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A NON-GAUSSIAN MODEL OF CONTINUOUS ATMOSPHERIC TURBULENCE FOR USE IN AIRCRAFT DESIGN

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16. Abstract
Although it is recognized that atmospheric turbulence is a non-gaussian process, mathematical models of turbulence for simulator work have assumed gaussian processes due to the lack of a practical non-gaussian model. This report

1) suggests a non-gaussian model of atmospheric turbulence,
2) presents a statistical analysis of this model,
3) shows that the statistical properties of this model are more nearly in agreement with observed atmospheric turbulence properties than are those of gaussian models, and
4) derives methods by which the model can be used to study linearized aircraft responses.

The model presented here is restricted to the regions of the atmosphere where the turbulence is "steady" or "continuous", and the assumptions of homogeneity and stationarity are justified. Also spatial distribution of turbulence has been neglected, so the model consists of three independent, stationary stochastic processes which represent the vertical, lateral, and logitudinal gust components.

The non-gaussian and gaussian models are compared with experimental data, and it is shown that the gaussian model underestimates the number of high velocity gusts which occur in the atmosphere, while the non-gaussian model can be adjusted to match the observed high velocity gusts more satisfactorily. Neither model represents velocity increment distributions very satisfactorily, although a possible solution to this difficulty is presented for the non-gaussian model.

Application of the proposed model to aircraft response is investigated, with particular attention to the response power spectral density, the probability distribution, and the level crossing frequency. A numerical example is presented which illustrates the application of the non-gaussian model to the study of an aircraft-autopilot system.

Listings and sample results of a number of computer programs used in working with the model are included.

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SYMBOLS AND NOTATION

In the following list, the subscript x will denote letter subscripts, n will denote integer subscripts, and specific numerical subscripts will be denoted by i followed by a list of possible values.

Symbols

a	random function of time
A_n	random variable
$\overline{A}$	random vector (Eq. (180)
$\tilde{A}$	covariance of $\overline{A}$ (Eq. 182)
b	random function of time
B_n	random variable
$\overline{B}$	random vector (Eq. 181)
$\tilde{B}$	covariance of $\overline{B}$ (Eq. 183)
c	random function of time
C	correlation function
C_n	random variable
$\tilde{C}$	matrix
d	random function of time
D_i	i = 1, 2, 3. constants
D_n	random variable
e	random function of time
$E\{\cdot\}$	expected value operator
f	frequency
F	cumulative probability of exceedance or cumulative probability distribution (Eqs. 139, 140, 141)

g	random function of time
$h(\cdot)$	random function of time
$h(\cdot, \cdot)$	kernel function (Eq. 147)
h_x	impulse response function
H_x	transfer function
i	$\sqrt{-1}$
j	integer index
k	random function of time
k_x	constant
K_0	modified Bessel function of second type and order zero
L_x	scale length
m	integer index
M	upper limit of index m
n	integer index
N	upper limit of index n
N_i	i = 1, 2, 3, 4, 5. constants
N_x	level crossing frequency, number of crossings with positive slope per unit distance (Eq. 81)
$\hat{N}_x$	special case of level crossing frequency (Eq. 109)
$p_{x_1, x_2, \ldots}$	joint probability density function of the random variables $x_1, x_2, \ldots$
$\hat{p}_x$	standardized probability density function (Eq. 31)
$\hat{p}_{x_1, x_2}$	special case of joint density function (Eq. 108)
P_x	probability distribution function (Eq. 10)
P_x^c	cumulative gust velocity distribution function (Eq. 11)
q	integral function (Eqs. 149, 150)

r	response time history
R_x	probability distribution parameter of the turbulence model (Fig. 13)
s	Laplace transform variable
t	time
t_c	truncation time limit (Fig. 24)
u	longitudinal gust velocity
U	mean true airspeed of aircraft
v	lateral gust velocity
w	vertical gust velocity
x	variable
$\overline{x}$	vector of variables
y	variable
$\overline{y}$	vector of variables
z	variable
α	variable
$\alpha(\cdot)$	function of correlation Eq. (28)
β	variable
γ_x	wavelength representing viscosity effect (Eqs. 82, 83, 84, 113, 114)
δ	variable
Δ_x	increment of x (Eq. 22)
ε_{ise}	goodness-of-fit criterion (Eq. 139)
ε_{log}	goodness-of-fit criterion (Eq. 141)
ε_{max}	goodness-of-fit criterion (Eq. 140)

η_x	wide band gaussian white noise time history
λ_n	eigenvalue of response
Λ_n	eigenvalue of response time derivative
ν_x	wavelength representing viscosity effect (Eq. 115)
ξ	interval of real line or region of plane
π	3.1415...
σ_x	standard deviation of x
τ	time lag
$\phi_{x_1, x_2, \ldots}$	joint characteristic function of variables $x_1, x_2, \ldots$
$\hat{\phi}_{x_1, x_2}$	special case of characteristic function (Eq. 103)
Φ_{xx}	power spectral density of random function x
χ_n	eigenfunction of vehicle response
X_n	eigenfunction of vehicle response time derivative
ψ_n	eigenfunction of vehicle response
Ψ_n	eigenfunction of vehicle response time derivative
Ω	spatial frequency
Ω_c	spatial frequency limit (Fig. 23)

Notation

$\int_{x_1}^{x_2}(N)\int$	denotes N integrations between limits x_1 and x_2
$\dot{x}$	time derivative of x
$\bar{x}$	column vector of elements x_1, x_2, etc.
$\bar{x}^T$	transpose of vector $\bar{x}$
A	matrix A
$\hat{x}$	standardized variable or special case of x
max	maximum value

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SUMMARY

This report considers the problem of modeling continuous atmospheric turbulence for the purposes of aircraft design. The discussion is limited to the representation of turbulence by three stationary, independent stochastic processes which are physically interpreted as the longitudinal, lateral, and vertical gust components at the vehicle center of gravity.

The gaussian model now in wide use is reviewed. A comparison of this model with experimental data shows that it underestimates the number of high velocity gusts which occur in the atmosphere. Furthermore, it cannot reproduce observed velocity increment distributions.

A class of non-gaussian processes is proposed as a turbulence model. Though previous publications have described the application of this model to flight simulator work, this report analyzes the model in greater detail, and is the first publication to apply it to analytical calculations. A comparison with experimental turbulence data is presented, and it is concluded that the non-gaussian model is superior to the gaussian model for the purposes of representing high velocity gusts. However, in the form presented, the new model does not improve upon the gaussian model in so far as the modeling of velocity increments is concerned.

The problem of applying the non-gaussian model to the calculation of vehicle response statistics is investigated. The specific statistics of interest in this report are the response power spectral density, probability distribution, and level crossing frequency. The first of these is easily handled by well known methods. The calculation of the second and third quantities, however, requires the development of an eigenfunction - eigenvalue expansion technique. A numerical example is presented and a number of computer programs useful for studying the model are included.

INTRODUCTION

The effects of atmospheric turbulence have been a continuing concern to the aircraft designer since the earliest days of powered flight. Typical turbulence related problems which must be solved during the design of any aircraft are:

1) determination of ultimate structural strength required to sustain peak loads induced by turbulence
2) effects of turbulence on the fatigue life of the structure
3) performance of control systems in turbulence
4) handling and ride qualities in turbulence.

In an effort to provide the designer with some practical means of solving these and other problems, a number of statistical models of turbulence have been developed over the years. These models attempt to describe, in terms of as few parameters as possible, those characteristics of turbulence which are most important for various aspects of the design problem. Several of the most widely used of these models will be described below. In general they fall into two complementary classes, discrete and continuous, both of which are in use at the present time.

Discrete Models

Historically, the discrete model of turbulence was the first to be developed beginning with the work of Rhode and Lundquist in 1931 (Ref. 1). This model was intended to describe only the extreme gusts encountered by an aircraft over its operational lifetime. Because of its emphasis on extreme gusts, the discrete model is typically used to estimate ultimate strength requirements. As its name implies, the discrete model treats turbulence as a series of isolated gusts. The principal assumptions used in deriving the discrete model are (Ref. 2, 3):

1) For purposes of aircraft design, atmospheric turbulence can be modeled as a collection of isolated gusts randomly distributed along the flight path of the aircraft.
2) These gusts have random magnitudes but fixed shape.
3) The aircraft is a deterministic linear system with sufficient damping that each encounter with one of these discrete gusts results in a single significant response peak.

Typical responses which may be studied by means of the discrete model are peak structural loads imposed by turbulence, extreme vertical accelerations of the aircraft, etc. Two discrete gust shapes which have been employed in the past are a ramp 30.48 meters (100 feet) in length used in the United Kingdom (Ref. 4) and a one-minus-cosine shape used in the United States (Ref. 5).

Since the gust shape is fixed and the vehicle is assumed to respond in a well damped linear manner, the relationship between gust magnitude and the peak value of response for a given aircraft is simply a constant of proportionality depending upon the flight condition (i.e., airspeed, gross weight, altitude, etc.) at the time of gust encounter. Thus, by recording response peaks and corresponding flight conditions, one can compute the magnitude of the discrete gust which caused each peak. These magnitudes, known as "equivalent gust velocities," have been extensively measured for various types of aircraft based on recordings of vertical accelerations by counting accelerometers. Statistics on the frequency of their occurrence have been compiled by many authors (e.g., Ref. 6).

The discrete model can be used to predict the responses of a proposed aircraft design by the following method (Ref. 7). First, using the experimentally determined distribution of equivalent gust velocities and the dynamic characteristics of the proposed vehicle, work backward through the procedure described above to obtain the distribution of vertical accelerations for the vehicle. Then assume a factor of proportionality relating these accelerations to the response magnitude, and finally convert the distribution of accelerations into the required response distribution.

Even though much data is available on the distribution of equivalent gusts, the discrete model is not satisfactory for all aspects of the aircraft design problem. For example, its application in areas such as structural fatigue, control system performance, ride qualities, or even extreme responses which involve lightly damped modes is highly questionable. The reason for this is that the assumptions on which the model is based are seldom realized in practice. In most instances turbulence does not occur as discrete gusts but as a continuous random disturbance. Furthermore, the conversion from measured responses to equivalent gust velocities neglects most of the dynamic characteristics of the vehicle, particularly the structural modes which tend to be lightly damped. As a result, success in calculating the response statistics of a proposed aircraft by means of the discrete gust model depends largely upon the proposed aircraft having very nearly the same response characteristics as the vehicle with which the original data were collected. Although this fortuitous circumstance may exist in some cases, it cannot always be assumed. Thus, although the necessity of evaluating the responses of a proposed aircraft to discrete gusts is still recognized as an important part of the design procedure (Ref. 8), the use of the discrete model to calculate most response statistics has largely given way in recent years to the use of turbulence models which attempt to take both the dynamic characteristics of the vehicle and the continuous nature of turbulence into consideration. These are known as "power spectral density," "power spectral," or (perhaps more correctly) "continuous" models. It is this type of model which is of primary interest in the present report.

Continuous or Power Spectral Models

As will be seen shortly, the underlying idea of this type of model is that atmospheric turbulence can be represented by a continuous stochastic process which acts as a disturbing influence on the vehicle. The name "power spectral" has frequently been associated with these models because their development originated from studies of the power spectral density of atmospheric turbulence (Ref. 9). In actuality, these models involve not only the power spectral density of turbulence, but also an assumed probabilistic structure which is consistent with the power spectrum. Hence, to call them "power spectral" models is to stress one aspect and neglect the other. Thus, although long usage has firmly established the name "power spectral" (and indeed the name will often be used in this report), the reader should be aware that power spectral models incorporate not only a power spectrum but also a probability structure.

Because a turbulence model will be of little value to aircraft design if it cannot be used to predict vehicle responses, the type of stochastic process employed in the model must be one for which it is possible to calculate these responses with a minimum of difficulty. Because of this constraint, it is usually assumed that turbulence can be represented by a gaussian process. In this report, continuous models which incorporate this assumption will be referred to as continuous gaussian models or gaussian power spectral models.

Gaussian Power Spectral Models

As mentioned above, power spectral models represent turbulence as a continuous disturbance. The aircraft is imagined to fly through large regions of turbulent air. (The adjective "large" is used here to mean that the time required for the aircraft to pass through one of these regions is very much greater than the response time constants of the vehicle.) It is usually assumed that within each region the turbulence is homogeneous and stationary, and is characterized by intensity and scale length parameters. More formally stated, the principal assumptions made in applying this type of model in its most simple form are (Refs. 3, 10):

1) Each encounter of an aircraft with continous atmospheric turbulence can be modeled as a deterministic linear system (the aircraft) perturbed by three independent, stationary stochastic processes. These processes represent the longitudinal, lateral, and vertical gust components occuring at the vehicle center of gravity as it moves through the gust field.
2) The power spectral density of each random process belongs to a family of spectral shapes characterized by two parameters, the gust velocity standard deviation (σ) and the gust scale length (L). The scale length is a deterministic function of altitude and the standard deviation is a random variable which changes for each encounter of

the aircraft with a turbulent region of air. Within each turbulent region σ is assumed constant. Both L and σ may take on different values for each of the three gust components.

3) Each of the three gust components is a gaussian process.

The assumptions of the gaussian model make it possible to calculate the statistics of any vehicle response for each region of turbulence as functions of the parameters L and σ of that region. The statistical quantities of most frequent interest are:

1) Power spectral density -- a description of the magnitude of the Fourier components present in the response.
2) Probability distribution -- a description of the probability that a given response magnitude is exceeded.
3) Level crossing frequency -- the expected number of times per unit distance of flight that a given level or response magnitude is exceeded.

Specific methods by which these quantities can be calculated will be discussed in a later part of this report; for the present it will merely be noted that the assumptions of vehicle linearity and the gaussian nature of turbulence permit their evaluation with a minimum of difficulty. These results will, as noted above, be dependent upon the assumed values of L and σ along with the characteristics of the vehicle dynamics. Another way to express this dependence is to say that the response statistics are "conditioned" on these parameters. Note that under the assumptions of the model only one of these parameters, σ, is random.

The response statistics which are to be expected over the lifetime of the vehicle for a given flight condition (i.e., fixed altitude and vehicle dynamics) can be computed by means of the gaussian power spectral model if the distribution of the random variable σ is known. Distributions of σ have been estimated based on extensive measurements of atmospheric turbulence. However, these distributions are not usually known for all altitudes of interest. Also, data presented in reference 11 shows that the scale length of turbulence is actually a random variable even at a fixed altitude; and the importance of neglecting this effect is not known. Furthermore, as will be discussed in later sections of this report, there is much evidence that atmospheric turbulence is not a gaussian process. As a result, the gaussian power spectral model is not entirely suitable for calculating lifetime statistics in a purely formal probabilistic manner. The procedures used for this purpose typically combine many of the ideas of the gaussian power spectral model with the response statistics of existing aircraft in a method not unlike that used with the discrete model (e.g., Ref. 12).

The power spectral model is used primarily to evaluate response statistics for selected flight conditions in continuous turbulence. For this application both the scale length and standard deviation of the turbulence as well as the vehicle characteristics are fixed at values representative of flight conditions which could reasonably be expected to occur in service. Thus the aircraft is imagined to be flying through an infinitely large region of stationary, homogeneous turbulence. The response statistics calculated for this case are then examined to determine whether or not the design is satisfactory. This approach is especially useful for control, handling, and ride qualities studies; and is the type of application for which the turbulence model developed in this report is intended.

For this restricted usage, sufficient data are available to develop a much more realistic model, and the three assumptions made above bear reconsideration. The first assumption stated that atmospheric turbulence could be represented by three independent, stationary stochastic processes. Physically, this is equivalent to requiring that: 1) the turbulence is stationary and homogeneous; and 2) the dimensions of the vehicle are much smaller than the scale lengths of the three gust components. These two conditions are, of course, not always satisfied and much research has been devoted to relaxing them. References 13 through 16, for example, describe how the power spectral model can be extended to account for such effects as the spatial distribution of the gust velocity field over the surface of the aircraft, correlation of the gust components, and the non-stationarity of the turbulence. Generalizations of this type will not be considered in this report because of the increased complication they would introduce. The reader should, however, be aware that such improvements are possible. For the simplified model described in this report the first assumption of the gaussian power spectral model will be considered valid.

The second assumption of the power spectral model concerned the specification of a family of power spectral densities for each gust component which depended upon only two parameters, L and σ. This assumption has been investigated at low altitude (Ref. 11) and found to be valid if both L and σ could be chosen freely for each sample. Representative values of the scale lengths to be used in the model can be expressed as a deterministic function of altitude by simply selecting the mean L measured at each altitude. The power spectral shapes usually assumed are those proposed by von Karman (Ref. 17) or Dyrden (Ref, 18). Thus at low altitudes it is possible to satisfy the second assumption,

At high altitudes the validity of this assumption is not so certain, the problem being made more difficult by a lack of data. Those results available (e.g., Refs. 19, 20) show that spectra often behave like $\Omega^{-5/3}$ over the full range of wavelengths measured. (Ω denotes spatial frequency in radians per meter.) This would indicate that scale lengths are either very long or nonexistant at these altitudes. In practice this difficulty might be overcome while retaining the validity of the second assumption by using the von Karman or Dryden spectra and merely choosing a value of L longer than the longest wavelength to which the vehicle in question will respond. The standard deviation could be selected so that the power spectral density is properly scaled in the range of wavelengths to which the vehicle does respond.

The second assumption of the power spectral model is thus valid at low altitudes, but becomes more questionable as altitude increases. The lack of a scale length at high altitudes may perhaps be overcome by choosing a very long scale length as described above, or the treatment of high altitude turbulence as a self-similar process in the manner to be described below may be another solution.

The third assumption of the power spectral model, that turbulence is a gaussian process, is known to be incorrect (Refs. 21, 22). Since this fact is of central interest in this report, it will be discussed in some detail.

The non-gaussian nature of atmospheric turbulence makes itself apparent in two ways which are of importance to aircraft design. Compared to a gaussian process, turbulence is characterized by:

1) an increased number of high velocity gusts

2) an increased number of large gust velocity increments.

These two effects will be discussed separately even though they are obviously somewhat related.

High gust velocities are of importance to aircraft because it is this type of disturbance which tends to produce significant rigid body motions of the vehicle. These large gusts displace the vehicle from its equilibrium flight path, disconcerting the passengers and requiring the pilot to take corrective control action. As mentioned above, experimental data indicates that the gaussian turbulence model significantly underestimates the frequency of occurrence of these high gust velocities.

Reference 23 reports the examination of a large number of turbulence samples, each recorded at low altitude over a flight path distance of approximately 37 kilometers. The average number of velocity peaks in each sample was found to be 495, while the magnitude of the highest peak in each sample was consistently found to be greater than five times the standard deviation of the sample. This frequency of occurrence of such high gust velocities is several orders of magnitude greater than predicted by the gaussian assumption. Furthermore, peak gust velocity cumulative probabilities of exceedance observed in atmospheric turbulence (Refs. 11, 24) behave like $\exp(-x)$ rather than $\exp(-x^2)$ as predicted by the gaussian assumption. This again indicates a serious underestimation of high velocity gusts by the gaussian model.

The second non-gaussian characteristic of atmospheric turbulence which is important to aircraft design is the occurrence of a greater number of large gust increments than predicted by the gaussian model. The increment of a process $u(t)$ is defined to be a running difference of the form

$u(t)$ - $(t-\tau)$, where τ is the constant time lag of the increment. Increments of $u(t)$ are necessarily gaussian if u itself is gaussian. Note that a large increment is not necessarily associated with high values of $u(t)$, only large changes of value. Large velocity increments, especially with short time lags, are of importance in causing loads on the aircraft structure because they contain high frequency components which tend to excite the elastic modes of the vehicle while having a minimal effect on its gross motions. Published evidence (Refs. 22, 25, 26) indicates that the gaussian model underestimates these large increments.

Gaussian Patch Model

The data described above which indicate that atmospheric turbulence is non-gaussian are typically composed of time histories recorded over a flight path distance of more than 30 or 40 kilometers. The apparently non-gaussian characteristics of these data which have been discussed above are sometimes explained within the framework of stationary gaussian processes by introducing a slightly modified form of the gaussian power spectral model known as the "quasi gaussian" or "gaussian patch" model (Ref. 21). The central idea of this model is that long, apparently non-gaussian samples of turbulence can be divided into a number of shorter gaussian segments with different intensities. The assumptions on which this model is based are the same as those of the conventional gaussian model described above, except that now each encounter of the aircraft with a large region of turbulence is imagined to be a number of encounters with smaller, independent patches of stationary, homogeneous turbulence. Though these patches are smaller than the turbulent region which they make up, they are nevertheless still assumed to be large in the sense that the vehicle response can be assumed stationary within each patch, and transient effects between patches can be neglected. The scale length of the turbulence in all patches is assumed to be the same, but the intensity is allowed to vary randomly from patch to patch. This random intensity is usually restricted to only two discrete values (Ref. 27), although more values can be admitted if required.

Because of the above mentioned assumptions, the order in which the vehicle encounters patches of differing intensities is immaterial, and all nonstationary effects can be neglected insofar as vehicle responses are concerned. Thus the response of an aircraft flying through a large region of turbulence is assumed divisible into a number of shorter, independent time histories, each of which is stationary and gaussian. The intensity of each of these shorter response time histories corresponds to one of the two discrete values of turbulence intensity which are allowed.

Under the assumptions of the gaussian patch model, measurements of the gust velocity and response standard deviations based on time averages over very large turbulent regions are invalid because the assumption of stationarity (and therefore ergodicity) is incorrect for time histories involving more than a single patch. For example, the turbulence intensity estimated by time averaging would be a weighted average of the true intensities which were encountered. Given that the patch model is correct, it is possible to estimate the true patch intensities by a simple curve fitting technique based on measured level crossing frequencies (Ref. 27).

The important result of assuming the gaussian patch model to be correct is that probability distributions and level crossing frequencies measured in large (30 km or larger) regions of turbulence will not appear gaussian because they will be based on samples from gaussian processes with differing intensities. If the gaussian patch model could be shown to correctly explain the non-gaussian characteristics of atmospheric turbulence it would greatly simplify the aircraft design problem because the conventional gaussian model described previously could still be used to model encounters with single patches of turbulence, and encounters with multiple patches could be described in terms of ensembles of stationary, independent gaussian processes. Thus the gaussian patch model would allow the apparently non-gaussian characteristics of atmospheric turbulence to be modeled without recourse to a non-gaussian turbulence model. Furthermore, responses of vehicles to the model could be expressed as collections of stationary gaussian processes and would therefore not require a nonstationary analysis.

Unfortunately, evidence indicates that the assumptions of the patch model are incorrect. Reference 27, for example, has examined the patch length implied by this model, and concluded that the most intense patches are only two or three kilometers in length. Because these patches are so short, the assumption of stationary vehicle responses becomes very questionable. This problem will be discussed in greater detail in a later section of this report.

It will also be shown later in this report that, if the assumptions of either the gaussian or the gaussian patch models are correct, then the standardized density function of the turbulence increments must be identical to that of the turbulence itself and independent of the time lag (at least for small time lags). However, references 22, 25, and 26 present data obtained at altitudes from sea level to 18,000 meters

(60,000 feet) showing that this is not the case, particularly at low altitudes. Thus it appears that neither the gaussian model nor the gaussian patch model can account for the frequent occurrence of large velocity increments in atmospheric turbulence.

To summarize the above comments, it appears that the first assumption of the gaussian power spectral model, regarding the representation of turbulence as a three component stochastic process, has been heavily researched and many improvements have been made. These possible improvements will not be considered in this report because of the increased complication they would introduce. The second assumption, regarding the use of a family of power spectral densities dependent only on the variables L and σ appears to be valid at low altitudes and can very probably be used at high altitudes for the purposes of modeling flight through turbulence. The third assumption, concerning the representation of atmospheric turbulence as a gaussian process, is not correct; and attempts to remedy the situation by introduction of the gaussian patch model do not appear to be justified.

Thus research on the subject of a non-gaussian turbulence model is a promising area in which to make a significant improvement in aircraft design procedures. The remainder of this report will concentrate on this aspect of turbulence modeling.

Associated Current Research

The only other current work in this area to the author's knowledge is that due to Jones (Refs. 3, 28) who is developing a model of turbulence based on the concept of turbulence as a self-similar process in the sense of Mandelbrot (Refs. 29, 30, 31). A process $u(t)$ is self-similar if it has the property that transformations of the form $h^k\ u(ht)$ (for any value of h and some fixed value of k) do not change its statistical properties.

Two important implications of self-similarity from the standpoint of turbulence modeling are:

1) a self-similar process has an intermittent structure.
2) the power spectral density of a self-similar process must behave like $\Omega^{-\nu}$ for all Ω, where ν is related to the constant k of the self-similar transformation.

The first of these is an observed characteristic of atmospheric turbulence (e.g., Ref. 32) and is a very desirable property of a turbulence model. The second characteristic is not true of low altitude turbulence but, as described above, is easier to justify at high altitudes where measurements of power spectra have indicated just such behavior.

In reference 28, Jones has suggested a discrete model of turbulence which employs both random magnitudes and random lengths of the assumed gust shape. These two variables are related so that the level crossing frequency exhibits a distribution of the form $\exp(-x)\sqrt{x}$, which is reasonably consistant with observed level crossing data. This discrete model is proposed as a means of investigating the well-damped modes of the vehicle response.

Jones also discusses the possibility of a power spectral model of atmospheric turbulence which utilizes the idea of self-similarity. This model is essentially the gaussian patch model discussed above, but the self-similarity assumption is used to derive a relationship between patch length and patch intensity such that the extreme gust velocities of the model exhibit a level crossing frequency of the form $\exp(-x)/\sqrt{x}$.

Again this form is consistent with observed data. Jones proposes the self-similar power spectral model as a means of investigating lightly damped modes of the aircraft response.

The Non-Gaussian Model

The turbulence model proposed in this report, unlike that of Jones, will consider all modes of the aircraft response simultaneously, and will apply to turbulence with finite scale lengths. The principal purpose is to provide a non-gaussian turbulence model for use in representing typical encounters of aircraft with continuous atmospheric turbulence.

The research reported here is an extension of that published in references 33 and 34, which have described the development and application of a "patchy" non-gaussian turbulence model for use in flight simulators. The contribution of the present report is the further analysis of this model and the development of analytical techniques for applying it to vehicle response studies.

Since the present problem is more difficult than that discussed in references 33 and 34, the model treated in this report has been simplified to consider only the three gust components acting at the vehicle center of gravity. Reference 34 indicates at least one technique which can be used to expand this model so as to include the effects of gusts distributed over the surface of the vehicle.

The non-gaussian model proposed here differs from the gaussian power spectral model described previously in that it assumes atmospheric turbulence to be modeled by a certain class of stochastic processes which are, in general, non-gaussian. Since this class contains gaussian processes as a subclass, it follows that the proposed model is not entirely distinct from the gaussian model, but can be viewed as a generalization which includes the gaussian model as a special case. (The name "non-gaussian" applied to this model is perhaps too general since the model is restricted to only a subclass of all non-gaussian processes. However, this name is conveniently short; and within this report there is no danger of confusion.)

The principal assumptions of the non-gaussian model as it is described here are (compare with those of the gaussian model above):

1) Each encounter of an aircraft with continuous atmospheric turbulence can be modeled as a deterministic linear system (the aircraft) perturbed by three independent, statistically stationary stochastic processes. These processes represent the longitudinal, lateral, and vertical gust components at the vehicle center of gravity as it moves through the gust field.

2) The power spectral density of each random process belongs to a family of spectral shapes characterized by two parameters, the gust velocity standard deviation (σ), and the gust scale length (L). The scale length is a deterministic function of altitude and σ is a random variable which changes for each encounter of the aircraft with a turbulent region of air. Within each turbulence region σ is assumed constant. Both L and σ may take on different values for each of the three gust components.

3) Each of the three gust components is a non-gaussian process of the form $u(t) = a(t)\,b(t) + c(t)$, where a, b, and c are independent, stationary gaussian processes.

The probability structure of the proposed model is implicit in the third assumption.

Outline of Report

The remainder of this work is divided into eight sections and an appendix. The principal topics treated in each section are:

Review of the Gaussian Model. Equations describing the gaussian power spectral model and its applications to vehicle response calculations are reviewed. This section also defines several statistical quantities which are used throughout the remainder of the report.

Validity of the Gaussian Model. This section shows that the assumption of a gaussian process is inconsistant with measured turbulence data.

Formulation of the Non-gaussian Model. The general ideas leading to the non-gaussian model proposed in this report are presented.

Analysis of the Non-gaussian Model. The specific form of the turbulence model is developed along with expressions for the power spectral density, probability distribution, and level crossing frequency of the model. The increment distribution of the model is also discussed.

Validity of the Non-Gaussian Model. This section presents probability distributions and level crossing statistics obtained for various low altitude flight conditions, and compares these with statistics predicted by the gaussian and non-gaussian models.

Calculation of Non-Gaussian Response Statistics. Methods for calculating the power spectral density, distribution function, and level crossing frequencies of aircraft responses to the non-gaussian turbulence model are developed.

Numerical Example. An example is presented showing the calculation of power spectral density, distribution function, and level crossing frequencies of the altitude error allowed by a simple autopilot.

Conclusions and Suggestions for Further Research. A brief summary of results is presented along with descriptions of a number of areas in which additional research is needed if the model described here is to become a useful tool for aircraft design.

Appendix A: Computer Programs. Listings and sample cases are presented for a number of computer programs used in working with the non-gaussian turbulence model.

Appendix B: Tabulated Functions. Tabulated values of the probability density, probability distribution, and level crossing frequency of the non-gaussian model are presented.

REVIEW OF THE GAUSSIAN MODEL

The gaussian model considered in this section is intended to represent large regions of homogeneous and stationary turbulence. The three principal assumptions on which it is based are:

1) Each encounter of the aircraft with continuous atmospheric turbulence can be modeled as a deterministic linear system (the aircraft) perturbed by three independent stationary stochastic process. These processes represent the longitudinal, lateral, and vertical gust components at the vehicle center of gravity as it moves through the gust field.

2) The power spectral density of each random process belongs to a family of spectral shapes characterized by two parameters, the gust velocity standard deviation (σ) and the gust scale length (L). The scale length is a deterministic function of altitude and the standard deviation is a random variable which changes from encounter to encounter.

3) Each of the three gust components is a gaussian process.

Also, the statistical quantities of major interest to aircraft design are:

1) power spectral density
2) probability distribution
3) level crossing frequency.

The purpose of this section is to familiarize the reader with the definitions and notation of this report as well as to acquaint him with the turbulence model which is now in wide use. The following discussion will show how the statistical quantities listed above may be calculated first for the gaussian model itself and then for the response of a linear system to the model. In addition, a result concerning the distribution of gust velocity increments and a simply physical interpretation of the model will be presented.

Power Spectral Density of the Turbulence Model

Two forms of power spectral densities are presently in common use, those proposed by von Karman (Ref. 17) and those by Dryden (Ref. 18). These spectra are compared in figure 1 below. The von Karman spectra (Eqs. 1, 2, and 3) are known to provide a more accurate representation of measured turbulence spectra because they properly model the $\Omega^{-5/3}$ behavior observed at high frequencies (Refs. 11, 35). However, the fractional exponents present mathematical difficulties which are avoided by use of the Dryden spectra (Eqs. 4, 5, and 6). The Dryden spectra will be assumed in this report.

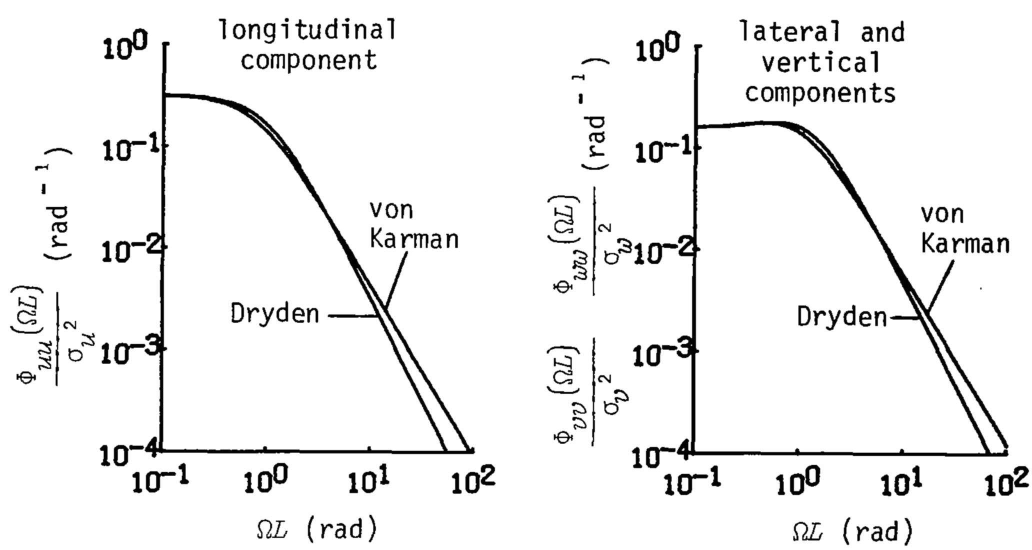


Figure 1.--Comparison of von Karman and Dryden power spectral densities. The Dryden spectra are assumed in this report.

$$\Phi_{uu}(\Omega) = \frac{L_u \sigma_u^2}{\pi} \frac{1}{[1 + (1.339 L_u \Omega)^2]^{5/6}} \tag{1}$$

$$\Phi_{vv}(\Omega) = \frac{L_v \sigma_v^2}{2\pi} \frac{[1 + \frac{8}{3}(1.339 L_v \Omega)^2]}{[1 + (1.339 L_v \Omega)^2]^{11/6}} \tag{2}$$

$$\Phi_{ww}(\Omega) = \frac{L_w \sigma_w^2}{2\pi} \frac{[1 + \frac{8}{3}(1.339 L_w \Omega)^2]}{[1 + (1.339 L_w \Omega)^2]^{11/6}} \tag{3}$$

$$\Phi_{uu}(\Omega) = \frac{L_u \sigma_u^2}{\pi} \frac{1}{[1 + (L_u \Omega)^2]} \tag{4}$$

$$\Phi_{vv}(\Omega) = \frac{L_v \sigma_v^2}{2\pi} \frac{[1 + 3(L_v \Omega)^2]}{[1 + (L_v \Omega)^2]^2} \tag{5}$$

$$\Phi_{ww}(\Omega) = \frac{L_w \sigma_w^2}{2\pi} \frac{[1 + 3(L_w \Omega)^2]}{[1 + (L_w \Omega)^2]^2} \tag{6}$$

The reader should be aware that these are "two sided" spectra defined for both positive and negative values of Ω. The standard deviation of each gust component is given by an expression of the form

$$\sigma^2 = \int_{-\infty}^{\infty} \Phi(\Omega)\, d\Omega \tag{7}$$

Probability Distribution of the Gust Velocity

The model discussed here is gaussian by definition, thus each component has a conditional probability density function of the form

$$p(x|\sigma) = \frac{1}{\sigma(2\pi)^{1/2}} \exp[-\frac{1}{2}(\frac{x}{\sigma})^2]. \tag{8}$$

The notation $(\cdot|\sigma)$ is used here to indicate the dependence or "conditioning" of the density function upon σ. In general, the σ in equation 8 corresponds to the σ's appearing in equations 4 through 6 above, and is different for each gust component. Figure 2-a below shows a graph of this density function for positive values of x.

An equivalent description of the gust velocity probability distribution is given by the distribution function. The distribution function is related to the density function by integration.

$$P(x|\sigma) = \int_{-\infty}^{x} p(y|\sigma)\, dy \tag{9}$$

Figure 2-b shows the gaussian distribution function plotted on a probability scale. Note that it is a straight line on this scale.

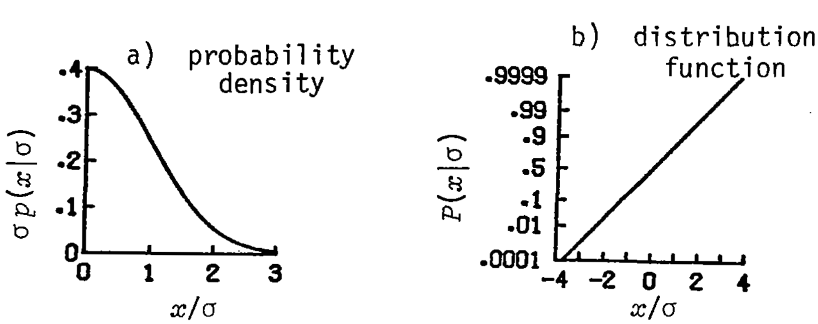


Figure 2.--Probability density and distribution functions of the gaussian turbulence model.

It will be of interest later to determine the distribution which would be obtained by combining data from a number of samples of the gaussian turbulence model. Note that the density function, equation 8, depends only on the standard deviation of the gust velocity. Thus, if the probability distribution of σ is known, the average or unconditional density function of the gust velocity can be found by integration

$$p(x) = \int_0^\infty p(x|y)\, p_\sigma(y)\, dy . \tag{10}$$

It should be noted that $p(x)$ is not a gaussian density.

In order to conform to conventional usage, the gust velocity distribution function for a number of data samples will be expressed as a complementary distribution of the absolute value of the gust velocity

$$P^c(x) = 1 - \int_{-x}^{x} p(y)\, dy . \tag{11}$$

$P^c(x)$ is the probability that the absolute value of the gust velocity is greater than x, and will be referred to as the cumulative gust velocity distribution in this report. The reader should be aware that P^c decreases monotonically even though the term "cumulative" is used to describe it.

Level Crossing Frequency of the Gust Velocity

The level crossing frequency of a gaussian process is given by the well known result due to Rice (Ref. 36). Rice's equation (divided by $2U$ to give the expected number of crossings with positive slope per unit distance of flight) applied to the longitudinal gust component gives the following expression for the conditional level crossing frequency.

$$N_u(x|\sigma_u , \sigma_{\dot{u}} , U) = \frac{\sigma_{\dot{u}}}{2\pi \sigma_u U} \exp[- \frac{1}{2}(\frac{x}{\sigma_u})^2] \qquad (12)$$

The quantity $\sigma_{\dot{u}}$ in this equation denotes the standard deviation of the first derivative of the u-gust time history, and U is the mean true airspeed of the aircraft. Similar equations apply to the lateral and vertical components of the model.

A problem arises in the application of equation 12 to the turbulence model described here because $\sigma_{\dot{u}}$ does not exist for a process with the Dryden power spectral density (Eq. 4), nor will the analogous standard deviations exist for the other components of the model. The reason for this is clear when the relationship between $\sigma_{\dot{u}}$ and the power spectral density of u is considered. The theory of continuous stochastic processes (Ref. 37) requires that

$$\sigma_{\dot{u}}^2 = U^2 \int_0^\infty \Omega^2 \, \Phi_{uu}(\Omega) \, d\Omega \, . \qquad (13)$$

But the u-gust spectrum (and for that matter the lateral and vertical gust spectra) behaves like Ω^{-2} for large values of Ω. Therefore, the integral in equation 13 does not converge and $\sigma_{\dot{u}}$ does not exist. This result stems from the fact that the model has not accounted for viscous dissipation, which causes the true power spectrum of turbulence to decay more rapidly for wavelengths shorter than about one centimeter. These extremely short wavelengths are of no importance when vehicle responses are being calculated, and are justifiably neglected when the model is used for this purpose. However, in the next section of this report it will be of interest to compare the level crossings of the model to those measured experimentally, and it is important to show that a level crossing frequency can be defined for the model without altering its characteristics as far as response calculations are concerned.

In practice this problem is sometimes overcome by truncating the power spectral densities at some convenient frequency. Another method is the addition of high frequency poles to the spectral forms as shown in equation 14 for the u-gust spectrum.

$$\Phi_{uu}(\Omega) = \frac{\sigma_u^2}{\pi} \frac{(L_u + \gamma_u)}{[1 + (L_u \Omega)^2][1 + (\gamma_u \Omega)^2]} \tag{14}$$

These poles can be physically interpreted as representing the effects of viscous dissipation, and γ_u can be thought of as representing the wavelength at which this dissipation becomes important. As will be seen shortly, however, the precise value chosen for γ_u will not be important for the purposes of this report.

Using equation 14 for Φ_{uu} allows one to compute the standard deviations required in Rice's equation. In particular,

$$\sigma_{\dot{u}} = U \sigma_u (\gamma_u L_U)^{-1/2}. \tag{15}$$

Substitution of this result into equation 12 gives the following result for N_u .

$$N_u(x|\sigma_u , L_u , \gamma_u) = \frac{1}{2\pi(\gamma_u L_u)^{1/2}} \exp[-\frac{1}{2}(\frac{x}{\sigma_u})^2] \tag{16}$$

Note that the zero crossing frequency is given by

$$N_u(0|\sigma_u, L_u, \gamma_u) = (2\pi)^{-1} (\gamma_u L_u)^{-1/2} \tag{17}$$

Thus the value of γ_u may be thought of as determining the zero crossing frequency of the u-gust component, and could be chosen so as to match equation 17 to some measured value.

The addition of high frequency poles to the vertical and lateral gust spectra in a manner similar to that of equation 14 will result in level crossing frequencies which differ from equation 16 by only a constant of proportionality.

$$N_v(x|\sigma, L, \gamma) = N_w(x|\sigma, L, \gamma) = (3/2)^{1/2} N_u(x|\sigma, L, \gamma) \tag{18}$$

Figure 3 shows a typical graph of level crossing frequencies for the gaussian model.

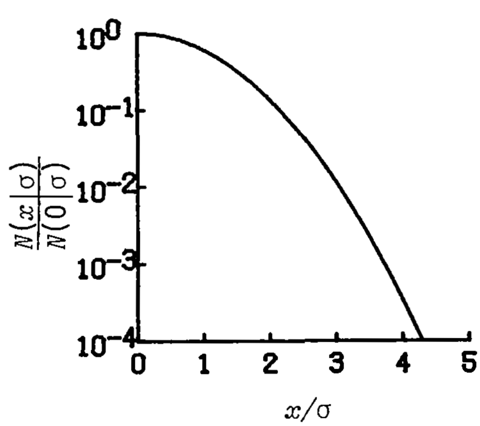


Figure 3.--Normalized level crossing frequency of the gaussian turbulence model.

It will be of interest later in this report to compute the level crossing frequencies which would be obtained by combining data from a large number of turbulence encounters. These theoretical results can then be compared with experimentally measured data to determine the validity of the gaussian model. In past analyses of this type (e.g., Ref. 23) it is common to assume that the zero crossing frequency is constant for all data samples. That is,

$$N_u(0) = N_u(0|\sigma_u, L_u, \gamma_u) \tag{19}$$

for all values of σ_u, L_u, and γ_u. Thus the level crossing frequency for a large number of data samples is

$$N_u(x) = N_u(0) \int_0^\infty \exp[-\frac{1}{2}(\frac{x}{\sigma})^2]\, p_{\sigma_u}(\sigma)\, d\sigma . \tag{20}$$

These level crossing curves are typically normalized with respect to $N_u(0)$ in order to obtain what is termed the normalized level crossing frequency or, perhaps more commonly, the cumulative probability of exceedance.

$$\frac{N_u(x)}{N_u(0)} = \int_0^\infty \exp[-\frac{1}{2}(\frac{x}{\sigma})^2]\, p_{\sigma_u}(\sigma)\, d\sigma \tag{21}$$

Note that this result depends only upon the density function of σ_u.

Equation 21 was obtained from equation 16 by assuming that $N_u(0|\cdot,\cdot,\cdot)$ was a constant for all data samples. However, it is readily verified (and will be proved later in this report, Eq. 111, 112) that precisely the same conclusion will be reached if it is only required that σ_u be a random variable independent of both L_u and σ_u. In this event σ_u acts only as a scale factor of the process and can have no influence on the frequency with which it changes sign. Thus $N_u(0|\cdot,\cdot,\cdot)$ need not be a constant, but only a random variable independent of σ_u.

Expressions completely analogous to equation 21 can be obtained for the vertical and lateral gust components. Note that these results depend only upon the standard deviation of the gusts. In particular, the value of γ has no effect on the normalized level crossing frequency.

Distribution of Velocity Increments

Consider now the velocity increments of the gaussian turbulence model. Since identical results will be obtained for all three components, only the longitudinal gusts will be considered here. Define the u-gust increment to be

$$\Delta_u(t) = u(t) - u(t-\tau) \tag{22}$$

Since Δ_u is a linear transformation of a gaussian process, it must be gaussian. The mean and variance are readily calculated from equation 22.

$$E\{\Delta_u\} = 0 \tag{23}$$

$$E\{\Delta_u{}^2\} = \sigma_\Delta{}^2 = 2[\sigma_u{}^2 - C_{uu}(\tau)] \tag{24}$$

where C_{uu} is the autocorrelation function of u. Thus the probability density of Δ_u is given by

$$p_\Delta(x|\sigma_\Delta) = \frac{1}{\sigma_\Delta(2\pi)^{1/2}} \exp[-\frac{1}{2}(\frac{x}{\sigma_\Delta})^2] \ , \tag{25}$$

and the average density of Δ_u computed from a number of samples of the gaussian model with differing intensities would be

$$p_\Delta(x) = \int_0^\infty \frac{1}{\sigma_\Delta(2\pi)^{1/2}} \exp[-\tfrac{1}{2}(\tfrac{x}{\sigma})^2]\, p_{\sigma_\Delta}(\sigma)\, d\sigma \;. \tag{26}$$

A relationship will now be derived relating the average density of Δ_u to the average density of u itself. First write σ_Δ in the form

$$\sigma_\Delta = \sigma_u\,\alpha(\tau) \tag{27}$$

where

$$\alpha^2(\tau) = 2[1 - \frac{C_{uu}(\tau)}{\sigma_u^{\,2}}] \;. \tag{28}$$

Then the probability density of σ_Δ is related to the density of σ_u by

$$p_{\sigma_\Delta}(\sigma) = \frac{1}{\alpha(\tau)}\, p_{\sigma_u}[\frac{\sigma}{\alpha(\tau)}] \;. \tag{29}$$

Introduce the change of variable $\sigma' = \sigma/\alpha(\tau)$, substitute equations 27 and 29 into equation 26, and compare the resulting expression with equation 10 to obtain

$$p_\Delta(x) = \frac{1}{\alpha(\tau)}\, p_u[\frac{x}{\alpha(\tau)}] \;. \tag{30}$$

Now define the standardized density function to be

$$\hat{p}(x) = \sigma p(\sigma x) \tag{31}$$

and note that the average standard deviation of Δ_u is related to the average standard deviation of σ_u by

$$E\{\sigma_\Delta\} = \alpha(\tau)\, E\{\sigma_u\} \,. \tag{32}$$

It then follows from equations 30, 31, and 32 that

$$\hat{p}_{\Delta}(x) = \hat{p}_{u}(x) \ . \tag{33}$$

Equation 33 implies that the standardized density function of Δ_u is identical to that of u itself, and this result is independent of both the time lag τ and the distribution of σ_u .

Power Spectral Density of Vehicle Response

The power spectral density of some response $r(t)$ of a stable linear vehicle to a single component of the gaussian turbulence model is given by the well known relationship (Ref. 38) written here for the longitudinal gust component in terms of the spatial frequency Ω.

$$\Phi_{rr}(\Omega) = |H(i\,U\Omega)|^2\,\Phi_{uu}(\Omega) \tag{34}$$

where $H(s)$ is the transfer function relation the response $r(t)$ to the input $u(t)$. Similar expressions hold for the other components of the turbulence model.

Probability Distribution of Vehicle Response

A linear response to a gaussian input is necessarily gaussian, thus the density function of any vehicle response $r(t)$ is given by

$$p_r(x) = \frac{1}{\sigma_r(2\pi)^{1/2}} \exp\left[-\frac{1}{2}\left(\frac{x}{\sigma_r}\right)^2\right] \tag{35}$$

where σ_r is obtained from the power spectral density, equation 34, in a manner similar to equation 7. The distribution function of the response is given by an expression analogous to equation 9.

Level Crossing Frequency of the Vehicle Response

Since the vehicle response is necessarily gaussian, the level crossing frequency must be given by equation 12. In the case of a stable aircraft, the parameter γ introduced in the above discussion of the gust velocity level crossings need not be used because the vehicle will act as a low pass filter and the derivative of the response will always exist. The level crossing frequency of a vehicle response $r(t)$ to the u-gust component of the turbulence model is thus given by

$$N_r(x|\sigma_r, \sigma_{\dot{r}}) = \frac{\sigma_{\dot{r}}}{2\pi\sigma_r} \exp[-\frac{1}{2}(\frac{x}{\sigma_r})^2], \tag{36}$$

here σ_r and $\sigma_{\dot{r}}$ are determined from the power spectral density of the response (Eq. 34) by means of equations 7 and 13 respectively. Entirely analogous relationships hold for the lateral and vertical components of the model.

Physical Interpretation of the Gaussian Model

Each component of the gaussian turbulence model can be interpreted as wide band gaussian white noise passed through a linear filter as shown in figure 4.

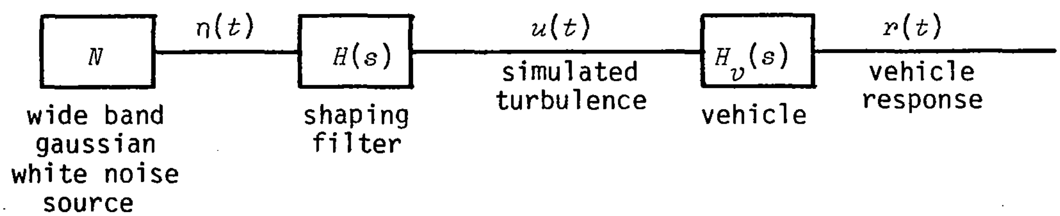


Figure 4.--Physical interpretation of a single component of the gaussian turbulence model and the response of a vehicle to this model.

The white noise source is assumed to generate gaussian noise with a power spectral density of unity over the range of frequencies passed by the shaping filter and the vehicle transfer function. That is,

$$\Phi_{\eta\eta}(\Omega) = 1.0 \tag{37}$$

The shaping filter transfer function is determined by means of equation 34 and the required turbulence power spectral densities discussed at the beginning of this section (Eqs. 4, 5, or 6).

VALIDITY OF THE GAUSSIAN MODEL

The gaussian turbulence model described in the previous section will now be compared with some experimentally measured gust statistics. It will be shown that neither the gaussian nor the gaussian patch turbulence model is able to reproduce these statistics.

Specific results to be presented are:

1) comparison of the gaussian probability distribution with gust velocity distributions estimated from single time histories of atmospheric turbulence
2) comparison of gust velocity distributions predicted by the gaussian model with distributions estimated from a large number of independent time histories
3) comparison of cumulative probability of exceedance distributions predicted by the gaussian model with distributions estimated from a large number of independent data samples
4) discussion of patch sizes implied by the gaussian patch model
5) comparison of measured velocity increment probability distributions with distributions predicted by the gaussian patch model.

Gust Velocity Distributions Estimated from Single Samples

Figure 5, originally published by Dutton (Ref. 21), presents estimated gust velocity distributions from two independent sources. In each case the gaussian distribution is indicated by a solid line. The curves

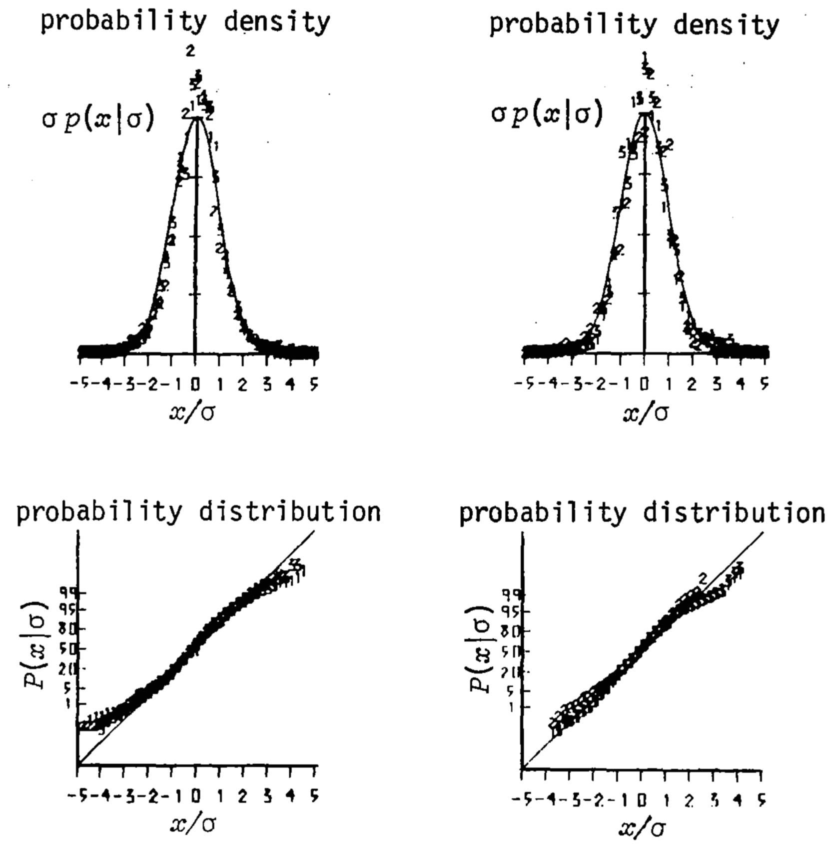


Figure 5.--Comparison of the gaussian turbulence model with gust velocity distributions estimated from single turbulence time histories.

on the left show three samples of the vertical gust component measured by G. K. Mather of the Canadian National Aeronautical Establishment (Ref. 39). These data were obtained at an altitude of approximately 9,000 meters (30,000 feet) over the Sierra Nevada mountains. The graphs on the right of figure 5 are from a single sample of severe turbulence encountered at an altitude of 18,000 meters (60,000 feet) during Project HI-CAT (Ref. 40, run number 264-16) sponsored by the United States Air Force. The numbers 1, 2, and 3 denote the vertical, lateral, and longitudinal gust components respectively. The data of figure 5 clearly depart from the gaussian distribution at both small and large gust velocities,

It must be pointed out that these results alone do not disprove the hypothesis that atmospheric turbulence can be modeled as a locally gaussian process. Another possible explanation of the behavior illustrated in the figure is that the time histories from which the data were derived contained patches with differing intensities. This interpretation would agree with the gaussian patch model discussed in the introduction of this report. Further remarks regarding this model will be found in the fourth and fifth parts of this section.

Gust Velocity Distributions Estimated from Many Samples

Figure 6 presents two vertical gust velocity cumulative distributions obtained during the United States Air Force sponsored Project LO-LOCAT (Ref. 11). The data on the left were obtained during 5,000

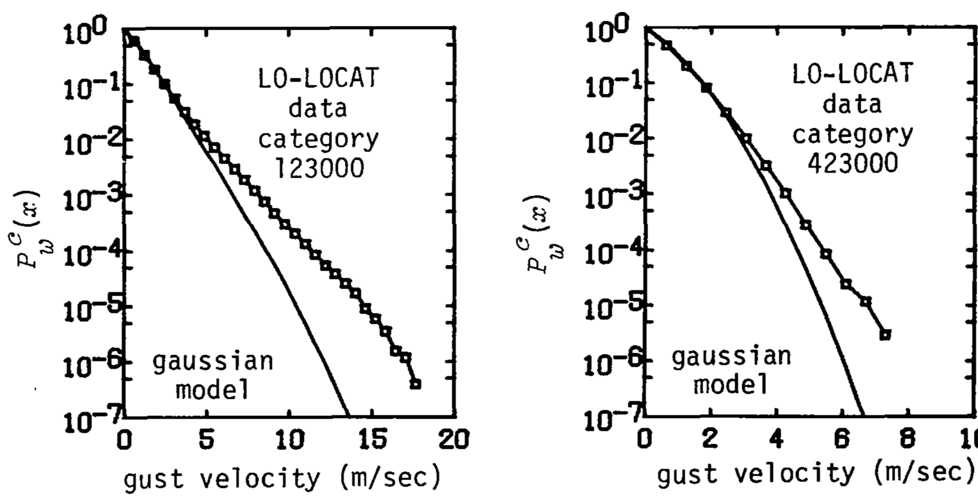


Figure 6.--Experimentally measured cumulative gust velocity distributions of atmospheric turbulence compared with those of the gaussian model.

kilometers (3,122 miles) of flight over high mountains at an average altitude of 230 meters (750 feet) above the surface in a neutrally stable atmosphere (data category 123000). The data on the right were collected during 2,000 kilometers (1,248 miles) of flight over plains at an altitude of 230 meters (750 feet) in neutrally stable conditions(data category 423000). In both cases the distribution predicted by the gaussian model has been calculated using equation 11 and the measured probability distributions of the standard deviations for each data category presented in reference 11. Note that in both cases the cumulative distribution estimated using the assumption of a gaussian process underestimates the occurrence of high gust velocities by substantial factors. Again, it should be pointed out that the apparently non-gaussian behavior of the data shown in figure 6 could be explained by the gaussian patch model.

Level Crossing Frequencies Estimated from Many Samples

Figure 7 presents two cumulative probability of exceedance distributions obtained during the LO-LOCAT project (Ref. 11). These data were

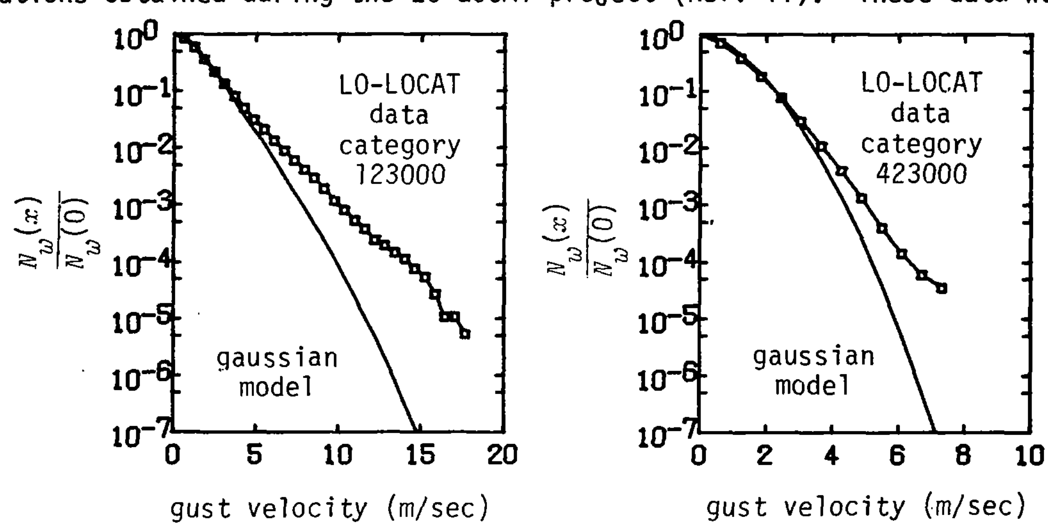


Figure 7.--Experimentally measured cumulative probability of exceedance distributions of atmospheric turbulence compared with those of the gaussian model.

derived from the same time histories which produced the data of figure 6, and a more detailed discussion of the test conditions will be found in the text describing that figure. The data on the left are from data category 123000 and those on the right are from category 423000. The distribution based on the gaussian model was derived using equation 21 and the distributions of gust velocity standard deviation presented in reference 11. Again the results based on the stationary gaussian turbulence model significantly underestimate the occurrences of high gust velocities.

Patch Sizes Implied by the Gaussian Patch Model

Figures 5, 6, and 7 have shown typical experimental observations which indicate that the stationary gaussian model underestimates the occurrences of high gust velocities. This apparently non-gaussian behavior can still be explained in terms of stationary gaussian processes by means of the gaussian patch model which was described in the introduction of this report.

If this model is a valid representation of atmospheric turbulence, the stationary gaussian model will provide a good representation of each turbulent patch and therefore will be a good model for aircraft design work. There is, however, evidence which indicates that the gaussian patch model is not valid.

A study of the patch size implied by the gaussian patch model carried out by Gould and MacPherson (Ref. 27) found that the more intense (and therefore the more important) patches are so small as to require a non-stationary analysis for vehicle response calculations. Figure 8 shows the relationship between patch size and intensity taken from reference 27. Intensity is measured in terms of vertical acceleration rms g's of the test aircraft. Note that patch size decreases rapidly with increasing intensity, and that the most intense patches are only two to three kilometers in length. An aircraft cruising at 200 m/sec true airspeed would pass through one of these patches in only 10 to 15 seconds. This interval is easily of the same order of magnitude as the significant response time constants of most vehicles, and encounters with such short disturbances would necessarily require a nonstationary analysis. It is also quite possible that the state of the vehicle at the time of initial entry into such short patches may have a significant effect on the response statistics, a result which would greatly complicate the analysis because the independence of patches could no longer be assumed.

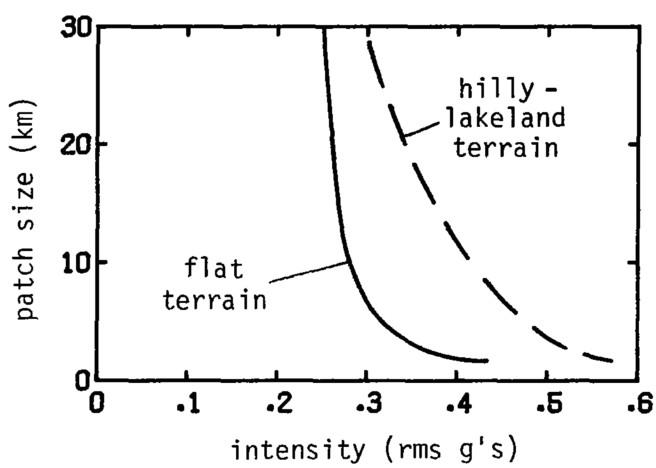


Figure 8.--Patch sizes implied by the gaussian patch model of atmospheric turbulence.

Thus, even though the gaussian patch model can explain the results shown in figures 5, 6, and 7, figure 8 indicates that the assumption of large patch sizes required to justify a stationary analysis of vehicle responses is not valid. In addition to these results, the distribution of gust velocity increments provides further evidence that the gaussian patch model is invalid.

Distribution of Gust Velocity Increments

Figure 9, taken from reference 24, shows comparisons of gust velocity probability densities with corresponding velocity increment density functions. The solid line in each graph is the gaussian distribution and the dashed lines indicate the limits of the original data which can be found in reference 24. Each graph is based on several time histories which were chosen for their stationary character. In all cases the density functions have been standardized as defined in equation 31. The data on the left of the figure were obtained during the HI-CAT program (Ref. 40). The center graphs show data collected during flight through severe storm turbulence (Ref. 41). The data on the right were obtained at low altitude during the Barbados Oceanographic and Meteorological Experiment (Ref. 42). In

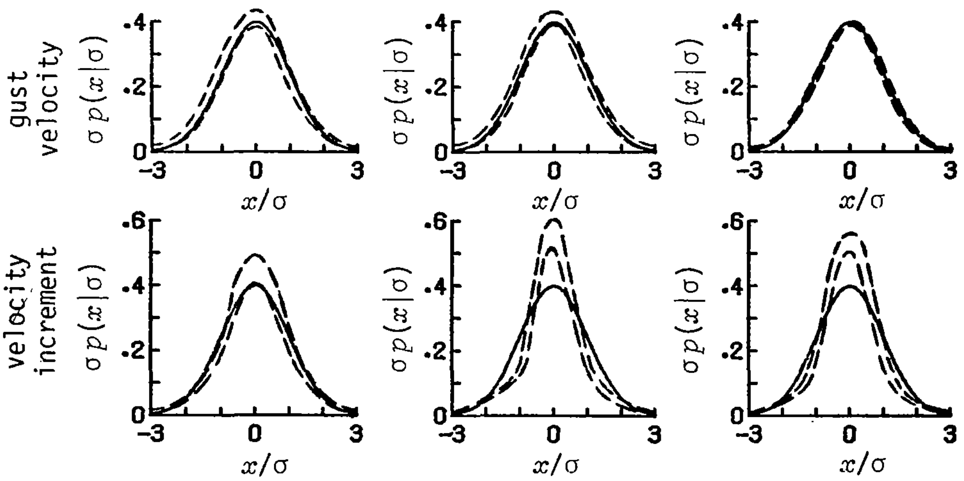


Figure 9.--Comparison of experimentally measured gust velocity distributions (top) with corresponding gust velocity increment distributions (bottom).

each case it is readily apparent that the increment distribution is not only non-gaussian but also quite different from the corresponding velocity distribution.

In the preceding section of this report (Eq. 33) it was shown that, if turbulence could be described as a collection of stationary gaussian patches, the standardized densities of the gust velocity and its increments were necessarily identical. Figure 9 clearly shows that this is not the case. Therefore, representation of atmospheric turbulence as a gaussian process is incorrect. Additional data presented in references 22 and 25 confirm this conclusion and also indicate that the gaussian model underestimates the occurrence of large gust increments.

Summary

This section has presented comparisons of the gaussian model with experimentally measured turbulence data from several independent sources. Examples of the gust velocity distributions and level crossing frequencies have indicated that the stationary gaussian model underestimates the occurrences of high gust velocities. The gaussian patch model, which could explain this behavior in terms of stationary gaussian processes, is not well suited to vehicle response calculations because it requires a nonstationary analysis of the most intense turbulence patches. It has also been shown that the distribution of velocity increments measured in turbulence is different from the distribution of velocity itself, a characteristic of turbulence which cannot be explained by the gaussian patch model. This last result constitutes a proof that neither the stationary gaussian nor the gaussian patch model is actually representative of the true structure of turbulence.

FORMULATION OF A NON-GAUSSIAN TURBULENCE MODEL

The preceding section has indicated that the gaussian turbulence model is not well suited for vehicle design work because it underestimates the occurrences of both high gust velocities and large gust increments. It is the purpose of this section to propose a non-gaussian turbulence model, and discuss the general line of reasoning leading to its formulation. Only a brief description will be presented here since a detailed analysis will be given in the following section.

The K_0 Model

The ideas which led to the turbulence model proposed here originated with the work of reference 33. The purpose of that report was to develop a "patchy" model of atmospheric turbulence for use with flight simulators. It is known that the gaussian model, generated by the system of figure 4, does not produce a realistic simulation of flight through turbulent air. Pilots complain of a lack of patchiness in the simulated turbulence (Ref. 43, 44). These patches are not associated with long term nonstationary changes of intensity in the sense of the gaussian patch model described previously, but are short bursts of activity which are sometimes only a few seconds in length.

No quantitative description of this type of patchiness has been given, but a qualitative measure can be obtained by observing the derivative of a turbulence time history. The top portion of figure 10 shows a comparison of derivatives from a sample of actual atmospheric turbulence and from the gaussian model. The gaussian time history was generated as shown in figure 4 and its power spectral density was chosen to match that of the actual turbulence as closely as possible. Observe that the derivative of the true turbulence contains distinct bursts of activity which are totally lacking in the gaussian time history.

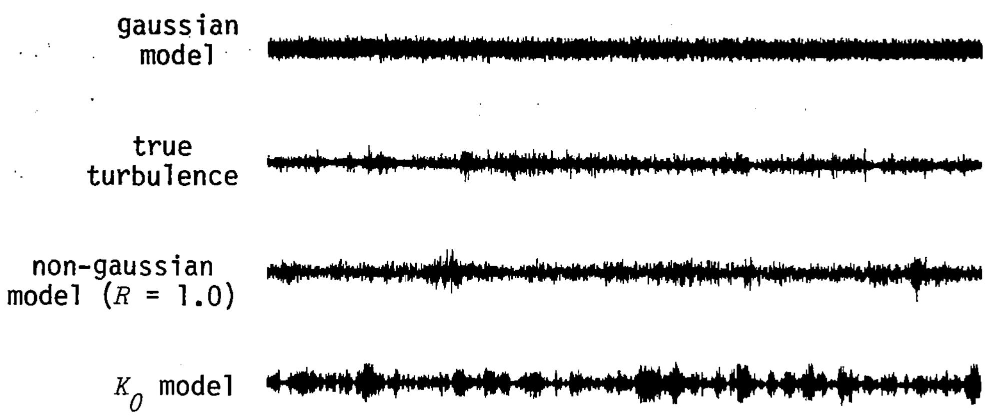


Figure 10.--The derivative of an atmospheric turbulence sample compared with derivative samples from several turbulence models. Sample length for these data is approximately 11 kilometers.

Reference 33 has suggested a product of independent gaussian processes as shown in figure 11 as a model of patchy turbulence. One of the two processes, say $a(t)$, can be imagined to represent a continuous gaussian time history without patches while $b(t)$ represents a modulating or "patch inducing" function. Reference 33 shows that the filter transfer functions

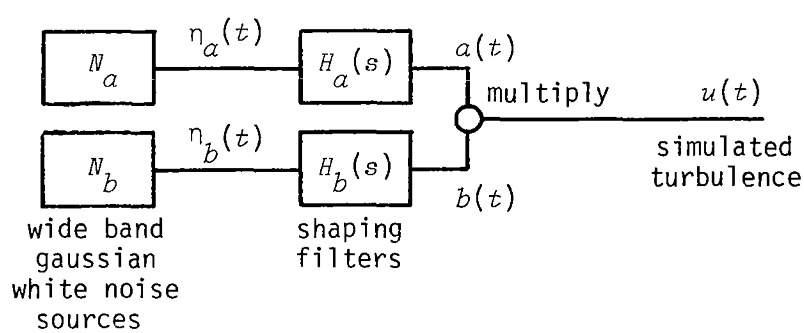


Figure 11.--Product of independent gaussian processes which has been suggested as a model of patchy turbulence.

H_a and H_b can be chosen so that the simulated turbulence time history has the Dryden spectral forms (Eq. 4, 5, or 6). The probability density of the simulated turbulence is proportional to a modified Bessel function of the second type and order zero, K_0 . For this reason the turbulence model represented by figure 11 is called the K_0 model. Figure 12 presents the probability density and distribution of this model along with those of the gaussian model.

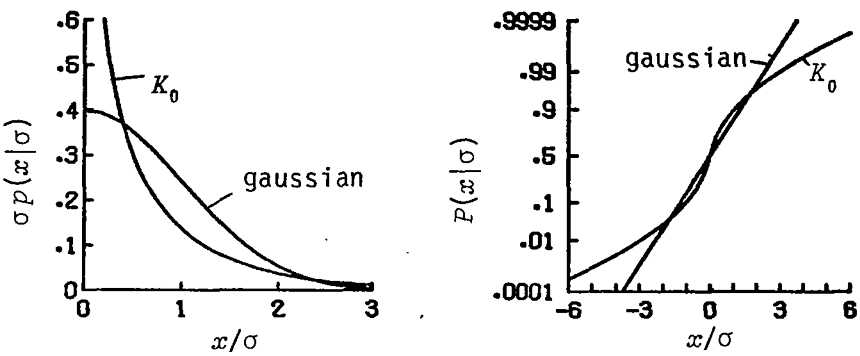


Figure 12.--Probability density and distribution functions of the K_0 turbulence model compared with those of the gaussian model.

A comparison of figure 12 with figure 5 on page 36 shows that the K_0 distribution departs from the gaussian distribution in a manner similar to that of actual turbulence. That is, the K_0 model is characterized by more small and large gusts than the gaussian model. However, it appears to be more severely non-gaussian than indicated by the experimental data.

The patchy characteristics of the K_0 model are shown at the bottom of figure 10. Note that they are much more severe than indicated by the true turbulence time history.

The Non-Gaussian Model

The fact that the K_0 and gaussian models seem to "bracket" the characteristics of atmospheric turbulence led, in reference 34, to the joining of these two models as shown in figure 13. For want of a better name this combination is called simply the non-gaussian turbulence model. It will be shown in the next section of this report that it is possible to choose the transfer functions H_a , H_b , and H_c of figure 13 so that the time histories $c(t)$ and $d(t)$ have identical power spectral densities of the Dryden form. When this choice is made the simulated turbulence will also have the Dryden spectral density. Furthermore, this result will be independent of the parameter R which appears in figure 13. Thus R does not affect the power spectral density of the non-gaussian model, it does however influence its probability distribution.

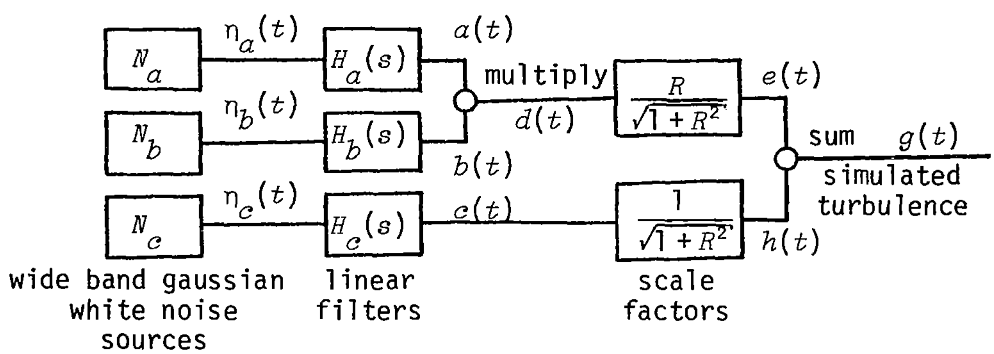


Figure 13.--Physical interpretation of a single component of the non-gaussian turbulence model. A similar interpretation applies to each of the three components.

To see the reason for this, suppose that the value of R is set to zero. Then the simulated turbulence will consist entirely of the gaussian time history, $c(t)$. As R is allowed to increase, more and more of the K_0 time history will be used. Finally, in the limit as the value of R

approaches infinity, the simulated turbulence will consist entirely of the K_0 process. Thus the parameter R is a control on the probability distribution and a continuum of distributions between gaussian and K_0 is available. Figure 14 shows the range of probability functions which can be obtained.

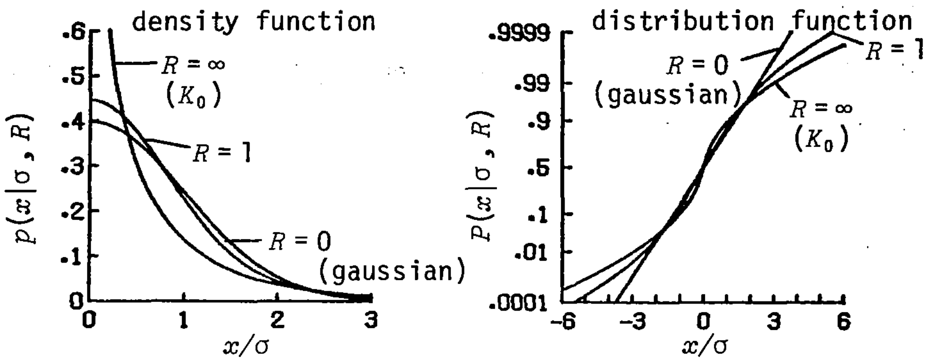


Figure 14.--Typical probability distributions attainable using the non-gaussian turbulence model.

As an indication of the patchy characteristics attainable using the non-gaussian model, figure 10 includes a sample of the derivative for the case R equal to unity. Note that it matches the characteristics of the true turbulence sample quite well.

Figure 13 is a physical interpretation of the non-gaussian model, and shows how it can be generated for numerical studies. The principal assumptions on which the model is based are the same as those of the gaussian model, except of course for the probability distribution. These assumptions are:

1) Each encounter of an aircraft with atmospheric turbulence can be modeled as a deterministic linear system (the aircraft) perturbed by three independent, stationary stochastic

processes. These processes represent the longitudinal, lateral, and vertical gust components at the vehicle center of gravity as it moves through the gust field.

2) The power spectral density of each random process belongs to a family of spectral shapes characterized by two parameters, the gust velocity standard deviation (σ), and the gust scale length (L). The scale length is a deterministic function of altitude and σ is a random variable which changes from encounter to encounter.

3) Each of the three gust components is a non-gaussian process of the form $u(t) = a(t)\,b(t) + c(t)$, where a, b, and c are independent, stationary gaussian processes.

The validity of the first two assumptions has been discussed in the introduction. The power spectral forms chosen for use in this report are those proposed by Dryden (Eq. 4, 5, 6). The third assumption is, of course, the central idea of the non-gaussian model. Its validity will be investigated in later sections of this report as the properties of the proposed model are compared with those of atmospheric turbulence.

ANALYSIS OF THE NON-GAUSSIAN MODEL

It is the purpose of this section to derive suitable expressions for the transfer functions used in the non-gaussian model, and to analyze the model's statistical properties. Specific results to be obtained for each component of the model are:

1) transfer functions
2) probability distribution
3) level crossing frequency
4) increment distribution.

Figure 13 presents the physical interpretation which can be applied to each of the model's three components. The notation of this figure will be used throughout the following development. The problem of vehicle response calculations will not be considered here, but will be taken up in a later section of this report.

Derivation of Transfer Functions

Recall from the introduction of this report that the turbulence models described here are to have the Dryden power spectral densities (Eq. 4, 5, 6). It will now be shown that it is possible to choose the filter transfer functions H_a, H_b, and H_c of figure 13 so that each component of the non-gaussian model has the appropriate spectral form. The method to be followed is:

1) Fourier transform the Dryden spectra to obtain the corresponding correlation functions
2) derive a general expression relating the correlation function of the non-gaussian model to the filter transfer functions

3) assume specific forms for the transfer functions involving unspecified constants

4) show that the constants can be chosen so that the correlation function of the non-gaussian model has the required Dryden form for each component.

In general, the correlation function is related to the power spectral density through a Fourier transform of the form

$$C(\tau) = \int_{-\infty}^{\infty} \Phi(\Omega) \exp(i\, U \,\Omega\, \tau)\, d\Omega \ . \tag{38}$$

Applying this transformation to the Dryden spectra, equations 4, 5, and 6 gives

$$C_{uu}(\tau) = \sigma_u{}^2 \exp\left(-\frac{U}{L_u}|\tau|\right) \tag{39}$$

$$C_{vv}(\tau) = \sigma_v{}^2 \left(1 - \frac{U|\tau|}{2L_v}\right) \exp\left(-\frac{U}{L_v}|\tau|\right) \tag{40}$$

$$C_{ww}(\tau) = \sigma_w{}^2 \left(1 - \frac{U|\tau|}{2L_w}\right) \exp\left(-\frac{U}{L_w}|\tau|\right) \ . \tag{41}$$

These equations complete the first step of the transfer function derivation. A general expression for the correlation function of the simulated turbulence time history of figure 13 will now be derived. By definition, this correlation function is

$$C_{gg}(\tau) = E\{g(t)\, g(t+\tau)\} \ . \tag{42}$$

From figure 13, $g(t)$ is

$$g(t) = a(t)\, b(t)\, R(1+R^2)^{-1/2} + c(t)(1+R^2)^{-1/2} \ . \tag{43}$$

Since a, b, and c are independent, zero mean random processes, the correlation function of $g(t)$ can be written as

$$C_{gg}(\tau) = C_{aa}(\tau)\, C_{bb}(\tau)\, R^2(1+R^2)^{-1} + C_{cc}(\tau)(1+R^2)^{-1} \quad . \qquad (44)$$

Now consider $a(t)$ alone. It follows from the general relationship between the input and output power spectral densities of a linear system (Ref. 38) that the spectral density of a has the form

$$\Phi_{aa}(\Omega) = |H_a(i\,U\Omega)|^2\ \Phi_{\eta_a\eta_a}(\Omega) \quad . \qquad (45)$$

If it is assumed that the power spectral density of the process $\eta_a(t)$ is unity over the range of frequencies for which H_a is not essentially zero, then

$$\Phi_{aa}(\Omega) = |H_a(i\,U\Omega)|^2 . \qquad (46)$$

The correlation function of $a(t)$ can be found by Fourier transforming equation 46 as shown in equation 38.

$$C_{aa}(\tau) = \int_{-\infty}^{\infty} |H_a(i\,U\Omega)|^2\ \exp(i\,U\Omega\tau)\, d\Omega \qquad (47)$$

Similar results apply to $b(t)$ and $c(t)$ of figure 13. This result, along with equation 44 completes the second step of the transfer function derivation.

For the third step of the procedure, suppose that the following general forms are assumed for the transfer functions of the model.

$$H_a(s) = \frac{N_1}{1+D_1\, s} \qquad (48)$$

$$H_b(s) = \frac{N_2 + N_3\, s}{(1 + D_2\, s)^2} \tag{49}$$

$$N_c(s) = \frac{N_4 + N_5\, s}{(1 + D_3\, s)^2} \tag{50}$$

These particular expressions have been chosen because they will lead to useful results. There is no reason why other general forms could not be chosen which might also give good results. (See the discussion of increment distributions in this section and suggestions for further research in a later section of this report.)

Transforming equations 48 through 50 as shown in equation 47 yields the following correlation functions.

$$C_{aa}(\tau) = \frac{N_1^{\,2}}{2D_1} \exp\left(-\frac{|\tau|}{D_1}\right) \tag{51}$$

$$C_{bb}(\tau) = \left(\frac{N_3}{2D_2}\right)^2 \left\{|\tau| \left[\left(\frac{N_2}{N_3}\right)^2 - \left(\frac{1}{D_2}\right)^2\right] + \left[\frac{1}{D_2} + D_2\left(\frac{N_2}{N_3}\right)^2\right]\right\} \exp\left(-\frac{|\tau|}{D_2}\right) \tag{52}$$

$$C_{cc}(\tau) = \left(\frac{N_5}{2D_3}\right)^2 \left\{|\tau| \left[\left(\frac{N_4}{N_5}\right)^2 - \left(\frac{1}{D_3}\right)^2\right] + \left[\frac{1}{D_3} + D_3\left(\frac{N_4}{N_5}\right)^2\right]\right\} \exp\left(-\frac{|\tau|}{D_3}\right) \tag{53}$$

Substitution of these results into equation 44 will give the general form of the correlation function of the non-gaussian model.

If now the following choices are made for the arbitrary constants in equations 51, 52, and 53

$$\left.\begin{array}{lll} N_1 = 4\sigma_u \dfrac{L_u}{U} & N_2 = 1.0 & N_3 = \dfrac{2L_u}{U} \\ N_4 = \sigma_u\left(\dfrac{2L_u}{U}\right)^{1/2} & N_5 = \sigma_u\left(\dfrac{2L_u^{\ 3}}{U^3}\right)^{1/2} & D_1 = \dfrac{2L_u}{U} \\ D_2 = \dfrac{2L_u}{U} & D_3 = \dfrac{L_u}{U} & \end{array}\right\} \quad (54)$$

the resulting correlation functions of the model become

$$C_{cc}(\tau) = C_{dd}(\tau) = C_{gg}(\tau) = \sigma_u^{\ 2} \exp\left(-\frac{U}{L_u}|\tau|\right) \quad (55)$$

which is the form of the u-gust correlation function, equation 39. Note that not only $g(t)$ but also $c(t)$ and $d(t)$ have the u-gust correlation function. The following result, which will be useful later in this section, is obtained directly from equation 55 by setting τ equal to zero.

$$\sigma_c^{\ 2} = \sigma_d^{\ 2} = \sigma_g^{\ 2} = \sigma_u^{\ 2} \quad (56)$$

If the constants of the filter transfer functions are chosen to be

$$\left.\begin{array}{lll} N_1 = \sigma_v(128)^{1/2}\left(\dfrac{L_v}{U}\right)^2 & N_2 = 0.0 & N_3 = 1.0 \\ N_4 = \sigma_v\left(\dfrac{L_v}{U}\right)^{1/2} & N_5 = \sigma_v\left(\dfrac{3L_v^{\ 3}}{U^3}\right)^{1/2} & D_1 = \dfrac{2L_v}{U} \\ D_2 = \dfrac{2L_v}{U} & D_3 = \dfrac{L_v}{U} & \end{array}\right\} \quad (57)$$

the resulting correlation functions are

$$C_{ee}(\tau) = C_{dd}(\tau) + C_{gg}(\tau) = \sigma_v^2 \left(1 - \frac{U|\tau|}{2L_v}\right) \exp\left(- \frac{U}{L_v} |\tau|\right) . \tag{58}$$

This is the desired form of the lateral gust correlation function, equation 40. Again, the time histories $e(t)$, $d(t)$ and $g(t)$ all have the same correlation function and

$$\sigma_e^2 = \sigma_d^2 = \sigma_g^2 = \sigma_v^2 . \tag{59}$$

Finally, if the constants are chosen to be

$$\begin{aligned} N_1 &= \sigma_w (128)^{1/2} \left(\frac{L_w}{U}\right)^2 & N_2 &= 0.0 & N_3 &= 1.0 \\ N_4 &= \sigma_w \left(\frac{L_w}{U}\right)^{1/2} & N_5 &= \sigma_w \left(\frac{3L_w^{\,3}}{U^3}\right)^{1/2} & D_1 &= \frac{2L_w}{U} \\ D_2 &= \frac{2L_w}{U} & D_3 &= \frac{L_w}{U} \end{aligned} \tag{60}$$

the correlation functions are

$$C_{ee}(\tau) = C_{dd}(\tau) = C_{gg}(\tau) = \sigma_w^2 \left(1 - \frac{U|\tau|}{2L_w}\right) \exp\left(- \frac{U}{L_w} |\tau|\right) . \tag{61}$$

This result is the Dryden form of the vertical gust correlation function. Again note that $e(t)$, $d(t)$ and $g(t)$ have the same correlation function, and therefore

$$\sigma_e^2 = \sigma_d^2 = \sigma_g^2 = \sigma_w^2 . \tag{62}$$

This completes the derivation of transfer functions for the non-gaussian model. The results are summarized in table 1.

Table 1.--Transfer functions of the non-gaussian model

Longitudinal Component		
$H_a(s)$	$H_b(s)$	$H_c(s)$
$\dfrac{4\sigma_u \dfrac{L_u}{U}}{1 + \dfrac{2L_u}{U} s}$	$\dfrac{1}{1 + \dfrac{2L_u}{U} s}$	$\dfrac{\sigma_u \left(\dfrac{2L_u}{U}\right)^{1/2}}{1 + \dfrac{L_u}{U} s}$
Lateral Component		
$H_a(s)$	$H_b(s)$	$H_c(s)$
$\dfrac{\sigma_u (128)^{1/2} \left(\dfrac{L_v}{U}\right)^2}{1 + \dfrac{2L_v}{U} s}$	$\dfrac{s}{\left[1 + \dfrac{2L_v}{U} s\right]^2}$	$\dfrac{\sigma_v \left(\dfrac{L_v}{U}\right)^{1/2} \left[1 + \sqrt{3}\dfrac{L_v}{U} s\right]}{\left[1 + \dfrac{L_v}{U} s\right]^2}$
Vertical Component		
$H_a(s)$	$H_b(s)$	$H_c(s)$
$\dfrac{\sigma_w (128)^{1/2} \left(\dfrac{L_w}{U}\right)^2}{1 + \dfrac{2L_w}{U} s}$	$\dfrac{s}{\left[1 + \dfrac{2L_w}{U} s\right]^2}$	$\dfrac{\sigma_w \left(\dfrac{L_w}{U}\right)^{1/2} \left[1 + \sqrt{3}\dfrac{L_w}{U} s\right]}{\left[1 + \dfrac{L_w}{U} s\right]^2}$

Two additional features of the above derivation justify comment. First, the forms of the transfer functions assumed here have been selected so that the correlation functions of the model do not depend upon the parameter R. Thus R will not influence the power spectral density of the model. As was discussed in the preceding section, this is a very convenient result which allows R to act solely as a control on the model's probability distribution. However, it will become apparent presently that the particular choices made here will have undesirable effects in so far as the distribution of increments is concerned. Further comment on this subject will be found in the discussion of increment distributions to be found in this section, and also in the suggestions for further research in a later section of this report.

A second point concerning the above derivation is that the choice of parameters made in equation 54 is somewhat arbitrary. In particular, the quantities D_1 and D_2 could be chosen to be any pair of numbers satisfying the relationship

$$\frac{1}{D_1} + \frac{1}{D_2} = \frac{U}{L_u} \tag{63}$$

This arbitrariness has been used in Reference 45 to define a "patchiness parameter" which can be included as an additional variable of the model. This complication is not considered in the present work, but is mentioned as an indication of the further generality which can be introduced into the non-gaussian model.

Probability Distribution

Attention is now turned to some aspects of the probabilistic structure of the non-gaussian model. The first topic to be discussed will be the model's probability distribution. The following remarks apply equally well to all three components of the proposed model; for the sake of definiteness, however, only the longitudinal gusts will be explicitly discussed.

Consider again the physical interpretation of the non-gaussian model presented in figure 13. The simulated turbulence time history is

$$g(t) = e(t) + h(t) \,, \tag{64}$$

where

$$e(t) = a(t)\,b(t)\,R(1+R^2)^{-1/2} \tag{65}$$

$$h(t) = c(t)(1+R^2)^{-1/2}. \tag{66}$$

The time histories $a(t)$, $b(t)$ and $c(t)$ are independent, zero mean, gaussian random processes. Thus each has a probability density of the form

$$p(x|\sigma) = \frac{1}{\sigma(2\pi)^{1/2}} \exp[-\tfrac{1}{2}(\tfrac{x}{\sigma})^2] \,. \tag{67}$$

The steps to be followed in deriving an expression for the probability distribution of the non-gaussian model are:

1) derive an expression for the probability density of the product $a(t)\,b(t)$ appearing in equation 65
2) using the density function from step 1, obtain an expression for the characteristic function of the first term of equation 64

3) find the characteristic function of the second term of equation 64

4) multiply the characteristic functions of steps 2 and 3 to obtain the characteristic function of the non-gaussian model

5) Fourier transform to obtain the density function of the non-gaussian model.

The first step is to find an expression for the probability density of the product $a(t)\,b(t)$. Since a and b are independent processes, it follows that the probability distribution function of their product is

$$P_{ab}(z|\sigma_a, \sigma_b) = \iint_{\xi} p_a(x|\sigma_a)\, p_b(y|\sigma_b)\, dx\, dy \quad , \tag{68}$$

where ξ denotes that region of the $x-y$ plane in which the condition $xy \leq z$ is satisfied. The density functions p_a and p_b are both of the gaussian form presented in equation 67. Substitution of equation 67 into equation 68 followed by differentiation with respect to z and finally integration gives the following result for the density function of $a(t)\,b(t)$.

$$p_{ab}(z|\sigma_a, \sigma_b) = \frac{1}{\pi\,\sigma_a\,\sigma_b} K_0\left(\frac{|z|}{\sigma_a\,\sigma_b}\right) , \tag{69}$$

where K_0 is the modified Bessel function of the second type and order zero. Equation 69 completes the first step of the derivation.

Now note that $\sigma_a\,\sigma_b$ is the standard deviation of the product $a(t)\,b(t)$ and, according to the notation of figure 13, this product is equivalent to $d(t)$. Also, equation 56 states that the standard deviation of $d(t)$ is equal to σ_u by virtue of the choice of transfer functions of the model.

Thus $\sigma_a \sigma_b$ is equal to σ_u, and equation 69 can be written

$$p_{ab}(z|\sigma_u) = \frac{1}{\pi\sigma_u} K_0\left(\frac{|z|}{\sigma_u}\right) . \tag{70}$$

The first term of equation 64 is, according to equation 65, the product of $a(t)\,b(t)$ scaled by the factor $(1+R^2)^{-1/2}$, so it follows immediately that

$$p_e(z|\sigma_u, R_u) = \frac{(1+R_u{}^2)^{1/2}}{\pi R_u \sigma_u} K_0\left[\frac{|z|(1+R_u{}^2)^{1/2}}{\pi R_u \sigma_u}\right] . \tag{71}$$

The characteristic function of $e(t)$ can be found by Fourier transforming equation 71.

$$\phi_e(f|\sigma_u, R_u) = \int_{-\infty}^{\infty} p_e(z|\sigma_u, R_u) \exp(i\, 2\pi f z)\, dz \tag{72}$$

The particular form of the Fourier transform employed in equation 72 was chosen for two reasons. First, it will lead to some simplification of the expressions to be derived presently. Secondly, and perhaps more importantly, a computer program implementing this transformation will be used for numerical studies later in this report.

Substitution of equation 71 into equation 72 and evaluation of the resulting integral gives

$$\phi_e(f|\sigma_u, R_u) = \{1 + [2\pi R_u \sigma_u (1+R_u{}^2)^{-1/2} f]^2\}^{-1/2}. \tag{73}$$

Equation 73 completes the second step of the derivation.

Now consider the second term on the right hand side of equation 64. The process $h(t)$ is clearly gaussian because $c(t)$ is gaussian, and its standard deviation is just that of $e(t)$ scaled by the factor $(1+R^2)^{-1/2}$. From equation 56 it is known that σ_e is identical to σ_u. Thus the probability density of $h(t)$ is

$$p_h(z|\sigma_u, R_u) = \frac{(1+R_u^2)^{1/2}}{\sigma_u(2\pi)^{1/2}} \exp\left[-\frac{z^2(1+R_u^2)}{2\sigma_u^2}\right]. \tag{74}$$

The characteristic function of h is found by Fourier transforming p_h in the same manner as equation 72.

$$\phi_h(f|\sigma_u, R_u) = \exp\{-2[\pi\sigma_u(1+R_u^2)^{-1/2} f]^2\} \tag{75}$$

This result completes the third step of the derivation.

Since $e(t)$ and $h(t)$ are independent random processes, it follows that the characteristic function of their sum is given by the product of their respective characteristic functions. Thus, for the case of the u-gust component of the non-gaussian model

$$\phi_g(f|\sigma_u, R_u) = \left[1 + \frac{(2\pi\sigma_u R_u f)^2}{1+R_u^2}\right]^{-1/2} \exp\left[-\frac{2(\pi\sigma_u f)^2}{1+R_u^2}\right]. \tag{76}$$

The characteristic functions of the vertical and lateral components of the model are obtained merely by replacing the subscript u of equation 76 with w and v respectively. This completes the fourth step of the derivation.

The probability density of the non-gaussian model can now be obtained by inverse Fourier transforming equation 76.

$$p_g(x|\sigma_u, R_u) = \int_{-\infty}^{\infty} \phi_g(f|\sigma_u, R_u) \exp(-i2\pi fx)\, dx \tag{77}$$

Equation 77 is easily evaluated numerically by means of the fast Fourier transform program *FFT* which will be found in the appendix of this report. Similar equations apply to each component of the model. The probability distribution function can be obtained directly from the density function of equation 77 by means of integration.

$$P_g(x|\sigma_u, R_u) = \int_{-\infty}^{x} p_g(y|\sigma_u, R_u)\, dy \qquad (78)$$

Figure 15 presents the density and distribution functions of the non-gaussian model for several values of the parameter R. Tabulated values of these functions will be found in tables B1 and B2 of Appendix B.

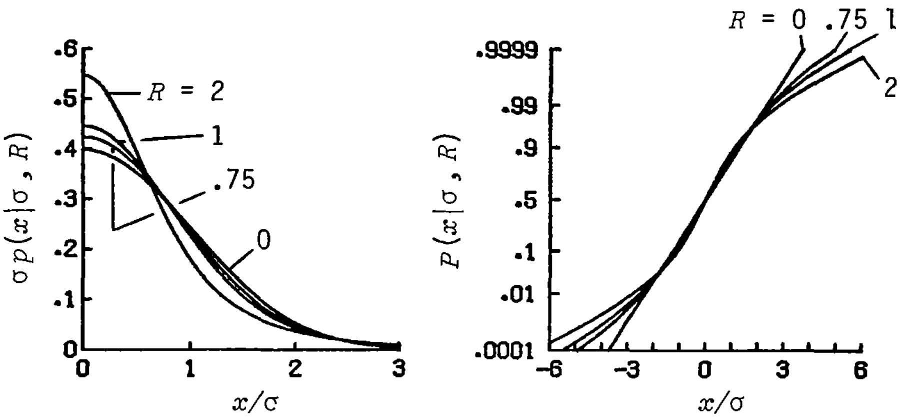


Figure 15.--Probability density and distribution functions of the non-gaussian turbulence model for various values of the parameter R.

In the next section of this report it will be of interest to compute the probability distributions which would be obtained by combining data from many samples of the non-gaussian model. It will be assumed that the parameter R is fixed, but σ will be allowed to vary randomly from sample to sample. If σ is distributed over the samples with the density function

p_σ, the probability density of the non-gaussian model based on all of the samples will be

$$p_g(x|R_u) = \int_0^\infty p_g(x|y, R_u)\, p_\sigma(y)\, dy \ , \tag{79}$$

and the cumulative gust velocity distribution of the model will be

$$P_g^{\,c}(x|R_u) = 1 - \int_{-\infty}^{\infty} p_g(y|R_u)\, dy \ . \tag{80}$$

Level Crossing Frequency

The level crossing frequency of the non-gaussian model will now be investigated. The basic relation used here will be the well known result of reference 36,

$$N_g(x_1|\sigma_g, \sigma_{\dot{g}}) = \frac{1}{U}\int_0^\infty x_2\, p_{g,\dot{g}}(x_1, x_2|\sigma_g, \sigma_{\dot{g}})\, dx_2 \ . \tag{81}$$

N_g is the number of crossings with positive slope of the level x_1 per unit distance of flight. The function $p_{g,\dot{g}}(x_1, x_2)$ is the joint density of the turbulence time history and its first derivative.

Just as in the case of the gaussian model considered previously in this report, a problem arises in the application of this equation to the non-gaussian model because the first derivative of a stochastic process having one of the Dryden spectral densities does not exist. This difficulty can be overcome in a manner similar to that of the gaussian model by adding high frequency poles to the transfer functions of the non-gaussian model. These new poles can be looked upon as the effect of viscosity at very short wavelengths. It will be shown that, similar to the case of the gaussian model treated previously in this report, these poles have no effect whatsoever on the probability distribution of the model, and act only as a scale factor of the level crossing frequency. Since vehicle responses virtually never involve such high frequencies, these poles need not be considered in response calculations.

The following discussion will at first be restricted to only the longitudinal gusts. The results will then be extended to the vertical and lateral components. The principal steps of the procedure are:

1) add high frequency poles to the transfer functions of the turbulence model
2) derive expressions for the standard deviation of the gust component and its first time derivative
3) examine the effect of these new poles on the spectral density of the model, showing that the spectra are still essentially the Dryden forms
4) use the results of step 2 to obtain an expression for the joint characteristic function of the gust component and its first derivative
5) inverse Fourier transform the joint characteristic function of step 4 to obtain the joint density of the gust time history and its first derivative
6) apply equation 81 to the joint density of step 5 to obtain the level crossing frequency of the non-gaussian model
7) review the results of steps 4 through 6 in order to determine the dependence of the level crossing frequency upon the poles added to the transfer functions in step 1 and the parameters σ_u , L_u ,

U, and R_u which appear in the result.

8) determine a set of universal curves from which the level crossing frequency for any set of parameters can be found without resorting to the complete process described above.

Level Crossing Frequency of the Longitudinal Component

Recall the three linear filters used to generate the longitudinal gusts (table 1, page 55). The spectrum produced by these filters is the Dryden u-gust form, equation 4. As was pointed out in the discussion of equation 13, this spectrum behaves like Ω^{-2} at high frequencies. Therefore the first derivative of the u-gust random process is not defined. In order to overcome this problem high frequency poles are added to the transfer functions as shown in equations 82, 83, and 84.

$$H_a(s) = \frac{4\sigma_u^2 \, k_d \frac{L_u}{U}}{\left(1 + \frac{2L_u s}{U}\right)\left(1 + \frac{\Upsilon_u s}{U}\right)} \tag{82}$$

$$H_b(s) = \frac{1}{\left(1 + \frac{2L_u s}{U}\right)\left(1 + \frac{\Upsilon_u s}{U}\right)} \tag{83}$$

$$H_c(s) = \frac{\sigma_u \, k_c \left(\frac{2L_u}{U}\right)^{1/2}}{\left(1 + \frac{L_u s}{U}\right)\left(1 + \frac{\Upsilon_u s}{U}\right)} \tag{84}$$

The constants k_c and k_d are scaling factors which will be chosen so as to correct the effects of γ_u on the standard deviation of $c(t)$ and $d(t)$. The power spectral density of $d(t)$ can be found by substituting first equation 82 and then 83 into equation 47, multiplying the results to obtain the correlation function of $d(t)$, and finally Fourier transforming to obtain the power spectral density. The result is

$$\Phi_{dd}(\Omega) = \frac{(k_d\,\sigma_u)^2}{\left[1 - \left(\frac{\gamma_u}{2L_u}\right)^2\right]^2} \left\{ \frac{L_u}{\pi\left[1 + (L_u\,\Omega)^2\right]} - \frac{\left[\frac{2\gamma_u^{\,2}}{\pi(2L_u + \gamma_u)}\right]}{1 + \left(\frac{2\gamma_u\,L_u\,\Omega}{2L_u + \gamma_u}\right)^2} + \frac{\frac{\gamma_u^{\,3}}{8\pi\,L_u^{\,2}}}{1 + \left(\frac{\gamma_u\Omega}{2}\right)^2} \right\}. \tag{85}$$

The power spectral density of $c(t)$ is obtained by substituting equation 82 into equation 47 and Fourier transforming.

$$\Phi_{cc}(\Omega) = \frac{(k_c\,\sigma_u)^2\,L_u}{\pi[1 + (L_u\,\Omega)^2][1 + (\gamma_u\,\Omega)^2]} \tag{86}$$

The constants k_c and k_d are now chosen so that the standard deviations of both $c(t)$ and $d(t)$ are equal to σ_u.

$$k_c = \left(1 + \frac{\gamma_u}{L_u}\right)^{1/2} \tag{87}$$

$$k_d = 1 + \frac{\gamma_u}{2L_u} \tag{88}$$

$$\sigma_c = \sigma_d = \sigma_u \tag{89}$$

It can also be verified that, with these values of k_c and k_d, the standard deviations of $\dot{c}(t)$ and $\dot{d}(t)$ are equal to

$$\sigma_{\dot{c}} = \sigma_{\dot{d}} = U\,\sigma_u(\gamma_u\,L_u)^{-1/2}. \tag{90}$$

These results imply that the standard deviation of the non-gaussian model and its derivative are

$$\sigma_g = \sigma_u \tag{91}$$

$$\sigma_{\dot{g}} = U\sigma_u(\gamma_u L_u)^{-1/2} . \tag{92}$$

Equations 89 through 92 complete the second step of the derivation.

The power spectral density of the model is

$$\Phi_{gg}(\Omega) = \Phi_{dd}(\Omega) R_u^{\,2}(1+R_u^{\,2})^{-1} + \Phi_{cc}(\Omega)(1+R_u^{\,2})^{-1} . \tag{93}$$

Substitution of equations 85 and 86 into equation 93 gives an expression for Φ_{gg} which is far more complicated than the simple Dryden spectral density. However, on the assumption that γ_u is very much smaller than L_u, the power spectral density of the non-gaussian model becomes

$$\Phi_{gg}(\Omega) = \frac{\dfrac{\sigma_u^{\,2} L_u}{\pi}\left(1 + \dfrac{\gamma_u}{L_u}\right)}{\left[1 + (L_u\,\Omega)^2\right]\left[1 + (\gamma_u\,\Omega)^2\right]} + O\left\{\left(\frac{\gamma_u}{L_u}\right)^2\right\} . \tag{94}$$

Equation 94 is, to second order in γ_u/L_u, the Dryden form of the u-gust power spectral density with an additional high frequency pole. This is precisely the result required in order to insure the existance of $\dot{g}(t)$, and equation 94 completes the third step of the level crossing derivation.

Note that the standard deviations of $c(t)$ and $d(t)$ have not been altered by the addition of γ_u. Thus the probability distributions discussed earlier in this section remain unchanged.

The level crossing frequency of the non-gaussian model will now be developed using the modified transfer functions, equations 82, 83, and 84. Consider the joint characteristic function of $g(t)$ and its first derivative. By definition this is

$$\Phi_{g,\dot{g}}(f_1, f_2) = E\{\exp[i2\pi f_1 g(t) + i2\pi f_2 \dot{g}(t)]\} \ . \tag{95}$$

In terms of the processes $a(t)$, $b(t)$, and $c(t)$ used in the non-gaussian model, $g(t)$ and $\dot{g}(t)$ are

$$g(t) = a(t)\,b(t)\,R_u(1 + R_u{}^2)^{-1/2} + c(t)(1 + R_u{}^2)^{-1/2} \tag{96}$$

$$\dot{g}(t) = [\dot{a}(t)\,b(t) + a(t)\,\dot{b}(t)]\,R_u(1 + R_u{}^2)^{-1/2} + \dot{c}(t)(1 + R_u{}^2)^{-1/2} \ . \tag{97}$$

Substitution of these two expressions into equation 95 will give an expression for $\Phi_{g,\dot{g}}$ in terms of a, $\dot{a}$, b, $\dot{b}$, c, and $\dot{c}$. If now the joint density of these random processes can be determined, the expected value appearing in equation 95 can be evaluated and the joint characteristic function found.

Note that the processes a, b, and c are zero mean, independent, and gaussian. Since the derivative of a gaussian process is always gaussian, it follows that $\dot{a}$, $\dot{b}$, and $\dot{c}$ are zero mean, independent, and gaussian. Furthermore, since a random process and its first derivative are always uncorrelated, it follows from their gaussian nature that a, $\dot{a}$, b, $\dot{b}$, c, and $\dot{c}$ are mutually independent. The joint density function of these processes can thus be written directly.

$$p_{a,\dot{a},b,\dot{b},c,\dot{c}}(x_1, x_2, x_3, x_4, x_5, x_6 | \sigma_a, \sigma_{\dot{a}}, \sigma_b, \sigma_{\dot{b}}, \sigma_c, \sigma_{\dot{c}}) =$$

$$\frac{\exp\left\{-\frac{1}{2}\left[\left(\frac{x_1}{\sigma_a}\right)^2 + \left(\frac{x_2}{\sigma_{\dot{a}}}\right)^2 + \left(\frac{x_3}{\sigma_b}\right)^2 + \left(\frac{x_4}{\sigma_{\dot{b}}}\right)^2 + \left(\frac{x_5}{\sigma_c}\right)^2 + \left(\frac{x_6}{\sigma_{\dot{c}}}\right)^2\right]\right\}}{(2\pi)^3 \sigma_a \sigma_{\dot{a}} \sigma_b \sigma_{\dot{b}} \sigma_c \sigma_{\dot{c}}} \tag{98}$$

The standard deviations appearing in equation 98 can be found from the filter transfer functions, equations 82, 83, and 84.

$$\left.\begin{aligned}
\sigma_a &= \frac{\sigma_u}{L_u}\left[\frac{U(2L_u+\gamma_u)}{2}\right]^{1/2} & \sigma_{\dot{a}} &= \frac{U\sigma_u}{2L_u}\left[\frac{U(2L_u+\gamma_u)}{\gamma_u L_u}\right]^{1/2} \\
\sigma_b &= L_u\left[\frac{2}{U(2L_u+\gamma_u)}\right]^{1/2} & \sigma_{\dot{b}} &= \left[\frac{U L_u}{\gamma_u(2L_u+\gamma_u)}\right]^{1/2} \\
\sigma_c &= \sigma_u & \sigma_{\dot{c}} &= U\sigma_u(\gamma_u L_u)^{-1/2}
\end{aligned}\right\} \tag{99}$$

Equations 96 through 99 can now be combined to evaluate the joint characteristic function of the non-gaussian model, equation 95.

$$\phi_{g,\dot{g}}(f_1, f_2 | \sigma_a, \sigma_{\dot{a}}, \sigma_b, \sigma_{\dot{b}}, \sigma_c, \sigma_{\dot{c}}, R_u) =$$

$$\int_{-\infty}^{\infty}(6)\int_{-\infty}^{\infty} \exp\left\{i2\pi f_1\left[\frac{R_u x_1 x_3 + x_5}{(1+R_u{}^2)^{1/2}}\right] + i2\pi f_2\left[\frac{R_u(x_1 x_4 + x_2 x_3) + x_6}{(1+R_u{}^2)^{1/2}}\right]\right\} \cdot$$

$$p_{a,\dot{a},b,\dot{b},c,\dot{c}}(x_1, x_2, x_3, x_4, x_5, x_6 | \sigma_a, \sigma_{\dot{a}}, \sigma_b, \sigma_{\dot{b}}, \sigma_c, \sigma_{\dot{c}})\, dx_1 \ldots dx_6 \tag{100}$$

This completes the fourth step of the derivation.

Equation 100 can be integrated to give a closed form algebraic expression for the joint characteristic function of the non-gaussian u-gust component and its derivative. The joint density function required for the evaluation of the level crossing frequency can then be found, at least in principle, by means of the Fourier transformation

$$p_{g,\dot{g}}(x_1, x_2 | \sigma_a, \sigma_{\dot{a}}, \sigma_b, \sigma_{\dot{b}}, \sigma_c, \sigma_{\dot{c}}, R_u) =$$
$$\iint_{-\infty}^{\infty} \phi_{g,\dot{g}}(f_1, f_2 | \sigma_a, \sigma_{\dot{a}}, \sigma_b, \sigma_{\dot{b}}, \sigma_c, \sigma_{\dot{c}}, R_u) \exp(-i2\pi f_1 x_1 - 2\pi f_2 x_2)\, df_1\, df_2 . \tag{101}$$

Evaluation of this expression would complete the fifth step of the derivation. Finally, the level crossing frequency can be found by means of equation 79.

In reality, of course, the series of integrations required to pass from equation 100 to the evaluation of the level crossing frequency are extremely tedious if not overwhelmingly difficult. Fortunately, a far more convenient means of solution is available through use of a digital computer and the appropriate numerical methods. Program LEVXNG, listed in the appendix of this report, is one means of performing these calculations. This program was, in fact, used to compute the level crossing results to be presented in this section.

Program LEVXNG permits the rapid calculation of N_g for any set of parameters σ_u, L_u, U, R_u, and γ_u which might be of interest. However, merely calculating results for various cases does not yield much useful information concerning the dependence of the solution on its parameters. Furthermore, every time any of the parameters is changed, it becomes necessary to repeat the entire calculation in order to obtain the new level crossing frequencies. Thus it would clearly be quite useful to have some idea as to how a change of parameters will affect the resulting level crossings of the model. With this in mind, a set of universal curves depending only upon the parameter R_u will now be derived. It will be shown that the level crossing frequency for any choice of the parameters σ_u, L_u, U, and γ_u can be found from these curves.

It is possible to integrate equation 100 with respect to x_5 and x_6 with a minimum of difficulty. The results of these integrations can be factored from the remaining integrals and identified as the joint characteristic function of c and $\dot{c}$. Equation 100 can then be rewritten in the form shown in equation 102. (For the sake of simplicity and clarity, the explicit parameter dependence notation will be temporarily suspended.)

$$\phi_{g,\dot{g}}(f_1, f_2) = \phi_{c,\dot{c}}(f_1, f_2) \int_{-\infty}^{\infty}(4)\int_{-\infty}^{\infty} \frac{\exp(\bar{x}^T \tilde{A}\,\bar{x})\, d\bar{x}}{(2\pi)^2 \sigma_a \sigma_{\dot{a}} \sigma_b \sigma_{\dot{b}}}$$

where

$$\phi_{g,\dot{g}}(f_1, f_2) = \exp\left[- \frac{2(\pi \sigma_c f_1)^2}{1 + R_u^{\,2}} - \frac{2(\pi \sigma_{\dot{c}} f_2)^2}{1 + R_u^{\,2}} \right] \qquad (102a)$$

$$\bar{x} = [x_1 \; x_2 \; x_3 \; x_4]^T$$

$$\tilde{A} = \begin{bmatrix} -(2\sigma_a^{\,2})^{-1} & 0 & i2\pi f_1 R_u^{\,*} & i2\pi f_2 R_u^{\,*} \\ 0 & -(2\dot{\sigma}_a^{\,2})^{-1} & i2\pi f_2 R_u^{\,*} & 0 \\ i2\pi f_2 R_u^{\,*} & i2\pi f_2 R_u^{\,*} & -(2\sigma_b^{\,2})^{-1} & 0 \\ i2\pi f_2 R_u^{\,*} & 0 & 0 & -(2\dot{\sigma}_b^{\,2})^{-1} \end{bmatrix}$$

$$R_u^{\,*} = R_u(1+R_u^{\,2})^{-1/2} \qquad (102b)$$

Now suppose that a special case of the joint characteristic function is computed. Let all of the parameters with the exception of R_u be set to unity.

$$\sigma_u = L_u = U = \Upsilon_u = 1.0 \qquad (103)$$

Denote this case by $\hat{\phi}_{g,\dot{g}}$. Note that $\hat{\phi}$ is a function only of f_1 , f_2 , and R_u .

Consider again equation 102 and introduce the following change of variables:

$$x_1' = \frac{x_1}{\sigma_a}\left(\frac{3}{2}\right)^{1/2} \qquad x_2' = \frac{x_2}{\dot{\sigma}_a}\left(\frac{3}{4}\right)^{1/2}$$

$$x_3' = \frac{x_3}{\sigma_b}\left(\frac{2}{3}\right)^{1/2} \qquad x_4' = \frac{x_4}{\dot{\sigma}_b}\left(\frac{1}{3}\right)^{1/2} \qquad (104)$$

A comparison of this transformed expression with the special case $\hat{\phi}$, taking note of the equalities

$$\left.\begin{aligned} \sigma_a \sigma_b &= \sigma_c \\ \sigma_{\dot a} \sigma_b 2^{1/2} &= \sigma_a \sigma_{\dot b} 2^{1/2} = \sigma_{\dot c} \end{aligned}\right\} \quad (105)$$

of equation 99, leads to the conclusion

$$\phi_{g,\dot g}(f_1, f_2) = \hat{\phi}_{g,\dot g}(f_1 \sigma_c, f_2 \sigma_{\dot c}) \ . \quad (106)$$

The joint density function of g and $\dot g$ is obtained by means of equation 101. Substitution of equation 106 for $\phi_{g,\dot g}$ in equation 101 and introduction of the variables

$$\left.\begin{aligned} f_1' &= f_1 \sigma_c \\ f_2' &= f_2 \sigma_{\dot c} \end{aligned}\right\} \quad (107)$$

gives

$$p_{g,\dot g}(x_1, x_2) = \frac{1}{\sigma_c \sigma_{\dot c}} \hat{p}_{g,\dot g}\left(\frac{x_1}{\sigma_c}, \frac{x_2}{\sigma_{\dot c}}\right) . \quad (108)$$

The notation $\hat{p}_{g,\dot g}$ is used to signify the joint density corresponding to the joint characteristic function $\hat{\phi}_{g,\dot g}$.

The level crossing frequency of the non-gaussian model can now be found by substituting equation 108 into equation 81 and performing the change of variable $x_2' = x_2/\sigma_{\dot c}$. The result is

$$N_g(x) = \frac{\sigma_{\dot c}}{U \sigma_c} \hat{N}_g\left(\frac{x}{\sigma_c}\right) \quad (109)$$

where $\hat{N}_g$ is the level crossing frequency of the special case.

The explicit parameter notation can now be reinstated. Substitute for σ_c and $\sigma_{\dot{c}}$ from equation 99 and recall that the special case $\hat{N}_g$ depends only on the parameter R_u . The final expression for N_g is

$$N_g(x|\sigma_u, L_u, R_u, \gamma_u) = (L_u \gamma_u)^{-1/2} \hat{N}_g\left(\frac{x}{\sigma_u}|R_u\right) . \tag{110}$$

Note the similarity of equation 110 to the level crossing frequency of the gaussian model, equation 16. Just as in equation 16, the crossing frequency is inversely proportional to the square root of γ_u . Thus γ_u can be interpreted as a control on the zero crossing frequency of the non-gaussian model, and could be chosen so as to match equation 110 to some observed data. For the purposes of the present report, however, the value of γ_u can be allowed to remain arbitrary.

Equation 110 implies that once the level crossing frequency of the special case has been computed for a given value of R_u , the result for any choice of σ_u , L_u , and γ_u can be found directly. Alternatively, the equation states that all u-gust level crossing frequencies of the non-gaussian model can be reduced to a single function through the indicated scaling. In order to show that these results apply only to the u-gust component of the model, the subscript g will now be replaced by u .

The function $\hat{N}_u(x|R_u)$ has been computed using program LEVXNG which is described in Appendix A of this report. Figure 16 and table B3 (of Appendix B) present the results for several values of the parameter R_u.

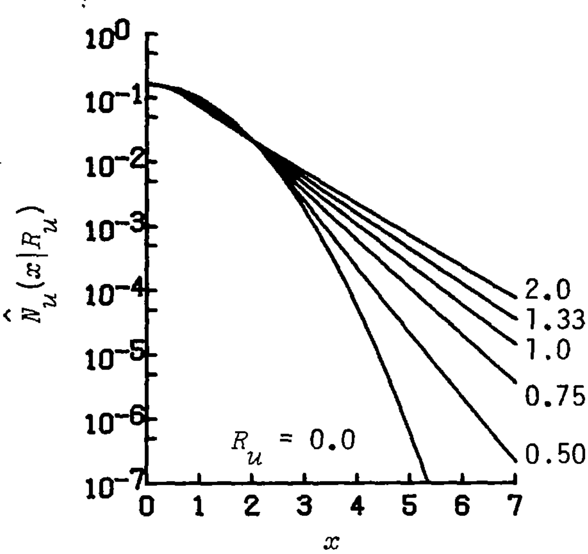


Figure 16.--Level crossing frequencies of the non-gaussian turbulence model longitudinal component for various values of the parameter R_u.

In the next section it will be of interest to evaluate the cumulative probability of exceedance which would result from combining data from a number of samples of the non-gaussian model. As in the case of the gaussian model described previously, it will be assumed that the parameters σ_u, L_u, and γ_u vary randomly from sample to sample. The standard deviation, σ_u, will be assumed independent of L_u and γ_u; and the parameter R_u will be required to remain constant over all the samples.

Recall from the discussion of the gaussian model that the cumulative probability of exceedance is calculated by first finding the average level crossing frequency for all of the samples, then normalizing with respect to the average zero crossing frequency. In view of the above assumptions, the cumulative probability of exceedance of the u-gust component is

$$\frac{N_u(x|R_u)}{N_u(0|R_u)} = \frac{\int\int\int_0^\infty N_u(x|\sigma, L, R_u, \gamma)\, p_{\sigma_u}(\sigma)\, p_{L_u, \gamma_u}(L, \gamma)\, d\sigma\, dL\, d\gamma}{\int\int\int_0^\infty N_u(0|\sigma, L, R_u, \gamma)\, p_{\sigma_u}(\sigma)\, p_{L_u, \gamma_u}(L, \gamma)\, d\sigma\, dL\, d\gamma}, \quad (111)$$

where p_{σ_u} and p_{L_u, γ_u} are the probability density functions of σ_u, L_u, and γ_u. Equation 111 can be greatly simplified by substituting equation 110 for N_u. The result is

$$\frac{N_u(x|R_u)}{N_u(0|R_u)} = \hat{N}_u(0|R_u)^{-1} \int_0^\infty \hat{N}_u(\tfrac{x}{\sigma}|R_u)\, p_{\sigma_u}(\sigma)\, d\sigma . \quad (112)$$

Note that it is not necessary to know the distribution of either L_u or γ_u in order to evaluate equation 112, a result analogous to that for the gaussian model (Eq. 21).

This completes the analysis of the longitudinal gust level crossings. The methods applied above will now be extended to the vertical and lateral gust components.

Level Crossing Frequency of Lateral and Vertical Components

The level crossings of the lateral gust component will now be considered. Since the vertical and lateral components of the non-gaussian model use linear filters of the same form, all results for the lateral gusts will apply directly to the vertical gusts.

Recall from table 1 the transfer functions to be used in generating the lateral gust component. The power spectral density produced by these filters is the Dryden v-gust form, equation 5. Just as in the u-gust case considered above, the derivative of a process with the Dryden spectrum is not defined. In order to avoid this problem, the spectrum is modified through the addition of high frequency poles to the transfer functions.

$$H_a(s) = \frac{\sigma_v (128)^{1/2} \left(\frac{L_v}{U}\right)^2 k_d}{\left(1 + \frac{2L_v s}{U}\right)\left(1 + \frac{\gamma_v s}{U}\right)} \tag{113}$$

$$H_b(s) = \frac{s\, k_d^{\,2}}{\left(1 + \frac{L_v s}{U}\right)^2 \left(1 + \frac{\gamma_v s}{U}\right)} \tag{114}$$

$$H_c(s) = \frac{\sigma_v \left(\frac{L_v}{U}\right)^{1/2} k_c \left[1 + 3^{1/2}\, \frac{L_v s}{U}\right]}{\left(1 + \frac{L_v s}{U}\right)^2 \left(1 + \frac{\nu_v s}{U}\right)} \tag{115}$$

Unlike the u-gust case, it is convenient to use two parameters, γ_v and ν_v, rather than only one. The constants k_c and k_d are scaling factors which will be chosen so as to correct the effects of γ_v and ν_v on the standard deviations of $c(t)$ and $d(t)$.

The power spectral densities of c and d can be derived using the above transfer functions. The condition

$$\sigma_c^{\,2} = \sigma_d^{\,2} = \sigma_v^{\,2} \tag{116}$$

is satisfied by choosing

$$k_c = \frac{1 + \frac{\nu_v}{L_v}}{\left(1 + \frac{\nu_v}{2L_v}\right)^{1/2}} \tag{117}$$

and

$$k_d = \left(1 + \frac{\gamma_v}{2L_v}\right)^{1/2} . \tag{118}$$

The standard deviations of c and d can be made equal by choosing ν_v to be a function of γ_v.

$$\nu_v = \frac{3\gamma_v L_v - 6L_v^2 + 6L_v^2\left(1 + \frac{\gamma_v}{L_v} + \frac{7\gamma_v^2}{12L_v}\right)^{1/2}}{6L_v + \gamma_v} \tag{119}$$

The resulting standard deviations are

$$\sigma_{\dot{c}}^2 = \sigma_{\dot{d}}^2 = \sigma_v^2\left(\frac{U}{2L_v}\right)^2\left(\frac{6L_v + \gamma_v}{\gamma_v}\right) . \tag{120}$$

With the above choices of parameters, and assuming the condition $\gamma_v \ll L_v$, the power spectral density of the lateral gust component becomes

$$\Phi_{vv}(\Omega) = \frac{\sigma_v^2 L_v\left(1 + \frac{3\gamma_v}{2L_v}\right)[1 + 3(L_v\Omega)^2]}{\pi[1 + (L_v\Omega)^2]^2[1 + (\gamma_v\Omega)^2]} + O\left\{\left(\frac{\gamma_v}{L_v}\right)^2\right\} . \tag{121}$$

Now consider the level crossing frequency of the lateral component. Just as in the treatment of the longitudinal gust, an expression for the joint characteristic function of v and $\dot{v}$ will be derived. This expression will be formally identical to equation 102, but in the present case the standard deviations are

$$
\left.\begin{aligned}
\sigma_a &= 2\sigma_v\left(\frac{2L_v}{U}\right)^{3/2} & \sigma_{\dot a} &= 4L_v\,\sigma_v\left(U\Upsilon_v\right)^{-1/2} \\
\sigma_b &= \frac{1}{2}\left(\frac{U}{2L_v}\right)^{3/2} & \sigma_{\dot b} &= \left(\frac{U}{2L_v}\right)^{5/2}\left(\frac{L_v}{\Upsilon_v}\right)^{1/2}\left(1+\frac{\Upsilon_v}{4L_v}\right)^{1/2} \\
\sigma_c &= \sigma_v & \sigma_{\dot c} &= \sigma_v\left(\frac{U}{2L_v}\right)\left(\frac{6L_v}{\Upsilon_v}\right)^{1/2}\left(1+\frac{\Upsilon_v}{6L_v}\right)^{1/2} .
\end{aligned}\right\} \qquad (122)
$$

Using equation 122 and computer program LEVXNG listed in the appendix, it is possible to calculate exact results for the level crossing frequency. However, a more convenient solution of the problem can be found by introducing an approximation for $\sigma_{\dot b}$ and $\sigma_{\dot c}$ of equation 122 which will allow the derivation of universal curves similar to those for the longitudinal case (Fig. 16). On the assumption that L_v is much larger than Υ_v , the expressions for $\sigma_{\dot b}$ and $\sigma_{\dot c}$ of equation 122 become

$$
\left.\begin{aligned}
\sigma_{\dot b} &\doteq \left(\frac{U}{2L_v}\right)^{5/2}\left(\frac{L_v}{\Upsilon_v}\right)^{1/2} \\
\sigma_{\dot c} &\doteq \sigma_v\left(\frac{U}{2L_v}\right)\left(\frac{6L_v}{\Upsilon_v}\right)^{1/2} .
\end{aligned}\right\} \qquad (123)
$$

Employing these approximate results and following the procedure described above for the longitudinal gust component, derive the joint characteristic function of the special case

$$
\sigma_v = L_v = U = \Upsilon_v = 1.0 , \qquad (124)
$$

and denote this result by $\hat{\Phi}_{v,\dot v}$. Return to equation 102 and introduce the change of variables

$$x_1' = \frac{x_1}{\sigma_a} 2^{5/2} \qquad x_2' = \frac{x_2}{\sigma_{\dot{a}}} 4$$

$$x_3' = \frac{x_3}{\sigma_b} 2^{-5/2} \qquad x_4' = \frac{x_4}{\sigma_{\dot{b}}} 2^{-5/2} . \tag{125}$$

Comparison of this result with $\hat{\phi}_{v,\dot{v}}$ gives

$$\phi_{v,\dot{v}}(f_1 , f_2) = \hat{\phi}_{v,\dot{v}}(f_1 \sigma_c , f_2 \sigma_{\dot{c}}) . \tag{126}$$

Continuation of the same procedure used for the u-gust case leads to the final result

$$N_v(x|\sigma_v , L_v , R_v , \gamma_v) = \left(\frac{3}{2L_v \gamma_v}\right)^{1/2} \hat{N}_v\left(\frac{x}{\sigma_v} |R_v\right) , \tag{127}$$

which is valid for $\gamma_v << L_v$. The level crossing frequency of the vertical gust component can be obtained by replacing the subscript v of equations 113 through 127 with the subecript w.

The special cases $\hat{N}_v$ and $\hat{N}_w$ have been computed by means of program LEVXNG. Figure 17 and table B4 (of Appendix B) present these results for several values of the parameter R.

A comparison of figures 16 and 17 reveals that the special cases appear to differ by only a constant of proportionality which depends weakly upon the parameter R. This suggests that a transformation exists such that the level crossing frequencies of all three components of the non-gaussian model can be expressed in terms of a single set of universal curves. This transformation is not derived here, but the possibility of its existence is mentioned as a point of interest.

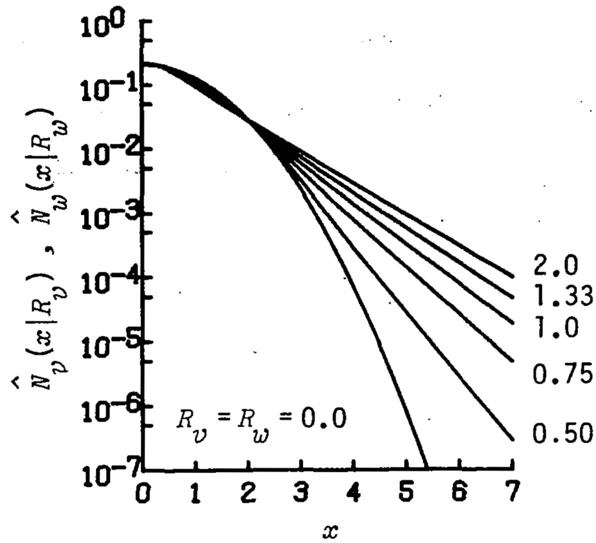


Figure 17.--Level crossing frequencies of the non-gaussian turbulence model lateral and vertical components for various values of the parameter R.

In the next section of this report it will be necessary to calculate cumulative probability of exceedance curves for the vertical and lateral gust components. Following the procedure described previously for the longitudinal components gives

$$\frac{N_v(x|R_v)}{N_v(0|R_v)} = \hat{N}_v(0|R_v)^{-1} \int_0^\infty \hat{N}_v\left(\frac{x}{\sigma}|R_v\right) p_{\sigma_v}(\sigma)\, d\sigma \ . \tag{128}$$

Just as in the u-gust case, the distribution of L_v and γ_v need not be known in order to evaluate this expression. The result for the vertical gusts can be obtained by replacing the subscript v of this equation with the subscript w. Equation 128 completes the analysis of the level crossing frequencies of the non-gaussian model.

Increment Distribution

The probability distribution of the non-gaussian model's velocity increments will now be derived. The following remarks apply to all three components of the model; however, only the u-gust component will be explicitly treated.

Consider the increment of the longitudinal gust component.

$$\Delta_u(t|\tau) = u(t) - u(t-\tau) \tag{129}$$

The parameter τ is a constant. According to figure 13, equation 129 can be rewritten as

$$\Delta_u(t|\tau) = \Delta_{ab}(t|\tau) + \Delta_c(t|\tau) \tag{130}$$

where

$$\Delta_{ab}(t|\tau) = [a(t)\,b(t) - a(t-\tau)\,b(t-\tau)]\,R_u(1+R_u{}^2)^{-1/2} \tag{131}$$

$$\Delta_c(t|\tau) = [c(t) - c(t-\tau)](1+R_u{}^2)^{-1/2} \quad . \tag{132}$$

The procedure to be followed in finding the probability distribution of Δ_u is

1) derive an expression for the characteristic function of Δ_c
2) derive an expression for the characteristic function of Δ_{ab}
3) multiply the results of steps 1 and 2 in order to obtain the characteristic function of Δ_u.
4) Fourier transform the result of step 3 to find the probability density of Δ_u.

Since Δ_c is a linear transformation of a gaussian process, it follows that it must also be gaussian. Its mean value is zero and its standard deviation is given by

$$\sigma_{\Delta_c}{}^2 = 2[\sigma_c{}^2 - C_{cc}(\tau)](1 + R_u{}^2)^{-1} , \tag{133}$$

where C_{cc} is the correlation function of $c(t)$, equation 55. The characteristic function of Δ_c is thus

$$\phi_{\Delta_c}(f|\sigma_u , L_u , U , R_u , \tau) = \tag{134}$$

$$\exp\{ - (2\pi\sigma_u f)^2 (1 + R_u{}^2)^{-1} [1 - \exp(\frac{U}{L_u} |\tau|)]\} .$$

The characteristic function of Δ_{ab} is more difficult to evaluate. Since a and b are gaussian processes with zero means and correlation functions given by equations 51, 52, and 54, $\phi_{\Delta_{ab}}$ can be written in the form

$$\phi_{\Delta_{ab}}(f|\sigma_u , L_u , U , R_u , \tau) = \int_{-\infty}^{\infty}(4)\int \exp\left[\frac{i(\pi f R_u x_1 x_2 - x_2 x_4)}{(1 + R_u{}^2)^{1/2}}\right] .$$

$$p_{a(t), a(t-\tau), b(t), b(t-\tau)}(x_1 , x_2 , x_3 , x_4 | \sigma_u , L_u , U , \tau)\, dx_1\, dx_2\, dx_3\, dx_4$$

where

$$p_{a(t), a(t-\tau), b(t), b(t-\tau)}(\bar{x}|\sigma_u , L_u , U , \tau) = \frac{\exp(- \frac{1}{2} \bar{x}^T \tilde{A}^{-1} \bar{x})}{(2\pi)^2 [\det(\tilde{A})]} \tag{135a}$$

$$
\left.
\begin{aligned}
&\bar{x} = [x_1\ x_2\ x_3\ x_4]^T \\
&\tilde{A} = \begin{bmatrix} C_{aa}(0) & C_{aa}(\tau) & 0 & 0 \\ C_{aa}(\tau) & C_{aa}(0) & 0 & 0 \\ 0 & 0 & C_{bb}(0) & C_{bb}(\tau) \\ 0 & 0 & C_{bb}(\tau) & C_{bb}(0) \end{bmatrix} \\
&C_{aa}(\tau) = \sigma_u^{\,2}\, 8\, \frac{L_u}{U} \exp\left(-\frac{U}{2L_u}|\tau|\right) \\
&C_{bb}(\tau) = \frac{U}{8L_u} \exp\left(-\frac{U}{2L_u}|\tau|\right) .
\end{aligned}
\right\} \qquad (135b)
$$

The characteristic function of Δ_u is given by

$$
\phi_{\Delta_u}(f|\sigma_u, L_u, U, R_u, \tau) = \phi_{\Delta_{ab}}(f|\sigma_u, L_u, U, R_u, \tau)\, \phi_{\Delta_c}(f|\sigma_u, L_u, U, R_u, \tau) \qquad (136)
$$

The probability density of Δ_u can be found by Fourier transforming equation 136. A closed form solution of this problem has not been derived, however program INCPD listed in the appendix of this report can perform all required computations for a numerical solution of the problem.

Figure 18 presents density and distribution functions of the u-gust and its increment for a typical case. Comparison of the density functions of figure 18 with those of figure 9 on page 41 leads to the conclusion that the non-gaussian model does not properly model the velocity increment

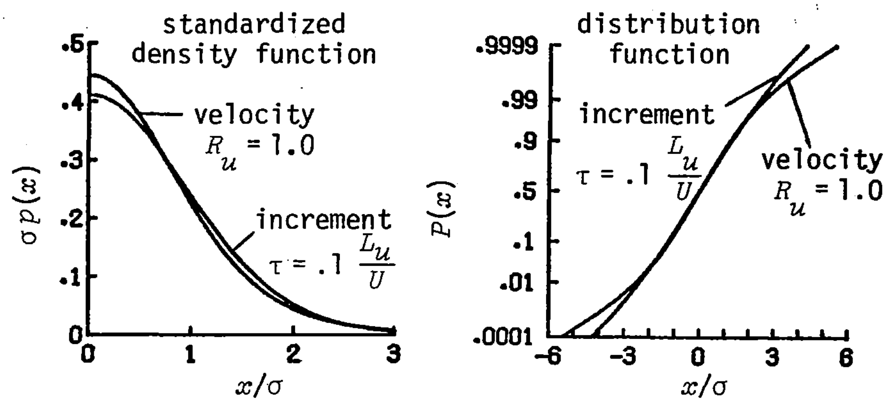


Figure 18.--Comparison of a typical non-gaussial model u-gust component probability distribution with that of the corresponding increment distribution.

distribution of atmospheric turbulence. Additional data in the form of probability distribution functions presented in references 22 and 25 confirm this conclusion. The increments of actual turbulence are sharply more non-gaussian than the velocity, while the increments of the model are less non-gaussian than the velocity. As a result, the non-gaussian model underestimates the occurrences of large increments.

The reason for this can be seen by considering the gaussian and non-gaussian portions of both the velocity and velocity increment. The velocity of the model is the sum of two independent parts; the product $a(t)\,b(t)$ which has a K_0 distribution (Fig. 12), and $c(t)$ which is gaussian. The relative intensity of these two parts can be shown to be [since $a(t)\,b(t)$ is equal to $d(t)$]

$$\frac{\sigma_{ab}}{\sigma_c} = R\left(\frac{C_{dd}(0)}{C_{cc}(0)}\right)^{1/2} = R\ . \tag{137}$$

The probability density of the gust velocity can be found by convolving the K_0 and gaussian densities to obtain a result which is, in general, non-gaussian. Similarly, the velocity increment of the model is the sum of two independent parts; the increment Δ_c which is gaussian, and the increment Δ_{ab} which is non-gaussian. Just as above, the probability density functions of these two parts, and their relative intensities can be shown to be

$$\frac{\sigma_{\Delta_{ab}}}{\sigma_{\Delta_c}} = R \frac{[C_{dd}(0) - C_{dd}(\tau)]}{[C_{cc}(0) - C_{cc}(\tau)]} = R \ . \qquad (138)$$

However, the probability distribution of Δ_{ab} is not sharply non-gaussian like the K_0 distribution. It is instead much smoother, tending to look like the exponential function, $\exp(-|x|)$. Because of this smoothness and because Δ_{ab} has the same intensity relative to Δ_c as has the product $a(t)\,b(t)$ relative to $c(t)$, it follows that when the density of Δ_{ab} is convolved with the gaussian density function, the result will be more nearly gaussian than the convolution of K_0 with the gaussian density. Thus, as shown in figure 18, the distribution of Δ_u is more nearly gaussian than the distribution of u itself.

There are two ways in which the increment distribution can be made more strongly non-gaussian than the velocity distribution; increase the intensity of Δ_{ab} relative to Δ_c while holding the ratio σ_{ab}/σ_c constant, or make the distribution of Δ_{ab} more non-gaussian than the K_0 distribution. The second of these does not appear to be possible. The first, however, could be achieved by modifying C_{cc} and C_{dd} of equation 138 so that the ratio $\sigma_{\Delta_{ab}}/\sigma_{\Delta_c}$ is greater than R. This would require changing the transfer functions of the model, and would invalidate many of the results obtained so far. For this reason, the possibility of modifying the model so as to more accurately model the increment distribution of atmospheric turbulence will not be discussed in this report. Instead, this subject will be suggested as a topic for further research.

Because the model in its present form does not properly reproduce the increment distributions of actual turbulence, its use should be restricted to those applications which require only the accurate modeling of the gust velocity distribution and level crossing frequency. These applications will typically involve responses of the rigid body modes of the aircraft which are not as excited by large increments with short time lags as are the structural modes.

Summary

This completes the statistical analysis of the proposed non-gaussian turbulence model. Before proceeding to the next section, a brief summary of this section is in order.

1) Equations 38 through 63 showed that the linear filters of the model could be chosen so as to produce the Dryden spectral densities. The results are contained in table 1 which summarizes the transfer functions for each of the three gust components.

2) Equations 64 through 80 were concerned with the probability distribution of the non-gaussian model. Figure 15 presents density and distribution functions attainable with the model.

3) Equations 81 through 128 derived the level crossing frequency of the model. The results of this derivation are presented in figures 16 and 17, which show sets of universal curves from which the level crossing frequency of the model can be derived for any set of parameters.

4) The last part of this section, equations 129 through 138 has discussed the increment distribution of the non-gaussian model. The results, presented in figure 18 for a typical case of the longitudinal gust component, show that the non-gaussian model in its present form does not properly model the increment distribution of atmospheric turbulence. The reason for this behavior has been discussed and a possible solution to this problem has been suggested as a topic for further research.

VALIDITY OF THE NON-GAUSSIAN MODEL

The results presented in the previous section have indicated that the probability distribution and level crossing frequency of the non-gaussian model exhibit characteristics very similar to those of single samples of atmospheric turbulence. In this section it will be shown that the model also fits the cumulative probability distributions and cumulative probability of exceedance distributions measured in atmospheric turbulence. In addition, it will be shown that the non-gaussian model fits these data better than the gaussian model.

Experimental Data

The experimental data used here will be a portion of those obtained during the LO-LOCAT project sponsored by the United States Air Force (Ref. 11). Although these data are exclusively from low altitude flight, they were selected for use in this report for two reasons.

1) The large body of data collected during the LO-LOCAT program can be divided into categories on the basis of flight altitude, atmospheric stability, and terrain roughness while retaining a large number of samples in each category. This will permit comparison of the models with turbulence data obtained under quite restricted conditions without the necessity of using small data samples.
2) These data are readily available to anyone wishing to either verify the results presented in this report or extend them to a wider variety of cases.

The disadvantage associated with using only this low altitude data is that any conclusions which might be drawn from them cannot be generalized to higher altitudes. Further work will be required in order to show the validity of the model at high altitudes.

All of the results presented in this section are based on vertical gusts. Computations for longitudinal and lateral gust components have indicated that the results presented here are representative of all three components.

The data categories used in this section are described in table 2. These particular categories were selected because they represent a wide range of terrain, altitude, and atmospheric stability conditions.

Goodness-of-Fit-Criteria

The object of this section is to compare both the gaussian and non-gaussian models with experimentally measured probability distributions and level crossing frequencies. Before any comparison can be made, however, it will be necessary to define some goodness-of-fit criterion. That is, some objective test which can be used as an indication of how well the theoretical curves of the turbulence models fit the measured data.

A problem arises in the application of standard statistical methods such as the chi-squared or Kolmogorov-Smirnov tests because they require that the experimental data be based on independent samples. Unfortunately, the data of reference 11 which are to be used here are not based entirely on independent samples. Hence these tests are not applicable.

There appears to be no easy solution to this problem. However, for the purposes of this report, it will be possible to avoid this difficulty by assuming that the data can be treated as essentially exact. Any possible differences between the true statistics and the measured

Table 2.--Description of data categories used in this report.

Category	Description
111000	Vertical gust data based on 5,700 kilometers (3,536 statute miles of flight at an altitude of 76 meters (250 feet) above the surface. All data collected during 109 flights over high mountains in very stable atmospheric conditions.
112000	Vertical gust data based on 7,350 kilometers (4,577 statute miles) of flight at an altitude of 76 meters (250 feet) above the surface. All data collected during 140 flights over high mountains in stable atmospheric conditions.
113000	Vertical gust data based on 6,800 kilometers (4,226 statute miles) of flight at an altitude of 76 meters (250 feet) above the surface. All data collected during 129 flights over high mountains in neutral atmospheric conditions.
121000	Vertical gust data based on 5,800 kilometers (3,620 statute miles) of flight at an altitude of 228 meters (750 feet) above the surface. All data collected during 112 flights over high mountains in very stable atmospheric conditions.
122000	Vertical gust data based on 7,800 kilometers (4,840 statute miles) of flight at an altitude of 228 meters (750 feet) above the surface. All data collected during 147 flights over high mountains in stable atmospheric conditions.
123000	Vertical gust data based on 5,000 kilometers (3,122 statute miles) of flight at an altitude of 228 meters (750 feet) above the surface. All data collected during 95 flights over high mountains in neutral atmospheric conditions.
413000	Vertical gust data based on 2,900 kilometers (1,811 statute miles) of flight at an altitude of 76 meters (250 feet) above the surface. All data collected during 55 flights over plains in neutral atmospheric conditions.
414000	Vertical gust data based on 2,300 kilometers (1,446 statute miles) of flight at an altitude of 76 meters (250 feet) above the surface. All data collected during 44 flights over plains in unstable atmospheric conditions.

statistics will be ignored. This is a common approach to the analysis of turbulence data (e.g., Ref. 23). This assumption is acceptable in the present work for two reasons.

1) The purpose of this section is only to indicate to the reader that the non-gaussian model produces a better fit of the experimental data than the gaussian model. The results to be presented will show that in every case the gaussian model underestimates the occurrences of the high velocity gusts. Even though this error may or may not be judged statistically significant by one of the standard tests, it is clearly significant for the purposes of aircraft design if it occurs in every case tested. Thus it is contended that a rigorous statistical test of significance is not required for the purposes of this report.

2) As indicated in reference 11, the LO-LOCAT data used here (which were selected for the reasons presented previously in this section) contain some nonstationary effects. Run tests of both the mean and mean square indicated that approximately 30% of the LO-LOCAT turbulence samples could not be accepted as stationary at the 0.02 level of significance. Unfortunately, the models used in this report assume turbulence to be a stationary process. Thus the presence of nonstationary effects in the experimental data can be expected to have some effect on the results to be presented here, and the magnitude of this effect is unknown.

For these reasons, a more careful analysis of the data is not warranted at this time.

Since the experimental data are assumed to represent the true statistics of atmospheric turbulence, the problem reduces to one of simple curve fitting. Three goodness-of-fit criteria have been investigated in the research reported here. The first of these is the integral of the squared error,

$$\varepsilon_{ise}(R) = \int_0^{x_{max}} |F_{data}(x) - F_{model}(x|R)|^2 \, dx \tag{139}$$

where x_{max} is the highest gust velocity measured. The functions F_{data} and F_{model} denote either the cumulative probability of exceedance or the cumulative probability distribution of the data and the model respectively. Note that since the model distribution depends upon the parameter R, the error criterion also depends upon R. The best fit of the experimental data is chosen to be the turbulence model with the value of R which minimizes ε_{ise} . Note that if setting R to zero minimizes ε_{ise} then the gaussian model is the best fit.

The second error criterion investigated was the maximum difference between the experimental data and the model.

$$\varepsilon_{max}(R) = \max_{0 \le x \le x_{max}} |F_{data}(x) - F_{model}(x|R)| \tag{140}$$

The quantities used in this equation are the same as those in equation 139. Again, the best fit of the data is chosen to be the model with that value of R which minimizes ε_{max} .

Application of both ε_{ise} and ε_{max} to a number of data samples revealed that both criteria produced essentially identical results. For this reason, and because it is a more difficult test to apply, ε_{ise} will not be used in the numerical calculations of this report.

The reader should note that both ε_{ise} and ε_{max} apply primarily at low gust velocities, where F is large, rather than at high gust velocities, where F is very small. Thus both criteria tend to ignore errors in modeling the occurrences of very high gust velocities and are therefore not ideally suited for determining the best fit of the data for the purposes of aircraft design.

A third criterion, which is more sensitive to the occurrences of high gust velocities, has been studied. This is the maximum absolute difference of logarithms.

$$\varepsilon_{\log}(R) = \max_{\substack{x \\ 0 \le x \le x_{max}}} \left| \log_{10}[F_{data}(x)] - \log_{10}[F_{model}(x|R)] \right| . \qquad (141)$$

This criterion seems to be quite sensitive to small errors at the high gust velocities and may therefore ignore more serious errors at lower gust velocities. It does however lead to a very good fit of the LO-LOCAT data, especially when plotted on logarithmic scales as is the usual practice.

The above discussion makes it apparent that the two error criteria, ε_{max} and $\varepsilon_{\log}$, emphasize different aspects of the curve fitting problem. For this reason, both criteria will be used in the following tests.

Generation of Model Data

The cumulative probability of exceedance and cumulative probability distribution of the non-gaussian model vertical component were generated according to equations 80 and 128 derived in the previous section of this report. The assumptions involved in these equations will be found in that section. The probability density of σ_ω for each case was obtained from distribution functions presented in reference 11.

It was also assumed that the parameter R of the model could be considered constant for each data category. Since these categories are defined by altitude, atmospheric stability, and terrain characteristics, this assumption is equivalent to requiring R to be in some sense determined by these factors. As R is merely a parameter which arises from the manner in which the non-gaussian model is physically interpreted (Fig. 13), there is no difficulty in assigning to it this functional dependence. The explicit form of the dependence is, of course, unknown.

Results were computed numerically for R values of 0.0, 0.5, 0.75, 1.0, 1.33, and 2.0. The first of these (R = 0.0) corresponds to the gaussian turbulence model. The five other cases correspond to the non-gaussian model with varying degrees of non-gaussian behavior.

Results

Table 3 presents the computed error criteria ε_{max} and $\varepsilon_{\log}$ for the various data categories and values of R. Error values for the cumulative probability of exceedance and cumulative probability distribution have been summed because the same value of R does not necessarily minimize a given error criterion for both functions simultaneously. In every case the non-gaussian model provides a better fit of the experimental data

Table 3.--Goodness-of-fit criteria for various LO-LOCAT data categories.

Category	Test	Turbulence Model Parameter					
		$R = 0.0$ Gaussian	$R = 0.5$	$R = 0.76$	$R = 1.0$	$R = 1.33$	$R = 2.0$
111000	ε_{max}	2.5E-1	2.3E-1	2.0E-1	1.6E-1	1.1E-1	4.8E-2
	ε_{log}	8.9E 0	4.5E 0	2.7E 0	1.6E 0	9.3E-1	3.9E-1
112000	ε_{max}	1.7E-1	1.6E-1	1.3E-1	1.0E-1	6.0E-2	1.0E-1
	ε_{log}	5.1E 0	2.5E 0	1.2E 0	4.3E-1	2.8E-1	5.8E-1
113000	ε_{max}	1.2E-1	1.0E-1	6.9E-2	2.5E-2	4.4E-2	1.4E-1
	ε_{log}	3.0E 0	1.4E 0	3.1E-1	5.0E-1	9.8E-1	1.4E 0
121000	ε_{max}	2.4E-1	2.2E-1	1.9E-1	1.5E-1	9.3E-2	3.1E-2
	ε_{log}	8.0E 0	4.5E 0	2.8E 0	2.0E 0	1.3E 0	8.2E-1
122000	ε_{max}	1.7E-1	1.5E-1	1.2E-1	8.0E-2	4.0E-2	8.4E-2
	ε_{log}	3.8E 0	2.3E 0	1.2E 0	6.2E-1	1.8E-1	4.0E-1
123000	ε_{max}	8.2E-2	6.5E-2	3.6E-2	4.1E-2	8.1E-2	1.7E-1
	ε_{log}	4.0E 0	2.1E 0	8.2E-1	2.4E-1	4.9E-1	8.9E-1
413000	ε_{max}	1.1E-1	9.2E-2	6.0E-2	6.6E-2	1.1E-1	1.9E-1
	ε_{log}	4.0E 0	1.5E 0	1.8E-1	7.3E-1	1.3E 0	1.8E 0
414000	ε_{max}	4.7E-2	3.2E-2	2.3E-2	6.1E-2	1.2E-1	2.2E-1
	ε_{log}	5.5E 0	2.9E 0	1.5E 0	7.2E-1	5.9E-1	8.4E-1

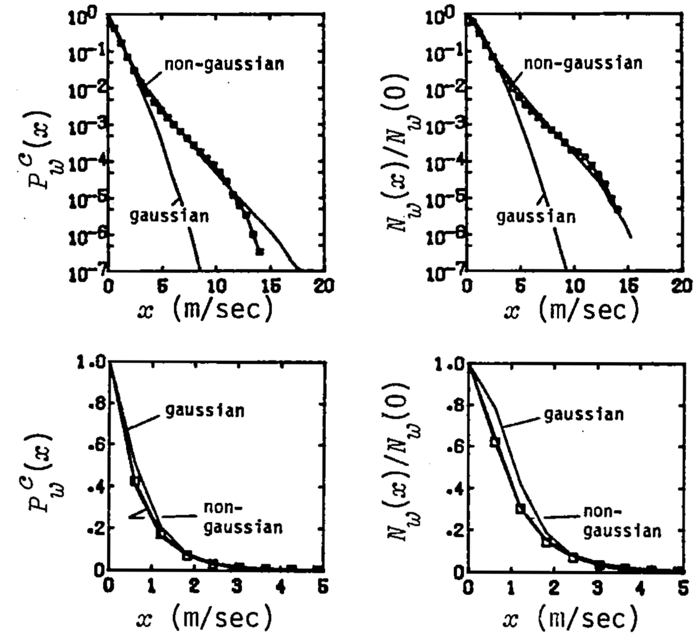
than does the gaussian model, although the best value of R for a given data category depends upon the error criterion used. This is especially true for those data collected over plains. Note, however, (Table 3) that in every case it is possible to choose an R value which simultaneously reduces both criteria below their values for the gaussian model.

Figures 19 and 20 presented on the next several pages compare the experimental data with both the indicated best fit models of table 3 and the gaussian model. It will be noted that in every case the gaussian model underestimates the occurrences of high gust velocities. In general, these results and others not presented here indicate that the two criteria agree much better for the high mountain data than they do for the plains data.

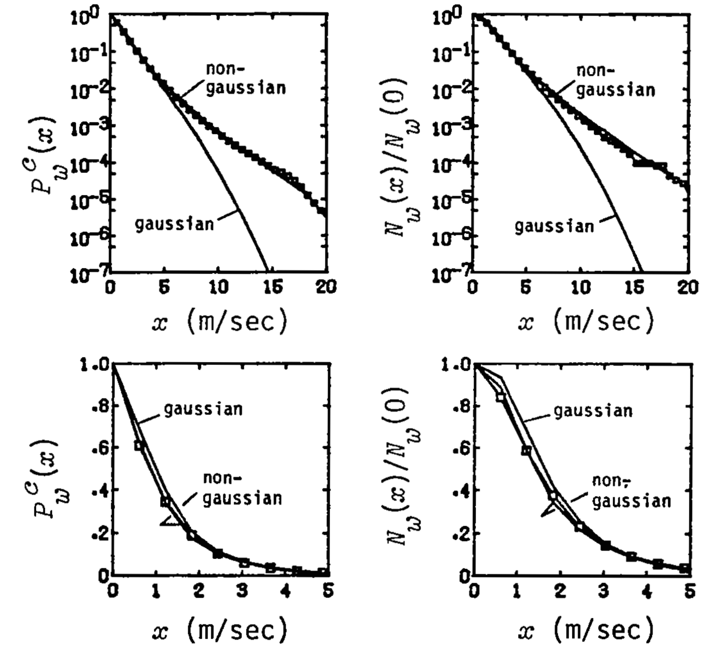
Table 4 summarizes the values of R which yield the best fit of the data for each of the error criteria. These results seem to indicate that the parameter R of the turbulence model may be related to the terrain and stability parameters of the LO-LOCAT data categories. It appears that R increases with atmospheric stability and terrain roughness, but is relatively unaffected by altitude. This result, however, cannot be verified without a much more careful investigation using more data and applying a regression analysis in order to objectively analyze the dependence of R upon the data parameters.

Summary

This section has presented a comparison of the gaussian and non-gaussian models with experimentally measured cumulative probability of exceedance and cumulative probability distributions of low altitude turbulence. Two simple error criteria have been used to select the best

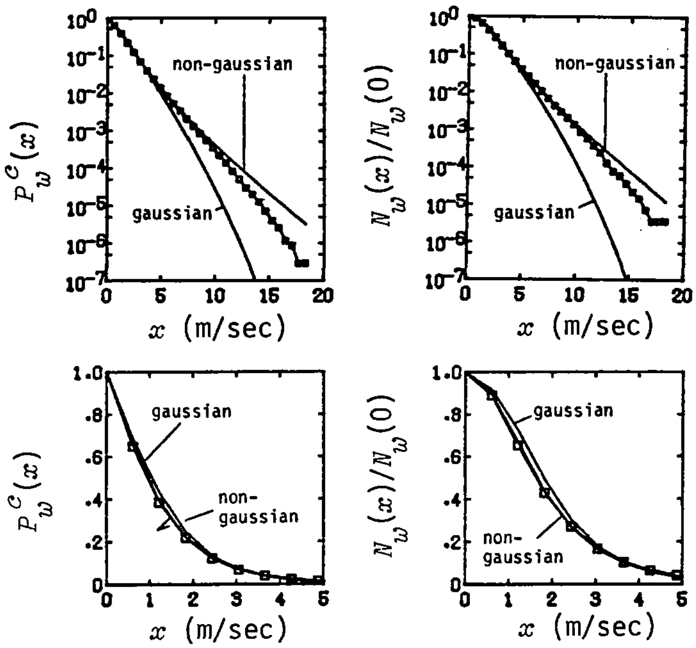


LO-LOCAT data category 111000



LO-LOCAT data category 112000

Figure 19.--Various LO-LOCAT data plotted on both logarithmic and linear scales compared with the gaussian model and the non-gaussian model selected according to the ε_{max} best fit criterion.



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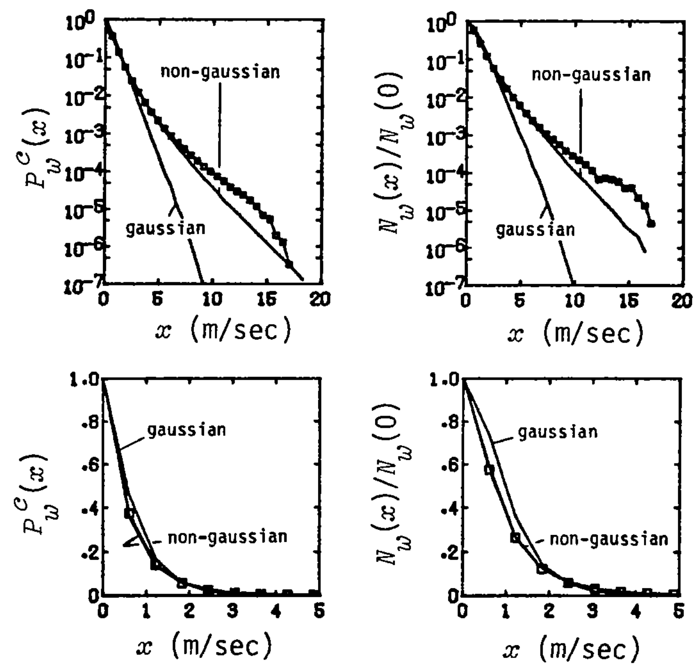
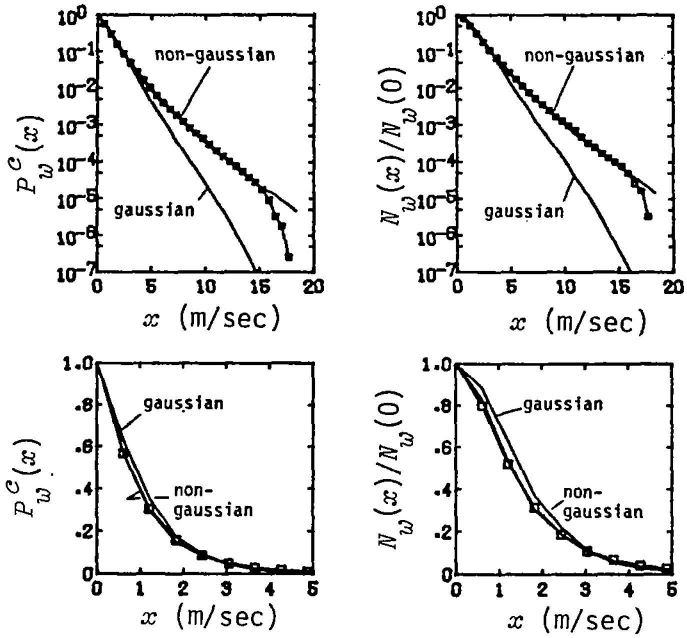


LO-LOCAT data category 121000

Figure 19 (cont.)--Various LO-LOCAT data plotted on both logarithmic and linear scales compared with the gaussian model and the non-gaussian model selected according to the ε_{max} best fit criterion.



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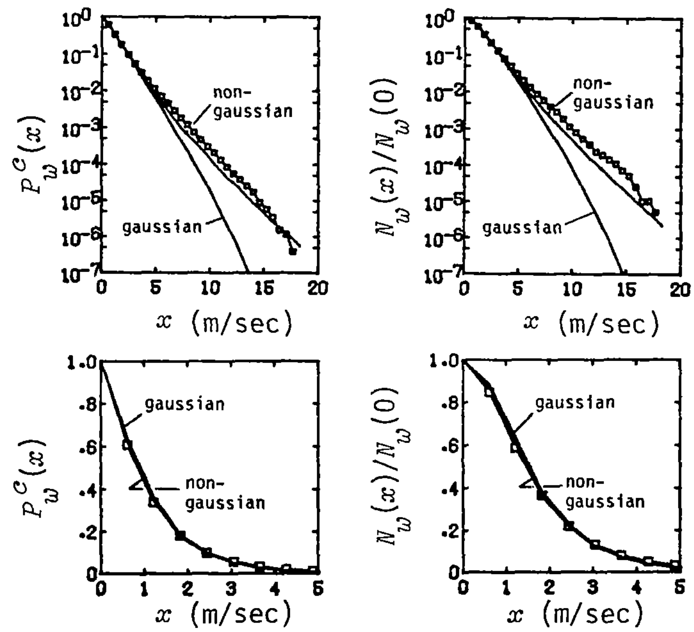
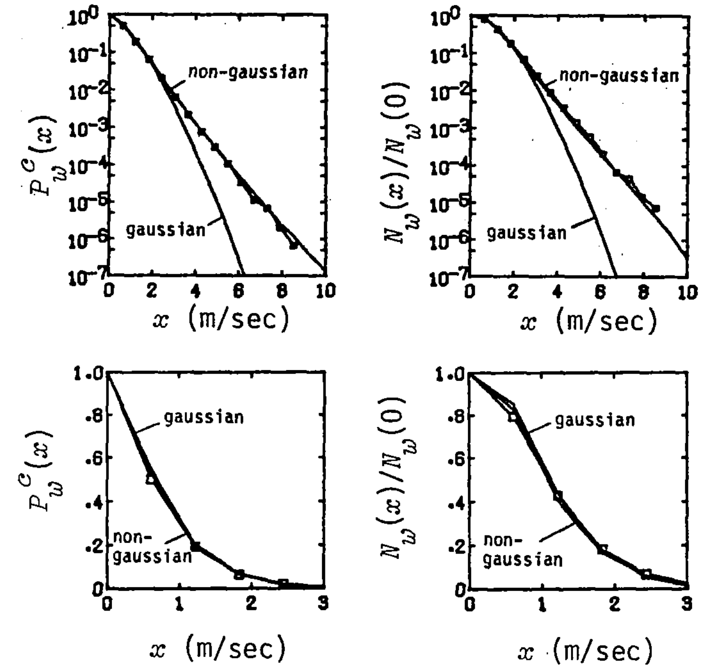


LO-LOCAT data category 123000

Figure 19 (cont.)--Various LO-LOCAT data plotted on both logarithmic and linear scales compared with the gaussian model and the non-gaussian model selected according to the ε_{max} best fit criterion.



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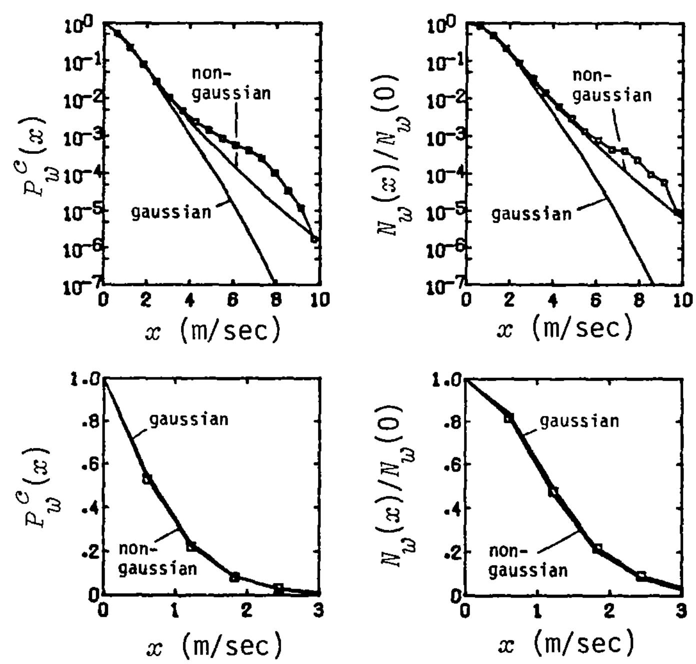
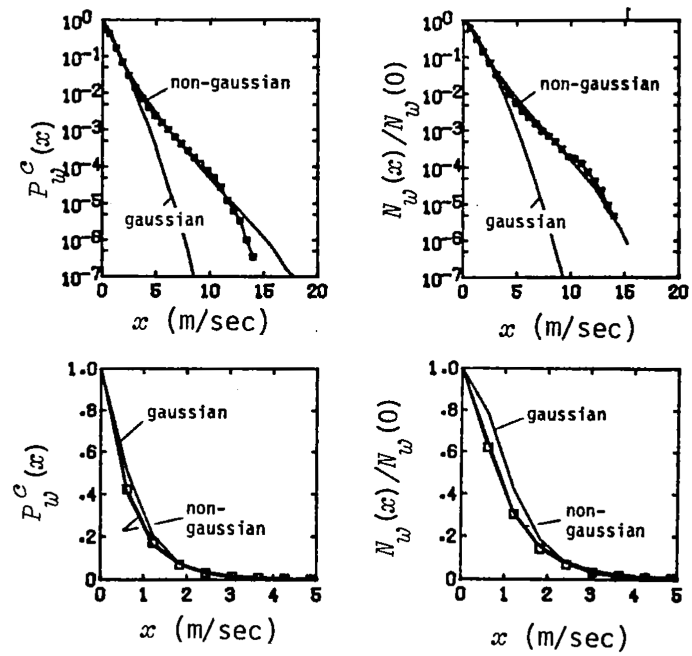


LO-LOCAT data category 414000

Figure 19 (cont.)--Various LO-LOCAT data plotted on both logarithmic and linear scales compared with the gaussian model and the non-gaussian model selected according to the ε_{max} best fit criterion.



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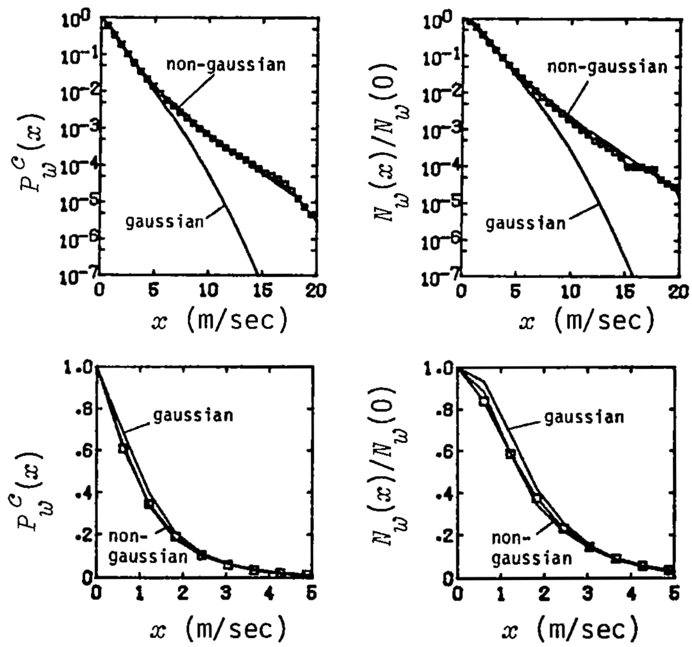
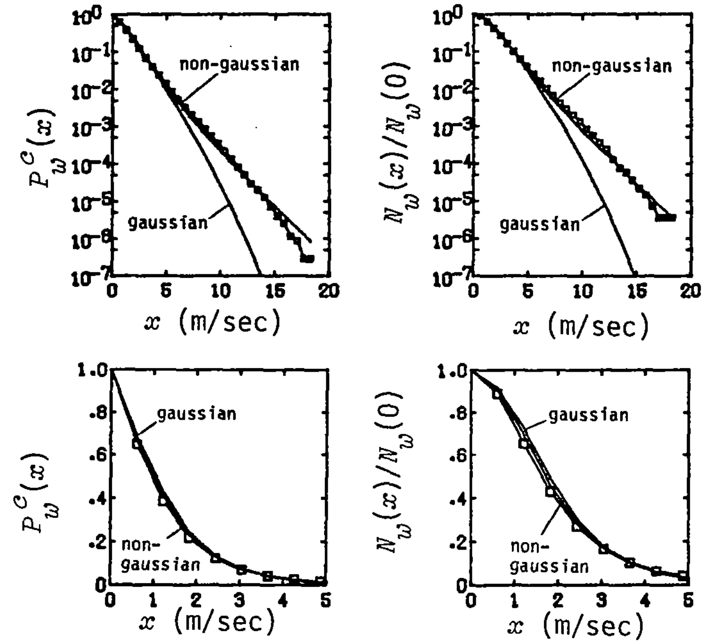


LO-LOCAT data category 112000

Figure 20.--Various LO-LOCAT data plotted on both logarithmic and linear scales compared with the gaussian model and the non-gaussian model selected according to the $\varepsilon_{\log}$ best fit criterion.



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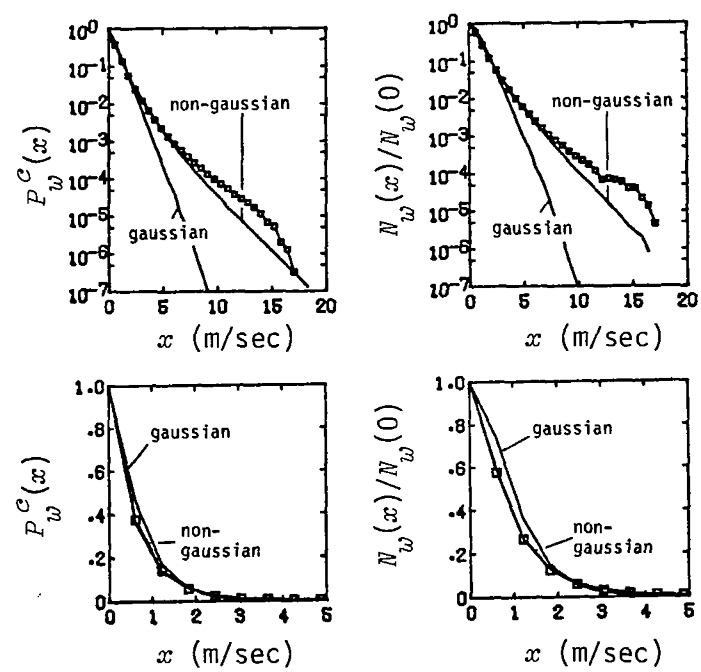
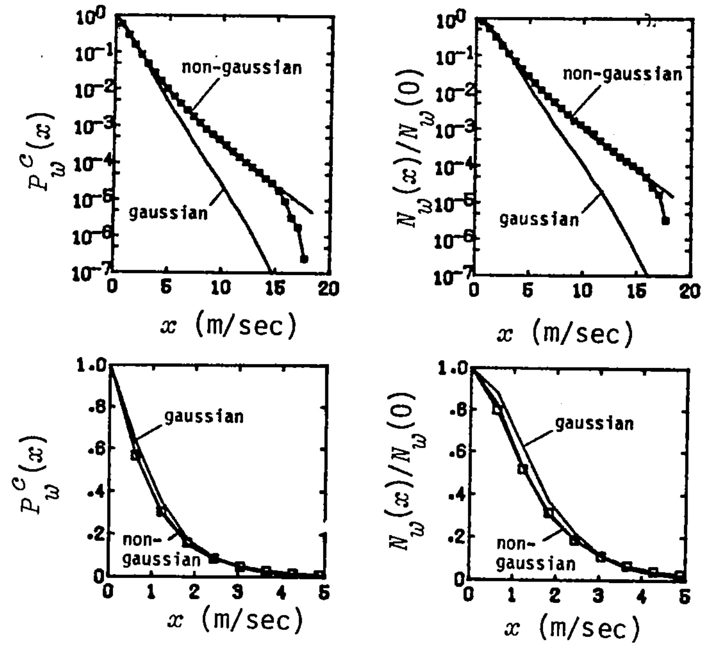


LO-LOCAT data category 121000

Figure 20 (cont.)--Various LO-LOCAT data plotted on both logarithmic and linear scales compared with the gaussian model and the non-gaussian model selected according to the $\varepsilon_{\log}$ best fit criterion.



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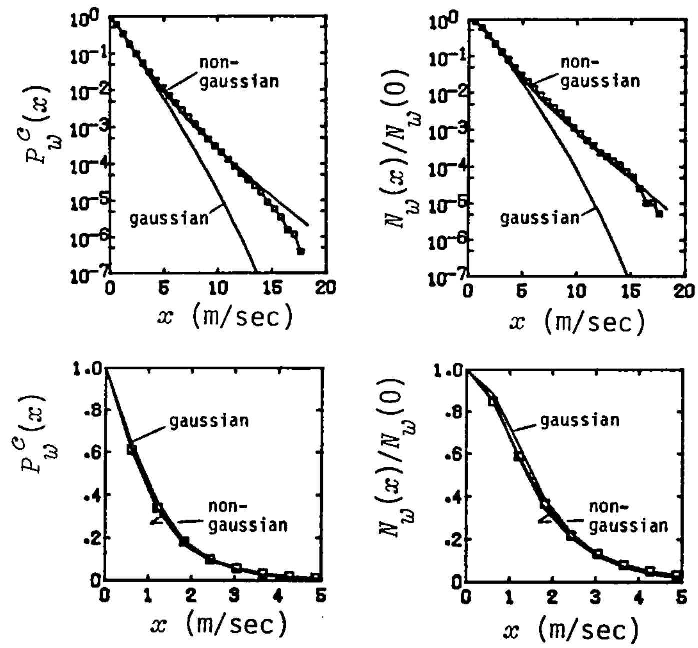
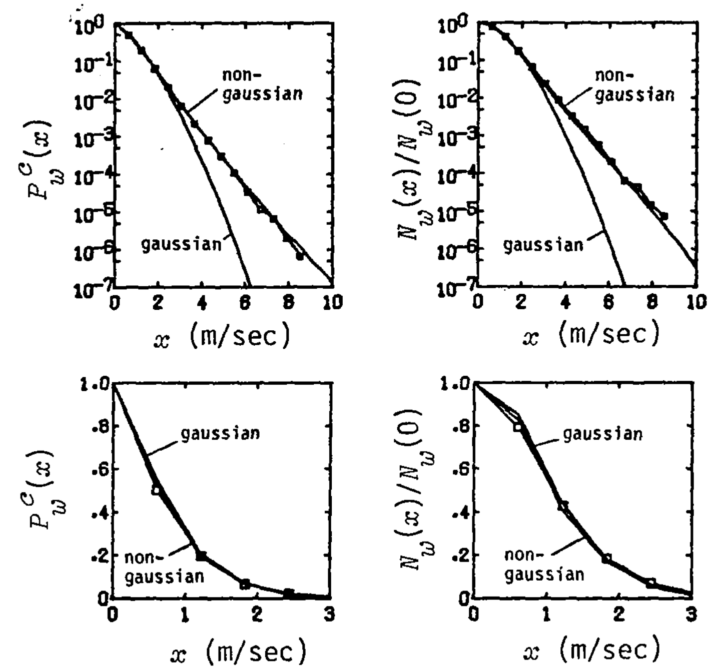


LO-LOCAT data category 123000

Figure 20 (cont.)--Various LO-LOCAT data plotted on both logarithmic and linear scales compared with the gaussian model and the non-gaussian model selected according to the $\varepsilon_{\log}$ best fit criterion.



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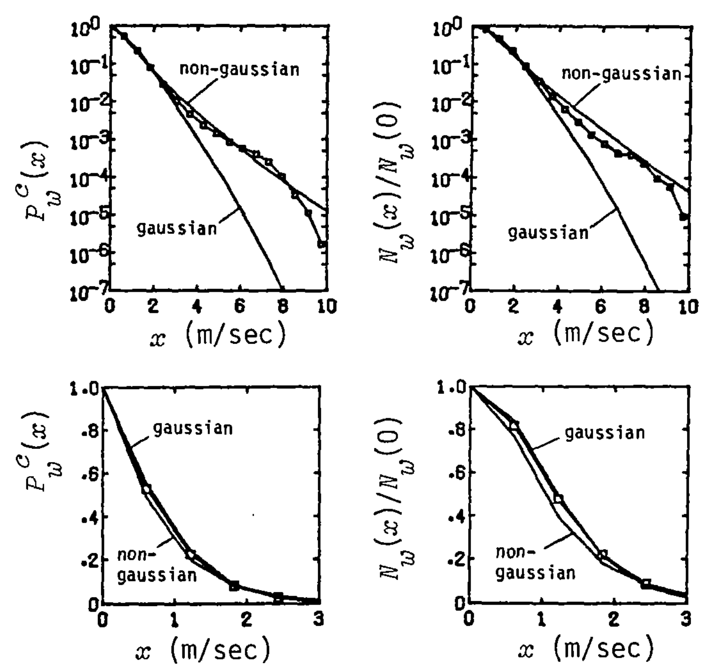

LO-LOCAT data category 414000

Figure 20 (cont.)--Various LO-LOCAT data plotted on both logarithmic and linear scales compared with the gaussian model and the non-gaussian model selected according to the $\varepsilon_{\log}$ best fit criterion.

values of the parameter R which fit the experimental data. Figures 19 and 20 present the resulting curves. Table 4 lists the values of R which minimize the error for each data category. These R values seem to exhibit a systematic increase with increasing atmospheric stability and terrain roughness. However, this result cannot be verified without a more careful analysis.

The results of this section indicate that the non-gaussian model is a better fit of experimental data than the gaussian model. The conclusion drawn is that the non-gaussian model is a better representation of atmospheric turbulence, at least as far as the modeling of distribution functions and level crossing frequencies for the purposes of aircraft design is concerned.

Table 4.--Values of the turbulence model parameter R which produce best fits of the experimental data.

Data Category	R which Minimizes ε_{max}	R which Minimizes ε_{log}
111000	2.0	2.0
112000	1.33	1.33
113000	1.0	0.75
121000	2.0	2.0
122000	1.33	1.33
123000	0.75	1.0
413000	0.75	0.75
414000	0.75	1.33

CALCULATION OF NON-GAUSSIAN RESPONSE STATISTICS

The preceding sections of this report have discussed the non-gaussian model in detail and shown that its statistics are consistent with those of experimentally measured turbulence data. The proposed model is of little value, however, if it cannot be used to study vehicle responses to turbulence. The purpose of this section is to investigate some methods by which response statistics of linear vehicles can be found using the non-gaussian model. The statistics of particular interest here are:

1) power spectral density
2) probability distribution
3) level crossing frequency.

The assumptions made regarding the nature of the turbulence are the same as those used in defining the non-gaussian model (page 47). The vehicles considered in this report will be required to satisfy the following conditions:

1) they must be stable linear systems
2) their transfer functions must have at least two more poles than zeros
3) their transfer functions must be rational
4) their transfer functions must have no multiple poles.

Only the first and second of these conditions is absolutely necessary. The third and fourth conditions are required because they permit simplification of the computer programs which will be used in the next section of this report. These two conditions could therefore be removed by writing more general programs.

The following discussion will first center on some standard techniques by which the above mentioned statistics might be calculated for responses to single components of the non-gaussian model. It will be shown that these methods are not entirely suitable for finding probability distributions or level crossing frequencies. Attention will then be turned to an approximate method of computing these statistics from the eigenvalues and eigenfunctions of a certain unsymmetric kernel. The problem of responses caused by two or all three components of the non-gaussian model will then be briefly discussed.

Standard Solution Techniques

Figure 21 shows the combined turbulence model - vehicle system which is to be analyzed. Note that this system is linear throughout with the exception of the single multiplication. It is this single nonlinearity which creates difficulties in response calculations. Now consider some standard techniques by which the three response statistics listed above might be computed.

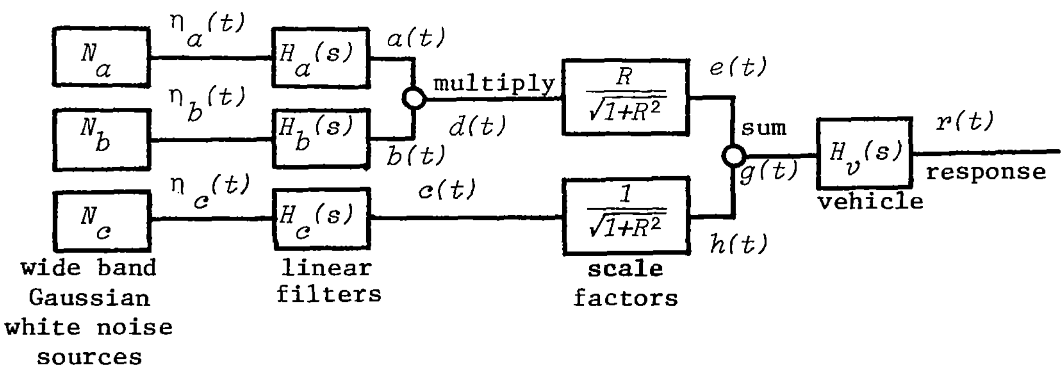

Figure 21.--Block diagram of the turbulence model-vehicle system which is to be analyzed in order to determine vehicle response statistics.

Power Spectral Density

Since the vehicle is assumed to be a linear system, and since the power spectral density of the turbulence time history is assumed known (Eq. 4, 5, or 6), the power spectral density of the vehicle response can be found from the well known input - output spectral relationship for linear systems (Ref. 38).

$$\phi_{rr}(\Omega) = |H(i\,U\Omega)|^2\ \Phi_{gg}(\Omega)\ , \tag{142}$$

where $H(s)$ is the transfer function relating the input $g(t)$ to the output $r(t)$. The calculation of response power spectral densities will therefore not present any difficulty, and will not be discussed further in this section.

Response Distribution and Level Crossing Frequency

The calculation of response distribution functions and level crossing frequencies provides somewhat more of a challenge than did the power spectral density discussed above. Two commonly used techniques are mentioned here as practical methods which may have application in some cases.

The first method discussed will be simulation. This is the direct approach to the problem of calculating response statistics, and may be the most convenient method of dealing with the non-gaussian model for many applications. Just as the name of this method suggests, the entire turbulence model - vehicle system is programmed on an analog or digital computer and the vehicle is "flown" through many miles of turbulence while its responses are recorded. After a sufficient amount of data have been collected, estimates are made of the response statistics.

The simulation method has both advantages and disadvantages. Among its advantages is the fact that it can be used to study both linear and nonlinear vehicles. It can also be applied to nonstationary problems such as the evaluation of control requirements during landing approaches. The disadvantage of this method is that it does not readily yield results concerning rare events such as encounters with very high velocity gusts. The computer time required to estimate these occurrences may be prohibitive.

The simulation approach to response calculations may thus be a very useful technique in some cases, but it is not a convenient method of estimating the occurrences of rare events. Thus it will probably not be satisfactory when information on the tails of the probability distribution or level crossing frequency is sought.

The second method of computing response statistics is the Gram-Charlier expansion. This is a technique for expanding probability density functions in an orthogonal series of Hermite polynomials. The method can be applied to both one-and two-dimensional density functions, and so could be used to obtain the density function of the response, as well as the joint density function of the response and its first derivative. The first of these is equivalent to the response distribution function, and the second can be used (through Eq. 81) to determine the response level crossing frequency. This method could, at least in principle, be used to obtain the response statistics which are of interest here.

Unfortunately, the Gram-Charlier technique cannot be applied in all cases because the coefficients of the expansion become very difficult to calculate if more than the first two or three terms are required.

Consequently, this method is only useful if a very minimal number of terms is needed. Since the first term of the expansion turns out to be the gaussian density function, it follows that the Gram-Charlier technique is useful only when the density function being expanded is very nearly gaussian. The method also becomes impractical if the tails of the distribution to be expanded do not decay as rapidly as those of the gaussian density function.

On the basis of preliminary analysis it appears that in many cases the non-gaussian turbulence model will produce response density functions which require an unreasonable number of terms to converge. Table 5 shows the number of terms required to expand the density function of the model itself for various values of the parameter R. The shape of these density functions can be inferred from figure 15 on page 61. These results make it clear that any distribution which differs from gaussian by more than a slight degree will not be suitable for analysis by the Gram-Charlier technique.

Table 5.--Number of non-zero terms of Gram-Charlier expansion required to represent density functions of the non-gaussian model to various accuracies over the range of zero to six standard deviations.

ACCURACY	MODEL PARAMETER R			
	.1	.15	.2	.25
± 10%	1	2	4	5
± 20%	1	1	3	5
± 50%	1	1	2	3

In the following parts of this section, an expansion method will be derived which works best in the case of a strongly non-gaussian response. This new method will thus compliment the Gram - Charlier expansion as a suitable technique for finding response statistics. Furthermore, unlike the Gram - Charlier technique, it can be shown that this expansion has the characteristic of approximating the tails of the distribution very well, even when only a few terms are used.

The Gram - Charlier method will not be discussed further in this report. However, references 24 and 46 provide good descriptions of the technique and its use in computing one- and two-dimensional density functions.

Decomposition of Response

The above remarks have considered two well known methods of dealing with system responses to non-gaussian inputs, and pointed to some of the shortcomings of each. A method will now be developed which specifically treats the response of linear systems to the non-gaussian turbulence model. Before beginning the derivation of this new method, however, some introductory remarks are in order.

Consider again the turbulence model - vehicle system of figure 21. Since the vehicle is assumed to be linear, the system can be redrawn as shown in figure 22. Note that the definition of $g(t)$ in this figure differs from that used previously. The process g is now gaussian because a gaussian process remains gaussian when passed through a linear filter. The process $k(t)$ on the other hand is the result of passing $d(t)$, which has a K_o probability density (Fig. 12), through a linear filter. In general, the distribution of $k(t)$ is unknown. The total vehicle response,

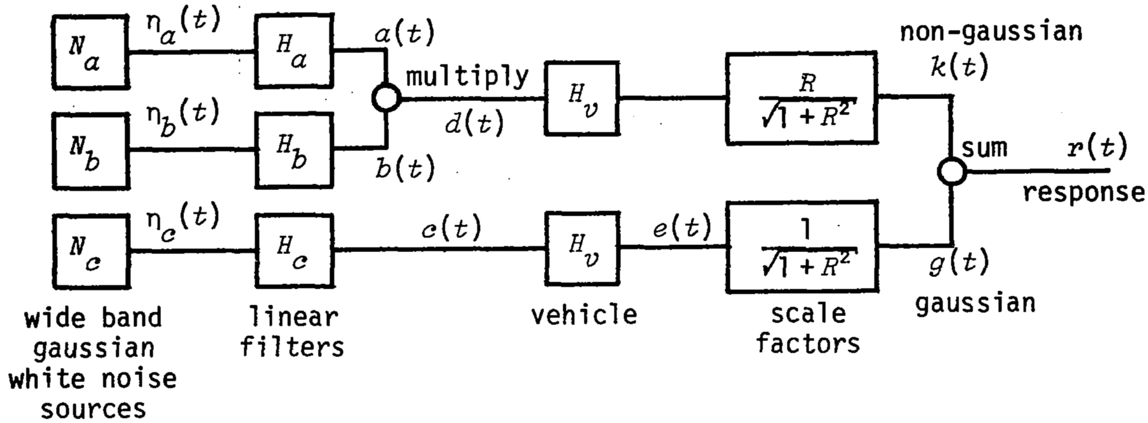

Figure 22.--Turbulence model - vehicle system with the vehicle response decomposed into gaussian and non-gaussian parts.

$r(t)$, is the sum of $g(t)$ and $k(t)$. Figure 22 has thus shown how the response of the vehicle can be decomposed into gaussian and non-gaussian parts.

Approximate Solution for Response Probability Distribution Function

Figure 22 has shown that the vehicle response can be written as a sum of independent gaussian and non-gaussian parts. This immediately suggests the possibility of finding the distribution function of the response from its characteristic function. The steps in this procedure are:

1) find the characteristic function of the gaussian portion of the response, ϕ_g
2) find the characteristic function of the non-gaussian portion of the response, ϕ_k
3) multiply ϕ_g and ϕ_k to obtain the characteristic function of the total response, ϕ_r

4) inverse Fourier transform ϕ_r to obtain the density function of the response, p_r.

The explicit parameter dependence notation used in previous discussions of characteristic functions and probability distributions will not be used in this section in order to simplify notation. The reader should be aware, however, that the results obtained here depend upon the parameters of the turbulence model as well as the dynamic characteristics of the vehicle.

As the first step in determining the response distribution function, the characteristic function of the gaussian portion of the response will be found. Recall the definition of the characteristic function used in this report (Eq. 72). For the gaussian portion of the response, the result is

$$\phi_g(f) = \exp[-2(\pi\sigma_e f)^2(1+R^2)^{-1}] \; . \qquad (143)$$

The standard deviation appearing in this expression, σ_e , can be found from the power spectral density of the response, equation 142. This completes the first step of the derivation.

Now consider the characteristic function of $k(t)$. Two very important assumptions will be made concerning this process. It is almost certain that neither of these assumptions is necessary in order to obtain the results presented here; however, their use will greatly simplify the derivation of these results.

1) It is assumed that negligible error will occur in the response statistics if the bandwidth of the white noise sources in figure 22 is fixed at some very high but finite limit as

shown in figure 23. By virtue of the second assumption regarding the vehicle transfer functions, only vehicles which act as low pass or band pass filters will be considered in this report. Furthermore, the filters used in the turbulence model itself (Table 1) are either low pass or band pass. Consequently, even when the nonlinear nature of the system is taken into account, it is clear that it will always be possible to select a value of Ω_c (e.g., 10^{10} or 10^{100}) such that the variance of the response and its first derivative will not be significantly affected by further increase of the noise bandwidth. In view of this, it is reasonable to expect that the response probability distribution and level crossing frequency will also be unaffected by increases in bandwidth.

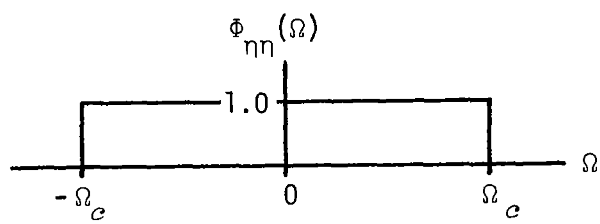

Figure 23.--Power spectral density of band limited gaussian white noise.

2) It is also assumed that the impulse response functions of the filters appearing in figure 22 can be truncated at some very large value of their argument, as shown in figure 24, without causing significant error in the vehicle response. This seems quite reasonable in view of the fact that the impulse response functions

of the vehicle and the turbulence model filters all decay exponentially. If this assumption were not true, it would follow that the response of the vehicle depended to a significant degree upon the infinite past. Thus the response could never reach a state of statistical equilibrium, and the assumption of stationarity which has been made throughout this report would not be valid.

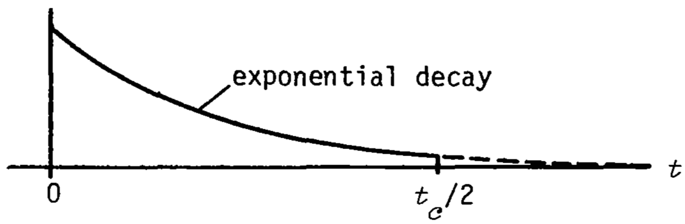

Figure 24.--Truncated impulse response function.

Once these two assumptions have been made, the eigenvalue expansion of the response distribution function follows quickly from a certain theorem due to Schmidt which will be stated shortly.

Consider again figure 22. From linear system theory it follows that the response time history $k(t)$ can be written as the iterated integral

$$k(t) = \frac{R}{(1+R^2)^{1/2}} \iiint_{-\infty}^{\infty} h_v(\delta)\, h_a(\alpha')\, h_b(\beta')\, \eta_a(t-\delta-\alpha')\, \eta_b(t-\delta-\beta')\, d\alpha'\, d\beta'\, d\delta \quad . \tag{144}$$

Now, because of the first assumption above limiting the bandwidth of

η_a and η_b, it follows (Ref. 47, page 170) that these processes are continuous with probability one. The integrand of equation 144 is therefore integrable and, by Fubini's theorem (Ref. 48), the order of integration can be freely interchanged. Substitute the change of variables

$$\left.\begin{aligned} \alpha &= t - \alpha' \\ \beta &= t - \beta' \end{aligned}\right\} \qquad (145)$$

and integrate with respect to δ. The resulting expression for $k(t)$ is

$$k(t) = R(1+R^2)^{-1/2} \int_{-\infty}^{\infty} \eta_a(t-\alpha)\left\{\int_{-\infty}^{\infty} \eta_b(t-\beta)\, h(\alpha\,,\beta)\, d\beta\right\} d\alpha \;, \qquad (146)$$

where the integration with respect to β is arbitrarily chosen to be performed first, and the kernel $h(\alpha\,,\beta)$ is given by

$$h(\alpha\,,\beta) = \int_{-\infty}^{\infty} h_v(\delta)\, h_a(\alpha-\delta)\, h_b(\beta-\delta)\, d\delta \;. \qquad (147)$$

Note that for the vertical and lateral components of the turbulence model, the impulse response functions h_a and h_b are not identical, thus $h(\alpha\,,\beta)$ is not generally symmetric.

As a consequence of the second assumption above concerning truncation of the impulse response functions, $h(\alpha\,,\beta)$ is non-zero only if α and β satisfy the condition

$$0 \le \alpha\,,\ \beta \le t_c \;, \qquad (148)$$

where $t_c/2$ is the truncation time defined in figure 24. Because $h(\alpha\,,\beta)$ is zero whenever α or β does not satisfy equation 148, it follows that finite limits of integration can be used in equation 146.

Now consider Schmidt's theorem concerning integral equations with unsymmetric kernels (Ref. 49, 50). This theorem states that every function q having one of the two forms

$$q(\alpha) = \int_{\xi} h(\alpha , \beta)\, \eta(\beta)\, d\beta \tag{149}$$

$$q(\beta) = \int_{\xi} h(\alpha , \beta)\, \eta(\alpha)\, d\alpha \tag{150}$$

where ξ is a finite interval, is the sum of its uniformly and absolutely convergent Fourier series with respect to the orthonormal system $\psi_n(\alpha)$ in the first case and with respect to the orthonormal system $\chi_n(\beta)$ in the second case. The functions ψ_n and χ_n are defined by the eigenfunction relationships

$$\lambda_n^{\,2}\, \psi_n(\alpha) = \int_{\xi} \left\{ \int_{\xi} h(\alpha , \delta)\, h(\beta , \delta)\, d\delta \right\} \psi_n(\beta)\, d\beta \qquad n = 1 , 2 , \ldots \tag{151}$$

$$\lambda_n^{\,2}\, \chi_n(\beta) = \int_{\xi} \left\{ \int_{\xi} h(\delta , \alpha)\, h(\delta , \beta)\, d\delta \right\} \chi_n(\alpha)\, d\alpha \qquad n = 1 , 2 , \ldots ; \tag{152}$$

and are related to each other by the equations

$$\lambda_n\, \psi_n(\alpha) = \int_{\xi} h(\alpha , \beta)\, \chi_n(\beta)\, d\beta \qquad n = 1 , 2 , \ldots \tag{153}$$

$$\lambda_n\, \chi_n(\alpha) = \int_{\xi} h(\alpha , \beta)\, \psi_n(\alpha)\, d\alpha \qquad n = 1 , 2 , \ldots . \tag{154}$$

The two sets of functions $\psi_n(\alpha)$ and $\chi_n(\beta)$ are known as the adjoint eigenfunctions of the kernel $h(\alpha, \beta)$, and the constants λ_n are said to be the eigenvalues of the kernel.

Schmidt's theorem holds if the following conditions are satisfied:

1) the functions $h(\alpha, \beta)$ and $h(\alpha, \beta)^2$ are integrable

2) the integrals $\int_\xi h(\alpha, \beta)^2 \, d\alpha$ and $\int_\xi h(\alpha, \beta)^2 \, d\beta$ are bounded

3) the functions η and η^2 are integrable.

All three of these conditions are satisfied for the inner integral of equation 146. The first and second conditions will always be satisfied by the well behaved linear systems of this report. The third condition is satisfied because of the bandlimited nature of the white noise functions (Ref. 47, page 170).

The set of eigenvalues λ_n is assumed to be ordered according to decreasing magnitude. It is also necessary that they satisfy Bessel's inequality,

$$\sum_{n=1}^{\infty} \lambda_n^2 \leq \iint_\xi h(\alpha, \beta)^2 \, d\alpha \, d\beta \, . \tag{155}$$

Schmidt's theorem will be applied to the inner integral of equation 146. Before doing so, however, it is useful to make the following observation. Note that equation 146 expresses k as a function of time. Since k is a stationary process however, its statistical properties cannot depend upon the time at which they are evaluated. Thus, for the purposes of calculating the probability distribution and level crossing frequency of $k(t)$, it must be permissible to fix t at any desired value. The value chosen in this report is zero. Therefore, upon taking equation 148 into account, equation 146 can be written

$$k(0) = R(1+R^2)^{-1/2} \int_0^{t_c} \eta_a(-\alpha) \left\{ \int_0^{t_c} \eta_b(-\beta)\, h(\alpha, \beta)\, d\beta \right\} d\alpha \,. \tag{156}$$

This is the equation to which Schmidt's theorem will be applied.

According to the theorem, the bracketed integral of equation 156 can be expressed as an absolutely and uniformly convergent series.

$$\int_0^{t_c} \eta_b(-\beta)\, h(\alpha, \beta)\, d\beta = \sum_{n=1}^{\infty} \psi_n(\alpha) \iint_0^{t_c} h(\delta, \beta)\, \eta_b(-\beta)\, \psi_n(\delta)\, d\beta\, d\delta \,. \tag{157}$$

All functions are continuous on $[0\,, t_c]$, so the order of integration can be interchanged and integration with respect to δ carried out first. By equation 154, the result is

$$\int_0^{t_c} \eta_b(-\beta)\, h(\alpha, \beta)\, d\beta = \sum_{n=1}^{\infty} \lambda_n \psi_n(\alpha) \int_0^{t_c} \eta_b(-\beta)\, \chi_n(\beta)\, d\beta \,. \tag{158}$$

Equation 158 can be substituted into equation 156 and, because the series is absolutely and uniformly convergent, integration with respect to α can be carried out term by term. The result is

$$k(0) = R(1+R^2)^{-1/2} \sum_{n=1}^{\infty} \lambda_n A_n B_n \,, \tag{159}$$

where

$$A_n = \int_0^{t_c} \eta_a(-\alpha)\, \psi_n(\alpha)\, d\beta \tag{160}$$

$$B_n = \int_0^{t_c} \eta_b(-\beta)\, \chi_n(\beta)\, d\beta \,. \tag{161}$$

A_n and B_n are random variables with the following properties

1) A_n and B_n are normally distributed because they are linear transformations of gaussian processes.
2) The mean value of each A_n and B_n is zero.
3) All A_n are independent of all B_n because η_a is independent of η_b .
4) For all n and m the correlation of A_n with A_m and B_n with B_m is given by

$$E\{A_n A_m\} = \iint_0^{t_c} \psi_n(\alpha)\psi_m(\beta)\; C_{\eta_a\eta_a}(\alpha - \beta)\; d\alpha\, d\beta \tag{162}$$

$$E\{B_n B_m\} = \iint_0^{t_c} \chi_n(\alpha)\chi_m(\beta)\; C_{\eta_b\eta_b}(\alpha - \beta)\; d\alpha\, d\beta \tag{163}$$

where $C_{\eta_a\eta_a}$ and $C_{\eta_b\eta_b}$ are the correlation functions of η_a and η_b respectively.

At this point a further approximation is introduced. Equation 159 expresses the non-gaussian portion of the vehicle response at an arbitrary instant (t = 0) as an infinite summation of random variables. In order to apply this equation to the practical evaluation of the response density function, it is going to be necessary to evaluate the eigenvalues, λ_n . Clearly it is unreasonable to expect to be able to evaluate more than the first few of these. Consequently, if equation 159 is to be of any practical use, it will be necessary that only the first few terms of the series be significant. Results to be derived shortly will show that the variance of the random variables A_n and B_n is approximately unity, and therefore the terms of equation 159 will decrease rapidly only if the eigenvalues decrease very rapidly. It will be assumed that this is the case.

Should this assumption prove invalid for some particular vehicle, it would follow that k could be represented by a summation of random variables with variances of comparable magnitude. These variables would be independent; and, though not identically distributed, it would not be unreasonable to expect that their sum would be nearly gaussian. In such a case the vehicle response would be almost gaussian, and the Graham - Charlier expansion mentioned earlier in this section would be a promising method of approach.

Under the assumption that only a few terms of the expansion are required to adequately represent k, equation 159 becomes

$$k(0) = R(1+R^2)^{-1/2} \sum_{n=1}^{N} \lambda_n A_n B_n . \qquad (164)$$

This assumption will also permit it to be shown that the random variables A_n and B_n can be considered independent and (0 , 1) normal.

Consider the correlation of A_n with A_m , equation 162. The correlation function of η_a can be written as the Fourier transform of its power spectral density. Noting the form of $\Phi_{\eta_a \eta_a}$ shown in figure 22, and using the transformation presented in equation 38, this relationship becomes

$$C_{\eta_a \eta_a}(\tau) = \int_{-\Omega_c}^{\Omega_c} \exp(i\, U \Omega \tau)\, d\Omega \qquad (165)$$

Substitution of this result into equation 162 and interchange of the order of integration gives

$$E\{A_n A_m\} = \int_{-\Omega_c}^{\Omega_c} \left\{ \int_0^{t_c} \psi_n(\alpha) \exp(i\, U \Omega\, \alpha)\, d\alpha \right\} \left\{ \int_0^{t_c} \psi_m(\beta) \exp(-i\, U \Omega\, \beta)\, d\beta \right\} d\Omega \tag{166}$$

where it is assumed that ψ_n and ψ_m are zero outside of the interval $[0\,, t_c]$. The bracketed terms of equation 166 can be identified as the Fourier transform of ψ_n and ψ_m.

Inspection of the kernel defined in equation 147 will now show that $h(\alpha\,,\beta)$ is zero if either α or β is zero, but $h(t_c\,,\beta)$ and $h(\alpha\,,t_c)$ are not necessarily zero. It follows therefore from equation 151 that $\psi_n(0)$ is equal to zero, but $\psi_n(t_c)$ is not necessarily zero. Thus $\psi_n(\alpha)$ and $\psi_m(\beta)$ in equation 166 are continuous everywhere except possibly at the points α and β equal to t_c. At these points, ψ_n and ψ_m may exhibit a simple step discontinuity. The theory of Fourier transforms (Ref. 51) then requires that the absolute value of the Fourier transforms appearing in equation 166 must decrease at least as rapidly as Ω^{-1} for large Ω. The product of transforms must therefore decrease at least as rapidly as Ω^{-2}. Thus, for all possible combinations of the indices n and m bounded by N, it must be possible to choose a single, finite value of Ω_c such that $E\{A_n A_m\}$ differs arbitrarily little from equation 166 with Ω_c replaced by infinity Equation 166 therefore reduces to

$$E\{A_n A_m\} \doteq \int_0^{t_c} \psi_n(\alpha)\, \psi_m(\alpha)\, d\alpha \tag{167}$$

Since the functions ψ_n and ψ_n are orthonormal, it follows that

$$E\{A_n A_m\} \doteq \begin{cases} 1 \text{ if } & n = m \\ 0 \text{ if } & n \neq m \end{cases} \tag{168}$$

A completely analogous result holds for equation 163.

Equation 168, along with the other properties of A_n and B_n listed previously, implies that the A_n and B_n or equation 164 can be assumed to be independent normal random variables with zero mean value and unit variance. It has been shown previously in this report that the product of independent gaussian random variables has a K_o probability distribution (Eq. 70). Hence, each term of equation 164 is a random variable with a density function of the form

$$p_n(x) = \frac{1}{\pi\sigma_n} K_o\left(\frac{|x|}{\sigma_n}\right) \qquad n = 1, 2, \ldots, N, \tag{169}$$

where

$$\sigma_n^2 = E\{(R\lambda_n A_n B_n)^2(1+R^2)^{-1}\} = (R\lambda_n)^2(1+R^2)^{-1}. \tag{170}$$

The reader will now recall that the object of all this is to obtain an approximate expression for the characteristic function of the non-gaussian portion of the vehicle response. It has just been shown that this response can be represented as a finite summation of independent random variables; therefore, the characteristic function in question must be given by a product of characteristic functions, one for each term of equation 164. It follows from equations 73, 169, and 170 that each term of this product is of the form

$$\phi_n(f) = [1 + (2\pi R \lambda_n f)^2 (1 + R^2)^{-1}]^{-1/2} , \qquad (171)$$

and the characteristic function of k is

$$\phi_k(f) = \prod_{n=1}^{N} [1 + (2\pi R \lambda_n f)^2 (1 + R^2)^{-1}]^{-1/2} . \qquad (172)$$

This completes the second step of the response density function derivation.

The characteristic function of the total vehicle response is now obtained by multiplying equations 143 and 172.

$$\phi_r(f) = \exp[-2(\pi \sigma_e f)^2 (1 + R^2)^{-1}] \prod_{n=1}^{N} [1 + (2\pi R \lambda_n f)^2 (1 + R^2)^{-1}]^{-1/2} \qquad (173)$$

This completes the third step of the derivation. The density function of r can now be found by numerically Fourier transforming equation 173. Program PDIST presented in Appendix A of this report performs this calculation.

Computer programs EIGU and EIGVW, which will also be found in Appendix A can be used to obtain the eigenfunctions and eigenvalues of the vehicle response to the longitudinal gust component and to the lateral and vertical gust components respectively. Note from equations 151 and 152 that, since the function $h(\alpha, \beta)$ is symmetric for the longitudinal gust case, the orthonormal sets ψ_n and χ_n will be identical. Thus program EIGU computes only one set of eigenfunctions. The eigenvalues obtained from either EIGU or EIGVW can then be used in program PDIST to obtain the probability density and distribution function of the vehicle response. The next section of this report will present a numerical example showing the application of these programs.

Approximate Solution for Response Level

Crossing Frequency

Recall from equation 81 that the level crossing frequency of a stochastic process can be found from a knowledge of the joint distribution of the process and its derivative,

$$N_r(x) = \int_0^\infty \dot{x}\, p_{r,\dot{r}}(x, \dot{x})\, d\dot{x} . \qquad (174)$$

The reader should note that the explicit parameter dependence notation used in earlier sections of this report is not used here in order to simplify notation. The results presented here will, of course, depend upon the parameters of the turbulence model as well as the dynamic characteristics of the vehicle.

Previously in this section it was shown that the distribution function of the response could be approximated by means of a characteristic function approach. It will now be shown that the joint distribution of the response and its derivative can be approximated by an analogous procedure. The steps to be followed are:

1) obtain the joint characteristic function of the gaussian portion of the response and its derivative, $\phi_{g,\dot{g}}$
2) obtain the joint characteristic function of the non-gaussian portion of the response and its derivative, $\phi_{k,\dot{k}}$

3) multiply these results to obtain the joint characteristic function of the complete response and its derivative, $\phi_{r,\dot{r}}$

4) inverse Fourier transform to obtain the joint density of the response and its derivative, $p_{r,\dot{r}}$

5) apply equation 174 to obtain the level crossing frequency of the response, $N_r(x)$.

Refer to figure 22 and note that the derivative of the response can be obtained by merely replacing the vehicle transfer function $H_v(s)$ with $sH_v(s)$.

The joint characteristic function of the gaussian portion of the response and its derivative can be found immediately.

$$\phi_{g,\dot{g}}(f_1, f_2) = \exp\{-2(1+R^2)^{-1}[(\pi\sigma_e f_1)^2 + (\pi\sigma_{\dot{e}} f_2)^2]\} \qquad (175)$$

where the standard deviations can be obtained from the power spectral density of $e(t)$. This completes the first step of the derivation.

The joint characteristic function of k and $\dot{k}$ will now be approximated by extending the results obtained above for the characteristic function of k alone. By equation 164, k can be represented as a finite summation of random variables.

$$k(0) = R(1+R^2)^{-1/2} \sum_{n=1}^{N} \lambda_n A_n B_n \qquad (176)$$

It will now be assumed that a similar approximation applies to $\dot{k}$,

$$\dot{k}(0) = R(1+R^2)^{-1/2} \sum_{m=1}^{M} \Lambda_m C_m D_m \,, \tag{177}$$

where Λ_m are the eigenvalues of the kernel analogous to $h(\alpha , \beta)$ of equation 147; and C_m , D_m are (0 , 1) normal, independent random variables. The adjoint eigenfunctions of the $\dot{k}$ kernel are assumed to be $\Psi_m(\alpha)$ and $X_m(\beta)$. It is further assumed that the noise bandwidths, Ω_c of figure 23, and the impulse response truncation times, $t_c/2$ of figures 24, are identical for both k and $\dot{k}$.

Now even though all of the random variables A_n and B_n are mutually independent, and all of the random variables C_m and D_m are mutually independent, it is readily verified that the variables A_n are correlated with the variables C_m and likewise for the variables B_n and D_m. These correlations can be shown to be

$$E\{A_n C_m\} = \int_0^{t_c} \psi_n(\alpha)\, \Psi_m(\alpha)\, d\alpha \tag{178}$$

$$E\{B_n D_m\} = \int_0^{t_c} \chi_n(\beta)\, X_m(\beta)\, d\beta \,. \tag{179}$$

In order to derive an approximate expression for the joint characteristic function of k and $\dot{k}$ it is convenient to introduce the following notation.

$$\overline{A} = [A_1\, A_2\, \ldots,\, A_N\, C_1\, C_2\, \ldots,\, C_M]^T \tag{180}$$

$$\overline{B} = [B_1\, B_2\, \ldots,\, B_N\, D_1\, D_2\, \ldots,\, D_M]^T \tag{181}$$

$$\tilde{A} = E\{\bar{A}\bar{A}^T\} \tag{182}$$

$$\tilde{B} = E\{\bar{B}\bar{B}^T\} \tag{183}$$

Since the elements of $\bar{A}$ are independent of those in $\bar{B}$, it follows that the joint density of all the random variables in both $\bar{A}$ and $\bar{B}$ is given by

$$p_{\bar{A},\bar{B}}(\bar{x},\bar{y}) = p_{\bar{A}}(\bar{x})\,p_{\bar{B}}(\bar{y}) \ , \tag{184}$$

where

$$p_{\bar{A}}(\bar{x}) = [(2\pi)^{\frac{M+N}{2}} \det(\tilde{A})]^{-1} \exp[-\bar{x}^T(\tfrac{1}{2}\tilde{A}^{-1})\bar{x}] \ , \tag{185}$$

and a similar expression applies to $p_{\bar{B}}$. The joint characteristic function of k and $\dot{k}$ is defined to be

$$\phi_{k,\dot{k}}(f_1, f_2) = E\{\exp(i2\pi\, k f_1 + i2\pi\, \dot{k} f_2)\} . \tag{186}$$

Substitution for k and $\dot{k}$ from equations 176 and 177 gives

$$\phi_{k,\dot{k}}(f_1, f_2) = E\{\exp(\bar{A}^T \tilde{C} \bar{B})\} \tag{187}$$

where

$$\tilde{C} = i2\pi R(1+R^2)^{-1/2}[\operatorname{diag}(f_1\lambda_1, \ldots, f_1\lambda_N, f_2\Lambda_1, \ldots, f_2\Lambda_M)] \ . \tag{188}$$

The joint characteristic function $\phi_{k,\dot{k}}$ can now be written in the integral form

$$\phi_{k,\dot{k}}(f_1, f_2) = \int_{-\infty}^{\infty} [2(M+N)] \int_{-\infty}^{\infty} \exp(\overline{x}^T \tilde{C} \overline{y}) \, p_{\overline{A}}(\overline{x}) \, p_{\overline{B}}(\overline{y}) \, d\overline{x} \, d\overline{y} \, , \tag{189}$$

This expression can be integrated numerically to obtain the value of $\phi_{k.\dot{k}}$ for any choice of f_1 and f_2. This completes the second step of the derivation.

The third, fourth, and fifth steps of the derivation are carried out by multiplying equations 175 and 189, applying a two-dimensional Fourier transform, and then integrating as indicated in equation 174. Program RLEVX, presented in Appendix A of this report performs all of the computations described here including the evaluation of the covariance matrices, equations 182 and 183. Use of this program in connection with the eigensolution programs EIGU and EIGVW as shown in the next section of this report thus provides an automated method of computing the level crossing frequency of a linear response to the non-gaussian turbulence model. This completes the discussion of the level crossing problem.

Response to Multiple Inputs

The above remarks have all dealt with the case of a single component of the turbulence model. Because the three components of the model are independent processes, however, these results can be easily extended to include the case of a vehicle disturbed simultaneously by two or all three gust components.

Recall that the three quantities of interest are the response power spectral density, distribution function, and level crossing frequency. The power spectral density of a linear response to several independent inputs is merely the sum of the spectral densities due to each input alone. Equation 142 can therefore be used to calculate a power spectrum for the response to each gust component and the total response power spectrum found by summing.

The distribution function of the response to multiple inputs can be found through its characteristic function. Since the response to each gust component is an independent process, the characteristic function of the total response will merely be the product of the characteristic functions corresponding to each component. Thus equation 173 could be used to find the characteristic function of the response to each component of the turbulence model, and the characteristic function of the total response found by multiplying. This product could then be Fourier transformed to yield the required distribution function.

A similar argument applies to the determination of response level crossings. In this case it is the joint characteristic function of the total response and its derivative which are found by multiplying the joint characteristic functions corresponding to each gust component alone. The joint density of the response and its derivative are then obtained by Fourier transforming, and equation 174 is used to calculate the level crossing frequency.

Summary

This section has investigated the response of a linear vehicle to the proposed non-gaussian turbulence model. An expression for the power spectral density of the response has been presented (Eq. 142), and approximate numerical procedures have been suggested for computing the response distribution function (Eq. 143 through 173) and level crossing frequency (Eq. 174 through 189). Finally, the problem of calculating response statistics induced by simultaneous gust components has been briefly discussed. In the next section of this report, the techniques developed above will be used to calculate the responses of an aircraft-autopilot system to the non-gaussian model.

NUMERICAL EXAMPLE

The previous section of this report has derived techniques which can be used to calculate the power spectral density, probability distribution, and level crossing frequency of a linear response to the proposed non-gaussian turbulence model. The purpose of this section is to illustrate the application of these techniques to a simple problem, and to compare the results with those obtained using the gaussian model. The computer programs mentioned will be found in Appendix A of this report.

Problem Statement

The aircraft to be studied is a STOL vehicle in cruising flight. Altitude is controled by a simple autopilot system using altitude error, its integral, and its derivative in a feedback loop driving the elevator so as to return the aircraft to the commanded altitude.

It is supposed that this aircraft - autopilot system is perturbed by the vertical component of the non-gaussian model, and it is desired to investigate the deviations from the commanded cruise altitude which the system experiences. The power spectral density, probability distribution, and level crossing frequency of the altitude error are to be calculated.

Model of the Vehicle

As mentioned above, the aircraft considered here is a STOL vehicle in cruising flight. The commanded altitude is 305 meters (1,000 feet) above sea level, and the true airspeed in equilibrium flight is 76 meters per second (249.7 feet per second). The transfer function of the altitude error due to vertical gust disturbances for this particular aircraft - autopilot system is

$$\frac{z}{w}(s) = \frac{40.92[s][s+(.0363 \pm .2083i)][s+2.912]}{[s+(.0443 \pm .0131i)][s+(.3035 \pm .2908i)][s+(2.112 \pm 2.404i)]} , \tag{190}$$

where the notation $[s+(a \pm ib)]$ has been used to indicate complex conjugate pairs of poles or zeros. Note that this transfer function satisfies all of the conditions stated at the beginning of the previous section (page 106). The derivative of the altitude error is

$$\frac{\dot{z}}{w}(s) = s\,\frac{z}{w}(s) . \tag{191}$$

In order to apply the computer programs presented in the appendix of this report, it will be necessary to express these two transfer functions in partial fraction form.

$$\frac{z}{w}(s) = \frac{[-1.946 \pm 5.538i]}{[s+(.0443 \pm .0131i)]} + \frac{[-4.016 \pm 8.026i]}{[s+(.03035 \pm .2908i)]} + \frac{[5.964 \mp 4.816i]}{[s+(2.112 \pm 2.404i)]} \tag{192}$$

$$\frac{\dot{z}}{w}(s) = \frac{[.158 \mp .220i]}{[s+(.0443 \pm .0131i)]} + \frac{[3.552 \mp 1.268i]}{[s+(.3035 \pm .2908i)]} + \frac{[24.18 \mp 4.164i]}{[s+(2.112 \pm 2.404i)]} \tag{193}$$

Turbulence Model Parameters

The turbulence model used here is the vertical component of the non-gaussian model. The parameters of the model are chosen to be

$$\left.\begin{aligned} L_w &= 142 \text{ m } (465 \text{ ft}) \\ \sigma_w &= 0.305 \text{ m/sec } (1 \text{ ft/sec}) \\ R_w &- 1.0 \;. \end{aligned}\right\} \quad (194)$$

These parameters are typical of those which would be encountered at an altitude of 228 meters (750 feet) over plains in unstable atmospheric conditions. With this choice of σ_w and L_w, the power spectral density of the turbulence becomes

$$\Phi_{ww}(\Omega) = 2.10 \frac{(1 + 60{,}500\,\Omega^2)}{(1 + 20{,}200\,\Omega^2)^2} \;. \quad (195)$$

The probability density and distribution functions of the gust velocity will be found in figure 15 on page 61, and the normalized level crossing frequency is given in figure 17 on page 80.

Response Power Spectral Density

The power spectral density of the altitude error is obtained by substituting equations 190 and 195 into equation 142. Figure 25 presents the resulting spectrum for positive values of Ω. The complete spectrum is, of course, symmetric about the origin. The variance of the altitude error can be calculated by integrating the power spectral density as indicated in equation 7. The result is

$$\sigma_r^{\;2} = 13.26 \text{ (m)}^2 \; [142.8 \text{ (ft)}^2] \tag{196}$$

Note that the power spectral density and variance are independent of the turbulence model parameter R.

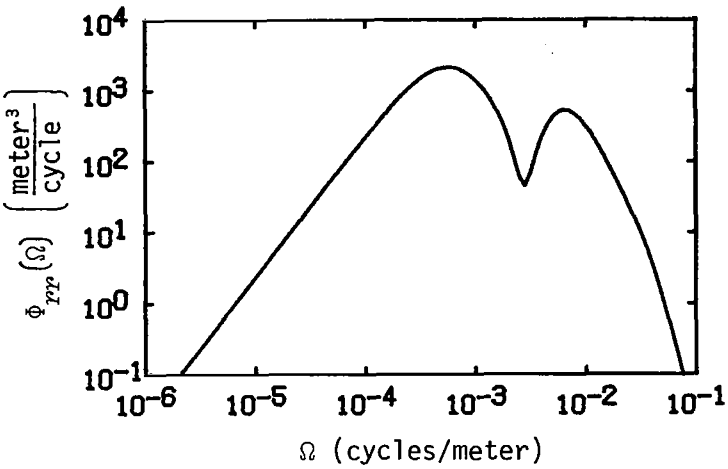

Figure 25.--Power spectral density of the altitude error.

Response Distribution Function

The probability distribution of the altitude error will now be calculated. This computation is in two parts,

1) use program EIGVW to compute the significant eigenvalues of the response
2) use these eigenvalues as input to program PDIST to obtain the response probability distribution function.

The first step of the procedure is the calculation of the altitude error eigensolutions. The transfer function of the system (Eq. 192) and the turbulence model parameters (Eq. 194), when used as input to program

EIGVW, result in the eigenvalues and adjoint eigenfunctions presented in figure 26. Note that program EIGVW cannot compute the eigenvalues without also computing the corresponding adjoint eigenfunctions.

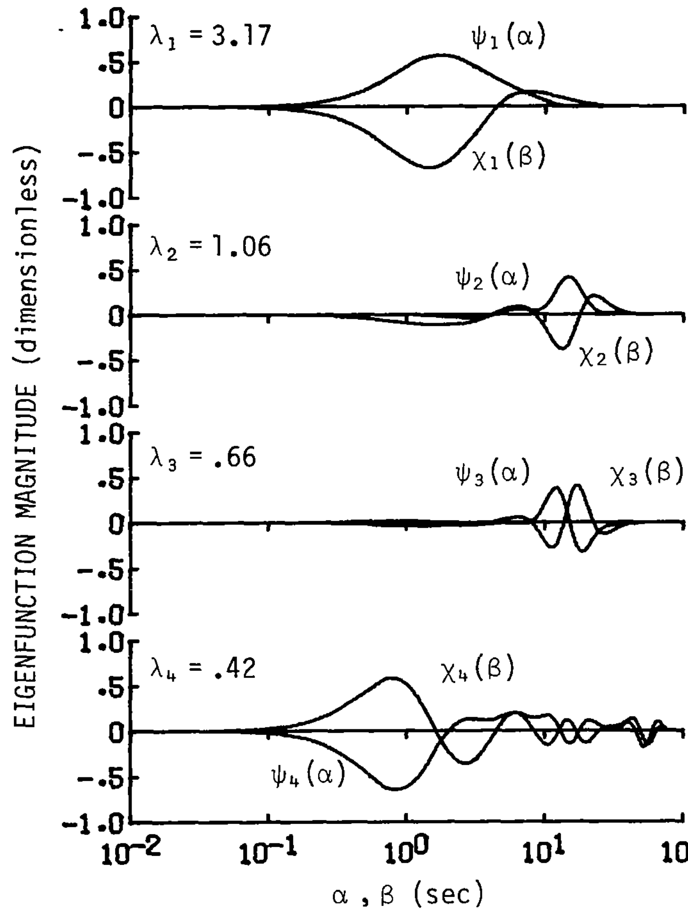

Figure 26.--Adjoint eigenfunctions and eigenvalues of the altitude error.

The second step in computing the response distribution is the use of the eigenvalues of figure 26 as input to program PDIST. This program has computed the probability density and distribution functions presented in figure 27. (Although used as input to PDIST, the eigenvalues of the response were not actually used in computing the gaussian distribution.

In fact, the gaussian result is merely the well known normal distribution which could be evaluated without the use of a computer

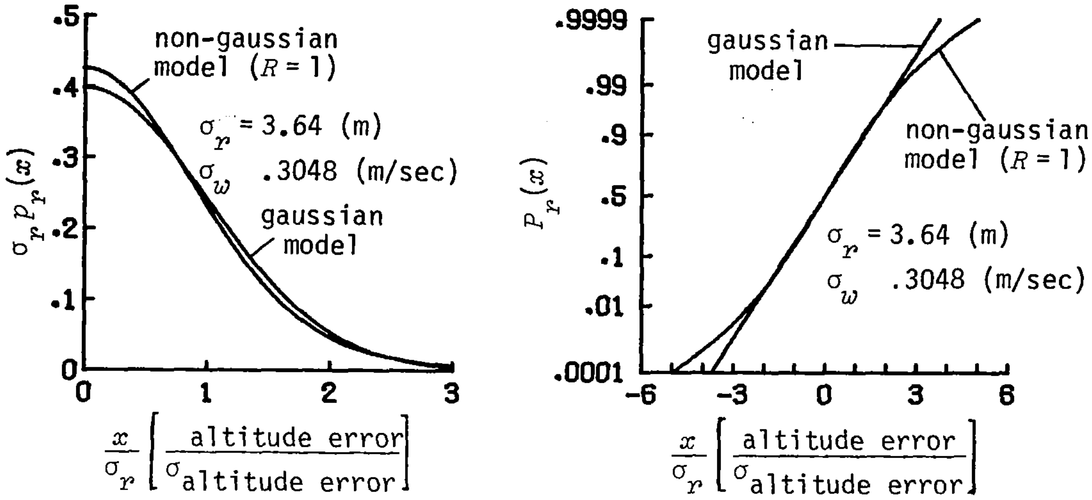

Figure 27.--Probability density and distribution functions of the altitude error in response to both gaussian and non-gaussian turbulence models.

Note that the non-gaussian model predicts that the occurrence of large altitude errors will occupy a much greater percentege of time than indicated by the gaussian model. For example, the computed values of the gaussian and non-gaussian distribution functions for a response magnitude of $3.5\,\sigma_r$ are 0.99977 and 0.99860 respectively. These numbers imply that the absolute value of the altitude error predicted by the gaussian model will exceed $3.5\,\sigma_r$ about 0.05% of the total flight time while the error predicted by the non-gaussian model will exceed this level about 0.28% of the time. These predictions differ by more than a factor of five.

Response Level Crossing Frequency

Consider now the level crossing frequency of the response. This calculation is in three parts,

1) compute the significant eigenvalues and eigenfunctions of the altitude error by means of program EIGVW
2) compute the significant eigenvalues and eigenfunctions of the altitude error derivative by means of program EIGVW
3) use the results of the first two steps as input to program RLEVX and obtain the level crossing frequency of the altitude error.

The eigensolutions of the altitude error have already been calculated, and are presented in figure 26. The eigensolutions of the altitude error derivative can be calculated using equations 193 and 194 as input data to program EIGVW. The results are presented in figure 28.

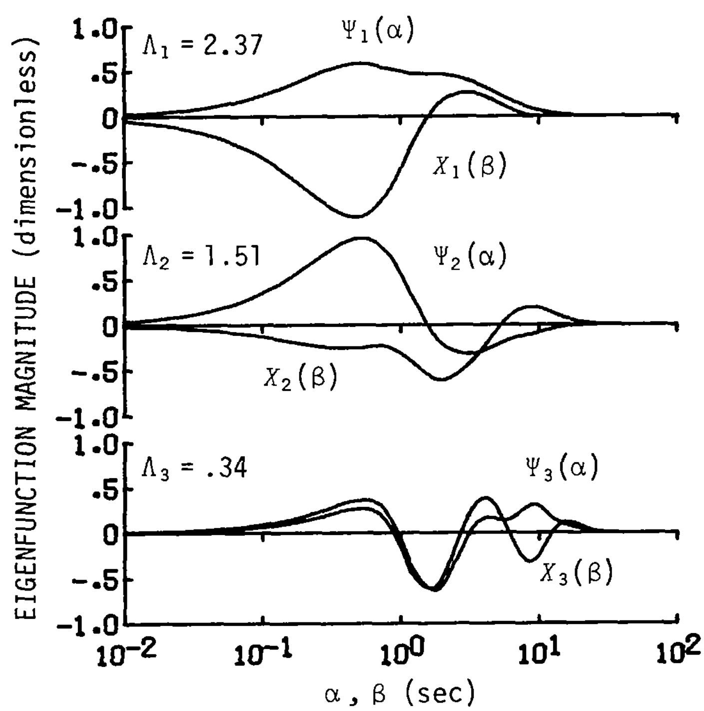

Figure 28.--Adjoint eigenfunctions and eigenvalues of the altitude error time derivative.

All of these eigenvalues and eigenfunctions (Figs. 26 and 28) are now used with program RLEVX to compute the level crossing frequency curves presented in figure 29. It should be noted that the gaussian result

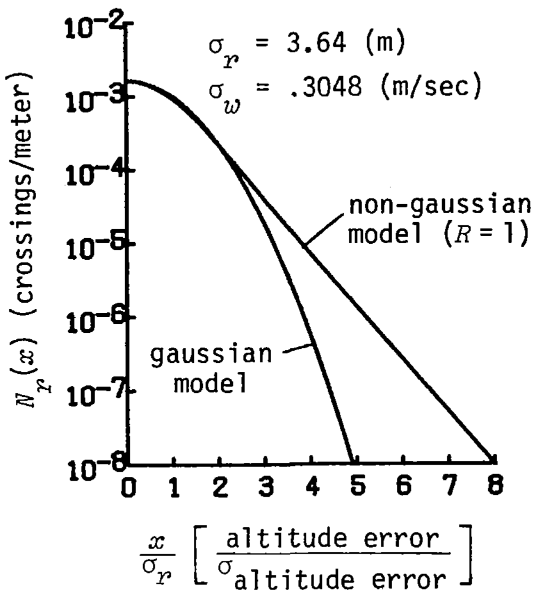

Figure 29.--Level crossing frequency of the altitude error in response to both gaussian and non-gaussian turbulence models.

follows directly from Rice's equation (Eq. 12), and could have been obtained without use of a computer.

Note that the non-gaussian model predicts that large altitude errors will occur much more frequently than indicated by the gaussian model. For example, the crossing frequencies of the $4\sigma_r$ response level predicted by the gaussian and non-gaussian models are 4.274×10^{-7} and 7.116×10^{-6} crossings per meter respectively. The gaussian value implies that the $4\sigma_r$ level will be exceeded once every 2,340 kilometers, while the non-gaussian value implies that this level will be exceeded once every 141 kilometers. These results differ by a factor of almost 17. Furthermore, the ratio of the two results increases rapidly with increasing response magnitude. At the $5\sigma_r$ level they differ by a factor of well over 100.

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

The previous sections of this report have developed a new model of atmospheric turbulence which is proposed for use in aircraft design work. In this section, the principal conclusions are summarized and some areas requiring further investigation are discussed.

Principal Conclusions

This report has reviewed the problem of modeling continuous atmospheric turbulence for the purposes of aircraft design. The model now in wide use; which assumes turbulence to be a homogeneous, stationary gaussian process with a specified power spectral density; has been discussed and its properties compared with experimentally measured characteristics of atmospheric turbulence. This comparison has shown that the gaussian model does not properly reproduce the number of high velocity gusts which are encountered in the atmosphere. Neither can it properly model the observed patchy character of turbulence or the experimentally measured distributions of velocity increments.

A modified form of the gaussian model, the gaussian patch model, has also been considered. This model is similar to the gaussian model, but assumes large regions of turbulence to be composed of independent patches which are homogeneous, stationary, and gaussian. The intensity of the turbulence varies randomly from patch to patch, but is assumed constant within each patch. The patch size is assumed to be sufficiently great that transient effects between patches can be neglected. It has been shown that this model cannot reproduce the distributions of velocity increments measured in turbulence. Furthermore, experimental measurement of patch sizes implied by this model has shown that the most intense patches are so short (on the order of 2-3 km) that in many cases the neglect of transient effects between patches cannot be justified.

A non-gaussian model of turbulence has been proposed. This model has been formulated from the idea of representing the patchy character of turbulence by a product of independent gaussian processes. The new model can be viewed as an extended or generalized form of the gaussian model, and the gaussian model is included as a special case.

The statistical properties of the proposed model have been derived and compared with some experimentally measured properties of low altitude atmospheric turbulence. The results have shown that the power spectral densities, probability distributions, and level crossing frequencies observed in the atmosphere can be modeled quite well. However, it has also been shown that this model, like the gaussian and gaussian patch models, does not reproduce the velocity increment distributions of turbulence. The reason for this difficulty has been discussed and a modification of the model has been suggested which may correct the problem. Until a solution is found, it is suggested that the proposed model (as well as the gaussian and gaussian patch models discussed above) is not a good representation of the high frequency components of turbulence. Hence, these models are probabily not suitable for studies of high frequency vehicle responses such as those involving structural modes.

Methods by which the non-gaussian model can be used to calculate vehicle response statistics have been investigated and techniques for computing the power spectral density, probability distribution, and level crossing frequency of linear responses to the model have been developed. These results have shown that the response power spectral density can be obtained by the usual methods of linear system theory, and therefore calculated without difficulty. Evaluation of response probability distributions and level crossing frequencies, on the other hand, has required calculation of the eigenfunctions and eigenvalues of a certain kernel function which is, in general, unsymmetric. Computer programs presented in Appendix A of this report have been developed as a means of automating these calculations.

The altitude response statistics of a simple, linear, STOL aircraft - autopilot system subjected to the vertical component of the non-gaussian model have been calculated. The results have shown that, compared to the gaussian model, the non-gaussian model model may predict more than a hundredfold increase in the occurences of large response magnitudes.

Suggestions for Further Research

The turbulence model and application methods developed in this report can be considered as only the first step in producing a practical tool for use in aircraft design work. Additional research in several areas will be required if this goal is ever to be attained. The following is a list of several areas which the author believes to be topics of useful research regarding this turbulence model.

Improved Data Fitting Procedures

It has been shown that the LO-LOCAT data presented in this report can apparently be fit better by the non-gaussian turbulence model than by the currently used gaussian model. The results of Table 4, however, indicates that it is important to choose the correct goodness-of-fit criterion if satisfactory results are to be obtained. Thus some rational prodedure for obtaining the best over-all fit of the data must be found. This procedure would have application not only to the non-gaussian turbulence model described here, but also to any other turbulence models which might be proposed in the future.

Relationship of the Non-gaussian Model to Meteorological Parameters

The proposed non-gaussian turbulence model contains a parameter which controls its probability distribution and level crossing frequency. Results presented previously in this report (Table 4) indicate that the value of this parameter which leads to the best fit of experimental turbulence data depends, at least partially, on variables such as atmospheric stability, surface roughness, and height above the surface. It should be possible to carry out a regression analysis to determine the relationship between these variables and the parameter. In order to be of practical value, this analysis will have to consider much more data than used in the present report.

Improved Mathematical Development

The results derived in this report have relied very heavily upon heuristic assumptions such as the use of band limited white noise and truncated impulse response functions. These assumptions can surely be relaxed in a rigorous treatment. Although increased rigor would not be expected to alter any of the results which have been presented, it would almost certainly lead to a better understanding of the non-gaussian model and a greater appreciation of its limitations.

An especially useful result of such an investigation might be a rapid procedure for obtaining approximate solutions for response statistics.

It is intuitively obvious, and easy to show from the development of equations 144 through 189, that the vehicle response statistics become identical to the statistics of the turbulence as the bandwidth of the vehicle transfer function becomes infinitely wide with respect to the bandwidth of the turbulence. On the other hand, as the vehicle bandwidth becomes indefinitely narrow with respect to that of the turbulence, the Gerschgorin Circle Theorem concerning upper bounds of eigenvalues (Ref. 49) requires that the non-gaussian portion of the vehicle response become the sum of an infinite number of independent, identically distributed random variables. Hence, the Central Limit Theorem (Ref. 37) requires that this response become gaussian, and therefore the total response of the vehicle will become gaussian. It would be extremely useful to develop a method for estimating (or at least bounding) response statistics for the case of intermediate bandwidth vehicles.

Extensions of the Model

Several possible improvements of the turbulence model proposed in this report merit further investigation.

1) Reference 45 has suggested the possibility of defining a patchiness parameter for the non-gaussian model. It would be interesting to see if this parameter could be related to meteorological variables of turbulence, and if this parameter has any significant effect upon response statistics.
2) Nothing in the mathematical development presented here has prohibited the use of linear filters with irrational transfer functions in the turbulence model. This

suggests the possibility of using such filters to obtain the von Karman power spectral densities rather than the Dryden spectra assumed in the present report.

3) The present results are concerned only with a "point" model of turbulence, that is, the turbulence field is represented by three orthogonal gust components which are assumed to act at the vehicle center of gravity. Turbulence, however, is a distributed phenomenon; and for large aircraft it is important to take into account the distribution of the gust field over the surface of the vehicle. It appears that the spatial distribution representation used in reference 34 can, in principal, be incorporated into the non-gaussian model of this report without difficulty. This could be accomplished by adding an independent rolling gust component to the present model and using the vertical and lateral gust components of the present model in conjunction with an appropriate time delay to represent gusts occuring at the vehicle tail. Unfortunately, the computational procedures required to calculate vehicle responses to this model would become very unwieldy because it would be necessary to consider correlated inputs to the system rather than independent inputs as assumed in this report. Thus, either a different approach to the spatial distribution problem will have to be found, or the computational procedures will have to be greatly speeded.

4) It has been shown that the velocity increment distribution of the non-gaussian model do not match those observed in the atmosphere. As discussed previously in this report (page 84), it appears that this problem might be corrected by modifying the transfer functions used in the model.

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APPENDIX A

COMPUTER PROGRAMS

The purpose of this appendix is to present a number of computer programs which are of use in working with the non-gaussian model, and to give examples of their use. Programs to be presented are:

1. PDIST - a program to compute probability distributions of either the non-gaussian model or vehicle responses to the model
2. LEVXNG - a program to compute universal level crossing frequency curves for the non-gaussian model
3. INCPD - a program to compute probability distributions of the non-gaussian model velocity increments
4. EIGU - a program to compute eigenvalues and eigenfunctions of vehicle responses to the longitudinal component of the non-gaussian model
5. EIGVW - a program to compute eigenvalues and adjoint eigenfunctions of vehicle responses to the vertical or lateral components of the non-gaussian model
6. RLEVX - a program to compute level crossing frequency curves of vehicle responses to the non-gaussian model.

Listings of all programs and subroutines will be found at the end of this appendix. All coding is in FORTRAN language, version 2.3 for Control Data 6000 Series Computer Systems. Modifications required for compatibility with other systems should be slight.

The sample programs to be presented here were run on the University of Washington CDC-6400 computer using the U of W version of the Control

Data RUN compiler, and the SCOPE 3.4 operating system. Storage requirements and execution times reflect this operating environment.

Numerous comment cards have been incorporated into each of the card decks presented here as an aid in understanding their operation. The input data required by each program is also described by comment cards.

Program PDIST

PDIST is a program which computes probability distributions for both the non-gaussian turbulence model and vehicle responses to the model. The example presented here calculates the probability distribution of the model for an *R* parameter value of 1.0.

Card Decks

The following card decks were required to produce the sample calculation presented here:

PDIST

FFT

FFTRS

Input Data

The following five data cards were used in producing this example. The meaning and format of these cards are explained by comments in the listing of program PDIST.

```
1         11        21        31        41        51        61        71
TEST OF PDIST, PROBABILITY DISTRIBUTION OF NON-GAUSSIAN MODEL FOR R PARAM. = 1.0
     1.0000000000E+00
 1
     1.0000000000E+00
1.0
```

Results

The above named card decks were compiled, loaded, and executed on the University of Washington CDC-6400 computer system. Storage requirement including I/O buffers and all system programs was $16,406_8$ words. Execution, including compilation and loading, required 3_{10} seconds of central processor time. The following output was generated.

```
TEST OF POIST, PROBABILITY DISTRIBUTION OF NON-GAUSSIAN MODEL FOR R PARAM. = 1.0

VARIANCE =    1.00000
NUMBER OF NON-GAUSSIAN EIGENVALUES = 1
EIGENVALUES ARE:
       1.0000000E+00

SIGMA RATIO OF TURBULENCE MODEL =    1.00000

SUM OF EIGENVALUES SQUARED =  1.0000000E+00

FOR VARIANCE OF  1.0000000E+00
NON-GAUSSIAN VARIANCE =  5.0000000E-01
GAUSSIAN VARIANCE INCLUDING CORRECTION FOR
NEGLECTED EIGENVALUES =  5.0000000E-01

FOR TOTAL VARIANCE OF UNITY
NON-GAUSSIAN VARIANCE =  5.0000000E-01
GAUSSIAN VARIANCE INCLUDING CORRECTION FOR
NEGLECTED EIGENVALUES =  5.0000000E-01

SQUARED EIGENVALUES SCALED TO GIVE CORRECT NON-GAUSSIAN
CONTRIBUTION TO UNIT VARIANCE:
  5.0000000E-01
```

TEST OF PDIST, PROBABILITY DISTRIBUTION OF NON-GAUSSIAN MODEL FOR R PARAM. = 1.0

NORMALIZED VARIABLE X/SIGMA X	STDIZED PROBABILITY DENSITY	DISTRIBUTION FUNCTION	UNNORMALIZED VARIABLE X	NONSTDIZED PROBABILITY DENSITY
0.000	4.455E-01	5.00000E-01	0.	4.455E-01
.050	4.447E-01	5.22262E-01	5.000E-02	4.447E-01
.100	4.423E-01	5.44445E-01	1.000E-01	4.423E-01
.150	4.384E-01	5.66469E-01	1.500E-01	4.384E-01
.200	4.330E-01	5.88260E-01	2.000E-01	4.330E-01
.250	4.261E-01	6.09741E-01	2.500E-01	4.261E-01
.300	4.178E-01	6.30844E-01	3.000E-01	4.178E-01
.350	4.083E-01	6.51503E-01	3.500E-01	4.083E-01
.400	3.976E-01	6.71656E-01	4.000E-01	3.976E-01
.450	3.859E-01	6.91249E-01	4.500E-01	3.859E-01
.500	3.733E-01	7.10231E-01	5.000E-01	3.733E-01
.550	3.598E-01	7.28561E-01	5.500E-01	3.598E-01
.600	3.457E-01	7.46202E-01	6.000E-01	3.457E-01
.650	3.311E-01	7.63125E-01	6.500E-01	3.311E-01
.700	3.161E-01	7.79306E-01	7.000E-01	3.161E-01
.750	3.008E-01	7.94730E-01	7.500E-01	3.008E-01
.800	2.854E-01	8.09387E-01	8.000E-01	2.854E-01
.850	2.700E-01	8.23274E-01	8.500E-01	2.700E-01
.900	2.547E-01	8.36392E-01	9.000E-01	2.547E-01
.950	2.396E-01	8.48749E-01	9.500E-01	2.396E-01
1.000	2.248E-01	8.60357E-01	1.000E+00	2.248E-01
1.050	2.103E-01	8.71234E-01	1.050E+00	2.103E-01
1.100	1.963E-01	8.81399E-01	1.100E+00	1.963E-01
1.150	1.828E-01	8.90876E-01	1.150E+00	1.828E-01
1.200	1.699E-01	8.99691E-01	1.200E+00	1.699E-01
1.250	1.575E-01	9.07872E-01	1.250E+00	1.575E-01
1.300	1.457E-01	9.15450E-01	1.300E+00	1.457E-01
1.350	1.346E-01	9.22455E-01	1.350E+00	1.346E-01
1.400	1.241E-01	9.28919E-01	1.400E+00	1.241E-01
1.450	1.142E-01	9.34874E-01	1.450E+00	1.142E-01
1.500	1.050E-01	9.40352E-01	1.500E+00	1.050E-01
1.550	9.638E-02	9.45384E-01	1.550E+00	9.638E-02
1.600	8.838E-02	9.50001E-01	1.600E+00	8.838E-02
1.650	8.096E-02	9.54232E-01	1.650E+00	8.096E-02
1.700	7.410E-02	9.58106E-01	1.700E+00	7.410E-02
1.750	6.777E-02	9.61650E-01	1.750E+00	6.777E-02
1.800	6.194E-02	9.64891E-01	1.800E+00	6.194E-02
1.850	5.659E-02	9.67852E-01	1.850E+00	5.659E-02
1.900	5.168E-02	9.70557E-01	1.900E+00	5.168E-02
1.950	4.719E-02	9.73027E-01	1.950E+00	4.719E-02
2.000	4.309E-02	9.75283E-01	2.000E+00	4.309E-02
2.050	3.934E-02	9.77342E-01	2.050E+00	3.934E-02
2.100	3.592E-02	9.79223E-01	2.100E+00	3.592E-02
2.150	3.281E-02	9.80940E-01	2.150E+00	3.281E-02
2.200	2.997E-02	9.82508E-01	2.200E+00	2.997E-02
2.250	2.739E-02	9.83941E-01	2.250E+00	2.739E-02
2.300	2.503E-02	9.85250E-01	2.300E+00	2.503E-02
2.350	2.289E-02	9.86448E-01	2.350E+00	2.289E-02
2.400	2.094E-02	9.87543E-01	2.400E+00	2.094E-02
2.450	1.917E-02	9.88545E-01	2.450E+00	1.917E-02
2.500	1.756E-02	9.89463E-01	2.500E+00	1.756E-02

TEST OF POIST, PROBABILITY DISTRIBUTION OF NON-GAUSSIAN MODEL FOR R PARAM. = 1.0

NORMALIZED VARIABLE X/SIGMA X	STDIZED PROBABILITY DENSITY	DISTRIBUTION FUNCTION	UNNORMALIZED VARIABLE X	NONSTDIZED PROBABILITY DENSITY
2.500	1.756E-02	9.89463E-01	2.500E+00	1.756E-02
2.550	1.608E-02	9.90303E-01	2.550E+00	1.608E-02
2.600	1.474E-02	9.91073E-01	2.600E+00	1.474E-02
2.650	1.352E-02	9.91779E-01	2.650E+00	1.352E-02
2.700	1.241E-02	9.92427E-01	2.700E+00	1.241E-02
2.750	1.139E-02	9.93022E-01	2.750E+00	1.139E-02
2.800	1.046E-02	9.93568E-01	2.800E+00	1.046E-02
2.850	9.612E-03	9.94069E-01	2.850E+00	9.612E-03
2.900	8.836E-03	9.94530E-01	2.900E+00	8.836E-03
2.950	8.126E-03	9.94954E-01	2.950E+00	8.126E-03
3.000	7.476E-03	9.95344E-01	3.000E+00	7.476E-03
3.050	6.881E-03	9.95702E-01	3.050E+00	6.881E-03
3.100	6.335E-03	9.96033E-01	3.100E+00	6.335E-03
3.150	5.835E-03	9.96337E-01	3.150E+00	5.835E-03
3.200	5.375E-03	9.96617E-01	3.200E+00	5.375E-03
3.250	4.954E-03	9.96875E-01	3.250E+00	4.954E-03
3.300	4.567E-03	9.97113E-01	3.300E+00	4.567E-03
3.350	4.211E-03	9.97332E-01	3.350E+00	4.211E-03
3.400	3.884E-03	9.97534E-01	3.400E+00	3.884E-03
3.450	3.583E-03	9.97721E-01	3.450E+00	3.583E-03
3.500	3.306E-03	9.97893E-01	3.500E+00	3.306E-03
3.550	3.052E-03	9.98052E-01	3.550E+00	3.052E-03
3.600	2.817E-03	9.98199E-01	3.600E+00	2.817E-03
3.650	2.601E-03	9.98334E-01	3.650E+00	2.601E-03
3.700	2.402E-03	9.98459E-01	3.700E+00	2.402E-03
3.750	2.219E-03	9.98574E-01	3.750E+00	2.219E-03
3.800	2.050E-03	9.98681E-01	3.800E+00	2.050E-03
3.850	1.894E-03	9.98780E-01	3.850E+00	1.894E-03
3.900	1.750E-03	9.98871E-01	3.900E+00	1.750E-03
3.950	1.618E-03	9.98955E-01	3.950E+00	1.618E-03
4.000	1.495E-03	9.99033E-01	4.000E+00	1.495E-03
4.050	1.383E-03	9.99104E-01	4.050E+00	1.383E-03
4.100	1.278E-03	9.99171E-01	4.100E+00	1.278E-03
4.150	1.182E-03	9.99232E-01	4.150E+00	1.182E-03
4.200	1.093E-03	9.99289E-01	4.200E+00	1.093E-03
4.250	1.011E-03	9.99342E-01	4.250E+00	1.011E-03
4.300	9.356E-04	9.99391E-01	4.300E+00	9.356E-04
4.350	8.657E-04	9.99436E-01	4.350E+00	8.657E-04
4.400	8.010E-04	9.99477E-01	4.400E+00	8.010E-04
4.450	7.412E-04	9.99516E-01	4.450E+00	7.412E-04
4.500	6.860E-04	9.99551E-01	4.500E+00	6.860E-04
4.550	6.349E-04	9.99584E-01	4.550E+00	6.349E-04
4.600	5.877E-04	9.99615E-01	4.600E+00	5.877E-04
4.650	5.441E-04	9.99643E-01	4.650E+00	5.441E-04
4.700	5.037E-04	9.99669E-01	4.700E+00	5.037E-04
4.750	4.664E-04	9.99694E-01	4.750E+00	4.664E-04
4.800	4.318E-04	9.99716E-01	4.800E+00	4.318E-04
4.850	3.999E-04	9.99737E-01	4.850E+00	3.999E-04
4.900	3.704E-04	9.99756E-01	4.900E+00	3.704E-04
4.950	3.430E-04	9.99774E-01	4.950E+00	3.430E-04
5.000	3.177E-04	9.99791E-01	5.000E+00	3.177E-04

TEST OF POIST, PROBABILITY DISTRIBUTION OF NON-GAUSSIAN MODEL FOR R PARAM. = 1.0

NORMALIZED VARIABLE X/SIGMA X	STDIZED PROBABILITY DENSITY	DISTRIBUTION FUNCTION	UNNORMALIZED VARIABLE X	NONSTDIZED PROBABILITY DENSITY
5.000	3.177E-04	9.99791E-01	5.000E+00	3.177E-04
5.050	2.943E-04	9.99806E-01	5.050E+00	2.943E-04
5.100	2.727E-04	9.99820E-01	5.100E+00	2.727E-04
5.150	2.526E-04	9.99833E-01	5.150E+00	2.526E-04
5.200	2.340E-04	9.99845E-01	5.200E+00	2.340E-04
5.250	2.169E-04	9.99857E-01	5.250E+00	2.169E-04
5.300	2.009E-04	9.99867E-01	5.300E+00	2.009E-04
5.350	1.862E-04	9.99877E-01	5.350E+00	1.862E-04
5.400	1.726E-04	9.99886E-01	5.400E+00	1.726E-04
5.450	1.599E-04	9.99894E-01	5.450E+00	1.599E-04
5.500	1.482E-04	9.99902E-01	5.500E+00	1.482E-04
5.550	1.374E-04	9.99909E-01	5.550E+00	1.374E-04
5.600	1.274E-04	9.99915E-01	5.600E+00	1.274E-04
5.650	1.181E-04	9.99921E-01	5.650E+00	1.181E-04
5.700	1.095E-04	9.99927E-01	5.700E+00	1.095E-04
5.750	1.015E-04	9.99932E-01	5.750E+00	1.015E-04
5.800	9.409E-05	9.99937E-01	5.800E+00	9.409E-05
5.850	8.724E-05	9.99942E-01	5.850E+00	8.724E-05
5.900	8.090E-05	9.99946E-01	5.900E+00	8.090E-05
5.950	7.501E-05	9.99950E-01	5.950E+00	7.501E-05
6.000	6.956E-05	9.99954E-01	6.000E+00	6.956E-05
6.050	6.451E-05	9.99957E-01	6.050E+00	6.451E-05
6.100	5.983E-05	9.99960E-01	6.100E+00	5.983E-05
6.150	5.549E-05	9.99963E-01	6.150E+00	5.549E-05
6.200	5.147E-05	9.99966E-01	6.200E+00	5.147E-05
6.250	4.774E-05	9.99968E-01	6.250E+00	4.774E-05
6.300	4.428E-05	9.99970E-01	6.300E+00	4.428E-05
6.350	4.107E-05	9.99973E-01	6.350E+00	4.107E-05
6.400	3.810E-05	9.99974E-01	6.400E+00	3.810E-05
6.450	3.535E-05	9.99976E-01	6.450E+00	3.535E-05
6.500	3.279E-05	9.99978E-01	6.500E+00	3.279E-05
6.550	3.042E-05	9.99980E-01	6.550E+00	3.042E-05
6.600	2.823E-05	9.99981E-01	6.600E+00	2.823E-05
6.650	2.619E-05	9.99982E-01	6.650E+00	2.619E-05
6.700	2.430E-05	9.99984E-01	6.700E+00	2.430E-05
6.750	2.255E-05	9.99985E-01	6.750E+00	2.255E-05
6.800	2.092E-05	9.99986E-01	6.800E+00	2.092E-05
6.850	1.942E-05	9.99987E-01	6.850E+00	1.942E-05
6.900	1.802E-05	9.99988E-01	6.900E+00	1.802E-05
6.950	1.672E-05	9.99989E-01	6.950E+00	1.672E-05
7.000	1.552E-05	9.99990E-01	7.000E+00	1.552E-05
7.050	1.440E-05	9.99990E-01	7.050E+00	1.440E-05
7.100	1.337E-05	9.99991E-01	7.100E+00	1.337E-05
7.150	1.241E-05	9.99992E-01	7.150E+00	1.241E-05
7.200	1.151E-05	9.99992E-01	7.200E+00	1.151E-05
7.250	1.069E-05	9.99993E-01	7.250E+00	1.069E-05
7.300	9.920E-06	9.99993E-01	7.300E+00	9.920E-06
7.350	9.208E-06	9.99994E-01	7.350E+00	9.208E-06
7.400	8.548E-06	9.99994E-01	7.400E+00	8.548E-06
7.450	7.935E-06	9.99995E-01	7.450E+00	7.935E-06
7.500	7.366E-06	9.99995E-01	7.500E+00	7.366E-06

TEST OF PDIST, PROBABILITY DISTRIBUTION OF NON-GAUSSIAN MODEL FOR R PARAM. = 1.0

NORMALIZED VARIABLE X/SIGMA X	STDIZED PROBABILITY DENSITY	DISTRIBUTION FUNCTION	UNNORMALIZED VARIABLE X	NONSTDIZED PROBABILITY DENSITY
7.500	7.366E-06	9.99995E-01	7.500E+00	7.366E-06
7.550	6.838E-06	9.99995E-01	7.550E+00	6.838E-06
7.600	6.348E-06	9.99996E-01	7.600E+00	6.348E-06
7.650	5.894E-06	9.99996E-01	7.650E+00	5.894E-06
7.700	5.472E-06	9.99996E-01	7.700E+00	5.472E-06
7.750	5.080E-06	9.99997E-01	7.750E+00	5.080E-06
7.800	4.717E-06	9.99997E-01	7.800E+00	4.717E-06
7.850	4.380E-06	9.99997E-01	7.850E+00	4.380E-06
7.900	4.067E-06	9.99997E-01	7.900E+00	4.067E-06
7.950	3.776E-06	9.99997E-01	7.950E+00	3.776E-06
8.000	3.506E-06	9.99998E-01	8.000E+00	3.506E-06
8.050	3.256E-06	9.99998E-01	8.050E+00	3.256E-06
8.100	3.023E-06	9.99998E-01	8.100E+00	3.023E-06
8.150	2.808E-06	9.99998E-01	8.150E+00	2.808E-06
8.200	2.607E-06	9.99998E-01	8.200E+00	2.607E-06
8.250	2.421E-06	9.99998E-01	8.250E+00	2.421E-06
8.300	2.249E-06	9.99998E-01	8.300E+00	2.249E-06
8.350	2.088E-06	9.99999E-01	8.350E+00	2.088E-06
8.400	1.939E-06	9.99999E-01	8.400E+00	1.939E-06
8.450	1.801E-06	9.99999E-01	8.450E+00	1.801E-06
8.500	1.673E-06	9.99999E-01	8.500E+00	1.673E-06
8.550	1.554E-06	9.99999E-01	8.550E+00	1.554E-06
8.600	1.443E-06	9.99999E-01	8.600E+00	1.443E-06
8.650	1.340E-06	9.99999E-01	8.650E+00	1.340E-06
8.700	1.245E-06	9.99999E-01	8.700E+00	1.245E-06
8.750	1.156E-06	9.99999E-01	8.750E+00	1.156E-06
8.800	1.074E-06	9.99999E-01	8.800E+00	1.074E-06
8.850	9.978E-07	9.99999E-01	8.850E+00	9.978E-07
8.900	9.269E-07	9.99999E-01	8.900E+00	9.269E-07
8.950	8.610E-07	9.99999E-01	8.950E+00	8.610E-07
9.000	7.999E-07	9.99999E-01	9.000E+00	7.999E-07
9.050	7.430E-07	9.99999E-01	9.050E+00	7.430E-07
9.100	6.903E-07	1.00000E+00	9.100E+00	6.903E-07
9.150	6.413E-07	1.00000E+00	9.150E+00	6.413E-07
9.200	5.957E-07	1.00000E+00	9.200E+00	5.957E-07
9.250	5.535E-07	1.00000E+00	9.250E+00	5.535E-07
9.300	5.142E-07	1.00000E+00	9.300E+00	5.142E-07
9.350	4.777E-07	1.00000E+00	9.350E+00	4.777E-07
9.400	4.438E-07	1.00000E+00	9.400E+00	4.438E-07
9.450	4.124E-07	1.00000E+00	9.450E+00	4.124E-07
9.500	3.831E-07	1.00000E+00	9.500E+00	3.831E-07
9.550	3.560E-07	1.00000E+00	9.550E+00	3.560E-07
9.600	3.308E-07	1.00000E+00	9.600E+00	3.308E-07
9.650	3.073E-07	1.00000E+00	9.650E+00	3.073E-07
9.700	2.856E-07	1.00000E+00	9.700E+00	2.856E-07
9.750	2.653E-07	1.00000E+00	9.750E+00	2.653E-07
9.800	2.466E-07	1.00000E+00	9.800E+00	2.466E-07
9.850	2.291E-07	1.00000E+00	9.850E+00	2.291E-07
9.900	2.129E-07	1.00000E+00	9.900E+00	2.129E-07
9.950	1.978E-07	1.00000E+00	9.950E+00	1.978E-07
10.000	1.838E-07	1.00000E+00	1.000E+01	1.838E-07

Program LEVXNG

LEVXNG is a program which computes universal level crossing frequency curves for the non-gaussian turbulence model. The example presented here shows the calculation of the u-gust curve for an R parameter value of 1.0.

Card Decks

The following car- decks were required to produce the sampel calculation presented here:

LEVXNG

CF2

INVR

FFT

Input Data

Program LEVXNG is written so as to require no input data from punched cards. The functioning of the program is determined by DATA statements as described by comment cards in the deck listing.

Results

The above named card decks were compiled, loaded, and executed on the University of Washington CDC-6400 computer system. Storage requirement including I/O buffers and all system programs was $40{,}500_8$ words. Execution, including compilation and loading, required 19_{10} seconds of central processor time. The following output was generated.

```
COVARIANCE MATRICIES OF FIRST AND SECOND
GAUSSIAN VECTORS ARE
 7.071E-01 0.
 0.        3.536E-01
 7.071E-01 0.
 0.        3.536E-01

FUNCTIONAL RELATIONSHIP MATRICIES FOR FIRST
AND SECOND TRANSFORM VARIABLES ARE
 1.000E+00 0.
 0.        0.
 0.        1.000E+00
 1.000E+00 0.

R PARAMETER = 1.000

DETERMINANTS OF COVARIANCE MATRICIES
DETA = 2.50000E-01
DETB = 2.50000E-01

X INCREMENTS:  DX(1) = 3.0000000E-01 DX(2) = 3.0000000E-01
F INCREMENTS:  DF(1) = 5.2083333E-02 DF(2) = 5.2083333E-02
```

FIRST ROW, FIRST COLUMN, AND DIAGONAL OF
JOINT CHARACTERISTIC FUNCTION

1.000E+00	0.	9.485E-01	0.	8.154E-01	0.	6.456E-01	0.
4.782E-01	0.	3.348E-01	0.	2.229E-01	0.	1.415E-01	0.
8.566E-02	0.	4.949E-02	0.	2.727E-02	0.	1.433E-02	0.
7.172E-03	0.	3.419E-03	0.	1.552E-03	0.	6.700E-04	0.
2.752E-04	0.	1.075E-04	0.	3.990E-05	0.	1.408E-05	0.
4.718E-06	0.	1.502E-06	0.	4.543E-07	0.	1.305E-07	0.
3.557E-08	0.	9.209E-09	0.	2.263E-09	0.	5.277E-10	0.
1.168E-10	0.	2.454E-11	0.	4.896E-12	0.	9.534E-13	0.
3.320E-13	0.	9.534E-13	0.	4.896E-12	0.	2.454E-11	0.
1.168E-10	0.	5.277E-10	0.	2.263E-09	0.	9.209E-09	0.
3.557E-08	0.	1.305E-07	0.	4.543E-07	0.	1.502E-06	0.
4.718E-06	0.	1.408E-05	0.	3.990E-05	0.	1.075E-04	0.
2.752E-04	0.	6.700E-04	0.	1.552E-03	0.	3.419E-03	0.
7.172E-03	0.	1.433E-02	0.	2.727E-02	0.	4.949E-02	0.
8.566E-02	0.	1.415E-01	0.	2.229E-01	0.	3.348E-01	0.
4.782E-01	0.	6.456E-01	0.	8.154E-01	0.	9.485E-01	0.

1.000E+00	0.	9.482E-01	0.	8.115E-01	0.	6.333E-01	0.
4.562E-01	0.	3.067E-01	0.	1.942E-01	0.	1.165E-01	0.
6.642E-02	0.	3.608E-02	0.	1.870E-02	0.	9.242E-03	0.
4.360E-03	0.	1.962E-03	0.	8.420E-04	0.	3.446E-04	0.
1.344E-04	0.	4.992E-05	0.	1.766E-05	0.	5.951E-06	0.
1.908E-06	0.	5.819E-07	0.	1.689E-07	0.	4.659E-08	0.
1.222E-08	0.	3.049E-09	0.	7.226E-10	0.	1.627E-10	0.
3.483E-11	0.	7.080E-12	0.	1.368E-12	0.	2.579E-13	0.
8.731E-14	0.	2.579E-13	0.	1.368E-12	0.	7.080E-12	0.
3.483E-11	0.	1.627E-10	0.	7.226E-10	0.	3.049E-09	0.
1.222E-08	0.	4.659E-08	0.	1.689E-07	0.	5.819E-07	0.
1.908E-06	0.	5.951E-06	0.	1.766E-05	0.	4.992E-05	0.
1.344E-04	0.	3.446E-04	0.	8.420E-04	0.	1.962E-03	0.
4.360E-03	0.	9.242E-03	0.	1.870E-02	0.	3.608E-02	0.
6.642E-02	0.	1.165E-01	0.	1.942E-01	0.	3.067E-01	0.
4.562E-01	0.	6.333E-01	0.	8.115E-01	0.	9.482E-01	0.

1.000E+00	0.	9.006E-01	0.	6.727E-01	0.	4.343E-01	0.
2.494E-01	0.	1.291E-01	0.	6.049E-02	0.	2.569E-02	0.
9.890E-03	0.	3.448E-03	0.	1.088E-03	0.	3.105E-04	0.
8.010E-05	0.	1.867E-05	0.	3.932E-06	0.	7.476E-07	0.
1.283E-07	0.	1.986E-08	0.	2.773E-09	0.	3.491E-10	0.
3.963E-11	0.	4.055E-12	0.	3.739E-13	0.	3.106E-14	0.
2.325E-15	0.	1.567E-16	0.	9.511E-18	0.	5.199E-19	0.
2.558E-20	0.	1.133E-21	0.	4.528E-23	0.	1.723E-24	0.
2.096E-25	0.	1.723E-24	0.	4.528E-23	0.	1.133E-21	0.
2.558E-20	0.	5.199E-19	0.	9.511E-18	0.	1.567E-16	0.
2.325E-15	0.	3.106E-14	0.	3.739E-13	0.	4.055E-12	0.
3.963E-11	0.	3.491E-10	0.	2.773E-09	0.	1.986E-08	0.
1.283E-07	0.	7.476E-07	0.	3.932E-06	0.	1.867E-05	0.
8.010E-05	0.	3.105E-04	0.	1.088E-03	0.	3.448E-03	0.
9.890E-03	0.	2.569E-02	0.	6.049E-02	0.	1.291E-01	0.
2.494E-01	0.	4.343E-01	0.	6.727E-01	0.	9.006E-01	0.

```
FIRST ROW, FIRST COLUMN, AND DIAGONAL OF
JOINT PROBABILITY DENSITY
2.010E-01 0.         1.876E-01 1.585E-14 1.530E-01 5.968E-14 1.098E-01 2.787E-14
7.043E-02-1.121E-14 4.124E-02-8.940E-14 2.272E-02-7.865E-15 1.219E-02 5.151E-14
6.565E-03-2.791E-15 3.615E-03 4.223E-14 2.044E-03 8.691E-15 1.182E-03-4.116E-16
6.955E-04-5.759E-16 4.142E-04-6.974E-14 2.490E-04-6.401E-14 1.508E-04-8.972E-15
9.190E-05-1.665E-16 5.629E-05-2.205E-14 3.462E-05-1.613E-13 2.138E-05-8.330E-14
1.324E-05-3.469E-17 8.230E-06 4.884E-14 5.127E-06 2.303E-14 3.202E-06 4.956E-15
2.005E-06-1.549E-17 1.258E-06 1.657E-14 7.924E-07 1.142E-14 5.016E-07-3.228E-14
3.209E-07-1.041E-17 2.102E-07 2.647E-14 1.448E-07-2.652E-14 1.106E-07 9.477E-15
9.992E-08 0.         1.106E-07-9.477E-15 1.448E-07 2.652E-14 2.102E-07-2.647E-14
3.209E-07 1.041E-17 5.016E-07 3.228E-14 7.924E-07-1.142E-14 1.258E-06-1.657E-14
2.005E-06 1.549E-17 3.202E-06-4.956E-15 5.127E-06-2.303E-14 8.230E-06-4.884E-14
1.324E-05 3.469E-17 2.138E-05 8.330E-14 3.462E-05 1.613E-13 5.629E-05 2.205E-14
9.190E-05 1.665E-16 1.508E-04 8.972E-15 2.490E-04 6.401E-14 4.142E-04 6.974E-14
6.955E-04 5.759E-16 1.182E-03 4.116E-16 2.044E-03-8.691E-15 3.615E-03-4.223E-14
6.565E-03 2.791E-15 1.219E-02-5.151E-14 2.272E-02 7.865E-15 4.124E-02 8.940E-14
7.043E-02 1.121E-14 1.098E-01-2.787E-14 1.530E-01-5.968E-14 1.876E-01-1.585E-14

2.010E-01 0.         1.889E-01 1.310E-14 1.573E-01 6.195E-14 1.166E-01 3.267E-14
7.766E-02-1.188E-14 4.709E-02-8.896E-14 2.643E-02-9.276E-15 1.399E-02 4.908E-14
7.106E-03-3.124E-15 3.523E-03 3.880E-14 1.725E-03 1.035E-14 8.404E-04 6.580E-15
4.097E-04-4.996E-16 2.004E-04-7.192E-14 9.849E-05-6.581E-14 4.872E-05-1.049E-14
2.427E-05 0.         1.217E-05-2.148E-14 6.144E-06-1.660E-13 3.121E-06-8.840E-14
1.595E-06 5.551E-17 8.192E-07 5.382E-14 4.227E-07 2.695E-14 2.190E-07 4.252E-15
1.138E-07-1.578E-17 5.937E-08 1.720E-14 3.106E-08 1.298E-14 1.630E-08-3.294E-14
8.601E-09 0.         4.599E-09 2.872E-14 2.559E-09-2.726E-14 1.605E-09 8.551E-15
1.326E-09 0.         1.605E-09-8.551E-15 2.559E-09 2.726E-14 4.599E-09-2.872E-14
8.601E-09 0.         1.630E-08 3.294E-14 3.106E-08-1.298E-14 5.937E-08-1.720E-14
1.138E-07 1.578E-17 2.190E-07-4.252E-15 4.227E-07-2.695E-14 8.192E-07-5.382E-14
1.595E-06-5.551E-17 3.121E-06 8.840E-14 6.144E-06 1.660E-13 1.217E-05 2.148E-14
2.427E-05 0.         4.872E-05 1.049E-14 9.849E-05 6.581E-14 2.004E-04 7.192E-14
4.097E-04 4.996E-16 8.404E-04-6.580E-15 1.725E-03-1.035E-14 3.523E-03-3.880E-14
7.106E-03 3.124E-15 1.399E-02-4.908E-14 2.643E-02 9.276E-15 4.709E-02 8.896E-14
7.766E-02 1.188E-14 1.166E-01-3.267E-14 1.573E-01-6.195E-14 1.889E-01-1.310E-14

2.010E-01 0.         1.765E-01 2.591E-14 1.206E-01 8.954E-14 6.625E-02 2.759E-14
3.097E-02-2.301E-14 1.335E-02-5.181E-14 5.719E-03-1.307E-14 2.533E-03-3.073E-15
1.163E-03-5.733E-15 5.471E-04-3.705E-15 2.618E-04-1.515E-15 1.268E-04-5.384E-16
6.193E-05-9.759E-16 3.047E-05-9.785E-16 1.508E-05-5.539E-16 7.501E-06-2.129E-16
3.747E-06-2.488E-16 1.878E-06-9.795E-17 9.444E-07-4.088E-17 4.763E-07-9.191E-17
2.408E-07-4.987E-17 1.220E-07-3.266E-17 6.198E-08 2.876E-17 3.156E-08 1.610E-17
1.612E-08-7.875E-18 8.275E-09 2.700E-17 4.284E-09 4.009E-18 2.256E-09-3.290E-18
1.227E-09-7.660E-18 7.088E-10-3.888E-17 4.537E-10-1.575E-17 3.380E-10-1.063E-17
3.050E-10 0.         3.380E-10 1.063E-17 4.537E-10 1.575E-17 7.088E-10 3.888E-17
1.227E-09 7.660E-18 2.256E-09 3.290E-18 4.284E-09-4.009E-18 8.275E-09-2.700E-17
1.612E-08 7.875E-18 3.156E-08-1.610E-17 6.198E-08-2.876E-17 1.220E-07 3.266E-17
2.408E-07 4.987E-17 4.763E-07 9.191E-17 9.444E-07 4.088E-17 1.878E-06 9.795E-17
3.747E-06 2.488E-16 7.501E-06 2.129E-16 1.508E-05 5.539E-16 3.047E-05 9.785E-16
6.193E-05 9.759E-16 1.268E-04 5.384E-16 2.618E-04 1.515E-15 5.471E-04 3.705E-15
1.163E-03 5.733E-15 2.533E-03 3.073E-15 5.719E-03 1.307E-14 1.335E-02 5.181E-14
3.097E-02 2.301E-14 6.625E-02-2.759E-14 1.206E-01-8.954E-14 1.765E-01-2.591E-14
```

COMPUTED VALUES OF X AND N(X)

0.	1.6183602E-01
3.0000000E-01	1.5262845E-01
6.0000000E-01	1.2847151E-01
9.0000000E-01	9.7523343E-02
1.2000000E+00	6.7960316E-02
1.5000000E+00	4.4515143E-02
1.8000000E+00	2.8145278E-02
2.1000000E+00	1.7584277E-02
2.4000000E+00	1.1020905E-02
2.7000000E+00	6.9694671E-03
3.0000000E+00	4.4460260E-03
3.3000000E+00	2.8545952E-03
3.6000000E+00	1.8407513E-03
3.9000000E+00	1.1904340E-03
4.2000000E+00	7.7143429E-04
4.5000000E+00	5.0066131E-04
4.8000000E+00	3.2530378E-04
5.1000000E+00	2.1155798E-04
5.4000000E+00	1.3768585E-04
5.7000000E+00	8.9662966E-05
6.0000000E+00	5.8419782E-05
6.3000000E+00	3.8080485E-05
6.6000000E+00	2.4833030E-05
6.9000000E+00	1.6201600E-05
7.2000000E+00	1.0576907E-05
7.5000000E+00	6.9123730E-06
7.8000000E+00	4.5273138E-06
8.1000000E+00	2.9793597E-06
8.4000000E+00	1.9817650E-06
8.7000000E+00	1.3499307E-06
9.0000000E+00	9.6704274E-07
9.3000000E+00	7.6236168E-07
9.6000000E+00	6.9808990E-07
9.9000000E+00	7.6236168E-07
1.0200000E+01	9.6704274E-07

LEVEL CROSSINGS
LEVEL CROSSING FREQUENCY OF THE NON-GAUSSIAN MODEL, R= 1.000

DIMENSIONAL LEVEL X	NON-DIMENSIONAL LEVEL X/SIGMA X	CROSSINGS PER UNIT TIME	CROSSINGS PER ZERO CROSSING
0.	0.	1.618E-01	1.000E+00
2.000E-01	2.000E-01	1.576E-01	9.741E-01
4.000E-01	4.000E-01	1.459E-01	9.014E-01
6.000E-01	6.000E-01	1.285E-01	7.938E-01
8.000E-01	8.000E-01	1.081E-01	6.677E-01
1.000E+00	1.000E+00	8.721E-02	5.389E-01
1.200E+00	1.200E+00	6.796E-02	4.199E-01
1.400E+00	1.400E+00	5.153E-02	3.184E-01
1.600E+00	1.600E+00	3.831E-02	2.367E-01
1.800E+00	1.800E+00	2.815E-02	1.739E-01
2.000E+00	2.000E+00	2.057E-02	1.271E-01
2.200E+00	2.200E+00	1.503E-02	9.290E-02
2.400E+00	2.400E+00	1.102E-02	6.810E-02
2.600E+00	2.600E+00	8.111E-03	5.012E-02
2.800E+00	2.800E+00	5.995E-03	3.704E-02
3.000E+00	3.000E+00	4.446E-03	2.747E-02
3.200E+00	3.200E+00	3.307E-03	2.044E-02
3.400E+00	3.400E+00	2.465E-03	1.523E-02
3.600E+00	3.600E+00	1.841E-03	1.137E-02
3.800E+00	3.800E+00	1.376E-03	8.504E-03
4.000E+00	4.000E+00	1.030E-03	6.364E-03
4.200E+00	4.200E+00	7.714E-04	4.767E-03
4.400E+00	4.400E+00	5.782E-04	3.573E-03
4.600E+00	4.600E+00	4.336E-04	2.679E-03
4.800E+00	4.800E+00	3.253E-04	2.010E-03
5.000E+00	5.000E+00	2.442E-04	1.509E-03
5.200E+00	5.200E+00	1.833E-04	1.133E-03
5.400E+00	5.400E+00	1.377E-04	8.508E-04
5.600E+00	5.600E+00	1.034E-04	6.392E-04
5.800E+00	5.800E+00	7.773E-05	4.803E-04
6.000E+00	6.000E+00	5.842E-05	3.610E-04
6.200E+00	6.200E+00	4.392E-05	2.714E-04
6.400E+00	6.400E+00	3.302E-05	2.040E-04
6.600E+00	6.600E+00	2.483E-05	1.534E-04
6.800E+00	6.800E+00	1.868E-05	1.154E-04
7.000E+00	7.000E+00	1.405E-05	8.684E-05
7.200E+00	7.200E+00	1.058E-05	6.536E-05
7.400E+00	7.400E+00	7.964E-06	4.921E-05
7.600E+00	7.600E+00	6.001E-06	3.708E-05
7.800E+00	7.800E+00	4.527E-06	2.797E-05
8.000E+00	8.000E+00	3.423E-06	2.115E-05

Program INCPD

INCPD is a program which computes velocity increment probability distributions of the non-gaussian turbulence model. The example given here shows the calculation of the u-gust increment distribution for a time lag of 0.1 L_u/U seconds. The correlation functions which were used in determining the input data are given in equations 51 through 54 of the report.

Card Decks

The following card decks were required to produce the sample calculation presented here:

INCPD

CF1

INVR

FFT

FFTRS

Input Data

The following nine data cards were used in producing this example. The meaning and format of these crards are explained by comments in the listing of program INCPD.

```
1         11        21        31        41        51        61        71
EXAMPLE USE OF INCPD TO COMPUTE U-GUST INCREMENT DIST., TAU=.1*L/U, R=1., SU=1.
2
.09516258
1.         .95122942
.95122942  1.0
1.0        .95122942
.95122942  1.0
.5         0.
0.        -.5
```

Results

The above named card decks were complied, loaded, and executed on the University of Washington CDC-6400 computer system. Storage requirement including I/O buffers and all system programs was $20,476_8$ words. Execution, including compilation and loading, required 5_{10} seconds of central processor time. The following output was generated.

```
EXAMPLE USE OF INCPD TO COMPUTE U-GUST INCREMENT DIST., TAU=.1*L/U, R=1., SU=1.

COVARIANCE MATRIX OF 1ST GAUSSIAN VECTOR
   1.00000    .95123
    .95123   1.00000

COVARIANCE MATRIX OF 2ND GAUSSIAN VECTOR
   1.00000    .95123
    .95123   1.00000

FUNCTIONAL RELATION MATRIX
    .50000   0.00000
   0.00000   -.50000

STANDARD DEVIATION OF PROCESS = 3.778E-01

DETERMINANTS OF COVARIANCE MATRICIES
DETA = 9.51626E-02
DETB = 9.51626E-02
```

EXAMPLE USE OF INCPD TO COMPUTE U-GUST INCREMENT DIST., TAU=.1*L/U, R=1., SU=1.

NORMALIZED VARIABLE X/SIGMA X	STDIZED PROBABILITY DENSITY	DISTRIBUTION FUNCTION	UNNORMALIZED VARIABLE X	NONSTDIZED PROBABILITY DENSITY
0.000	4.118E-01	5.00000E-01	0.	2.885E+00
.050	4.112E-01	5.20579E-01	7.137E-03	2.881E+00
.100	4.095E-01	5.41100E-01	1.427E-02	2.869E+00
.150	4.066E-01	5.61506E-01	2.141E-02	2.849E+00
.200	4.027E-01	5.81742E-01	2.855E-02	2.821E+00
.250	3.976E-01	6.01754E-01	3.569E-02	2.786E+00
.300	3.916E-01	6.21487E-01	4.282E-02	2.743E+00
.350	3.845E-01	6.40893E-01	4.996E-02	2.694E+00
.400	3.766E-01	6.59924E-01	5.710E-02	2.638E+00
.450	3.678E-01	6.78536E-01	6.423E-02	2.576E+00
.500	3.582E-01	6.96687E-01	7.137E-02	2.509E+00
.550	3.479E-01	7.14341E-01	7.851E-02	2.437E+00
.600	3.370E-01	7.31465E-01	8.565E-02	2.361E+00
.650	3.255E-01	7.48030E-01	9.278E-02	2.280E+00
.700	3.136E-01	7.64010E-01	9.992E-02	2.197E+00
.750	3.013E-01	7.79385E-01	1.071E-01	2.111E+00
.800	2.888E-01	7.94139E-01	1.142E-01	2.023E+00
.850	2.760E-01	8.08259E-01	1.213E-01	1.934E+00
.900	2.631E-01	8.21737E-01	1.285E-01	1.843E+00
.950	2.502E-01	8.34569E-01	1.356E-01	1.753E+00
1.000	2.373E-01	8.46755E-01	1.427E-01	1.662E+00
1.050	2.245E-01	8.58298E-01	1.499E-01	1.572E+00
1.100	2.118E-01	8.69203E-01	1.570E-01	1.484E+00
1.150	1.994E-01	8.79481E-01	1.642E-01	1.397E+00
1.200	1.872E-01	8.89144E-01	1.713E-01	1.311E+00
1.250	1.754E-01	8.98207E-01	1.784E-01	1.228E+00
1.300	1.639E-01	9.06686E-01	1.856E-01	1.148E+00
1.350	1.528E-01	9.14601E-01	1.927E-01	1.070E+00
1.400	1.421E-01	9.21972E-01	1.998E-01	9.956E-01
1.450	1.319E-01	9.28820E-01	2.070E-01	9.240E-01
1.500	1.221E-01	9.35169E-01	2.141E-01	8.557E-01
1.550	1.129E-01	9.41042E-01	2.213E-01	7.907E-01
1.600	1.041E-01	9.46463E-01	2.284E-01	7.290E-01
1.650	9.575E-02	9.51457E-01	2.355E-01	6.708E-01
1.700	8.793E-02	9.56046E-01	2.427E-01	6.160E-01
1.750	8.058E-02	9.60257E-01	2.498E-01	5.645E-01
1.800	7.370E-02	9.64112E-01	2.569E-01	5.163E-01
1.850	6.729E-02	9.67635E-01	2.641E-01	4.714E-01
1.900	6.132E-02	9.70848E-01	2.712E-01	4.296E-01
1.950	5.578E-02	9.73774E-01	2.784E-01	3.908E-01
2.000	5.066E-02	9.76433E-01	2.855E-01	3.549E-01
2.050	4.593E-02	9.78846E-01	2.926E-01	3.217E-01
2.100	4.157E-02	9.81032E-01	2.998E-01	2.912E-01
2.150	3.757E-02	9.83009E-01	3.069E-01	2.632E-01
2.200	3.391E-02	9.84795E-01	3.140E-01	2.375E-01
2.250	3.056E-02	9.86405E-01	3.212E-01	2.141E-01
2.300	2.750E-02	9.87855E-01	3.283E-01	1.927E-01
2.350	2.472E-02	9.89160E-01	3.354E-01	1.732E-01
2.400	2.219E-02	9.90331E-01	3.426E-01	1.555E-01
2.450	1.990E-02	9.91383E-01	3.497E-01	1.394E-01
2.500	1.782E-02	9.92325E-01	3.569E-01	1.249E-01

EXAMPLE USE OF INCPO TO COMPUTE U-GUST INCREMENT DIST., TAU=.1*L/U, R=1., SU=1.

NORMALIZED VARIABLE X/SIGMA X	STDIZED PROBABILITY DENSITY	DISTRIBUTION FUNCTION	UNNORMALIZED VARIABLE X	NONSTDIZED PROBABILITY DENSITY
5.000	4.341E-05	9.99982E-01	7.137E-01	3.041E-04
5.050	3.841E-05	9.99984E-01	7.209E-01	2.691E-04
5.100	3.398E-05	9.99986E-01	7.280E-01	2.381E-04
5.150	3.007E-05	9.99988E-01	7.351E-01	2.106E-04
5.200	2.660E-05	9.99989E-01	7.423E-01	1.863E-04
5.250	2.353E-05	9.99990E-01	7.494E-01	1.649E-04
5.300	2.082E-05	9.99991E-01	7.565E-01	1.459E-04
5.350	1.842E-05	9.99992E-01	7.637E-01	1.290E-04
5.400	1.630E-05	9.99993E-01	7.708E-01	1.142E-04
5.450	1.442E-05	9.99994E-01	7.780E-01	1.010E-04
5.500	1.276E-05	9.99995E-01	7.851E-01	8.937E-05
5.550	1.129E-05	9.99995E-01	7.922E-01	7.907E-05
5.600	9.985E-06	9.99996E-01	7.994E-01	6.995E-05
5.650	8.834E-06	9.99996E-01	8.065E-01	6.189E-05
5.700	7.816E-06	9.99997E-01	8.136E-01	5.475E-05
5.750	6.915E-06	9.99997E-01	8.208E-01	4.844E-05
5.800	6.118E-06	9.99998E-01	8.279E-01	4.286E-05
5.850	5.413E-06	9.99998E-01	8.351E-01	3.792E-05
5.900	4.789E-06	9.99998E-01	8.422E-01	3.355E-05
5.950	4.237E-06	9.99998E-01	8.493E-01	2.968E-05
6.000	3.748E-06	9.99998E-01	8.565E-01	2.626E-05
6.050	3.316E-06	9.99999E-01	8.636E-01	2.323E-05
6.100	2.934E-06	9.99999E-01	8.707E-01	2.055E-05
6.150	2.596E-06	9.99999E-01	8.779E-01	1.818E-05
6.200	2.297E-06	9.99999E-01	8.850E-01	1.609E-05
6.250	2.032E-06	9.99999E-01	8.921E-01	1.423E-05
6.300	1.798E-06	9.99999E-01	8.993E-01	1.259E-05
6.350	1.590E-06	9.99999E-01	9.064E-01	1.114E-05
6.400	1.407E-06	9.99999E-01	9.136E-01	9.857E-06
6.450	1.245E-06	9.99999E-01	9.207E-01	8.721E-06
6.500	1.101E-06	1.00000E+00	9.278E-01	7.716E-06
6.550	9.744E-07	1.00000E+00	9.350E-01	6.826E-06
6.600	8.621E-07	1.00000E+00	9.421E-01	6.039E-06
6.650	7.627E-07	1.00000E+00	9.492E-01	5.343E-06
6.700	6.748E-07	1.00000E+00	9.564E-01	4.727E-06
6.750	5.970E-07	1.00000E+00	9.635E-01	4.182E-06
6.800	5.282E-07	1.00000E+00	9.707E-01	3.700E-06
6.850	4.673E-07	1.00000E+00	9.778E-01	3.274E-06
6.900	4.134E-07	1.00000E+00	9.849E-01	2.896E-06
6.950	3.658E-07	1.00000E+00	9.921E-01	2.563E-06
7.000	3.236E-07	1.00000E+00	9.992E-01	2.267E-06
7.050	2.863E-07	1.00000E+00	1.006E+00	2.006E-06
7.100	2.533E-07	1.00000E+00	1.013E+00	1.775E-06
7.150	2.241E-07	1.00000E+00	1.021E+00	1.570E-06
7.200	1.983E-07	1.00000E+00	1.028E+00	1.389E-06
7.250	1.754E-07	1.00000E+00	1.035E+00	1.229E-06
7.300	1.552E-07	1.00000E+00	1.042E+00	1.087E-06
7.350	1.373E-07	1.00000E+00	1.049E+00	9.619E-07
7.400	1.215E-07	1.00000E+00	1.056E+00	8.511E-07
7.450	1.075E-07	1.00000E+00	1.063E+00	7.530E-07
7.500	9.509E-08	1.00000E+00	1.071E+00	6.662E-07

EXAMPLE USE OF INCPD TO COMPUTE U-GUST INCREMENT DIST., TAU=.1*L/U, R=1., SJ=1.

NORMALIZED VARIABLE X/SIGMA X	STDIZED PROBABILITY DENSITY	DISTRIBUTION FUNCTION	UNNORMALIZED VARIABLE X	NONSTDIZED PROBABILITY DENSITY
2.500	1.782E-02	9.92325E-01	3.569E-01	1.249E-01
2.550	1.595E-02	9.93168E-01	3.640E-01	1.117E-01
2.600	1.425E-02	9.93922E-01	3.711E-01	9.986E-02
2.650	1.273E-02	9.94596E-01	3.783E-01	8.918E-02
2.700	1.136E-02	9.95198E-01	3.854E-01	7.958E-02
2.750	1.013E-02	9.95735E-01	3.925E-01	7.095E-02
2.800	9.023E-03	9.96213E-01	3.997E-01	6.321E-02
2.850	8.033E-03	9.96639E-01	4.068E-01	5.628E-02
2.900	7.147E-03	9.97018E-01	4.140E-01	5.007E-02
2.950	6.355E-03	9.97355E-01	4.211E-01	4.452E-02
3.000	5.648E-03	9.97655E-01	4.282E-01	3.957E-02
3.050	5.017E-03	9.97921E-01	4.354E-01	3.515E-02
3.100	4.455E-03	9.98158E-01	4.425E-01	3.121E-02
3.150	3.954E-03	9.98368E-01	4.496E-01	2.770E-02
3.200	3.507E-03	9.98554E-01	4.568E-01	2.457E-02
3.250	3.111E-03	9.98719E-01	4.639E-01	2.179E-02
3.300	2.758E-03	9.98866E-01	4.711E-01	1.932E-02
3.350	2.445E-03	9.98996E-01	4.782E-01	1.713E-02
3.400	2.166E-03	9.99111E-01	4.853E-01	1.518E-02
3.450	1.919E-03	9.99213E-01	4.925E-01	1.345E-02
3.500	1.700E-03	9.99303E-01	4.996E-01	1.191E-02
3.550	1.506E-03	9.99383E-01	5.067E-01	1.055E-02
3.600	1.333E-03	9.99454E-01	5.139E-01	9.341E-03
3.650	1.181E-03	9.99517E-01	5.210E-01	8.270E-03
3.700	1.045E-03	9.99572E-01	5.282E-01	7.322E-03
3.750	9.251E-04	9.99622E-01	5.353E-01	6.481E-03
3.800	8.189E-04	9.99665E-01	5.424E-01	5.737E-03
3.850	7.248E-04	9.99704E-01	5.496E-01	5.077E-03
3.900	6.414E-04	9.99738E-01	5.567E-01	4.494E-03
3.950	5.677E-04	9.99768E-01	5.638E-01	3.977E-03
4.000	5.023E-04	9.99795E-01	5.710E-01	3.519E-03
4.050	4.445E-04	9.99818E-01	5.781E-01	3.114E-03
4.100	3.933E-04	9.99839E-01	5.852E-01	2.756E-03
4.150	3.480E-04	9.99858E-01	5.924E-01	2.438E-03
4.200	3.080E-04	9.99874E-01	5.995E-01	2.157E-03
4.250	2.725E-04	9.99889E-01	6.067E-01	1.909E-03
4.300	2.411E-04	9.99902E-01	6.138E-01	1.689E-03
4.350	2.133E-04	9.99913E-01	6.209E-01	1.494E-03
4.400	1.887E-04	9.99923E-01	6.281E-01	1.322E-03
4.450	1.670E-04	9.99932E-01	6.352E-01	1.170E-03
4.500	1.477E-04	9.99940E-01	6.423E-01	1.035E-03
4.550	1.307E-04	9.99947E-01	6.495E-01	9.157E-04
4.600	1.156E-04	9.99953E-01	6.566E-01	8.102E-04
4.650	1.023E-04	9.99958E-01	6.638E-01	7.168E-04
4.700	9.052E-05	9.99963E-01	6.709E-01	6.342E-04
4.750	8.009E-05	9.99967E-01	6.780E-01	5.611E-04
4.800	7.086E-05	9.99971E-01	6.852E-01	4.964E-04
4.850	6.269E-05	9.99974E-01	6.923E-01	4.392E-04
4.900	5.546E-05	9.99977E-01	6.994E-01	3.886E-04
4.950	4.907E-05	9.99980E-01	7.066E-01	3.438E-04
5.000	4.341E-05	9.99982E-01	7.137E-01	3.041E-04

EXAMPLE USE OF INCPD TO COMPUTE U-GUST INCREMENT DIST., TAU=.1*L/U, R=1., SU=1.

NORMALIZED VARIABLE X/SIGMA X	STDIZED PROBABILITY DENSITY	DISTRIBUTION FUNCTION	UNNORMALIZED VARIABLE X	NONSTDIZED PROBABILITY DENSITY
7.500	9.509E-08	1.00000E+00	1.071E+00	6.662E-07
7.550	8.413E-08	1.00000E+00	1.078E+00	5.894E-07
7.600	7.443E-08	1.00000E+00	1.085E+00	5.214E-07
7.650	6.585E-08	1.00000E+00	1.092E+00	4.613E-07
7.700	5.826E-08	1.00000E+00	1.099E+00	4.081E-07
7.750	5.154E-08	1.00000E+00	1.106E+00	3.611E-07
7.800	4.560E-08	1.00000E+00	1.113E+00	3.195E-07
7.850	4.035E-08	1.00000E+00	1.121E+00	2.826E-07
7.900	3.570E-08	1.00000E+00	1.128E+00	2.501E-07
7.950	3.158E-08	1.00000E+00	1.135E+00	2.212E-07
8.000	2.794E-08	1.00000E+00	1.142E+00	1.957E-07
8.050	2.472E-08	1.00000E+00	1.149E+00	1.732E-07
8.100	2.187E-08	1.00000E+00	1.156E+00	1.532E-07
8.150	1.935E-08	1.00000E+00	1.163E+00	1.356E-07
8.200	1.712E-08	1.00000E+00	1.170E+00	1.199E-07
8.250	1.515E-08	1.00000E+00	1.178E+00	1.061E-07
8.300	1.340E-08	1.00000E+00	1.185E+00	9.387E-08
8.350	1.185E-08	1.00000E+00	1.192E+00	8.305E-08
8.400	1.049E-08	1.00000E+00	1.199E+00	7.348E-08
8.450	9.279E-09	1.00000E+00	1.206E+00	6.501E-08
8.500	8.210E-09	1.00000E+00	1.213E+00	5.751E-08
8.550	7.263E-09	1.00000E+00	1.220E+00	5.088E-08
8.600	6.426E-09	1.00000E+00	1.228E+00	4.502E-08
8.650	5.685E-09	1.00000E+00	1.235E+00	3.983E-08
8.700	5.030E-09	1.00000E+00	1.242E+00	3.524E-08
8.750	4.450E-09	1.00000E+00	1.249E+00	3.118E-08
8.800	3.937E-09	1.00000E+00	1.256E+00	2.758E-08
8.850	3.483E-09	1.00000E+00	1.263E+00	2.440E-08
8.900	3.082E-09	1.00000E+00	1.270E+00	2.159E-08
8.950	2.727E-09	1.00000E+00	1.278E+00	1.910E-08
9.000	2.412E-09	1.00000E+00	1.285E+00	1.690E-08
9.050	2.134E-09	1.00000E+00	1.292E+00	1.495E-08
9.100	1.888E-09	1.00000E+00	1.299E+00	1.323E-08
9.150	1.671E-09	1.00000E+00	1.306E+00	1.170E-08
9.200	1.478E-09	1.00000E+00	1.313E+00	1.035E-08
9.250	1.308E-09	1.00000E+00	1.320E+00	9.160E-09
9.300	1.157E-09	1.00000E+00	1.328E+00	8.104E-09
9.350	1.023E-09	1.00000E+00	1.335E+00	7.170E-09
9.400	9.055E-10	1.00000E+00	1.342E+00	6.343E-09
9.450	8.011E-10	1.00000E+00	1.349E+00	5.612E-09
9.500	7.088E-10	1.00000E+00	1.356E+00	4.966E-09
9.550	6.270E-10	1.00000E+00	1.363E+00	4.393E-09
9.600	5.548E-10	1.00000E+00	1.370E+00	3.887E-09
9.650	4.908E-10	1.00000E+00	1.377E+00	3.438E-09
9.700	4.343E-10	1.00000E+00	1.385E+00	3.042E-09
9.750	3.841E-10	1.00000E+00	1.392E+00	2.691E-09
9.800	3.398E-10	1.00000E+00	1.399E+00	2.381E-09
9.850	3.007E-10	1.00000E+00	1.406E+00	2.107E-09
9.900	2.660E-10	1.00000E+00	1.413E+00	1.864E-09
9.950	2.353E-10	1.00000E+00	1.420E+00	1.648E-09
10.000	2.083E-10	1.00000E+00	1.427E+00	1.459E-09

Program EIGU

EIGU is a program which computes eigenvalues and eigenfunctions of linear system responses to the longitudinal component of the non-gaussian turbulence model. The example presented here shows the calculation of the first eigensolution for a system having the transfer function

$$H(s) = \frac{1+i}{s-(1+i)} + \frac{1-i}{s-(1-i)} \cdot \qquad \text{(A1)}$$

The parameters of the problem are:

$$\begin{aligned} \sigma_u &= .3048 \text{ (m/sec)} \\ L_u &= 200. \text{ (m)} \\ U &= 100. \text{ (m/sec)} \end{aligned} \qquad \text{(A2)}$$

Card Decks

The following card decks were required to produce the sample calculation presented here:

EIGU

attached subroutines.

(All of the decks required by program EIGU have been given the identification tag EIGU and will be found listed consecutively with program EIGU at the end of this appendix.)

Input Data

The following five data cards were used to produce the example presented here. The meaning and format of these cards is described by comments in program EIGU.

```
1         |11        |21        |31        |41        |51        |61        |71
F
1
     1.0000000000E+00     1.0000000000E+00    -1.0000000000E+00    -1.0000000000E+00
.3048     100.      200.
1
```

Results

The above named card decks were compiled, loaded, and executed on the University of Washington CDC-6400 computer system. Storage requirement including I/O buffers and all system programs was $17{,}664_8$ words. Execution, including dompilation and loading, required 11_{10} seconds of central processor time. The following printer output was generated.

```
BEGIN ITERATION FOR EIGENVALUE AND EIGENFUNCTION NO. 1

TURBULENCE PARAMETERS:
STD. DEV. =   .305      MTAS = 100.000      SCALE LENGTH =  200.000

VEHICLE PARAMETERS:
      COEFFICIENTS                           POLES
 1.0000E+00    1.0000E+00           1.0000E+00    1.0000E+00

SYSTEM RESPONSE VARIANCE = 3.144411E-01


ITERATION NO.  1.  ESTIMATED EIGENVALUE IS  1.1279131E-01.  (DEVP = 1.000E+00
ITERATION NO.  2.  ESTIMATED EIGENVALUE IS  2.3169420E-01.  (DEVP = 1.054E+00
ITERATION NO.  3.  ESTIMATED EIGENVALUE IS  5.5995409E-01.  (DEVP = 1.417E+00
ITERATION NO.  4.  ESTIMATED EIGENVALUE IS  5.6586583E-01.  (DEVP = 1.056E-02
ITERATION NO.  5.  ESTIMATED EIGENVALUE IS  5.6579235E-01.  (DEVP = 1.299E-04
ITERATION NO.  6.  ESTIMATED EIGENVALUE IS  5.6578378E-01.  (DEVP = 1.513E-05
ITERATION NO.  6.  ESTIMATED EIGENVALUE IS  9.9804141E-01.  (DEVP = 7.640E-01
ITERATION NO.  7.  ESTIMATED EIGENVALUE IS  5.5910282E-01.  (DEVP = 4.398E-01
ITERATION NO.  8.  ESTIMATED EIGENVALUE IS  5.5924255E-01.  (DEVP = 2.499E-04
ITERATION NO.  9.  ESTIMATED EIGENVALUE IS  5.5924316E-01.  (DEVP = 1.078E-06


EI = 8.866E+50)
EI = 1.720E-02)
EI = 1.183E-02)
EI = 3.261E-04)
EI = 2.309E-05)
EI = 1.754E-06)
EI = 1.758E-06)
EI = 2.277E-02)
EI = 2.842E-05)
EI = 1.799E-06)
```

EIGENVALUE NO. 1 IS 5.5924316E-01

FRACTION OF RESPONSE VARIANCE DUE TO THIS EIGENVALUE IS .99463
FRACTION OF RESPONSE VARIANCE DUE TO FIRST 1 EIGENVALUES IS .99463

X	EIGENFUNCTION
0.0000	0.
.0100	7.0285872829E-03
.0200	1.4055442114E-02
.0300	2.1078851805E-02
.0450	3.1603814298E-02
.0688	4.8233469059E-02
.1588	1.1050040718E-01
.3925	2.6064371720E-01
.7532	4.3826169502E-01
1.2436	5.5772357582E-01
1.8600	5.6060854671E-01
2.6071	4.7467481916E-01
3.4827	3.6776402814E-01
4.4830	2.7839939679E-01
5.6153	2.0852833382E-01
6.8713	1.5266666688E-01
8.2603	1.0799291389E-01
9.7720	7.4001491704E-02
11.4176	4.9038754620E-02
13.1918	3.1470795013E-02
15.0874	1.9593129325E-02
17.1183	1.1792418679E-02
19.2695	6.8870511724E-03
21.5571	3.8874500871E-03
23.9732	2.1248914355E-03
26.5083	1.1274420928E-03
29.1811	5.7796094088E-04
31.9719	2.8766772923E-04
34.9013	1.3830176211E-04
37.9594	6.4387963348E-05
41.1340	2.9115511114E-05
44.4487	1.2712580531E-05
47.8791	5.3924807584E-06
51.4505	2.2081584150E-06
55.1365	8.7865881544E-07
58.9646	3.3743817791E-07
62.9212	1.2548984137E-07
66.9911	4.5365498615E-08
71.2044	1.5822400221E-08
75.5300	5.3656784205E-09
80.0000	1.7551071753E-09

FIRST 1 EIGENVALUES ARE:
1 5.5924316E-01

REMAINING EIGENVALUES SQUARED SUM TO 1.6881512E-03
THE ABOVE EIGENVALUES ACCOUNT FOR .99463 OF THE RESPONSE VAR.
LARGEST POSSIBLE REMAINING EIGENVALUE IS 4.1087118E-02

Program EIGVW

EIGVW is a program which computes eigenvalues and adjoint eigenfunctions of linear system responses to the vertical or lateral components of the non-gaussian turbulence model. The example presented here shows the calculation of the first eigensolution for a system having the transfer function

$$H(s) = \frac{1+i}{s-(1+i)} + \frac{1-i}{s-(1-i)} \tag{A3}$$

The parameters of the problem are

$$\begin{aligned} \sigma_\omega &= .3048 \text{ (m/sec)} \\ L_\omega &= 200. \text{ (m)} \\ U &= 100. \text{ (m/sec)} \end{aligned} \tag{A4}$$

Card Decks

The following card decks were required to produce the sample calculation presented here:

EIGVW

attached subroutines.

(All of the decks required by program EIGVW have been given the identification tag EIGVW and will be found listed consecutively with program EIGVW at the end of this appendix.)

Input Data

The following five data cards were used to produce the example presented here. The meaning and format of these cards is described by comments in program EIGVW.

```
|1        |11       |21       |31       |41       |51       |61       |71
|F
|1
|  1.0000|000000E+00|  1.0000|000000E+00|  -1.0000|000000E+00|  -1.0000|000000E+00
|.3048    |100.     |200.
|1
```

<u>Results</u>

The above named card decks were compiled, loaded, and executed on the University of Washington CDC-6400 computer system. Storage requirement including I/O buffers and all system programs was $23{,}165_8$ words. Execution, including compilation and loading, required 18_{10} seconds of central processor time. The following printer output was generated.

```
BEGIN ITERATION FOR EIGENVALUE AND EIGENFUNCTION NO. 1

TURBULENCE PARAMETERS:
STD. DEV. =  .305     KTAS = 100.000     SCALE LENGTH =  200.000

VEHICLE PARAMETERS:
      COEFFICIENTS                          POLES
 1.0000E+00   1.0000E+00            1.0000E+00   1.0000E+00

SYSTEM RESPONSE VARIANCE = 2.847561E-01


ITERATION NO.  1.  ESTIMATED P EIGENVALUE IS  1.1279131E-01.  (DEVP = 1.000E+00
ITERATION NO.  2.  ESTIMATED Q EIGENVALUE IS  2.2656324E-01.  (DEVQ = 1.600E+00
ITERATION NO.  3.  ESTIMATED P EIGENVALUE IS  5.2661024E-01.  (DEVP = 3.669E+00
ITERATION NO.  4.  ESTIMATED Q EIGENVALUE IS  5.4074826E-01.  (DEVQ = 1.387E+00
ITERATION NO.  5.  ESTIMATED P EIGENVALUE IS  5.3623562E-01.  (DEVP = 1.828E-02
ITERATION NO.  6.  ESTIMATED Q EIGENVALUE IS  5.4059088E-01.  (DEVQ = 2.910E-04
ITERATION NO.  7.  ESTIMATED P EIGENVALUE IS  5.3619030E-01.  (DEVP = 8.452E-05
ITERATION NO.  8.  ESTIMATED Q EIGENVALUE IS  5.4058727E-01.  (DEVQ = 6.682E-06
ITERATION NO.  8.  ESTIMATED Q EIGENVALUE IS  9.9717091E-01.  (DEVQ = 8.446E-01
ITERATION NO.  9.  ESTIMATED P EIGENVALUE IS  5.2998707E-01.  (DEVP = 1.157E-02
ITERATION NO. 10.  ESTIMATED Q EIGENVALUE IS  5.3070587E-01.  (DEVQ = 4.678E-01
ITERATION NO. 11.  ESTIMATED P EIGENVALUE IS  5.3037573E-01.  (DEVP = 7.333E-04
ITERATION NO. 12.  ESTIMATED Q EIGENVALUE IS  5.3070909E-01.  (DEVQ = 6.053E-06
ITERATION NO. 13.  ESTIMATED P EIGENVALUE IS  5.3037603E-01.  (DEVP = 5.765E-07

EI = 8.866E+50)
EI = 2.282E-02)
EI = 1.836E-02)
EI = 1.243E-02)
EI = 6.531E-04)
EI = 6.816E-05)
EI = 7.757E-06)
EI = 7.889E-07)
EI = 7.912E-07)
EI = 7.785E-04)
EI = 2.695E-02)
EI = 8.203E-05)
EI = 7.583E-06)
EI = 8.100E-07)
```

EIGENVALUE NO. 1 IS 5.3054256E-01
EVP = 5.3037603E-01   EVQ = 5.3070909E-01   RELATIVE DIFFERENCE = 6.278E-04

FRACTION OF RESPONSE VARIANCE CUE TO THIS EIGENVALUE IS .98848
FRACTION OF RESPONSE VARIANCE DUE TO FIRST 1 EIGENVALUES IS .98848

X	FUNCTION OF 1ST ARG	FUNCTION OF 2ND ARG
0.0000	0.	0.
.0100	6.7324511024E-03	1.0461591109E-02
.0200	1.3482343875E-02	2.0894505585E-02
.0300	2.0247587919E-02	3.1296288211E-02
.0450	3.0419692969E-02	4.6835310860E-02
.0688	4.6571979135E-02	7.1268468976E-02
.1588	1.0784141104E-01	1.6144901928E-01
.3925	2.5942980425E-01	3.6958022178E-01
.7532	4.4146159714E-01	5.9018308049E-01
1.2436	5.5866644871E-01	6.8767121324E-01
1.8600	5.5238306103E-01	5.9129268019E-01
2.6071	4.6575693583E-01	3.7291355406E-01
3.4827	3.6835976265E-01	1.5671227976E-01
4.4830	2.8544936694E-01	4.9083324351E-03
5.6153	2.1516724290E-01	-8.6185086779E-02
6.8713	1.5702144891E-01	-1.3371281284E-01
8.2603	1.1086763418E-01	-1.5010589158E-01
9.7720	7.5981065944E-02	-1.4449874249E-01
11.4176	5.0358752968E-02	-1.2579504215E-01
13.1918	3.2317565084E-02	-1.0150526678E-01
15.0874	2.0120179993E-02	-7.7014503724E-02
17.1183	1.2109644039E-02	-5.5263694482E-02
19.2695	7.0723185987E-03	-3.7788198842E-02
21.5571	3.9920256440E-03	-2.4638804756E-02
23.9732	2.1820527423E-03	-1.5377986136E-02
26.5083	1.1577712014E-03	-9.2228821484E-03
29.1811	5.9350855992E-04	-5.3027323990E-03
31.9719	2.9540622495E-04	-2.9380489023E-03
34.9013	1.4202219191E-04	-1.5632781479E-03
37.9594	6.6120051893E-05	-8.0106806971E-04
41.1340	2.9898742026E-05	-3.9662702230E-04
44.4487	1.3054559279E-05	-1.8885713453E-04
47.8791	5.5375428735E-06	-8.6993337604E-05
51.4505	2.2675596710E-06	-3.8557179289E-05
55.1365	9.0229545170E-07	-1.6547612754E-05
58.9646	3.4651553914E-07	-6.8355563809E-06
62.9212	1.2886562009E-07	-2.7268259085E-06
66.9911	4.6585867398E-08	-1.0544685683E-06
71.2044	1.6248035647E-08	-3.9257897507E-07
75.5300	5.5190195309E-09	-1.4176732168E-07
80.0000	1.8023209848E-09	-4.9291119477E-08

FIRST 1 EIGENVALUES ARE*
    1   5.3054256E-01

REMAINING EIGENVALUES SQUARED SUM TO 3.2806552E-03
THE ABOVE EIGENVALUES ACCOUNT FOR .98848 OF THE RESPONSE VAR.
LARGEST POSSIBLE REMAINING EIGENVALUE IS 5.7277004E-02

Program RLEVX

RLEVX is a program which computes the level crossing frequency of a linear system response to the non-gaussian turbulence model. The example presented here, in order to be as simple and compact as possible, utilizes the first and second eigensolutions of the response and its first derivative which were presented in the numerical example section of this report. These solutions were assumed to represent the total non-gaussian portion of the vehicle response.

Card Decks

The following card decks were required to produce the sample calculation presented here:

RLEVX

CF2

COEF

FFT

INRPDT

INVR

SCALE

Input Data

The 175 data cards listed on the next three pages of this appendix were used to produce the example presented here. The meaning and format of these cards is described by comments in program RLEVX.

Results

The above named card decks were compiled, loaded, and executed on the University of Washington CDC 6400 computer system. Storage requirement including I/O buffers and all system programs was $44{,}550_8$ words. Execution, including compilation and loading, required 53_{10} seconds of central processor time. Printer output generated by the program is presented following the listing of input data on the next few pages.

```
1         11        21        31        41        51        61        71

EXAMPLE OF PROGRAM RLEVX USING 4 EIGENSOLUTIONS FROM NUMERICAL EXAMPLE OF REPORT
41
2
EIGENVALUE AND CONJUGATE EIGENFUNCTIONS NO. 1    3.17000000000E+00
      0.                            0.                          0.
      1.00000000000000E-02          1.03380453484144E-04       -1.72991714847168E-04
      2.00000000000000E-02          4.12091576661881E-04       -6.87407578743960E-04
      3.00000000000000E-02          9.23896341785153E-04       -1.53631177532787E-03
      4.49999999999999E-02          2.06720167699628E-03       -3.42136455657666E-03
      6.81428683949825E-02          4.69721641553622E-03       -7.71843570801392E-03
      1.53285815554841E-01          2.28626732999262E-02       -3.66087563417696E-02
      3.72590895436845E-01          1.17667551347099E-01       -1.77411931788454E-01
      7.10029610127151E-01          3.15235689867819E-01       -4.41306397306692E-01
      1.16810150566051E+00          5.06897531032767E-01       -6.52671271597395E-01
      1.74342597764184E+00          5.67526255624035E-01       -6.49494662899258E-01
      2.44026468882656E+00          5.19197439608305E-01       -4.85131711607991E-01
      3.25670747957044E+00          4.24516551805992E-01       -2.63762803987694E-01
      4.18908044493492E+00          3.27771101732861E-01       -5.85279676770651E-02
      5.24429005133021E+00          2.49079990695487E-01        8.41333427699911E-02
      6.41454877398570E+00          1.86462553522839E-01        1.52833075683970E-01
      7.70852519603235E+00          1.31750567381850E-01        1.66427982508313E-01
      9.11666967597887E+00          8.35926886090079E-02        1.51907809431097E-01
      1.06494129136769E+01          4.41659736179496E-02        1.29219411572128E-01
      1.23017602309341E+01          1.60219729957691E-02        1.06441974604521E-01
      1.40669532042637E+01         -5.48702502495294E-04        8.44795620987124E-02
      1.59580673371723E+01         -7.66173407083739E-03        6.31065303437834E-02
      1.79611460677929E+01         -8.63893748285316E-03        4.38186274857890E-02
      2.00910270163529E+01         -6.82849982042144E-03        2.81514669994369E-02
      2.23405120444720E+01         -4.46049667514021E-03        1.68717553016897E-02
      2.47006392684760E+01         -2.55025463732295E-03        9.55632383823490E-03
      2.71688911122464E+01         -1.29766408007102E-03        5.15262972619396E-03
      2.97869040935412E+01         -5.90696512745958E-04        2.68500152170235E-03
      3.25139227529630E+01         -2.33464304622154E-04        1.35755209985893E-03
      3.53605454919441E+01         -7.40024255770607E-05        6.71330037414992E-04
      3.83156069666220E+01         -1.27131811524769E-05        3.25952700191411E-04
      4.14009965212542E+01          5.83078018676558E-06        1.53963513657933E-04
      4.45939437532236E+01          8.23567267220282E-06        7.05683858937510E-05
      4.79181001235074E+01          6.09255924741253E-06        3.08558623572342E-05
      5.13489331127671E+01          3.60552463379568E-06        1.27738830239340E-05
      5.49118562987023E+01          1.85513385142841E-06        4.88950261394951E-06
      5.85943835641965E+01          8.57487222683428E-07        1.69291201392801E-06
      6.23822650468401E+01          3.61763259176069E-07        5.08360093577582E-07
      6.63235591279859E+01          1.38814280111638E-07        1.15665820127314E-07
      7.03293263679197E+01          4.87069025402085E-08        8.57255448554467E-09
      7.44893872647162E+01          1.58616695717021E-08       -1.10905148192891E-08
EIGENVALUE AND CONJUGATE EIGENFUNCTIONS NO. 2    1.06000000000E+00
      0.                            0.                          0.
      1.00000000000000E-02         -8.69519530805193E-06       -2.52815052205637E-05
      2.00000000000000E-02         -3.39523493750350E-05       -1.00459167608672E-04
      3.00000000000000E-02         -7.45416537300987E-05       -2.24520718071720E-04
      4.49999999999999E-02         -1.61536342906318E-04       -5.00016835807367E-04
      6.81428683949825E-02         -3.48931473193238E-04       -1.12808927923015E-03
      1.53285815554841E-01         -1.39148673268830E-03       -5.35521121283800E-03
      3.72590895436845E-01         -3.93038362627093E-03       -2.61890546807597E-02
      7.10029610127151E-01         -3.83589175539736E-03       -6.73452998806918E-02
      1.16810150566051E+00         -6.78939134014542E-03       -1.06560156451943E-01
      1.74342597764184E+00         -2.14490278886953E-02       -1.17581773707561E-01
      2.44026468282656E+00         -3.83616057015668E-02       -1.01221403202409E-01
      3.25670747957044E+00         -3.91568443981112E-02       -6.20101607081083E-02
      4.18908044493492E+00         -7.63839777097708E-03        9.92956566809651E-04
```

1 11 21 31 41 51 61 71

5.24429005133021E+00	3.97177922571035E-02	6.57123429445154E-02
6.41454877398570E+00	6.06885777644510E-02	9.03789788032769E-02
7.70852519603235E+00	4.50805539134367E-02	4.98002894663261E-02
9.11666967597887E+00	4.71604951463764E-02	-5.94319491016844E-02
1.06494129136769E+01	1.35591320566842E-01	-2.22167775229246E-01
1.23017602309341E+01	2.94485384218987E-01	-3.63392695078556E-01
1.40669532042637E+01	4.08035325734462E-01	-3.68788789597764E-01
1.59580673371723E+01	3.89297788794009E-01	-2.11617096227440E-01
1.79611460677929E+01	2.71570497335306E-01	1.93767290027855E-03
2.00910270163529E+01	1.44661071328695E-01	1.52781694393144E-01
2.23405120444720E+01	6.21005711383047E-02	2.03027824667086E-01
2.47006392684760E+01	2.49596625136401E-02	1.83762358792857E-01
2.71888911122464E+01	1.36646736902045E-02	1.37931865751325E-01
2.97869040935412E+01	1.19885313164789E-02	9.26571524561961E-02
3.25139227529630E+01	1.17493632958510E-02	5.74945188032219E-02
3.53605454919441E+01	1.04811163248553E-02	3.33096610587029E-02
3.83156069666220E+01	8.22929508429632E-03	1.78103196585160E-02
4.14009965212542E+01	5.66523355253881E-03	8.37137425128397E-03
4.45939437532236E+01	3.39998958075259E-03	3.10800793111082E-03
4.79181001235074E+01	1.72421306391670E-03	5.22663851835921E-04
5.13489331127671E+01	6.91156204792816E-04	-4.51110542564829E-04
5.49118562987023E+01	1.65606254513518E-04	-6.07999512135440E-04
5.85943835641965E+01	-3.42273569713914E-05	-4.51907425055079E-04
6.23822650468401E+01	-7.21755676678982E-05	-2.53514867514097E-04
6.63035591279859E+01	-5.36292503554574E-05	-1.11782167060860E-04
7.03293263679197E+01	-2.83773231187326E-05	-3.69582286494101E-05
7.44893872647162E+01	-1.11984106958818E-05	-6.06986937417661E-06

11.1725

2

EIGENVALUE AND CONJUGATE EIGENFUNCTIONS NO. 1 2.37000000000E+00

0.	0.	0.
1.00000000000000E-02	2.67833760736946E-02	-5.15251505856056E-02
2.00000000000000E-02	5.30625583253603E-02	-1.01874912801769E-01
3.00000000000000E-02	7.88074957739657E-02	-1.51024263681225E-01
4.49999999999999E-02	1.16363508544427E-01	-2.22449038406235E-01
6.81428683949825E-02	1.71640383766994E-01	-3.27104143804830E-01
1.53285815954841E-01	3.44118671938734E-01	-6.52202540402399E-01
3.725908955436845E-01	5.63937095410314E-01	-1.07314077204266E+00
7.10029610127151E-01	5.42510706785940E-01	-9.31857530905500E-01
1.16810150566051E+00	4.74721832016675E-01	-3.44354249413065E-01
1.74342597764184E+00	4.73797897477160E-01	9.31581358611679E-02
2.44026468882656E+00	4.43802726360362E-01	2.43951203309891E-01
3.25670747957044E+00	3.86559791690964E-01	2.63555657191302E-01
4.18908044493492E+00	3.12958534477563E-01	2.13692339808374E-01
5.24429005133021E+00	2.35792396071104E-01	1.36507751973840E-01
6.41454877398570E+00	1.69581710798080E-01	6.99000293998200E-02
7.70852519603235E+00	1.18287975089846E-01	2.74275187917110E-02
9.11666967597887E+00	8.06506340959054E-02	7.11411209514043E-03
1.06494129136769E+01	5.34101524019166E-02	2.79028766096090E-04
1.23017602309341E+01	3.42755108763746E-02	-7.30310853090402E-04
1.40669532042637E+01	2.13381902207938E-02	-3.29130633516623E-04
1.59580673371723E+01	1.28467170758664E-02	-7.49930794329804E-05
1.79611460677929E+01	7.50617661303865E-03	-1.15083437043135E-04
2.00910270163529E+01	4.23652766018562E-03	-2.38522654276351E-04
2.23405120444720E+01	2.31319394395975E-03	-3.05141127892416E-04
2.47006392684760E+01	1.22477581756603E-03	-2.93680566403198E-04
2.71888911122464E+01	6.25973554386275E-04	-2.36295385150925E-04
2.97869040935412E+01	3.10409732592443E-04	-1.68059995933396E-04
3.25139227529630E+01	1.48608698680192E-04	-1.08633922450165E-04

```
1         11        21        31        41        51        61        71

      3.536054549194410E+01    6.88880832720551E-05     -6.503796894485066E-05
      3.831560696662200E+01    3.10271683566852E-05     -3.654388710717166E-05
      4.140099652125422E+01    1.35044074250793E-05     -1.932676332778244E-05
      4.459394375322366E+01    5.71725849849897E-06     -9.713880734499066E-06
      4.791810012350744E+01    2.34019483343113E-06     -4.633346572453122E-06
      5.134893311276711E+01    9.32443248999940E-07     -2.114723329250644E-06
      5.491185629870233E+01    3.59237586111701E-07     -9.202588693350097E-07
      5.859438356419655E+01    1.34270644205201E-07     -3.835909362887366E-07
      6.238226504684011E+01    4.88683144455833E-08     -1.539108837197772E-07
      6.630355912798599E+01    1.71882282036868E-08     -5.911613000581411E-08
      7.032932636791977E+01    5.88997291890889E-09     -2.191239498438899E-08
      7.448938726471622E+01    1.94660630625993E-09     -7.785650742273011E-09
EIGENVALUE ANC CONJUGATE EIGENFUNCTIONS NO. 2    1.5100000000E+00
      0.                       0.                        0.
      1.000000000000000E-02    3.88063033243247E-02     -1.533759860192788E-02
      2.000000000000000E-02    7.69740139635919E-02     -3.030470075999500E-02
      3.000000000000000E-02    1.14474934736473E-01     -4.487484523920911E-02
      4.499999999999999E-02    1.69419351382761E-01     -6.593388784741222E-02
      6.814286839498255E-02    2.50947334564264E-01     -9.640785980230855E-02
      1.532858155548411E-01    5.13737271069587E-01     -1.851017146994466E-01
      3.725910895436845E-01    9.04404597705405E-01     -2.558526191595322E-01
      7.100296101271511E-01    8.68124792815401E-01     -2.251530914921611E-01
      1.168101505660511E+00    3.45374241005247E-01     -3.986564998540333E-01
      1.743425977641844E+00   -1.19196350875066E-01     -5.964106658520277E-01
      2.440264688826566E+00   -2.89263748237742E-01     -5.566034897280833E-01
      3.256707479570444E+00   -3.16233332528343E-01     -4.013001531507399E-01
      4.189080444934922E+00   -2.70586456071879E-01     -2.020566782397822E-01
      5.244290051330211E+00   -2.04741585567772E-01     -1.224351142093455E-02
      6.414548773985700E+03   -1.54446341675700E-01      1.205904820746966E-01
      7.708525196032355E+00   -1.21557713204129E-01      1.858389825278433E-01
      9.116669675978877E+00   -9.56485908048741E-02      1.950294389641099E-01
      1.064941291367699E+01   -7.02727245954553E-02      1.705584671816111E-01
      1.230176023093411E+01   -4.65354077224842E-02      1.336111766889844E-01
      1.406695320426377E+01   -2.76058455786113E-02      9.780850873469144E-02
      1.595806733717233E+01   -1.46736948501758E-02      6.830938144549366E-02
      1.796114606779299E+01   -7.07182723300015E-03      4.605214383933222E-02
      2.009102701635299E+01   -3.08817934231163E-03      2.990683029483855E-02
      2.234051204447200E+01   -1.21779445260214E-03      1.870031401593766E-02
      2.470063926847600E+01   -4.20119922575086E-04      1.126431250568044E-02
      2.718891112246644E+01   -1.10100155145177E-04      6.509129883554900E-03
      2.978690409354122E+01   -7.32254500619177E-06      3.623846526483322E-03
      3.251392275296300E+01    1.70496169813352E-05      1.936392458625399E-03
      3.536054549194410E+01    1.61936227279608E-05      9.958008457758222E-04
      3.831560696662200E+01    1.03182196228569E-05      4.944648482438266E-04
      4.140099652125422E+01    5.45605183054567E-06      2.359899386407999E-04
      4.459394375322366E+01    2.55315094911578E-06      1.089085742043722E-04
      4.791810012350744E+01    1.07570451409445E-06      4.834746321755099E-05
      5.134893311276711E+01    4.13740633153342E-07      2.077878705767266E-05
      5.491185629870233E+01    1.43811405615230E-07      8.594955979455433E-06
      5.859438356419655E+01    4.46115937752297E-08      3.433237634667899E-06
      6.238226504684011E+01    1.18891375860138E-08      1.329438579453844E-06
      6.630355912798599E+01    2.33059539609191E-09      4.956070237686811E-07
      7.032932636791977E+01    3.44089527994805E-11      1.792121572901311E-07
      7.448938726471622E+01   -2.33971875959158E-10      6.239181347848711E-08
7.8970
1.0
```

```
RESULTS OF SCALING
EXAMPLE OF PROGRAM RLEVX USING 4 EIGENSOLUTIONS FROM NUMERICAL EXAMPLE OF REPORT

VARIABLE NO. 1
GAUSSIAN VARIANCE INCLUDING CORRECTION FOR NEGLECTED EIGENVALUES = 5.58625E+00
SCALED EIGENVALUES
     2.24153E+00
     7.49533E-01

VARIABLE NO. 2
GAUSSIAN VARIANCE INCLUDING CORRECTION FOR NEGLECTED EIGENVALUES = 3.94850E+00
SCALED EIGENVALUES
     1.67584E+00
     1.06773E+00

SIGMA RATIO OF TURBULENCE MODEL = 1.000

VARIANCE CHECK, VARIABLE NO. 1
CORRECT TOTAL VARIANCE = 1.11725E+01
SUM OF SCALED EIGENVALUES SQUARED = 5.58625E+00
GAUSSIAN VARIANCE = 5.58625E+00
TOTAL VARIANCE = 1.11725E+01

VARIANCE CHECK, VARIABLE NO. 2
CORRECT TOTAL VARIANCE = 7.89700E+00
SUM OF SCALED EIGENVALUES SQUARED = 3.94850E+00
GAUSSIAN VARIANCE = 3.94850E+00
TOTAL VARIANCE = 7.89700E+00
```

COVARIANCE MATRIX FOR P VARIABLES
EXAMPLE OF PROGRAM RLEVX USING 4 EIGENSOLUTIONS FROM NUMERICAL EXAMPLE OF REPORT

1.000000	0.000000	.943717	-.261535
0.000000	1.000000	.058157	-.090917
.943717	.058157	1.000000	0.000000
-.261535	-.090917	0.000000	1.000000

COVARIANCE MATRIX FOR Q VARIABLES
EXAMPLE OF PROGRAM RLEVX USING 4 EIGENSOLUTIONS FROM NUMERICAL EXAMPLE OF REPORT

1.000000	0.000000	.171608	.940547
0.000000	1.000000	.032251	-.063324
.171608	.032251	1.000000	0.000000
.940547	-.063324	0.000000	1.000000

FUNCTIONAL DEPENDENCE MATRIX
EXAMPLE OF PROGRAM RLEVX USING 4 EIGENSOLUTIONS FROM NUMERICAL EXAMPLE OF REPORT
VARIABLE NO. 1

2.241528	0.000000
0.000000	.749533

FUNCTIONAL DEPENDENCE MATRIX
EXAMPLE OF PROGRAM RLEVX USING 4 EIGENSOLUTIONS FROM NUMERICAL EXAMPLE OF REPORT
VARIABLE NO. 2

1.675843	0.000000
0.000000	1.067731

CORRELATION COEFFICIENT OF RESPONSE AND ITS FIRST DERIVATIVE = 2.831E-03
THIS COEFFICIENT SHOULD BE MUCH LESS THAN 1.0

DETERMINANTS OF COVARIANCE MATRICIES
DETA = 3.43320E-02
DETB = 8.25690E-02

INCREMENTS:
DX(1) = 1.0027587E+00   DX(2) = 8.4304804E-01
DF(1) = 1.5582014E-02   DF(2) = 1.8533938E-02

FIRST ROW, FIRST COLUMN, AND DIAGONAL OF JOINT PROBABILITY DENSITY FUNCTION

```
2.098E-02 0.        1.966E-02 1.366E-15 1.622E-02 6.143E-15 1.186E-02 2.935E-15
7.777E-03-1.412E-15 4.654E-03-9.462E-15 2.604E-03-1.001E-15 1.401E-03 5.158E-15
7.448E-04-3.567E-16 3.997E-04 4.152E-15 2.187E-04 9.766E-16 1.222E-04 3.305E-16
6.959E-05-7.763E-17 4.019E-05-7.471E-15 2.347E-05-6.798E-15 1.383E-05-1.047E-15
8.202E-06-3.123E-17 4.894E-06-2.298E-15 2.934E-06-1.712E-14 1.766E-06-9.027E-15
1.067E-06-1.778E-17 6.469E-07 5.386E-15 3.933E-07 2.638E-15 2.397E-07 4.678E-16
1.464E-07-6.335E-18 8.969E-08 1.778E-15 5.510E-08 1.304E-15 3.402E-08-3.395E-15
2.120E-08 4.337E-19 1.351E-08 2.901E-15 9.047E-09-2.811E-15 6.740E-09 9.358E-16
6.028E-09 0.        6.740E-09-9.358E-16 9.047E-09 2.811E-15 1.351E-08-2.901E-15
2.120E-08-4.337E-19 3.402E-08 3.395E-15 5.510E-08-1.304E-15 8.969E-08-1.778E-15
1.464E-07 6.335E-18 2.397E-07-4.678E-16 3.933E-07-2.638E-15 6.469E-07-5.386E-15
1.067E-06 1.778E-17 1.766E-06 9.027E-15 2.934E-06 1.712E-14 4.894E-06 2.298E-15
8.202E-06 3.123E-17 1.383E-05 1.047E-15 2.347E-05 6.798E-15 4.019E-05 7.471E-15
6.959E-05 7.763E-17 1.222E-04-3.305E-16 2.187E-04-9.766E-16 3.997E-04-4.152E-15
7.448E-04 3.567E-16 1.401E-03-5.158E-15 2.604E-03 1.001E-15 4.654E-03 9.462E-15
7.777E-03 1.412E-15 1.186E-02-2.935E-15 1.622E-02-6.143E-15 1.966E-02-1.366E-15

2.098E-02 0.        1.972E-02 1.411E-15 1.641E-02 6.560E-15 1.216E-02 3.507E-15
8.102E-03-1.114E-15 4.927E-03-9.203E-15 2.785E-03-8.792E-16 1.495E-03 5.169E-15
7.781E-04-2.525E-16 3.998E-04 4.046E-15 2.053E-04 1.093E-15 1.062E-04 6.551E-16
5.546E-05-2.776E-17 2.929E-05-7.559E-15 1.563E-05-6.848E-15 8.432E-06-1.066E-15
4.590E-06 0.        2.520E-06-2.209E-15 1.394E-06-1.730E-14 7.765E-07-9.233E-15
4.351E-07 0.        2.451E-07 5.604E-15 1.387E-07 2.835E-15 7.883E-08 4.363E-16
4.497E-08-2.670E-18 2.574E-08 1.811E-15 1.478E-08 1.365E-15 8.529E-09-3.438E-15
4.960E-09-3.469E-18 2.938E-09 3.016E-15 1.824E-09-2.862E-15 1.270E-09 8.893E-16
1.102E-09 0.        1.270E-09-8.893E-16 1.824E-09 2.862E-15 2.938E-09-3.016E-15
4.960E-09 3.469E-18 8.529E-09 3.438E-15 1.478E-08-1.365E-15 2.574E-08-1.811E-15
4.497E-08 2.670E-18 7.883E-08-4.363E-16 1.387E-07-2.835E-15 2.451E-07-5.604E-15
4.351E-07 0.        7.765E-07 9.233E-15 1.394E-06 1.730E-14 2.520E-06 2.209E-15
4.590E-06 0.        8.432E-06 1.066E-15 1.563E-05 6.848E-15 2.929E-05 7.559E-15
5.546E-05 2.776E-17 1.062E-04-6.551E-16 2.053E-04-1.093E-15 3.998E-04-4.046E-15
7.781E-04 2.525E-16 1.495E-03-5.169E-15 2.785E-03 8.792E-16 4.927E-03 9.203E-15
8.102E-03 1.114E-15 1.216E-02-3.507E-15 1.641E-02-6.560E-15 1.972E-02-1.411E-15

2.098E-02 0.        1.847E-02 2.419E-15 1.270E-02 9.314E-15 7.025E-03 2.918E-15
3.291E-03-2.514E-15 1.406E-03-5.611E-15 5.909E-04-1.444E-15 2.558E-04-3.050E-16
1.151E-04-6.330E-16 5.332E-05-3.841E-16 2.521E-05-1.870E-16 1.210E-05-8.934E-17
5.866E-06-1.209E-16 2.869E-06-1.199E-16 1.413E-06-8.085E-17 7.004E-07-4.182E-17
3.488E-07-3.944E-17 1.745E-07-2.694E-17 8.762E-08-3.187E-17 4.414E-08-2.184E-17
2.231E-08-2.148E-17 1.130E-08-1.316E-17 5.742E-09-1.261E-17 2.924E-09-9.779E-18
1.493E-09-6.662E-18 7.649E-10-4.123E-18 3.938E-10-2.271E-18 2.044E-10 2.075E-18
1.079E-10 2.958E-18 5.886E-11-5.050E-19 3.445E-11 5.997E-18 2.325E-11 6.116E-18
2.002E-11 0.        2.325E-11-6.116E-18 3.445E-11-5.997E-18 5.886E-11 5.050E-19
1.079E-10-2.958E-18 2.044E-10-2.075E-18 3.938E-10 2.271E-18 7.649E-10 4.123E-18
1.493E-09 6.662E-18 2.924E-09 9.779E-18 5.742E-09 1.261E-17 1.130E-08 1.316E-17
2.231E-08 2.148E-17 4.414E-08 2.184E-17 8.762E-08 3.187E-17 1.745E-07 2.694E-17
3.488E-07 3.944E-17 7.004E-07 4.182E-17 1.413E-06 8.085E-17 2.869E-06 1.199E-16
5.866E-06 1.209E-16 1.210E-05 8.934E-17 2.521E-05 1.870E-16 5.332E-05 3.841E-16
1.151E-04 6.330E-16 2.558E-04 3.050E-16 5.909E-04 1.444E-15 1.406E-03 5.611E-15
3.291E-03 2.514E-15 7.025E-03-2.918E-15 1.270E-02-9.314E-15 1.847E-02-2.419E-15
```

FIRST ROW, FIRST COLUMN, AND DIAGONAL OF JOINT CHARICTERISTIC FUNCTION

1.000E+00 0.	9.484E-01 0.	8.140E-01 0.	6.410E-01 0.
4.698E-01 0.	3.238E-01 0.	2.111E-01 0.	1.307E-01 0.
7.693E-02 0.	4.310E-02 0.	2.298E-02 0.	1.167E-02 0.
5.639E-03 0.	2.594E-03 0.	1.136E-03 0.	4.730E-04 0.
1.874E-04 0.	7.064E-05 0.	2.531E-05 0.	8.626E-06 0.
2.794E-06 0.	8.603E-07 0.	2.517E-07 0.	6.999E-08 0.
1.849E-08 0.	4.640E-09 0.	1.106E-09 0.	2.504E-10 0.
5.383E-11 0.	1.099E-11 0.	2.132E-12 0.	4.035E-13 0.
1.370E-13 0.	4.035E-13 0.	2.132E-12 0.	1.099E-11 0.
5.383E-11 0.	2.504E-10 0.	1.106E-09 0.	4.640E-09 0.
1.849E-08 0.	6.999E-08 0.	2.517E-07 0.	8.603E-07 0.
2.794E-06 0.	8.626E-06 0.	2.531E-05 0.	7.064E-05 0.
1.874E-04 0.	4.730E-04 0.	1.136E-03 0.	2.594E-03 0.
5.639E-03 0.	1.167E-02 0.	2.298E-02 0.	4.310E-02 0.
7.693E-02 0.	1.307E-01 0.	2.111E-01 0.	3.238E-01 0.
4.698E-01 0.	6.410E-01 0.	8.140E-01 0.	9.484E-01 0.

1.000E+00 0.	9.483E-01 0.	8.122E-01 0.	6.354E-01 0.
4.599E-01 0.	3.112E-01 0.	1.985E-01 0.	1.200E-01 0.
6.892E-02 0.	3.769E-02 0.	1.965E-02 0.	9.765E-03 0.
4.628E-03 0.	2.091E-03 0.	9.007E-04 0.	3.697E-04 0.
1.446E-04 0.	5.383E-05 0.	1.909E-05 0.	6.441E-06 0.
2.068E-06 0.	6.319E-07 0.	1.836E-07 0.	5.071E-08 0.
1.332E-08 0.	3.324E-09 0.	7.886E-10 0.	1.777E-10 0.
3.806E-11 0.	7.742E-12 0.	1.497E-12 0.	2.823E-13 0.
9.561E-14 0.	2.823E-13 0.	1.497E-12 0.	7.742E-12 0.
3.806E-11 0.	1.777E-10 0.	7.886E-10 0.	3.324E-09 0.
1.332E-08 0.	5.071E-08 0.	1.836E-07 0.	6.319E-07 0.
2.068E-06 0.	6.441E-06 0.	1.909E-05 0.	5.383E-05 0.
1.446E-04 0.	3.697E-04 0.	9.007E-04 0.	2.091E-03 0.
4.628E-03 0.	9.765E-03 0.	1.965E-02 0.	3.769E-02 0.
6.892E-02 0.	1.200E-01 0.	1.985E-01 0.	3.112E-01 0.
4.599E-01 0.	6.354E-01 0.	8.122E-01 0.	9.483E-01 0.

1.000E+00 0.	9.002E-01 0.	6.709E-01 0.	4.308E-01 0.
2.452E-01 0.	1.253E-01 0.	5.780E-02 0.	2.410E-02 0.
9.086E-03 0.	3.095E-03 0.	9.527E-04 0.	2.648E-04 0.
6.646E-05 0.	1.505E-05 0.	3.077E-06 0.	5.675E-07 0.
9.441E-08 0.	1.417E-08 0.	1.917E-09 0.	2.338E-10 0.
2.571E-11 0.	2.549E-12 0.	2.277E-13 0.	1.833E-14 0.
1.329E-15 0.	8.683E-17 0.	5.110E-18 0.	2.709E-19 0.
1.293E-20 0.	5.558E-22 0.	2.153E-23 0.	7.820E-25 0.
8.065E-26 0.	7.820E-25 0.	2.153E-23 0.	5.558E-22 0.
1.293E-20 0.	2.709E-19 0.	5.110E-18 0.	8.683E-17 0.
1.329E-15 0.	1.833E-14 0.	2.277E-13 0.	2.549E-12 0.
2.571E-11 0.	2.338E-10 0.	1.917E-09 0.	1.417E-08 0.
9.441E-08 0.	5.675E-07 0.	3.077E-06 0.	1.505E-05 0.
6.646E-05 0.	2.648E-04 0.	9.527E-04 0.	3.095E-03 0.
9.086E-03 0.	2.410E-02 0.	5.780E-02 0.	1.253E-01 0.
2.452E-01 0.	4.308E-01 0.	6.709E-01 0.	9.002E-01 0.

COMPUTED VALUES OF X AND N(X)

0.	1.3474656E-01
1.0027587E+00	1.2746600E-01
2.0055174E+00	1.0803034E-01
3.0082761E+00	8.2746242E-02
4.0110348E+00	5.8122990E-02
5.0137935E+00	3.8195555E-02
6.0165522E+00	2.4032236E-02
7.0193109E+00	1.4803461E-02
8.0220696E+00	9.0793143E-03
9.0248283E+00	5.5959660E-03
1.0027587E+01	3.4756023E-03
1.1030346E+01	2.1737180E-03
1.2033104E+01	1.3666137E-03
1.3035863E+01	8.6235751E-04
1.4038622E+01	5.4557255E-04
1.5041380E+01	3.4580208E-04
1.6044139E+01	2.1948558E-04
1.7046898E+01	1.3945934E-04
1.8049657E+01	8.8685815E-05
1.9052415E+01	5.6435779E-05
2.0055174E+01	3.5933260E-05
2.1057933E+01	2.2889875E-05
2.2060691E+01	1.4587248E-05
2.3063450E+01	9.3000929E-06
2.4066209E+01	5.9323970E-06
2.5068967E+01	3.7874490E-06
2.6071726E+01	2.4222495E-06
2.7074485E+01	1.5552427E-06
2.8077243E+01	1.0078336E-06
2.9080002E+01	6.6737529E-07
3.0082761E+01	4.6386707E-07
3.1085520E+01	3.5549707E-07
3.2088278E+01	3.2001740E-07
3.3091037E+01	3.5016234E-07
3.4093796E+01	4.5215746E-07

LEVEL CROSSINGS

EXAMPLE OF PROGRAM RLEVX USING 4 EIGENSOLUTIONS FROM NUMERICAL EXAMPLE OF REPORT

DIMENSIONAL LEVEL X	NON-DIMENSIONAL LEVEL X/SIGMA X	CROSSINGS PER UNIT TIME	CROSSINGS PER ZERO CROSSING
0.	0.	1.347E-01	1.000E+00
6.685E-01	2.000E-01	1.315E-01	9.755E-01
1.337E+00	4.000E-01	1.221E-01	9.059E-01
2.006E+00	6.000E-01	1.080E-01	8.017E-01
2.674E+00	8.000E-01	9.140E-02	6.783E-01
3.343E+00	1.000E+00	7.421E-02	5.507E-01
4.011E+00	1.200E+00	5.812E-02	4.314E-01
4.680E+00	1.400E+00	4.420E-02	3.280E-01
5.348E+00	1.600E+00	3.285E-02	2.438E-01
6.017E+00	1.800E+00	2.403E-02	1.784E-01
6.685E+00	2.000E+00	1.742E-02	1.292E-01
7.354E+00	2.200E+00	1.257E-02	9.332E-02
8.022E+00	2.400E+00	9.079E-03	6.738E-02
8.691E+00	2.600E+00	6.570E-03	4.876E-02
9.359E+00	2.800E+00	4.770E-03	3.540E-02
1.003E+01	3.000E+00	3.476E-03	2.579E-02
1.070E+01	3.200E+00	2.540E-03	1.885E-02
1.136E+01	3.400E+00	1.861E-03	1.381E-02
1.203E+01	3.600E+00	1.367E-03	1.014E-02
1.270E+01	3.800E+00	1.005E-03	7.460E-03
1.337E+01	4.000E+00	7.402E-04	5.493E-03
1.404E+01	4.200E+00	5.456E-04	4.049E-03
1.471E+01	4.400E+00	4.025E-04	2.987E-03
1.538E+01	4.600E+00	2.972E-04	2.205E-03
1.604E+01	4.800E+00	2.195E-04	1.629E-03
1.671E+01	5.000E+00	1.622E-04	1.204E-03
1.738E+01	5.200E+00	1.199E-04	8.900E-04
1.805E+01	5.400E+00	8.869E-05	6.582E-04
1.872E+01	5.600E+00	6.561E-05	4.869E-04
1.939E+01	5.800E+00	4.855E-05	3.603E-04
2.006E+01	6.000E+00	3.593E-05	2.667E-04
2.072E+01	6.200E+00	2.660E-05	1.974E-04
2.139E+01	6.400E+00	1.970E-05	1.462E-04
2.206E+01	6.600E+00	1.459E-05	1.083E-04
2.273E+01	6.800E+00	1.081E-05	8.019E-05
2.340E+01	7.000E+00	8.005E-06	5.941E-05
2.407E+01	7.200E+00	5.932E-06	4.403E-05
2.473E+01	7.400E+00	4.398E-06	3.264E-05
2.540E+01	7.600E+00	3.262E-06	2.421E-05
2.607E+01	7.800E+00	2.422E-06	1.798E-05
2.674E+01	8.000E+00	1.802E-06	1.337E-05

DECK LISTINGS

```
      SUBROUTINE CF1(ANS,F)
C
C A SUBROUTINE TO EVALUATE THE CHARACTERISTIC FUNCTION OF THE
C NON-GAUSSIAN RESPONSE FOR THE CASE OF CORRELATED GAUSSIAN COMPONENTS
C SUBROUTINE INTEGRATES FUNCTION OF THE FORM
C (X(T)*A(-1)*X/2 + Y(T)*B(-1)*Y/2 + I*PI2*F*X(T)*C*Y)*DX*DY
C FROM -INF TO +INF, WHERE X,Y ARE N VECTORS, (T) DENOTES TRANSPOSE,
C (-1) DENOTES INVERSE, A,B ARE SYMMETRIC COVARIANCE MATRICIES,
C AND I DENOTES SQRT(-1.).
C
C ANS = ANSWER
C A = INPUT CORRELATION MATRIX FOR FIRST SET OF GAUSSIAN PROCESSES
C A1,B1,C1 = SCRATCH MATRICIES USED IN RECURSIVE COMPUTATION
C A2 = .5*(INVERSE OF CORRELATION MATRIX A)
C B = INPUT CORRELATION MATRIX FOR SECOND SET OF GAUSSIAN PROCESSES
C B2 = .5*(INVERSE OF CORRELATION MATRIX B)
C C = FUNCTIONAL MATRIX
C DETA = DETERMINANT OF A
C DETB = DETERMINANT OF B
C F = INDEPENDENT VARIABLE OF CHARACTERISTIC FUNCTION
C N = NUMBER OF MODIFIED BESSEL PROCESSES
      COMMON /COR/ A(8,8),B(8,8),C(8,8),N,ISIZE,DETA,DETB,
     1A1(8,8),B1(8,8),C1(8,8),A2(8,8),B2(8,8)
      DATA PI2/6.28318530717958/
C
C
C COPY MATRICIES
      DO 5 I=1,N
      DO 5 J=I,N
      A1(I,J)=A2(I,J)
    5 B1(I,J)=B2(I,J)
      F2=F*PI2
      DO 6 I=1,N
      DO 6 J=1,N
    6 C1(I,J)=C(I,J)*F2
C
C COMPUTE ANSWER FOR FIRST PAIR OF INTEGRATIONS
      A11=A1(1,1)  $  B11=B1(1,1)  $  C11=C1(1,1)
      DIV=4.*A11*B11+C11**2  $  ANS=1./(DETA*DETB*DIV)
      IF (N-1) 106,105,10
C
C COMPUTE ANSWER FOR ADDITIONAL INTEGRATIONS
   10 DO 100 NI=2,N
      K=NI-1
      DO 60 I=NI,N
      A1KI=A1(K,I)  $  B1KI=B1(K,I)  $  C1KI=C1(K,I)  $  C1IK=C1(I,K)
      DO 60 J=NI,N
      A1KJ=A1(K,J)  $  B1KJ=B1(K,J)  $  C1KJ=C1(K,J)  $  C1JK=C1(J,K)
      IF (J.LT.I) GO TO 50
      A1(I,J)=A1(I,J)-(4.*B11*A1KI*A1KJ+(C11*(A1KI*C1JK+C1IK*A1KJ)-
     1A11*C1IK*C1JK))/DIV
      B1(I,J)=B1(I,J)-(4.*A11*B1KI*B1KJ+(C11*(B1KI*C1KJ+C1KI*B1KJ)-
     1B11*C1KI*C1KJ))/DIV
   50 C1(I,J)=C1(I,J)-(4.*(A11*C1IK*B1KJ+B11*A1KI*C1KJ-C11*A1KI*B1KJ)+
     1C11*C1IK*C1KJ)/DIV
   60 CONTINUE
      A11=A1(NI,NI)  $  B11=B1(NI,NI)  $  C11=C1(NI,NI)
      DIV=4.*A11*B11+C11**2
  100 ANS=ANS/DIV
  105 ANS=SQRT(ANS)
  106 RETURN
C
C
```

```
C COMPUTE INVERSE AND DETERMINANT OF CORRELATION MATRICIES
      ENTRY INVRS
      IF (N-1) 160,110,110
  110 DO 130 I=1,N
      DO 130 J=1,N
      A2(I,J)=A(I,J)
  130 B2(I,J)=B(I,J)
      CALL INVR(A2,N,DETA,ISIZE,ISIZE)
      CALL INVR(B2,N,DETB,ISIZE,ISIZE)
      DO 140 I=1,N
      DO 140 J=1,N
      A2(I,J)=A2(I,J)/2.
  140 B2(I,J)=B2(I,J)/2.
      IF ((DETA.GT.0.).AND.(DETB.GT.0.)) RETURN
      WRITE (6,150) DETA,DETB
  150 FORMAT (1H ,5X,*DETERMINANT .LE. ZERO, DETA =*,E11.3,*, DETB =*,
     1E11.3)
      GO TO 180
  160 WRITE (6,170) N
  170 FORMAT (1H ,5X,*MATRIX DIMENSION .LE. ZERO, N =*,I3)
  180 STOP $ END
```

```
      SUBROUTINE CF2(ANS,F1,F2)
C
C A SUBROUTINE TO EVALUATE THE 2-D CHARACTERISTIC FUNCTION OF THE
C NON-GAUSSIAN RESPONSE FOR THE CASE OF CORRELATED GAUSSIAN COMPONENTS
C SUBPOUTINE INTEGRATES FUNCTION OF THE FORM
C (X(T)*A(-1)*X + Y(T)*B(-1)*Y + I*PI2*(X(T)*C*Y*F1 + X(T)*D*Y*F2)*DX*DY
C FROM -INF TO +INF, WHERE X,Y ARE N VECTORS, (T) DENOTES TRANSPOSE,
C (-1) DENOTES INVERSE/2, A,B, ARE COVARIANCE MATRICIES, I IS SQRT(-1.)
C
      COMMON /COR/ A(8,8),B(8,8),C(8,8),D(8,8),N,ISIZE,DETA,DETB,
     1A1(8,8),B1(8,8),C1(8,8),A2(8,8),B2(8,8)
      DATA PI2/6.28318530717958/
C
C A = INPUT CORRELATION MATRIX FOR FIRST SET OF GAUSSIAN PROCESSES
C ANS = ANSWER
C A2 = .5*(INVERSE OF CORRELATION MATRIX A)
C A1,B1,C1 = SCRATCH ARRAYS USED FOR RECURSIVE COMPUTATION
C B = INPUT CORRELATION MATRIX FOR SECOND SET OF GAUSSIAN PROCESSES
C B2 = .5*(INVERSE OF CORRELATION MATRIX B)
C C = FUNCTIONAL MATRIX FOR F1
C D = FUNCTIONAL MATRIX FOR F2
C DETA = DETERMINANT OF A
C DETB = DETERMINANT OF B
C F1,F2 = INDEPENDENT VARIABLES OF CHARACTERISTIC FUNCTION
C N = NUMBER OF MODIFIED BESSEL PROCESSES
C
C
      IF (N) 160,107,4
C COPY MATRICIES
    4 DO 5 I=1,N
      DO 5 J=I,N
      A1(I,J)=A2(I,J)
    5 B1(I,J)=B2(I,J)
      DO 6 I=1,N
      DO 6 J=1,N
    6 C1(I,J)=(C(I,J)*F1+D(I,J)*F2)*PI2
C
C COMPUTE ANSWER FOR FIRST PAIR OF INTEGRATIONS
```

```
      A11=A1(1,1)  $  B11=B1(1,1)  $  C11=C1(1,1)
      DIV=4.*A11*B11+C11**2  $  DIV1=DETA*DETB*DIV
      IF (N-1) 160,105,10
C
C COMPUTE ANSWER FOR ADDITIONAL INTEGRATIONS
   10 DO 100 NI=2,N
      K=NI-1
      DO 60 I=NI,N
      A1KI=A1(K,I)  $  C1IK=C1(I,K)  $  B1KI=B1(K,I)  $  C1KI=C1(K,I)
      DO 60 J=NI,N
      C1KJ=C1(K,J)  $  B1KJ=B1(K,J)
      IF (J.LT.I) GO TO 55
      A1KJ=A1(K,J)  $  C1JK=C1(J,K)
      A1(I,J)=A1(I,J)-(4.*B11*A1KI*A1KJ+(C11*(A1KI*C1JK+C1IK*A1KJ)
     1-A11*C1IK*C1JK))/DIV
      B1(I,J)=B1(I,J)-(4.*A11*B1KI*B1KJ+(C11*(B1KI*C1KJ+C1KI*B1KJ)
     1-B11*C1KI*C1KJ))/DIV
   55 C1(I,J)=C1(I,J)-(4.*A11*C1IK*B1KJ+4.*B11*A1KI*C1KJ-4.*C11*A1KI*
     1B1KJ+C11*C1IK*C1KJ)/DIV
   60 CONTINUE
      A11=A1(NI,NI)  $  B11=B1(NI,NI)  $  C11=C1(NI,NI)
      DIV=4.*A11*B11+C11**2
  100 DIV1=DIV1*DIV
  105 ANS=1./SQRT(DIV1)
  106 RETURN
C
  107 ANS=1.0
      GO TO 106
C
C
C COMPUTE INVERSE AND DETERMINANT OF CORRELATION MATRICIES
      ENTRY INVRS
      IF (N-1) 160,110,110
  110 DO 130 I=1,N
      DO 130 J=1,N
      A2(I,J)=A(I,J)
  130 B2(I,J)=B(I,J)
      CALL INVR(A2,N,DETA,ISIZE,ISIZE)
      CALL INVR(B2,N,DETB,ISIZE,ISIZE)
      DO 140 I=1,N
      DO 140 J=1,N
      A2(I,J)=A2(I,J)/2.
  140 B2(I,J)=B2(I,J)/2.
      IF ((DETA.GT.0.).AND.(DETB.GT.0.)) GO TO 106
      WRITE (6,150) DETA,DETB
  150 FORMAT (1H ,5X,*DETERMINANT .LE. ZERO, DETA =*,E11.3,*, DETB =*,
     1E11.3)
      GO TO 180
  160 WRITE (6,170) N
  170 FORMAT (1H ,5X,*MATRIX DIMENSION .LE. ZERO, N =*,I3)
  180 STOP  $  END
```

```
      SUBROUTINE COEF(I,A,C)
C A SUBROUTINE TO FILL C ARRAY WITH THE COEFFICIENTS OF THE
C POLYNOMIAL REPRESENTING A(X) BETWEEN X(I) AND X(I+1).  NPOLY IS THE
C THE ORDER OF THE POLYNOMIAL
C
      DIMENSION A(1),C(1)
      COMMON /ARRAYS/ E(8),EFP(45,8),EFQ(45,8),X(45),FF1(45),FF2(45),
     1NP,NPOLY
      DATA NPOLY/3/
      IF (I.GT.1) GO TO 10
      N=1  $  GO TO 30
   10 IF (I.LE.NP-2) GO TO 20
      N=NP-3  $  GO TO 30
   20 N=I-1
   30 F1=A(N)  $  X1=X(N)  $  N=N+1  $  F2=A(N)  $  X2=X(N)
      N=N+1  $  F3=A(N)  $  X3=X(N)
      N=N+1  $  F4=A(N)  $  X4=X(N)
      F1=F1/(X1-X2)/(X1-X3)/(X1-X4)
      F2=F2/(X2-X1)/(X2-X3)/(X2-X4)
      F3=F3/(X3-X1)/(X3-X2)/(X3-X4)
      F4=F4/(X4-X1)/(X4-X2)/(X4-X3)
      C(4)=F1+F2+F3+F4
      C(3)=-F1*(X2+X3+X4)-F2*(X1+X3+X4)-F3*(X1+X2+X4)-F4*(X1+X2+X3)
      C(2)=F1*(X2*(X3+X4)+X3*X4)+F2*(X1*(X3+X4)+X3*X4)+F3*(X1*(X2+X4)+
     1X2*X4)+F4*(X1*(X2+X3)+X2*X3)
      C(1)=-F1*X2*X3*X4-F2*X1*X3*X4-F3*X1*X2*X4-F4*X1*X2*X3
      RETURN  $  END
```

```
      PROGRAM EIGU(INPUT,OUTPUT,PUNCH,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7=
     1PUNCH)
C
C THIS IS A PROGRAM TO GENERATE EIGENVALUES AND EIGENFUNCTIONS FOR THE
C NON-GAUSSIAN TURBULENCE MODEL RESPONSE CALCULATIONS:  THE EIGEN-
C FUNCTIONS ARE REPRESENTED BY PIECEWISE POLYNOMIAL FUNCTIONS OVER
C THE RANGE OF INTEGRATION, AND ALL INTEGRATIONS ARE EXACT WITHIN
C THIS APPROXIMATION.
C THIS PROGRAM IS FOR LONGITUDINAL GUSTS ONLY
C P,Q ARE EIGENFUNCTION ARRAYS FOR ITERATIVE PROCESS
C EFP IS ARRAY FOR STORAGE OF EIGENFUNCTIONS
C ER1 IS ERROR TOLERANCE ARRAY FOR EIGENVALUES USING 21 POINT EIGFN APP.
C ER2 IS ERROR TOLERANCE ARRAY FOR EIGENVALUES USING 41 POINT EIGFN APP.
C X(I) IS ARRAY CONTAINING ABCISSAE OF EIGENFUNCTIONS
C NP IS THE MAXIMUM NUMBER OF ABCISSAE POINTS FOR EIGENFUNCTIONS.
C    CURRENTLY SET TO 41, VALUES .GT. 45 REQUIRE REDIMENSIONING, NP MUST
C    BE AN ODD NUMBER.
C  SHOULD COMPLEX POLES OCCUR, ONLY ONE OF EACH CONJUGATE PAIR
C  IS TO BE USED
C
C INPUT DATA:
C     RESTRT (L1)  LOGICAL INDICATOR FOR RESTART OPTION.  IF RESTRT IS
C                  TRUE DATA DECK PUNCHED BY PREVIOUS RUN CAN BE USED
C                  TO RESTART COMPUTATION.
C
C     IF RESTRT = .FALSE. DATA DECK IS AS FOLLOWS
C     NPOL (I1)  NUMBER OF SYSTEM POLES TO BE READ.  IF COMPLEX POLES
C                OCCUR, ONLY ONE OF EACH CONJUGATE PAIR IS USED.
C     (CI(I),AI(I),I=1,NPOL) (4E20.10) NUMERATORS AND POLES RESPECTIVELY
C                OF SYSTEM TRANSFER FUNCTION PARTIAL FRACTION EXPANSION
C                IN COMPLEX FORM.  IF COMPLEX POLES OCCUR, ONLY ONE OF
C                THE PARTIAL FRACTION COMPLEX CONJUGATE TERMS IS USED.
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C              REAL PART OF POLES MUST BE NEGATIVE FOR STABLE SYSTEM.
C     SU,U,UL (3F10.5)  TURBULENCE VELOCITY STANDARD DEVIATION,
C              VEHICLE MEAN TRUE AIRSPEED, AND TURBULENCE GUST
C              VELOCITY SCALE LENGTH.
C     NEV (I1)  TOTAL NUMBER OF EIGENVALUES/EIGENFUNCTIONS REQUIRED.
C              NEV MUST BE LESS THAN OR EQUAL TO 5.
C
C     IF RESTRT = .TRUE. DATA DECK IS AS FOLLOWS
C     (ALL BUT LAST THREE CARDS (EOR,CONT,NEV) ARE PUNCHED BY
C     PREVIOUS RUN)
C     SU,U,UL (FORMAT 17)  SEE DESCRIPTION ABOVE
C     B,B1,NPOL,(CI(I),AI(I),I=1,NPOL) (FORMAT 6)
C     NP,LO,HI (FORMAT 7)
C     J,E(J),(X(K),EFP(K,J),K=1,NP)  (FORMAT 13)
C     END OF RECORD INDICATES END OF EIGENVALUE/EIGENFUNCTION DATA.
C     CONT (L1)  LOGICAL VARIABLE, IF TRUE IT IS ASSUMED THAT ITERATION
C     OF THE LAST EIGENSOLUTION IS TO CONTINUE, IF FALSE IT IS ASSUMED
C     THAT THE LAST EIGENSOLUTION IS ACCURATE AND ITERATION OF THE
C     NEXT SOLUTION IS TO BEGIN.
C     NEV (I1) TOTAL NUMBER OF EIGENVALUES/EIGENFUNCTIONS SOUGHT (.LE.5)
C
      DIMENSION E(5),ER1(5),ER2(5)
      COMMON /ARRAYS/ X(45),P(45),Q(45),NP,NSKIP,NPOLY,EFP(45,5),C(4),
     1D(4)
      COMMON /CONST/ CI(5),AI(5),NPOL,B,B1
      COMMON /EVAL/ EVP
      COMPLEX CI,AI  $  REAL LO  $  LOGICAL RESTRT,CONT
      DATA ERI/1.E-3/
      DATA ER1,ER2,NP/1.E-4,1.E-4,5.E-4,1.E-3,1.E-2,1.E-5,1.E-5,1.E-4,
     15.E-4,1.E-3,41/
      DATA (X(I),I=1,41)/0.,.01,.02,.03,.045,.06,.08,.1071,.1341,
     1.1612,.1882,.2153,.2424,.2694,.2965,.3235,.3506,.3776,.4047,
     2.4318,.4588,.4859,.5129,.5400,.5671,.5941,.6212,.6482,.6753,
     3.7024,.7294,.7565,.7835,.8106,.8376,.8647,.8918,.9188,.9459,
     4.9729,1./
C
    1 FORMAT (I1)
    2 FORMAT (4E20.10)
    3 FORMAT (3F10.5)
    4 FORMAT (I10)
    5 FORMAT (L1)
    6 FORMAT (3X,E20.12,5X,E20.12,11X,I2,/,(4E20.12))
    7 FORMAT (4X,I3,27X,2E20.11)
    8 FORMAT (8F10.5)
   11 FORMAT (*B =*,E20.12,1X,*B1 =*,E20.12,5X,*NPOL =*,I2,/,(4E20.12))
   12 FORMAT (*NP =*,I3,5X,*RANGE OF INTEGRATION =*,2E20.11)
   13 FORMAT (43X,I2,E20.11,/,(2E25.14))
   14 FORMAT (*SU =*,E15.7,1X,*MTAS =*,E15.7,1X,*UL =*,E15.7)
   17 FORMAT (4X,E15.7,7X,E15.7,5X,E15.7)
C
      READ (5,5) RESTRT
      IF(.NOT.RESTRT) GO TO 20
C
C RESTART IS TRUE, READ DATA FROM PREVIOUS RUN
      READ (5,17) SU,U,UL
      READ (5,6) B,B1,NPOL,(CI(I),AI(I),I=1,NPOL)
      READ (5,7) NP,LO,HI
      DO 15 I1=1,6
      READ (5,13) J,E(J),(X(K),EFP(K,J),K=1,NP)
      IF (EOF,5) 16,15
   15 CONTINUE
      GO TO 800
   16 READ (5,5) CONT
```

```
      READ (5,1) NEV
      I=I1-1
      DO 18 K=1,NP
   18 P(K)=EFP(K,I)
      IF (.NOT.CONT) GO TO 95
      I1=I1-1  $  I=I1-1
      GO TO 95
C
C RESTART IS FALSE, READ DATA DEFINING NEW PROBLEM.
   20 READ (5,1) NPOL
      IF (NPOL) 900,900,21
   21 READ (5,2) (CI(I),AI(I),I=1,NPOL)
C CHANGE SIGN OF EXPONENT SINCE ALL SUBROUTINES EXPECT NEGATIVE OF
C TRUE EXPONENT.
      DO 22 I=1,NPOL
   22 AI(I)=-AI(I)
      READ (5,3) SU,U,UL
      B=U/2./UL  $  HI=20./B  $  B1=SU*U/UL
      IF (HI.LE.1.) GO TO 26
C
C SCALE X VALUES TO COVER RANGE OF INTEGRATION, USE EI,EVP,EROR FOR
C TEMPORARY STORAGE.
C EI IS LOWER LIMIT OF POWER LAW SCALING.
C NPWR IS POWER OF SCALING, HIGHER VALUES WILL PLACE MORE X VALUES NEAR
C ORIGIN.  LOW VALUES WILL PROVIDE MORE EVEN DISTRIBUTION OF X VALUES
C OVER RANGE OF INTEGRATION.
      EI=.05  $  NPWR=1
      EROR=((HI-EI)/(1.-EI)-1.)/(1.-EI)**NPWR
      DO 25 I=1,NP
      EVP=X(I)
      IF (EVP.LT.EI) GO TO 25
      EVP=EVP-EI  $  EVP=EVP*(1.+EROR*EVP**NPWR)+EI
   25 X(I)=EVP
C
C PUNCH PARAMETERS ON CARDS
   26 WRITE (7,14) SU,U,UL
      WRITE (7,11) B,B1,NPOL,(CI(I),AI(I),I=1,NPOL)
      WRITE (7,12) NP,X(1),X(NP)
      I=0  $  I1=1
   90 READ (5,1) NEV
      CONT=.F.
   95 IF (NEV-I1) 800,100,100
C
C COMPUTE EXACT SYSTEM RESPONSE VARIANCE
  100 CALL VARU(SU,UL,U,SR2)
      CALL STARAY(LO,LO)
      IF (.NOT.CONT) GO TO 110
      EROR=ER2(I1)  $  NSKIP=1  $  GO TO 111
C
C GENERATE INITIAL GUESS OF EIGENFUNCTION P(ALPHA)
  110 EROR=ER1(I1)  $  NSKIP=2
      CALL GUESS(I1)
C
  111 WRITE (6,101) I1,SU,U,UL,(CI(J),AI(J),J=1,NPOL)
  101 FORMAT (1H1,5X,*BEGIN ITERATION FOR EIGENVALUE AND EIGENFUNCTION N
     1O.*,I2,/,/,6X,*TURBULENCE PARAMETERS:*,/,6X,*STD. DEV. =*,F6.3,5X,
     2*MTAS =*,F8.3,5X,*SCALE LENGTH =*,F9.3,/,/,6X,*VEHICLE PARAMETERS:
     3*,/,12X,*COEFFICIENTS*,26X,*POLES*,/,(4X,2E13.4,8X,2E13.4))
      WRITE (6,102) SR2
  102 FORMAT (1H0,5X,*SYSTEM RESPONSE VARIANCE =*,E13.6,/,1H0)
      J=1  $  EVP=EI=1.E50  $  SEV=1.  $  GO TO 210
C
  200 CALL FILL
```

```
      EROR=ER2(I1)
C
C IF (I.GT.0) EIGENFUNCTION MUST BE ORTHOGONAL TO ALL PREVIOUS FNS
  210 IF (I.LE.0) GO TO 300
      CALL ORTHOG(I)
C
C COMPUTE NORMALIZATION FACTOR
  300 CALL NORMF(F)
C
C ESTIMATE EIGENVALUE
  400 EI=EI*F
  401 DEVP=ABS(ABS(1./EVP/F)-1.)
      EVP=SEV/F
      WRITE (6,405) J,EVP,DEVP,EI
  405 FORMAT (1H ,5X,*ITERATION NO.*,I3,*.  ESTIMATED EIGENVALUE IS*,
     1E15.7,*.  (DEVP =*,E10.3,2X,*EI =*,E10.3,*)*)
C
C NORMALIZE EIGENFUNCTION
  500 CALL NORM(F)
      IF ((J.LT.30).OR.((J/5)*5.NE.J)) GO TO 520
      WRITE (6,510) (X(JJ),P(JJ),JJ=1,NP)
  510 FORMAT (1H ,2E25.14)
      WRITE (7,511) I1
  511 FORMAT (43X,I2)
      WRITE (7,707) (X(JJ),P(JJ),P(JJ),JJ=1,NP)
C
C IF DEV IS SUFFICIENTLY SMALL, ACCEPT EIGENFUNCTION
  520 IF ((DEVP.LE.EROR).AND.(EI.LE.ERI)) GO TO 700
      IF ((J.GE.25).AND.(NSKIP.NE.1)) GO TO 200
C
C EIGENFUNCTION NOT YET FOUND, CONTINUE ITERATION PROCESS
  600 CALL ITERAT(EI,SEV)
      J=J+1
      GO TO 210
C
C ERROR IS BELOW TOLERANCE, CHECK TO SEE THAT MAX. NO. OF POINTS USED
  700 IF (NSKIP.NE.1) GO TO 200
C ACCEPT EIGENFUNCTION
      I=I+1  $  I1=I+1
      E(I)=EVP  $  FS=EVP**2/SR2  $  TFS=0.
      DO 702 J=1,I
  702 TFS=TFS+E(J)**2  $  TFS=TFS/SR2
      WRITE (7,706) I,EVP
  706 FORMAT (*EIGENVALUE AND CONJUGATE EIGENFUNCTIONS NO.*,I2,E20.11)
      WRITE (7,707) (X(J),P(J),P(J),J=1,NP)
  707 FORMAT (3E25.14)
      WRITE (6,708) I,EVP,FS,I,TFS,(X(J),P(J),J=1,NP)
  708 FORMAT (1H1,5X,*EIGENVALUE NO.*,I2,* IS*,E14.7,2(/),6X,*FRACTION O
     2F RESPONSE VARIANCE DUE TO THIS EIGENVALUE IS*,F7.5,/,6X,
     3*FRACTION OF RESPONSE VARIANCE DUE TO FIRST*,I2,* EIGENVALUES IS*,
     4F7.5,/,/,9X,*X*,11X,*EIGENFUNCTION*,/,(6X,F8.4,5X,E17.10))
      IF (I.GE.NEV) GO TO 800
      CALL STOR(I)
      CONT=.F.
      GO TO 110
C
  800 WRITE (6,805) NEV,(J,E(J),J=1,I)
  805 FORMAT (1H1,5X,*FIRST*,I2,* EIGENVALUES ARE:*,/,(10X,I2,2X,E15.7))
      FS=SR2*(1.-TFS)
      WRITE (6,810) FS,TFS
  810 FORMAT (1H0,5X,*REMAINING EIGENVALUES SQUARED SUM TO*,E15.7,/,6X,
     1*THE ABOVE EIGENVALUES ACCOUNT FOR*,F7.5,* OF THE RESPONSE VAR.*)
      IF (FS.LT.0.) GO TO 900
```

```
      FS=SQRT(FS)                                                       EIGU0222
      WRITE (6,812) FS                                                  EIGU0223
  812 FORMAT (1H ,5X,*LARGEST POSSIBLE REMAINING EIGENVALUE IS*,E14.7)  EIGU0224
  900 STOP  $  END                                                      EIGU0225
      SUBROUTINE VARU(SU,UL,U,S2)                                       EIGU0226
C                                                                       EIGU0227
C A SUBROUTINE TO COMPUTE THE RESPONSE VARIANCE OF A LINEAR SYSTEM      EIGU0228
C SUBJECTED TO RANDOM INPUT WITH DRYDEN U-GUST SPECTRUM                 EIGU0229
C                                                                       EIGU0230
C SU = TURBULENCE STANDARD DEVIATION                                    EIGU0231
C UL = TURBULENCE SCALE LENGTH                                          EIGU0232
C U = VEHICLE MEAN TRUE AIRSPEED                                        EIGU0233
C AI = VEHICLE IMPULSE RESPONSE EXPONENTS (MUST HAVE POSITIVE REAL PART)EIGU0234
C CI = VEHICLE IMPULSE RESPONSE CONSTANTS                               EIGU0235
C  IF CI OR AI IS COMPLEX, ONLY ONE OF EACH CONJUGATE PAIR IS TO BE USEDEIGU0236
C                                                                       EIGU0237
      COMPLEX CI,AI,AJ,AK,CJ,CK,C2,C3,C4                                EIGU0238
      COMMON /CCNST/ CI(5),AI(5),NPOL                                   EIGU0239
      LOGICAL RL1,RL2                                                   EIGU0240
      C=U/UL  $  S2=0.                                                  EIGU0241
      DO 20 K=1,NPOL                                                    EIGU0242
      C4=C.  $  AK=AI(K)  $  CK=CI(K)                                   EIGU0243
      RL1=AIMAG(AK).EQ.0..AND.AIMAG(CK).EQ.0.                           EIGU0244
    5 DO 10 J=1,NPOL                                                    EIGU0245
      C3=C.  $  AJ=AI(J)  $  CJ=CI(J)                                   EIGU0246
      RL2=AIMAG(AJ).EQ.0..AND.AIMAG(CJ).EQ.0.                           EIGU0247
    6 C2=1./(C-AJ)                                                      EIGU0248
      C2=C2*(1./(AJ+AK)/(AJ+C)-.5/C/(AK+C))                             EIGU0249
      C2=C2*CJ*CK                                                       EIGU0250
      C3=C3+C2                                                          EIGU0251
      IF (RL2) GO TO 10                                                 EIGU0252
      AJ=CONJG(AJ)  $  CJ=CONJG(CJ)  $  RL2=.T.  $  GO TO 6             EIGU0253
   10 C4=C4+C3                                                          EIGU0254
      IF (RL1) GO TO 20                                                 EIGU0255
      AK=CONJG(AK)  $  CK=CONJG(CK)  $  RL1=.T.  $  GO TO 5             EIGU0256
   20 S2=S2+REAL(C4)                                                    EIGU0257
      S2=S2*2.*C*SU**2                                                  EIGU0258
      RETURN  $  END                                                    EIGU0259
      SUBROUTINE FILL                                                   EIGU0260
C                                                                       EIGU0261
C THIS SUBROUTINE INTERPOLATES THE 21 POINT EIGENFUNCTION TO FORM THE   EIGU0262
C FIRST APPROXIMATION TO THE 41 POINT EIGENFUNCTION.                    EIGU0263
C                                                                       EIGU0264
      COMMON /ARRAYS/ X(45),P(45),I,I1,K,XP,C(41),NP,NSKIP              EIGU0265
      I1=NSKIP/2+1                                                      EIGU0266
      DO 10 I=I1,NP,NSKIP                                               EIGU0267
      XP=X(I)  $  K=I+1-I1                                              EIGU0268
      CALL COEF(K,P,C)                                                  EIGU0269
   10 P(I)=((C(4)*XP+C(3))*XP+C(2))*XP+C(1)                             EIGU0270
      NSKIP=NSKIP/2                                                     EIGU0271
      RETURN  $  END                                                    EIGU0272
      SUBROUTINE GUESS(I)                                               EIGU0273
C                                                                       EIGU0274
C THIS SUBROUTINE GUESSES THE NEXT EIGENFUNCTION TO BE THE              EIGU0275
C DERIVATIVE OF THE PROVIOUS RESULT OR X*EXP(-3.*X) FOR THE FIRST.      EIGU0276
C                                                                       EIGU0277
      COMMON /ARRAYS/ X(45),P(45),J,XJ,K,DX,DP,Q(40),NP,NSKIP           EIGU0278
      IF (NSKIP.LE.1) GO TO 70                                          EIGU0279
      IF (I.GT.1) GO TO 50                                              EIGU0280
      DO 10 J=1,NP,NSKIP                                                EIGU0281
      XJ=X(J)                                                           EIGU0282
   10 P(J)=XJ*EXP(-3.*XJ)                                               EIGU0283
      RETURN                                                            EIGU0284
```

```
C
   50 K=NSKIP+1
      DO 60 J=K,NP,NSKIP
      DP=P(J)-P(J-1)   $   DX=X(J)-X(J-1)   $   XJ=X(J)
      XJ=XJ*EXP(-3.*XJ)
   60 P(J)=XJ*DP/DX
      RETURN
C
   70 WRITE (6,71) NSKIP
   71 FORMAT (1H ,5X,*NSKIP =*,I10,* NOT ALLOWED IN GUESS*)
      STOP  $  END
      SUBROUTINE NORMF(F)
C
C COMPUTES NORM OF THE FUNCTION P(X)
C THE NORM IS THE ESTIMATED EIGENVALUE, THE INVERSE OF THE NORM IS THE
C NORMALIZATION FACTOR
C
      COMMON /ARRAYS/ X(45),P(45),C(4),F1,NP1,J,I,X1,X2,II,CII,IJ,K,CK,
     1X12,X22,JJ,C(27),NP,NSKIP,NPOLY
      F1=0.  $  NP1=NPOLY+1  $  J=NP-1
      DO 60 I=1,J,NSKIP
      F = 0.
      CALL COEF(I,P,C)
   12 X1=X(I)  $  X2=X(I+NSKIP)
      IF (X2.GT.X(NP)) X2=X(NP)
      DO 40 II=1,NPOLY
      CII=C(II)  $  IJ=II+1  $  K=II+IJ-1  $  CK=FLOAT(K)
      X12=X1**K  $  X22=X2**K
      DO 40 JJ=IJ,NP1
      F = F+CII*C(JJ)*(X22-X12)/CK
      CK=CK+1.  $  X12=X12*X1
   40 X22=X22*X2
      F=F*2.  $  X12=X1**2  $  X22=X2**2  $  CK=1.
      DO 50 II=1,NP1
      F=F+(X2-X1)*(C(II)**2)/CK  $  CK=CK+2.  $  X2=X2*X22
   50 X1=X1*X12
   60 F1=F1+F
      F=SQRT(1./F1)
      RETURN
C
C NORMALIZE EIGENFUNCTION P
      ENTRY NORM
      DO 100 I=1,NP,NSKIP
  100 P(I)=P(I)*F
      RETURN  $  END
      SUBROUTINE ORTHOG(I)
C
C A SUBROUTINE TO ORTHOGONALIZE P(X) WITH RESPECT TO THE FIRST I EGNFNS
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),NP,NSKIP,NPOLY,EFP(45,5),C(4),
     1D(4)
      NP1 = NPOLY+1
      DO 50 J=1,I
      DO 10 JJ=1,NP,NSKIP
   10 Q(JJ)=EFP(JJ,J)
      K=NP-1  $  F=0.
      DO 40 JJ=1,K,NSKIP
      CALL COEF(JJ,P,C)
      CALL COEF(JJ,Q,D)
      X1=X(JJ)  $  II=JJ+NSKIP  $  IF (II.GT.NP) II=NP  $  X2=X(II)
      DO 40 II=1,NP1
      X12=X1**II  $  X22=X2**II  $  CII=C(II)  $  CL=FLOAT(II)
      DO 40 KK=1,NP1
```

```
      F=F+CII*D(KK)*(X22-X12)/CL  $  CL=CL+1.  $  X12=X12*X1
   40 X22 = X22*X2
      DO 50 JJ=1,NP,NSKIP
   50 P(JJ)=P(JJ)-F*Q(JJ)
      RETURN  $  END
      FUNCTION SUM(I,J)
C
C FORMS PRODUCT  (J-I+1)*(J-I+2)......(J)
C
      N=1
      IF(I) 20,20,10
   10 K=J-I+1
      DO 15 L=K,J
   15 N=N*L
   20 SUM=FLOAT(N)
      RETURN  $  END
      SUBROUTINE COEF(I,A,C)
C A SUBROUTINE TO FILL C ARRAY WITH THE COEFFICIENTS OF THE
C POLYNOMIAL REPRESENTING A(X) BETWEEN X(I) AND X(I+NSKIP).  NPOLY IS
C THE ORDER OF THE POLYNOMIAL
C POLYNOMIAL IS C(1) + C(2)*X + C(3)*X**2 + C(4)*X**3
C
      DIMENSION A(1),C(1)
      COMMON /ARRAYS/ X(45),P(45),Q(45),NP,NSKIP,NPOLY
      DATA NPOLY/3/
      IF (I.GT.NSKIP) GO TO 10
      N=1  $  GO TO 30
   10 IF (I.LE.NP-2*NSKIP) GO TO 20
      N=NP-3*NSKIP  $  GO TO 30
   20 N=I-NSKIP
   30 F1=A(N)  $  X1=X(N)  $  N=N+NSKIP  $  F2=A(N)  $  X2=X(N)
      N=N+NSKIP  $  F3=A(N)  $  X3=X(N)
      N=N+NSKIP  $  F4=A(N)  $  X4=X(N)
      F1=F1/(X1-X2)/(X1-X3)/(X1-X4)
      F2=F2/(X2-X1)/(X2-X3)/(X2-X4)
      F3=F3/(X3-X1)/(X3-X2)/(X3-X4)
      F4=F4/(X4-X1)/(X4-X2)/(X4-X3)
      C(4)=F1+F2+F3+F4
      C(3)=-F1*(X2+X3+X4)-F2*(X1+X3+X4)-F3*(X1+X2+X4)-F4*(X1+X2+X3)
      C(2)=F1*(X2*(X3+X4)+X3*X4)+F2*(X1*(X3+X4)+X3*X4)+F3*(X1*(X2+X4)+
     1X2*X4)+F4*(X1*(X2+X3)+X2*X3)
      C(1)=-F1*X2*X3*X4-F2*X1*X3*X4-F3*X1*X2*X4-F4*X1*X2*X3
      RETURN  $  END
      SUBROUTINE CIFC
C INTEGRATE COMPLEX TERMS OF TRANSFER FUNCTION
C
      EQUIVALENCE (E,ER),(XE,XER),(XE1,XE1R),(D1,D1R)
      COMMON /DIF/ X,E,N,D1
      COMPLEX E,XE,XE1,D1
      N1=N+1  $  C1=0.
      IF (N) 70,10,20
C
C CONSTANT TERM OF POLYNOMIAL
   10 D1=1.  $  GO TO 30
C
C POWER TERM OF POLYNOMIAL
   20 IF (X.NE.0.) GO TO 25
      D1=(-1./E)**N*SUM(N,N)
      GO TO 30
   25 XN=X**N
      XE1=X*E  $  XE=1.  $  S1=1.  $  S2=FLOAT(N)
      DO 28 I=1,N1
      D1=XN/XE*S1+D1  $  XE=XE*XE1  $  S1=S1*S2  $  S2=S2-1.
```

```
   28 XN=-XN
   30 D1=C1*CEXP(E*X)/E
   70 RETURN
C
C INTEGRATE REAL TERMS OF TRANSFER FUNCTION
      ENTRY DIFR
      N1=N+1  $  D1=0.
      IF (N) 170,110,120
C
C CONSTANT TERM OF POLYNOMIAL
  110 D1R=1.  $  GO TO 130
C
C POWER TERM OF POLYNOMIAL
  120 IF (X.NE.0.) GO TO 125
      D1R=(-1./ER)**N*SUM(N,N)
      GO TO 130
  125 XN=X**N
      XE1R=X*ER  $  XER=1.  $  S1=1.  $  S2=FLOAT(N)
      DO 128 I=1,N1
      D1R=XN/XER*S1+D1R  $  XER=XER*XE1R  $  S1=S1*S2  $  S2=S2-1.
  128 XN=-XN
  130 D1=CMPLX(D1R*EXP(ER*X)/ER,0.0)
  170 RETURN  $  END
      SUBROUTINE NTGRAL(V,ALPHA)
C
C INTEGRATES P(X)*H(ALPHA,X)
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),NP,NSKIP,NPOLY,EFP(225),C(4)
      COMMON /CONST/ CI(5),AI(5),NPOL,B,B1
      COMMON /DIF/ X1,E,NPOM,D1C
      DIMENSION D1S(4),D2S(4)
      DIMENSION D1A(45,4),D1CA(45,4,2),CA(180)
      COMPLEX D1CA
      COMPLEX AI,CI,E,C1,C2,V1,V2,E2,D2S,D1C
      LOGICAL RL
      EQUIVALENCE (E,ER),(D1,D1C)
      V=0.  $  E1=EXP(-B*ALPHA)
C
C CYCLE THROUGH POLES OF TRANSFER FUNCTION
      DO 100 IPOL=1,NPOL
      C1=CI(IPOL)/(AI(IPOL)-2.*B)
      E2=CEXP(-ALPHA*(AI(IPOL)-B))
      V1=0.  $  V2=0.
C
C CYCLE THROUGH X INTERVALS
      DO 10 NX=1,NPO1
      D1S(NX)=D1A(1,NX)
   10 D2S(NX)=D1CA(1,NX,IPOL)
      I=NSKIP+1  $  NXI=1
      DO 80 NX=I,NP,NSKIP
      X1=X(NX)
C
C CYCLE THROUGH TERMS OF POLYNOMIAL
      DO 60 NPO=1,NPO1
      C1=D1A(NX,NPO)  $  E=D1-D1S(NPO)  $  V1=E1*E
   20 D1S(NPO)=C1
      IF (X1-ALPHA) 21,21,40
   21 D1C=D1CA(NX,NPO,IPOL)
      V1=V1-E1*(D1C-D2S(NPO))
   25 D2S(NPO)=D1C
      GO TO 59
   40 V1=V1-E2*E
   59 V2=V1*CA(NXI)+V2
```

```
   60 NXI=NXI+1
   80 CONTINUE
      V2=V2*C1
      IF ((AIMAG(C1).EQ.0.).AND.(AIMAG(E2).EQ.0.)) GO TO 100
      V2=2.*REAL(V2)
  100 V=V+V2
      V=V*B1
      GO TO 500
C
C
      ENTRY STARAY
      NPO1=NPOLY+1  $  ER=-B
      DO 200 NX=1,NP
      X1=X(NX)
      DO 200 NPO=1,NPO1
      NPOM=NPO-1
      CALL DIFR
  200 D1A(NX,NPO)=D1
      DO 300 IPCL=1,NPOL
      E=B-AI(IPCL)  $  RL=AIMAG(E).EQ.0.
      DO 300 NPO=1,NPO1
      NPOM=NPO-1
      DO 300 NX=1,NP
      X1=X(NX)  $  IF(RL) GO TO 250  $  CALL DIFC  $  GO TO 300
  250 CALL DIFR
  300 D1CA(NX,NPO,IPOL)=D1C
      GO TO 500
C
C GENERATE TABLE OF POLYNOMIAL COEFFICIENTS
      ENTRY GENCOE
      NPO=0
      DO 400 NXI=1,NP,NSKIP
      IF (NXI-NP) 350,400,400
  350 CALL COEF(NXI,P,C)
      DO 380 NPO1=1,4
      NPO=NPO+1
  380 CA(NPO)=C(NPO1)
  400 CONTINUE
      NPO1=NPOLY+1
  500 RETURN  $  END
      SUBROUTINE ITERAT(E,SEV)
C
C A SUBROUTINE TO ITERATE THE EIGENFUNCTION P(X)
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),NP,NSKIP,NPOLY
      COMMON /EVAL/ EVP
      CALL GENCOE(Q,X)
      Q(1)=0.  $  E=0.  $  I1=NSKIP+1
      DO 50 I=I1,NP,NSKIP
   30 QI=P(I)*EVP
      CALL NTGRAL(Q(I),X(I))
   50 E=E+(ABS(QI)-ABS(Q(I)))**2
      E=SQRT(E)/FLOAT(NP)  $  S1=S2=0.
      DO 60 I=1,NP,NSKIP
      S1=S1+ABS(P(I)+Q(I))  $  S2=S2+ABS(P(I)-Q(I))
   60 P(I)=Q(I)
      IF (S1-S2) 70,80,90
   70 SEV=-1.  $  GO TO 100
   80 WRITE (6,81)  $  STOP
   81 FORMAT (1H ,5X,*ZERO EIGFN. COMPUTED BY ITERAT*)
   90 SEV=1.
  100 RETURN  $  END
      SUBROUTINE STOR(I)
```

```
C                                                                       EIGU0537
C STORES ITH EIGENFUNCTION                                              EIGU0538
C                                                                       EIGU0539
      COMMON /ARRAYS/ X(45),P(45),J,DUM(44),NP,NSKIP,NPOLY,EFP(45,5)    EIGU0540
      DO 10 J=1,NP                                                      EIGU0541
   10 EFP(J,I)=P(J)                                                     EIGU0542
      RETURN $ END                                                      EIGU0543


      PROGRAM EIGVW(INPUT,OUTPUT,PUNCH,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7=  EIGVW000
     1PUNCH)                                                            EIGVW001
C                                                                       EIGVW002
C THIS IS A PROGRAM TO GENERATE EIGENVALUES AND EIGENFUNCTIONS FOR THE  EIGVW003
C NON-GAUSSIAN TURBULENCE MODEL RESPONSE CALCULATIONS.  THE EIGEN-      EIGVW004
C FUNCTIONS ARE REPRESENTED BY PIECEWISE POLYNOMIAL FUNCTIONS OVER      EIGVW005
C THE RANGE OF INTEGRATION, AND ALL INTEGRATIONS ARE EXACT WITHIN       EIGVW006
C THIS APPROXIMATION.                                                   EIGVW007
C THIS PROGRAM IS FOR VERTICAL AND LATERAL GUSTS ONLY                   EIGVW008
C P(ALPHA),Q(BETA) ARE EIGENFUNCTION ARRAYS FOR ITERATIVE PROCESS       EIGVW009
C EFP,EFQ ARE ARRAYS FOR STORAGE OF EIGENFUNCTIONS                      EIGVW010
C ER1 IS ERROR TOLERANCE ARRAY FOR EIGENVALUES USING 21 POINT EIGFN APP.EIGVW011
C ER2 IS ERROR TOLERANCE ARRAY FOR EIGENVALUES USING 41 POINT EIGFN APP.EIGVW012
C X(I) IS ARRAY CONTAINING ABCISSAE OF EIGENFUNCTIONS                   EIGVW013
C NP IS THE MAXIMUM NUMBER OF ABCISSAE POINTS FOR EIGENFUNCTIONS.       EIGVW014
C    CURRENTLY SET TO 41, VALUES .GT. 45 REQUIRE REDIMENSIONING, NP MUSTEIGVW015
C    BE AN ODD NUMBER.                                                  EIGVW016
C E IS THE EIGENVALUE ARRAY                                             EIGVW017
C HI IS THE UPPER LIMIT OF INTEGRATION                                  EIGVW018
C NPOL IS THE NUMBER OF POLES IN THE LINEAR SYSTEM TRANSFER FUNCTION    EIGVW019
C  SHOULD COMPLEX POLES OCCUR, ONLY ONE OF EACH CONJUGATE PAIR          EIGVW020
C  IS TO BE USED                                                        EIGVW021
C                                                                       EIGVW022
C INPUT DATA.                                                           EIGVW023
C     RESTRT (L1)  LOGICAL INDICATOR FOR RESTART OPTION.  IF RESTRT IS  EIGVW024
C                  TRUE DATA DECK PUNCHED BY PREVIOUS RUN CAN BE USED   EIGVW025
C                  TO RESTART COMPUTATION.                              EIGVW026
C                                                                       EIGVW027
C     IF RESTRT = .FALSE. DATA DECK IS AS FOLLOWS                       EIGVW028
C     NPOL (I1)  NUMBER OF SYSTEM POLES TO BE READ.  IF COMPLEX POLES   EIGVW029
C                OCCUR, ONLY ONE OF EACH CONJUGATE PAIR IS USED.        EIGVW030
C     (CI(I),AI(I),I=1,NPOL) (4E20.10) NUMERATORS AND POLES RESPECTIVELYEIGVW031
C                OF SYSTEM TRANSFER FUNCTION PARTIAL FRACTION EXPANSION EIGVW032
C                IN COMPLEX FORM.  IF COMPLEX POLES OCCUR, ONLY ONE OF  EIGVW033
C                THE PARTIAL FRACTION COMPLEX CONJUGATE TERMS IS USED.  EIGVW034
C                REAL PART OF POLES MUST BE NEGATIVE FOR STABLE SYSTEM. EIGVW035
C     SW,U,WL (3F10.5)  TURBULENCE VELOCITY STANDARD DEVIATION,         EIGVW036
C                VEHICLE MEAN TRUE AIRSPEED, AND TURBULENCE GUST        EIGVW037
C                VELOCITY SCALE LENGTH.                                 EIGVW038
C     NEV (I1)  TOTAL NUMBER OF EIGENVALUES/EIGENFUNCTIONS REQUIRED.    EIGVW039
C                NEV MUST BE LESS THAN OR EQUAL TO 5.                   EIGVW040
C                                                                       EIGVW041
C     IF RESTRT = .TRUE. DATA DECK IS AS FOLLOWS                        EIGVW042
C     (ALL BUT LAST THREE CARDS (EOR,CONT,NEV) ARE PUNCHED BY           EIGVW043
C     PREVIOUS RUN)                                                     EIGVW044
C     SW,U,WL (FORMAT 17)  SEE DESCRIPTION ABOVE.                       EIGVW045
C     B,B1,NPOL,(CI(I),AI(I),I=1,NPOL) (FORMAT 6)                       EIGVW046
C     NP,LO,HI (FORMAT 7)                                               EIGVW047
C     J,E(J),(X(K),EFP(K,J),EFQ(K,J),K=1,NP) (FORMAT 13)                EIGVW048
C     END OF RECORD INDICATES END OF EIGENVALUE/EIGENFUNCTION DATA.     EIGVW049
C     CONT (L1)  LOGICAL VARIABLE, IF TRUE IT IS ASSUMED THAT ITERATION EIGVW050
C     OF THE LAST EIGENSOLUTION IS TO CONTINUE, IF FALSE IT IS ASSUMED  EIGVW051
```

```
C     THAT THE LAST EIGENSOLUTION IS ACCURATE AND ITERATION OF THE
C     NEXT SOLUTICN IS TO BEGIN.
C     NEV (I1) TOTAL NUMBER OF EIGENVALUES/EIGENFUNCTIONS SOUGHT (.LE.5)
C
      DIMENSICN E(5),ER1(5),ER2(5)
      COMMON /ARRAYS/ X(45),P(45),Q(45),DUM(45),NP,NSKIP,NPOLY,USEP,
     1EFP(45,5),EFQ(45,5),C(4),D(4)
      COMMON /CCNST/ CI(5),AI(5),NPOL,B,B1
      CCMMON /EVAL/ EVP,EVQ
      COMPLEX CI,AI  $  REAL LO  $  LOGICAL RESTRT,USEP,CONT
      DATA ERI/1.E-3/
      DATA ER1,ER2,NP/1.E-4,1.E-4,5.E-4,1.E-3,1.E-2,1.E-5,1.E-5,1.E-4,
     15.E-4,1.E-3,41/
      DATA (X(I),I=1,41)/0.,.01,.02,.03,.045,.06,.08,.1071,.1341,
     1.1612,.1882,.2153,.2424,.2694,.2965,.3235,.3506,.3776,.4047,
     2.4318,.4588,.4859,.5129,.5400,.5671,.5941,.6212,.6482,.6753,
     3.7024,.7294,.7565,.7835,.8106,.8376,.8647,.8918,.9188,.9459,
     4.9729,1./
C
    1 FORMAT (I1)
    2 FORMAT (4E20.10)
    3 FORMAT (3F10.5)
    4 FORMAT (I10)
    5 FORMAT (L1)
    6 FORMAT (3X,E20.12,5X,E20.12,11X,I2,/,(4E20.12))
    7 FORMAT (4X,I3,27X,2E20.11)
    8 FORMAT (8F10.5)
   11 FORMAT (*E =*,E20.12,1X,*B1 =*,E20.12,5X,*NPOL =*,I2,/,(4E20.12))
   12 FORMAT (*NP =*,I3,5X,*RANGE OF INTEGRATION =*,2E20.11)
   13 FORMAT (43X,I2,E20.11,/,(3E25.14))
   14 FORMAT (*SW =*,E15.7,1X,*MTAS =*,E15.7,1X,*WL =*,E15.7)
   17 FORMAT (4X,E15.7,7X,E15.7,5X,E15.7)
C
      READ (5,5) RESTRT
      IF(.NOT.RESTRT) GO TO 20
C
C RESTART IS TRUE, READ DATA FROM PREVIOUS RUN
      READ (5,17) SW,U,WL
      READ (5,6) B,B1,NPOL,(CI(I),AI(I),I=1,NPOL)
      READ (5,7) NP,LO,HI
      DO 15 I1=1,6
      READ (5,13) J,E(J),(X(K),EFP(K,J),EFQ(K,J),K=1,NP)
      IF (EOF,5) 16,15
   15 CONTINUE
      GO TO 800
   16 READ (5,5) CONT
      READ (5,1) NEV
      I=I1-1
      DO 18 K=1,NP
      P(K)=EFP(K,I)
      IF (.NOT.CONT) GO TO 18
      Q(K)=EFQ(K,I)
   18 CONTINUE
      IF (.NOT.CONT) GO TO 95
      I1=I1-1  $  I=I1-1
      GO TO 95
C
C RESTART IS FALSE, READ DATA DEFINING NEW PROBLEM.
   20 READ (5,1) NPOL
      IF (NPOL) 900,900,21
   21 READ (5,2) (CI(I),AI(I),I=1,NPOL)
C CHANGE SIGN OF EXPONENT SINCE ALL SUBROUTINES EXPECT NEGATIVE OF
C TRUE EXFONENT.
```

```
      DO 22 I=1,NPOL
   22 AI(I)=-AI(I)
      READ (5,3) SW,U,WL
      B=U/2./WL  $  HI=20./B  $  B1=SW*U/WL*SQRT(2.)
      IF (HI.LE.1.) GO TO 26
C
C SCALE X VALUES TO COVER RANGE OF INTEGRATION, USE EI,EVP,EVQ FOR
C TEMPORARY STORAGE.
C EI IS LOWER LIMIT OF POWER LAW SCALING.
C NPWR IS POWER OF SCALING, HIGHER VALUES WILL PLACE MORE X VALUES NEAR
C ORIGIN.  LOW VALUES WILL PROVIDE MORE EVEN DISTRIBUTION OF X VALUES
C OVER RANGE OF INTEGRATION.
      EI=.05  $  NPWR=1
      EVQ=((HI-EI)/(1.-EI)-1.)/(1.-EI)**NPWR
      DO 25 I=1,NF
      EVP=X(I)
      IF (EVP.LT.EI) GO TO 25
      EVP=EVP-EI  $  EVP=EVP*(1.+EVQ*EVP**NPWR)+EI
   25 X(I)=EVP
C
C PUNCH PARAMETERS ON CARDS
   26 WRITE (7,14) SW,U,WL
      WRITE (7,11) B,B1,NPOL,(CI(I),AI(I),I=1,NPOL)
      WRITE (7,12) NP,X(1),X(NP)
      I=0  $  I1=1
   90 READ (5,1) NEV
      CONT=.F.
   95 IF (NEV-I1) 800,100,100
C
C COMPUTE EXACT SYSTEM RESPONSE VARIANCE
  100 CALL VARW(SW,WL,U,SR2)
      CALL STARAY(LO,LO)
      IF (.NOT.CONT) GO TO 110
      EROR=ER2(I1)  $  NSKIP=1  $  USEP=.T.  $  GO TO 111
C
C GENERATE INITIAL GUESS OF EIGENFUNCTION P(ALPHA)
  110 EROR=ER1(I1)  $  NSKIP=2  $  USEP=.T.
      CALL GUESS(I1)
C
  111 WRITE (6,101) I1,SW,U,WL,(CI(J),AI(J),J=1,NPOL)
  101 FORMAT (1H1,5X,*BEGIN ITERATION FOR EIGENVALUE AND EIGENFUNCTION N
     10.*,I2,/,/,6X,*TURBULENCE PARAMETERS:*,/,6X,*STD. DEV. =*,F6.3,5X,
     2*MTAS =*,F8.3,5X,*SCALE LENGTH =*,F9.3,/,/,6X,*VEHICLE PARAMETERS:
     3*,/,12X,*COEFFICIENTS*,26X,*POLES*,/,(4X,2E13.4,8X,2E13.4))
      WRITE (6,102) SR2
  102 FORMAT (1H0,5X,*SYSTEM RESPONSE VARIANCE =*,E13.6,/,1H0)
      J=1  $  EVP=EVQ=EI=DEVQ=1.E50  $  GO TO 210
C
  200 CALL FILL
      EROR=ER2(I1)
C
C IF (I.GT.0) EIGENFUNCTION MUST BE ORTHOGONAL TO ALL PREVIOUS FNS
  210 IF (I.LE.0) GO TO 300
      CALL ORTHOG(I)
C
C COMPUTE NORMALIZATION FACTOR
  300 CALL NORMF(F)
C
C ESTIMATE EIGENVALUE
  400 EI=EI*F
      IF(USEP) 401,410
  401 DEVP=ABS(1./EVP/F-1.)
      EVP=1./F
```

```
      WRITE (6,405) J,EVP,DEVP,EI
  405 FORMAT (1H ,5X,*ITERATION NO.*,I3,*.  ESTIMATED P EIGENVALUE IS*,
     1E15.7,*.  (DEVP =*,E10.3,2X,*EI =*,E10.3,*)*)
      GO TO 500
  410 DEVQ= ABS(1./EVQ/F-1.)
      EVQ=1./F
      WRITE (6,415) J,EVQ,DEVQ,EI
  415 FORMAT (1H ,5X,*ITERATION NO.*,I3,*.  ESTIMATED Q EIGENVALUE IS*,
     1E15.7,*.  (DEVQ =*,E10.3,2X,*EI =*,E10.3,*)*)
C
C NORMALIZE EIGENFUNCTION
  500 CALL NORM(F)
      IF ((J.LT.30).OR.((J/5)*5.NE.J)) GO TO 520
      WRITE (6,510) (X(JJ),P(JJ),Q(JJ),JJ=1,NP)
  510 FORMAT (1H ,3E25.14)
      WRITE (7,511) I1
  511 FORMAT (43X,I2)
      WRITE (7,707) (X(JJ),P(JJ),Q(JJ),JJ=1,NP)
C
C IF DEV IS SUFFICIENTLY SMALL, ACCEPT EIGENFUNCTION
  520 IF ((DEVP.LE.EROR).AND.(DEVQ.LE.EROR).AND.(EI.LE.ERI)) GO TO 700
      IF ((J.GE.25).AND.(NSKIP.NE.1)) GO TO 200
C
C EIGENFUNCTION NOT YET FOUND, CONTINUE ITERATION PROCESS
  600 CALL ITERAT(EI)
      J=J+1
      GO TO 210
C
C ERROR IS BELOW TOLERANCE, CHECK TO SEE THAT MAX. NO. OF POINTS USED
  700 IF (NSKIP.NE.1) GO TO 200
C ACCEPT EIGENFUNCTIONS
      I=I+1  $  I1=I+1
      DEVP=ABS(EVP-EVQ)  $  DEVQ=(EVP+EVQ)/2.  $  DEVP=DEVP/DEVQ
      E(I)=DEVQ  $  FS=DEVQ**2/SR2  $  TFS=0.
      DO 702 J=1,I
  702 TFS=TFS+E(J)**2  $  TFS=TFS/SR2
      WRITE (7,706) I,DEVQ
  706 FORMAT (*EIGENVALUE AND CONJUGATE EIGENFUNCTIONS NO.*,I2,E20.11)
      WRITE (7,707) (X(J),P(J),Q(J),J=1,NP)
  707 FORMAT (3E25.14)
      WRITE (6,708) I,DEVQ,EVP,EVQ,DEVP,FS,I,TFS,(X(J),P(J),Q(J),J=1,NP)
  708 FORMAT (1H1,5X,*EIGENVALUE NO.*,I2,* IS*,E14.7,/,6X,*EVP =*,E14.7,
     1*  EVQ =*,E14.7,*  RELATIVE DIFFERENCE =*,E10.3,/,/,6X,*FRACTION O
     2F RESPONSE VARIANCE DUE TO THIS EIGENVALUE IS*,F7.5,/,6X,
     3*FRACTION OF RESPONSE VARIANCE DUE TO FIRST*,I2,* EIGENVALUES IS*,
     4F7.5,/,/,9X,*X*,8X,*FUNCTION OF 1ST ARG*,3X,*FUNCTION OF 2ND ARG*,
     5/,(6X,F8.4,5X,E17.10,5X,E17.10))
      IF (I.GE.NEV) GO TO 800
      CALL STOR(I)
      CONT=.F.
      GO TO 110
C
  800 WRITE (6,805) NEV,(J,E(J),J=1,I)
  805 FORMAT (1H1,5X,*FIRST*,I2,* EIGENVALUES ARE$*,/,(10X,I2,2X,E15.7))
      FS=SR2*(1.-TFS)
      WRITE (6,810) FS,TFS
  810 FORMAT (1H0,5X,*REMAINING EIGENVALUES SQUARED SUM TO*,E15.7,/,6X,
     1*THE ABOVE EIGENVALUES ACCOUNT FOR*,F7.5,* OF THE RESPONSE VAR.*)
      IF (FS.LT.0.) GO TO 900
      FS=SQRT(FS)
      WRITE (6,812) FS
  812 FORMAT (1H ,5X,*LARGEST POSSIBLE REMAINING EIGENVALUE IS*,E14.7)
  900 STOP  $  END
```

```
      SUBROUTINE VARW(SW,WL,U,S2)
C
C A SUBROUTINE TO COMPUTE THE RESPONSE VARIANCE OF A LINEAR SYSTEM
C SUBJECTED TO RANDOM INPUT WITH DRYDEN W-GUST SPECTRUM
C
C SW = TURBULENCE STANDARD DEVIATION
C WL = TURBULENCE SCALE LENGTH
C U = VEHICLE MEAN TRUE AIRSPEED
C AI = NEGATIVE OF VEHICLE IMPULSE RESPONSE EXPONENTS (POSITIVE REAL PT)
C CI = VEHICLE IMPULSE RESPONSE CONSTANTS
C  IF CI OR AI IS COMPLEX, ONLY ONE OF EACH CONJUGATE PAIR IS TO BE USED
C
      COMMON /CONST/ CI(5),AI(5),NPOL
      COMPLEX CI,AI,AJ,AK,CJ,CK,C2,C3,C4,D1,D2
      LOGICAL RL1,RL2
      C=U/WL  $  S2=0.
      DO 20 K=1,NPOL
      AK=AI(K)  $  CK=CI(K)  $  C4=0.
      RL1=AIMAG(AK).EQ.0..AND.AIMAG(CK).EQ.0.
    5 D1=C+AK
      DO 10 J=1,NPOL
      AJ=AI(J)  $  CJ=CI(J)  $  C3=0.
      RL2=AIMAG(AJ).EQ.0..AND.AIMAG(CJ).EQ.0.
    6 D2=C-AJ
      C2=(C*(C+(AK-AJ)/2.)/D1/D2-1.)/D1/D2
      C2=C2+C*(C**2-3.*AJ**2)/(AK+AJ)/(AJ+C)**2/(AJ-C)**2
      C2=C2*CJ*CK  $  C3=C3+C2  $  IF (RL2) GO TO 10
      AJ=CONJG(AJ)  $  CJ=CONJG(CJ)  $  RL2=.T.  $  GO TO 6
   10 C4=C4+C3
      IF (RL1) GO TO 20
      AK=CONJG(AK)  $  CK=CONJG(CK)  $  RL1=.T.  $  GO TO 5
   20 S2=S2+REAL(C4)  $  S2=S2*SW**2  $  RETURN  $  END
      SUBROUTINE FILL
C
C THIS SUBROUTINE INTERPOLATES THE 21 POINT EIGENFUNCTION TO FORM THE
C FIRST APPROXIMATION TO THE 41 POINT EIGENFUNCTION.
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),I,I1,K,XP,C(41),NP,NSKIP
      I1=NSKIP/2+1
      DO 10 I=I1,NP,NSKIP
      XP=X(I)  $  K=I+1-I1
      CALL COEF(K,P,C)
      P(I)=((C(4)*XP+C(3))*XP+C(2))*XP+C(1)
      CALL COEF(K,Q,C)
   10 Q(I)=((C(4)*XP+C(3))*XP+C(2))*XP+C(1)
      NSKIP=NSKIP/2
      RETURN  $  END
      SUBROUTINE GUESS(I)
C
C THIS SUBROUTINE GUESSES THE NEXT EIGENFUNCTION TO BE THE
C DERIVATIVE OF THE PROVIOUS RESULT OR X*EXP(-3.*X) FOR THE FIRST.
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),J,XJ,K,DX,DP,DUM(40),NP,NSKIP
      IF (NSKIP.LE.1) GO TO 70
      IF (I.GT.1) GO TO 50
      DO 10 J=1,NP,NSKIP
      Q(J)=0.  $  XJ=X(J)
   10 P(J)=XJ*EXP(-3.*XJ)
      RETURN
C
   50 K=NSKIP+1
      DO 60 J=K,NP,NSKIP
      Q(J)=0.  $  DP=P(J)-P(J-1)  $  DX=X(J)-X(J-1)  $  XJ=X(J)
```

```
      XJ=XJ*EXP(-3.*XJ)
   60 P(J)=XJ*DP/DX
      RETURN
   70 WRITE (6,71) NSKIP
   71 FORMAT (1H ,5X,*NSKIP =*,I10,*NOT ALLOWED IN GUESS*)
      STOP  $  END
      SUBROUTINE NORMF(F)
C
C COMPUTES NORM OF P(X) OR Q(X) DEPENDING UPON USEP TRUE OR FALSE
C THE NORM OF P OR Q IS THE ESTIMATED EV., THE INVERSE IS THE NORM. FAC.
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),C(4),F1,NP1,J,I,X1,X2,II,CII,
     1IJ,K,CK,X12,X22,JJ,DUM(27),NP,NSKIP,NPOLY,USEP
      LOGICAL USEP
      F1=0.  $  NP1=NPOLY+1  $  J=NP-1
      DO 60 I=1,J,NSKIP
      F = 0.
      IF(USEP) 10,11
   10 CALL COEF(I,P,C)
      GO TO 12
   11 CALL COEF(I,Q,C)
   12 X1=X(I)  $  X2=X(I+NSKIP)
      IF (X2.GT.X(NP)) X2=X(NP)
      DO 40 II=1,NPOLY
      CII=C(II)  $  IJ=II+1  $  K=II+IJ-1  $  CK=FLOAT(K)
      X12=X1**K  $  X22=X2**K
      DO 40 JJ=IJ,NP1
      F = F+CII*C(JJ)*(X22-X12)/CK
      CK=CK+1.  $  X12=X12*X1
   40 X22=X22*X2
      F=F*2.  $  X12=X1**2  $  X22=X2**2  $  CK=1.
      DO 50 II=1,NP1
      F=F+(X2-X1)*(C(II)**2)/CK  $  CK=CK+2.  $  X2=X2*X22
   50 X1=X1*X12
   60 F1=F1+F
      F=SQRT(1./F1)
      RETURN
C
C NORMALIZE EIGENFUNCTION P OR Q DEPENDING UPON USEP TRUE OR FALSE.
      ENTRY NORM
      DO 100 I=1,NP,NSKIP
      IF (USEP) 95,96
   95 P(I)=P(I)*F
      GO TO 100
   96 Q(I)=Q(I)*F
  100 CONTINUE
      RETURN  $  END
      SUBROUTINE ORTHOG(I)
C
C A SUBROUTINE TO ORTHOGONALIZE P(X) OR Q(X) WITH RESPECT TO THE FIRST I
C EIGENFUNCTIONS DEPENDING UPON USEP TRUE OR FALSE.
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),DUM(45),NP,NSKIP,NPOLY,USEP,
     1EFP(45,5),EFQ(45,5),C(4),D(4)
      LOGICAL USEP
      NP1 = NPOLY+1
      DO 50 J=1,I
      DO 10 JJ=1,NP,NSKIP
      IF (USEP) 5,6
    5 DUM(JJ)=EFP(JJ,J)
      GO TO 10
    6 DUM(JJ)=EFQ(JJ,J)
   10 CONTINUE
```

```
      K=NP-1  $  F=0.
      DO 40 JJ=1,K,NSKIP
      IF (USEP) 20,21
   20 CALL COEF(JJ,P,C)
      GO TO 22
   21 CALL COEF(JJ,Q,C)
   22 CALL COEF(JJ,DUM,D)
      X1=X(JJ)  $  II=JJ+NSKIP  $  IF (II.GT.NP) II=NP  $  X2=X(II)
      DO 40 II=1,NP1
      X12=X1**II  $  X22=X2**II  $  CII=C(II)  $  CL=FLOAT(II)
      DO 40 KK=1,NP1
      F=F+CII*D(KK)*(X22-X12)/CL  $  CL=CL+1.  $  X12=X12*X1
   40 X22 = X22*X2
      DO 50 JJ=1,NP,NSKIP
      IF (USEP) 45,46
   45 P(JJ)=P(JJ)-F*DUM(JJ)
      GO TO 50
   46 Q(JJ)=Q(JJ)-F*DUM(JJ)
   50 CONTINUE
      RETURN  $  END
      FUNCTION SUM(I,J)
C
C FORMS PRODUCT  (J-I+1)*(J-I+2)......(J)
C
      N=1
      IF(I) 20,20,10
   10 K=J-I+1
      DO 15 L=K,J
   15 N=N*L
   20 SUM=FLOAT(N)
      RETURN  $  END
      SUBROUTINE COEF(I,A,C)
C A SUBROUTINE TO FILL C ARRAY WITH THE COEFFICIENTS OF THE
C POLYNOMIAL REPRESENTING A(X) BETWEEN X(I) AND X(I+NSKIP).  NPOLY IS
C THE ORDER OF THE POLYNOMIAL
C POLYNOMIAL IS C(1) + C(2)*X + C(3)*X**2 + C(4)*X**3
C
      DIMENSION A(1),C(1)
      COMMON /ARRAYS/ X(45),P(45),Q(45),DUM(45),NP,NSKIP,NPOLY
      DATA NPOLY/3/
      IF (I.GT.NSKIP) GO TO 10
      N=1  $  GO TO 30
   10 IF (I.LE.NP-2*NSKIP) GO TO 20
      N=NP-3*NSKIP  $  GO TO 30
   20 N=I-NSKIP
   30 F1=A(N)  $  X1=X(N)  $  N=N+NSKIP  $  F2=A(N)  $  X2=X(N)
      N=N+NSKIP  $  F3=A(N)  $  X3=X(N)
      N=N+NSKIP  $  F4=A(N)  $  X4=X(N)
      F1=F1/(X1-X2)/(X1-X3)/(X1-X4)
      F2=F2/(X2-X1)/(X2-X3)/(X2-X4)
      F3=F3/(X3-X1)/(X3-X2)/(X3-X4)
      F4=F4/(X4-X1)/(X4-X2)/(X4-X3)
      C(4)=F1+F2+F3+F4
      C(3)=-F1*(X2+X3+X4)-F2*(X1+X3+X4)-F3*(X1+X2+X4)-F4*(X1+X2+X3)
      C(2)=F1*(X2*(X3+X4)+X3*X4)+F2*(X1*(X3+X4)+X3*X4)+F3*(X1*(X2+X4)+
     1X2*X4)+F4*(X1*(X2+X3)+X2*X3)
      C(1)=-F1*X2*X3*X4-F2*X1*X3*X4-F3*X1*X2*X4-F4*X1*X2*X3
      RETURN  $  END
      SUBROUTINE DIFC
C INTEGRATE COMPLEX TERMS OF TRANSFER FUNCTION
C
      EQUIVALENCE (E,ER),(XE,XER),(XE1,XE1R),(D1,D1R)
      COMMON /DIF/ X,E,N,D1
```

```
      COMPLEX E,XE,XE1,D1
      N1=N+1  $  D1=0.
      IF (N) 70,10,20
C
C CONSTANT TERM OF POLYNOMIAL
   10 D1=1.  $  GC TO 30
C
C POWER TERM CF PCLYNOMIAL
   20 IF (X.NE.0.) GO TO 25
      D1=(-1./E)**N*SUM(N,N)
      GO TO 30
   25 XN=X**N
      XE1=X*E  $  XE=1.  $  S1=1.  $  S2=FLOAT(N)
      DO 28 I=1,N1
      D1=XN/XE*S1+D1  $  XE=XE*XE1  $  S1=S1*S2  $  S2=S2-1.
   28 XN=-XN
   30 D1=D1*CEXP(E*X)/E
   70 RETURN
C
C INTEGRATE REAL TERMS OF TRANSFER FUNCTION
      ENTRY DIFR
      N1=N+1  $  D1=0.
      IF (N) 170,110,120
C
C CONSTANT TERM OF POLYNOMIAL
  110 D1R=1.  $  GO TO 130
C
C POWER TERM CF PCLYNOMIAL
  120 IF (X.NE.C.) GO TO 125
      D1R=(-1./ER)**N*SUM(N,N)
      GO TO 130
  125 XN=X**N
      XE1R=X*ER  $  XER=1.  $  S1=1.  $  S2=FLOAT(N)
      DO 128 I=1,N1
      D1R=XN/XER*S1+D1R  $  XER=XER*XE1R  $  S1=S1*S2  $  S2=S2-1.
  128 XN=-XN
  130 D1=CMPLX(D1R*EXP(ER*X)/ER,0.0)
  170 RETURN  $  END
      SUBROUTINE NTGRAL(V,XAB)
C
C INTEGRATES P(X)*H(X,XAB) OR Q(X)*H(XAB,X) DEPENDING UPON USEP TRUE
C OR FALSE.
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),E1,E2,C1,C2,C02,C03,C04,C06,IPOL
     1,NX,NPO,NPO1,IA,ICA,D1S(4),D2S(4),D1S1(4),D1S2(4),NP,NSKIP,NPOLY,
     2USEP,EPP(450),C(4)
      COMMON /CCNST/ CI(5),AI(5),NPOL,B,B1
      COMMON /DIF/ X1,E,NPOM,D1C
      DIMENSICN D1A(225),D1CA(900),CA(180)
      COMPLEX D1CA,AI,CI,E,C1,C2,V1,V2,E2,D1C,D1S1,D1S2,C01,C02,C03,
     1C04,C05,CC6,D1C1,D1C2
      LOGICAL RL,USEP
      EQUIVALENCE (E,ER),(D1,D2,D1C),(C01,C02),(C04,C05),(D1C1,D1C2)
      V=0.  $  E1=EXP(-B*XAB)  $  ICA=0
C
C CYCLE THROUGH PCLES OF TRANSFER FUNCTION
      DO 100 IPCL=1,NPOL
      E2=CEXP(-XAB*(AI(IPOL)-B))  $  C1=B1*CI(IPOL)/(AI(IPOL)-2.*B)
      C2=C1*B  $  C1=C1*(AI(IPOL)-B)/(AI(IPOL)-2.*B)  $  C03=C2*E1
      IF (USEP) 5,6
    5 C01=(C1-XAB*C2)*E1  $  C05=C1*E2  $  GO TO 7
    6 C02=(C1+XAB*C2)*E2  $  C04=C2*(E2-E1)  $  C06=C1*E1
    7 V1=0.  $  V2=0.  $  IA=1  $  ICA=ICA+1
```

```
C
C CYCLE THROUGH X INTERVALS
      DO 15 NPO=1,NPO1
      D1S(NPO)=D1A(IA)  $  IA=IA+1  $  D1S1(NPO)=D1CA(ICA)  $  ICA=ICA+1
      IF (USEP) 12,13
   12 D1S2(NPO)=D1CA(ICA)
      GO TO 15
   13 D2S(NPO)=D1A(IA)
   15 CONTINUE
      I=NSKIP+1  $  NXI=1
      DO 80 NX=I,NP,NSKIP
      X1=X(NX)  $  IA=IA+IINC  $  ICA=ICA+IINC
C
C CYCLE THROUGH TERMS OF POLYNOMIAL
      IF (USEP) 16,25
C
C COMPUTE NEW Q(BETA) EIGENFUNCTION
   16 DO 20 NPO=1,NPO1
      D1=D1A(IA)  $  IA=IA+1  $  E=D1-D1S(NPO)  $  V1=C01*E
      D1S(NPO)=D1
      IF (X1-XAB) 17,17,18
   17 D1C1=D1CA(ICA)
      ICA=ICA+1  $  E=D1C1-D1S1(NPO)  $  V1=V1-C01*E  $  D1S1(NPO)=D1C1
      D1C2=D1CA(ICA)  $  E=D1C2-D1S2(NPO)  $  V1=V1-C03*E
      D1S2(NPC)=D1C2
      GO TO 19
   18 V1=V1-C05*E
      ICA=ICA+1
   19 V2=V2+V1*CA(NXI)
   20 NXI=NXI+1
      GO TO 80
C
C COMPUTE NEW P(ALPHA) EIGENFUNCTION
   25 DO 40 NPO=1,NPO1
      D1=D1A(IA)  $  IA=IA+1  $  E=D1-D1S(NPO)  $  V1=C06*E
   30 D1S(NPO)=D1
      IF (X1-XAB) 31,31,38
   31 D1C1=D1CA(ICA)
      ICA=ICA+1  $  E=D1C1-D1S1(NPO)  $  V1=V1-C06*E
   35 D1S1(NPO)=D1C1
      D2=D1A(IA)  $  E=D2-D2S(NPO)  $  V1=V1-C03*E  $  D2S(NPO)=D2
      GO TO 39
   38 V1=V1-C02*E
      D2=D1A(IA)  $  E=D2-D2S(NPO)  $  V1=V1+C04*E  $  D2S(NPO)=D2
      ICA=ICA+1
   39 V2=V2+V1*CA(NXI)
   40 NXI=NXI+1
   80 CONTINUE
      IF ((AIMAG(C1).EQ.0.).AND.(AIMAG(E2).EQ.0.)) GO TO 100
      V2=2.*REAL(V2)
  100 V=V+V2
      GO TO 500
C
C
C SET UP D1A AND D1CA ARRAYS
      ENTRY STARAY
      NPO1=NPOLY+2  $  ER=-8  $  I=1
      DO 200 NX=1,NP
      X1=X(NX)
      DO 200 NPO=1,NPO1
      NPOM=NPO-1  $  CALL DIFR  $  D1A(I)=D1
  200 I=I+1  $  I=1
      DO 300 IPOL=1,NPOL
```

```
      E=B-AI(IPCL)  $  RL=AIMAG(E).EQ.0.
      DO 300 NX=1,NP
      X1=X(NX)
      DO 300 NPO=1,NPO1
      NFOM=NPO-1  $  IF (RL) GO TO 250
      CALL DIFC
      GO TO 290
  250 CALL DIFR
  290 D1CA(I)=D1C
  300 I=I+1
      GO TO 500
C
C GENERATE TABLE CF POLYNOMIAL COEFFICIENTS, CA, FOR EIGENFUNCTION P OR
C Q DEPENDING UFON USEP TRUE OR FALSE
      ENTRY GENCOE
      NPO=0
      DO 400 NXI=1,NP,NSKIP
      IF (NXI-NP) 350,400,400
  350 IF (USEP) 360,370
  360 CALL COEF(NXI,P,C)
      GO TO 375
  370 CALL COEF(NXI,Q,C)
  375 DO 380 NPO1=1,4
      NPO=NPO+1
  380 CA(NPO)=C(NPO1)
  400 CONTINUE
      NPO1=NPOLY+1
      IINC=(NSKIP-1)*(NPOLY+2)+1
  500 RETURN  $  END
      SUBROUTINE ITERAT(E)
C
C A SUBROUTINE TO ITERATE THE EIFENFUNCTION P(X) OR Q(X) DEPENDING
C UPON USEP TRUE CR FALSE.
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),DUM(45),NP,NSKIP,NPOLY,USEP
      COMMON /EVAL/ EVP,EVQ
      EQUIVALENCE (PI,QI)
      LOGICAL USEP
      CALL GENCOE(Q,X)
      Q(1)=0.  $  P(1)=0.  $  E=0.  $  I1=NSKIP+1
      DO 50 I=I1,NP,NSKIP
      IF (USEP) 30,40
   30 QI=Q(I)*EVQ
      CALL NTGRAL(Q(I),X(I))
      E=E+(ABS(QI)-ABS(Q(I)))**2
      GO TO 50
   40 PI=P(I)*EVP
      CALL NTGRAL(P(I),X(I))
      E=E+(ABS(PI)-ABS(P(I)))**2
   50 CONTINUE
      E=SCRT(E)/FLOAT(NP)
      USEP=.NOT.USEP
      RETURN  $  END
      SUBROUTINE STOR(I)
C
C STORES ITH EIGENFUNCTION
C P(X) IN EFP(X,I), Q(X) IN EFQ(X,I)
C
      COMMON /ARRAYS/ X(45),P(45),Q(45),J,DUM(44),NP,NSKIP,NPOLY,USEP,
     1EFP(45,5),EFQ(45,5)
      DO 10 J=1,NP
      EFP(J,I)=P(J)
   10 EFQ(J,I)=Q(J)
```

```
      RETURN  $  END                                                     EIGVH619


      SUBROUTINE FFT (A,M,INV,S,IFSET,IFERR)                             FFT00001
C                                                                        FFT00002
C FAST FOURIER TRANSFORM FOR COMPLEX FUNCTIONS OF UP TO THREE DIMENSIONSFFT00003
C WRITTEN BY DUANE HARDER, FEB. 1969                                     FFT00004
C AVAILABLE THROUGH VIM INC. (CDC USER ORGANIZATION)                     FFT00005
C                                                                        FFT00006
      DIMENSION A(1),INV(1),S(1),N(3),M(3),NP(3),W(2),W2(2),W3(2)        FFT00007
      EQUIVALENCE (N1,N(1)), (N2,N(2)), (N3,N(3))                        FFT00008
10    IF (IABS(IFSET)-1) 610,610,20                                      FFT00009
20    MTT=MAX0(M(1),M(2),M(3))-2                                         FFT00010
      ROOT2=SQRT(2.)                                                     FFT00011
      IF (MTT-MT) 40,40,30                                               FFT00012
30    IFERR=1                                                            FFT00013
      RETURN                                                             FFT00014
40    IFERR=0                                                            FFT00015
      M1=M(1)                                                            FFT00016
      M2=M(2)                                                            FFT00017
      M3=M(3)                                                            FFT00018
      N1=2**M1                                                           FFT00019
      N2=2**M2                                                           FFT00020
      N3=2**M3                                                           FFT00021
      IF (IFSET) 50,50,70                                                FFT00022
50    NX=N1*N2*N3                                                        FFT00023
      FN=NX                                                              FFT00024
      DO 60 I=1,NX                                                       FFT00025
      A(2*I-1)=A(2*I-1)/FN                                               FFT00026
60    A(2*I)=-A(2*I)/FN                                                  FFT00027
70    NP(1)=N1*2                                                         FFT00028
      NP(2)=NP(1)*N2                                                     FFT00029
      NP(3)=NP(2)*N3                                                     FFT00030
      DO 330 ID=1,3                                                      FFT00031
      IL=NP(3)-NP(ID)                                                    FFT00032
      IL1=IL+1                                                           FFT00033
      MI=M(ID)                                                           FFT00034
      IF (MI) 330,330,80                                                 FFT00035
80    IDIF=NP(ID)                                                        FFT00036
      KBIT=NP(ID)                                                        FFT00037
      MEV=2*(MI/2)                                                       FFT00038
      IF (MI-MEV) 120,120,90                                             FFT00039
90    KBIT=KBIT/2                                                        FFT00040
      KL=KBIT-2                                                          FFT00041
      DO 100 I=1,IL1,IDIF                                                FFT00042
      KLAST=KL+I                                                         FFT00043
      DO 100 K=I,KLAST,2                                                 FFT00044
      KD=K+KBIT                                                          FFT00045
      T=A(KD)                                                            FFT00046
      A(KD)=A(K)-T                                                       FFT00047
      A(K)=A(K)+T                                                        FFT00048
      T=A(KD+1)                                                          FFT00049
      A(KD+1)=A(K+1)-T                                                   FFT00050
100   A(K+1)=A(K+1)+T                                                    FFT00051
      IF (MI-1) 330,330,110                                              FFT00052
110   LFIRST=3                                                           FFT00053
      JLAST=1                                                            FFT00054
      GO TO 130                                                          FFT00055
120   LFIRST=2                                                           FFT00056
      JLAST=0                                                            FFT00057
130   DO 320 L=LFIRST,MI,2                                               FFT00058
```

```
      JJDIF=KBIT
      KBIT=KBIT/4
      KL=KBIT-2
      DO 140 I=1,IL1,IDIF
      KLAST=I+KL
      DO 140 K=I,KLAST,2
      K1=K+KBIT
      K2=K1+KBIT
      K3=K2+KBIT
      T=A(K2)
      A(K2)=A(K)-T
      A(K)=A(K)+T
      T=A(K2+1)
      A(K2+1)=A(K+1)-T
      A(K+1)=A(K+1)+T
      T=A(K3)
      A(K3)=A(K1)-T
      A(K1)=A(K1)+T
      T=A(K3+1)
      A(K3+1)=A(K1+1)-T
      A(K1+1)=A(K1+1)+T
      T=A(K1)
      A(K1)=A(K)-T
      A(K)=A(K)+T
      T=A(K1+1)
      A(K1+1)=A(K+1)-T
      A(K+1)=A(K+1)+T
      R=-A(K3+1)
      T=A(K3)
      A(K3)=A(K2)-R
      A(K2)=A(K2)+R
      A(K3+1)=A(K2+1)-T
  140 A(K2+1)=A(K2+1)+T
      IF (JLAST) 310,310,150
  150 JJ=JJDIF+1
      ILAST=IL+JJ
      DO 160 I=JJ,ILAST,IDIF
      KLAST=KL+I
      DO 160 K=I,KLAST,2
      K1=K+KBIT
      K2=K1+KBIT
      K3=K2+KBIT
      R=-A(K2+1)
      T=A(K2)
      A(K2)=A(K)-R
      A(K)=A(K)+R
      A(K2+1)=A(K+1)-T
      A(K+1)=A(K+1)+T
      AWR=A(K1)-A(K1+1)
      AWI=A(K1+1)+A(K1)
      R=-A(K3)-A(K3+1)
      T=A(K3)-A(K3+1)
      A(K3)=(AWR-R)/ROOT2
      A(K3+1)=(AWI-T)/ROOT2
      A(K1)=(AWR+R)/ROOT2
      A(K1+1)=(AWI+T)/ROOT2
      T=A(K1)
      A(K1)=A(K)-T
      A(K)=A(K)+T
      T=A(K1+1)
      A(K1+1)=A(K+1)-T
      A(K+1)=A(K+1)+T
      R=-A(K3+1)
```

```
      T=A(K3)
      A(K3)=A(K2)-R
      A(K2)=A(K2)+R
      A(K3+1)=A(K2+1)-T
160   A(K2+1)=A(K2+1)+T
      IF (JLAST-1) 310,310,170
170   JJ=JJ+JJDIF
      DO 300 J=2,JLAST
      I=INV(J+1)
      IC=NT-I
      W(1)=S(IC)
      W(2)=S(I)
      I2=2*I
      I2C=NT-I2
      IF (I2C) 200,190,180
180   W2(1)=S(I2C)
      W2(2)=S(I2)
      GO TO 210
190   W2(1)=0.
      W2(2)=1.
      GO TO 210
200   I2CC=I2C+NT
      I2C=-I2C
      W2(1)=-S(I2C)
      W2(2)=S(I2CC)
210   I3=I+I2
      I3C=NT-I3
      IF (I3C) 240,230,220
220   W3(1)=S(I3C)
      W3(2)=S(I3)
      GO TO 280
230   W3(1)=0.
      W3(2)=1.
      GO TO 280
240   I3CC=I3C+NT
      IF (I3CC) 270,260,250
250   I3C=-I3C
      W3(1)=-S(I3C)
      W3(2)=S(I3CC)
      GO TO 280
260   W3(1)=-1.
      W3(2)=0.
      GO TO 280
270   I3CCC=NT+I3CC
      I3CC=-I3CC
      W3(1)=-S(I3CCC)
      W3(2)=-S(I3CC)
280   ILAST=IL+JJ
      DO 290 I=JJ,ILAST,IDIF
      KLAST=KL+I
      DO 290 K=I,KLAST,2
      K1=K+KBIT
      K2=K1+KBIT
      K3=K2+KBIT
      R=A(K2)*W2(1)-A(K2+1)*W2(2)
      T=A(K2)*W2(2)+A(K2+1)*W2(1)
      A(K2)=A(K)-R
      A(K)=A(K)+R
      A(K2+1)=A(K+1)-T
      A(K+1)=A(K+1)+T
      R=A(K3)*W3(1)-A(K3+1)*W3(2)
      T=A(K3)*W3(2)+A(K3+1)*W3(1)
      AWR=A(K1)*W(1)-A(K1+1)*W(2)
```

```
      AWI=A(K1)*W(2)+A(K1+1)*W(1)
      A(K3)=AWR-R
      A(K3+1)=AWI-T
      A(K1)=AWR+R
      A(K1+1)=AWI+T
      T=A(K1)
      A(K1)=A(K)-T
      A(K)=A(K)+T
      T=A(K1+1)
      A(K1+1)=A(K+1)-T
      A(K+1)=A(K+1)+T
      R=-A(K3+1)
      T=A(K3)
      A(K3)=A(K2)-R
      A(K2)=A(K2)+R
      A(K3+1)=A(K2+1)-T
  290 A(K2+1)=A(K2+1)+T
  300 JJ=JJDIF+JJ
  310 JLAST=4*JLAST+3
  320 CONTINUE
  330 CONTINUE
      NTSQ=NT*NT
      M3MT=M3-MT
      IF (M3MT) 350,340,340
  340 IGO3=1
      N3VNT=N3/NT
      MINN3=NT
      GO TO 360
  350 IGO3=2
      N3VNT=1
      NTVN3=NT/N3
      MINN3=N3
  360 JJD3=NTSQ/N3
      M2MT=M2-MT
      IF (M2MT) 380,370,370
  370 IGO2=1
      N2VNT=N2/NT
      MINN2=NT
      GO TO 390
  380 IGO2=2
      N2VNT=1
      NTVN2=NT/N2
      MINN2=N2
  390 JJD2=NTSQ/N2
      M1MT=M1-MT
      IF (M1MT) 410,400,400
  400 IGO1=1
      N1VNT=N1/NT
      MINN1=NT
      GO TO 420
  410 IGO1=2
      N1VNT=1
      NTVN1=NT/N1
      MINN1=N1
  420 JJD1=NTSQ/N1
      JJ3=1
      J=1
      DO 570 JPP3=1,N3VNT
      IPP3=INV(JJ3)
      DO 560 JP3=1,MINN3
      GO TO (430,440), IGO3
  430 IP3=INV(JP3)*N3VNT
      GO TO 450
```

```
440   IP3=INV(JP3)/NTVN3
450   I3=(IPP3+IP3)*N2
      JJ2=1
      DO 560 JPP2=1,N2VNT
      IPP2=INV(JJ2)+I3
      DO 550 JP2=1,MINN2
      GO TO (460,470), IGO2
460   IP2=INV(JP2)*N2VNT
      GO TO 480
470   IP2=INV(JP2)/NTVN2
480   I2=(IPP2+IP2)*N1
      JJ1=1
      DO 550 JPP1=1,N1VNT
      IPP1=INV(JJ1)+I2
      DO 540 JP1=1,MINN1
      GO TO (490,500), IGO1
490   IP1=INV(JP1)*N1VNT
      GO TO 510
500   IP1=INV(JP1)/NTVN1
510   I=2*(IPP1+IP1)+1
      IF (J-I) 520,530,530
520   T=A(I)
      A(I)=A(J)
      A(J)=T
      T=A(I+1)
      A(I+1)=A(J+1)
      A(J+1)=T
530   CONTINUE
540   J=J+2
550   JJ1=JJ1+JJD1
560   JJ2=JJ2+JJD2
570   JJ3=JJ3+JJD3
      IF (IFSET) 580,600,600
580   DO 590 I=1,NX
590   A(2*I)=-A(2*I)
600   RETURN
610   MT=MAX0(M(1),M(2),M(3))-2
      MT=MAX0(2,MT)
      IF (MT-20) 630,630,620
620   IFERR=1
      GO TO 600
630   IFERR=0
      NT=2**MT
      NTV2=NT/2
      THETA=.7853981634
      JSTEP=NT
      JDIF=NTV2
      S(JDIF)=SIN(THETA)
      DO 660 L=2,MT
      THETA=THETA/2.
      JSTEP2=JSTEP
      JSTEP=JDIF
      JDIF=JSTEP/2
      S(JDIF)=SIN(THETA)
      JC1=NT-JDIF
      S(JC1)=COS(THETA)
      JLAST=NT-JSTEP2
      IF (JLAST-JSTEP) 660,640,640
640   DO 650 J=JSTEP,JLAST,JSTEP
      JC=NT-J
      JD=J+JDIF
650   S(JD)=S(J)*S(JC1)+S(JDIF)*S(JC)
660   CONTINUE
```

```
C
C     SET UP INV(J) TABLE
      MTLEXP=NTV2
      LM1EXP=1
      INV(1)=0
      DO 680 L=1,MT
      INV(LM1EXP+1)=MTLEXP
      DO 670 J=2,LM1EXP
      JJ=J+LM1EXP
670   INV(JJ)=INV(J)+MTLEXP
      MTLEXP=MTLEXP/2
680   LM1EXP=LM1EXP*2
      IF (IFSET) 20,600,20
      END
```

```
      SUBROUTINE FFTRS(X,DX,DF,M,INV,S,IFS,IFERR)
C
C FOURIER TRANSFORM OF REAL SYMMETRIC DATA STORED AS COMPLEX ARRAY X
C
      DIMENSION X(1),M(1),INV(1),S(1)
C
      DF=1./DX/2.**M(1)  $  X(1)=X(1)*DX  $  X(2)=0.  $ I1=3
      I2=2**(M(1)+1)-1  $  M1=2**(M(1)-1)
      DO 10 I=1,M1
      X(I1)=(X(I1)+X(I2))*DX  $  X(I2)=X(I1)
      X(I1+1)=0.  $  X(I2+1)=0.  $  I1=I1+2
   10 I2=I2-2
      IF=IABS(IFS)
      CALL FFT(X,M,INV,S,IF,IFERR)
      RETURN  $  END
```

```
      PROGRAM INCPD(INPUT,OUTPUT,PUNCH,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7=
     1PUNCH)
C A PROGRAM TO GENERATE AND FOURIER TRANSFORM ONE-DIMENSIONAL CHARACTER-
C ISTIC FUNCTIONS OF THE GENERAL NON-GAUSSIAN MODEL.  CORRELATION
C MATRICIES AND FUNCTIONAL MATRIX ARE READ FROM DATA CARDS.
C PROGRAM PRINTS PROBABILITY DENSITY FUNCTION IN BOTH STANDARDIZED
C AND NONSTANDARDIZED FORM IN INCREMENTS OF DX*SR FROM ZERO TO
C 200*DX*SR.  RESULTS ARE PUNCHED ON CARDS IF LOGICAL VARIABLE PNCH
C IS SET TRUE.
C
C INPUT DATA$
C     TITLE (8A10) DESCRIPTIVE TITLE TO ACCOMPANY OUTPUT
C     NDIM (I1) SIZE OF MATRICIES (.LE.8)
C     SG2 (F10.5) VARIANCE OF GAUSSIAN PROCESS
C     A ((A(I,J),J=1,NDIM),I=1,NDIM) BY ROWS (UP TO 8 COLUMNS)
C         CORRELATION MATRIX OF FIRST GAUSSIAN VECTOR
C     B (SAME FORMAT AS A) CORRELATION MATRIX OF 2ND GAUSSIAN VECTOR
C     C (SAME FORMAT AS A) FUNCTIONAL MATRIX
C
      COMMON /CCR/ A(8,8),B(8,8),C(8,8),NDIM,ISIZE,DETA,DETB,
     1A1(8,8),B1(8,8),C4(8,8),A2(8,8),B2(8,8)
      COMMON / / S(128),INV(128),PD(1026)
      DIMENSION M(3),PHI(1026),TITLE(8)
      EQUIVALENCE (PD(1),PHI(1))  $  LOGICAL PNCH
      DATA PNCH/.F./
      DATA M,IFS/9,0,0,1/
```

```
C MINIMUN VALUE OF M(1) FOR DX OF .05 AND XMAX OF 10. IS 9 (512)
      DATA DX/.05/
C NORMALIZED DENSITY IS TO BE EVALUATED AT X/SIGMA = MULTIPLES OF DX
      DATA C1,C2,C3/7.6388888889E-03,-6.4583333333E-02,5.5694444444E-01/
C C1,C2,C3 ARE CONSTANTS USED IN NUMERICAL INTEGRATION OF PROB. DENSITY
C
      DATA PI22/39.478417604/
      DATA ISIZE/8/
    1 FORMAT (8A10,/,I1,/,F10.5)
    2 FORMAT (8F10.5)
    3 FORMAT (1H ,5X,8F10.5)
    4 FORMAT (1H0)
    5 FORMAT (1H1,5X,8A10)
    6 FORMAT (1H0,5X,*COVARIANCE MATRIX OF 1ST GAUSSIAN VECTOR*)
    7 FORMAT (1H0,5X,*COVARIANCE MATRIX OF 2ND GAUSSIAN VECTOR*)
    8 FORMAT (1H0,5X,*FUNCTIONAL RELATION MATRIX*)
C READ INPUT DATA
   15 READ (5,1) TITLE,NDIM,SG2
      IF (EOF,5) 400,16
   16 WRITE (6,5) TITLE
      WRITE (6,6)
      DO 17 I=1,NDIM
      READ (5,2) (A(I,J),J=1,NDIM)
   17 WRITE (6,3) (A(I,J),J=1,NDIM)
      WRITE (6,7)
      DO 18 I=1,NDIM
      READ (5,2) (B(I,J),J=1,NDIM)
   18 WRITE (6,3) (B(I,J),J=1,NDIM)
      WRITE (6,8)
      DO 19 I=1,NDIM
      READ (5,2) (C(I,J),J=1,NDIM)
   19 WRITE (6,3) (C(I,J),J=1,NDIM)
C
C CALCULATE STANDARD DEVIATION OF TOTAL PROCESS, SR
      SR2=SG2
      DO 60 I=1,NDIM  $  DO 60 J=1,NDIM
      CIJ=C(I,J)  $  IF (CIJ.EQ.0.) GO TO 60
      DO 50 K=1,NDIM  $  DO 50 L=1,NDIM
      CKL=C(K,L)  $  IF (CKL.EQ.0.) GO TO 50
      SR2=SR2+A(I,K)*B(J,L)*CIJ*CKL
   50 CONTINUE
   60 CONTINUE
      SR=SQRT(SR2)  $  WRITE (6,65) SR
   65 FORMAT (1H0,5X,*STANDARD DEVIATION OF PROCESS =*,E10.3)
C
C GENERATE CHARACTERISTIC FUNCTION
      CALL INVRS
      WRITE (6,70) DETA,DETB
   70 FORMAT (1H0,5X,*DETERMINANTS OF COVARIANCE MATRICIES*,/,6X,
     1*DETA =*,E12.5,/,6X,*DETB =*,E12.5)
      EVG2=SG2/2.  $  F=0.  $  DF=1./DX/2.**M(1)  $  NPTS=2**(M(1)+1)
      DF1=DF/SR
      DO 150 I=1,NPTS,2
      F2=PI22*F**2
      CFR=1.
      CALL CF1(CFR,F)
      CFR=CFR*EXP(-EVG2*F2)
      PHI(I)=CFR  $  PHI(I+1)=0.
  150 F=F+DF1
C
C FOURIER TRANSFORM TO OBTAIN PROBABILITY DENSITY
      CALL FFTRS(PHI,DF,DX,M,INV,S,IFS,IFER)
      IFS=2
```

```
      IF (IFER.NE.0) WRITE (6,200) IFER                                 INCPD090
  200 FORMAT (1H0,*----ERROR IN FFT, ERROR FLAG =*,I2,/,5X,*RESULTS INVAINCPD091
     1LID FOR THIS CASE*)                                               INCPD092
      IF (PNCH) WRITE (7,219) TITLE                                     INCPD093
  219 FORMAT (8A10)                                                     INCPD094
      WRITE (6,220) TITLE                                               INCPD095
  220 FORMAT (1H1,5X,8A10)                                              INCPD096
      WRITE (6,221)                                                     INCPD097
  221 FORMAT (1H0,5X,*NORMALIZED*,4X,*STDIZED*,4X,*DISTRIBUTION*,5X,    INCPD098
     1*UNNORMALIZED*,3X,*NONSTDIZED*,/,7X,*VARIABLE*,3X,*PROBABILITY*,  INCPD099
     24X,*FUNCTION*,9X,*VARIABLE*,5X,*PROBABILITY*,/,7X,*X/SIGMA X*,    INCPD100
     34X,*DENSITY*,27X,*X*,10X,*DENSITY*,/)                             INCPD101
      DX1=DX*SR2                                                        INCPD102
      CD=.5  $  X=0.  $  X1=0.                                          INCPD103
      LINCNT=0                                                          INCPD104
      N=INT(10./DX)+2                                                   INCPD105
      IF (N.GT.NPTS/4) N=NPTS/4  $  N=N*2                               INCPD106
      JMAX=N                                                            INCPD107
      DO 250 J=1,N,2                                                    INCPD108
      IF (PD(J).LT.0.) GO TO 255                                        INCPD109
  250 CONTINUE                                                          INCPD110
      GO TO 265                                                         INCPD111
  255 JMAX=J                                                            INCPD112
      DO 260 I=J,N,2                                                    INCPD113
  260 PD(I)=0.0                                                         INCPD114
  265 CONTINUE                                                          INCPD115
      IF (.NOT.PNCH) N=JMAX                                             INCPD116
      DO 300 J=1,N,2                                                    INCPD117
      UPD=PD(J)/SR2                                                     INCPD118
      WRITE (6,222) X,PD(J),CD,X1,UPD                                   INCPD119
  222 FORMAT (1H ,7X,F6.3,4X,E10.3,3X,E12.5,6X,E10.3,4X,E10.3)          INCPD120
      IF (PNCH) WRITE (7,223) X,PD(J),CD,X1,UPD                         INCPD121
  223 FORMAT (5E16.8)                                                   INCPD122
      LINCNT=LINCNT+1                                                   INCPD123
C RESULTS ARE PRINTED AT 51 LINES PER PAGE                              INCPD124
      IF (LINCNT.LT.51) GO TO 290                                       INCPD125
      IF (J.GE.N-1) GO TO 300                                           INCPD126
      WRITE (6,220) TITLE                                               INCPD127
      WRITE (6,221)                                                     INCPD128
      WRITE (6,222) X,PD(J),CD,X1,UPD                                   INCPD129
      LINCNT=1                                                          INCPD130
  290 IF (J-3) 291,292,293                                              INCPD131
  291 CD=CD+(C1*PD(J+4)+C2*PD(J+2))*DX                                  INCPD132
      GO TO 294                                                         INCPD133
  292 CD=CD+(C1*PD(J)+C2*PD(J-2))*DX                                    INCPD134
      GO TO 294                                                         INCPD135
  293 CD=CD+(C1*PD(J-4)+C2*PD(J-2))*DX                                  INCPD136
  294 CD=CD+(C1*PD(J+6)+C2*PD(J+4)+C3*(PD(J)+PD(J+2)))*DX               INCPD137
      X=X+DX                                                            INCPD138
  300 X1=X1+DX1                                                         INCPD139
      GO TO 15                                                          INCPD140
  400 STOP  $  END                                                      INCPD141
```

```
      SUBROUTINE INRPDT(V,I,J,KK)
C
C SUBROUTINE TO COMPUTE THE INNER PRODUCT OF TWO EIGENFUNCTIONS
C
C E = ARRAY OF EIGENVALUES
C EFP = ARRAY OF P EIGENFUNCTIONS (EIGENFUNCTIONS OF 1ST VARIABLE)
C EFQ = ARRAY OF Q EIGENFUNCTIONS (EIGENFUNCTIONS OF 2ND VARIABLE)
C F1,F2 = TEMPORARY STORAGE ARRAYS
C I,J = ITH AND JTH EIGENFUNCTIONS ARE TO BE USED IN COMPUTATION
C KK = + IF P EIGENFUNCTIONS ARE TO BE USED IN COMPUTATION
C      - IF Q EIGENFUNCTIONS ARE TO BE USED
C NPOLY = DEGREE OF INTERPOLATING POLYNOMIALS
C NPT = NUMBER OF ABCISSAE POINTS
C V = VALUE OF INNER PRODUCT RETURNED BY SUBROUTINE
C X = ARRAY OF ABCISSAE
C
      COMMON /ARRAYS/ E(8),EFP(45,8),EFQ(45,8),X(45),F1(45),F2(45),
     1NPT,NPOLY,C(4),D(4)
C
      NC=NPOLY+1  $  IF (KK) 10,100,15
C
C INNER PRODUCT (Q(I),Q(J))
   10 DO 11 L=1,NPT
      F1(L)=EFQ(L,I)
   11 F2(L)=EFQ(L,J)
      GO TO 20
C
C INNER PRODUCT (P(I),P(J))
   15 DO 16 L=1,NPT
      F1(L)=EFP(L,I)
   16 F2(L)=EFP(L,J)
C
   20 V=0.0  $  K=NPT-1
      DO 40 JJ=1,K
      CALL COEF(JJ,F1,C)  $  CALL COEF(JJ,F2,D)
      X1=X(JJ)  $  X2=X(JJ+1)
      DO 40 II=1,NC
      X12=X1**II  $  X22=X2**II
      CII=C(II)  $  CL=FLOAT(II)
      DO 40 LL=1,NC
      V=V+CII*D(LL)*(X22-X12)/CL  $  CL=CL+1.  $  X12=X12*X1
   40 X22=X22*X2
      RETURN
C
  100 WRITE (6,101)
  101 FORMAT (1H0,5X,*FATAL ERROR IN INRPDT, KK=0*)
      STOP  $  END
```

```
      SUBROUTINE INVR(A,N,DETERM,ISIZE,JSIZE)
C
C SUBROUTINE TO INVERT MATRIX A AND COMPUTE ITS DETERMINANT
C
      DIMENSION IPIVOT(25),A(ISIZE,JSIZE),INDEX(25,2),PIVOT(25)
      EQUIVALENCE (IROW,JROW),(ICOLUM,JCOLUM),(AMAX,T,SWAP)
C
   10 DETERM=1.0
   15 DO 20 J=1,N
   20 IPIVOT(J)=0
   30 DO 550 I=1,N
C
```

```
C     SEARCH FOR PIVOT ELEMENT                                          INVR0013
   40 AMAX=0.  $  ICOLUM=0                                              INVR0014
   45 DO 105 J=1,N                                                      INVR0015
   50 IF (IPIVOT(J)-1) 60, 105, 60                                      INVR0016
   60 DO 100 K=1,N                                                      INVR0017
   70 IF (IPIVOT(K)-1) 80, 100, 740                                     INVR0018
   80 IF(ABS(AMAX)-ABS(A(J,K)))85,100,100                               INVR0019
   85 IROW=J  $  ICOLUM=K  $  AMAX=A(J,K)                               INVR0020
  100 CONTINUE                                                          INVR0021
  105 CONTINUE                                                          INVR0022
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1                                   INVR0023
C                                                                       INVR0024
C     INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL                 INVR0025
  130 IF (IROW-ICOLUM) 140, 260, 140                                    INVR0026
  140 DETERM=-DETERM                                                    INVR0027
  150 DO 200 L=1,N                                                      INVR0028
  160 SWAP=A(IROW,L)  $  A(IROW,L)=A(ICOLUM,L)                          INVR0029
  200 A(ICOLUM,L)=SWAP                                                  INVR0030
  260 INDEX(I,1)=IROW                                                   INVR0031
  270 INDEX(I,2)=ICOLUM  $  PIVOT(I)=A(ICOLUM,ICOLUM)                   INVR0032
  320 DETERM=DETERM*PIVOT(I)                                            INVR0033
C                                                                       INVR0034
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT                                 INVR0035
  330 A(ICOLUM,ICOLUM)=1.0                                              INVR0036
  340 DO 350 L=1,N                                                      INVR0037
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)                                  INVR0038
C                                                                       INVR0039
C     REDUCE NON-PIVOT ROWS                                             INVR0040
  380 DO 550 L1=1,N                                                     INVR0041
  390 IF(L1-ICOLUM) 400, 550, 400                                       INVR0042
  400 T=A(L1,ICOLUM)                                                    INVR0043
  420 A(L1,ICOLUM)=0.0                                                  INVR0044
  430 DO 450 L=1,N                                                      INVR0045
      A(L1,L)=A(L1,L)-A(ICOLUM,L)*T                                     INVR0046
  450 CONTINUE                                                          INVR0047
  550 CONTINUE                                                          INVR0048
C                                                                       INVR0049
C     INTERCHANGE COLUMNS                                               INVR0050
  600 DO 710 I=1,N                                                      INVR0051
  610 L=N+1-I                                                           INVR0052
  620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630                          INVR0053
  630 JROW=INDEX(L,1)                                                   INVR0054
  640 JCOLUM=INDEX(L,2)                                                 INVR0055
  650 DO 705 K=1,N                                                      INVR0056
  660 SWAP=A(K,JROW)  $  A(K,JROW)=A(K,JCOLUM)  $  A(K,JCOLUM)=SWAP     INVR0057
  705 CONTINUE                                                          INVR0058
  710 CONTINUE                                                          INVR0059
  740 RETURN  $  END                                                    INVR0060
```

```
      PROGRAM LEVXNG(OUTPUT,PUNCH,TAPE6=OUTPUT,TAPE7=PUNCH)             LEVXG001
C                                                                       LEVXG002
C PROGRAM TO COMPUTE SPECIAL CASE LEVEL CROSSING FREQUENCIES OF THE NON-LEVXG003
C GAUSSIAN MODEL FOR GIVEN LIST OF R PARAMETER VALUES.                  LEVXG004
C                                                                       LEVXG005
C INPUT DATA - NONE (PROGRAM IS CONTROLLED THROUGH DATA STATEMENTS)     LEVXG006
C                                                                       LEVXG007
      COMMON /CCR/ C1(8,8),C2(8,8),C3(8,8),C4(8,8),NN,ISIZE,DETA,DETB,  LEVXG008
     1C(320)                                                            LEVXG009
      DIMENSION X(8192),S(1024),INV(1024),M(3),DF(2),DX(2),TITLE(8)     LEVXG010
      DIMENSION RARRAY(6)                                               LEVXG011
```

```
      COMPLEX SUM,A(64,64)  $  LOGICAL PNCH                             LEVXG012
      EQUIVALENCE(X(1),A(1,1))                                          LEVXG013
C IFS AND M ARE VARIABLES USED BY FFT                                   LEVXG014
      DATA IFS,M/1,6,6,0/                                               LEVXG015
C RARRAY CONTAINS THE R PARAMETER VALUES TO BE USED                     LEVXG016
      DATA RARRAY/1.,0.,.5,.75,1.3333333333,2./                         LEVXG017
C NCASE IS THE NUMBER OF R PARAMETER VALUES TO BE USED, .LE.6           LEVXG018
      DATA NCASE/1/                                                     LEVXG019
C DXOS IS INCREMENT OF TABULATION (IN STANDARD DEVIATIONS)              LEVXG020
C NTAB IS NUMBER OF TABULATED VALUES DESIRED                            LEVXG021
      DATA DXOS,NTAB/.2,41/                                             LEVXG022
C ISIZE IS SIZE OF MATRICIES USED BY INVR                               LEVXG023
      DATA ISIZE/8/                                                     LEVXG024
C PNCH IS LOGICAL VARIABLE, IF SET TRUE RESULTS ARE PUNCHED ON CARDS    LEVXG025
      DATA PNCH/.F./                                                    LEVXG026
C C11,C12, AND C13 ARE CONSTANTS USED IN NUMERICAL INTEGRATION OF THE   LEVXG027
C JOINT DENSITY FUNCTION                                                LEVXG028
      DATA C11,C12,C13/7.6388888889E-03,-6.4583333333E-02,5.5694444444E-LEVXG029
     101/                                                               LEVXG030
      DATA TWOPI2/19.739208802/                                         LEVXG031
C                                                                       LEVXG032
      DO 140 NRATIO=1,NCASE                                             LEVXG033
      R=RARRAY(NRATIO)                                                  LEVXG034
C GENERATE TITLE ARRAY TO ACCOMPANY OUTPUT                              LEVXG035
      ENCODE(80,5,TITLE)R                                               LEVXG036
    5 FORMAT (*        LEVEL CROSSING FREQUENCY OF THE NON-GAUSSIAN MODLEVXG037
     1EL, R=*,F7.3,*          *)                                        LEVXG038
C CONSTRUCT CORRELATION AND FUNCTIONAL DEPENDENCE MATRICIES             LEVXG039
      NN=2  $  IF (R.LE.0.)NN=0                                         LEVXG040
      C1(1,1)=C2(1,1)=R/SQRT(1.+R**2)                                   LEVXG041
      C1(2,1)=C1(1,2)=C2(1,2)=C2(2,1)=0.                                LEVXG042
      C3(1,1)=1.  $  C3(2,2)=C3(2,1)=C3(1,2)=0.                         LEVXG043
      C4(1,1)=C4(2,2)=0.  $  C4(1,2)=C4(2,1)=1.                         LEVXG044
      S1G2=1./(1.+R**2)                                                 LEVXG045
C                                                                       LEVXG046
C CODING USED HERE IS FOR U-GUST COMPONENT UNIVERSAL CURVES             LEVXG047
      C1(2,2)=C1(1,1)/2.  $  C2(2,2)=C2(1,1)/2.  $  S2G2=S1G2           LEVXG048
C                                                                       LEVXG049
C CODING FOR THE V-W GUST UNIVERSAL CURVES WOULD BE                     LEVXG050
C     C1(2,2)=C1(2,2)/2.  $  C2(2,2)=C2(1,1)  $  S2G2=1.5*S1G2          LEVXG051
C                                                                       LEVXG052
      S12=S1G2+C1(1,1)*C2(1,1)                                          LEVXG053
      S22=S2G2+C1(1,1)*C2(2,2)+C1(2,2)*C2(1,1)                          LEVXG054
      IF (NN.NE.0) WRITE (6,30) ((C1(I,J),J=1,2),I=1,2),((C2(I,J),J=1,2)LEVXG055
     1,I=1,2)                                                           LEVXG056
   30 FORMAT (1H1,5X,*COVARIANCE MATRICIES OF FIRST AND SECOND*,/,      LEVXG057
     16X,*GAUSSIAN VECTORS ARE*,/,(6X,2E10.3))                          LEVXG058
      IF (NN.NE.0) WRITE (6,31) ((C3(I,J),J=1,2),I=1,2),((C4(I,J),J=1,2)LEVXG059
     1,I=1,2)                                                           LEVXG060
   31 FORMAT (1H0,5X,*FUNCTIONAL RELATIONSHIP MATRICIES FOR FIRST*,/,   LEVXG061
     16X,*AND SECOND TRANSFORM VARIABLES ARE*,/,(6X,2E10.3))            LEVXG062
      WRITE (6,35) R                                                    LEVXG063
   35 FORMAT (1H0,5X,*R PARAMETER =*,F6.3)                              LEVXG064
      IF (PNCH) WRITE (7,40) R                                          LEVXG065
   40 FORMAT (*R =*,F6.3)                                               LEVXG066
C                                                                       LEVXG067
C GENERATE CHARACTERISTIC FUNCTION                                      LEVXG068
   56 CALL INVRS(ANS,F1,F2)                                             LEVXG069
      WRITE (6,54) DETA,DETB                                            LEVXG070
   54 FORMAT (1H0,5X,*DETERMINANTS OF COVARIANCE MATRICIES*,/,6X,*DETA =LEVXG071
     1*,E12.5,/,6X,*DETB =*,E12.5)                                      LEVXG072
   57 DX(1)=.3*SQRT(S12)  $  DX(2)=.3*SQRT(S22)                         LEVXG073
      M1=2**M(1)  $  M2=2**M(2)                                         LEVXG074
```

```
      DF(1)=1./DX(1)/M1  $  DF(2)=1./DX(2)/M2
      WRITE (6,102) DX,DF
  102 FORMAT (1H0,5X,*X INCREMENTS!  DX(1) =*,E14.7,* DX(2) =*,E14.7,/,
     16X,*F INCREMENTS!  DF(1) =*,E14.7,* DF(2) =*,E14.7)
      DFDF=DF(1)*DF(2)
      S1G2=-S1G2*TWOPI2  $  S2G2=-S2G2*TWOPI2
      A(1,1)=CMPLX(1.,0.)  $  F1=DF(1)  $  F2=0.
      F11=FLOAT(1-M1)*DF(1)
      DO 200 I=2,M1
      CALL CF2(ANS,F1,F2)  $  ANS=ANS*EXP(S1G2*F1**2)
      CALL CF2(ANS1,F11,F2)  $  ANS=ANS+ANS1*EXP(S1G2*F11**2)
      A(I,1)=CMPLX(ANS,0.)  $  F11=F11+DF(1)
  200 F1=F1+DF(1)
      F1=0.  $  F2=DF(2)  $  F22=FLOAT(1-M2)*DF(2)
      DO 300 J=2,M2
      CALL CF2(ANS,F1,F2)  $  ANS=ANS*EXP(S2G2*F2**2)
      CALL CF2(ANS1,F1,F22)  $  ANS=ANS+ANS1*EXP(S2G2*F22**2)
      A(1,J)=CMPLX(ANS,0.)  $  F22=F22+DF(2)
  300 F2=F2+DF(2)
      F11=FLOAT(M1)*DF(1)  $  F1=0.  $  M22=M2/2+1
      DO 410 I=2,M1
      F1=F1+DF(1)  $  F11=F11-DF(1)
      F2=0.  $  F22=FLOAT(M2)*DF(2)
      DO 400 J=2,M22
      F2=-F2+DF(2)  $  CALL CF2(ANS,F1,F2)
      ANS=ANS*EXP(S1G2*F1**2+S2G2*F2**2)
      F2=-F2  $  CALL CF2(ANS1,F11,F2)
      ANS=ANS+ANS1*EXP(S1G2*F11**2+S2G2*F2**2)
      F22=-F22+DF(2)  $  CALL CF2(ANS1,F1,F22)
      ANS=ANS+ANS1*EXP(S1G2*F1**2+S2G2*F22**2)
      F22=-F22  $  CALL CF2(ANS1,F11,F22)
      ANS=ANS+ANS1*EXP(S1G2*F11**2+S2G2*F22**2)
      A(I,J)=CMPLX(ANS,0.)
  400 A(M1+2-I,M2+2-J)=CMPLX(ANS,0.)
  410 CONTINUE
      WRITE (6,61) (A(J,1),J=1,M1)
      WRITE (6,62) (A(1,J),J=1,M2)
      WRITE (6,62) (A(J,J),J=1,M2)
   61 FORMAT (1H1,5X,*FIRST ROW, FIRST COLUMN, AND DIAGONAL OF*,/,
     16X,*JOINT CHARACTERISTIC FUNCTION*,/,(5X,8E10.3))
   62 FORMAT (1H ,/,(5X,8E10.3))
C
C FOURIER TRANSFORM TO OBTAIN PROBABILITY DENSITY
      DO 71 J=1,8192,2
   71 X(J)=X(J)*DFDF
      CALL FFT(X,M,INV,S,IFS,IFER)
      IF (IFER.EQ.0) GO TO 73
      WRITE (6,72) IFER  $  STOP
   72 FORMAT (1H0,5X,*FATAL ERROR IN FFTRS2, IFER =*,I5)
C
C INTEGRATE TO OBTAIN LEVEL CROSSINGS
   73 C1X=C11*DX(2)  $  C2X=C12*DX(2)  $  C3X=C13*DX(2)
      WRITE (6,74) (A(J,1),J=1,M1)
      WRITE (6,62) (A(1,J),J=1,M2)
      WRITE (6,62) (A(J,J),J=1,M2)
   74 FORMAT (1H1,5X,*FIRST ROW, FIRST COLUMN, AND DIAGONAL OF*,/,
     16X,*JOINT PROBABILITY DENSITY*,/,(5X,8E10.3))
      X2=0.  $  M21=M2/2+1  $  M11=M1/2+3
      DO 80 J=1,M21
      DO 75 I=1,M11
   75 A(I,J)=A(I,J)*X2
   80 X2=X2+DX(2)
      M21=M2/2-2
```

```
      DO 86 I=1,M11
      SUM=C1X*(-A(I,3)+A(I,4)-A(I,2)+A(I,5))+C2X*(A(I,1)-A(I,2)+A(I,3)+
     1A(I,4))+C3X*(A(I,1)+2.*A(I,2)+A(I,3))
      DO 85 J=1,M21
   85 SUM=SUM+C1X*(A(I,J)+A(I,J+5))+C2X*(A(I,J+1)+A(I,J+4))+C3X*(A(I,J+2
     1)+A(I,J+3))
      X(I)=FLOAT(I-1)*DX(1)
   86 S(I)=REAL(SUM)
      WRITE (6,87) (X(I),S(I),I=1,M11)
   87 FORMAT (1H1,5X,*COMPUTED VALUES OF X AND N(X)*,/,(4X,2E15.7))
C
C INTERPOLATE TO FIND LEVEL CROSSINGS AT SPECIFIED VALUES OF X
      X1=0.  $  X2=0.  $  DIVISR=S(1)
      DX1=DXOS*SQRT(S12)  $  M21=M2/2+1  $  NINDEX=1
      WRITE (6,90) TITLE
   90 FORMAT (1H1,36X,*LEVEL CROSSINGS*,/,5X,8A10,/,/,16X,*DIMENSIONAL
     1NON-DIMENSIONAL  CROSSINGS PER  CROSSINGS PER*,/,19X,*LEVEL*,10X,
     2*LEVEL*,9X,*UNIT TIME*,4X,*ZERO CROSSING*,/,21X,*X*,10X,*X/SIGMA X
     3*,/)
      DO 105 I=1,NTAB
   91 IF (X(NINDEX)-X1) 92,99,93
   92 NINDEX=NINDEX+1  $  IF (NINDEX-M21) 91,91,150
   93 NINDEX=NINDEX-1
      H=(X1-X(NINDEX))/DX(1)
      FI3=S(NINDEX)  $  FI4=S(NINDEX+1)  $  FI5=S(NINDEX+2)
      FI6=S(NINDEX+3)  $  IF (NINDEX-2) 94,96,97
   94 IF (NINDEX-1) 150,95,96
   95 FI1=FI5  $  FI2=FI4  $  GO TO 98
   96 FI1=FI3  $  FI2=S(1)  $  GO TO 98
   97 FI1=S(NINDEX-2)  $  FI2=S(NINDEX-1)
   98 F1=-FI1*H*(H**2-1.)*(H-2.)*(H-3.)/120.+FI2*H*(H-1.)*(H**2-4.)*
     1(H-3.)/24.-FI3*(H**2-1.)*(H**2-4.)*(H-3.)/12.+FI4*H*(H+1.)*(H**2
     2-4.)*(H-3.)/12.-FI5*H*(H**2-1.)*(H+2.)*(H-3.)/24.+FI6*H*(H**2-4.)
     3*(H**2-1.)/120.
      GO TO 100
   99 F1=S(NINDEX)
  100 F2=F1/DIVISR
      WRITE (6,120) X1,X2,F1,F2
      IF (PNCH) WRITE (7,121) X2,F2,F1
      IF (F2.LE.1.E-10) GO TO 140
      X1=X1+DX1
  105 X2=X2+DXOS
  120 FORMAT (1H ,15X,E10.3,5X,E10.3,6X,E10.3,5X,E10.3)
  121 FORMAT (3E20.10)
  140 CONTINUE
  150 STOP  $  END
```

```
      PROGRAM PDIST(INPUT,OUTPUT,PUNCH,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7=
     1PUNCH)
C
C A PROGRAM TO GENERATE PROBABILITY DISTRIBUTIONS FOR THE NON-GAUSSIAN
C MODEL OR THE RESPONSE OF A LINEAR SYSTEM TO THE MODEL.
C
C INPUT DATA:
C     TITLE(I),I=1,8  (8A10)  DESCRIPTIVE TITLE TO ACCOMPANY OUTPUT
C     SR2  (E20.10)  SYSTEM RESPONSE VARIANCE FOR UNIT VARIANCE INPUT
C          SR2=1.0 FOR TURBULENCE MODEL DISTRIBUTION CALCULATIONS
C     NEV  (I2)  NUMBER OF NON-GAUSSIAN EIGENVALUES TO BE USED (.LE.20)
C          NEV=1 FOR TURBULENCE MODEL DISTRIBUTION CALCULATIONS
C     EV(I),I=1,NEV  (E20.10)  NON-GAUSSIAN EIGENVALUES FOR RESPONSE TO
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C        UNIT VARIANCE MODIFIED BESSEL FUNCTION INPUT.  EV(1)=1. FOR      POIST014
C        TURBULENCE MODEL DISTRIBUTION CALCULATIONS.                      POIST015
C     R  (F10.5)  SIGMA RATIO OF THE TURBULENCE MODEL (R.GE.0, R=0        POIST016
C        FOR GAUSSIAN MODEL, R=INF FOR MODIFIED BESSEL FUNCTION MODEL,    POIST017
C        IN ORDER TO RETAIN COMPUTATIONAL ACCURACY R SHOULD BE LESS       POIST018
C        THAN 2)                                                          POIST019
C                                                                         POIST020
C IF THE SUM OF THE NON-GAUSSIAN EIGENVALUES SQUARED IS FOUND TO BE       POIST021
C MORE THAN 5 PERCENT GREATER THAN THE SYSTEM RESPONSE VARIANCE, SR2,     POIST022
C EXECUTION IS ENDED.                                                     POIST023
C                                                                         POIST024
C IF THE SUM IS FOUND TO BE LESS THAN THE RESPONSE VARIANCE, AN           POIST025
C INDEPENDENT GAUSSIAN PROCESS IS INTRODUCED TO CORRECT THE DEFICIT.      POIST026
C                                                                         POIST027
C PROGRAM PRINTS THE RESPONSE PROBABILITY DENSITY FUNCTION IN BOTH        POIST028
C NORMALIZED AND UNNORMALIZED FORM IN INCREMENTS OF DX*SR2 FROM ZERO      POIST029
C TO 200*DX*SR2.  DX IS GIVEN IN DATA STATEMENT BELOW.                    POIST030
C RESULTS ARE PUNCHED ON CARDS IF LOGICAL PARAMETER PNCH IS SET TRUE.     POIST031
C                                                                         POIST032
      COMMON / / S(128),INV(128),PD(1026)                                 POIST033
      DIMENSION M(3),PHI(1026)                                            POIST034
      DIMENSION TITLE(8),EV(20),EV2(20)                                   POIST035
      EQUIVALENCE (PD(1),PHI(1))                                          POIST036
      LOGICAL PNCH                                                        POIST037
C RESULTS ARE PUNCHED ON CARDS IF PNCH IS TRUE                            POIST038
      DATA PNCH/.F./                                                      POIST039
      DATA PI22/39.478417604/                                             POIST040
      DATA M,IFS/9,0,0,1/                                                 POIST041
C MINIMUM VALUE OF M(1) FOR DX OF .05 AND XMAX OF 10. IS 9 (512)          POIST042
      DATA DX/.05/                                                        POIST043
C DX*SR2 IS THE INCREMENT OF EVALUATION                                   POIST044
      DATA C1,C2,C3/7.6388888889E-03,-6.4583333333E-02,5.5694444444E-01/POIST045
C C1,C2,C3 ARE CONSTANTS USED IN NUMERICAL INTEGRATION OF PROB. DENSITY POIST046
C                                                                         POIST047
C READ INPUT DATA                                                         POIST048
   15 READ (5,1) TITLE,SR2,NEV                                            POIST049
      IF (EOF,5) 400,16                                                   POIST050
   16 READ (5,2) (EV(I),I=1,NEV)                                          POIST051
      READ (5,3) R                                                        POIST052
    1 FORMAT (8A10,/,E20.10,/,I2)                                         POIST053
    2 FORMAT (E20.10)                                                     POIST054
    3 FORMAT (F10.5)                                                      POIST055
C                                                                         POIST056
C WRITE INPUT DATA                                                        POIST057
      WRITE (6,11) TITLE,SR2,NEV,(EV(I),I=1,NEV)                          POIST058
      WRITE (6,12) R                                                      POIST059
   11 FORMAT (1H1,5X,8A10,/,/,6X,*VARIANCE =*,F10.5,/,6X,*NUMBER OF NON-POIST060
     1GAUSSIAN EIGENVALUES =*,I2,/,6X,*EIGENVALUES ARE:*,/,(12X,E15.8)) POIST061
   12 FORMAT (1H0,5X,*SIGMA RATIO OF TURBULENCE MODEL =*,F10.5)           POIST062
C                                                                         POIST063
C COPY EIGENVALUES TO WORKING ARRAY, CHECK SUM OF SQUARES                 POIST064
      SE2=0.                                                              POIST065
      DO 20 I=1,NEV                                                       POIST066
      EV2(I)=EV(I)**2                                                     POIST067
   20 SE2=SE2+EV2(I)                                                      POIST068
      IF (SE2.LE.SR2) GO TO 100                                           POIST069
      IF (SE2.LT.SR2*1.05) GO TO 90                                       POIST070
C                                                                         POIST071
C     SUM OF SQUARES IS TOO LARGE                                         POIST072
      WRITE (6,30) SR2,SE2                                                POIST073
   30 FORMAT (1H0,5X,*-----ERROR IN DATA, SUM OF SQUARED EV(I) MORE THANPOIST074
     1 5 PERCENT GREATER THAN SR2*,/,6X,*SR2 =*,E12.5,/,6X,*SUM OF EV(I)POIST075
     2 SQUARED =*,E12.5)                                                  POIST076
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      GO TO 15                                                          PDIST077
C                                                                       PDIST078
C SUM OF SQUARED EIGENVALUES EXCEEDS SYSTEM VARIANCE BY LESS THAN 5 PCNTPDIST079
   90 SE2=SE2/SR2                                                       PDIST080
      DO 92 I=1,NEV                                                     PDIST081
      EV2(I)=EV2(I)/SE2                                                 PDIST082
   92 EV(I)=SQRT(EV2(I))                                                PDIST083
      WRITE (6,94) SE2,(EV(I),EV2(I),I=1,NEV)                           PDIST084
   94 FORMAT (1H0,5X,*SUM OF EIGENVALUES SQUARED EXCEEDS VARIANCE BY A FPDIST085
     1ACTOR OF*,F8.5,/,6X,*EIGENVALUES WILL BE SCALED TO GIVE CORRECT VAPDIST086
     2RIANCE*,/,6X,*SCALED EIGENVALUES AND SQUARED EIGENVALUES ARE*,/,  PDIST087
     3(6X,2E15.7))                                                      PDIST088
      SE2=SR2                                                           PDIST089
C                                                                       PDIST090
C SCALE GAUSSIAN AND NON-GAUSSIAN VARIANCES ACCORDING TO SIGMA RATIO    PDIST091
C OF TURBULENCE MODEL                                                   PDIST092
  100 RC=R**2/(1.+R**2)                                                 PDIST093
      SES2=SE2*RC  $  SE2G=(SR2-SE2)*RC                                 PDIST094
      EVG2=SR2/(1.+R**2)+SE2G                                           PDIST095
C EVG2 IS THE VARIANCE OF THE GAUSSIAN PORTION OF THE SYSTEM RESPONSE   PDIST096
C INCLUDING CORRECTION FOR NEGLECTED NON-GAUSSIAN EIGENVALUES           PDIST097
C                                                                       PDIST098
      SC=1./(EVG2+SES2) $  SE=SES2*SC  $  SG=EVG2*SC                    PDIST099
      WRITE (6,130) SE2,SR2,SES2,EVG2,SE,SG                             PDIST100
  130 FORMAT (1H0,5X,*SUM OF EIGENVALUES SQUARED =*,E15.7,/,/,6X,       PDIST101
     1*FOR VARIANCE OF*,E15.7,/,6X,*NON-GAUSSIAN VARIANCE =*,E15.7,/,6X,PDIST102
     2*GAUSSIAN VARIANCE INCLUDING CORRECTION FOR*,/,6X,*NEGLECTED EIGENPDIST103
     3VALUES =*,E15.7,/,/,6X,*FOR TOTAL VARIANCE OF UNITY*,/,6X,*NON-GAUPDIST104
     4SSIAN VARIANCE =*,E15.7,/,6X,*GAUSSIAN VARIANCE INCLUDING CORRECTIPDIST105
     5ON FOR*,/,6X,*NEGLECTED EIGENVALUES =*,E15.7)                     PDIST106
      SC=SC*RC                                                          PDIST107
      DO 135 I=1,NEV                                                    PDIST108
  135 EV2(I)=EV2(I)*SC                                                  PDIST109
      WRITE (6,137) (EV2(I),I=1,NEV)                                    PDIST110
  137 FORMAT (1H0,5X,*SQUARED EIGENVALUES SCALED TO GIVE CORRECT NON-GAUPDIST111
     1SSIAN*,/,6X,*CONTRIBUTION TO UNIT VARIANCE:*,/,(6X,E15.7))        PDIST112
C GENERATE RESPONSE CHARACTERISTIC FUNCTION                             PDIST113
      EVG2=SG/2.  $  F=0.  $  DF=1./DX/2.**M(1)  $  NPTS=2**(M(1)+1)    PDIST114
      DO 150 I=1,NPTS,2                                                 PDIST115
      F2=PI22*F**2  $  CFR=1.                                           PDIST116
      DO 140 J=1,NEV                                                    PDIST117
  140 CFR=CFR/(1.+EV2(J)*F2)                                            PDIST118
      CFR=SQRT(CFR)  $  CFR=CFR*EXP(-EVG2*F2)                           PDIST119
      PHI(I)=CFR  $  PHI(I+1)=0.                                        PDIST120
  150 F=F+DF                                                            PDIST121
C                                                                       PDIST122
C TRANSFORM CHARACTERISTIC FUNCTION                                     PDIST123
      CALL FFTRS(PHI,DF,DX,M,INV,S,IFS,IFER)                            PDIST124
      IFS=2                                                             PDIST125
      IF (IFER.EQ.0) GO TO 210                                          PDIST126
      WRITE (6,200) IFER  $  STOP                                       PDIST127
  200 FORMAT (1H0,*----ERROR IN FFT, ERROR FLAG =*,I2,/,5X,*RESULTS INVAPDIST128
     1LID FOR THIS CASE*)                                               PDIST129
C                                                                       PDIST130
C INTERPOLATE DENSITY FUNCTION TO OBTAIN FINAL RESULTS                  PDIST131
  210 IF (PNCH) WRITE (7,219) TITLE                                     PDIST132
  219 FORMAT (8A10)                                                     PDIST133
      WRITE (6,220) TITLE                                               PDIST134
  220 FORMAT (1H1,5X,8A10)                                              PDIST135
      WRITE (6,221)                                                     PDIST136
  221 FORMAT (1H0,5X,*NORMALIZED*,4X,*STDIZED*,4X,*DISTRIBUTION*,5X,    PDIST137
     1*UNNORMALIZED*,3X,*NONSTDIZED*,/,7X,*VARIABLE*,3X,*PROBABILITY*,  PDIST138
     24X,*FUNCTION*,9X,*VARIABLE*,5X,*PROBABILITY*,/,7X,*X/SIGMA X*,    PDIST139
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     34X,*DENSITY*,27X,*X*,10X,*DENSITY*,/)                              PDIST140
      DX1=DX*SR2  $  CD=.5  $  X=0.  $  X1=0.                            PDIST141
      LINCNT=0  $  N=INT(10./DX+.99)+1  $  IF (N.GT.NPTS/4) N=NPTS/4     PDIST142
      DO 300 I=1,N                                                       PDIST143
      J=2*I-1  $  UPD=PD(J)/SR2                                          PDIST144
      WRITE (6,222) X,PD(J),CD,X1,UPD                                    PDIST145
  222 FORMAT (1H ,7X,F6.3,4X,E10.3,3X,E12.5,6X,E10.3,4X,E10.3)           PDIST146
      IF (PNCH) WRITE (7,223) X,PD(J),CD,X1,UPD                          PDIST147
  223 FORMAT (5E16.8)                                                    PDIST148
      LINCNT=LINCNT+1                                                    PDIST149
C LISTING PRODUCED AT 51 LINES PER PAGE                                  PDIST150
      IF (LINCNT.LT.51) GO TO 290  $  IF (I.EQ.N) GO TO 15               PDIST151
      WRITE (6,220) TITLE                                                PDIST152
      WRITE (6,221)                                                      PDIST153
      WRITE (6,222) X,PD(J),CD,X1,UPD                                    PDIST154
      LINCNT=1                                                           PDIST155
  290 IF (I-2) 291,292,293                                               PDIST156
  291 CD=CD+(C1*PD(J+4)+C2*PD(J+2))*DX  $  GO TO 294                     PDIST157
  292 CD=CD+(C1*PD(J)+C2*PD(J-2))*DX $  GO TO 294                        PDIST158
  293 CD=CD+(C1*PD(J-4)+C2*PD(J-2))*DX                                   PDIST159
  294 CD=CD+(C1*PD(J+6)+C2*PD(J+4)+C3*(PD(J)+PD(J+2)))*DX                PDIST160
      X=X+DX                                                             PDIST161
  300 X1=X1+DX1                                                          PDIST162
      GO TO 15                                                           PDIST163
  400 STOP  $  END                                                       PDIST164
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      PROGRAM RLEVX(INPUT,OUTPUT,PUNCH,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7=  RLEVX001
     1PUNCH)                                                             RLEVX002
C                                                                        RLEVX003
C PROGRAM TO COMPUTE LEVEL CROSSINGS/UNIT TIME OF LINEAR SYSTEM RESPONSERLEVX004
C TO NON-GAUSSIAN MODEL.  CROSSINGS/DISTANCE CAN BE OBTAINED BY DIVISIONRLEVX005
C BY MEAN TRUE AIRSPEED.                                                 RLEVX006
C                                                                        RLEVX007
C INPUT DATA                                                             RLEVX008
C TITLE  (8A10)  DESCRIPTIVE TITLE TO ACCOMPANY OUTPUT                   RLEVX009
C NPT  (I2)  NUMBER OF ABCISSAE POINTS AT WHICH EIGENFUNCTIONS ARE       RLEVX010
C     TABULATED                                                          RLEVX011
C NR1  (I1)  NUMBER OF EIGENSOLUTIONS DEFINING FIRST VARIABLE            RLEVX012
C E(I),(X(J),EFP(J,I),EFQ(J,I),J=1,NPT)  (45X,E20.11,/,(3E25.14))        RLEVX013
C     EIGENVALUE, ABCISSAE POINTS, P EIGENFUNCTION, AND Q EIGENFUNCTION RLEVX014
C     RESPECTIVELY AS PUNCHED BY GENERATING PROGRAM.  REPEATED FOR       RLEVX015
C     EACH OF THE NR1 EIGENSOLUTIONS. (I=1,NR1)                          RLEVX016
C S12  (F10.5)  TOTAL VARIANCE OF FIRST VARIABLE                         RLEVX017
C NR2  (I1)  NUMBER OF EIGENSOLUTIONS DEFINING SECOND VARIABLE           RLEVX018
C E(I),(X(J),EFP(J,I),EFQ(J,I),J=1,NPT)  (45X,E20.11),/,(3E25.14))       RLEVX019
C     EIGENVALUE, ABCISSAE POINTS, P EIGENFUNCTION, AND Q EIGENFUNCTION RLEVX020
C     RESPECTIVELY AS PUNCHED BY GENERATING PROGRAM.  REPEATED FOR       RLEVX021
C     EACH OF THE NR2 EIGENSOLUTIONS (I=1,NR2)                           RLEVX022
C S22  (F10.5)  TOTAL VARIANCE OF SECOND VARIABLE                        RLEVX023
C R  (F10.5)  SIGMA RATIO OF TURBULENCE MODEL                            RLEVX024
C                                                                        RLEVX025
C NR1+NR2 MUST BE .LE.8, OTHERWISE REDIMENSIONING IS REQUIRED.           RLEVX026
C RESULTS ARE PUNCHED ON CARDS IF LOGICAL VARIABLE PNCH IS SET TRUE.     RLEVX027
C                                                                        RLEVX028
      COMMON /ARRAYS/ E(8),EFP(45,8),EFQ(45,8),XA(45),G1(45),G2(45),     RLEVX029
     1NPT,NPOLY,COEFA(8)                                                 RLEVX030
      COMMON /CCR/ C1(8,8),C2(8,8),C3(8,8),C4(8,8),NN,ISIZE,DETA,DETB,   RLEVX031
     1C(320)                                                             RLEVX032
      DIMENSION X(8192),S(1024),INV(1024),M(3),DF(2),DX(2),A(64,64),     RLEVX033
     1TITLE(8)                                                           RLEVX034
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      COMPLEX A,SUM  $  LOGICAL PNCH                                    RLEVX035
      EQUIVALENCE (X(1),E(1),A(1,1))                                    RLEVX036
C PNCH SET .F. SUPPRESSES PUNCHED OUTPUT                                RLEVX037
      DATA PNCH/.F./                                                    RLEVX038
C IFS AND M ARE VARIABLES USED BY SUBROUTINE FFT                        RLEVX039
      DATA IFS,M/1,6,6,0/                                               RLEVX040
      DATA TWOPI2/19.739208802/                                         RLEVX041
C C11,C12,C13 ARE CONSTANTS USED IN NUMERICAL INTEGRATION OF JOINT      RLEVX042
C PROBABILITY DENSITY FUNCTION                                          RLEVX043
      DATA C11,C12,C13/7.6388888889E-03,-6.4583333333E-02,5.5694444444E-RLEVX044
     101/                                                               RLEVX045
C DXOS IS TABULATION INCREMENT (IN STD. DEVIATIONS OF THE RESPONSE)     RLEVX046
C NTAB IS NUMBER OF TABULATED VALUES TO BE COMPUTED                     RLEVX047
      DATA DXOS,NTAB/.2,41/                                             RLEVX048
C ISIZE IS DIMENSION OF MATRICIES USED BY INVERSION ROUTINE INVR        RLEVX049
      DATA ISIZE/8/                                                     RLEVX050
C                                                                       RLEVX051
C READ INPUT DATA                                                       RLEVX052
    1 FORMAT (I2,/,I1)                                                  RLEVX053
    2 FORMAT (45X,E20.11,/,(3E25.14))                                   RLEVX054
    3 FORMAT (E20.11,/,I1)                                              RLEVX055
    4 FORMAT (E20.11,/,F10.5)                                           RLEVX056
    5 FORMAT (8A10)                                                     RLEVX057
      READ (5,5) TITLE                                                  RLEVX058
      READ (5,1) NPT,NR1                                                RLEVX059
      IF (NR1.LE.0) GO TO 7                                             RLEVX060
      DO 6 I=1,NR1                                                      RLEVX061
    6 READ (5,2) E(I),(XA(J),EFP(J,I),EFQ(J,I),J=1,NPT)                 RLEVX062
    7 READ (5,3) S12,NR2                                                RLEVX063
      J=NR1+1  $  NN=NR1+NR2  $  IF (NR2.LE.0) GO TO 9                  RLEVX064
      DO 8 I=J,NN                                                       RLEVX065
    8 READ (5,2) E(I),(XA(J),EFP(J,I),EFQ(J,I),J=1,NPT)                 RLEVX066
    9 READ (5,4) S22,R                                                  RLEVX067
C                                                                       RLEVX068
C SCALE EIGENVALUES AND COMPUTE GAUSSIAN VARIANCES                      RLEVX069
      CALL SCALE(E,NR1,NR2,S12,S22,S1G2,S2G2,R,TITLE)                   RLEVX070
C                                                                       RLEVX071
C GENERATE COVARIANCE MATRICIES                                         RLEVX072
      IF (NN.LE.0) GO TO 57                                             RLEVX073
      DO 20 I=1,NN                                                      RLEVX074
      C1(I,I)=1.  $  C2(I,I)=1.  $  I1=I+1                              RLEVX075
      DO 20 J=I1,NN                                                     RLEVX076
      IF ((I-NR1)*(J-NR1)) 10,11,12                                     RLEVX077
   10 CALL INRPOT(C1(I,J),I,J,1)  $  C1(J,I)=C1(I,J)                    RLEVX078
      CALL INRPOT(C2(I,J),I,J,-1)  $  C2(J,I)=C2(I,J)                   RLEVX079
      GO TO 20                                                          RLEVX080
   11 IF (I.EQ.NR1) GO TO 10                                            RLEVX081
   12 C1(J,I)=0.  $  C1(I,J)=0.  $  C2(J,I)=0.  $  C2(I,J)=0.           RLEVX082
   20 CONTINUE                                                          RLEVX083
      WRITE (6,21) TITLE                                                RLEVX084
   21 FORMAT (1H1,5X,*COVARIANCE MATRIX FOR P VARIABLES*,/,6X,8A10)     RLEVX085
      DO 25 I=1,NN                                                      RLEVX086
   25 WRITE (6,30) (C1(I,J),J=1,NN)                                     RLEVX087
   30 FORMAT (1H0,5X,10F11.6)                                           RLEVX088
      WRITE (6,31) TITLE                                                RLEVX089
   31 FORMAT (1H0,5X,*COVARIANCE MATRIX FOR Q VARIABLES*,/,6X,8A10)     RLEVX090
      DO 35 I=1,NN                                                      RLEVX091
   35 WRITE (6,30) (C2(I,J),J=1,NN)                                     RLEVX092
C                                                                       RLEVX093
C GENERATE FUNCTIONAL DEPENDENCE MATRICIES                              RLEVX094
   36 IF (NR1.LE.0) GO TO 51                                            RLEVX095
      I=1                                                               RLEVX096
      WRITE (6,40) TITLE,I                                              RLEVX097
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   40 FORMAT (1H0,5X,*FUNCTIONAL DEPENDENCE MATRIX*,/,6X,8A10,/,6X,*VARIRLEVX098
     1ABLE NO.*,I2)                                                     RLEVX099
      DO 50 I=1,NR1                                                     RLEVX100
      C3(I,I)=E(I)  $  C4(I,I)=0.  $  I1=I+1                            RLEVX101
      DO 45 J=I1,NN                                                     RLEVX102
      C3(I,J)=0.  $  C3(J,I)=0.  $  C4(I,J)=0.  $  C4(J,I)=0.           RLEVX103
   45 CONTINUE                                                          RLEVX104
   50 WRITE (6,30) (C3(I,J),J=1,NR1)                                    RLEVX105
   51 IF (NR2.LE.0) GO TO 56                                            RLEVX106
      I=2  $  M1=NR1+1                                                  RLEVX107
      WRITE (6,40) TITLE,I                                              RLEVX108
      DO 55 I=M1,NN                                                     RLEVX109
      C3(I,I)=0.  $  C4(I,I)=E(I)  $  I1=I+1                            RLEVX110
      DO 53 J=I1,NN                                                     RLEVX111
      C3(I,J)=0.  $  C3(J,I)=0.  $  C4(I,J)=0.  $  C4(J,I)=0.           RLEVX112
   53 CONTINUE                                                          RLEVX113
   55 WRITE (6,30) (C4(I,J),J=M1,NN)                                    RLEVX114
      IF (NR1.LE.0) GO TO 56  $  ERRD=0.                                RLEVX115
      DO 65 I=1,NR1                                                     RLEVX116
      DO 65 J=M1,NN                                                     RLEVX117
   65 ERRD=ERRD+C1(I,J)*C2(I,J)*C3(I,I)*C4(J,J)                         RLEVX118
      ERRD=ERRD/SQRT(S12*S22)                                           RLEVX119
      WRITE (6,66) ERRD                                                 RLEVX120
   66 FORMAT (1H0,5X,*CORRELATION COEFFICIENT OF RESPONSE AND ITS FIRST RLEVX121
     1DERIVATIVE =*,E10.3,/,6X,*THIS COEFFICIENT SHOULD BE MUCH LESS THARLEVX122
     2N 1.0*)                                                           RLEVX123
C                                                                       RLEVX124
C GENERATE CHARACTERISTIC FUNCTION                                      RLEVX125
   56 CALL INVRS(ANS,F1,F2)                                             RLEVX126
      WRITE (6,54) DETA,DETB                                            RLEVX127
   54 FORMAT (1H0,5X,*DETERMINANTS OF COVARIANCE MATRICIES*,/,6X,*DETA =RLEVX128
     1*,E12.5,/,6X,*DETB =*,E12.5)                                      RLEVX129
   57 DX(1)=.3*SQRT(S12)  $  DX(2)=.3*SQRT(S22)                         RLEVX130
      M1=2**M(1)  $  M2=2**M(2)                                         RLEVX131
      DF(1)=1./DX(1)/M1  $  DF(2)=1./DX(2)/M2                           RLEVX132
      WRITE (6,102) DX,DF                                               RLEVX133
  102 FORMAT (1H0,5X,*INCREMENTS:*,/,6X,*DX(1) =*,E14.7,5X,*DX(2) =*,   RLEVX134
     1E14.7,/,6X,*DF(1) =*,E14.7,5X,*DF(2) =*,E14.7)                    RLEVX135
      DFDF=DF(1)*DF(2)                                                  RLEVX136
      S1G2=-S1G2*TWOPI2  $  S2G2=-S2G2*TWOPI2                           RLEVX137
      A(1,1)=CMPLX(1.,0.)  $  F1=DF(1)  $  F2=0.                        RLEVX138
      F11=FLOAT(1-M1)*DF(1)                                             RLEVX139
      DO 200 I=2,M1                                                     RLEVX140
      CALL CF2(ANS,F1,F2)  $  ANS=ANS*EXP(S1G2*F1**2)                   RLEVX141
      CALL CF2(ANS1,F11,F2)  $  ANS=ANS+ANS1*EXP(S1G2*F11**2)           RLEVX142
      A(I,1)=CMPLX(ANS,0.)  $  F11=F11+DF(1)                            RLEVX143
  200 F1=F1+DF(1)                                                       RLEVX144
      F1=0.0  $  F2=DF(2)  $  F22=FLOAT(1-M2)*DF(2)                     RLEVX145
      DO 300 J=2,M2                                                     RLEVX146
      CALL CF2(ANS,F1,F2)  $  ANS=ANS*EXP(S2G2*F2**2)                   RLEVX147
      CALL CF2(ANS1,F1,F22)  $  ANS=ANS+ANS1*EXP(S2G2*F22**2)           RLEVX148
      A(1,J)=CMPLX(ANS,0.)  $  F22=F22+DF(2)                            RLEVX149
  300 F2=F2+DF(2)                                                       RLEVX150
      F11=FLOAT(M1)*DF(1)  $  F1=0.  $  M22=M2/2+1                      RLEVX151
      DO 410 I=2,M1                                                     RLEVX152
      F1=F1+DF(1)  $  F11=F11-DF(1)                                     RLEVX153
      F2=0.  $  F22=FLOAT(M2)*DF(2)                                     RLEVX154
      DO 400 J=2,M22                                                    RLEVX155
      F2=-F2+DF(2)  $  CALL CF2(ANS,F1,F2)                              RLEVX156
      ANS=ANS*EXP(S1G2*F1**2+S2G2*F2**2)                                RLEVX157
      F2=-F2  $  CALL CF2(ANS1,F11,F2)                                  RLEVX158
      ANS=ANS+ANS1*EXP(S1G2*F11**2+S2G2*F2**2)                          RLEVX159
      F22=-F22+DF(2)  $  CALL CF2(ANS1,F1,F22)                          RLEVX160
```

```
      ANS=ANS+ANS1*EXP(S1G2*F1**2+S2G2*F22**2)
      F22=-F22  $  CALL CF2(ANS1,F11,F22)
      ANS=ANS+ANS1*EXP(S1G2*F11**2+S2G2*F22**2)
      A(I,J)=CMPLX(ANS,0.)
  400 A(M1+2-I,M2+2-J)=CMPLX(ANS,0.)
  410 CONTINUE
      WRITE (6,61) (A(J,1),J=1,M1)
      WRITE (6,62) (A(1,J),J=1,M2)
      WRITE (6,62) (A(J,J),J=1,M2)
   61 FORMAT (1H1,5X,*FIRST ROW, FIRST COLUMN, AND DIAGONAL OF JOINT CHA
     1RICTERISTIC FUNCTION*,/,(5X,8E10.3))
   62 FORMAT (1H ,/,(5X,8E10.3))
C
C FOURIER TRANSFORM TO OBTAIN PROBABILITY DENSITY
      DO 71 J=1,8192,2
   71 X(J)=X(J)*DFDF
      CALL FFT(X,M,INV,S,IFS,IFER)  $  IF (IFER.EQ.0) GO TO 73
      WRITE (6,72) IFER
   72 FORMAT (1H0,5X,*FATAL ERROR IN FFTRS2, IFER =*,I5)
      STOP
C
C INTEGRATE TO OBTAIN LEVEL CROSSINGS
   73 C1X=C11*DX(2)  $  C2X=C12*DX(2)  $  C3X=C13*DX(2)
      WRITE (6,74) (A(J,1),J=1,M1)
      WRITE (6,62) (A(1,J),J=1,M2)
      WRITE (6,62) (A(J,J),J=1,M2)
   74 FORMAT (1H1,5X,*FIRST ROW, FIRST COLUMN, AND DIAGONAL OF JOINT PRO
     1BABILITY DENSITY FUNCTION*,/,(5X,8E10.3))
      X2=0.  $  M21=M2/2+1  $  M11=M1/2+3
      DO 80 J=1,M21
      DO 75 I=1,M11
   75 A(I,J)=A(I,J)*X2
   80 X2=X2+DX(2)  $  M21=M2/2-2
      DO 86 I=1,M11
      SUM=C1X*(-A(I,3)+A(I,4)-A(I,2)+A(I,5))+C2X*(A(I,1)-A(I,2)+A(I,3)+
     1A(I,4))+C3X*(A(I,1)+2.*A(I,2)+A(I,3))
      DO 85 J=1,M21
   85 SUM=SUM+C1X*(A(I,J)+A(I,J+5))+C2X*(A(I,J+1)+A(I,J+4))+C3X*(A(I,J+2
     1)+A(I,J+3))
      X(I)=FLOAT(I-1)*DX(1)
   86 S(I)=REAL(SUM)
      WRITE (6,87) (X(I),S(I),I=1,M11)
   87 FORMAT (1H1,5X,*COMPUTED VALUES OF X AND N(X)*,/,(4X,2E15.7))
C
C INTERPOLATE TO FIND LEVEL CROSSINGS AT SPECIFIED VALUES OF X
      X1=0.  $  X2=0.  $  DIVISR=S(1)
      DX1=DXOS*SQRT(S12)  $  M21=M2/2+1  $  NINDEX=1
      WRITE (6,90) TITLE
   90 FORMAT (1H1,36X,*LEVEL CROSSINGS*,/,5X,8A10,/,/,16X,*DIMENSIONAL
     1NON-DIMENSIONAL  CROSSINGS PER  CROSSINGS PER*,/,19X,*LEVEL*,10X,
     2*LEVEL*,9X,*UNIT TIME*,4X,*ZERO CROSSING*,/,21X,*X*,10X,*X/SIGMA X
     3*,/)
      DO 105 I=1,NTAB
   91 IF (X(NINDEX)-X1) 92,99,93
   92 NINDEX=NINDEX+1  $  IF (NINDEX-M21) 91,91,150
   93 NINDEX=NINDEX-1  $  H=(X1-X(NINDEX))/DX(1)
      FI3=S(NINDEX)  $  FI4=S(NINDEX+1)  $  FI5=S(NINDEX+2)
      FI6=S(NINDEX+3)  $  IF (NINDEX-2) 94,96,97
   94 IF (NINDEX-1) 150,95,96
   95 FI1=FI5  $  FI2=FI4  $  GO TO 98
   96 FI1=FI3  $  FI2=S(1)  $  GO TO 98
   97 FI1=S(NINDEX-2)  $  FI2=S(NINDEX-1)
   98 F1=-FI1*H*(H**2-1.)*(H-2.)*(H-3.)/120.+FI2*H*(H-1.)*(H**2-4.)*
```

```
      1(H-3.)/24.-FI3*(H**2-1.)*(H**2-4.)*(H-3.)/12.+FI4*H*(H+1.)*(H**2
      2-4.)*(H-3.)/12.-FI5*H*(H**2-1.)*(H+2.)*(H-3.)/24.+FI6*H*(H**2-4.)
      3*(H**2-1.)/120.
       GO TO 100
   99 F1=S(NINDEX)
  100 F2=F1/DIVISR
      WRITE (6,120) X1,X2,F1,F2
      IF (F1.LE.1.E-11) GO TO 150
      X1=X1+DX1
  105 X2=X2+DXOS
  120 FORMAT (1H ,15X,E10.3,5X,E10.3,6X,E10.3,5X,E10.3)
  150 STOP $ END


      SUBROUTINE SCALE(E,N1,N2,S12,S22,S1G2,S2G2,R,TITLE)
C
C SUBROUTINE TO SCALE EIGENVALUES AND CALCULATE GAUSSIAN VARIANCES
C
C E = EIGENVALUE ARRAY
C N1 = NO. OF EIGENVALUES DESCRIBING FIRST VARIABLE
C N2 = NO. OF EIGENVALUES DESCRIBING SECOND VARIABLE
C S12 = VARIANCE OF 1ST VARIABLE IN RESPONSE TO UNIT VARIANCE INPUT
C S22 = VARIANCE OF 2ND VARIABLE IN RESPONSE TO UNIT VARIANCE INPUT
C S1G2 = GAUSSIAN VARIANCE OF 1ST VARIABLE
C S2G2 = GAUSSIAN VARIANCE OF 2ND VARIABLE
C
      DIMENSION E(1),TITLE(8)
C
      S=S12 $ N=N1 $ N0=1
      DO 100 I=1,2
      SE2=0. $ IF (N.GT.N0) GO TO 9 $ S2G2=S
      IF (I.EQ.1) S1G2=S2G2 $ GO TO 60
    9 DO 10 J=N0,N
   10 SE2=SE2+E(J)**2
      IF (SE2.LE.S)GO TO 40 $ IF (SE2.LE.S*1.05)GO TO 30
      WRITE (6,15) SE2,S,I,N $ STOP
   15 FORMAT (1H0,5X,*FATAL ERROR IN SCALE, SE2 EXCEEDS S BY MORE THAN 5
     1 PERCENT*,/,6X,*SE2 =*,E15.8,/,6X,*S =*,E15.8,/,6X,*I =*,I2,/,6X,*
     2N =*,I2)
   30 SE2=SQRT(S/SE2)
      DO 35 J=N0,N
   35 E(J)=E(J)*SE2 $ SE2=S
   40 S2G2=S-SE2*(R**2)/(1.+R**2)
      IF (I.EQ.1) S1G2=S2G2 $ SE2=(R**2)/(1.+R**2) $ SE2=SQRT(SE2)
      DO 50 J=N0,N
   50 E(J)=E(J)*SE2
   60 N0=N1+1 $ N=N1+N2
  100 S=S22
      IF (N.LE.0) E(1)=0.
      WRITE (6,110) TITLE $ J=1 $ WRITE (6,115) J,S1G2,(E(I),I=1,N1)
  110 FORMAT (1H1,5X,*RESULTS OF SCALING*,/,6X,8A10)
      J=2 $ WRITE (6,115) J,S2G2,(E(I),I=N0,N) $ WRITE (6,120) R
  115 FORMAT (1H0,5X,*VARIABLE NO.*,I2,/,6X,*GAUSSIAN VARIANCE INCLUDING
     1 CORRECTION FOR NEGLECTED EIGENVALUES =*,E12.5,/,6X,*SCALED EIGENV
     2ALUES*,/,(10X,E12.5))
  120 FORMAT (1H0,5X,*SIGMA RATIO OF TURBULENCE MODEL =*,F6.3)
      SE2=0.
      DO 125 I=1,N1
  125 SE2=SE2+E(I)**2
      S=SE2+S1G2 $ I=1
      WRITE (6,130) I,S12,SE2,S1G2,S
```

```
130 FORMAT (1H0,5X,*VARIANCE CHECK, VARIABLE NO.*,I2,/,6X,*CORRECT TOT
   1AL VARIANCE =*,E12.5,/,6X,*SUM CF SCALED EIGENVALUES SQUARED =*,E1
   22.5,/,6X,*GAUSSIAN VARIANCE =*,E12.5,/,6X,*TOTAL VARIANCE =*,E12.5
   3)
    SE2=0.
    DO 135 I=NO,N
135 SE2=SE2+E(I)**2  $  S=SE2+S2G2  $  I=2
    WRITE (6,130) I,S22,SE2,S2G2,S
140 RETURN  $  END
```

APPENDIX B

TABULATED FUNCTIONS

The purpose of this appendix is to present tabulated values of certain results which were presented in graphic form within the main body of this report. Specific results to be presented are:

Table B1 - standardized probability densities of the non-gaussian turbulence model for various values of the probability distribution parameter R

Table B2 - probability distribution functions of the non-gaussian turbulence model for various values of the probability distribution parameter R

Table B3 - universal level crossing frequencies of the non-gaussian turbulence model longitudinal component for various values of the probability distribution parameter R

Table B4 - universal level crossing frequencies of the non-gaussian turbulence model vertical and lateral components for various values of the probability distribution parameter R.

Table B1.--Standardized probability densities of the non-gaussian turbulence model for various values of the probability distribution parameter $\bar{R}$. (Graphical presentation of most of those data is given in figure 15 on page 61 of this report.)

$\hat{p}(x|R)$

x	R = 0.0	0.5	0.75	1.0	1.33	2.0
0.0	3.989E-01	4.073E-01	4.234E-01	4.455E-01	4.792E-01	5.478E-01
.2	3.910E-01	3.986E-01	4.132E-01	4.330E-01	4.623E-01	5.182E-01
.4	3.683E-01	3.738E-01	3.843E-01	3.976E-01	4.156E-01	4.409E-01
.6	3.332E-01	3.359E-01	3.407E-01	3.457E-01	3.496E-01	3.434E-01
.8	2.897E-01	2.894E-01	2.884E-01	2.854E-01	2.774E-01	2.514E-01
1.0	2.420E-01	2.391E-01	2.335E-01	2.248E-01	2.096E-01	1.786E-01
1.2	1.942E-01	1.898E-01	1.814E-01	1.699E-01	1.530E-01	1.266E-01
1.4	1.497E-01	1.447E-01	1.356E-01	1.241E-01	1.093E-01	9.100E-02
1.6	1.109E-01	1.063E-01	9.802E-02	8.838E-02	7.748E-02	6.662E-02
1.8	7.895E-02	7.527E-02	6.885E-02	6.194E-02	5.510E-02	4.953E-02
2.0	5.399E-02	5.153E-02	4.728E-02	4.309E-02	3.952E-02	3.724E-02
2.2	3.547E-02	3.420E-02	3.195E-02	2.997E-02	2.865E-02	2.822E-02
2.4	2.239E-02	2.208E-02	2.138E-02	2.094E-02	2.098E-02	2.151E-02
2.6	1.358E-02	1.392E-02	1.425E-02	1.474E-02	1.549E-02	1.648E-02
2.8	7.915E-03	8.603E-03	9.511E-03	1.046E-02	1.150E-02	1.266E-02
3.0	4.432E-03	5.240E-03	6.373E-03	7.476E-03	8.585E-03	9.764E-03
3.2	2.384E-03	3.160E-03	4.295E-03	5.375E-03	6.432E-03	7.548E-03
3.4	1.232E-03	1.896E-03	2.912E-03	3.884E-03	4.834E-03	5.848E-03
3.6	6.119E-04	1.135E-03	1.987E-03	2.817E-03	3.643E-03	4.539E-03
3.8	2.919E-04	6.812E-04	1.362E-03	2.050E-03	2.751E-03	3.530E-03
4.0	1.338E-04	4.103E-04	9.382E-04	1.495E-03	2.081E-03	2.749E-03

Table B2.--Probability distribution functions of the non-gaussian turbulence model for various values of the probability distribution parameter R.

(Graphical presentation of most of these data is given in figure 15 on page 61 of this report.)

$P(-\sigma x|R)^{\dagger}$

x	R = 0.0	0.5	0.75	1.0	1.33	2.0
0.0	5.000E-01	5.000E-01	5.000E-01	5.000E-01	5.000E-01	5.000E-01
.2	4.207E-01	4.191E-01	4.160E-01	4.117E-01	4.053E-01	3.924E-01
.4	3.446E-01	3.416E-01	3.360E-01	3.284E-01	3.171E-01	2.959E-01
.6	2.743E-01	2.705E-01	2.633E-01	2.538E-01	2.404E-01	2.174E-01
.8	2.119E-01	2.078E-01	2.003E-01	1.906E-01	1.776E-01	1.582E-01
1.0	1.587E-01	1.550E-01	1.481E-01	1.396E-01	1.291E 01	1.156E-01
1.2	1.151E-01	1.121E-01	1.067E-01	1.003E-01	9.305E-02	8.537E-02
1.4	8,076E-02	7.878E-02	7.511E-02	7.108E-02	6.704E-02	6.384E-02
1.6	5.480E-02	5.380E-02	5.189E-02	5.000E-02	4.854E-02	4.823E-02
1.8	3.593E-02	3.578E-02	3.535E-02	3.511E-02	3.542E-02	3.671E-02
2.0	2.275E-02	2.322E-02	2.385E-02	2.472E-02	2.605E-02	2.810E-02
2.2	1.390E-02	1.474E-02	1.602E-02	1.750E-02	1.930E-02	2.160E-02
2.4	8.198E-03	9.188E-03	1.075E-02	1.246E-02	1.438E-02	1.666E-02
2.6	4.661E-03	5.643E-03	7.237E-03	8.931E-03	1.076E-02	1.288E-02
2.8	2.555E-03	3.429E-03	4.892E-03	6.436E-03	8.085E-03	9.987E-03
3.0	1.350E-03	2.069E-03	3.324E-03	4.659E-03	6.091E-03	7.757E-03
3.2	6.871E-04	1.245E-03	2.271E-03	3.386E-03	4.599E-03	6.036E-03
3.4	3.369E-04	7.491E-04	1.559E-03	2.468E-03	3.480E-03	4.704E-03
3.6	1.591E-04	4.517E-04	1.075E-03	1.803E-03	2.638E-03	3.671E-03
3.8	7.235E-05	2.735E-04	7.433E-04	1.321E-03	2.003E-03	2.868E-03
4.0	3.167E-05	1.663E-04	5.156E-04	9.687E-04	1.523E-03	2.243E-03
4.2	1.335E-05	1.016E-04	3.585E-04	7.117E-04	1.159E-03	1.756E-03
4.4	5.412E-06	6.230E-05	2.498E-04	5.236E-04	8.830E-04	1.376E-03
4.6	2.112E-06	3.840E-05	1.744E-04	3.857E-04	6.735E-04	1.079E-03
4.8	7.933E-07	2.370E-05	1.219E-04	2.844E-04	5.141E-04	8.472E-04
5.0	2.867E-07	1.470E-05	8.530E-05	2.100E-04	3.928E-04	6.655E-04

† $P(\sigma x) = 1 - P(-\sigma x)$

Table B3.--Universal level crossing frequencies of the non-gaussian turbulence model longitudinal component for various values of the probability distribution parameter R. (Graphical presentation of these data is given in figure 16 on page 74 of this report.)

$\hat{N}_u(x|R)$

x	R = 0.0	0.5	0.75	1.0	1.33	2.0
0.0	1.591E-01	1.597E-01	1.606E-01	1.618E-01	1.637E-01	1.674E-01
.2	1.506E-01	1.563E-01	1.569E-01	1.576E-01	1.587E-01	1.603E-01
.4	1.469E-01	1.468E-01	1.465E-01	1.459E-01	1.448E-01	1.417E-01
.6	1.329E-01	1.322E-01	1.307E-01	1.285E-01	1.249E-01	1.172E-01
.8	1.156E-01	1.142E-01	1.116E-01	1.081E-01	1.027E-01	9.300E-02
1.0	9.653E-02	9.481E-02	9.152E-02	8.721E-02	8.126E-02	7.221E-02
1.2	7.747E-02	7.564E-02	7.221E-02	6.796E-02	6.262E-02	5.595E-02
1.4	5.973E-02	5.809E-02	5.506E-02	5.153E-02	4.755E-02	4.355E-02
1.6	4.425E-02	4.301E-02	4.075E-02	3.831E-02	3.591E-02	3.415E-02
1.8	3.150E-02	3.077E-02	2.944E-02	2.815E-02	2.717E-02	2.691E-02
2.0	2.154E-02	2.133E-02	2.088E-02	2.057E-02	2.063E-02	2.128E-02
2.2	1.415E-02	1.436E-02	1.462E-02	1.503E-02	1.575E-02	1.687E-02
2.4	8.934E-03	9.432E-03	1.017E-02	1.102E-02	1.207E-02	1.339E-02
2.6	5.417E-03	6.063E-03	7.051E-03	8.111E-03	9.281E-03	1.064E-02
2.8	3.157E-03	3.833E-03	4.896E-03	5.995E-03	7.154E-03	8.465E-03
3.0	1.768E-03	2.395E-03	3.409E-03	4.446E-03	5.524E-03	6.738E-03
3.2	9.508E-04	1.484E-03	2.383E-03	3.307E-03	4.271E-03	5.367E-03
3.4	4.916E-04	9.164E-04	1.672E-03	2.465E-03	3.306E-03	4.276E-03
3.6	2.441E-04	5.654E-04	1.177E-03	1.841E-03	2.560E-03	3.409E-03
3.8	1.166E-04	3.496E-04	8.310E-04	1.376E-03	1.984E-03	2.718E-03
4.0	5.354E-05	2.168E-04	5.880E-04	1.030E-03	1.539E-03	2.168E-03
4.2	2.351E-05	1.348E-04	4.167E-04	7.714E-04	1.194E-03	1.730E-03
4.4	1.004E-05	8.425E-05	2.959E-04	5.782E-04	9.265E-04	1.380E-03
4.6	4.090E-06	5.278E-05	2.102E-04	4.336E-04	7.192E-04	1.102E-03
4.8	1.580E-06	3.315E-05	1.495E-04	3.253E-04	5.585E-04	8.793E-04
5.0	6.109E-07	2.089E-05	1.064E-04	2.442E-04	4.338E-04	7.020E-04
5.2	2.206E-07	1.318E-05	7.581E-05	1.833E-04	3.370E-04	5 605E-04
5.4	7.410E-08	8.328E-06	5.402E-05	1.377E-04	2.619E-04	4.476E-04
5.6	2.668E-08	5.274E-06	3.852E-05	1.034E-04	2.035E-04	3.574E-04
5.8	8.484E-09	3.341E-06	2.748E-05	7.773E-05	1.582E-04	2.855E-04
6.0	2.424E-09	2.117E-06	1.960E-05	5.842E-05	1.230E-04	2.281E-04
6.2	8.545E-10	1.344E-06	1.399E-05	4.392E-05	9.563E-05	1.822E-04
6.4	2.359E-10	8.534E-07	9.990E-06	3.302E-05	7.436E-05	1.456E-04
6.6	5.528E-11	5.419E-07	7.134E-06	2.483E-05	5.784E-05	1.163E-04
6.8	2.059E-11	3.446E-07	5.096E-06	1.868E-05	4.500E-05	9.301E-05
7.0	4.792E-12	2.191E-07	3.641E-06	1.405E-05	3.502E-05	7.439E-05

Table B4.--Universal level crossing frequencies of the non-gaussian turbulence model vertical and lateral components for various values of the probability parameter R.

(Graphical presentation of these data is given in figure 17 on page 80 of this report.)

$\hat{N}_w(x|R)$, $\hat{N}_v(x|R)$

x	R					
	0.0	0.5	0.75	1.0	1.33	2.0
0.0	1.949E-01	1.955E-01	1.965E-01	1.979E-01	2.000E-01	2.042E-01
.2	1.911E-01	1.914E-01	1.920E-01	1.928E-01	1.938E-01	1.954E-01
.4	1.799E-01	1.797E-01	1.792E-01	1.784E-01	1.768E-01	1.728E-01
.6	1.628E-01	1.618E-01	1.599E-01	1.571E-01	1.525E-01	1.429E-01
.8	1.415E-01	1.399E-01	1.366E-01	1.321E-01	1.254E-01	1.134E-01
1.0	1.182E-01	1.161E-01	1.120E-01	1.066E-01	9.924E-02	8.806E-02
1.2	9.488E-02	9.260E-02	8.834E-01	8.308E-02	7.647E-02	6.825E-02
1.4	7.316E-02	7.112E-02	6.736E-02	6.299E-02	5.807E-02	5.313E-02
1.6	5.420E-02	5.266E-02	4.985E-02	4.682E-02	4.386E-02	4.167E-02
1.8	3.857E-02	3.767E-02	3.601E-02	3.440E-02	3.318E-02	3.284E-02
2.0	2.638E-02	2.611E-02	2.554E-02	2.514E-02	2.520E-02	2.598E-02
2.2	1.733E-02	1.758E-02	1.788E-02	1.838E-02	1.923E-02	2.059E-02
2.4	1.094E-02	1.155E-02	1.244E-02	1.347E-02	4.474E-02	1.635E-02
2.6	6.635E-03	7.421E-03	8.624E-03	9.914E-03	1.134E-02	1.300E-02
2.8	3.866E-03	4.692E-03	5.988E-03	7.328E-03	8.742E-03	1.034E-02
3.0	2.165E-03	2.931E-03	4.170E-03	5.435E-03	6.751E-03	8.231E-03
3.2	1.164E-03	1.816E-03	2.914E-03	4.043E-03	5.220E-03	6.556E-03
3.4	6.021E-04	1.122E-03	2.045E-03	3.014E-03	4.040E-03	5.225E-03
3.6	2.990E-04	6.920E-04	1.439E-03	2.251E-03	3.130E-03	4.165E-03
3.8	1.429E-04	4.279E-04	1.016E-03	1.683E-03	2.426E-03	3.322E-03
4.0	6.557E-05	2.653E-04	7.192E-04	1.259E-03	1.881E-03	2.650E-03
4.2	2.880E-05	1.650E-04	5.098E-04	9.434E-04	1.459E-03	2.114E-03
4.4	1.229E-05	1.031E-04	3.619E-04	7.071E-04	1.133E-03	1.687E-03
4.6	5.009E-06	6.460E-05	2.572E-04	5.303E-04	8.794E-04	1.347E-03
4.8	1.935E-06	4.057E-05	1.829E-04	3.978E-04	6.829E-04	1.075E-03
5.0	7.482E-07	2.557E-05	1.302E-04	2.986E-04	5.304E-04	8.582E-04
5.2	2.702E-07	1.614E-05	9.275E-05	2.242E-04	4.121E-04	6.852E-04
5.4	9.075E-08	1.019E-05	6.610E-05	1.684E-04	3.202E-04	5.472E-04
5.6	3.267E-08	6.455E-06	4.713E-05	1.265E-04	2.489E-04	4.370E-04
5.8	1.039E-08	4.089E-06	3.362E-05	9.508E-05	1.935E-04	3.491E-04
6.0	2.968E-09	2.591E-06	2.399E-05	7.146E-05	1.504E-04	2.789E-04
6.2	1.047E-09	1.645E-06	1.712E-05	5.373E-05	1.170E-04	2.228E-04
6.4	2.890E-10	1.044E-06	1.222E-05	4.040E-05	9.095E-05	1.780E-04
6.6	6.774E-11	6.633E-06	8.729E-06	3.038E-05	7.074E-05	1.423E-04
6.8	2.526E-11	4.218E-07	6.236E-06	2.285E-05	5.504E-05	1.137E-04
7.0	5.911E-12	2.682E-07	4.455E-06	1.719E-05	4.283E-05	9.098E-05

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