}_o$). From the dimensional form of $f_j$, it could be recognized that contribution to $\frac{dA_1}{dz}$ due to absorption is the same as obtianed in the linear theory by Doak and Vaidya (7). For other harmonics, equation (53) can be rewritten as

$$\varepsilon f_j = \frac{\bar{k}\bar{a}}{\sqrt{\bar{k}^2\bar{a}^2 - \bar{m}^2\pi^2}} \quad K_t(j) \tag{54}$$

where

$$K_t(j) = \frac{\bar{\rho}_o\bar{c}_o\bar{R}_o}{\bar{R}_o^2 + j^2\bar{\chi}_o{}^2} \tag{55}$$

is the admittance ratio for the $j^{th}$ harmonic. For the first harmonic, its definition coincides with the coventional definition. Other harmonics have their admittance ratio computed after enlarging the duct reactance by $j^2$.

Equations (50) through (54) were obtained by assuming that the resistance of the linear was very large. Similar analysis could be carried out by assuming that the reactance was very large. Detailed analysis shows that the form of equations (50) and (52) is valid as long as the addmittance ratio is not an order of magnitude greater than $\varepsilon$. The analysis is also valid above cut off of the exciting mode.

As the admittance ratio gets smaller and smaller, the duct approaches the hard wall conditions. In that case the equation (50) reduces to the one obtained by Wang and Vaidya (1). Therefore, there is a continuous transition from the hard to the slightly soft case. As the softness increases the generation of harmonics slows down. In that case, depending upon the type of boundary conditions, simple multiple scale expansion or a straightforward expansion is adequate.

## 3.3 The Solution of the Second Order System

Having obtained $A_j$'s we could proceed with the second order solution. The condition that this solution be valid would lead to an equation in $a_j$ with $dA_j/dz$, replaced by equation (50) and from equations (35), (38) and (49).

$$\tilde{\Psi}_j(x,z_1) = -(\alpha/m)f_jA_j\ x \sin j m x + \frac{jY_2}{4m^2}\left[\sum_{\ell=1}^{j_1} A_{j-\ell}A_\ell \cos(j-2\ell)mx + \sum_{\ell=1}^{j_2} A_{j+\ell}A_\ell \cos(j+2\ell)mx + \frac{\eta}{2}\left(A^2_{j/2} - \delta_{j/2,1}\right)\right] \quad j = 1,2,3,\ldots \tag{56}$$

where

$$\delta_{j/2,1} = \begin{cases} 1 & j = 2 \\ 0 & j \neq 2 \end{cases} \tag{57}$$

Substituting the form of $\tilde{\Psi}_j$ into the condition (48) gives

$$\text{at } x = 1,\ \Psi_j' = j^2\ \alpha\ \chi_o f_j A_j \cos jm \tag{58}$$

Hence, for the system of $\Psi_j$ (equations 46,47, and 58) the condition of solvability is

$$a_j' = j\ \chi_o\ f_j \tag{59}$$

Therefore

$$a_j = j\ \chi_o\ f_j\ z_1 \tag{60}$$

For small value of z (i.e. not far away from the inlet) the order of $a_j$ is $\varepsilon$. This agrees with the assumption we made at Section 3.2. The solution of $\Psi_j$ is

$$\Psi_j(x,z_1) = (\alpha/m)j\ \chi_o\ f_jA_j\ x \sin jmx\ ,\ j = 1,2,3,\ldots \tag{61}$$

From equations (37), (56) and (61) the complete solution of the second order could be written as

$$\phi_1 = \sum_{j=1}^{N} \{ -(\alpha/m)\ f_j\ A_j\ x \sin jmx + \frac{jY_2}{4m^2} [ \sum_{j=1}^{j_1} A_{j-\ell}A_\ell \cos(j-2\ell)mx +$$

$$\sum_{j=1}^{j_2} A_{j+\ell}A_\ell \cos(j+2\ell)\ mx + \eta/2(A^2_{j/2} - \delta_{j/2,1})]\} \quad \sin j\theta +$$

$$\sum_{j=1}^{N} \{(\alpha/m) j\ \chi_o\ f_j\ A_j\ x \sin jmx \cos j\theta\} \tag{62}$$

This solution is a generalized form of the one obtained for the case of the rigid walled duct. Equation (62) gives the correction, $O(\varepsilon)$, of the input condition. That is

$$\phi_1\Big|_{z=0} = (\alpha/m) f_1\ x \sin mx\ (\chi_o \cos \omega t - \sin \omega t) \tag{63}$$

## IV. NONLINEAR PROPAGATION IN PRESENCE OF MEAN FLOW

The governing equation in presence of mean flow was derived in reference 1.

$$\mu\nabla^2\phi - M^2\,\phi_{zz} - 2M\phi_{zt} - \phi_{tt} = \varepsilon\,N_1(\phi) + \varepsilon^2 N_2(\phi) \qquad (64)$$

where $\mu = 1 - \frac{\gamma - 1}{2} M^2$, and $N_1(\phi)$ becomes (1)

$$N_1(\phi) = (\gamma-1)\,(\phi_t - M\phi_z)\,\nabla^2\phi + 2\nabla\phi\cdot(\nabla\phi_t - M\nabla\phi_z) \qquad (4)$$

Therefore equation (21) should be changed to

$$\mu\phi_{jo,zz} - \mu j^2m^2\phi_{jo} - M^2\phi_{jo,zz} - 2M\phi_{jo,zt} - \phi_{jo,tt} = 0$$

$$j = 1,2,3,\ldots\ldots \qquad (65)$$

Assume again that solutions are the form in equation (25). Substituting that into equation (65) yields

$$\alpha = \frac{1}{\mu-M^2}\left[-M\omega \pm [\mu(\omega^2 + M^2m^2 - \mu m^2)]^{\frac{1}{2}}\right] \qquad (66)$$

Where the positive sign is to be used for waves traveling in mean flow direction and the negative sign is to be used for waves traveling against mean flow direction. $\mu$ is the coefficient that takes into account compressibility. Note that in the conventional convective wave equation $\mu$ is assumed to be unity. The soltuions of $\phi_o$ and $\phi_1$ are of the same form as that in the previous section, except for the following modifications:

$$Y_1 = Y_{1M} = \tfrac{1}{4}(\gamma+1)(\alpha^2+m^2)(\omega-M\alpha) \tag{67}$$

$$Y_2 = Y_{2M} = \tfrac{1}{4}[(3-\gamma)m^2 - (1+\gamma)\alpha^2](\omega-M\alpha) \tag{68}$$

$$\beta_{1M} = Y_{1M}/[\alpha(\mu-M^2) + \omega M] \tag{69}$$

$$\beta_{2M} = Y_{2M}/\mu m^2 \tag{70}$$

$A_j$'s are, once again, to be obtained by solving the equation (50). The mean flow changes the transmission parameter $\alpha$ and $Y_1$. These changes alter the propagation. The change depends upon the mode, the frequency of operation, magnitude and direction of the mean flow.

## V. NUMERICAL RESULTS

Equation (50) was solved using N = 25, $Y_1/\alpha$ = 15.6, $\omega$ = 3.685, $\alpha$ = 1.926, and four cases of wall impedance parameters shown in the table below:

| No. | $\rho_o C_o/\bar{R}_o \varepsilon$ | $\bar{X}_o/\bar{R}_o$ | $K_t$ |
|---|---|---|---|
| 1 | 0.0 | 0.0 | 0.0 |
| 2 | 1.0 | 1.0 | $0.75 \times 10^{-3}$ |
| 3 | 10.0 | 1.0 | $7.5 \times 10^{-3}$ |
| 4 | 1.0 | 0.1 | $1.48 \times 10^{-3}$ |
| 5 | 50.0 | 1.0 | $3.75 \times 10^{-2}$ |

In this table the values of $K_t$ were obtained for j=1 and assuming $\varepsilon = 1.5 \times 10^{-3}$. It represents the admittance ratio for the principal mode.

Figure 2 shows the effect of nonlinearity on propagation of first harmonic. The curves by dot line are obtained from linear theory with the boundary conditions given above. Which diminish only by the absorption of the wall. The nonlinear theory (solid line) shows more reduction in amplitude due to the transfer of energy into higher modes.

Figure 3 shows $A_1$ for various impedances. As the admittance ratio $K_t$ goes up, so does the attenuation. Figure 4 and 5 show the second and the third harmonic respectively.

Figure 6 shows the ratio of $A_2$ to $A_1$ as wall admittance ratio changes. It should be noted that as the wall gets softer the generation of the higher harmonics is slowed down. For a very soft wall, the resonance would disappear and a straight forward multiple scale prediction would be adequate.

## VII. DISCUSSION AND CONCLUSIONS

This report has outlined a method to obtain the sound field in a slightly soft rectangular duct. It was shown in an earlier paper that the rigid rectangular duct demonstrates a resonance phenomenon. The method of resonance expansion was proposed to solve that problem. In this report, the results are extended to the case when the wall admittance is small.

It is seen that the phenomenon of resonance persists. However, it gets weaker as the wall gets softer. The analysis shows that the criteria of softness is through parameters $\rho_o C_o / \bar{R}_o \varepsilon$ and $\varepsilon_o C_o / \bar{X}_o \varepsilon$. This implies that at high sound intensities even a moderately soft wall would act as a hard one. These parameters show the classical antithesis between nonlinear convective terms and the dissipative terms. When the nonlinear terms predominate over the dissipative terms, the resonance persists.

It is also to be noticed that the same parameters which control nonlinearity for the rigid duct control nonlinearity in a soft duct. Thus the parameter $Y_1/\alpha$ shows that the resonance is stronger near the cut-off frequency. Effect of mean flow is again qualitatively similar to that for the hard wall.

It is interesting that for the principal mode the dissipation function is same as the one which could be obtained by linear considerations. For higher harmonics the function is lower than would be expected by the linear theory.

The results also show that a small amount of absorption redices the resonance and hinders the process of higher harmonic generation. In a generalized form, this conclusion could lead to some practical applications. (see for example Reference 8)

## VIII. ACKNOWLEDGMENTS

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## APPENDIX A: COMPATABILITY CONDITIONS

For the non-homogeneous linear system

$$L(u) = r$$

$$U_i(u) = u'(b_i) = \gamma_i \qquad i = 1,2 \qquad (A.1)$$

where L is a second order linear differential operator and $U_i$ is some algebric operator indicating the boundary condition.

The necessary and sufficient condition that this system may have a solution is that every solution, v, of the homogeneous adjoint system (defined below)

$$\bar{L}(v) = 0$$

$$V_1(v) = v'(1) = 0,\ V_2(v) = v'(0) = 0 \qquad (A.2)$$

Satisfies the relation (9) which could be obtained from the Green's formula,

$$\int_{b_1}^{b_2} vr dx = -\gamma_1 V(b_1) + \gamma_2 V(b_2) \qquad (A.3)$$

In second order systems if $L \equiv P_o \frac{d^2}{dx^2} + (2P_o'-P_1) + (P_o''-P_1' + P_2)$

where the prime denote derivation with respect to x, in the particular case described in this appendix $L = \bar{L}$, that is the equation is self-adjoint. In any case, any given equation can be converted into the self-adjoint form $L = [\frac{d}{dx}(k\frac{d}{dx}) - G]$. (see p. 215 of Ref.9) The boundary conditions $V_i$ for the adjoint equation are obtained from $V_i$ and K as illustrated below.

Consider the particular system defined as $L = \frac{d^2}{dx^2} + j^2m^2$.

In confirmity with the rest of the paper let the argument u of this operator be $\tilde{\Psi}_j$ and v, the argument of the adjoint operator be $\Phi_j$.

Therefore let

$$L(\tilde{\Psi}_j) = \frac{d^2}{dx^2} = (\tilde{\Psi}_j) + j^2m^2 \; (\tilde{\Psi}_j) \tag{A.4}$$

$$U_1(\tilde{\Psi}_j') = \tilde{\Psi}_j(0) = \gamma_1 = 0 \tag{A.5}$$

$$U_2(\tilde{\Psi}_j') = \tilde{\Psi}_j(1) = \gamma_2 \tag{A.6}$$

$$\gamma_2 = -(\eta_1/\varepsilon)\; j\omega\; A_j \cos jm \Big/ (1+j^2\hat{\chi}_o{}^2) \tag{A.7}$$

$$r = 2j\alpha A_j' \cos jmx + \sum_{\ell=1}^{j_1} j\ell(j-\ell)\; A_{j-\ell}A_\ell\; [-Y_1 \cos jmx + Y_2 \cos(j-2\ell)mx]$$

$$+ \eta[-Y_1 \cos jmx + Y_2](j/2)^3 A_{j/2}^2$$

$$- \sum_{\ell=1}^{j_2} j\ell(j+\ell)\; A_{j+\ell}A_\ell \;\; [-Y_1 \cos jmx + Y_2 \cos\,(j+2\ell)mx] \tag{A.8}$$

Therefore, the adjoint system corresponding to (A.4), (A.5), and (A.6) is

$$\bar{L}(\Phi_j) = \frac{d^2}{dx^2}\,(\Phi_j) + j^2m^2\Phi_j = 0 \tag{A.9}$$

$$V_1(\Phi_j) = \Phi_j'(1) = 0 \tag{A.10}$$

$$V_2(\Phi_j) = \Phi_j'(0) = 0 \tag{A.11}$$

The adjoint system has a nontrival solution that is

$$\Phi_j = \cos jmx \tag{A.12}$$

By the relation (A.3), we have

$$\int_0^1 [2j\alpha A_j' \cos jmx + \sum_{=1}^{j_1} j\ell(j-\ell)A_{j-\ell}A_\ell(-Y_1 \cos jmx + Y_2 \cos(j-2\ell)mx)$$

$$+ \eta(-Y_1 \cos jmx + Y_2)\ (j/2)^3\ A^2_{j/2}$$

$$- \sum_{\ell=1}^{j_2} j\ell(j+\ell)\ A_{j+\ell}A_\ell\ (-Y_1 \cos jmx + Y_2 \cos\ (j+2\ell)mx)]\cos jmx\ dx$$

$$= \cos jm(\frac{-\eta_1}{\varepsilon})j\omega A_j \cos jm/(1+j^2\chi_o{}^2) \tag{A.13}$$

After integration, it yields

$$A_j' + f_jA_j - \frac{Y_1}{2\alpha}\ [\sum_{\ell=1}^{j_1}\ell(j-\ell)\ A_{j-\ell}A_\ell + \frac{\eta}{2}(j/2)^2\ A^2_{j/2}]$$

$$+ \frac{Y_1}{2\alpha}\sum_{\ell=1}^{j_1}\ (j+\ell)\ A_{j+\ell}A_\ell = 0 \tag{A.14}$$

## APPENDIX B: GENERAL EQUATIONS OF $A_j$ AND $a_j$

Form of the equation (50) when the phase angles $a_j$ may or may not be small.

Equation (60) provides meaningful insight about the order of magnitude of the phases $a_j$. Rewriting it in the form

$$a_j' = \frac{(\delta_{j,1}-A_j)\bar{\omega}\bar{a}\rho_o}{A_j\sqrt{\bar{k}^2\bar{a}^2-\bar{m}^2\bar{\pi}^2}} \frac{j\bar{\chi}_o}{(R_o^{\;2}+j^2\chi_o^{\;2})} \frac{1}{\varepsilon} \qquad \text{(B.1)}$$

Order of magnitude analysis on this equation shows that when $\frac{j\chi_o}{R_o} >> 1$ or $<< 1$, $a_j$ would be of the order smaller than $\varepsilon$.

When $\frac{j\chi_o}{R_o} = 1$, $a_j$ would be of the order of $\varepsilon$ in the nearfield of the solution. In the farfield solution, a modified form of the first order solution is required.

Working out the procedure outlined in Appendix A, when $a_j$'s are not assumed to be arbitrarily small, equations of compatability become

$$\frac{d(A_j \cos a_j)}{dZ_1} + f_j A_j \,(\cos a_j - j\chi_o \sin a_j) - \frac{\gamma_1}{2\alpha} \Big[ \sum_{\ell=1}^{j_1} \ell(j-\ell)$$

$$A_{j-\ell}A_\ell \cos(a_{j-\ell}+ a_\ell) + \frac{\eta}{2} (j/2)^2 A_{j/2}^2 \cos (2a_{j/2})$$

$$- \sum_{\ell=1}^{j_2} \ell(j+\ell) A_{j+\ell}A_\ell \cos(a_{j+\ell}-a_\ell)\Big] = 0, \; j = 1,2,3,.. \qquad \text{(B.2)}$$

and

$$\frac{d(A_j \sin a_j)}{dZ_1} + f_j A_j (\sin a_j - j\chi_o \cos a_j) - \frac{Y_1}{2\alpha} \sum_{\ell=1}^{j_1} \ell(j+\ell)$$

$$A_{j-\ell}A_\ell \sin(a_{j-\ell} + a_\ell) + \frac{\eta}{2} (j/2)^2 A_{j/2}^2 \sin(2a_{j/2}) -$$

$$\sum_{\ell=1}^{j_2} \ell(j+\ell) A_{j+\ell}A_\ell \sin(a_{j+\ell} - a_\ell)] = 0,\ j = 1,2,3,\ldots \qquad \text{(B.3)}$$

Note that equation (B.2) changes into equation (50) as $a_j \rightarrow 0$. Equation (B.3) changes into equation (60) as $a_j \rightarrow 0$. These equations will be discussed in a coming paper (10) which gives detail about the couplex **sound** wave input.

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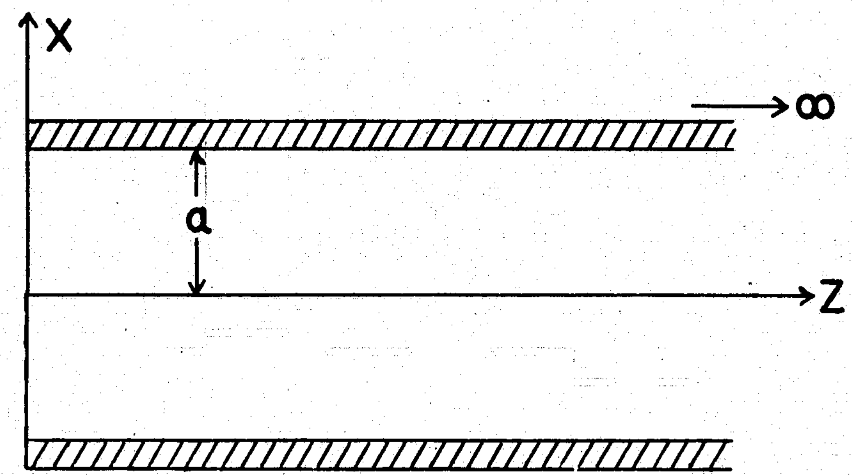
# LIST OF FIGURES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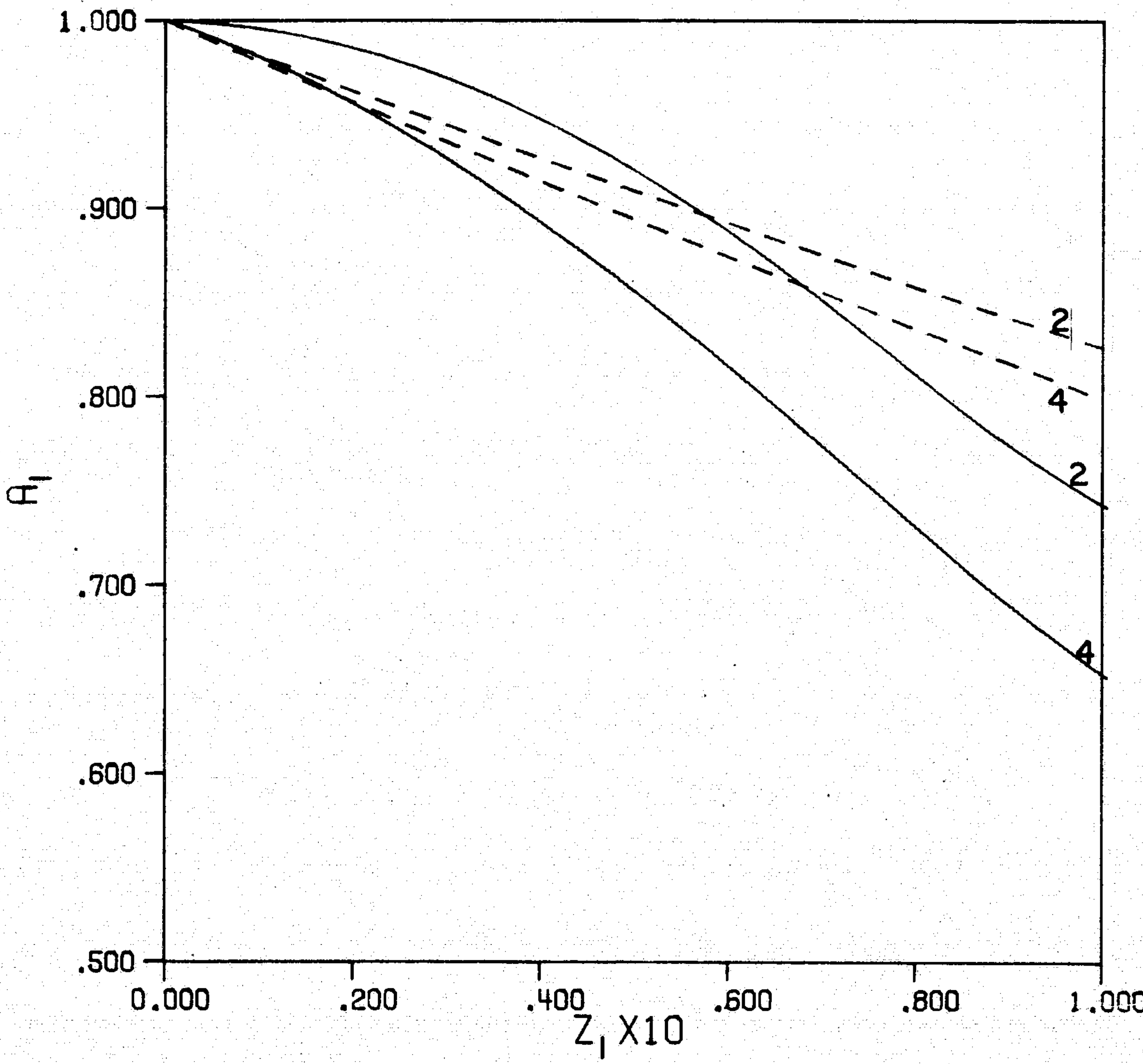
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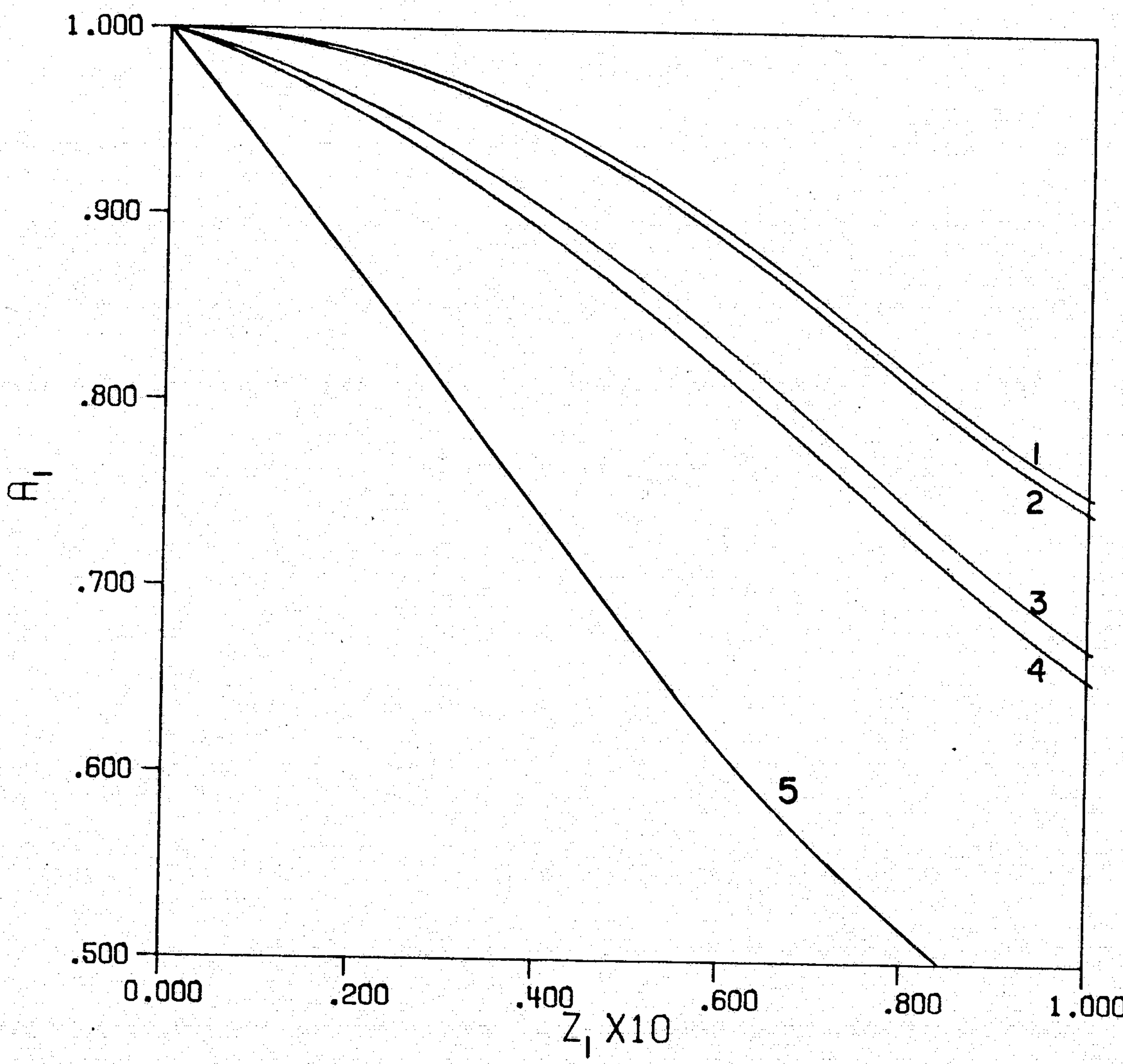
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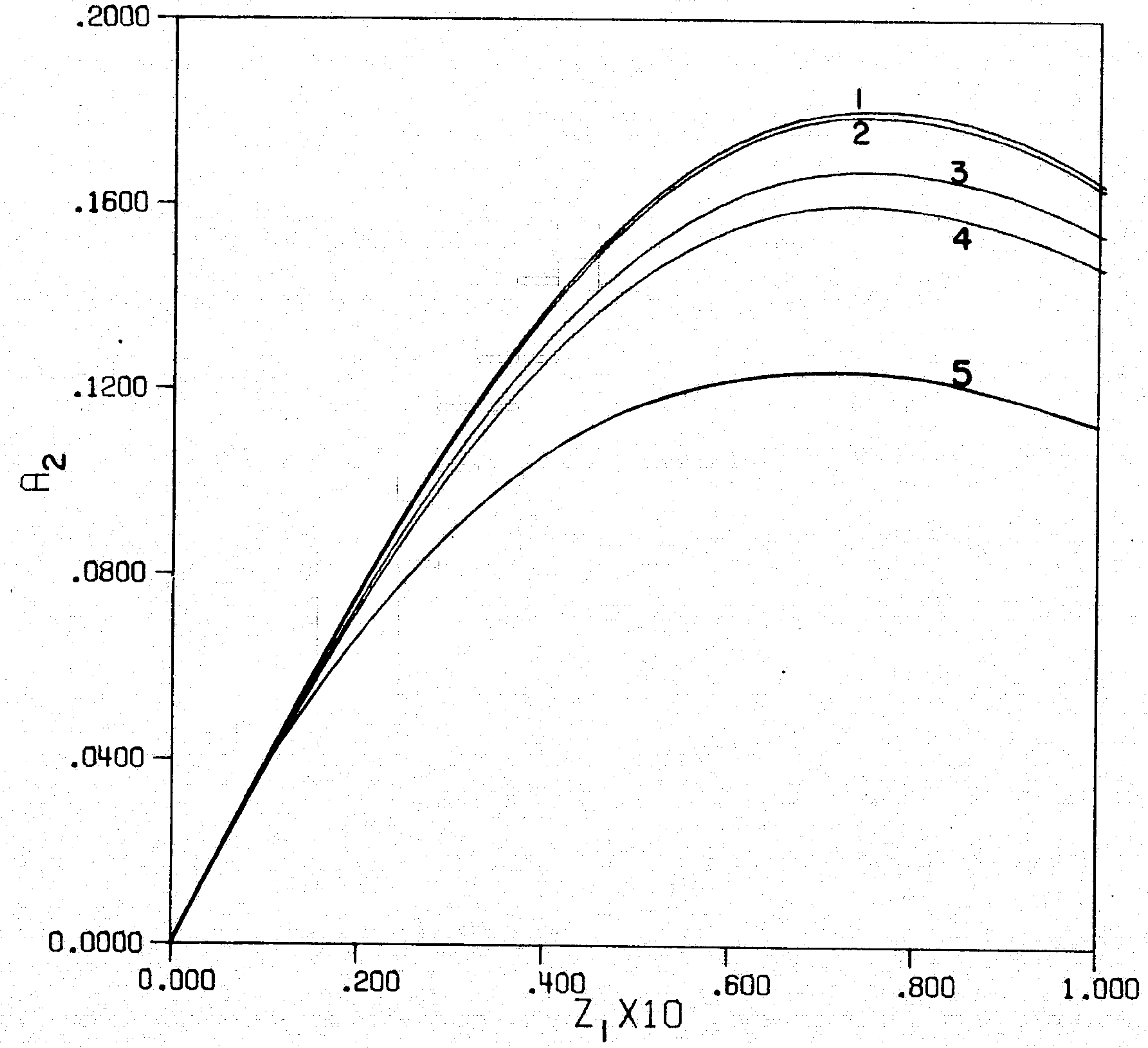

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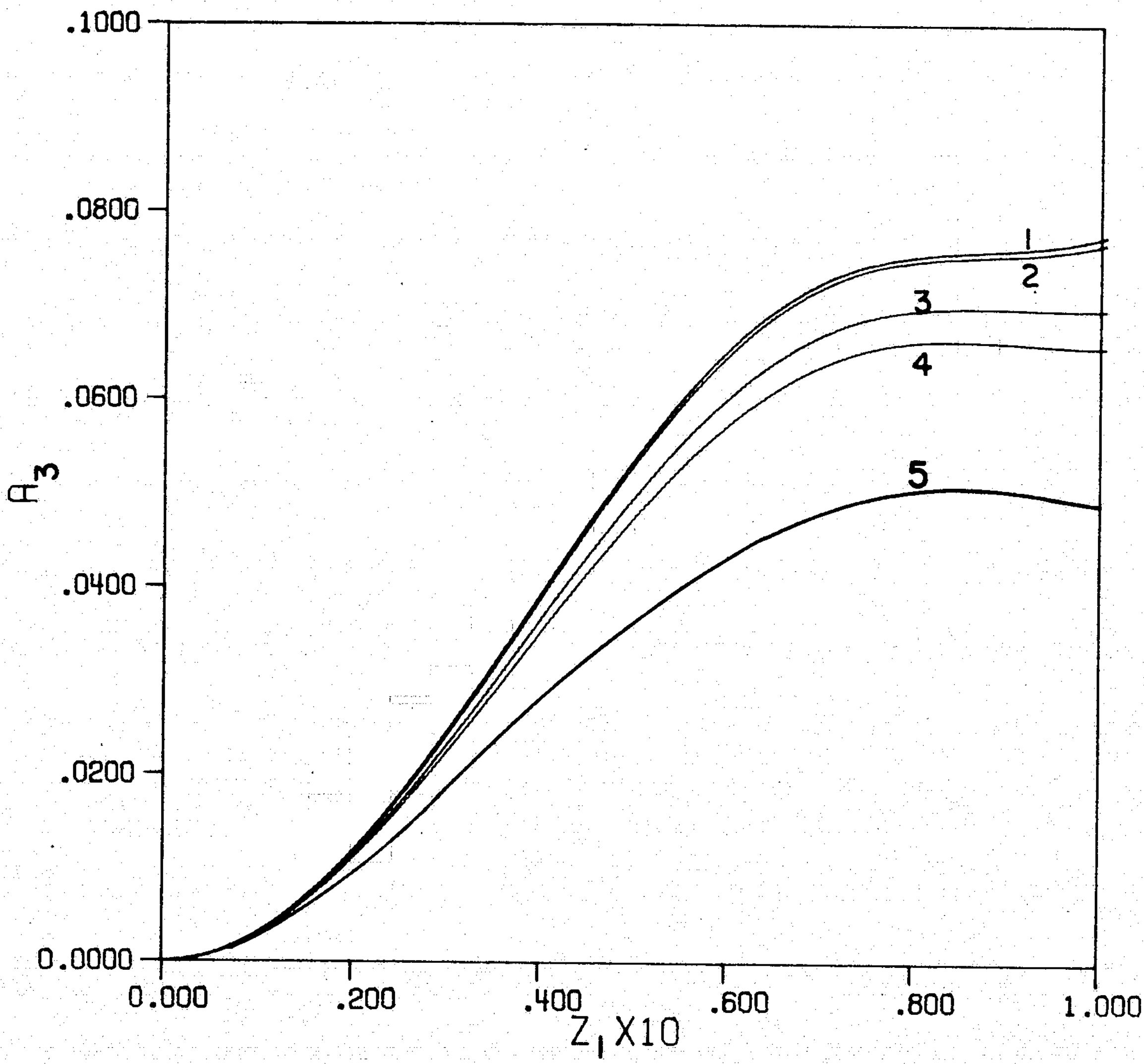
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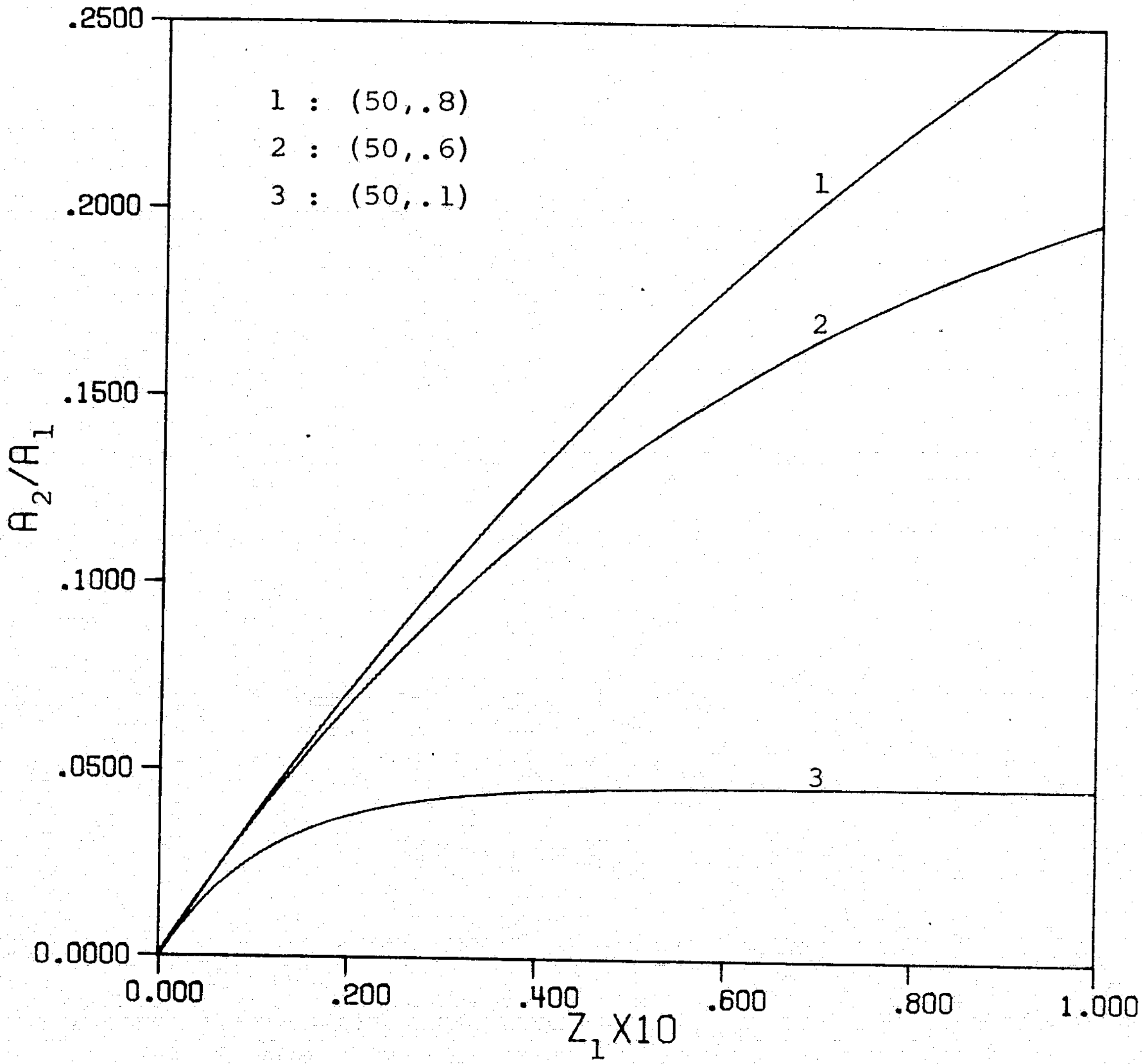

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