SURFACE FITTING THREE-DIMENSIONAL BODIES

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23

SUMMARY

The geometry of general three-dimensional bodies is generated from coordinates of points in several cross sections. Since these points may not be smooth, they are divided into segments and general conic sections are curve fit in a least-squares sense to each segment of a cross section. The conic sections are then blended in the longitudinal direction through longitudinal curves which may be used to define the conic sections in the cross-sectional planes. Both the cross-sectional and longitudinal curves may be modified by specifying particular segments as straight lines or specifying slopes at selected points. Slopes may be continuous or discontinuous and finite or infinite.

This method was used to surface fit a 70° slab delta wing and the HL-10 Lifting Body. The results for the delta wing were very close to the exact geometry. Although there is no exact solution for the lifting body, the surface fit generated a smooth surface with cross-sectional planes very close to prescribed coordinate points.

INTRODUCTION

Many disciplines require a mathematical description of three-dimensional surfaces which cannot be represented by simple mathematical expressions. The location and slope of points on a body are needed in the analysis of structures and inviscid flow fields. Viscous flow-field analyses require the body curvature in addition to location and slopes. The geometrical properties of simple shapes like spheres, cones, ellipsoids, and paraboloids can be described by relatively simple mathematical equations. However, many configurations of interest today, such as the space shuttle, are complex three-dimensional shapes whose geometry cannot be described very easily. In many cases all the information that is known is a drawing with a plan view, a side view, and several cross sections of the body. Sometimes models of a vehicle are available and coordinate positions can be accurately measured on them. On the other hand, slopes cannot be accurately measured on a drawing or model, and the measured radii of curvature are even less accurate. This report develops a method for surface fitting mathematical relations to complex three-dimensional bodies. The method yields accurate coordinates and slopes and reasonably accurate radii of curvature at any position on the body.

Previous approaches to surface fitting three-dimensional bodies generally divided the surface into "patches" and represented each patch by flat surfaces, cubic or higher order polynomials, or conic sections (e.g. see refs. 1-5).

Each of these methods has undesirable features in the forms used previously. Flat surfaces are completely unacceptable if radii of curvature or continuous slopes are needed. Cubic or higher order polynomials often lead to unwanted wiggles and bulges since they allow points of inflection. In ref. 1, DeJarnette applied the method of double splines, which used bi-cubic interpolation, to surface fit the coordinates of points in several cross-sectional planes of three-dimensional bodies. However, it was found that bulges and/or dimples occurred in these surfaces, particularly when the thickness was much smaller than the span.

Reference 2 approximated the shape of a space shuttle with elliptical cross sections with different ellipticity on the windward and leeward sides. Cubic polynomials were used to define segments of the plan and thickness distributions. The coefficients of these polynomials were chosen to make the slopes continuous across boundaries of the longitudinal segments. However, points of inflections were found to occur inside the segments and thus gave undesirable bumps and wiggles, which in turn significantly affected the surface pressure and heating distributions calculated with this geometry.

For sometime conic sections have been used to describe segments of aircraft contours (ref. 3). Both longitudinal and transverse contours were represented by conic sections, but the slopes at the ends of each segment had to be measured. As mentioned earlier, slopes are difficult to measure accurately. On the other hand, conic sections cannot have inflection points, and this feature is useful in eliminating undesirable wiggles.

Coons (ref. 4) developed a technique to describe three-dimensional surfaces by using blending functions to blend the surface between the boundary curves of each patch. A major difficulty of applying Coons' method is that the user must supply the coordinates, slopes, and twists (cross derivatives) at all four corners of each patch. This information is generally difficult to determine, particularly the cross derivatives (See ref. 5). Craidon (ref. 6) used this method with the twists arbitrarily chosen to be zero to surface fit three-dimensional geometries.

The method presented here uses data points in cross-sectional planes to curve fit segments of general conic sections in a least-squares sense. The conic sections are then blended in the longitudinal direction by use of coordinate points which define the conic sections in the cross-sectional planes. This technique has the advantage of allowing the user to continually modify the cross-sectional curves and the longitudinal curves until the body shape has the desired features. Also, discontinuous slopes in both the circumferential and longitudinal directions may be specified.

SYMBOLS

$a_{m,i}$, $b_{m,j}$	parameters defined by Eqs. (B5) - (B16)
A_1 thru A_5	coefficients of conic section given by Eq. (18) in global coordinates

A_j, B_j, C_j D_j, E_j, F_j	coefficients of conic section given by Eq. (1) in local coordinates
d_p	constant vector in Eq. (19)
G_{pq}	matrix used in Eq. (19)
K_j	data point number of first control point in segment j
m_j, n_j	slopes defined by Eqs. (5) and (6)
N	number of segments in a cross section
P_j, Q_j	terms defined by Eqs. (A2) and (A3)
$r_{j,k}$	residual in Eqs. (B1) and (B2)
R_j	term defined by Eq. (A5)
$\bar{x}$, $\bar{y}$, $\bar{z}$	Cartesian coordinates, see figure 2
y, z	local coordinates, see figure 3
α_j, β_j, γ_j	coefficients defined by Eqs. (15), (16), and (17)
θ_j	slope of segment j, see figure 5
$\Delta\theta_j$	$\theta_j - \theta_{j-1}$

Subscripts:

h	intermediate point in a cross-sectional segment
j	segment number in a cross section
k	data point number in a cross section

ANALYSIS

In a typical application, the geometry of a body must be determined from a model or a three-view drawing. Generally several cross sections are obtainable from the model or drawing, and the coordinates of data points on the boundary of these cross sections can be measured or calculated. A three-dimensional surface must then be fit through all the cross sections. In the method presented here, the data points in each cross section are divided into segments and general conic sections are fit to the data points in each segment as shown in figure 1. A three-dimensional surface is then generated by "blending" the cross-sectional curves in the longitudinal direction. Consider first the curve-fitting of data points in a cross-sectional plane.

Curve-Fit in a Cross-Sectional Plane

The data points in a cross-sectional plane are generally not completely smooth, and in these cases a smooth curve cannot be made to pass through every data point. Therefore, the data points are divided into segments, and a general conic section is fit in a least-squares sense through the data points in that segment. The conic section is constrained to go through the control points (end points of a segment) and also have a continuous slope at each control point unless otherwise specified (See figure 1).

A three-dimensional coordinate system $\bar{x}$, $\bar{y}$, $\bar{z}$ is used with $\bar{x}$ in the longitudinal direction and $\bar{x}$ = constant is a cross-sectional plane (See figure 2). Let j denote the segment number in a cross-sectional plane as shown in figure 1, with the first segment starting on the positive $\bar{y}$ - axis and j increasing clockwise. It is convenient to use a local coordinate system y,z for each segment with the origin on the first control point and the positive y - axis passing through the control point at the other end of the segment (See figure 3). In this local coordinate system it is easy to investigate possibilities of complex roots and interpret the coefficients of a general conic section geometrically, whereas it is difficult to interpret them geometrically in the global coordinates $\bar{y},\bar{z}$. It is assumed that the local coordinate z is single-valued in each segment.

The equation for a general conic section for the j^{th} segment is given by ref. 7 as

$$A_j y^2 + B_j yz + C_j z^2 + D_j y + E_j z + F_j = 0 \ . \tag{1}$$

Only five of the six coefficients are independent since the equation may be divided by any non-zero coefficient. The constraints that the conic sections pass through the two control points

$$y = 0, \ z = 0 \quad \text{and} \quad y = y_j, \quad z = 0 \quad \text{yields}$$

$$F_j = 0 \tag{2}$$

$$D_j = - A_j y_j \tag{3}$$

By differentiating Eq. (1) with respect to y, the slope in the local coordinates is given by

$$\frac{dz}{dy} = \frac{A_j y_j - 2A_j y - B_j z}{B_j y + 2C_j z + E_j} \tag{4}$$

The slopes at the control points are generally not known, so define

$$m_j = \left(\frac{dz}{dy}\right) \text{ at } y = 0,\ z = 0 \tag{5}$$

and

$$n_j = \left(\frac{dz}{dy}\right) \text{ at } y = y_j,\ z = 0\ . \tag{6}$$

By using these two equations in Eq. (4) it follows that

$$E_j = A_j\, y_j/m_j \tag{7}$$

and

$$B_j = -\,A_j\left(\frac{1}{m_j} + \frac{1}{n_j}\right)\ . \tag{8}$$

These equations show that the end slopes do not affect the coefficient C_j. On the other hand, the product A_jC_j determines the nature of the general conic section (ref. 7). If the discriminant $(B_j^2 - 4A_jC_j) = 0$ the conic is a parabola, if the discriminant < 0 the conic is an ellipse, and if the discriminate > 0 the conic is hyperbola. Of particular interest is the possibility of complex roots when solving for z as a function of y in the region of interest. It is shown in Appendix A that z will not be complex in the region $0 \leq y \leq y_j$ if

$$A_jC_j \geq \left(A_j/m_j\right)\left(A_j/n_j\right)\ . \tag{9}$$

For prescribed slopes m_j, n_j figure 4 shows how the product A_jC_j affects a conic section.

Unless a slope is specified at a control point, the method used here constrains the slope in global coordinates $(\bar{y},\bar{z})$ to be continuous at a control point. This makes the conic section in one segment dependent on data points in other segments as well as its own. From figure 5 it can be seen that continuity of slope at control point j requires

$$\tan^{-1} n_{j-1} = \tan^{-1} m_j + \Delta\theta_j\ . \tag{10}$$

This equation can be expanded and rearranged into the form

$$\frac{A_{j-1}}{(A/n)_{j-1}} = \frac{(A/m)_j \sin\Delta\theta_j + A_j \cos\Delta\theta_j}{(A/m)_j \cos\Delta\theta_j - A_j \sin\Delta\theta_j}\ . \tag{11}$$

Because this last equation is non-linear in the coefficients A_j, it would cause difficulty in obtaining a solution for them. However, as mentioned previously only 5 of 6 coefficients in a given segment are independent; therefore, an additional constraint can be imposed without affecting the resulting equation for the general conic section. The additional constraint used is to equate separately the numerators and denominators of both sides of Eq. (11). This procedure yields two equations, both linear in the coefficient A_j, which may be rearranged to give

$$(A/n)_j = \left\{A_j \cos \Delta\theta_{j+1} - A_{j+1}\right\}/\sin \Delta\theta_{j+1} \tag{12}$$

and

$$(A/m)_j = \left\{A_{j-1} - A_j \cos \Delta\theta_j\right\}/\sin \Delta\theta_j \; . \tag{13}$$

For the first segment $(j = 1)$, m_1 is specified and Eq. (13) is not needed, and also for the last segment $(j = N)$, n_N is specified and Eq. (12) is not needed.

Now substitute Eqs. (2), (3), (7), (8), (12), and (13) into Eq. (1) to obtain the equation for the general conic sections as

$$\alpha_j A_{j-1} + \beta_j A_j + \gamma_j A_{j+1} + z^2 C_j = 0 \tag{14}$$

where

$$\alpha_j = \begin{cases} \left\{zy_j - zy\right\}/\sin \Delta\theta_j & \text{for } j > 1 \\ 0 & \text{for } j = 1 \end{cases} \tag{15}$$

$$\beta_j = \begin{cases} y^2 - (1/m_1 + \cot\Delta\theta_2)\, yz + y_j z/m_1 - y_j y & \text{for } j = 1 \\ y^2 + (\cot\Delta\theta_j - \cot\Delta\theta_{j+1})\, yz - \cot\Delta\theta_j y_j z - y_j y & \text{for } 1 < j < N \\ y^2 + (\cot\Delta\theta_N - 1/n_N)\, yz - \cot\Delta\theta_N y_N z - y_N y & \text{for } j = N \end{cases} \tag{16}$$

$$\gamma_j = \begin{cases} yz/\sin\Delta\theta_{j+1} & \text{for } j < N \\ 0 & \text{for } j = N \end{cases} \; . \tag{17}$$

These equations indicate that there are only two unknowns in each segment, A_j and C_j. However, there is one more segment than interior control points which means that one of these coefficients is arbitrary. Here $A_1 = 1$ is used arbitrarily (unless the conic section requires $A_1 = 0$) and the independent unknowns become C_1, C_j, $A_j (j = 2, \cdots, N)$. If there were as many data points as unknowns, and if some conic section could be made to pass through these data points, then Eq. (14) could be applied to all the data points to give (2N-1) linear equations for the (2N-1) coefficients C_1, C_j, A_j for $j = 2, \cdots, N$. However, there are generally more than (2N-1) data points, and if Eq. (14) were applied to all of them an overdetermined system of linear equations (See ref. 8) would result. Therefore, a least-squares solution of the overdetermined system is used to determine the coefficients C_1, C_j, A_j for $j = 2, \cdots, N$. This procedure is described in Appendix B. After obtaining the coefficients in this manner, the inequality of Eq. (9) is checked for the possibility of complex roots for z in each segment. If the inequality of Eq. (9) is not satisfied, the coefficient C_j in that section is replaced by the value obtained using the equality sign in Eq. (9). As mentioned earlier, the coefficient C_j does not affect the slopes of the curve at the end points of the segment, and hence C_j does not affect other segments. The equality sign in Eq. (9) gives a hyperbola of 2 straight lines as shown in figure 4.

In order to solve for z as a function of y from Eq. (14), a quadratic equation must be solved, and the proper choice of the + or - sign must be determined beforehand for each segment. It is shown in Appendix A that in order to make $z = 0$ at $y = 0$ (a control point at the beginning of segment j), the + sign must be used if $(A/m)_j > 0$ and the - sign must be used if $(A/m)_j < 0$.

Once the coefficients A_j and C_j are determined, all the conic sections for that cross section are completely defined. In order to put these results into a form suitable for "blending" the cross sections in the longitudinal direction, the conic sections for each segment are redefined in terms of 4 points: the two control points at the end of the segment, a slope point which determines the slope at the end points, and finally an intermediate point on the curve between the end points (See figure 6). The 3 points on the curve and the two slopes at the end points of a segment are sufficient to determine new coefficients A_1, A_2, A_3, A_4, and A_5 for the general conic section*,

$$A_1\bar{y}^2 + A_2\bar{y}\bar{z} + A_3\bar{z}^2 + A_4\bar{y} + A_5\bar{z} + 1 = 0 \tag{18}$$

in global coordinates $\bar{y},\bar{z}$. This process is applied to each segment in a cross section, and therefore the 5 coefficients become functions of the longitudinal coordinate $\bar{x}$ when the segments of a cross section are blended with corresponding segments in the other cross sections.

Longitudinal Variation of Cross Sections

In order to determine the longitudinal variation of the coefficients in Eq. (18), a three-dimensional curve is fit through each of the 4 points used to define the conic section of corresponding segments (See figure 7). In contrast to the cross-sectional data points, these curves must pass through each of the points in the longitudinal direction. They are represented by their projections

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*Note that the conic section given by Eq. (18) must have the constant 1 replaced by 0 if the curve is to pass through the origin.

in the $\bar{x}$, $\bar{y}$ and $\bar{x}$, $\bar{z}$ planes; hence, two planar curves are used to represent each three-dimensional curve. The parameteric method of cubic splines (ref. 9) is used to curve-fit each planar curve, with the chordal distance between the coordinate points as the parameter. The parametric spline allows infinite slopes whereas the regular spline does not.

This method worked very well for some test cases but had trouble with others because of the bumps and wiggles inherent to cubic splines. These bumps and wiggles can be corrected by specifying slopes in the longitudinal direction at certain cross sections. Also parts of the longitudinal curves can be specified as straight lines.

Consider now the longitudinal variation of a conic section. In each cross-sectional plane, Eq. (18) will hold but the coefficients A_1, A_2, A_3, A_4, and A_5 will vary with $\bar{x}$. As mentioned previously, these coefficients are determined by 4 defining points (the two control points, an intermediate point, and the slope point). For each segment, the 5 equations used to determine the coefficients A_q ($q = 1, \cdots, 5$) are formed by applying Eq. (18) to the 3 points on the cross-sectional curve (the two control points $\bar{y}_o$, $\bar{z}_o$ and $\bar{y}_1$, $\bar{z}_1$, the intermediate point $\bar{y}_h$, $\bar{z}_h$) and the slopes at the ends of the segment using the slope point $\bar{y}_s$, $\bar{z}_s$ (See figure 6). This procedure yields the following 5 equations:

$$\sum_{q=1}^{5} G_{pq} A_q = d_p \qquad p = 1, \cdots, 5 . \tag{19}$$

At any longitudinal position, Eq. (19) can be solved by any standard matrix inversion routine to determine the coefficients A_q. The derivatives $dA_q/d\bar{x}$ and $d^2A_q/d\bar{x}^2$ can be obtained by differentiating Eq. (19) and successively solving the resulting system of linear equations. The elements of G_{pq} and their derivatives with respect to $\bar{x}$ are obtained from the three-dimensional curves which were spline-fit through the 4 points used to define the conic section for that segment in each cross-sectional plane.

COMPUTATIONAL ALGORITHM

Given the set of data points (y_k, z_k) in cross-sectional planes at several longitudinal stations,

(1) For each cross-sectional plane, divide the data points into segments so that a conic section can be curve-fit to the data points in each segment by the least-squares technique developed herein.

(2) If the curves fit to the cross-sectional data points are not satisfactory, modify them by one or more of the following methods: (a) define new boundaries (control points) for segments, (b) specify slope(s) at control point(s) (slopes may be finite or infinite and continuous or discontinuous), (c) specify selected segments as straight lines, (d) a specific conic section can be

specified for a segment by prescribing the slopes at the ends of the segment and using only one datum point between the end control points.

(3) Represent the conic section for each segment in a cross-sectional plane in terms of the two control points at the ends of the segment, an intermediate point, and the slope point (See figure 6).

(4) For each point found in step (3), spline-fit a three-dimensional longitudinal curve through it and the corresponding points in other cross-sectional planes (See figure 7).

(5) If the longitudinal curves are not satisfactory, modify them by one or more of the following methods: (a) specify slope(s) at longitudinal station(s) (slopes may be finite or infinite and continuous or discontinuous), (b) specify selected longitudinal segments as straight lines, (c) redefine the boundaries (control points) of the segments in the cross-sectional planes so that the points used for the longitudinal spline-fit form a smooth curve.

(6) The geometrical properties of the surface may be computed in polar or Cartesian coordinates at any position $\bar{x}$, $\bar{y}$ by the following steps:

   a. Locate $\bar{x}$ between two consecutive longitudinal stations, and then locate the cross-sectional segment which contains $\bar{y}$.

   b. Use the spline function to calculate the coordinates, slopes, and second derivatives of the 4 longitudinal curves for this segment at $\bar{x}$ (See figure 7).

   c. Calculate the coefficients $A_q (q = 1, \cdots, 5)$ of the conic section at this location by use of Eq. (19). Determine the first and second derivatives of A_q with respect to $\bar{x}$ from the first and second derivatives of Eq. (19).

   d. Calculate the body position $\bar{z}$ from Eq. (18), and the derivatives of $\bar{z}$ with respect to $\bar{x}$ and $\bar{y}$ from derivatives of Eq. (18).

(7) After a satisfactory surface fit has been obtained, the data which must be retained for a geometry subroutine package are the coordinates and longitudinal slopes of the longitudinal curves at those longitudinal stations where cross-sectional data points were given. Then, the geometrical properties of the surface can be calculated at any position by the method outlined in step (6) above.

APPLICATION TO 70° DELTA WING

The surface fitting method was applied to the 70° slab delta wing shown in figure 8. This example was chosen because it illustrates many of the options available to modify the longitudinal curves and because the results can be compared with an exact solution.

Cross-sectional data was used as shown in figure 8, and due to symmetry only the first quadrant is used. Two segments (three control points) are needed to represent the cross section of $\bar{x} = 10$. The first segment is a straight line and the second is one-fourth of an ellipse. The least-squares curve-fit technique represents the ellipse exactly by specifying a zero slope at control point $j = 2$, an infinite slope at control point $j = 3$, and one datum point between these two control points. Although two segments must also be used for the other two cross sections, only one segment is necessary to specify the circle at $\bar{x} = 0.65798$ and the ellipse at $\bar{x} = 1.0$. Therefore, the first two data points, which are also control points, are made coincident. Then the exact curves are calculated from the least-squares curve-fit by specifying a zero slope at control point $j = 2$, an infinite slope at control point $j = 3$, and one datum point between these two control points. The exact location of this datum point is irrelevant except that it must lie on the desired curve.

The three-dimensional longitudinal curve is represented by its projections in the $\bar{x},\bar{y}$ and $\bar{x},\bar{z}$ planes. Therefore, 16 longitudinal planar curves are used for this example (8 for each cross-sectional segment). Modifications were made to the initial spline-fits to 14 of these curves since exact slopes are readily obtained from figure 8.

The geometrical properties and their derivatives were calculated at four circumferential positions for $\bar{x} = 0.3$, 1.0, 2.0, and 5. The results at $\bar{x} = 1.0$, 2.0, and 5.0 are exact (within the accuracy of single precision on the IBM 360/175 computer), whereas some inaccuracies were noted at $\bar{x} = 0.3$.

APPLICATION TO HL-10 LIFTING BODY

The surface fitting method was also applied to part of the HL-10 lifting body. Figure 9 shows a view of this body and the six cross sections used to generate the surface fit. Tabulated values of the coordinates of surface points were available for many cross sections in addition to the six actually used, and these additional cross sections were used as a check on the accuracy of the surface fit. Due to symmetry about the $\bar{x},\bar{y}$ plane, only half of the body was used.

Figure 10 illustrates the first attempt to curve fit the cross-sectional data at $\bar{x} = 128.32$. Only two segments were used, and hence the resulting curve-fit is not satisfactory. Figure 11 shows the revised curve-fit to this same cross section when three segments are used, and the results are very good.

After satisfactory curve-fits were obtained for all six cross sections, using three segments in each one, the cross sections were blended in the longitudinal direction by use of the parametric cubic splines. Since each three-dimensional longitudinal curve is represented by its projections in the $\bar{x},\bar{y}$ and $\bar{x},\bar{z}$ planes, a total of 24 planar curves were used to represent the four longitudinal curves required for each of the three segments in a cross section. Of these 24 curves, seven of the initial spline-fits were found to have wiggles, and hence they were modified by specifying their slopes at the nose ($\bar{x} = 0$). Some of the revised longitudinal curves are shown in figure 9, and they are actually smooth although the plotter makes them appear to have wiggles.

As a check on the accuracy of the final surface fit, cross sections were calculated at axial stations in between those used to generate the surface fit. Figure 12 compares the calculated cross section at $\bar{x} = 43.656$ with the actual coordinates for this body. Other cross sections were found to compare even more closely than this one with the actual coordinates.

CONCLUDING REMARKS

An algorithm has been developed for surface fitting three-dimensional bodies from data points in several cross-sectional planes. This method was found to give satisfactory surface fits to the 70° slab delta wing and HL-10 lifting body. The surface geometry can best be analyzed through the use of an interactive computer graphics environment.

After a satisfactory surface fit to a body has been obtained, a relatively simple geometry subroutine package can be formed to use in other computer programs requiring a mathematical model of the geometry. It will calculate the body coordinates, slopes, and second derivatives at any position on the body. This method could be used to generate the input data for Coons' pathing method if desired. Computational time is short, and the amount of storage required is relatively small. A copy of the computer program can be obtained from the authors.

ACKNOWLEDGMENTS

This research was supported by NASA Langley Grant No. NGR 34-002-193. The NASA Technical Officer for this grant was Mr. H. Harris Hamilton of the Advanced Entry Analysis Branch, Space Systems Division.

APPENDIX A

Solution for Conic Section Equation

Substitute Eqs. (2), (3), (7), and (8) into Eq. (1) to obtain the conic section in the local coordinate system in the form

$$C_j z^2 + P_j z + Q_j = 0 \tag{A1}$$

where

$$P_j = - \left[(A/m)_j + (A/n)_j \right] y + (A/m)_j \; y_j \tag{A2}$$

and

$$Q_j = A_j y \left(y - y_j \right) \; . \tag{A3}$$

The solution of Eq. (A1) is

$$\begin{aligned} z &= \left(- P_j \pm R_j^{\,1/2} \right) / 2C_j \qquad &&\text{for } C_j \neq 0 \\ z &= - Q_j / P_j &&\text{for } C_j = 0,\; P_j \neq 0 \end{aligned} \tag{A4}$$

where the discriminate (R_j) is given by

$$R_j = P_j^{\,2} - 4C_j Q_j \; . \tag{A5}$$

In order to obtain real roots for z from Eq. (A4), Eq. (A5) must give $R_j \geq 0$ for $0 \leq y \leq y_j$. Note that Eq. (A5) gives $R_j \geq 0$ when $A_j C_j \geq 0$.

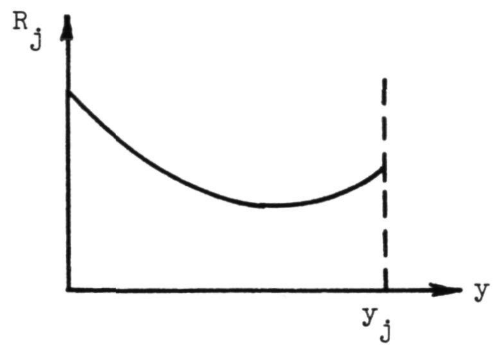
Also,

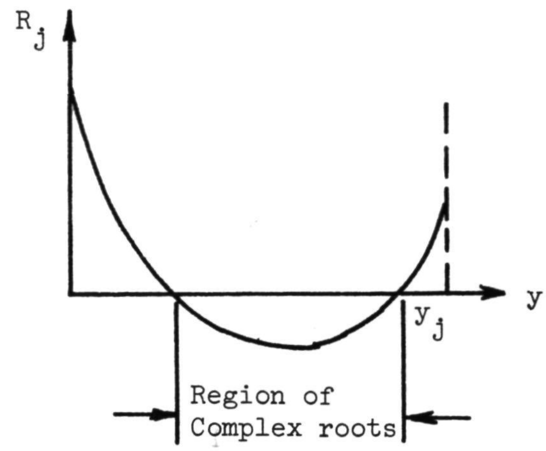
$$R_j = (A/m)_j^{\,2} \; y_j^{\,2} \geq 0 \qquad \text{at } y = 0$$

and

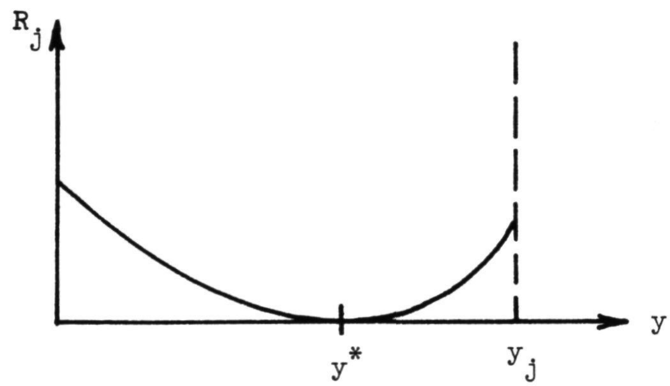
$$R_j = (A/n)_j^{\,2} \; y_j^{\,2} \geq 0 \qquad \text{at } y = y_j \; .$$

The possible variations for R_j are illustrated on the following page.





The minimum value of A_jC_j which still gives real roots corresponds to the limiting case shown below.



The minimum value of R_j occurs at $y = y^*$ where both $R_j = 0$ and $dR_j/dy = 0$. The solution of $R_j = 0$ and $dR_j/dy = 0$, using Eq. (A5), gives the minimum value of A_jC_j as

$$\left(A_jC_j\right)^* = (A/m)_j\ (A/n)_j \tag{A6}$$

Thus, Eq. (A4) will give real roots for z if

$$A_jC_j \geq (A/m)_j\ (A/n)_j\ . \tag{A7}$$

The choice of the sign to be used in Eq. (A4) is determined by requiring the equation to be satisfied at the control points ($z = 0$, $y = 0$) and ($z = 0$, $y = y_j$). For the first control point, $y = 0$, Eq. (A4) yields

$$z = \frac{-(A/m)_j y_j \pm \left[(A/m)_j^2 {y_j}^2\right]^{1/2}}{2C_j} \tag{A8}$$

and thus $z = 0$ requires the + sign if $(A/m)_j > 0$ and the - sign if $(A/m)_j < 0$. At the second control point, $y = y_j$, Eq. (A4) yields

$$z = \frac{(A/n)_j y_j \pm \left[(A/n)_j^2 y_j^2\right]^{1/2}}{2C_j} \tag{A9}$$

and $z = 0$ requires the - sign if $(A/n)_j > 0$ and the + sign if $(A/n)_j < 0$. These conditions are all compatible because a conic section passing through the two control points and single-valued in z will have $n_j < 0$ if $m_j > 0$ and $n_j > 0$ if $m_j < 0$. The sign given by these conditions can be used in Eq. (A4) for all values of y in the range $0 \le y \le y_j$.

APPENDIX B

Least-Squares Solution for Conic Section Coefficients

Equation (14) cannot be satisfied at all data points because there would be more equations than unknowns. It is also not desirable, in general, to have the curve go through all the data points because there may be scatter in the data. Accordingly, Eq. (14) is applied at data point k and rewritten as

$$\alpha_{j,k} A_{j-1} + \beta_{j,k} A_j + \gamma_{j,k} A_{j+1} + z_k^2 C_j = r_{j,k} \tag{B1}$$

where $\alpha_{j,k}$, $\beta_{j,k}$, and $\gamma_{j,k}$ are the values of α_j, β_j, and γ_j evaluated at $y = y_k$, $z = z_k$ in segment j; and $r_{j,k}$ is called the residual. The least-squares solution of the overdetermined system of equations determines the coefficients C_1, C_j, A_j ($j = 2, \cdots, N$) which minimize the sum of the residuals squared (See ref. 8). Define K_j as the data point number which corresponds to the first control point in segment j. Square Eq. (B1) and sum over all the data points in continuous segments $j = 1, \cdots, N$ to obtain

$$\sum_{j=1}^{N} \sum_{k=K_j}^{K_{j+1}} \left[\alpha_{j,k} A_{j-1} + \beta_{j,k} A_j + \gamma_{j,k} A_{j+1} + z_k^2 C_j\right]^2 = \sum_{j=1}^{N} \sum_{k=K_j}^{K_{j+1}} r^2_{j,k} \; . \tag{B2}$$

The right side of Eq. (B2) is minimized by the system of equations obtained by setting partial derivatives of Eq. (B2) with respect to the independent coefficients equal to zero. The result of setting partial derivatives with respect to $A_2, \cdots, A_N$ equal to zero yields the following set of equations:

$$a_{m,1}A_{m-2} + a_{m,2}A_{m-1} + a_{m,3}A_m + a_{m,4}A_{m+1} + a_{m,5}A_{m+2} + a_{m,6}C_{m-1} + a_{m,7}C_m$$
$$+ a_{m,8}C_{m+1} = 0 \qquad m = 2, \cdots, N \tag{B3}$$

The result of setting partial derivatives with respect to $C_1, \cdots, C_N$ equal to zero gives the following additional system of equations:

$$b_{m,1}A_{m-1} + b_{m,2}A_m + b_{m,3}A_{m+1} + b_{m,4}C_m = 0$$

$$m = 1, \cdots, N . \qquad (B4)$$

The combined system of Eqs. (B3) and (B4) gives a system of (2N - 1) linear equations for (2N - 1) coefficients. The parameters used in Eqs. (B3) and (B4) are defined as follows:

$$a_{m,1} = \sum_{k=K_{m-1}}^{K_m} \alpha_{m-1,k}\, \gamma_{m-1,k} \qquad (B5)$$

$$a_{m,2} = \sum_{k=K_{m-1}}^{K_m} (\beta\gamma)_{m-1,k} + \sum_{k=K_m}^{K_{m+1}} \alpha_{m,k}\, \beta_{m,k} \qquad (B6)$$

$$a_{m,3} = \sum_{k=K_{m-1}}^{K_m} \gamma^2_{m-1,k} + \sum_{k=K_m}^{K_{m+1}} \beta^2_{m,k} + \sum_{k=K_{m+1}}^{K_{m+2}} \alpha^2_{m+1,k} \qquad (B7)$$

$$a_{m,4} = a_{m+1,2} \qquad (B8) \qquad\qquad a_{m,5} = a_{m+2,1} \qquad (B9)$$

$$a_{m,6} = \sum_{k=K_{m-1}}^{K_m} z_k^2\, \gamma_{m-1,k} \qquad (B10) \qquad\qquad a_{m,7} = \sum_{k=K_m}^{K_{m+1}} z_k^2\, \beta_{m,k} \qquad (B11)$$

$$a_{m,8} = \sum_{k=K_{m+1}}^{K_{m+2}} z_k^2\, \alpha_{m+1,k} \qquad (B12) \qquad\qquad b_{m,1} = a_{m-1,8} \qquad (B13)$$

$$b_{m,2} = a_{m,7} \qquad (B14) \qquad\qquad b_{m,3} = a_{m+1,6} \qquad (B15)$$

$$b_{m,4} = \sum_{k=K_m}^{K_{m+1}} z_k^4 \qquad (B16)$$

Note that if slopes should be specified at selected control points, then the least-squares solution described above is applied to the segments between two consecutive control points with specified slopes. In the solution of the combined system of Eqs. (B3) and (B4) the term $\alpha_{N+1,k}$ and $\gamma_{N+1,k}$ must be interpreted as zero. Hence, the coefficients A_o, A_{N+1}, and A_{N+2} do not appear in the resulting system, and recall that $A_1 = 1$ unless the conic section requires $A_1 = 0$.

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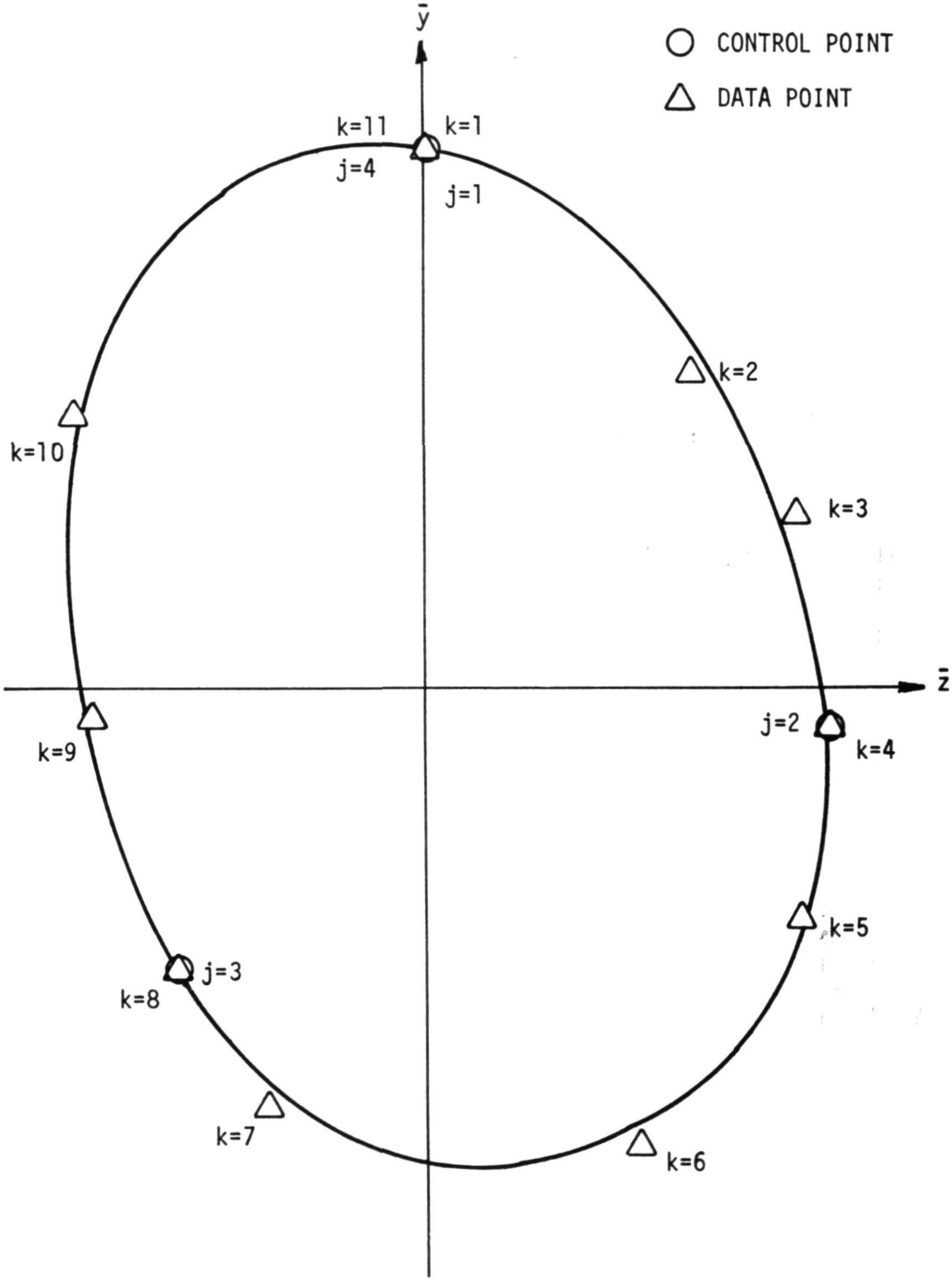

Figure 1. Control points and data points in a cross section.

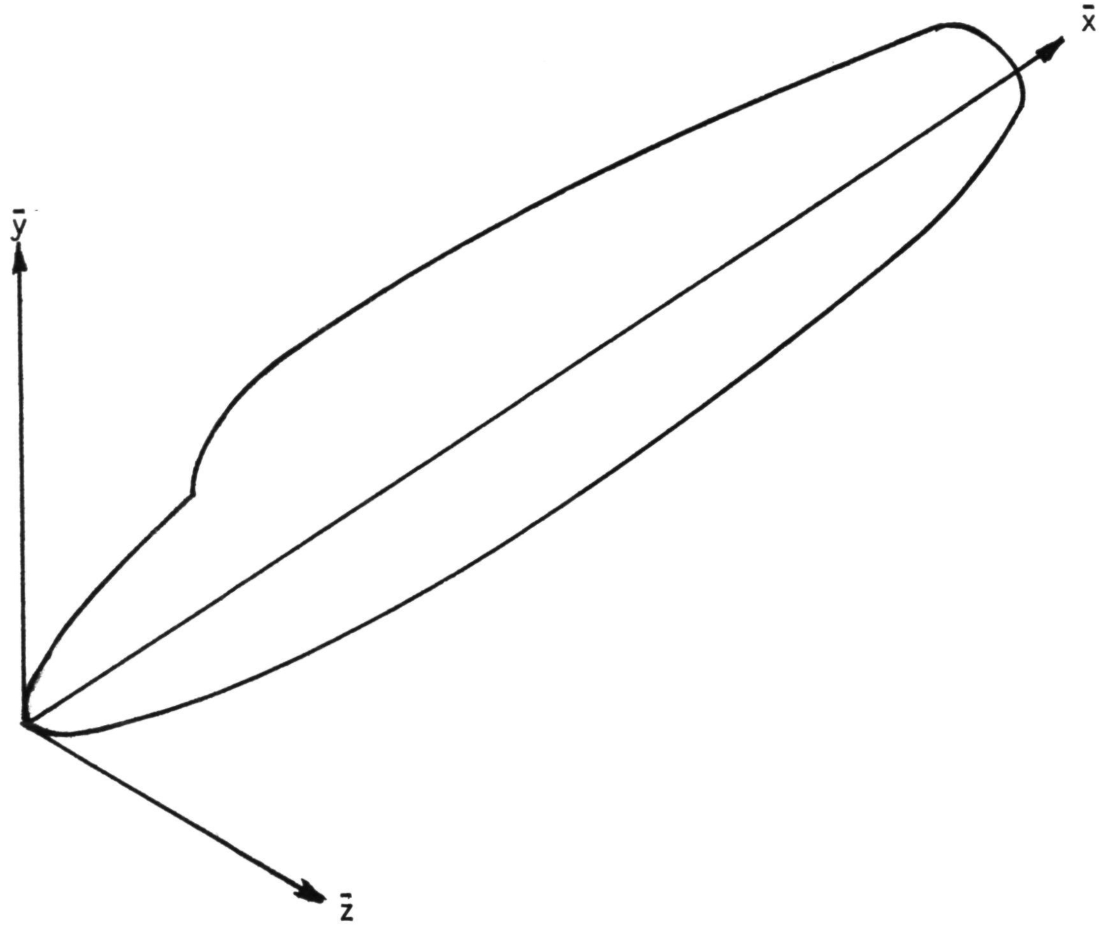

Figure 2. Cartesian coordinate system.

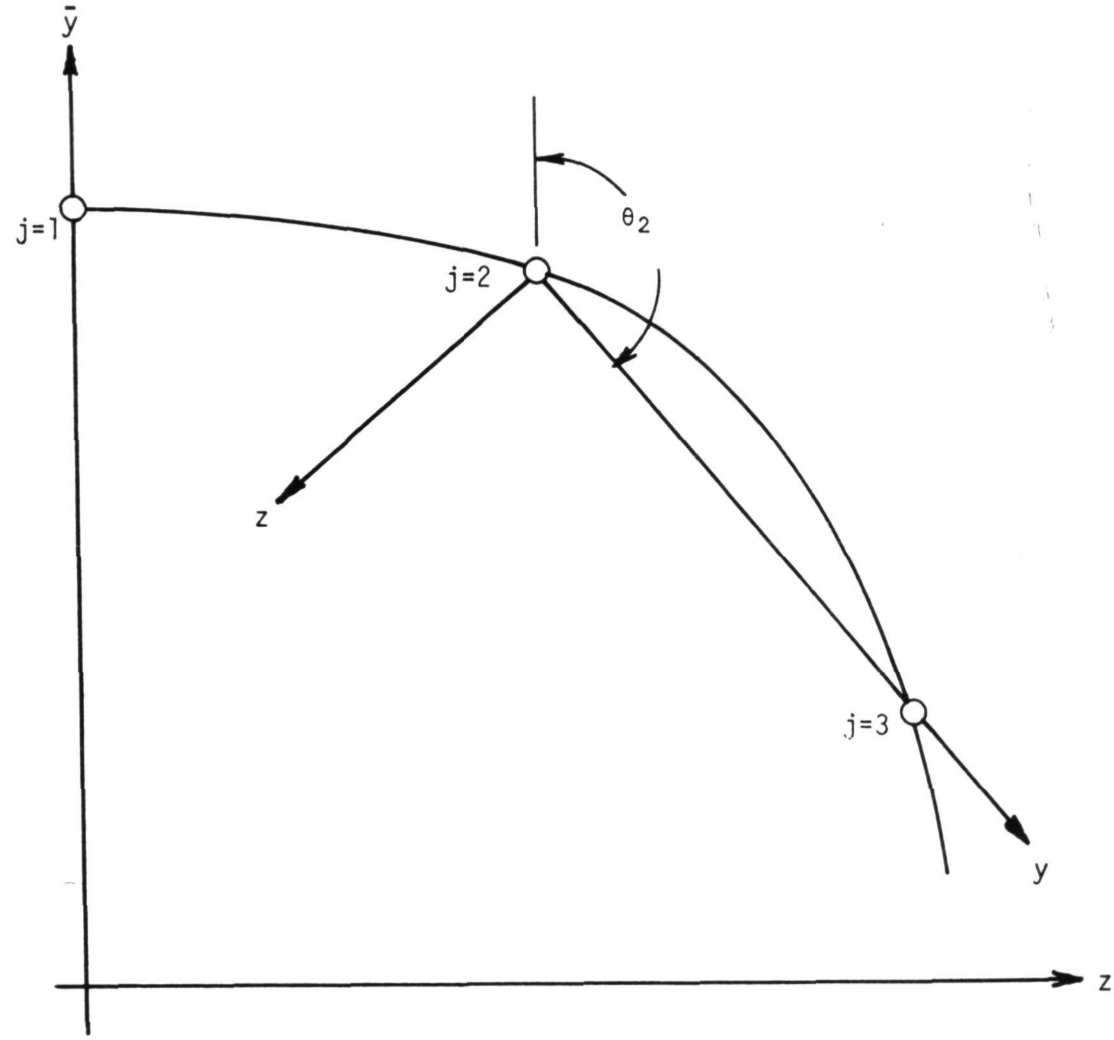

Figure 3. Local coordinate system, illustrated for segment j=2.

Equation for General Conic Section

$$A_j\, y^2 + B_j\, yz + C_j\, z^2 + D_j\, y + E_j\, z + F_j = 0$$

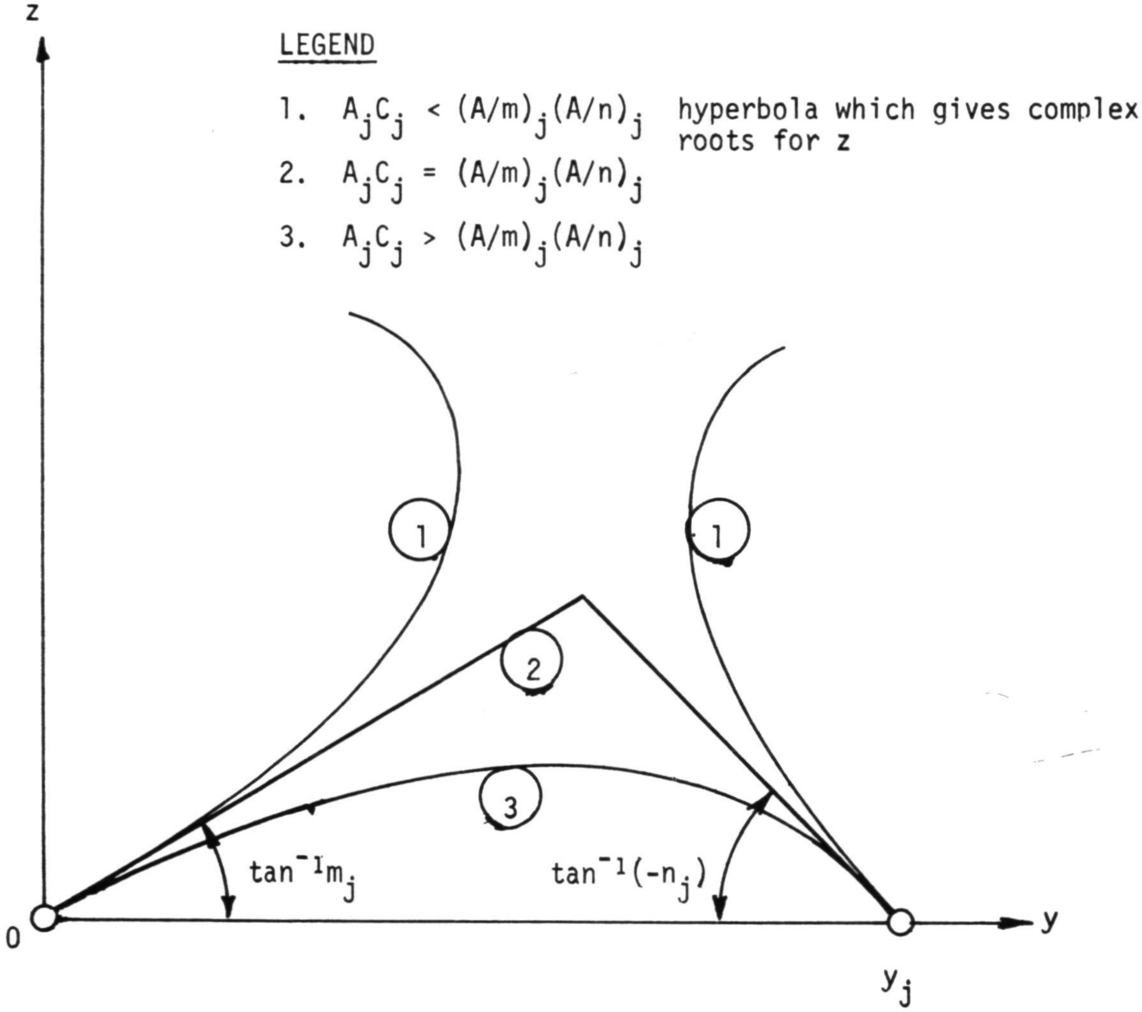

Figure 4. Effect of A_jC_j on conic section.

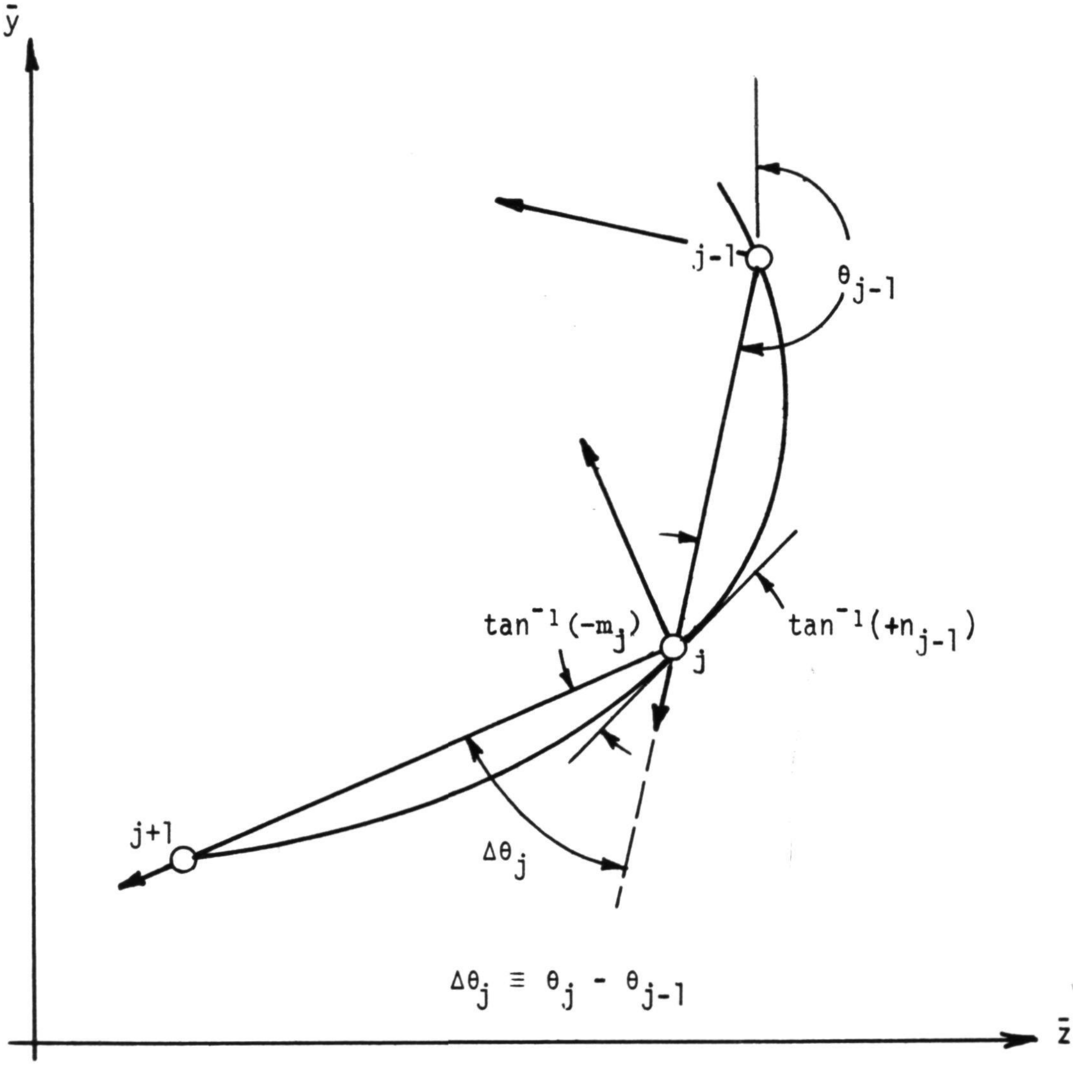

Figure 5. Continuity of slope at a control point.

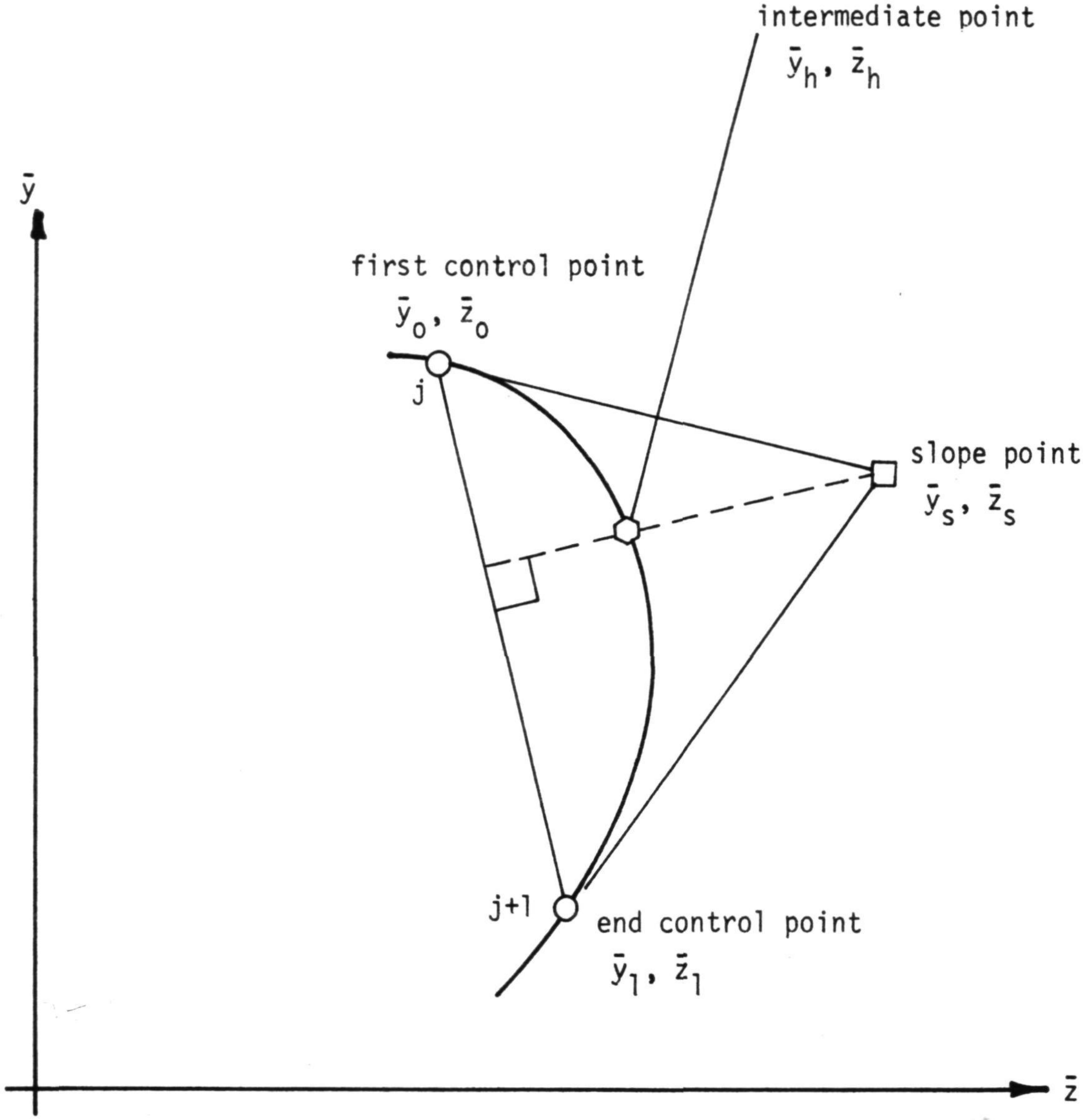

Figure 6. Four points used to define a segment of a conic section.

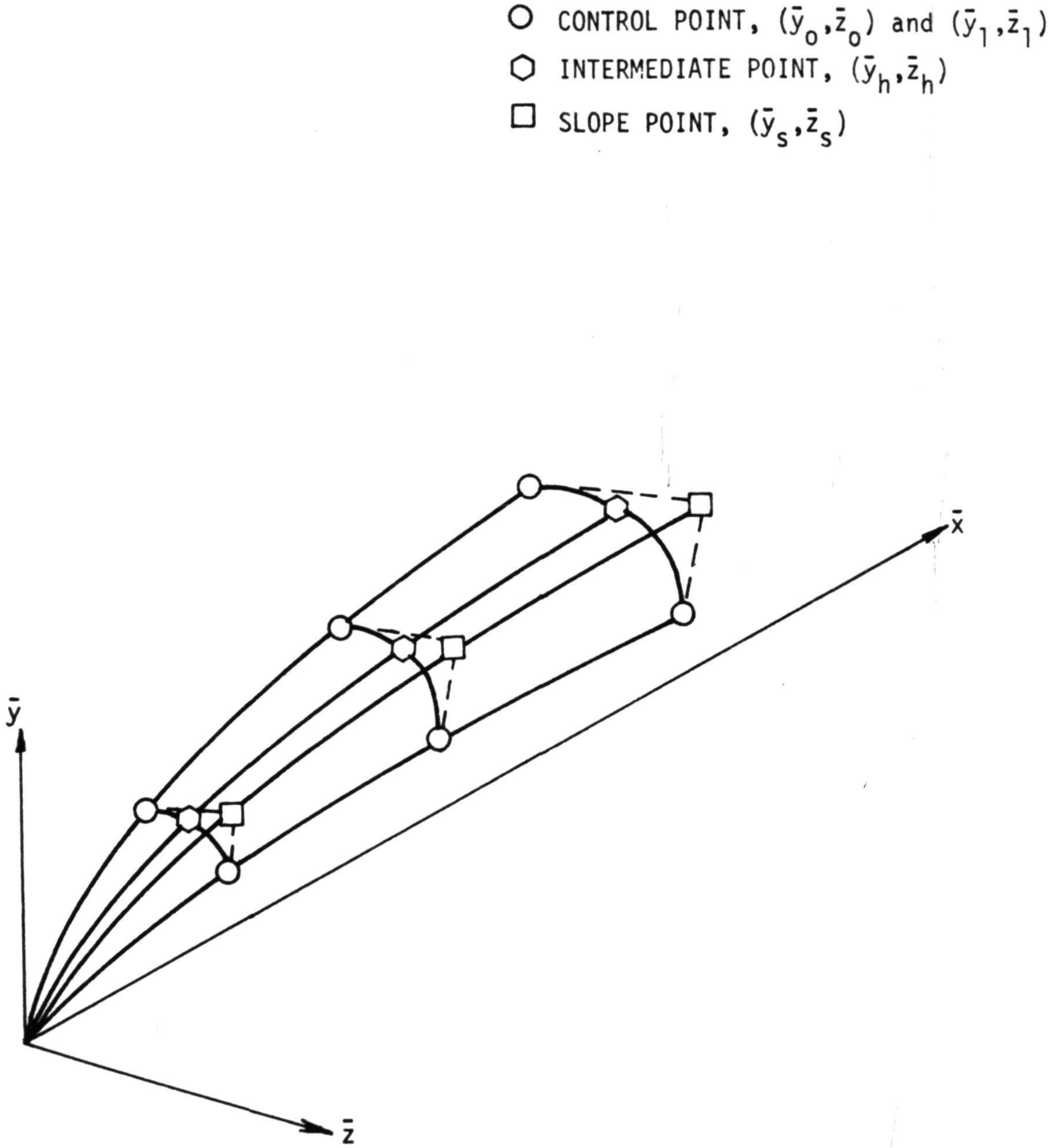

Figure 7. Longitudinal curves through the four points used to define a segment of a conic section.

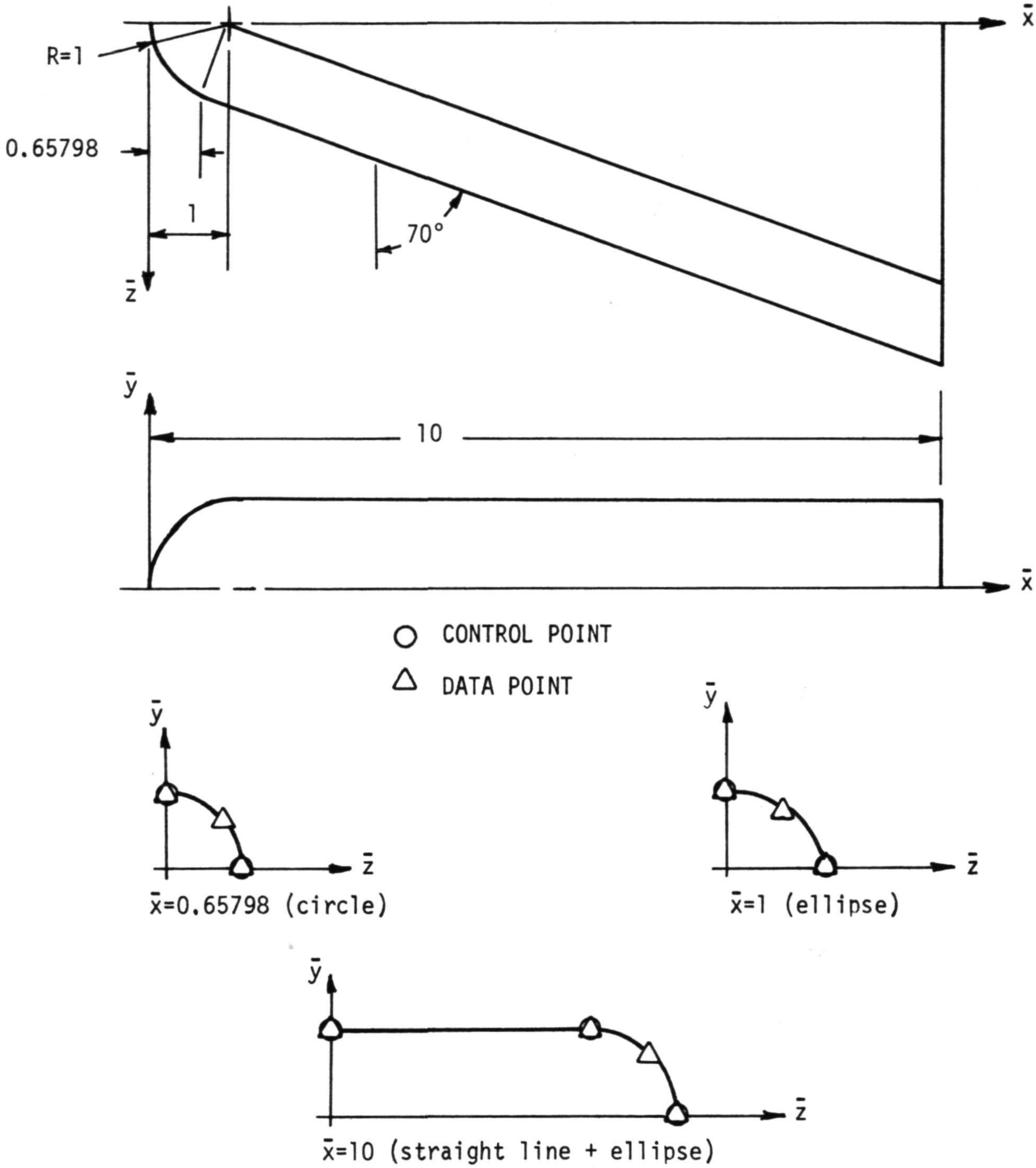

Figure 8. Geometry of 70° delta wing.

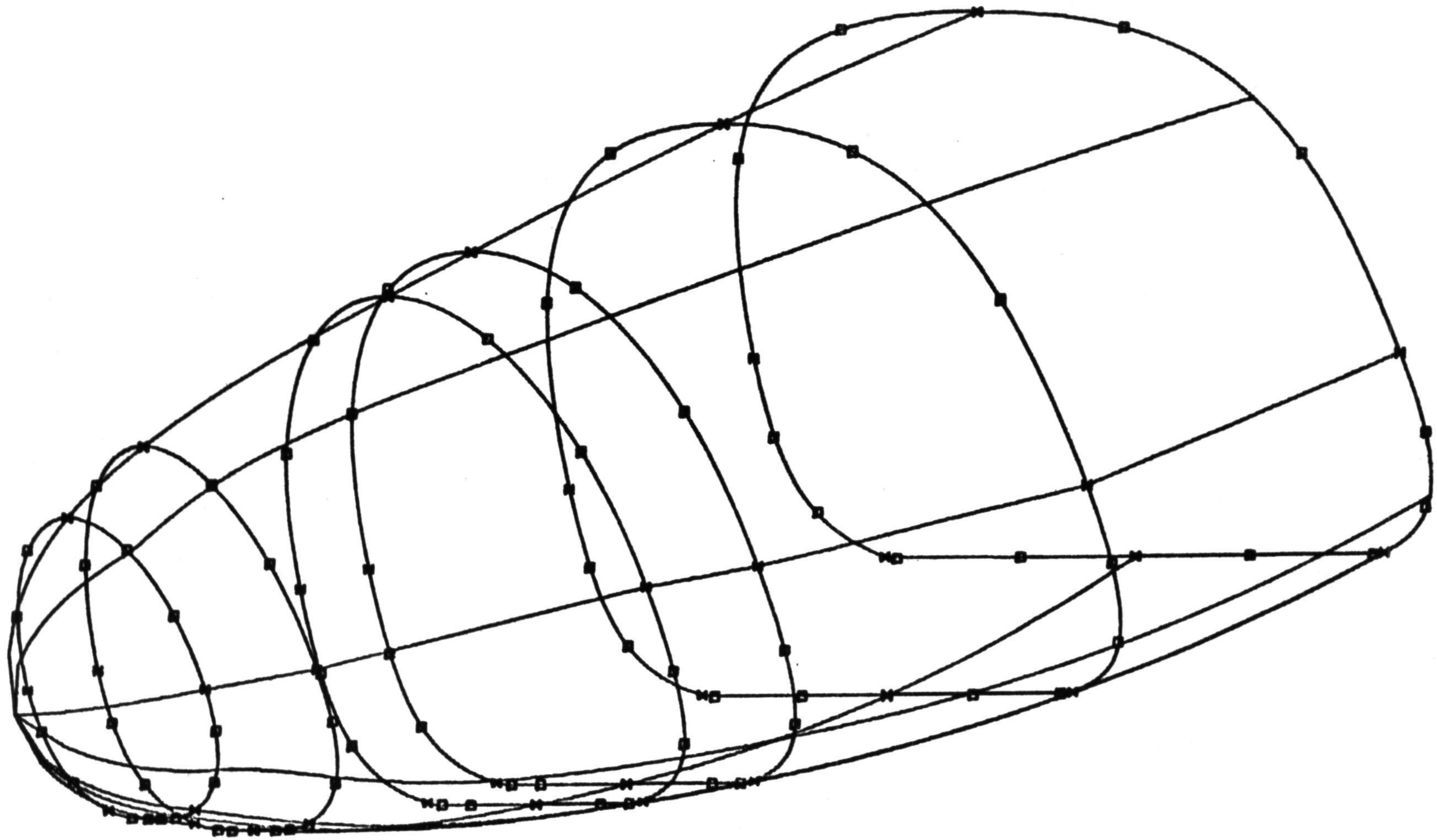

Figure 9. View of cross sections and longitudinal curves for HL-10 body.

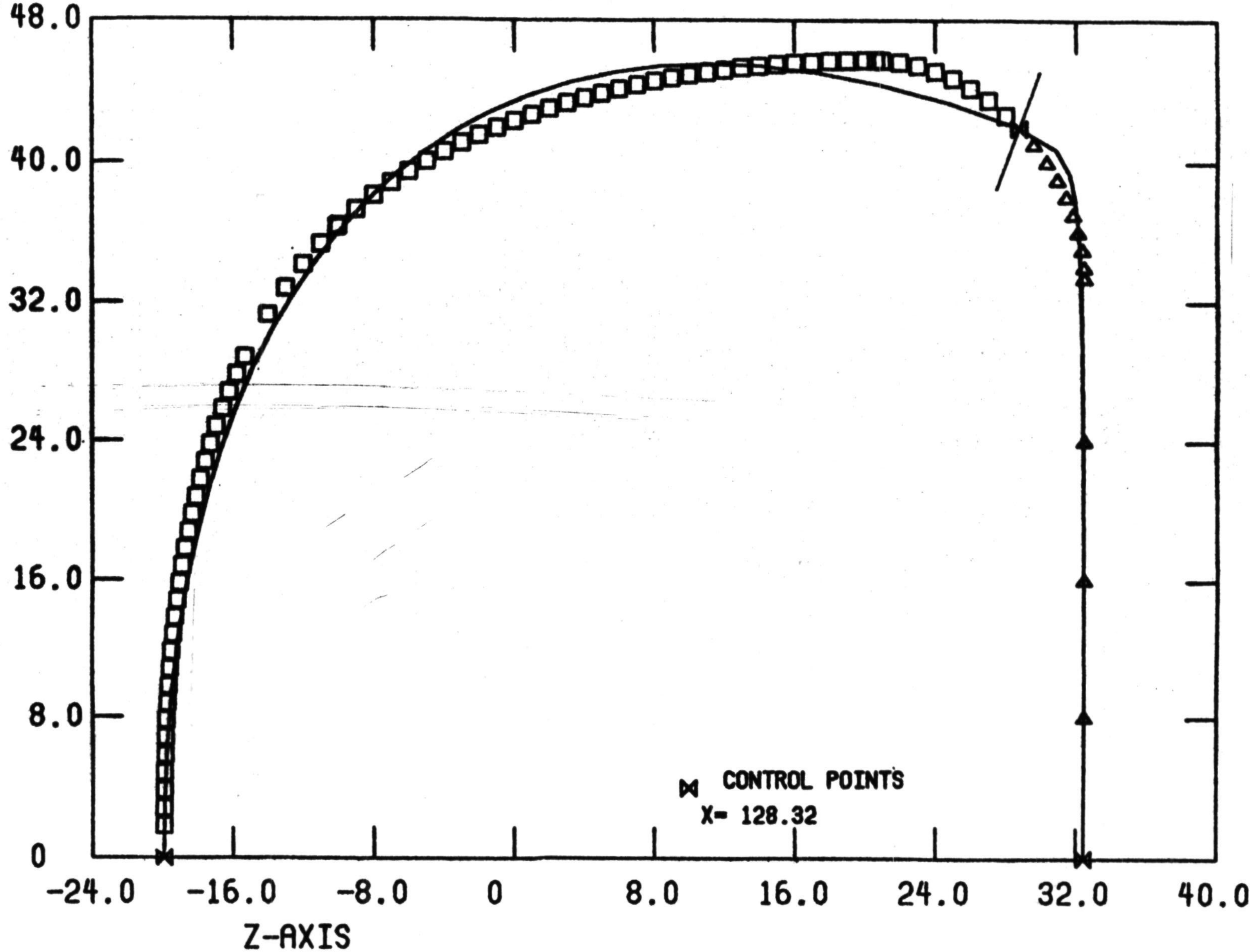

Figure 10. First attempt to curve-fit cross section of HL-10 body using two segments.

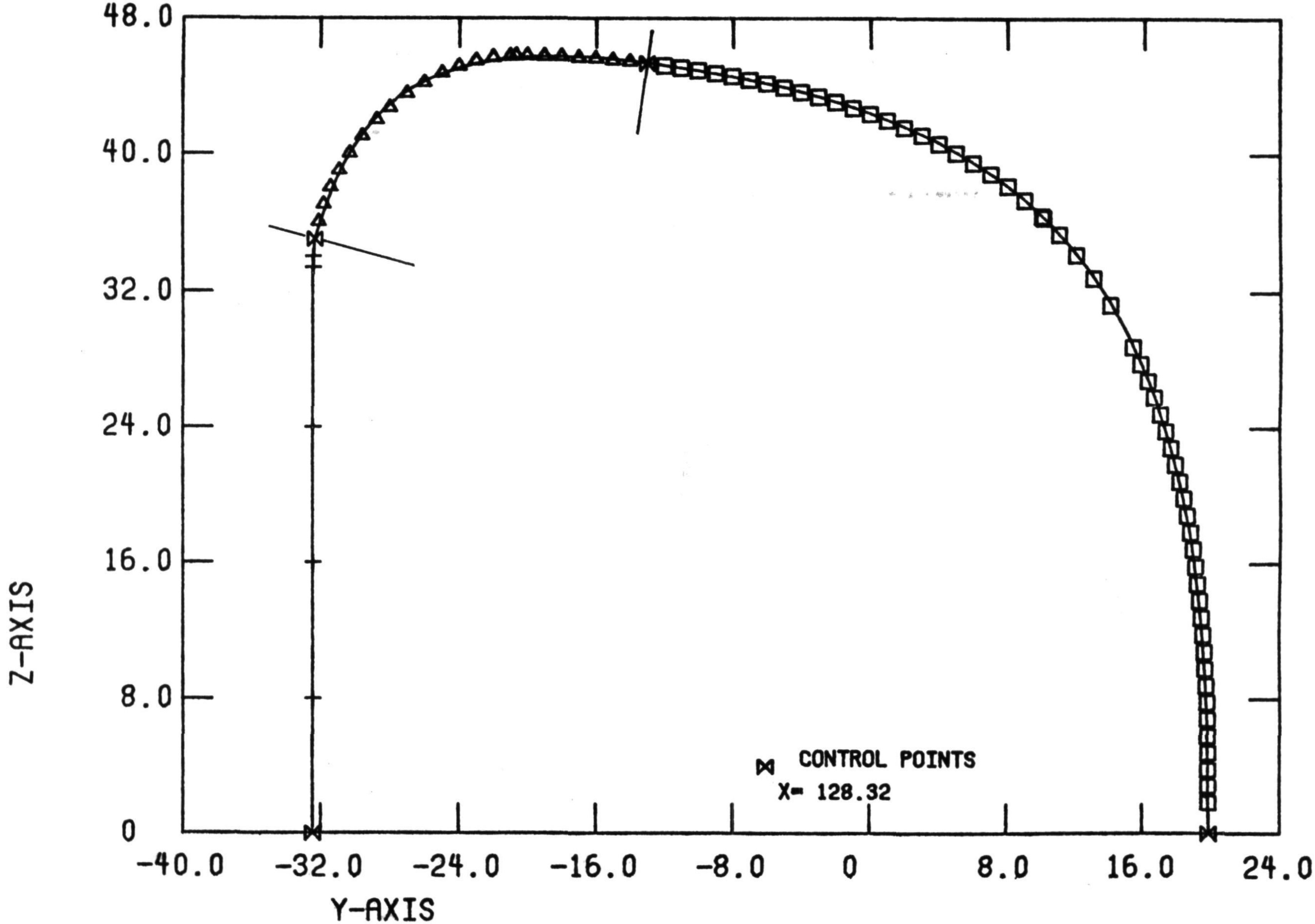

Figure 11. Revised curve-fit of cross section of HL-10 body using three segments.

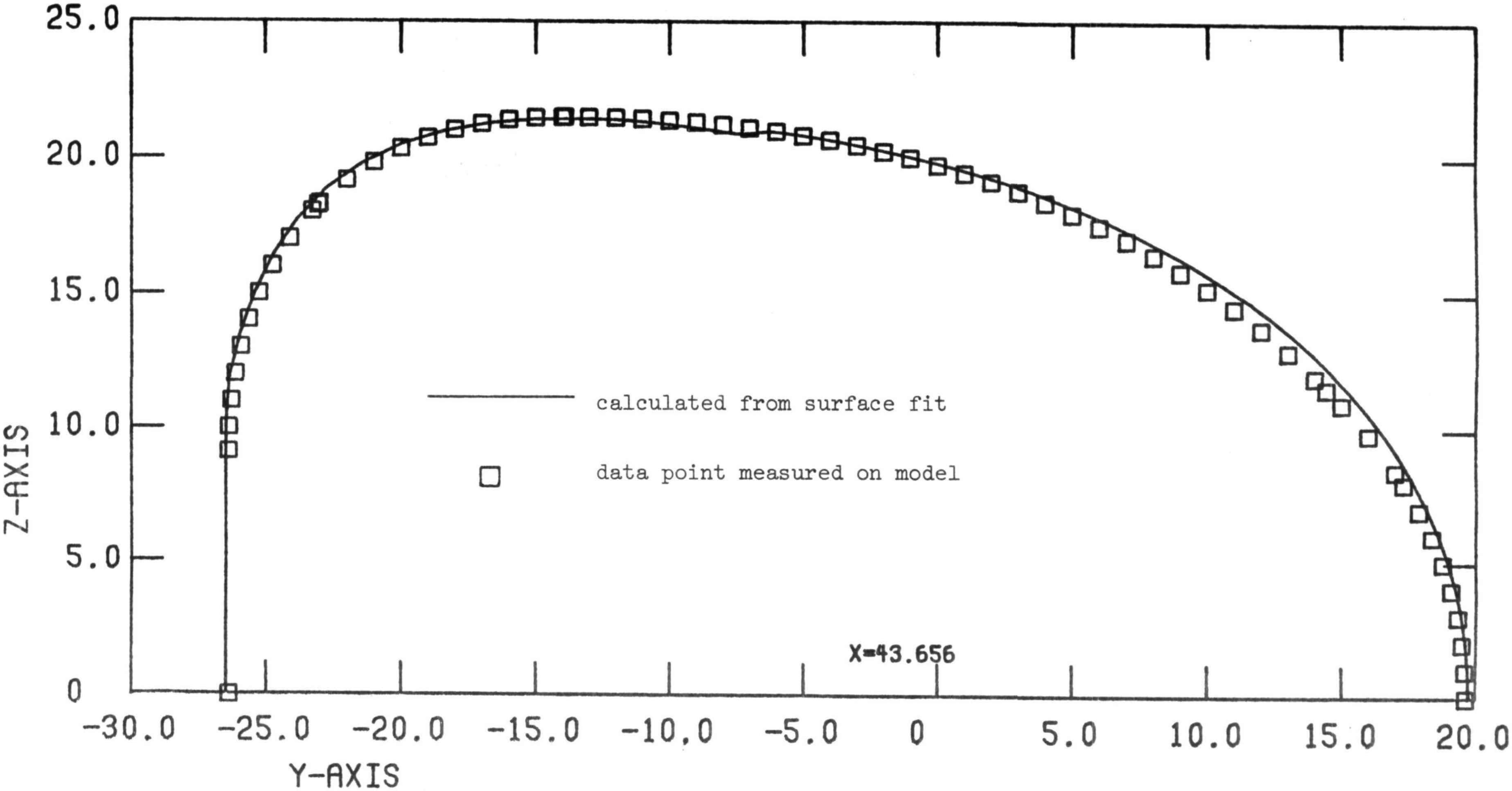

Figure 12. Comparison of cross section calculated in between input cross sections with measured data on HL-10 body.