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THE EFFECT OF REMOTE ZONES

ON THE ACCURACY OF EVALUATING

THE MOLODENSKY INTEGRAL

BY

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16. Abstract

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THE EFFECT OF REMOTE ZONES ON THE ACCURACY OF EVALUATING THE MOLODENSKY INTEGRAL

James J. Buglia

An analytical method is presented for determining the error in computing the gravity anomaly from the Molodensky integral if geoidal undulation data in remote zones are neglected. The error is given in terms of the usual degree variance and a set of parameters which are functions only of the size of the area in which undulation data are taken. A numerical calculation using Kaula's degree variances indicates that it is necessary to integrate the Molodensky integral over approximately a hemisphere in order to reduce the effect of neglected data to the order of 1 milligal in the computation of gravity anomaly.

INTRODUCTION

The Molodensky integral, which relates the gravity anomaly at a specific point to a global distribution of geoidal undulation measurements, has long been of considerable theoretical use to the geodetic community. This equation has recently begun receiving renewed practical attention because direct geoidal measurements are now available, through altimetry data obtained from the GEOS-3 spacecraft, thus permitting the direct calculation of the anomalous gravity field.

Theoretically, the use of the Molodensky equation to compute gravity anomalies requires a continuous, global distribution of geoidal undulation measurements. This ideal situation is, of course, never realized in practice. The integral must be replaced by sums over finite areas, and interpolation and/or extrapolation of measured data are generally required to produce a sufficiently dense network of data.

It is thus of current interest to inquire if one can, as in the case of the Stokes equation, replace the global integration by an integration over a small spherical cap centered at the computation point. Two related questions are thereby raised: What is the expected error incurred in the computation of the gravity anomaly by neglecting undulation measurements in the remote zone outside a spherical cap of specified size; and, how large should the spherical cap be in order to contain the error within a specified accuracy?

A study which addresses these questions has been made and the results are the subject of this paper. The approach used is similar to that taken for addressing these questions for the related Stokes problem. An expression for the mean-square error incurred in neglecting data beyond a spherical cap of given size is derived in terms of a set of universal functions of only the cap size, and of the degree variances associated with the gravity field.

An illustration is provided indicating the mean-square error of omission for two different sets of degree variances, one using Kaula's rule of thumb, and the other for the GEM-4 gravity model. It is shown that a cap size of approximately one hundred degrees in radius is required to reduce the mean-square of omission to about 1 milligal.

ANALYSIS

The Molodensky integral (Molodensky, et al., 1962; Pick et al, 1973) is generally written as

$$\Delta g_p = - \frac{\gamma N_p}{R_e} - \frac{\gamma}{2\pi} \iint_S \frac{N - N_p}{r_p^3} \, d\sigma \qquad (1)$$

in which the reference figure to which all quantities are referred is an ellipsoid of revolution with equatorial radius, a, and flattening, f. Δg_p and N_p are the gravity anomaly and geoidal undulation, respectively, at the point p, γ and R_e are mean values of the acceleration of gravity and radius of the reference ellipsoid, and $d\sigma$ is the differential area element. N is the variable geoidal undulation associated with $d\sigma$ and is measured along the normal to the reference ellipsoid. r_p is the chord distance between point p and $d\sigma$, and the subscript S on the integral denotes a global integration.

Equation (1) relates measured geoidal undulations about the reference ellipsoid, N and N_p, to gravity anomalies calculated with respect to the ellipsoid. Hence, all quantities in (1) should properly be computed on the ellipsoid. The chord distance, r_p, and the differential area, $d\sigma$, are in this case rather complex functions of position on the ellipsoidal surface. Fortunately, however, the Earth is very nearly spherical in shape with a flattening of approximately 1/300. Consequently, the expressions for r_p and $d\sigma$ can be replaced by their spherical approximations with an acceptably small loss in accuracy. Thus, if Ψ is defined as the central angle between the point p and the area element $d\sigma$, then

$$r_p \approx 2 R_e \sin \frac{\Psi}{2}$$

Further, if p is taken as the pole of a spherical coordinate system, and α is the azimuth angle of $d\sigma$ referred to any convenient meridian through p, the area element $d\sigma$ can be written

$$d\sigma \approx R_e^2 \sin \Psi d\Psi d\alpha$$

As a final preliminary, a new function $f(\Psi)$ is defined by

$$f(\Psi) = \frac{1}{\sin^3 \frac{\Psi}{2}}$$

such that

$$\frac{1}{r_p^{\,3}} = \frac{f(\Psi)}{8\,R_e^{\,3}} \tag{2}$$

Equation (1) is now written as the sum of two parts - an integral over a spherical cap of angular radius Ψ_o centered at the point p, and a second integral covering the rest of the globe.

$$\Delta g_p = -\frac{\gamma N_p}{R_e} - \frac{\gamma}{16\pi R_e}\int_{\Psi=0}^{\Psi_o}\int_{\alpha=0}^{2\pi} f(\Psi)(N - N_p)\sin\Psi\, d\Psi\, d\alpha$$

$$\qquad - \frac{\gamma}{16\pi\, R_e}\int_{\Psi=\Psi_o}^{\pi}\int_{\alpha=0}^{2\pi} f(\Psi)(N - N_p)\sin\Psi\, d\Psi\, d\alpha \tag{3}$$

The second integral of (3) gives the error that would be incurred in the computation of Δg_p by neglecting the undulation data in the region beyond the spherical cap of radius Ψ_o. We call this integral ε_p, and refer to it as the "error of omission." It is the evaluation of ε_p, and, more importantly, the global RMS value of ε_p, which is the topic of the following analysis.

Introducing a new function $\overline{f}(\Psi)$ with the following properties:

$$\overline{f}(\Psi) = 0 \text{ for } \Psi \leq \Psi_o$$

$$\overline{f}(\Psi) = f(\Psi) \text{ for } \Psi > \Psi_o \quad ,$$

the expression for ε_p becomes

$$\varepsilon_p = -\frac{\gamma}{16\pi R_e} \int_{\Psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \overline{f}(\Psi)(N - N_p) \sin\Psi \, d\Psi \, d\alpha \qquad (4)$$

where now the integration covers the entire globe.

Since $\overline{f}(\Psi)$ is piecewise continuous, it can be expanded in a series of Legendre polynomials (Macmillan, 1958, p 386; Heiskanen and Moritz, 1967, p 28).

$$\bar{f}(\Psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} R_n P_n(\cos\Psi) \qquad (5)$$

where the constants are defined by $[(2n + 1)/2]R_n$ to simplify the algebra later, and $P_n(\cos\Psi)$ are the Legendre polynomials with argument $\cos\Psi$. The constants are evaluated by utilizing the orthogonality relations for the Legendre polynomials (Heiskanen and Moritz, 1967, p. 30).

$$\frac{2n+1}{2} R_n = \frac{2n+1}{4\pi} \int_{\Psi=0}^{\pi} \int_{\alpha=0}^{2\pi} \bar{f}(\Psi) P_n(\cos\Psi) \sin\Psi \, d\Psi \, d\alpha \qquad (6)$$

or

$$R_n = \int_{\Psi=0}^{\pi} \bar{f}(\Psi) P_n(\cos\Psi) \sin\Psi \, d\Psi$$

$$= \int_{\Psi=\Psi_o}^{\pi} f(\Psi) P_n(\cos\Psi) \sin\Psi \, d\Psi \qquad (7)$$

Substitute (5) into (4), interchange the order of summation and integration, and expand the integrals,

$$\epsilon_p = - \frac{\gamma}{16\pi R_e} \sum_{n=0}^{\infty} \frac{2n+1}{2} R_n \left[\int_{\Psi=0}^{\pi} \int_{\alpha=0}^{2\pi} N\, P_n(\cos\Psi)\sin\Psi \, d\Psi \, d\alpha - N_p \int_{\Psi=0}^{\pi} \int_{\alpha=0}^{2\pi} P_n(\cos\Psi)\sin\Psi \, d\Psi \, d\alpha \right] \tag{8}$$

N can also be expanded in a series of surface harmonics

$$N = \sum_{n=2}^{\infty} N_n$$

Where the N_n are given by equations similar to eq (6) of this text

$$N_n = \frac{2n+1}{4\pi} \int_{\Psi=0}^{\pi} \int_{\alpha=0}^{2\pi} N\, P_n(\cos\Psi)\sin\Psi \, d\Psi \, d\alpha$$

Hence, the first integral of (8) is simply

$$\frac{4\pi N_n}{2n+1}$$

The second integral of (8) is also easily evaluated by recalling that $P_o(\cos\Psi) = 1$, and by making use of the orthogonality relations

$$\int_{\Psi=0}^{\pi}\int_{\alpha=0}^{2\pi} P_n(\cos\Psi)P_m(\cos\Psi)\sin\Psi \, d\Psi \, d\alpha = 0 \quad , n \neq m$$

$$= \frac{4\pi}{2n+1} \, , n = m$$

and, hence, the second integral is equal to 4π.

Eq (8) is written in final form, then, as

$$\varepsilon_p = \frac{\gamma}{8R_e}\left(R_o N_p - \sum_{n=2}^{\infty} R_n N_n\right) \tag{10}$$

Eq (10) gives the error in the calculation of the gravity anomaly Δg_p at a specific point p, caused by neglecting the undulation data beyond a spherical cap of angular radius Ψ_o centered at p. Thus, ε_p is a function of the coordinates of p.

While this is of considerable interest in its own right, a more useful parameter is one which has global applicability and is not identified with a specific point.

One such parameter is the global RMS values of ε_p, and is found by squaring (10) and evaluating the mean of the result of the entire globe.

By definition, $M\left\{\varepsilon_p^{\,2}\right\}$ is the value of $\varepsilon_p^{\,2}$ integrated over the unit sphere, divided by the area of the unit sphere, or

$$M\left\{\varepsilon_p^{\,2}\right\} = \frac{1}{4\pi} \iint_S \varepsilon_p^{\,2}\, d\sigma \tag{11}$$

If we write $N_p = \sum_{n=2}^{\infty} N_n$, then ε_p can be written

$$\varepsilon_p = \frac{\gamma}{8R_e} \sum_{n=2}^{\infty} (R_o - R_n)\, N_n$$

and thus (11) is

$$M\left\{\varepsilon_p^{\,2}\right\} = \frac{\gamma^2}{64R_e^{\,2}} \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} (R_o - R_n)(R_o - R_m) \left[\frac{1}{4\pi} \iint_S N_n N_m \, d\sigma\right] \tag{12}$$

The bracketed term of (12) is the mean value of the product of two Laplace harmonics. This integral vanishes unless $n = m$. Heiskanen and Moritz

(1967) gives the following relations (pp. 256 and 259), and show that the mean value of the product of two gravity anomaly harmonics can be written in terms of the degree variances - i.e.

$$M\left\{\Delta g_n\ \Delta g_m\right\} = 0 \quad , \quad n \neq m$$

$$= c_n \quad , \quad n = m \tag{13}$$

Also (p. 97), the general undulation harmonic can be written as a function of the gravity anomaly harmonic as

$$N_n = \frac{R_e}{\gamma} \quad \frac{\Delta g_n}{n-1} \tag{14}$$

Using (13) and (14), then, we can write

$$M\left\{N_n\ N_m\right\} = 0 \quad , \quad n \neq m$$

$$= \frac{R_e^{\ 2}}{\gamma^2} \quad \frac{c_n}{(n-1)^2} \quad , \quad n = m$$

and (12) can be written in final form

$$M\left\{\varepsilon_p^{\,2}\right\} = \frac{1}{64} \sum_{n=2}^{n} (R_o - R_n)^2 \; \frac{c_n}{(n-1)^2} \qquad (15)$$

This is the desired equation and expresses the mean square value of the error of omission in Δg_p, in terms of only the degree variances, c_n, and the R_n terms, which are functions only of the cap size, Ψ_o.

NUMERICAL RESULTS

Expressions for R_n can be found from a direct integration of eq (7) for any Ψ_o. R_o and R_1 are thus found to be

$$R_o = 4 \left(\frac{1 - \Gamma}{\Gamma} \right)$$

$$R_1 = R_o - 8 (1 - \Gamma)$$

Where $\Gamma = \sin(\Psi_o/2)$. For larger n, the direct integration of (7) becomes cumbersome. However, Meissl (1971) has derived a recursion formula for Molodensky-type kernals. With a slight modification of notation this formula is

$$R_n = \frac{2\,[\,P_{n-2}\,(\cos\Psi_o) - P_n\,(\cos\Psi_o)\,]}{(n - 1/2)\,\sin(\Psi_o/2)} + 2R_{n-1} - R_{n-2}$$

values of R_n are presented in table I for $n = 0$ to 20, and for $\Psi_o = 5°$ to $180°$ in 5-degree increments. R_o through R_6 are plotted against Ψ_o in figure 1. All of the R_n tend towards zero as Ψ_o goes to 180°, naturally, since the error of omission is zero if one integrates eq (1) over the entire globe. All of the R_n display a rather heavily damped oscillatory character, with R_n crossing the zero-axis n times. For moderate to large values of Ψ_o, the magnitude of R_n decreases rapidly as n increases, indicating that the high frequency components of the gravity field contribute little to the error of omission for these cap sizes. For small values of Ψ_o the contribution of the higher frequency terms appears to be more significant. However, the degree variances themselves drop off quite rapidly with increasing n. For example, using the well-known Kaula rule of thumb (Kaula, 1962), one can write

$$c_n \approx \frac{96\,(n-1)^2\,(2n+1)}{n^4}$$

for the Earth's gravity field, and it can be seen that the c_n decreases approximately as $1/n$. It has been found that, for Ψ_o greater than about 10°, seven or eight terms of (15) are sufficient for convergence to less than 1% error in $M\left\{\varepsilon_p^{\,2}\right\}$.

For illustrative purposes, eq (15) was evaluated for two sets of degree variances -- the Kaula approximation above, and a set of degree variances derived from the GEM-4 gravity field (Lerch, et al., 1972). These results are displayed in fig. 2, where the quantity $\sqrt{M\left\{\varepsilon_p^{\,2}\right\}}$ is plotted for a range of Ψ_o. The general trend of the curves is that expected, namely, that the error of omission drops off rather rapidly with increasing size of the spherical cap, asymptotically approaching zero error as Ψ_o goes to 180°. Both sets of degree variances yield quantitatively similar results, indicating that a cap size of the order of 90° - 100° is necessary to reduce the error of omission to the order of 1 milligal.

CONCLUSIONS

A rather simple formula has been derived which predicts the mean-square error incurred in the computation of gravity anomaly by the Molodensky formula when one uses only measured data in a small spherical cap surrounding the computation point, and neglects data outside this cap. If one uses representative degree variances for the Earth it is found that integration of geoidal undulation data over approximately a hemisphere centered at the computation point is required to yield the gravity anomaly at the computation point to an accuracy of 1 milligal. It is evident that rather extensive interpolation and/or extrapolation of measured data would i general be necessary to provide a sufficiently dense network of data for direct utilization of the Molodensky integral.

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ψ_0	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}
5	87.70234	80.05130	72.74854	65.79276	59.18195	52.91350	46.98414	41.38996	36.12646	31.18854	26.57052
10	41.89485	34.59210	27.98129	22.05193	16.78843	12.17041	8.17296	4.76704	1.91990	-.40447	-2.24474
15	26.64519	19.68940	13.76003	8.82210	4.82465	1.70242	-.62196	-2.23540	-3.23160	-3.70808	-3.76321
20	19.03508	12.42427	7.16075	3.16328	.31609	-1.52422	-2.52045	-2.84562	-2.67363	-2.17038	-1.48597
25	14.48091	8.21242	3.59434	.47203	-1.36834	-2.18046	-2.23568	-1.80049	-1.11612	-.38230	.25328
30	11.45481	5.52537	1.52777	-.79679	-1.79321	-1.84854	-1.34545	-.62016	.06881	.55724	.77582
35	9.30204	3.70768	.30145	-1.31238	-1.63794	-1.20303	-.47787	.18637	.58811	.67209	.49730
40	7.69522	2.43138	-.41637	-1.41328	-1.24195	-.55803	.13073	.53245	.57379	.34641	.02133
45	6.45250	1.51397	-.81144	-1.28908	-.78681	-.05356	.43470	.51877	.29150	-.03320	-.25710
50	5.46431	.84575	-.99621	-1.05310	-.36998	.26505	.48370	.30926	-.01555	-.23939	-.24763
55	4.66272	.35671	-1.04291	-.77550	-.03947	.40605	.36731	.05811	-.20053	-.23052	-.07295
60	4.00000	-.00000	-1.00000	-.50000	.18750	.40625	.17969	-.12891	-.22583	-.09167	.09073
65	3.44464	-.25697	-.90108	-.25304	.31323	.31426	-.00214	-.20636	-.13515	.05528	.13857
70	2.97379	-.43760	-.76999	-.04929	.35183	.17881	-.12909	-.18133	-.00720	.12552	.07571
75	2.57072	-.55919	-.62382	.10491	.32398	.04105	-.18196	-.09373	.08795	.10244	-.02204
80	2.22290	-.63480	-.47488	.20906	.25232	-.06976	-.16668	.00662	.11543	.02544	-.07810
85	1.92075	-.67453	-.33192	.26674	.15883	-.13772	-.10565	.07923	.08036	-.04707	-.06493
90	1.65685	-.68529	-.20101	.28427	.06245	-.15938	-.02765	.10408	.01483	-.07441	-.00897
95	1.42537	-.67641	-.08612	.26955	-.02224	-.14118	.04101	.08319	-.04301	-.05141	.04060
100	1.22163	-.65002	.01043	.23118	-.08565	-.09591	.08276	.03470	-.06747	-.00282	.05046
105	1.04189	-.61128	.03762	.17776	-.12318	-.03890	.09136	-.01719	-.05405	.03738	.02217
110	.88310	-.56369	.14547	.11726	-.13470	.01543	.07114	-.05216	-.01696	.04723	-.01639
115	.74276	-.51011	.18485	.05663	-.12368	.05610	.03365	-.06026	.02130	.02691	-.03575
120	.61880	-.45299	.20726	.00149	-.09603	.07708	-.00690	-.04353	.04205	-.00578	-.02575
125	.50953	-.39439	.21467	-.04408	-.05888	.07755	-.03810	-.01296	.03850	-.02907	.00100
130	.41351	-.33603	.20941	-.07754	-.01943	.06112	-.05250	.01719	.01676	-.03075	.02227
135	.32957	-.27939	.19403	-.09790	.01605	.03434	-.04884	.03539	-.00933	-.01368	.02371
140	.25671	-.22575	.17118	-.10554	.04294	.00487	-.03144	.03673	-.02626	.00856	.00782
145	.19412	-.17615	.14350	-.10200	.05870	-.02037	-.00793	.02372	-.02739	.02169	-.01082
150	.14110	-.13149	.11356	-.08969	.06295	-.03658	.01352	.00408	-.01509	.01948	-.01825
155	.09712	-.09251	.08374	-.07161	.05724	-.04188	.02681	-.01322	.00200	.00625	-.01128
160	.06171	-.05983	.05620	-.05102	.04460	-.03732	.02959	-.02182	.01442	-.00774	.00205
165	.03452	-.03393	.03276	-.03107	.02891	-.02634	.02345	-.02035	.01711	-.01385	.01066
170	.01528	-.01516	.01493	-.01459	.01414	-.01360	.01296	-.01224	.01145	-.01060	.00970
175	.00381	-.00380	.00379	-.00377	.00374	-.00370	.00366	-.00361	.00356	-.00349	.00343
180	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	.00000	.00000	.00000	.00000	.00000

TABLE I(a) R_n coefficients for ψ_0 = 5 to 180 degrees

n = 0 to 10

ψ_0	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	R_{16}	R_{17}	R_{18}	R_{19}	R_{20}
5	22.26617	18.26873	14.57092	11.16501	8.04278	5.19564	2.61456	.29020	-1.78713	-3.62740
10	-3.64176	-4.63795	-5.27671	-5.60184	-5.65696	-5.48493	-5.12735	-4.62407	-4.01269	-3.32825
15	-3.49331	-2.99004	-2.33794	-1.61250	-.87863	-.18957	.41362	.90208	1.25871	1.47728
20	-.74642	-.05937	.50818	.90996	1.12853	1.15978	1.05803	.83047	.53129	.20605
25	.69913	.91771	.91783	.74297	.45741	.13213	-.16744	-.39146	-.51082	-.51897
30	.73662	.50911	.19137	-.11794	-.34195	-.43889	-.40542	-.27078	-.08457	.09827
35	.18750	-.12178	-.32644	-.37848	-.28918	-.11403	.07294	.20615	.24827	.19804
40	-.23758	-.33451	-.26282	-.08772	.09506	.20449	.20513	.11411	-.01464	-.11896
45	-.28703	-.15252	.03850	.17165	.18597	.09499	-.03391	-.12419	-.13227	-.06562
50	-.08994	.09271	.17610	.12595	.00070	-.10355	-.12133	-.05471	.03902	.09326
55	.10415	.16017	.07832	-.04963	-.11487	-.07746	.01567	.08186	.07264	.00633
60	.14674	.05598	-.06740	-.10458	-.03808	.05233	.07918	.02781	-.04205	-.06255
65	.05693	-.06632	-.09382	-.01637	.06433	.06223	-.00634	-.05743	-.03891	.01914
70	-.05181	-.08396	-.01163	.06460	.04857	-.02375	-.05521	-.01414	.03823	.03608
75	-.08705	-.02091	.05956	.04358	-.02877	-.04922	.00125	.04214	.01871	-.02735
80	-.04111	.04373	.04673	-.02529	-.04572	.00698	.04035	.00662	-.03229	-.01583
85	.02721	.05394	-.01402	-.04533	.00486	.03816	.00164	-.03195	-.00628	.02646
90	.05646	.00589	-.04468	-.00410	.03647	.00299	-.03049	-.00226	.02598	.00175
95	.03183	-.03689	-.01883	.03282	.00978	-.02875	-.00331	.02481	-.00136	-.02107
100	-.01417	-.03445	.02235	.02040	-.02485	-.00874	.02360	-.00036	-.01997	.00687
105	-.03366	.00114	.02996	-.01521	-.01713	.02079	.00428	-.01960	.00587	.01397
110	-.02558	.02850	.00244	-.02440	.01336	.01170	-.01857	.00195	.01456	-.01099
115	.00666	.02257	-.02253	-.00076	.01894	-.01410	-.00480	.01565	-.00824	-.00693
120	.02724	-.00450	-.01742	.01944	-.00357	-.01276	.01475	-.00290	-.00986	.01168
125	.02091	-.02221	.00645	.01111	-.01687	.00858	.00502	-.01256	.00910	.00105
130	-.00209	-.01461	.01876	-.00995	-.00363	.01248	-.01173	.00341	.00591	-.01000
135	-.01924	.00591	.00767	-.01454	.01247	-.00413	-.00503	.01006	-.00890	.00309
140	-.01700	.01712	-.01012	.00027	.00786	-.01124	.00926	-.00361	-.00275	.00701
145	-.00076	.00947	-.01329	.01207	-.00714	.00073	.00486	-.00796	.00794	-.00527
150	.01309	-.00603	-.00097	.00635	-.00920	.00931	-.00712	.00354	.00039	-.00366
155	.01325	-.01260	.00999	-.00622	.00211	.00164	-.00450	.00613	-.00646	.00562
160	.00246	-.00568	.00761	-.00834	.00802	-.00689	.00518	-.00317	.00111	.00079
165	-.00762	.00482	-.00233	.00018	.00158	-.00293	.00387	-.00443	.00462	-.00450
170	-.00877	.00782	-.00686	.00590	-.00496	.00405	-.00317	.00235	-.00158	.00088
175	-.00335	.00327	-.00319	.00310	-.00300	.00291	-.00280	.00270	-.00259	.00248
180	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

Table I(b) R_n coefficients for ψ_0 = 5 to 180 degrees

n = 11 to 20

ψ_0	R_{21}	R_{22}	R_{23}	R_{24}	R_{25}	R_{26}	R_{27}	R_{28}	R_{29}	R_{30}
5	-5.24082	-6.63785	-7.82907	-8.82526	-9.63727	-10.27603	-10.75250	-11.07762	-11.26232	-11.31743
10	-2.60281	-1.85524	-1.14097	-.45191	.18372	.75128	1.23974	1.64152	1.95232	2.17089
15	1.56121	1.52197	1.37739	1.14983	.86424	.54645	.22147	-.08788	-.36219	-.58621
20	-.10330	-.36231	-.54668	-.64362	-.65177	-.58001	-.44535	-.27015	-.07910	.10376
25	-.42964	-.27170	-.08230	.10056	.24457	.32801	.34231	.29206	.19285	.06743
30	.23150	.28755	.26179	.17073	.04533	-.07835	-.16869	-.20591	-.18577	-.11906
35	.08540	-.04275	-.14040	-.17761	-.14861	-.07084	.02353	.10010	.13411	.11794
40	-.15617	-.11868	-.03256	.05805	.11232	.11017	.05867	-.01401	-.07321	-.09373
45	.02870	.09494	.10000	.04852	-.02429	-.07550	-.07892	-.03763	.02075	.06185
50	.07711	.01075	-.05475	-.07525	-.04217	.01604	.05713	.05494	.01536	-.03096
55	-.05618	-.06530	-.02050	.03559	.05641	.02910	-.01899	-.04668	-.03353	.00574
60	-.02135	.03470	.05099	.01701	-.02925	-.04258	-.01393	.02507	.03625	.01167
65	.04859	.02134	-.02582	-.03915	-.00808	.02843	.02981	-.00176	-.02825	-.02101
70	-.01037	-.03797	-.01501	.02390	.02856	-.00297	-.02742	-.01500	.01502	.02327
75	-.02916	.00991	.03029	.00583	-.02401	-.01680	.01330	.02154	-.00148	-.02019
80	.02292	.02107	-.01335	-.02284	.00446	.02175	.00307	-.01852	-.00880	.01386
85	.00954	-.02154	-.01175	.01711	.01313	-.01312	-.01384	.00953	.01400	-.00634
90	-.02247	-.00139	.01969	.00113	-.01743	-.00093	.01558	.00077	-.01403	-.00065
95	.00473	.01754	-.00709	-.01425	.00868	.01121	-.00965	-.00843	.01011	.00591
100	.01501	-.01093	-.00953	.01278	.00421	-.01276	.00048	.01128	-.00419	-.00880
105	-.01138	-.00535	.01351	-.00103	-.01145	.00656	.00700	-.00933	-.00169	.00927
110	-.00553	.01313	-.00363	-.00921	.00918	.00224	-.00965	.00428	.00590	-.00772
115	.01261	-.00398	-.00787	.00978	-.00085	-.00802	.00719	.00141	-.00761	.00486
120	-.00240	-.00791	.00954	-.00203	-.00652	.00799	-.00175	-.00550	.00681	-.00152
125	-.00899	.00375	-.00157	-.00602	.00790	-.00323	-.00355	.00676	-.00419	-.00153
130	.00690	.00039	-.00652	.00754	-.00338	-.00259	.00618	-.00522	.00082	.00372
135	.00362	-.00748	.00676	-.00242	-.00276	.00584	-.00536	.00196	.00219	-.00472
140	-.00765	.00487	-.00031	-.00385	.00583	-.00502	.00208	.00148	-.00403	.00454
145	.00119	.00276	-.00529	.00572	-.00415	.00135	.00162	-.00373	.00434	-.00341
150	.00558	-.00586	.00463	-.00239	-.00020	.00245	-.00384	.00411	-.00331	.00175
155	-.00392	.00175	.00045	-.00231	.00355	-.00402	.00374	-.00282	.00149	-.00002
160	-.00234	.00342	-.00398	.00403	-.00361	.00283	-.00181	.00069	.00040	-.00134
165	.00411	-.00350	.00275	-.00192	.00105	-.00022	-.00054	.00119	-.00169	.00204
170	-.00025	-.00031	.00079	-.00119	.00151	-.00175	.00191	-.00201	.00203	-.00200
175	-.00236	.00225	-.00213	.00201	-.00189	.00177	-.00165	.00153	-.00141	.00130
180	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

Table I(c) R_n coefficients for ψ_0 = 5 to 180 degrees

n = 21 to 30

ψ_0	R_{31}	R_{32}	R_{33}	R_{34}	R_{35}	R_{36}	R_{37}	R_{38}	R_{39}	R_{40}
5	-11.25366	-11.08165	-10.81174	-10.45416	-10.01888	-9.51559	-8.95371	-8.34234	-7.69023	-7.00576
10	2.29879	2.34003	2.30078	2.18902	2.01413	1.78657	1.51745	1.21827	.90047	.57522
15	-.74948	-.84842	-.87630	-.84281	-.75347	-.61880	-.45146	-.26522	-.07404	.10882
20	.25768	.36714	.42320	.42382	.37356	.28249	.16455	.03573	-.08796	-.19233
25	-.05897	-.16333	-.22864	-.24642	-.21740	-.15069	-.06141	.03243	.11340	.16769
30	-.02741	.06323	.12951	.15638	.14020	.08868	.01799	-.05205	-.10329	-.12382
35	.06194	-.01075	-.07335	-.10492	-.09706	-.05575	.00193	.05454	.08407	.08187
40	-.07004	-.01683	.03934	.07295	.07076	.03669	-.01135	-.05057	-.06393	-.04729
45	.06428	.03022	-.01792	-.05185	-.05364	-.02493	.01565	.04427	.04564	.02101
50	-.05173	-.03510	.00420	.03727	.04193	.01731	-.01732	-.03734	-.03007	-.00254
55	.03667	.03478	.00455	-.02684	-.03358	-.01218	.01756	.03057	.01741	-.00914
60	-.02180	-.03134	-.00995	.01918	.02744	.00861	-.01704	-.02428	-.00754	.01527
65	.00876	.02516	.01303	-.01343	-.02279	-.00606	.01609	.01863	.00022	-.01709
70	.00146	-.02021	-.01448	.00906	.01918	.00421	-.01492	-.01366	.00484	.01585
75	-.00355	.01417	.01475	-.00569	-.01634	-.00284	.01365	.00938	-.00798	-.01266
80	.01252	-.00350	-.01421	.00309	.01405	.00181	-.01235	-.00576	.00951	.00850
85	-.01373	.00352	.01310	-.00107	-.01219	-.00102	.01106	.00277	-.00977	-.00419
90	.01272	.00056	-.01160	-.00048	.01064	.00042	-.00980	-.00037	.00907	.00032
95	-.01016	-.00366	.00989	.00167	-.00935	.00005	.00860	-.00150	-.00770	.00270
100	.00673	.00575	-.00806	-.00257	.00825	-.00041	-.00746	.00288	.00594	-.00467
105	-.00307	-.00691	.00623	.00322	-.00731	.00069	.00639	-.00383	-.00400	.00555
110	-.00030	.00722	-.00446	-.00367	.00650	-.00090	-.00539	.00439	.00210	-.00545
115	.00297	-.00682	.00282	.00396	-.00579	.00107	.00447	-.00462	-.00037	.00459
120	-.00472	.00590	-.00134	-.00411	.00517	-.00119	-.00362	.00458	-.00107	-.00322
125	.00547	-.00461	.00007	.00414	-.00462	.00128	.00284	-.00432	.00214	.00163
130	-.00530	.00313	.00098	-.00407	.00412	-.00134	-.00214	.00388	-.00283	-.00009
135	.00438	-.00163	-.00180	.00392	-.00367	.00138	.00151	-.00332	.00313	-.00119
140	-.00296	.00024	.00237	-.00370	.00326	-.00139	-.00095	.00269	-.00306	.00206
145	.00139	.00093	-.00272	.00342	-.00288	.00138	.00048	-.00232	.00275	-.00247
150	.00012	-.00178	.00284	-.00309	.00252	-.00135	-.00006	.00137	-.00222	.00243
155	-.00131	.00229	-.00277	.00271	-.00218	.00129	-.00024	-.00077	.00156	-.00202
160	.00203	-.00243	.00252	-.00230	.00184	-.00121	.00048	.00025	-.00089	.00138
165	-.00222	.00224	-.00212	.00186	-.00151	.00108	-.00062	.00016	.00029	-.00068
170	.00191	-.00177	.00159	-.00139	.00116	-.00091	.00066	-.00041	.00016	.00007
175	-.00116	.00107	-.00095	.00085	-.00074	.00064	-.00054	.00044	-.00035	.00026
180	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

Table I(d) R_n coefficients for ψ_0 = 5 to 180 degrees

n = 31 to 40

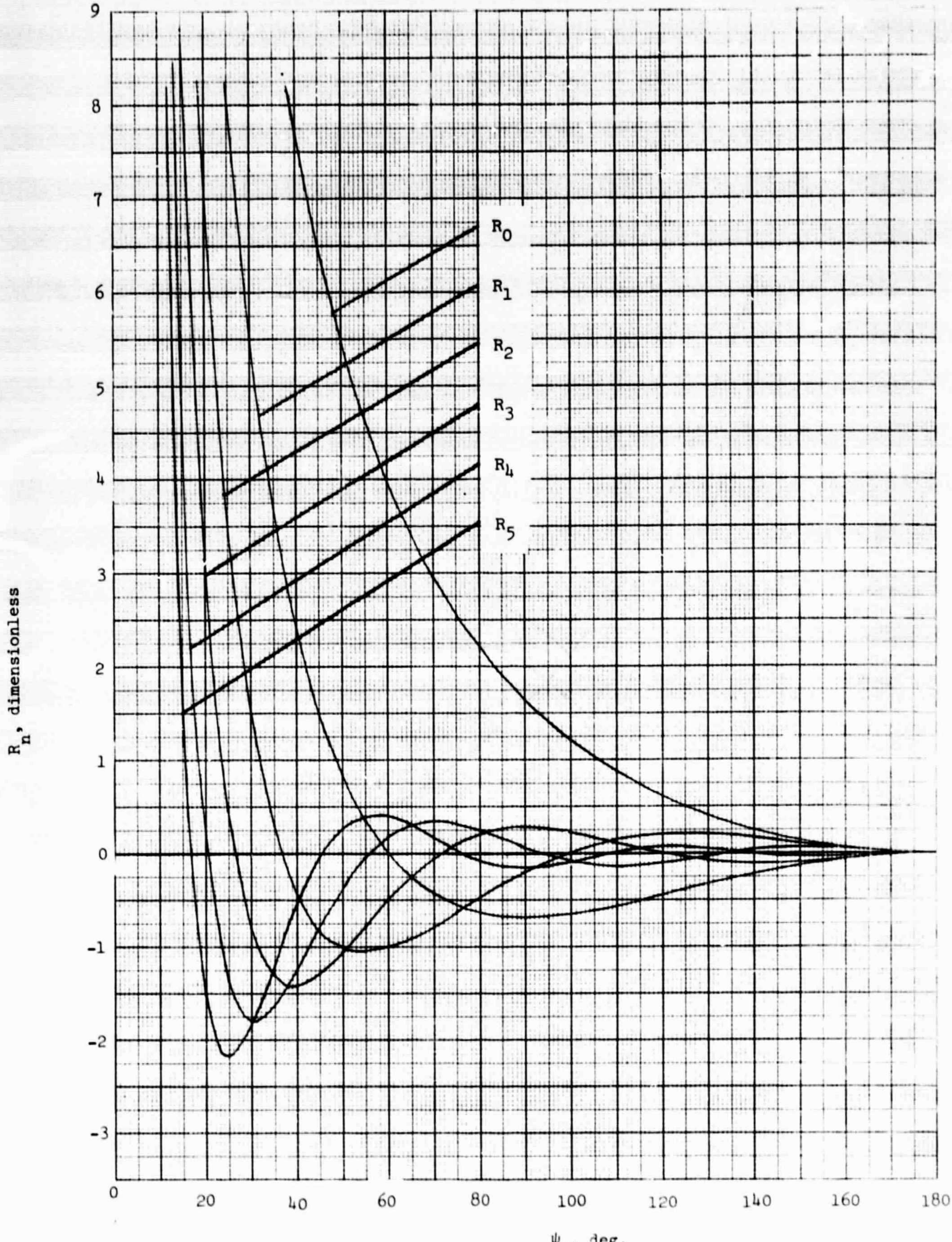

Figure 1. Dimensionless R_n coefficients plotted against ψ_0 for n = 0 to 5

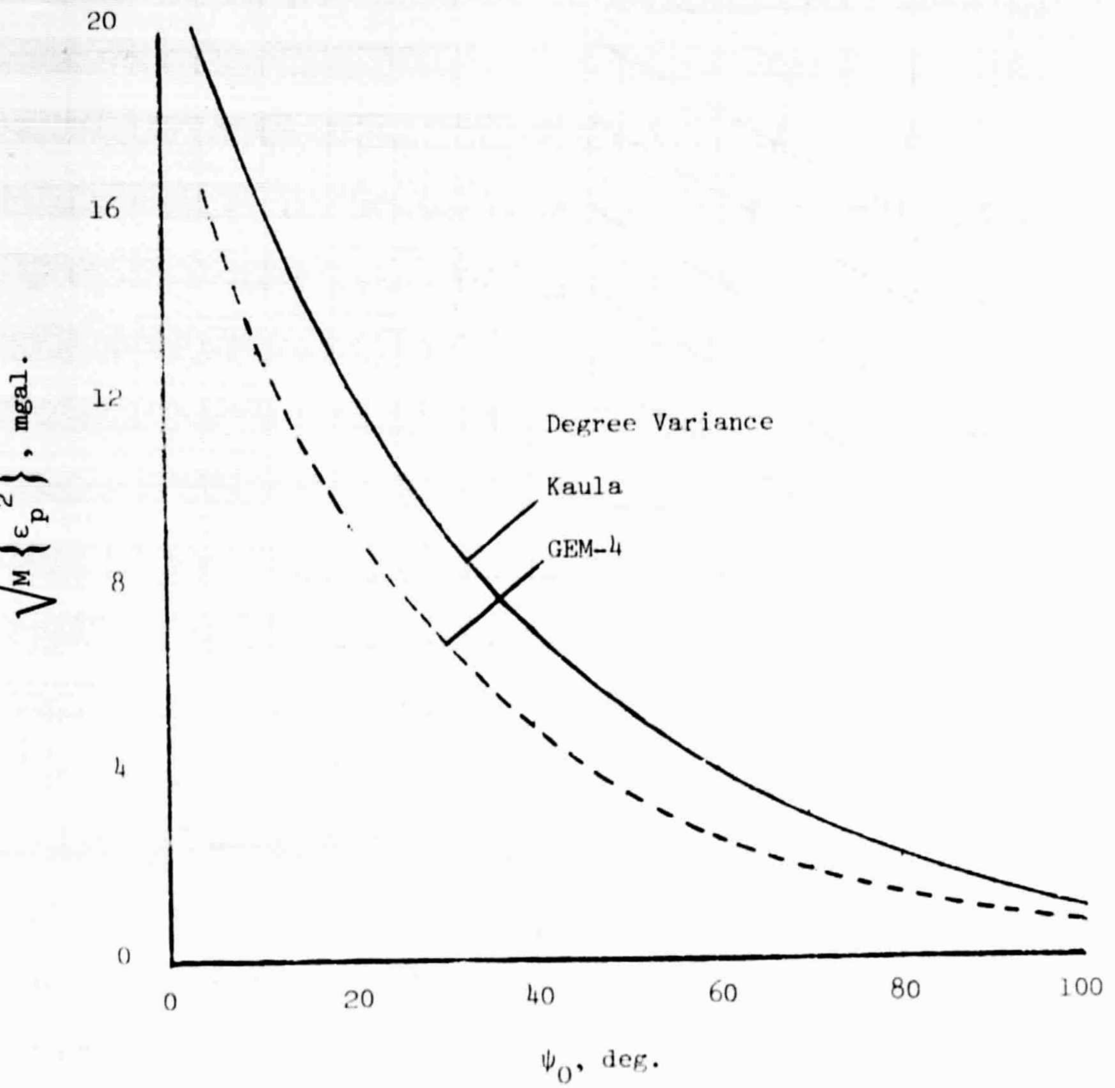

Figure 2. Mean value of error of omission in gravity anomaly as a function of distance from the computation point for two sets of degree variances.