Approximate Solutions of Radiative Transfer in Dusty Nebulae: II. Hydrogen and Helium

by

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APPROXIMATE SOLUTIONS OF RADIATIVE TRANSFER IN DUSTY NEBULAE
II. HYDROGEN AND HELIUM

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ABSTRACT

In this paper we discuss the ionization structure of hydrogen and helium in dusty nebulae. The general equations of Paper I (Petrosian and Dana 1975) for pure hydrogen nebulae have been modified to include helium. We first present an approximate analytic solution for Strömgren radii of hydrogen and helium in the absence of dust. We then show that these results, with simple modifications, are also applicable to dusty nebulae where the effective absorption cross section of dust grains varies slowly with frequency in the 1000 to 200Å range. No analytic solutions are possible if this cross section varies rapidly with frequency. In this case, however, we have derived simple coupled differential equations which can easily be solved numerically. We present approximate analytic expressions for evaluation of the variation of the fraction of ionizing radiation absorbed by dust and the ratio of the volume emission measures of He II to H II regions with the spectrum of the ionizing source, helium abundance and absorption properties of dust. The effects of dust on the He III zone are discussed in the Appendix. As in Paper I our results are restricted to spherically symmetric nebulae, but non-uniform gas and dust distributions and clumpiness can be taken into account by our general results.

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I. INTRODUCTION

In Paper I we discussed various approximate analytic solutions of radiative transfer equations in nebulae contained hydrogen and dust. In this paper we consider dusty nebulae containing hydrogen and helium. Our aim is to obtain analytic solution or simple differential equations which could be solved numerically easily.

In section II we present modification of the general equations in Paper I so that they would be applicable to the present case. As shown here, with very few simple approximations (approximation usually made even in detailed numerical calculations such as the on-the-spot approximation) the problem of transfer of ionizing photons can be reduced to two simple coupled differential equations. These are equations (16) and (17).

These equations form the basis for the rest of this paper. In section III we reconsider the pure hydrogen nebulae with and without dust. This allows us to simplify the treatment of the H plus He case which is discussed in section IV. Here we show that in absence of dust one can describe the ionization of H and He analytically. To our knowledge no such solution has been presented before. The generalization of these results in dusty nebulae is simple and is presented in section IVb. In the concluding section (V) we apply these results to two observationally interesting parameters: the fraction of ionizing photons absorbed by gas (or dust) and the ratio of He to H volume emission measures. We give simple formulae for calculation of these quantities for the general problem of dusty nebulae.

II. GENERAL EQUATIONS OF TRANSFER OF IONIZING PHOTONS

The general equations of radiative transfer and ionization equilibrium described in section II.1 of Paper I can be applied to nebulae with hydrogen, helium and dust with following modifications.

In equations (I.2) and (I.3) (all figures and equations of Paper I will be identified with the prefix I) we must include absorption and emission coefficients of helium,

$$\kappa_{\nu,He} = (1-y)Yn\sigma_{He}(\nu) \quad , \quad 4\pi j_{\nu,He} = ynn_e[\alpha'_{1,\nu} + \zeta_1\alpha_\nu^{(2)'}] \tag{1}$$

where n is the total hydrogen (H I and H II) density,

$$Y = \frac{n(He)}{n} \quad , \quad y = \frac{n(He\ II)}{n(He)} \quad , \quad n_e = (x + yY)n \quad , \tag{2}$$

ζ_1 is fraction of photons from recombination to excited states of helium capable of ionizing hydrogen (ζ_1 varies between 0.8 and 0.96, cf. e.g. Mathis 1971), and α'_1 and $\alpha^{(2)'}$ are the recombination coefficients to the ground and excited states of helium ($\alpha' = \alpha'_1 + \alpha^{(2)'}$). Similarly equation (I.6) must be supplemented by the ionization equilibrium of helium:

$$4\pi\int_{\nu_0}^{4\nu_0} J_\nu\kappa_{\nu,H}d\nu = xnn_e\alpha \quad , \quad 4\pi\int_{1.8\nu_0}^{4\nu_0} J_\nu\kappa_{\nu,He}d\nu = yYnn_e\alpha' \quad . \tag{3}$$

It should also be noted that in the presence of helium the upper limit of integrals over frequency will be $4\nu_0$ instead of ∞, because photons with $\nu > 4\nu_0$ will be absorbed in the inner zone where He will be doubly ionized. For H II regions this zone is in general negligible. For nebulae with central sources emitting considerable numbers of He II ionizing photons our results will apply to the regions beyond this zone with minor modifications (cf. Hummer and Seaton 1964). The effect of dust on the He III zone is discussed in the Appendix.

As in Paper I, we shall be dealing with intensities and fluxes integrated over frequency. We must however distinguish between photons capable of ionizing only hydrogen and those capable of ionizing both hydrogen and helium. We therefore define net fluxes crossing spherical shells

$$S_1 = \int_{\nu_0}^{1.8\nu_0} S(\nu)d\nu \quad , \qquad S_2 = \int_{1.8\nu_0}^{4\nu_0} S(\nu)d\nu \quad ,$$
$$S = S_1 + S_2 \quad , \qquad \gamma \equiv S_2/S \tag{4}$$

and similar expressions for intensities I and J.

If S_2 photons were absorbed by helium alone, the equations for H and He ionizing photons would be decoupled and the results of Paper I would apply to S_1 and S_2 photons separately. However, because H and He compete for S_2 photons, the solution of the problem is more complicated. Instead of equation (I.9) we must define separate average hydrogen cross sections for $\nu < 1.8\nu_0$ and $\nu > 1.8\nu_0$ photons:

$$J_1\sigma_{H,1} = \int_{\nu_0}^{1.8\nu_0} \sigma_0(\nu/\nu_0)^3 \, J(\nu)d\nu \quad , \qquad J_2\sigma_{H,2} = \int_{1.8\nu_0}^{4\nu_0} \sigma_0(\nu/\nu_0)^3 \, J(\nu)d\nu \tag{5}$$

so that the total average hydrogen cross section is

$$\sigma_H = \sigma_{H,1} \frac{1 + (1-\beta) J_2/J_1}{1 + J_2/J_1} \quad , \quad \beta \equiv 1 - \sigma_{H,2}/\sigma_{H,1} \quad . \tag{6}$$

Similarly for dust we define $\sigma_{d,1}$ and $\sigma_{d,2}$ so that

$$\sigma_d = \sigma_{d,1} \frac{1 + (1+\alpha) J_2/J_1}{1 + J_2/J_1} \quad , \quad 1 + \alpha = \sigma_{d,2}/\sigma_{d,1} \quad . \tag{7}$$

For helium we have one average cross section σ_{He} defined as

$$J_2 \sigma_{He} = \int_{1.8\nu_0}^{4\nu_0} \sigma_{He}(\nu) J(\nu) d\nu \quad , \quad \sigma_{He}(\nu) = 1.17\, \sigma_0 \exp\{0.73(1.8 - \nu/\nu_0)\} \quad ,$$

$$\sigma_0 = 6.3 \times 10^{-18} \text{ cm}^2 \quad . \tag{8}$$

As in Paper I, we can define separate cross sections σ_x^* and σ_x^D for the stellar and diffuse radiation. For σ_x^* the average intensities J_2 and J_1 can be replaced by S_2^* and S_1^*, respectively.

We shall use the on-the-spot approximation and set the dust albedo $\omega(\nu) = 0$. As we have shown in Paper I, the effects of absorption by dust of diffuse radiation and scattering by dust can be accounted for by defining effective dust absorption optical depths. The relationship between the effective and the true absorption optical depths can be found in figures I.4 and I.7.

With these approximations the diffuse radiation satisfies the relation

$$\kappa_{He}^D (1+\rho) 4\pi J_2^D = Yynn_e \alpha_1' \quad , \quad \kappa_{H,1}^D \, 4\pi J_1^D = xnn_e \alpha_1 + \zeta_1 Yynn_e \alpha^{(2)'} \quad , \tag{9}$$

where

$$\rho = \kappa_{H,2}^D / \kappa_{He}^D \simeq (1 - x) \sigma_H(1.8\nu_0) / [Y(1 - y) \sigma_{He}(1.8\nu_0)] \tag{10}$$

is the fraction of recombination photons to ground state of helium absorbed by hydrogen.[1] Substitution of equation (9) in equation (3) for ionization equilibrium gives

$$(1-x) = A_1(x+yY)[x-y\tilde{Y}(\zeta_1+\zeta_2)] \quad , \quad (1-y) = A_2 y(x+yY)(1+\zeta_2) \qquad (11)$$

where we have defined

$$\tilde{Y} = Y\alpha^{(2)\prime}/\alpha^{(2)} \quad , \quad \zeta_2 = \alpha_1' \rho/[\alpha^{(2)\prime}(1+\rho)] \quad , \qquad (12)$$

and

$$A_1 = 4\pi r^2 n\,\alpha^{(2)}/(S^* \sigma_H^*) \quad , \quad A_2 = A_1 \sigma_H^* \alpha^{(2)\prime}/(Y\sigma_{He}^* \alpha^{(2)}) \quad . \qquad (13)$$

Substitution of these in the equations governing transfer of stellar photons $(dS_i^*/dr = -\kappa_{tot,i} S_i^*)$ gives

$$dS_1^*/dr = -\kappa_{d,1}^* S_1^* - 4\pi r^2 n n_e \alpha^{(2)}[x-\tilde{Y}y(\zeta_1+\zeta_2)]\sigma_{H,1}^* S_1^*/\sigma_H^* S^* \quad ,$$

$$dS_2^*/dr = -\kappa_{d,2}^* S_2^* - 4\pi r^2 n n_e \alpha^{(2)}\{(1+\zeta_2)y\tilde{Y} + [x-\tilde{Y}y(\zeta_1+\zeta_2)]\sigma_{H,2}^* S_2^*/\sigma_H^* S^*\} \quad . \qquad (14)$$

Addition of these two equations gives

$$dS^*/dr = -\kappa_d^* S^* - 4\pi r^2 n n_e \alpha^{(2)}[x+y\tilde{Y}(1-\zeta_1)] \quad , \qquad (15)$$

[1] In most treatments of the problem this fraction (and consequently the quantity ζ_2 in eq. [12]) is set equal to zero. If this was the case the absorption of S_2 photons by hydrogen could also be neglected. As we shall see below, neglecting ρ or ζ_2 with respect to unity will cause up to 30 percent under-estimation of γ_0 or a similar over-estimation of the relative He abundance Y.

which for $\zeta_1 = 1$ is identical to equation (I.25) except that here n_e has a contribution from helium (note also that $\kappa_d^*/\kappa_{d,1}^* = 1 + \alpha^* Y$ depends on Y).

From here on we shall be dealing only with stellar photons, therefore we shall eliminate the * notation in what follows. With this modification and with elimination of S_1 and S_2 in favor of $Y = S_2/S$ we find the general expressions for stellar photons are

$$dS/d\tau = -(1+\alpha Y)S - f(r) \quad ,$$
$$dY/d\tau = -\alpha Y(1-Y) + (1-Y)(\beta Y - Y')(1-\beta Y)^{-1} f(r)/S \quad , \tag{16}$$

where

$$f(r) = [x + y\tilde{Y}(1-\zeta_1)]4\pi r^2 n n_e \alpha^{(2)}/\kappa_{d,1} \quad , \quad d\tau = \kappa_{d,1} dr \quad ,$$
$$Y' = \frac{y\tilde{Y}(1+\zeta_2)}{x + y\tilde{Y}(1-\zeta_1)} \quad . \tag{17}$$

Another approximation from Paper I which we employ here also is to ignore the $(1-x)$ and $(1-y)$ terms in equations (14) to (17) with respect to unity. This is true everywhere (in particular for large dust optical depths) except very near the ionization fronts (cf. fig. I.2). This amounts to setting $x = y = 1$ in equations (14) to (17).

III. PURE HYDROGEN NEBULA

(a) Nebulae Without Dust

In the absence of dust and helium ($\kappa_d = y = Y = 0$) equations (11) and (15), with the approximations stated in section I, give

$$S/S_0 = 1 - \xi \quad , \qquad (1 - x) = \left(1 + \frac{1}{2A_1}\right) - \left[\left(1 + \frac{1}{2A_1}\right)^2 - 1\right]^{1/2} \qquad (18)$$

where A_1 is defined in equation (13),

$$\xi' = \frac{d\xi}{dr} = 4\pi r^2 n^2 \alpha^{(2)}/S_0 \quad , \qquad \xi(r) = \int_{r_0}^{r} \xi' \, dr \qquad (19)$$

and

$$S_0 = \int_{r_0}^{r_1} 4\pi r^2 n^2 \alpha^{(2)} dr \qquad (20)$$

is the total number of ionizing photons emitted by the central source(s). Here r_0 and r_1 are the inner and the outer radii of the ionized region respectively; $\xi(r_1) = 1$ and in general $r_0 \ll r_1$. These are identical to equations (I.21) and (I.22).

(b) Nebulae With Dust

In this case $\kappa_d \neq 0$. It can be shown that results identical to equations (18) to (20) are obtained if we replace S and ξ' by

$$\tilde{S} = S e^{\tau} \qquad \text{and} \qquad \tilde{\xi}' = \xi' e^{\tau} \qquad (21)$$

and if we include e^{τ} in the integrand of equation (20); $d\tau = n_d \sigma_d dr$. The results thus obtained are identical to that of equations (I.26) to (I.32).

IV. DUSTY NEBULAE CONTAINING HYDROGEN AND HELIUM

Since to our knowledge there are no approximate analytic solutions for nebulae with H and He (but without dust) we consider this case first.

(a) Nebulae Without Dust

With $\kappa_d = 0$, equations (16) and (17) reduce to

$$S/S_0 = (1 - \xi) \quad , \quad d\gamma/dS = -\frac{1}{S}\frac{(1-\gamma)(\beta\gamma - Y')}{(1-\beta\gamma)} \quad , \qquad (22)$$

where S_0 and ξ are now

$$S_0 = \int_{r_0}^{r_1} [1+Y][1+\tilde{Y}(1-\zeta_1)]\; 4\pi r^2 n^2 \alpha^{(2)} dr \quad ,$$

$$\xi = \int_{r_0}^{r} [1+Y][1+\tilde{Y}(1-\zeta_1)]\; 4\pi r^2 n^2 \alpha^{(2)} dr/S_0 \quad . \qquad (23)$$

Figure 1 shows the position of r_2', r_2 and r_1, the Strömgren radii of He^{++}, He^{+} and H^{+}, respectively. Within our approximation $r_2 \le r_1$. In cases where $r_2 < r_1$ the parameters Y and $\tilde{Y}$ must be set equal to zero for $r_1 > r > r_2$.

The parameters β and Y' in equation (22) vary throughout the nebula. Variation of β is due to change in the spectrum of ionizing radiation. However, since the absorption cross section of hydrogen decreases rapidly with frequency $(\sigma \propto \nu^{-3})$, $\beta \approx 1 - (1/1.8)^3 \sim 0.8$ and changes by a few percent for a variety of plausible spectra (cf. table 1). We therefore neglect variation of β. The parameter Y', on the other hand, varies because of the variation of ζ_2 which is primarily due to the change of the ratio $(1-x)/(1-y)$ in equation (10). In general, if $Y_0 < Y$, $\zeta_2 \ll 1$ and it can be ignored. However, for $Y_0 > Y$, $\zeta_2 > 0.1$ in the inner regions, and near the edge of the nebula where $Y \to 1$, $\zeta_2 \to \alpha_1'/\alpha^{(2)\prime} \sim 0.6$ (for electron temperature of 10^4 °K). As we shall see below, even in this case the variation of ζ_2 is negligible. Thus we shall assume that Y' is a constant.

Ionizing Flux

With the above assumptions, equation (22) is readily solved:

$$\left(\frac{S}{S_0}\right)^{\beta-Y'} = \left(\frac{1-Y}{1-Y_0}\right)^{1-\beta}\left(\frac{\beta Y_0-Y'}{\beta Y-Y'}\right)^{1-Y'} \quad . \qquad (24)$$

Figure 2 shows the variation of Y with S for various values of Y_0 (solid lines). For $\beta Y_0 = Y'$, $Y = Y_0 =$ constant. For $\beta Y_0 < Y'$ Y (and therefore S_2) becomes zero at

$$S = S_{cr} \equiv S_0(1-\beta Y_0/Y')^{(1-Y')/(\beta-Y')}\,(1-Y_0)^{(\beta-1)/(\beta-Y')} \qquad (25)$$

or at $r = r_2$, where r_2 is obtained from equation (23) with $\xi = \xi_2 \equiv 1 - S_{cr}/S_0$. Thus, in these cases helium ionization stops before the edge of the nebula. For $\beta Y_0 > Y'$, Y increases toward the outer edge and approaches unity at the edge of the ionized region (at $r = r_1$), where $S = 0$. As is evident, the shape of these curves is determined primarily by the value of Y_0 (actually by the value of $\beta Y_0/Y'$).

Ionization Structure

Once the variations of S and Y are known the ionization structure can be calculated according to equations (11) to (13). Since $1.0 <$ $(\zeta_1 + \zeta_2) \le 1.4$, equation (11) can be approximated as $(1 - x) \simeq A_1 x^2$ so that $(1 - x)$ is given by equation (18) with $\sigma_H = \sigma_{H,1}(1 - \beta Y)$ (cf. eq. [6]). A few values of $\sigma_{H,1}/\sigma_0$ and σ_{He}/σ_0 are given in table 1. The slow variations of these quantities due to changes of the spectrum throughout the nebula are neglected in the above treatment.

Equation (11) can now be solved for $(1 - y)$ using the above values of x and assuming ζ_2 = constant. This assumption is clearly justified since A_2 varies much more rapidly than any expected variation of ζ_2. The results of this calculation are shown on figure 3 for various values of Y_0, Y = 0.1 and uniform nebula. Here we also show variation of ζ_2. As evident, the assumption of constancy of ζ_2 is a good approximation for small values of Y_0. Fo[illegible] values of Y_0, $\zeta_2 \ll 1$ so that $Y' \approx \tilde{Y} \approx 1.05Y$ (for electro[illegible] ures of about 10^4 °K, cf. Burgess and Seaton 1960). For large valu[illegible], ζ_2 varies slowly but it is no longer negligible compared to [illegible]. Therefore, neglecting ζ_2, as is commonly done, will imply incorrect determination of the value of Y_0/Y' from observations. For example, for a given value of Y and observed value of r_2/r_1, the required value of Y_0 is underestimated when ζ_2 is neglected. In general for small r_2/r_1, ζ_2 is negligible and it rarely exceeds 0.4 (cf. figs. 5 and 7).

(b) Nebulae With Dust

In this case simple analytic solutions to the coupled differential equations (16) and (17) are possible only if α is zero or negligible, i.e. only if $\sigma_d(\nu)$ varies slowly so that $\sigma_d \approx \sigma_{d,2} \approx \sigma_{d,1}$. We discuss this case first.

(i) $\underline{\alpha \approx 0}$

In this case, as in section II.b, we can define $\tilde{S}$ and $\tilde{\xi}'$ as in equation (21) and obtain results identical to those presented by equations (22) to (25) with replacement of S and ξ' by $\tilde{S}$ and $\tilde{\xi}'$ and with inclusion of e^{τ} in the integrand for S_0. Thus, the solid lines in figure 2 give the variation of γ with $\tilde{S}$. However, variation of γ with S will be different (dashed lines). Using these results we have calculated variations of $(1 - x)$, $(1 - y)$ and ζ_2 for uniform nebulae ($n_d \propto n$ = constant) and for a dust to gas ratio and gas column density such that $\tau_1 = n_d\sigma_d r_1 \approx 1.0$ and $n_d\sigma_d/n\sigma_0 \equiv \epsilon = 2.5 \times 10^{-4}$. These results are presented in figure 4. Comparison of figures 3 and 4 show that for a given size r_1 of the nebula, introduction of dust increases the fractional ionization of both hydrogen and helium. This is primarily due to increased values of S throughout the nebula.

(ii) $\underline{\alpha \neq 0}$

In this case simple analytic solutions are not possible. However, equations (16) can readily be solved numerically. We have solved these equations for few values of α, γ_0 and Y. As above, these results can be used to obtain the ionization structure of the nebulae. Some of these results are presented in figures 2 and 5 where we show the variation of ζ_2 with α for various values of γ_0 and Y.

In general we find the shapes of $(1 - x)$ and $(1 - y)$ curves are nearly independent of the details of the problem and are determined primarily by the values of r_2 and r_1. Consequently, we do not present the variation of $(1-x)$ and $(1-y)$ for values of $\alpha \neq 0$. Instead we concentrate on the dependence of observable quantities on α and other parameters in the next section.

V. SUMMARY AND RESULTS

We have derived approximate analytic solutions to the equations of radiative transfer and ionization structure of nebulae containing hydrogen and helium. These solutions can easily be extended to nebulae containing dust if the effective dust absorption cross section is a slow varying function of frequency in the 1000 to 200Å wavelength range. For dust with a widely varying cross section in this wavelength range we have derived simple differential equations which can readily be solved numerically. The main approximation of our treatment is neglect of albedo of the dust grains. The other approximations are of minor consequence.

Since we treat fluxes integrated over wide ranges of frequency, we lose most of the information on the variation of the spectrum of ionizing radiation throughout the nebulae. Consequently, we cannot calculate the ionization structure of heavier elements. The main result of these solutions is the ionization structure of He and H. We summarize these results below.

(a) Fraction of Ionizing Radiation Absorbed by Gas

A useful parameter for comparison with observations is the fraction of stellar ionizing photons absorbed by gas (or dust). The fraction f_{net} absorbed by gas is

$$f_{net} = 4\pi\alpha^{(2)} \int_{r_0}^{r_1} (1+Y)[1+\tilde{Y}(1-\zeta_1)]r^2n^2dr/S_0 \quad . \qquad (26)$$

For $\alpha = 0$ and $\zeta_1 \approx 1$ equation (26) reduces to

$$f_{net} = \frac{f(\tau_1) + Yf(\tau_2)g(\tau_2)/g(\tau_1)}{1 + Yg(\tau_2)/g(\tau_1)} \quad , \tag{27}$$

where (cf. eqs. [I.27] and [I.32])

$$g(\tau) = \frac{\kappa_{d,1}^3}{n_0^2} \int_0^\tau n^2 r'^2 \exp(\tau') d\tau' / \kappa_{d,1} \quad ,$$

$$f(\tau) = \frac{\kappa_{d,1}^3}{g(\tau) n_0^2} \int_0^\tau n^2 r'^2 d\tau' / \kappa_{d,1} \quad . \tag{28}$$

In these equations n_0 is the value of n at $r = r_0$, and τ_1 and τ_2 are the dust optical depths for S_1 photons up to the H and He Strömgren radii r_1 and r_2, respectively:

$$\tau_1 = \int_{r_0}^{r_1} n_d \sigma_{d,1} dr \quad , \quad \tau_2 = \int_{r_0}^{r_2} n_d \sigma_{d,1} dr \quad . \tag{29}$$

The fraction f_{net} calculated from equation (27) is nearly equal to $f(\tau_1)$, the fraction obtained in Paper I for $Y = 0$. This equality is exact for $\tau_2/\tau_1 = 0$ and 1.0. In the intermediate range the difference between the two cases is less than three percent (for $\tau_1 \leq 5$) because $Yg(\tau_2)/g(\tau_1) < Y \approx 0.1$.

For $\alpha \neq 0$ equation (27) is not valid because S_0 is no longer a simple function of τ_1 and τ_2. In this case f_{net} is given by (cf. also Mathis 1971)

$$f_{net} = (1 - \gamma_0) f_{S_1} + \gamma_0 f_{S_2} \tag{30}$$

where f_{S_1} and f_{S_2} are fractions of S_1 and S_2 photons absorbed by the gas. From equation (14) these are (for $\zeta_1 \approx 1$)

$$f_{s_1} = \frac{\int_{r_0}^{r_2} (1+\gamma)(1-\gamma')\left(\frac{1-\gamma}{1-\beta\gamma}\right) n^2r^2dr + \int_{r_2}^{r_1} n^2r^2dr}{\int_{r_0}^{r_2} (1+\gamma)(1-\gamma')\left(\frac{1-\gamma}{1-\beta\gamma}\right) n^2r^2e^{\tau}dr + \int_{r_2}^{r_1} n^2r^2e^{\tau}dr} ,$$

and (31)

$$f_{s_2} = \frac{\int_{r_0}^{r_2} \left[\frac{1-\gamma}{1-\beta\gamma} + (1-\beta)\frac{\gamma/\gamma'}{1-\beta\gamma}\right] n^2r^2dr}{\int_{r_0}^{r_2} \left[\frac{1-\gamma}{1-\beta\gamma} + (1-\beta)\frac{\gamma/\gamma'}{1-\beta\gamma}\right] n^2r^2e^{(1+\alpha)\tau}dr} .$$

Examination of the results presented in figure 2 will show that the quantity $(1-\gamma)/(1-\beta\gamma)$ is nearly a constant (approximately equal to one) except for large values of γ_0 where it goes to zero rapidly at the very edge of the nebula. The quantity in the square brackets of the expression for f_{s_2} is constant for $\gamma_0 < 0.1$ and for large values of γ_0 it varies slowly from 2 (at $r \simeq r_0$) to 6 (at $r = r_1$). Because these variations are much slower than the variation of the remaining quantities in the integrands, the variations of γ can be neglected so that $f_{s_1} \approx f(\tau_1)$ and $f_{s_2} = f[(1+\alpha)\tau_2]$ [cf. eq. (28); note that we have neglected $\gamma - \gamma' \approx \gamma\zeta_2$ in comparison with unity for cases when $r_2 < r_1$ which is justified since in these cases $\gamma\zeta_2 < 0.03$]. Equation (30) then becomes

$$f_{net} = (1-\gamma_0)f(\tau_1) + \gamma_0 f[(1+\alpha)\tau_2] \quad . \tag{32}$$

The results obtained from this equation are within 2% of the results from the exact equation (26).

(b) He to H Line Intensity Ratio

Another important parameter for comparison with observation is the expected ratio of He to H line intensities. These ratios are easily obtained from the ratios of He to H volume emission measures

$$R = \int_{r_2'}^{r_2} r^2 n n_e dr \Big/ \int_{r_0}^{r_1} r^2 n n_e dr \quad , \qquad R' = \int_{r_0}^{r_2'} r^2 n n_e dr \Big/ \int_{r_0}^{r_1} r^2 n n_e dr \ . \tag{33}$$

If the He III zone is negligible $r_2' \rightarrow r_0$ and $R' = 0$. We shall assume this to be the case (cf., however appendix). On figure 6 we plot R versus γ_0/Y for nebulae without dust (dotted line) and for dusty nebulae with various values of the dust cross section parameter α. As explained previously (Petrosian 1973, 1974) for given γ_0/Y the presence of dust with $\sigma_d(\nu) \simeq$ constant (i.e. $\alpha = 0$) increases the value of R with increasing values of the total optical depth τ_1 (eq. [29]). For negative values of α ($\sigma_{d,1} > \sigma_{d,2}$) fewer He ionizing photons compared with H ionizing photons are absorbed by the dust and R is larger. The reverse occurs for positive value of α. It should be noted that R depends on γ_0/Y and is insensitive to the value of Y. R is also fairly independent of non-uniformities or inhomogeneities in the gas and dust distributions as long as the dust to gas ratio is constant. However R changes for dust and gas distributions which are not the same.[2]

[2] Note that because of the approximations $x = 1$ for $r \le r_1$ and $y = 1$ for $r \le r_2$ the ratio $R \le 1$. However, for large values of γ_0 (see figs. 3 and 4) where $r_2 = r_1$ and $(1 - y) < (1 - x)$ the line intensity ratio R could be slightly larger than one.

There are no exact analytic expressions for R for the general case. However, the following recipe seems to give fairly accurate results (cf. also Mathis 1971).

In the absence of dust, equations (22), (23), (25) and (33) give for $\beta\gamma_0/Y' < 1$,

$$R = \xi_2 = h(\gamma_0) \equiv 1 - (1 - \beta\gamma_0/Y')^{(1-Y')/(\beta - Y')}(1-\gamma_0)^{(\beta-1)/(\beta - Y')} \quad (34)$$

and $R = 1$ for $\beta\gamma_0/Y' \geq 1$.

In general in the presence of dust with arbitrary values of α it can be shown that the ratio R is given by

$$R_\alpha = \xi_2/f_{net} \quad . \qquad (35)$$

For $\alpha = 0$ there exists an exact analytic solution since in this case $\xi_2 = f(\tau_2)\,\tilde{\xi}_2$, $f_{net} \approx f(\tau_1)$ and $\tilde{\xi}_2 = h(\gamma_0)$ (cf. the discussions in part (a) above and Section III-b-i), so that

$$R_{\alpha=0} = f(\tau_2)h(\gamma_0)/f(\tau_1) \quad . \qquad (36)$$

For $\alpha \neq 0$ there is no simple analytic expression for $\tilde{\xi}_2$ in terms of γ_0. However, from equation (31) we can write $\xi_2 = \gamma_0 f_{s_2}\langle X^{-1}\rangle/Y'$ where $\langle X^{-1}\rangle$ is the average value of the inverse of the quantity in the square brackets. As mentioned before, $f_{s_2} \approx f[(1+\alpha)\tau_2]$ and for small values of γ, $X \approx 1$, so that for $\beta\gamma_0/Y' \ll 1$ (i.e. for $R \ll 1$) we have

$$R_\alpha = \gamma_0 f[(1+\alpha)\tau_2]/Y' f_{net} \quad . \qquad (37)$$

Furthermore, for small $\beta\gamma_0/Y'$ equation (34) gives $R = \gamma_0/Y'$ so that we can write $R_\alpha = \gamma_0/Y'$ if we define

$$\gamma_0' = \gamma_0 \, f[(1+\alpha)\tau_2]/f_{net} \quad . \tag{38}$$

This relationship, when generalized to all values of γ_0, suggests that for dusty nebulae with arbitrary α

$$R_\alpha = h(\gamma_0') \quad . \tag{39}$$

Comparison of R_α from equations (38) and (39) with our calculations presented on figure 6, shows that for low values of optical depth ($\tau_1 \leq 2$) and for $(\gamma_0/Y) \leqslant 3$, these two results agree to within a few percent. Ten to fifty percent difference is obtained for large values of α and for $\gamma_0/Y' \geqslant 3$.

Finally examination of figures 5 and 6 shows that the quantity ζ_2 depends primarily on R. This is shown on figure 7. This simplifies comparison with observations since ζ_2, which enters as a parameter in our discussion, can be determined purely observationally. Application of these results to observations will be discussed in Paper III of this series.

ACKNOWLEDGMENT

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APPENDIX

If the central source emits He II ionizing photons $(\nu > 4\nu_0)$

$$S_3 = \int_{4\nu_0}^{\infty} S(\nu)d\nu \quad , \quad Y' = S_3/S \tag{A.1}$$

then there will be an inner region of doubly ionized helium ($r_2' > r > r_0$ in figure 1). Assuming on-the-spot approximation for the He II recombination photons we find

$$dS_3/dr = -\kappa_{d,3}S_3 - 4\pi r^2 Yy' nn_e\alpha^{(2)''} \tag{A.2}$$

where $\kappa_{d,3}$ is the average dust absorption coefficient for S_3 photons, $y' = n(\text{He III})/n(\text{He})$ is the fraction of doubly ionized helium and $\alpha^{(2)''}$ is the recombination coefficient to the excited states of He II. In writing equation (A.2) we have neglected absorption coefficient of H and He I with respect to that of He II at these frequencies. This is justified since for $Y \simeq 0.1$, $\kappa_{H,3}/\kappa_{He\ II} \approx 0.16\ (1-x)/(1-y') \ll 1$ and $\kappa_{He\ I,3}/\kappa_{He\ II} \approx 0.2\ (1-y)/(1-y') \ll 1$.

Clearly the effect of dust is to absorb some of the S_3 photons and reduce the size of the He III zone as in the case of pure hydrogen nebulae. The details of this are obtained from equations similar to equations (18) to (21) with replacement of S by S_3, r_1 by r_2', x by y' and τ by an optical depth appropriate for S_3 photons.

There will also be absorption of S_1 and S_2 photons in this region. Absorption of S_2 photons by He I will be negligible. However because of recombination of hydrogen (whose total number in this region is equal to the integral from r_0 to r_2' of $4\pi r^2 xnn_e\alpha^{(2)}$) there will be some absorption of

S_1 and S_2 photons by hydrogen. On the other hand, there will be some emission of such photons from recombination of He III. The number of such photons have been calculated by Hummer and Seaton (1964). For relative helium abundance $Y \approx 0.1$ this number is approximately equal to the number of hydrogen recombinations. Consequently, absorption by hydrogen can also be neglected. The remaining absorption by dust of S_1 and S_2 photons can then easily be taken into account once the effective dust absorption coefficients $\kappa_{d,1}$ and $\kappa_{d,2}$ are known. Thus, if $S_{1,0}$ and $S_{2,0}$ are total number of S_1 and S_2 photons emitted by the central source, then at r_2' their numbers will be reduced by factors of $\exp\left\{\int_{r_0}^{r_2'} \kappa_{d,1} dr\right\}$ and $\exp\left\{\int_{r_0}^{r_2'} \kappa_{d,2} dr\right\}$, respectively. Substitution of these reduced fluxes for $S_{i,0}$ and replacement of r_0 by r_2' in the equations of Sections I and III will give the correct result.

For $Y' \neq 0$ the ratio R' (eq. [33]) which is related to the He II to H line intensity ratio will no longer be zero. If we define

$$\alpha' = \frac{\sigma_{d,3}}{\sigma_{d,1}} - 1 \quad \text{and} \quad \tau_2' = \int_{r_0}^{r_2'} \kappa_{d,1} dr' \tag{A.3}$$

then it can be shown [eqs. (26), (28) and (29)] that

$$R' = \frac{Y_0'}{Y} \frac{f[(1+\alpha')\tau_2']}{f(\tau_1)} \tag{A.4}$$

where we have used the approximation $S_0 \simeq (1+Y)4\pi\alpha^{(2)} g(\tau_1)/\kappa_{d,1}^3$ and assumed $r_2' \ll r_1$ so that $n \sim$ constant for $r < r_2'$. This is similar to equation (37) for R.

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TABLE 1

AVERAGE CROSS SECTIONS AND STELLAR PARAMETERS

$T(^{o}K)^{\dagger}$	γ_0	β	$\sigma_{H,1}/\sigma_0$	σ_{He}/σ_0
90300.	0.500	0.837	0.452	0.688
62200.	0.300	0.808	0.487	0.832
50000.	0.194	0.793	0.514	0.900
40000.	0.108	0.787	0.555	0.959
40000.++	0.271	0.825	0.519	0.797
37500.	0.0887	0.786	0.564	0.979
30900.	0.0445	0.785	0.602	1.023
30000.++	0.0039	0.664	0.628	1.138
$n = 2^{*}$	0.553	0.916	0.507	0.410
$n = 4^{*}$	0.169	0.855	0.585	0.754

*Averaged over power law distribution $S_{\nu} = S_0(\nu_0/\nu)^n$.

†Averaged over black body distribution $B_{\nu}(T)$.

††Averaged over model atmosphere (Auer and Mihalas 1972)

FIGURE CAPTIONS

Fig. 1. The Strömgren radii of He^{++}, He^{+} and H^{+}. r_0 is the inner boundary of the nebula.

Fig. 2. Y, the number ratio of helium ionizing photons to total ionizing photons, as a function of the normalized stellar flux S/S_0. Y_0 = 0.50, 0.194 and 0.0445 correspond to black body stellar temperatures of 90,300., 50,000. and 30,900. °K respectively. The solid lines are for nebulae without dust ($\tau_1 = 0.0$). Four different dust cross-sections were used: $\alpha = -1$ (dash-dot line), $\alpha = 0$ (short dash), $\alpha = 1$ (dot) and $\alpha = 5$ (long dash) where $1+\alpha = \sigma_{d,2}/\sigma_{d,1}$. Two cases where $\tau_1 \approx 2.0$ are plotted and labeled. All other cases are for $\tau_1 = 1.0$. All curves are for a uniform dust and gas distribution.

Fig. 3. The neutral fraction of hydrogen (solid lines) and helium (dashed lines) and ζ_2 (dash-dot) for uniform dustless nebulae as a function of the normalized radius r/r_1. r_1 is the radius of the Strömgren sphere of hydrogen. The open circles denote the volume average of ζ_2. $Y = 0.1$.

Fig. 4. The neutral fraction of hydrogen and helium and ζ_2 as a function of the optical depth of dust τ for uniform dusty nebulae with a constant dust cross section ($\alpha = 0$). $\tau_1 = n_d \sigma_d r_1 \approx 1.0$ and $Y = 0.1$ for all cases. The symbols are the same as in Fig. 3.

Fig. 5. The variation of average (over volume) ζ_2 with α for $Y = 0.1$. For $Y_0 = 0.30$, we show the variation of ζ_2 when $Y = 0.15$ (filled circles) and for two cases where the gas and dust densities are not constant. The open circles and squares are

for non-uniform distributions of dust and gas. The open squares are for a uniform dust to gas ratio with $n(\tau) \propto n_d(\tau) \propto e^{-\tau}$ (or as functions of the radius r, $n(r) \propto n_d(r) \propto (1+\mathcal{R})^{-1}$, $\mathcal{R} = n_d(r=r_0)\sigma_d r$). The open circles are for dust to gas ratio increasing toward outer regions with $n(\tau) \propto e^{-2\tau}$ and $n_d(\tau) \propto e^{-\tau}$ ($n(r) \propto (1+\mathcal{R})^{-2}$ and $n_d(r) \propto (1+\mathcal{R})^{-1}$).

Fig. 6. Variation of R, the ratio of He to H volume emission measures, with Y_0/Y for various values of α (as labeled) and τ_1: $\tau_1 = 0$, no dust (dotted line), $\tau_1 = 1.0$ (solid lines) and $\tau_1 = 2.0$ (dashed lines). The other symbols are as in fig. 5. The open circles are connected to their corresponding lines for uniform nebulae by the vertical dashed lines.

Fig. 7. The variation of R with the average value of ζ_2 for various values of Y_0 and τ_1.

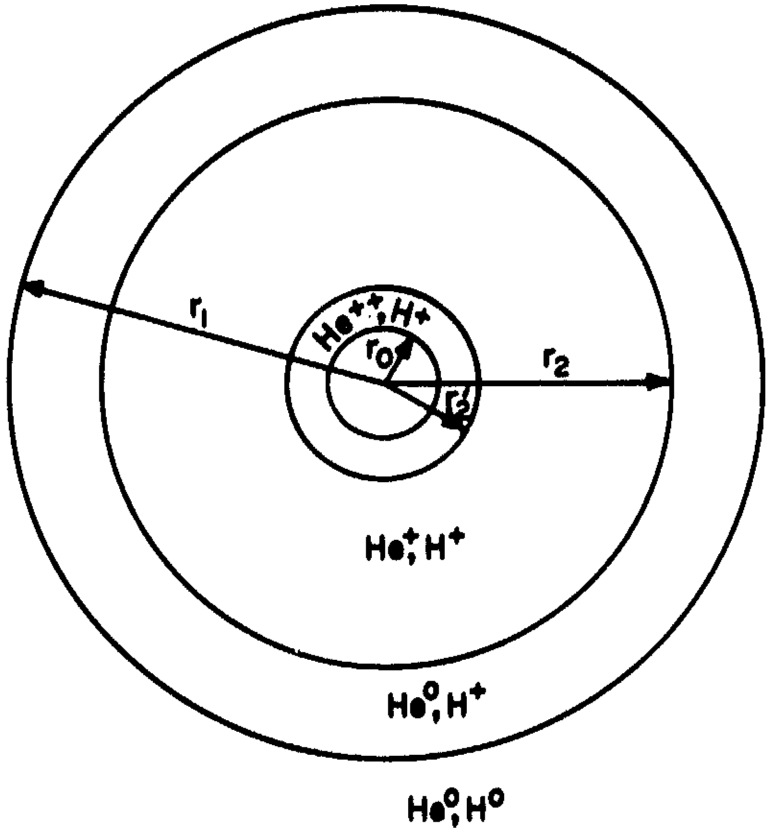


Figure 1

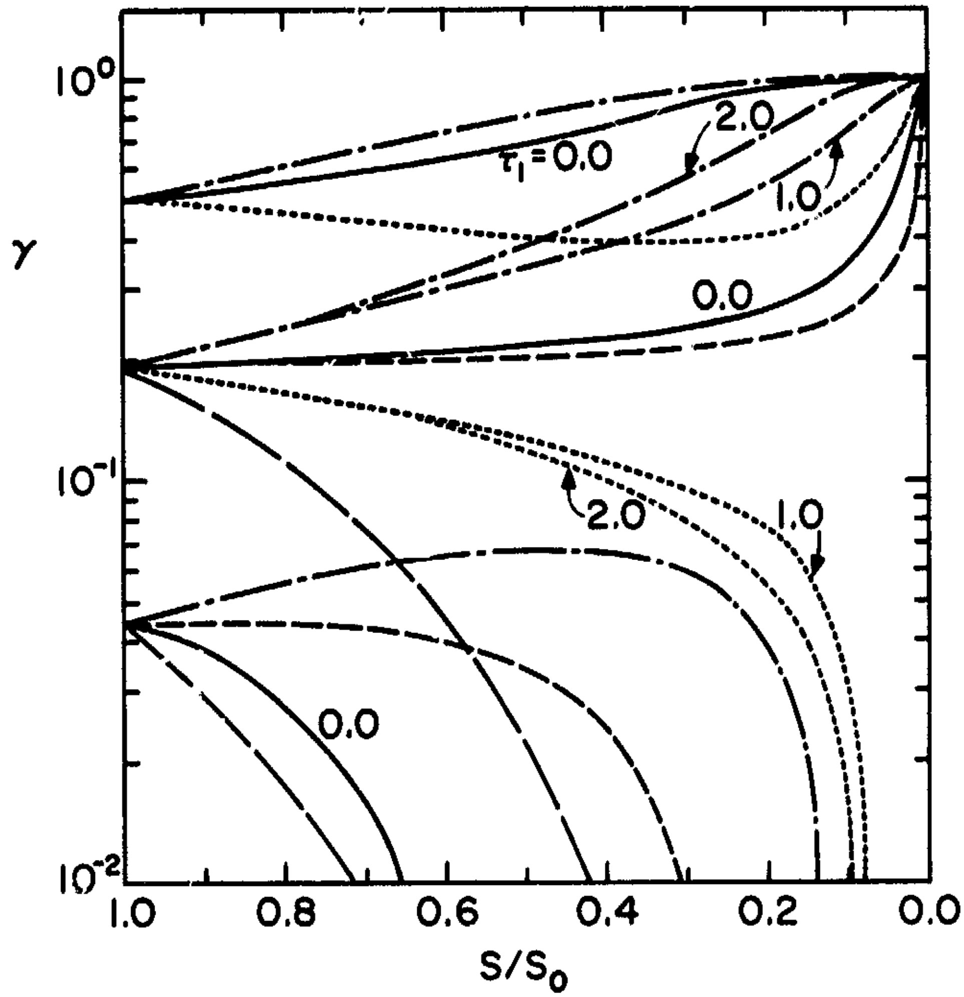
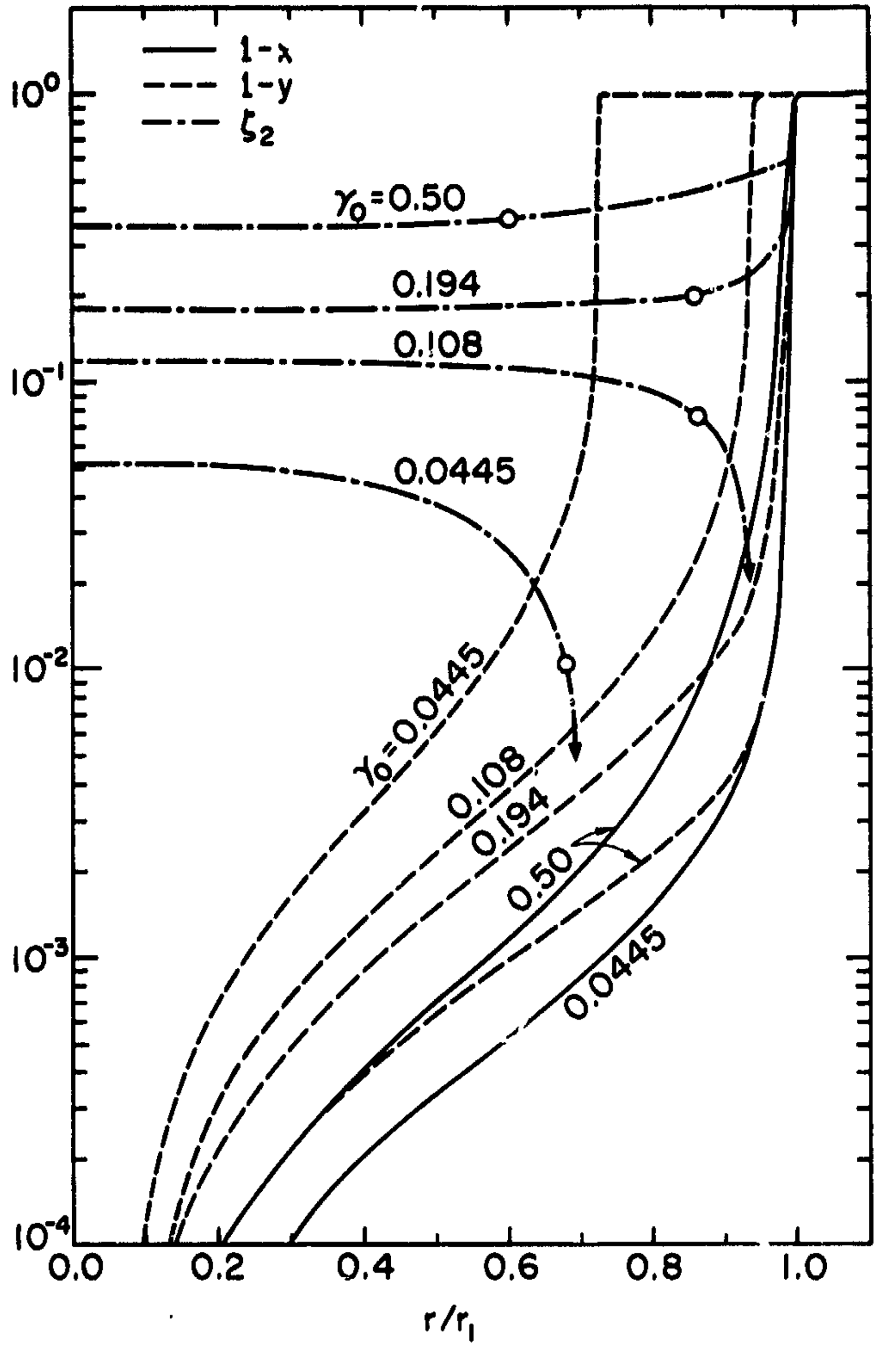
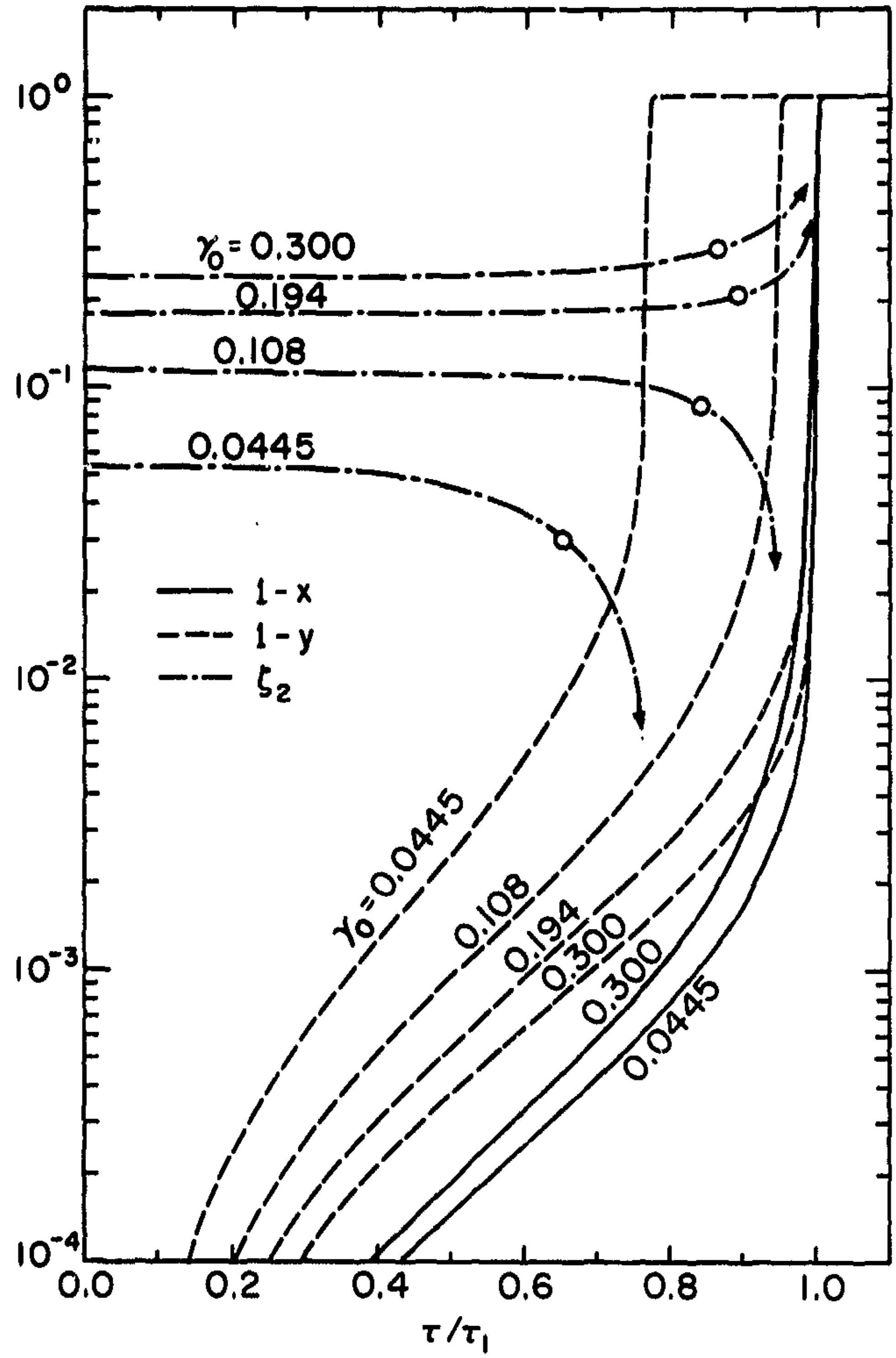


Figure 2



1-x
1-y
ζ2
10⁰
10⁻¹
10⁻²
10⁻³
10⁻⁴
γ0=0.50
0.194
0.108
0.0445
γ0=0.0445
0.108
0.194
0.50
0.0445
0.0
0.2
0.4
0.6
0.8
1.0
r/r1

Figure 3



10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\gamma_0 = 0.300$
0.194
0.108
0.0445
1 - x
1 - y
ζ_2
$\gamma_0 = 0.0445$
0.108
0.194
0.300
0.300
0.0445
0.0
0.2
0.4
0.6
0.8
1.0
τ/τ_1

Figure 4

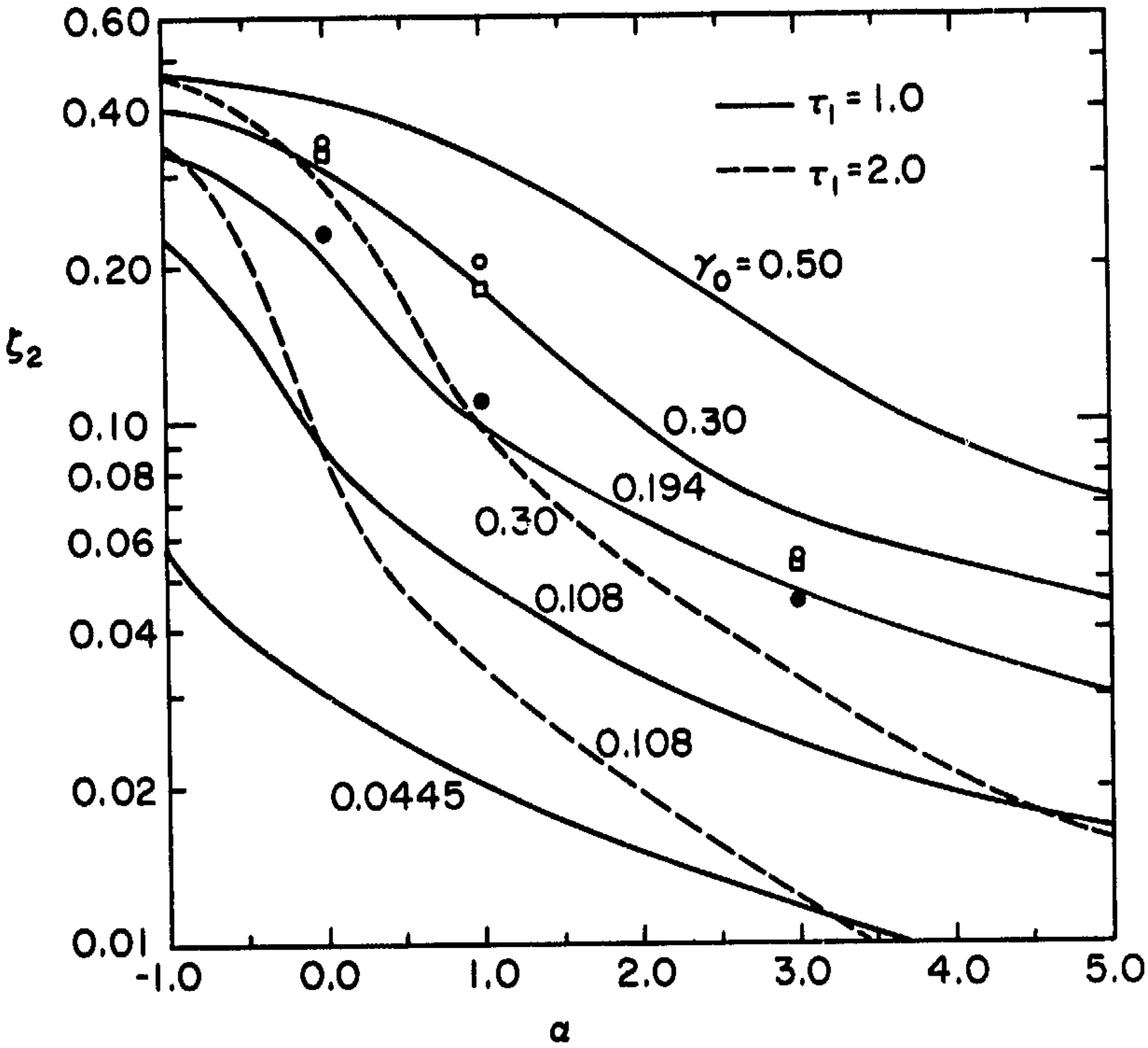

0.60
0.40
0.20
0.10
0.08
0.06
0.04
0.02
0.01
ζ_2
-1.0
0.0
1.0
2.0
3.0
4.0
5.0
α
τ_1 = 1.0
τ_1 = 2.0
γ_0 = 0.50
0.30
0.194
0.30
0.108
0.108
0.0445

Figure 5

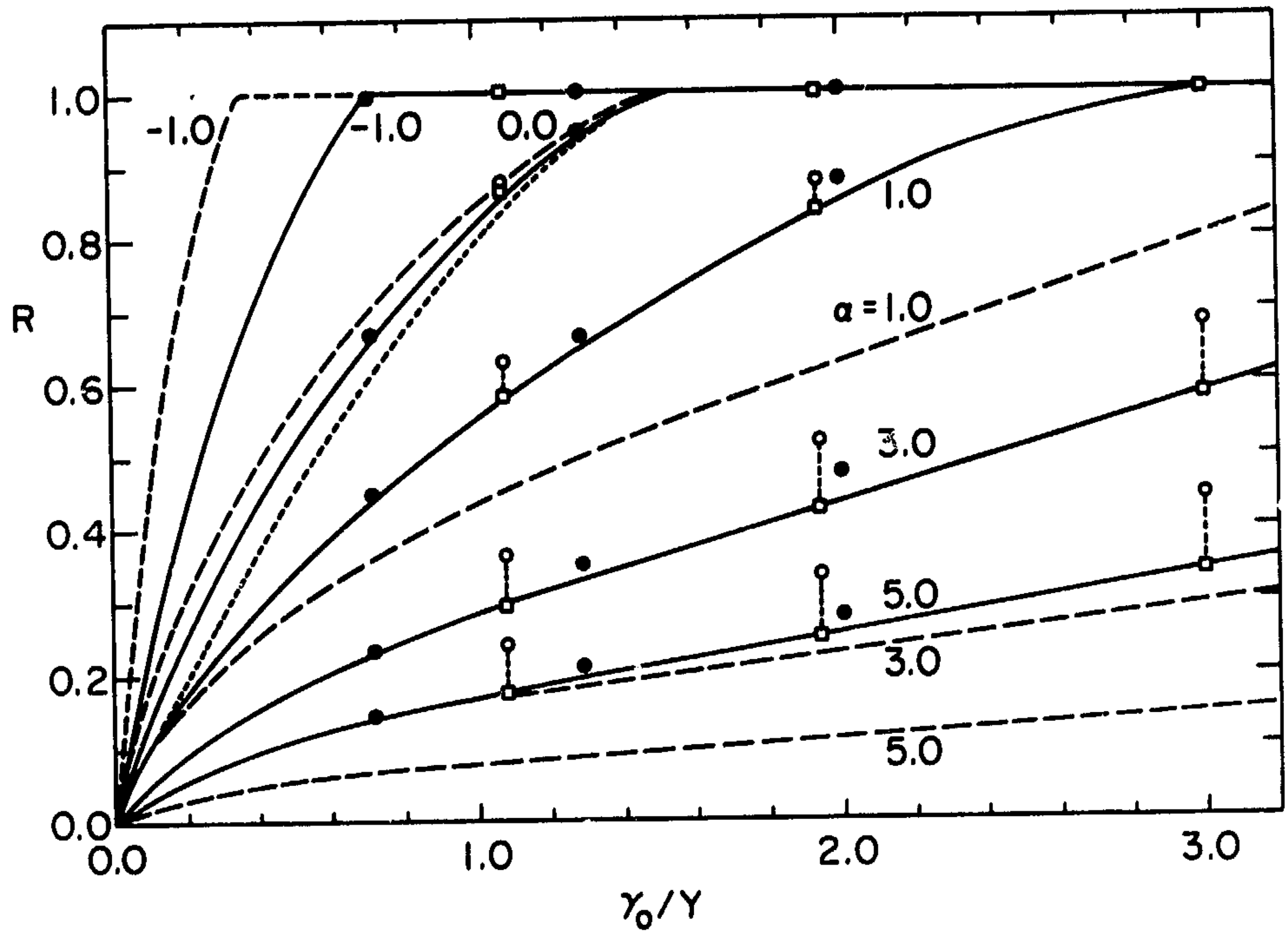
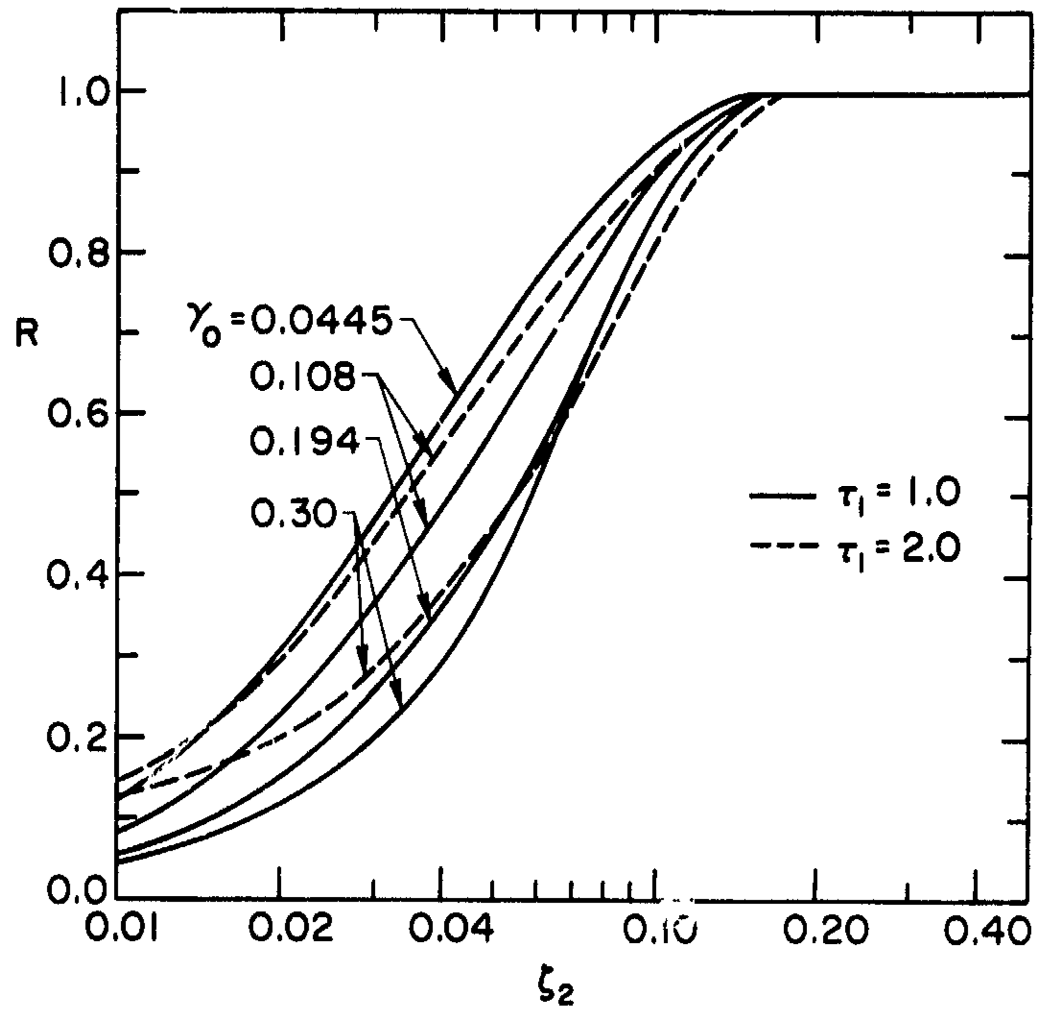


Figure 6



1.0
0.8
0.6
0.4
0.2
0.0
R
γ_0 = 0.0445
0.108
0.194
0.30
τ_1 = 1.0
τ_1 = 2.0
0.01
0.02
0.04
0.10
0.20
0.40
ζ_2

Figure 7