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OPTIMUM DESIGN CONSIDERATIONS OF A GUST ALLEVIATOR FOR AIRCRAFT

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16. Abstract

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Application of the analysis was illustrated by a short take-off and landing (STOL) airplane in flight through turbulent air. Effects of varying the noise characteristics of the vane sensor and of a weighting matrix in the performance-index function have been determined.

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SUMMARY

Optimum design considerations of a gust alleviation system for aircraft flying in turbulent air are presented in this paper. A vane sensor (with noise) was used to measure vertical gusts, and elevators and flaps were used to reduce the root-mean-square (rms) value of the normal accelerations associated with the aircraft response to gusts. Since turbulence has stochastic properties, stochastic control theory was used in the analysis. A performance-index function involving normal acceleration and control deflections with a weighting matrix was used.

Application of the analysis was illustrated by a short take-off and landing (STOL) airplane in flight through turbulent air. Effects of varying the noise characteristics of the vane sensor and of the weighting matrix in the performance-index function were determined.

Stochastic control theory was applied to calculate the rms response of the airplane to turbulence. The calculations showed that a weighting number appearing in the performance index should be less than 10. A value of 3 was selected for subsequent calculations. Normal acceleration could be reduced by 92 percent when the intensity of the measurement noise had a very small value. However, the filter gains were large, and large control deflection angles were required. Good alleviation was calculated when the intensity of the measurement noise was about 3.6 percent of the vane deflection angle. Normal acceleration was reduced by 63 percent for moderate values of the gains.

INTRODUCTION

One of the purposes of gust alleviation systems for aircraft is the reduction of normal accelerations caused by gusts. Analyses of such systems have made use of various mathematical techniques to represent the gust spectra and to evaluate system response. For example, reference 1 describes atmospheric turbulence as sinusoids having unit amplitude and varying frequencies. The system was analyzed by using the amplitude and phase angle of the airplane response to the turbulence as a function of gust frequency. In references 2 and 3, atmospheric turbulence was assumed to have a von Kármán power spectral density function. Stochastic control theory was used to evaluate the root-mean-square (rms) responses of the airplane.

Since a turbulent atmosphere is stochastic, the application of stochastic control theory to gust alleviation problems appears to be natural. Reference 4 gives the first results obtained in the application of stochastic control theory to flight control problems and emphasizes the problem of stability derivative identification during flight in turbulence. Results concerning gust alleviation were primarily qualitative.

In this study, stochastic control theory (see ref. 5) is applied directly to the optimum design of a gust alleviation system for aircraft. Atmospheric turbulence, in this study, is assumed to be a random process characterized by a Dryden power spectral density function. An angle-of-attack vane was mounted ahead of the wing. The deflection angle of the vane was measured and the resulting signal, which had superimposed noise, was used to actuate a wing flap and the elevator. The vane signal and the control deflections were the inputs to a Kalman-Bucy filter. This filter gives the best estimate of the state of the system. Linear combinations of the estimated state variables were used to actuate the controls. The performance of the system was evaluated by the percent reduction of the normal acceleration of the airplane. Quantitative results are presented to show the influence of two unknown parameters on the system performance. One parameter is a weighting number appearing in the performance index; the second parameter is the intensity of the measurement noise.

SYMBOLS

A	system matrix, 4×4
$a_Z(t)$	normal acceleration, m/sec^2
$a_0, a_1, \gamma_1, \gamma_2$	parameters used in appendix A
B	input matrix, 4×2
C	measurement matrix, 1×4
C_m	pitching-moment coefficient, $\dfrac{\text{Pitching moment}}{q_\infty S_W \bar{c}}$
C_Z	Z-force coefficient, $\dfrac{\text{Force in Z-direction}}{q_\infty S_W}$
$\bar{c}$	wing mean aerodynamic chord, m

D	controlled variable matrix, 1×4
E	controlled variable matrix, 1×2
$\overrightarrow{e(t)}$	error vector, 4×1
F	feedback gain matrix, 2×4
G	noise gain matrix, 4×1
g	standard free-fall acceleration, 9.80665 m/sec^2
H	matrix solution of equation (18)
$H(\omega)$	frequency response function
I	identity matrix, 2×2
i	$= \sqrt{-1}$
J	performance index
K	Kalman-Bucy filter gain matrix, 4×1
k_Y	radius of gyration about Y-axis, m
L	scale of turbulence, m
ℓ_V	vane distance forward of airplane center of gravity, m
M	mass, kg
M_q	$= \frac{\partial C_m}{\partial q} \frac{q_\infty S_w \bar{c}}{M k_Y^2}$, $m\text{-}N\text{-}sec^2/rad^2\text{-}kg\text{-}m^2$
M_α	$= \frac{\partial C_m}{\partial \alpha} \frac{q_\infty S_w \bar{c}}{M k_Y^2}$, $m\text{-}N/rad\text{-}kg\text{-}m^2$

M_{δ_e} $= \frac{\partial C_m}{\partial \delta_e} \frac{q_\infty S_w \bar{c}}{M k_Y^2}$, m-N/rad-kg-m^2

M_{δ_f} $= \frac{\partial C_m}{\partial \delta_f} \frac{q_\infty S_w \bar{c}}{M k_Y^2}$, m-N/rad-kg-m^2

$N_I(t)$ input white-noise process, rad/sec^2

$N_O(t)$ measurement white-noise process, rad

$n_Z(t)$ normal acceleration, $a_Z(t)/g$

P matrix solution of equation (14), 4×4

Q,R,S matrices defined by equation (11), 4×4, 2×2, 2×2, respectively

q(t) pitch rate, rad/sec

q_∞ dynamic pressure, N/m^2

R_1 diagonal weighting matrix, 2×2

r_1, r_2 diagonal elements of matrix R_1

S_w wing area, m^2

t time, sec

$\overrightarrow{u(t)}$ input vector, 2×1

V airspeed, m/sec

V_1 matrix defined following equation (18), 4×4

v_I constant intensity of input noise process, $N_I(t)$, m^2/sec^5

v_O constant intensity of measurement noise process, $N_O(t)$, rad^2

$w_g(t)$ vertical component of gust velocity (positive upward), m/sec

X,Y,Z	body axes
$\overrightarrow{x(t)}$	state vector, 4×1
$\overrightarrow{\hat{x}(t)}$	estimate of $\overrightarrow{x(t)}$, 4×1
Z_α	$= \frac{\partial C_Z}{\partial \alpha} \frac{q_\infty S_W}{MV}$, N-sec/rad-kg-m
Z_{δ_e}	$= \frac{\partial C_Z}{\partial \delta_e} \frac{q_\infty S_W}{MV}$, N-sec/rad-kg-m
Z_{δ_f}	$= \frac{\partial C_Z}{\partial \delta_f} \frac{q_\infty S_W}{MV}$, N-sec/rad-kg-m
$\alpha(t)$	angle of attack, rad
β	scalar multiplier
$\delta_e(t)$	elevator deflection angle, rad
$\delta_f(t)$	flap deflection angle, rad
$\delta_v(t)$	vane deflection angle, rad
$\eta(t)$	variable appearing in equation (4), rad/sec
λ	variable of polynomial, rad/sec
$\lambda_{1,2,3,4}$	zeros of polynomial, rad/sec
$\xi(t)$	variable appearing in equation (4), rad
$\sigma_{w_g}{}^2$	variance of w_g, m^2/sec^2
σ_{n_Z}	rms normal acceleration
σ_q	rms pitch rate, rad/sec
σ_α	rms angle of attack, rad

σ_{δ_e}	rms elevator deflection angle, rad
σ_{δ_f}	rms flap deflection angle, rad
σ_{δ_v}	rms vane deflection angle, rad
τ	time increment, sec
$\Phi_{N_I(t)}(\omega)$	power spectral density function for input noise process $\left(=v_I\right)$, m^2/sec^5
$\Phi_{N_O(t)}$	power spectral density function for measurement noise process $\left(=v_O\right)$, rad^2
$\Phi_{w_g(t)}(\omega)$	power spectral density function for w_g, rad^2/sec
ω	circular frequency, rad/sec
$\| \; \|$	absolute value of a quantity
$[\;]$	rectangular matrix
$E\{\;\}$	expected value

Subscripts:

e	elevator
f	flap
g	gust
I	input quantity
max	maximum value
O	measurement quantity
v	vane

w wing

∞ free-stream value

Superscripts:

T transpose

$\longrightarrow$ vector

$\hat{}$ estimate

$\tilde{}$ transformed quantity

Dot over a quantity denotes derivative with respect to time.

THEORY

Optimum design considerations of a gust alleviation system for an airplane cruising in turbulent air are studied in this paper. The elevator and a wing flap (that is, a trailing-edge control capable of positive and negative deflection angles) were used as controls to produce the force and moment necessary for reduction of normal acceleration caused by gusts. A measured vane deflection angle actuated the elevator and flap. A quadratic performance index should be minimized. Since atmospheric turbulence is random, the performance index was the variance of the normal acceleration and the control deflection angles. In addition, the airplane with the alleviation system in operation must be stable. Stochastic control theory was applied to obtain the desired alleviation system.

Mathematical Model of Airplane Motion

Linear equations of longitudinal motion that approximate the short period mode were used in this study. The frame of reference for the airplane motion is the system of body axes illustrated in figure 1. The equations are similar to those given in reference 3. The resulting differential equations follow:

$$\left.\begin{aligned} \dot{\alpha}(t) &= Z_\alpha \alpha(t) + q(t) + \frac{1}{V} Z_\alpha w_g(t) + Z_{\delta_e}\delta_e(t) + Z_{\delta_f}\delta_f(t) \\ \dot{q}(t) &= M_\alpha \alpha(t) + M_q q(t) + \frac{1}{V} M_\alpha w_g(t) + M_{\delta_e}\delta_e(t) + M_{\delta_f}\delta_f(t) \end{aligned}\right\} \tag{1}$$

For simplicity, the controls $\delta_e(t)$ and $\delta_f(t)$ were assumed to respond instantaneously when actuated. The influence of turbulence on the airplane motion was the vertical gust velocity $w_g(t)$. Since $w_g(t)$ is a random variable, it is only known statistically. Its mathematical representation as a function of time is discussed in a later subsection.

Vane Deflection Angle

The vane, located ahead of the airplane wing, is shown in figure 2. It is positioned so that it is not appreciably influenced by the flow field generated by the fuselage and wing. The sign convention used is also shown in figure 2. The angular displacement of the vane is represented by the following equation:

$$\delta_V(t) = -\alpha(t) + \frac{\ell_V}{V} q(t) - \frac{1}{V} w_g(t) + N_O(t) \qquad (2)$$

The response of the vane was considered to be instantaneous.

In equation (2), the term $N_O(t)$ has been included to account for the noise (uncertainty) in the measurement of $\delta_V(t)$. The noise $N_O(t)$ is assumed to be a stationary white-noise process having intensity v_O and zero mean. Additional properties of $N_O(t)$ necessary for this study are the covariance function

$$E\left\{N_O(t)\, N_O(\tau)\right\} = v_O \delta(t - \tau)$$

where $\delta(t - \tau)$ is a Dirac-delta function; and the power spectral density function for $N_O(t)$

$$\Phi_{N_O(t)}(\omega) = v_O$$

which is constant for all frequencies ω. Further discussion of white-noise processes may be found in references 6 and 7.

Normal Acceleration

Normal acceleration at the airplane center of gravity is

$$a_Z(t) = -V\left[q(t) - \dot{\alpha}(t)\right]$$

Using the first of equations (1), the normal acceleration is expressed by the equation,

$$a_Z(t) = V\left[Z_\alpha \alpha(t) + \frac{1}{V} Z_\alpha w_g(t) + Z_{\delta_e}\delta_e(t) + Z_{\delta_f}\delta_f(t)\right]$$

For convenience, the normal acceleration $a_Z(t)$ is nondimensionalized by dividing by the acceleration of gravity g and is called normal acceleration $n_Z(t)$ where

$$n_Z(t) = \frac{a_Z(t)}{g}$$

and

$$n_Z(t) = \frac{V}{g} Z_\alpha \alpha(t) + \frac{1}{g} Z_\alpha w_g(t) + \frac{V}{g} Z_{\delta_e} \delta_e(t) + \frac{V}{g} Z_{\delta_f} \delta_f(t) \tag{3}$$

Mathematical Model of Atmospheric Turbulence

Atmospheric turbulence is assumed to have the following characteristics:

(a) It is one-dimensional (vertical gusts);

(b) The vertical gust velocity $w_g(t)$, a random variable, has a normal distribution function and a zero mean;

(c) Its power spectral density may be adequately represented by the Dryden function:

$$\Phi_{w_g}(\omega) = \frac{L\sigma_{w_g}^2}{V} \frac{1 + \frac{3L^2}{V^2}\omega^2}{\left(1 + \frac{L^2}{V^2}\omega^2\right)^2}$$

(d) It is stationary, homogeneous, and isotropic.

The random variable $w_g(t)$ can be mathematically represented (suggested in ref. 5) as the output of a linear system which has a stationary white-noise process $N_I(t)$ as its input. This representation has been derived in appendix A and is given by the following equations:

$$\left.\begin{aligned} \dot{\xi}(t) &= \eta(t) \\ \dot{\eta}(t) &= -\frac{V^2}{L^2}\xi(t) - \frac{2V}{L}\eta(t) + N_I(t) \end{aligned}\right\} \tag{4}$$

$$w_g(t) = \xi(t) + \frac{L}{V}\sqrt{3}\,\eta(t) \tag{5}$$

where the intensity of $N_I(t)$ is $v_I = \frac{\sigma_{w_g}^2 V^3}{L^3}$.

Statement of Problem

The formulation of the problem is obtained by manipulating equations (1) to equation (5). Equation (5) is substituted into equations (1), (2), and (3). The resulting equations are combined with equation (4), and matrix notation is used to obtain the following equations:

$$\overrightarrow{\dot{x}(t)} = [A]\overrightarrow{x(t)} + [B]\overrightarrow{u(t)} + [G]N_I(t) \tag{6}$$

$$\delta_V(t) = [C]\overrightarrow{x(t)} + N_O(t) \tag{7}$$

$$n_Z(t) = [D]\overrightarrow{x(t)} + [E]\overrightarrow{u(t)} \tag{8}$$

where

$$\overrightarrow{x(t)} = \begin{bmatrix} \alpha(t) \\ q(t) \\ \xi(t) \\ \eta(t) \end{bmatrix}$$

is the system state vector;

$$\overrightarrow{u(t)} = \begin{bmatrix} \delta_e(t) \\ \delta_f(t) \end{bmatrix}$$

is the input vector;

$$[A] = \begin{bmatrix} Z_\alpha & 1.0 & \frac{1}{V} Z_\alpha & \frac{L\sqrt{3}}{V^2} Z_\alpha \\ M_\alpha & M_q & \frac{1}{V} M_\alpha & \frac{L\sqrt{3}}{V^2} M_\alpha \\ 0 & 0 & 0 & 1.0 \\ 0 & 0 & -\frac{V^2}{L^2} & -\frac{2V}{L} \end{bmatrix}$$

is the system matrix;

$$[B] = \begin{bmatrix} Z_{\delta_e} & Z_{\delta_f} \\ M_{\delta_e} & M_{\delta_f} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is the output matrix;

$$[G] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is the system noise gain matrix;

$$[C] = \begin{bmatrix} -1.0 & \frac{\ell_V}{V} & \frac{-1.0}{V} & -\frac{L\sqrt{3}}{V^2} \end{bmatrix}$$

is the measurement matrix; and

$$[D] = \begin{bmatrix} \frac{V}{g} Z_{\alpha} & 0 & \frac{1}{g} Z_{\alpha} & \frac{\sqrt{3}\,L}{gV} Z_{\alpha} \end{bmatrix}$$

and

$$[E] = \begin{bmatrix} \frac{V}{g} Z_{\delta_e} & \frac{V}{g} Z_{\delta_f} \end{bmatrix}$$

are the controlled variable matrices.

Equation (6) is a system of linear, constant coefficient, differential equations with the controls $\overrightarrow{u(t)}$ and white noise $N_I(t)$ as inputs. The vane deflection angle $\delta_V(t)$ given by equation (7) is assumed to be a measured quantity that is corrupted by white noise $N_O(t)$. Normal acceleration $n_Z(t)$ given by equation (8) is the variable to be controlled. Since the input $N_I(t)$ is assumed to be a Gaussian random variable, the system variables $\overrightarrow{x(t)}$, $\delta_V(t)$, and $n_Z(t)$ are also Gaussian random variables. The problem is to determine a control functional $\overrightarrow{u(t)} = f\left(\delta_V(t)\right)$ which minimizes the variances of normal acceleration and the required control. That is, the quadratic performance index to be minimized is

$$J = E\left\{n_Z^{\,2}(t) + \overrightarrow{u(t)}^T \left[R_1\right] \overrightarrow{u(t)}\right\} \tag{9}$$

where $[R_1]$ is a 2×2 weighting matrix which is to be specified.

Thus,

$$n_Z^2(t) = \left[\overrightarrow{x(t)}^T [D]^T + \overrightarrow{u(t)}^T [E]^T\right]\left[[D]\overrightarrow{x(t)} + [E]\overrightarrow{u(t)}\right]$$

so that equation (9) may be rewritten as

$$J = E\left\{\overrightarrow{x(t)}^T [Q]\overrightarrow{x(t)} + 2\overrightarrow{x(t)}^T [S]\overrightarrow{u(t)} + \overrightarrow{u(t)}^T [R]\overrightarrow{u(t)}\right\} \tag{10}$$

In equation (10) the matrices $[Q]$, $[S]$, and $[R]$ are defined as

$$\left.\begin{aligned} [Q] &= [D]^T [D] \\ [S] &= [D]^T [E] \\ [R] &= [R_1] + [E]^T [E] \end{aligned}\right\} \tag{11}$$

Equation (10) may be put into a form more suitable for calculations by applying the transformations given in appendix B. The resulting equivalent form for equation (10) is the following:

$$J = E\left\{\overrightarrow{x(t)}^T [\tilde{Q}]\overrightarrow{x(t)} + \tilde{u}(t)^T [R]\tilde{u}(t)\right\}$$

The solution to the problem depends on a knowledge of both the intensity of the measurement noise v_O and the value of the elements of the weighting matrix $[R_1]$. A perfect measuring instrument is one for which $v_O = 0$, and an undesirable instrument is one for which v_O is greater than the quantity to be measured. The matrix $[R_1]$ must be positive definite and symmetric; that is, $[R_1] > 0$ and $[R_1]^T = [R_1]$, respectively.

Solution of the Problem

The solution of the preceding problem has two interacting parts. In one part, the optimum regulator, a feedback gain matrix is calculated that gives $\overrightarrow{u(t)}$ as a linear function of an estimate of the state vector. This estimate is denoted $\overrightarrow{\hat{x}(t)}$. The second part of the solution, the Kalman-Bucy filter, gives the structure and gain of an estimator. The estimator gives $\overrightarrow{\hat{x}(t)}$ from the measurement $\delta_V(t)$ and the control $\overrightarrow{u(t)}$.

First the control input $\overrightarrow{u(t)}$ is chosen according to the linear control law that minimizes the performance index. (See eq. (10).) This optimum control law is written

$$\overrightarrow{u(t)} = -[F]\overrightarrow{\hat{x}(t)} \tag{12}$$

where the feedback gain matrix $[F]$ is

$$[F] = [R]^{-1}\left([B]^T[P] + [S]^T\right) \tag{13}$$

(See eqs. (11) and appendix B.) The matrix $[P]$ is the positive, semidefinite solution of the following matrix Riccati equation. (See eqs. (11) and appendix B for matrix definitions.)

$$[P][\tilde{A}] + [\tilde{A}]^T[P] + [\tilde{Q}] - [P][B][R]^{-1}[B]^T[P] = 0 \tag{14}$$

The equations needed to obtain the feedback gain matrix are independent of the stochastic characteristics of the problem. In addition, the measurement matrix $[C]$ is not used in the equations.

Second, the matrix $[K]$ is chosen in the system

$$\dot{\hat{\vec{x}}}(t) = [A]\vec{\hat{x}}(t) + [B]\vec{u}(t) + [K]\left[\delta_V(t) - [C]\vec{\hat{x}}(t)\right] \tag{15}$$

so that the variance of the error $\vec{e}(t) = \vec{x}(t) - \vec{\hat{x}}(t)$ is minimized. Thus,

$$E\left\{\vec{e}(t)^T\,\vec{e}(t)\right\} \tag{16}$$

is a minimum. For this part of the solution, it is assumed that the white-noise processes $N_I(t)$ and $N_O(t)$ are uncorrelated.

The system described by equation (15) is a Kalman-Bucy filter for the system given by equations (6) and (7). The filter gain matrix $[K]$ which minimizes the variance of the error (see eq. (16)) is

$$[K] = \frac{1}{v_O}[H][C]^T \tag{17}$$

where $[H]$ is the positive, semidefinite solution of the following steady-state matrix Riccati equation

$$[H][A] + [A]^T[H] + [V_1] - \frac{1}{v_O}[H][C]^T[C][H] = 0 \tag{18}$$

where

$$[V_1] = v_I[G][G]^T$$

Although the input matrix $[B]$ is included in the Kalman-Bucy filter equations, it is not needed to determine the filter gain matrix $[K]$.

The feedback control input given by equation (12), the Kalman-Bucy filter given by equation (15), and the filter gain matrix given by equation (17) constitute the optimum gust alleviator. Figure 3 shows a diagram of the closed-loop system.

The weighting matrix $\left[R_1\right]$ and the measurement noise intensity v_O are arbitrary parameters. These parameters are varied to evaluate the performance of the gust alleviator.

SAMPLE APPLICATION OF THEORY

The theory outlined in this paper was illustrated by applying it to the optimum design of a gust alleviation system for a STOL airplane. The airplane mass, dimensions, flight condition, and aerodynamic characteristics used in the calculations are presented in table I. The flight condition was for cruise at an airspeed of 109 m/sec and at an altitude of 3048 m. The scale of turbulence was chosen to be $L = 305$ m and the mean square gust intensity was chosen to be $\sigma_{w_g}{}^2 = 1\ m^2/sec^2$; the corresponding noise intensity was $v_I = 0.04559\ m^2/sec^5$.

A wide range of values for the weighting matrix $\left[R_1\right]$ and the measurement noise intensity v_O were used to calculate the following quantities: rms angle of attack σ_α, rms pitch rate σ_q, rms elevator deflection angle σ_{δ_e}, rms flap deflection angle σ_{δ_f}, rms normal acceleration σ_{n_Z}, estimated rms angle of attack $\hat{\sigma}_\alpha$, estimated pitch rate $\hat{\sigma}_q$, estimated rms gust velocity $\hat{\sigma}_{w_g}$, gain matrices $\left[F\right]$ and $\left[K\right]$, poles of the closed-loop system, and percent alleviation r. The rms of the variables α, q, n_Z, $\hat{\alpha}$, $\hat{q}$, $\hat{w}_g$, δ_e, and δ_f were directly proportional to the rms gust velocity. Therefore, the results are valid for values of the rms gust velocity other than unity. A computer subroutine package entitled ORACLS (see ref. 7) was used for the necessary computations.

Weighting Matrix Selection

Selection of the weighting matrix $\left[R_1\right]$ reflected the desire of the designer to keep the rms normal acceleration as near zero as possible with rms control deflections as small as possible. In most practical applications, $\left[R_1\right]$ was chosen to be diagonal. This choice permitted individually weighting δ_e and δ_f relative to normal acceleration. As an initial selection for this example, the elements r_1 and r_2 were defined as the square of the ratio of the normal acceleration to the maximum deflection angle of the specific control. Thus,

$$r_1 = \left(\frac{n_Z}{\delta_{e,max}}\right)^2$$

and

$$r_2 = \left(\frac{n_Z}{\delta_{f,max}}\right)^2$$

In these expressions, the acceleration n_Z results whenever the maximum deflection angle of either δ_e or δ_f is the only input to the airplane. Adjustments can be made in r_1 and r_2 to satisfy the designer's specifications as nearly as possible. The values of r_1 and r_2 computed for the airplane of this study are:

$$r_1 = (22.07)^2/\text{rad}^2$$

$$r_2 = (16.60)^2/\text{rad}^2$$

It is not unreasonable to choose $r_1 = r_2$ so that $[R_1]$ may be written as

$$[R_1] = \beta[I]$$

where β is a scalar and $[I]$ is the identity matrix. Initially, the value of β was $\beta = (20)^2/\text{rad}^2$. However, calculations were made for a wide range of values of β and for two values of measurement noise intensity v_O. Results are presented in figure 4. The plots of σ_{n_Z}, σ_{δ_e}, and σ_{δ_f} in figure 4 are of interest. The plot of σ_q is presented for convenience. It is evident that the value $\beta = 400/\text{rad}^2$ was large enough to limit the control movements to small rms deflection angles and, consequently, rms normal acceleration was relatively large. A much smaller value of β $\left(\beta < 10/\text{rad}^2\right)$ would be more desirable for this problem since the rms control deflection angles would not be excessive for large gusts, and normal acceleration could be reduced substantially. For very large values of β, the penalty for using the controls is so great that the alleviation system cannot produce large reductions in the rms normal acceleration. A value of $\beta = 3/\text{rad}^2$ was arbitrarily chosen for the system.

Design Performance

The performance of the gust alleviation system was measured by the reduction in rms normal acceleration. The unalleviated airplane, at the flight condition of this study, had an rms normal acceleration of 0.07928. Percent alleviation r may be calculated by using the following formula:

$$r = \frac{0.07928 - \sigma_{n_Z}}{0.07928} 100 \qquad (19)$$

where σ_{n_Z} is the rms normal acceleration.

The Kalman-Bucy filter provided the estimate of the state $\overrightarrow{\hat{x}(t)}$ needed in the optimum control law given by equation (12). Calculation of the Kalman-Bucy filter gain matrix $[K]$ depended on the value of the intensity v_O of the measurement noise. (See eq. (17).) The gain matrix $[K]$ has been calculated for a wide range of values of the measurement noise intensity v_O with $\beta = 3/\text{rad}^2$. The corresponding percent alleviation r is presented in figure 5 as a function of the measurement noise intensity. The ratio of the measurement noise intensity to the rms vane deflection angle v_O/σ_{δ_V} is also shown in figure 5.

Reference 1 showed that perfect alleviation theoretically can be achieved, with finite gains, for an airplane using a vane sensor and two controls $(\delta_e$ and $\delta_f)$. Perfect alleviation was not possible, under the assumptions of the present theory, when noise was superimposed on the vane angle measurement. The measurement noise was propagated through the feedback loop and influenced the percent alleviation. However, the Kalman-Bucy filter reduced the effect of the measurement noise since the variance of the error vector $\overrightarrow{e(t)}$ was minimized. (See eq. (16).) Figure 5 shows that alleviation was 92 percent for a very small value of the measurement noise intensity $\left(v_O = 4.56 \times 10^{-7} \text{ rad}\right.$ and $\left.v_O/\sigma_{\delta_V} \approx 4 \times 10^{-5}\right)$. The corresponding filter gain is

$$[K] = \begin{bmatrix} -5.4964 \\ 11.5172 \\ 7.1569 \\ -316.0585 \end{bmatrix} \tag{20}$$

and the required rms control deflection angles are

$$\left.\begin{aligned} \sigma_{\delta_e} &= 0.003485 \text{ rad} \\ \sigma_{\delta_f} &= 0.008179 \text{ rad} \end{aligned}\right\} \tag{21}$$

A more realistic instrument for measuring the vane deflection angle could have a noise intensity $v_O = 4.56 \times 10^{-4}$ rad or a ratio $v_O/\sigma_{\delta_V} = 0.036$. This ratio means that the noise is about 3.6 percent of the vane deflection angle. The optimum gust alleviation system then provides about 63 percent alleviation of normal acceleration. Table II gives a complete set of calculated characteristics for this optimum gust alleviation system. The Kalman-Bucy filter produced very good estimates of the state variables which are reflected by the good percent alleviation. The gains $[F]$ and $[K]$ are moderate and the required rms control deflection angles are smaller than required for better alleviation. (See eq. (21).)

If the possibility of the existence of ±20 percent tolerance in the rms measurement noise intensity is considered, this characteristic of the instrument could result in reduced performance of the alleviation system having the gains $[F]$ and $[K]$ fixed. However, the calculated reduction in percent alleviation for this "off-design" condition was negligible.

Table II includes the zeros of the polynomials in λ given by the following equations:

$$\det\Big[\lambda[I] - [A] + [B][F]\Big] = 0 \tag{22}$$

and

$$\det\Big[\lambda[I] - [A] + [K][C]\Big] = 0 \tag{23}$$

The zeros of equations (22) and (23) are the poles of the closed-loop systems. (See fig. 3.) The zeros of equation (22) are the optimum regulator poles; the zeros of equation (23) are the Kalman-Bucy filter poles. Since the real parts of all the zeros are negative, the closed-loop system is stable. This result is not surprising since the regulator can always be stabilized with an additional feedback gain matrix, and the Kalman-Bucy filter is automatically stable.

It should be emphasized that only one flight condition has been considered. In an actual design study, consideration of several flight conditions is essential. Also, in the problem formulation, control servodynamics must be included for a more accurate model of the system. The effect on alleviation associated with servo lag times can be partially offset by proper location of the vane.

CONCLUDING REMARKS

A study has been made of optimum design considerations of a gust alleviation system for aircraft flying in turbulent air. A vane sensor (with noise) was used to measure vertical gusts, and elevators and flaps were used to reduce the root-mean-square value of the normal accelerations associated with the aircraft response to gusts. Since turbulence has stochastic properties, stochastic control theory was used in the analysis. A performance-index function involving normal acceleration and control deflections with a weighting matrix was used.

A short take-off and landing (STOL) airplane in flight through turbulent air was used as an example to illustrate the application of the analysis. Effects of varying the noise characteristics of the vane sensor and the weighting matrix in the performance-index function were determined.

Calculations were performed as required by stochastic control theory to obtain the root-mean-square response of the airplane to turbulence. The calculations showed that a weighting number appearing in the performance index should be less than 10. A value of 3 was selected for subsequent calculations. Normal acceleration could be reduced by 92 percent when the intensity of the measurement noise had a very small value. However, the filter gains were large and large control deflection angles were required. Good alleviation was calculated when the intensity of the measurement noise was about 3.6 percent of the vane deflection angles. Normal acceleration was reduced by 63.2 percent for moderate values of the gains.

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APPENDIX A

MODEL OF ATMOSPHERIC TURBULENCE

Development of a mathematical representation for the one-dimensional vertical gust having a power spectral density represented by the Dryden function was desired. The Dryden function has a scale length L and a mean square gust intensity $\sigma_{w_g}{}^2$. The desired representation can be developed as follows:

In a linear system given by the following equations

$$\left.\begin{aligned} \dot{\xi}(t) &= \eta(t) \\ \dot{\eta}(t) &= -a_0\xi(t) - a_1\eta(t) + N_I(t) \end{aligned}\right\} \tag{A1}$$

the function $N_I(t)$ is the input. An output is defined as

$$w_g(t) = \gamma_1\xi(t) + \gamma_2\eta(t) \tag{A2}$$

It is desired to determine γ_1, γ_2, a_0, and a_1 to obtain the desired form of the gust power spectral density function. If $N_I(t)$ is a stationary white-noise process, then $\xi(t)$, $\eta(t)$, and $w_g(t)$ are stochastic variables. Further, $N_I(t)$ is assumed to be a stationary white-noise process having the following characteristics:

The mean of $N_I(t)$ is

$$E\left\{N_I(t)\right\} = 0$$

and the convariance function is

$$E\left\{N_I(t)\ N_I(\tau)\right\} = v_I\delta(t - \tau)$$

where v_I is the intensity of the process and $\delta(t - \tau)$ is a Dirac-delta function.

The power spectral density function is

$$\Phi_{N_I(t)}(\omega) = v_I$$

which is constant for all frequencies ω.

The power spectral density function of the output can be calculated by the following relationship:

$$\Phi_{w_g(t)}(\omega) = \left|H(\omega)\right|^2 \Phi_{N_I(t)}(\omega)$$

or

$$\Phi_{w_g(t)}(\omega) = \left|H(\omega)\right|^2 v_I \tag{A3}$$

where $\left|H(\omega)\right|$ is the absolute value of the complex frequency response function $H(\omega)$ of the output $w_g(t)$. The frequency response function of $w_g(t)$ is (from eqs. (A1) and (A2))

$$H(\omega) = \frac{\gamma_1 + i\omega\gamma_2}{\left(a_0 - \omega^2\right) + i\omega a_1}$$

and

$$\left|H(\omega)\right|^2 = \frac{\gamma_1^2 + \gamma_2^2\omega^2}{\omega^4 + \left(a_1^2 - 2a_0\right)\omega^2 + a_0^2}$$

Consequently, by using equation (A3)

$$\Phi_{w_g(t)}(\omega) = \frac{\left(\gamma_1^2 + \gamma_2^2\omega^2\right)v_I}{\omega^4 + \left(a_1^2 - 2a_0\right)\omega^2 + a_0^2} \tag{A4}$$

Since $w_g(t)$ may be considered the vertical component of gust velocity in atmospheric turbulence, evaluation of the parameters γ_1, γ_2, a_0, and a_1 may be done in the following way. Atmospheric turbulence may be assumed to be a one-dimensional, isotropic, stationary, and homogeneous random process. Also, the Dryden power spectral density function

$$\Phi_{w_g(t)}(\omega) = \frac{\sigma_{w_g}{}^2 L}{V} \frac{1 + \frac{3L^2}{V^2}\omega^2}{\left(1 + \frac{L^2}{V^2}\omega^2\right)^2} \tag{A5}$$

may be considered to represent atmospheric turbulence. The functions given by equations (A4) and (A5) are equated and the constants and coefficients of ω are equated; the following values for the unknown parameters result. If the constant γ_1 is set equal to 1.0, then

$$\gamma_1 = 1$$

$$\gamma_2 = \frac{L\sqrt{3}}{V}$$

$$a_0 = \frac{V^2}{L^2}$$

$$a_1 = \frac{2V}{L}$$

$$v_I = \frac{\sigma_{w_g}{}^2 V^3}{L^3}$$

Note that $a_0 > 0$ and $a_1 > 0$ are used to assure stability of the system of equations (A1). These parameters are substituted into equations (A1) and (A2) and the following model of the assumed atmospheric turbulence is obtained:

$$\left.\begin{aligned} \dot{\xi}(t) &= \eta(t) \\ \dot{\eta}(t) &= -\frac{V^2}{L^2}\,\xi(t) - \frac{2V}{L}\,\eta(t) + N_I(t) \end{aligned}\right\} \tag{A6}$$

and

$$w_g(t) = \xi(t) + \frac{L\sqrt{3}}{V}\,\eta(t) \tag{A7}$$

One-dimensional atmospheric turbulence has been modeled as the output of a linear system with a white-noise process as the input. The power spectral density function for the vertical gust velocity is the Dryden function.

APPENDIX B

SYSTEM EQUIVALENCE

In optimum control problems, the quadratic performance index often contains cross products of the states and the controls. A transformation is presented here to show that such a performance index is equivalent to another one which does not contain the cross-product terms. The dynamics of a plant are assumed to be represented by the following constant linear differential equation:

$$\overrightarrow{\dot{x}(t)} = [\tilde{A}]\overrightarrow{x(t)} + [B]\overrightarrow{\tilde{u}(t)} \tag{B1}$$

where $\overrightarrow{x(t)}$ is the state vector, $\overrightarrow{\tilde{u}(t)}$ is the input vector, $[\tilde{A}]$ is the system matrix, and $[B]$ is the input matrix. A well-known result (see ref. 5) of optimum control theory is that the feedback control law given by

$$\overrightarrow{\tilde{u}(t)} = -[R]^{-1}[B]^T[P]\overrightarrow{x(t)} \tag{B2}$$

minimizes a performance index given by

$$\tilde{J} = E\left\{\overrightarrow{x(t)}^T[\tilde{Q}]\overrightarrow{x(t)} + \overrightarrow{\tilde{u}(t)}^T[R]\overrightarrow{\tilde{u}(t)}\right\} \tag{B3}$$

and the symbol $E\{\ \}$ indicates the expected value when $\vec{x}$ and $\vec{\tilde{u}}$ are random variables.

The symmetric matrix $[P]$ in equation (B2) is the positive-definite solution of the following matrix Riccati equation:

$$0 = -[P][\tilde{A}] - [\tilde{A}]^T[P] - [\tilde{Q}] + [P][B][R]^{-1}[B]^T[P] \tag{B4}$$

Equations (B1) to (B4) are called System I.

The following transformations are applied to System I:

$$\begin{aligned} \overrightarrow{u(t)} &= \overrightarrow{\tilde{u}(t)} - [R]^{-1}[S]^T\overrightarrow{x(t)} \\ [A] &= [\tilde{A}] + [B][R]^{-1}[S]^T \\ [Q] &= [\tilde{Q}] + [S][R]^{-1}[S]^T \end{aligned} \tag{B5}$$

and the result is System II, given by the following formulas:

$$\dot{\overrightarrow{x(t)}} = [A]\overrightarrow{x(t)} + [B]\overrightarrow{u(t)} \tag{B6}$$

$$\overrightarrow{u(t)} = -[R]^{-1}\left[[B]^T[P] + [S]^T\right]\overrightarrow{x(t)} \tag{B7}$$

$$J = E\left\{\overrightarrow{x(t)}^T[Q]\overrightarrow{x(t)} + 2\overrightarrow{x(t)}[S]\overrightarrow{u(t)} + \overrightarrow{u(t)}^T[R]\overrightarrow{u(t)}\right\} \tag{B8}$$

The corresponding matrix Riccati equation results:

$$0 = -[P][A] - [A]^T[P] - [Q] + \left[[P][B] + [S]\right][R]^{-1}\left[[B]^T[P] + [S]^T\right] \tag{B9}$$

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TABLE I.- AIRPLANE MASS, DIMENSIONS, FLIGHT CONDITION, AND AERODYNAMIC CHARACTERISTICS

Mass, M, kg	5670
Wing area, S_W, m^2	39
Mean aerodynamic chord, $\bar{c}$, m	1.981
Radius of gyration about Y-axis, k_Y, m	2.572
Vane distance ahead of center of gravity, ℓ_v, m	2.972
True airspeed, V, m/sec	109
Altitude, m	3048
Dynamic pressure, q_∞, N/m^2	5364
Z_α, N-sec/rad-kg-m	-1.969
Z_{δ_e}, N-sec/rad-kg-m	-0.156
Z_{δ_f}, N-sec/rad-kg-m	-0.746
M_α, m-N/rad-kg-m^2	-14.597
M_q, m-N-sec^2/rad^2-kg-m^2	-2.095
M_{δ_e}, m-N/rad-kg-m^2	-20.042
M_{δ_f}, m-N/rad-kg-m^2	8.672
$\ell_v/\bar{c}$	1.5

TABLE II.- CHARACTERISTICS OF AN OPTIMUM GUST ALLEVIATION SYSTEM

β per rad^2	3
v_O, rad	4.56×10^{-4}
v_O/σ_{δ_v}	0.036
σ_α, rad	0.008629
$\hat{\sigma}_\alpha$, rad	0.007375
σ_q, rad/sec	0.01527
$\hat{\sigma}_q$, rad/sec	0.01522
σ_{w_g}, m/sec	1.0
$\hat{\sigma}_{w_g}$, m/sec	0.8880
σ_{δ_e}, rad	0.003268
σ_{δ_f}, rad	0.007643
σ_{n_Z}	0.02914
r, percent	63.2

$$[F] = \begin{bmatrix} -1.0405 & -0.2920 & -0.8337 & -4.2172 \\ 2.7328 & 0.0611 & 2.6892 & 13.0734 \end{bmatrix}$$

$$[K] = \begin{bmatrix} -4.6441 \\ 12.2582 \\ 4.3912 \\ -9.4190 \end{bmatrix}$$

Zeros of det $[\lambda I - A + BF] = 0$:

$\lambda_1 = -0.3573$

$\lambda_2 = -0.3572$

$\lambda_{3,4} = -4.2838 \pm 16.4486$

Zeros of det $[\lambda I - A + KC] = 0$:

$\lambda_1 = -4.8199$

$\lambda_2 = -2.5355$

$\lambda_3 = -1.0100$

$\lambda_4 = -1.9362$

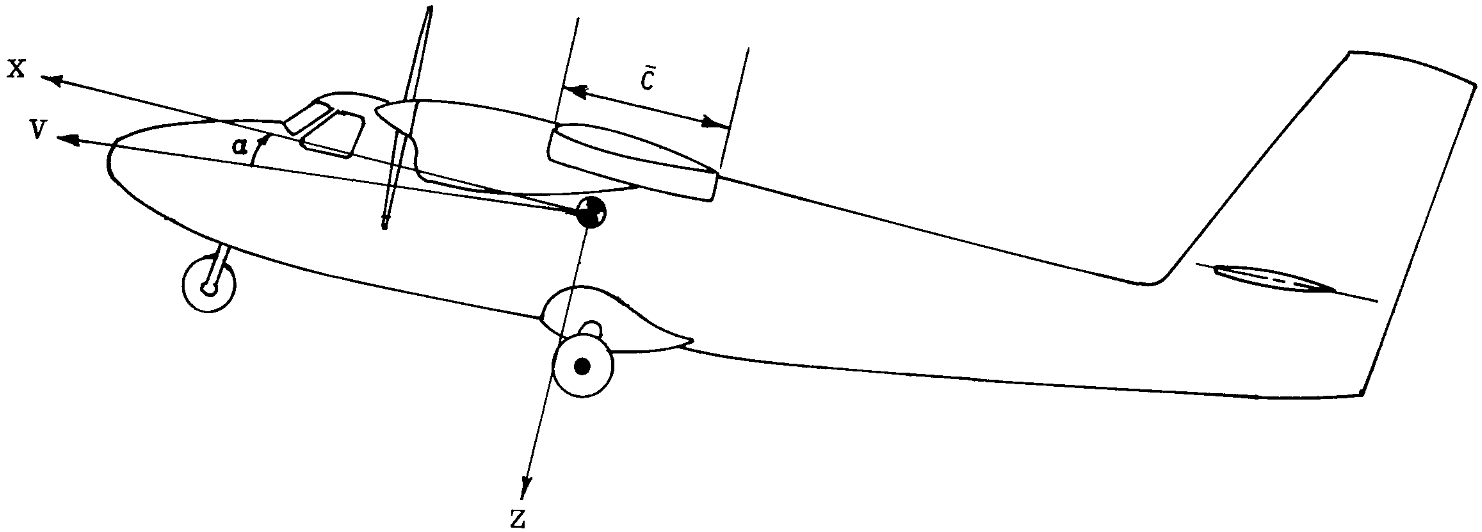


Figure 1.- Axis system.

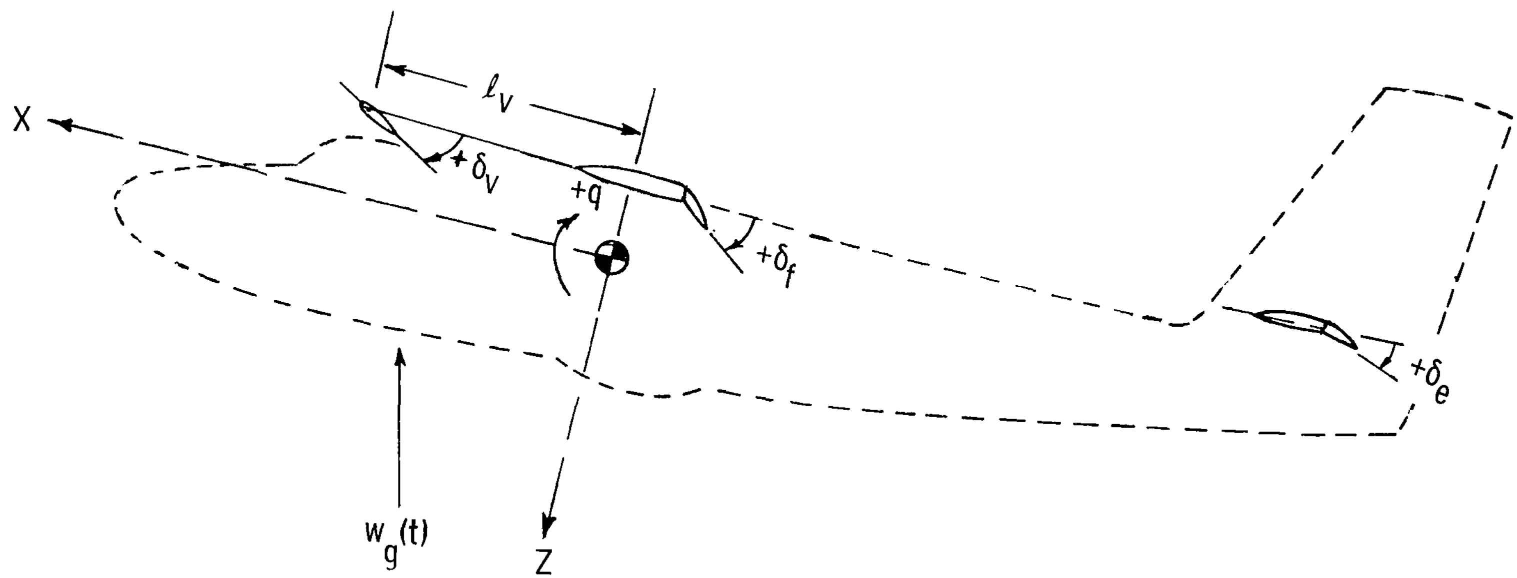


Figure 2.- Vane location and sign convention.

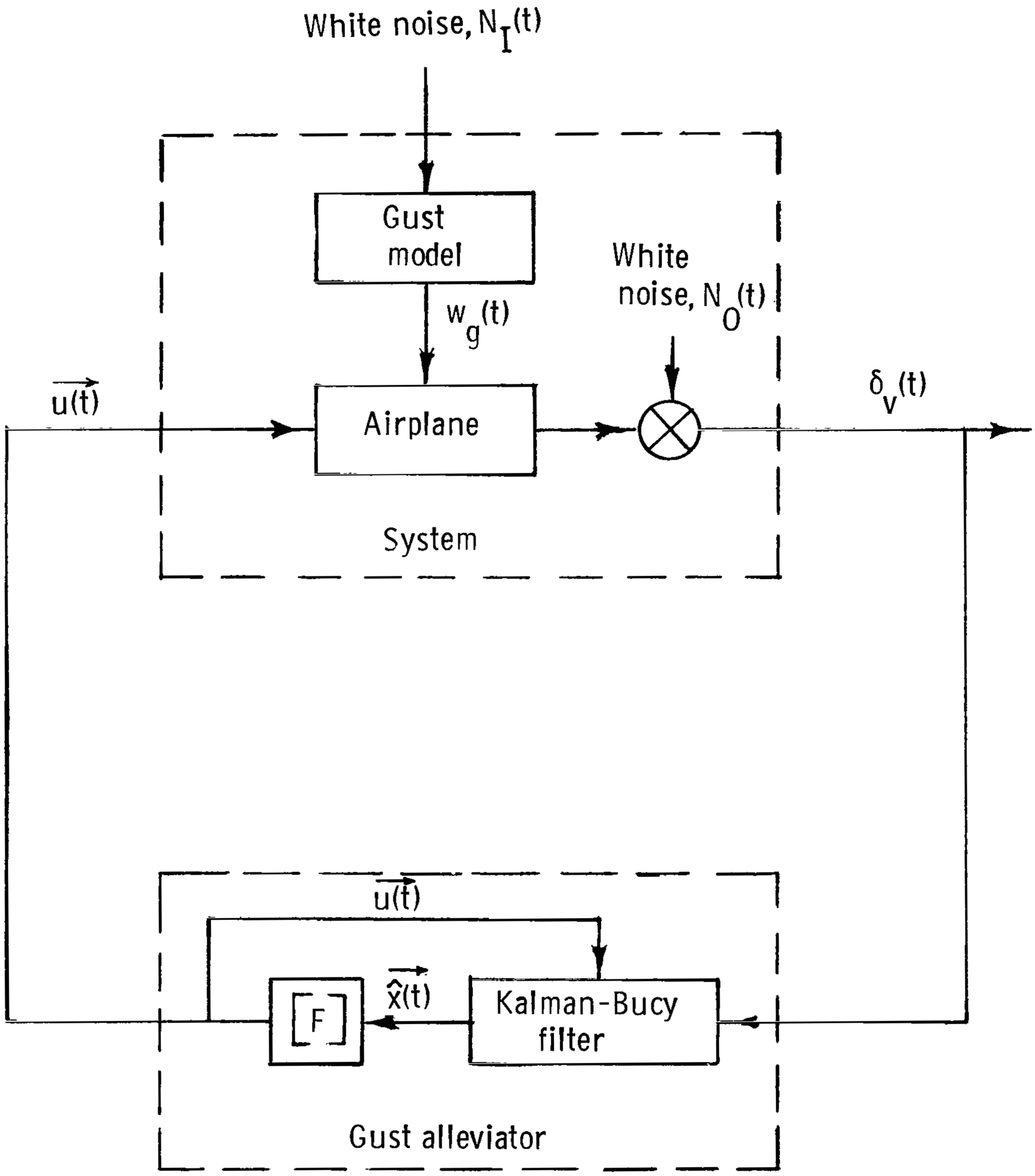
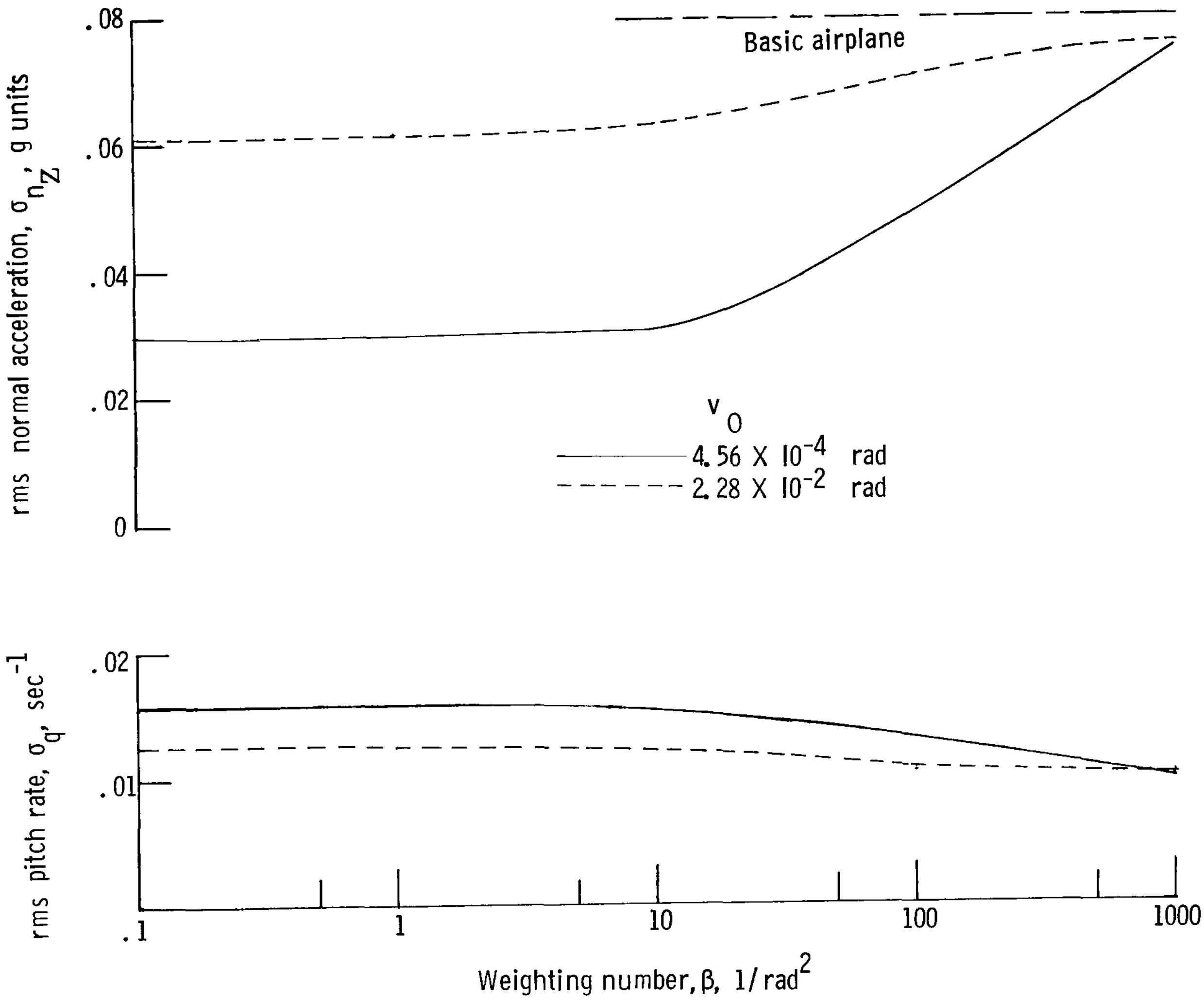
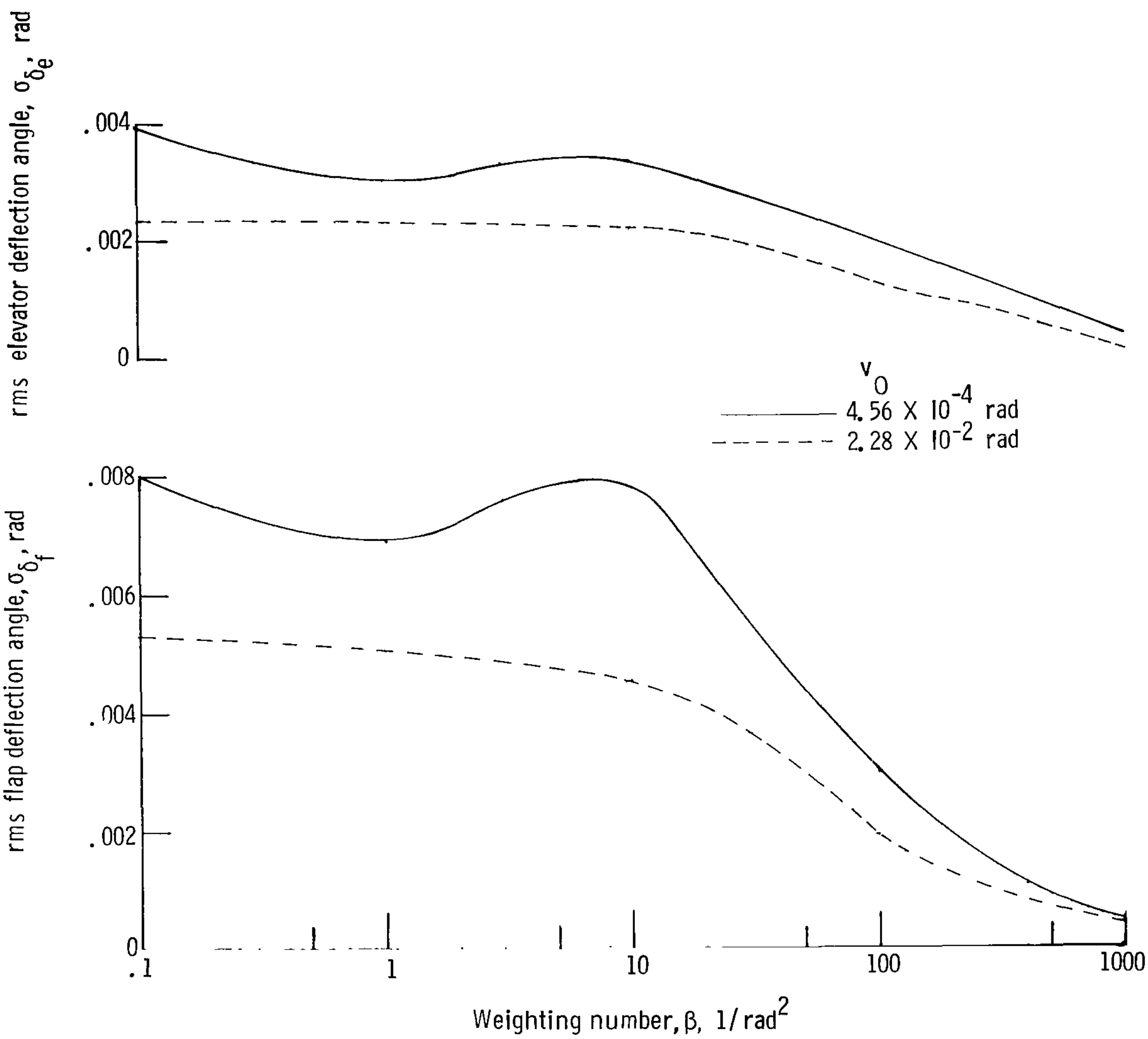


Figure 3.- Block diagram.



(a) Root-mean-square (rms) normal acceleration and pitch rate.

Figure 4.- Airplane and gust alleviator response to turbulence.



(b) Root-mean-square (rms) control deflection angles.

Figure 4.- Concluded.

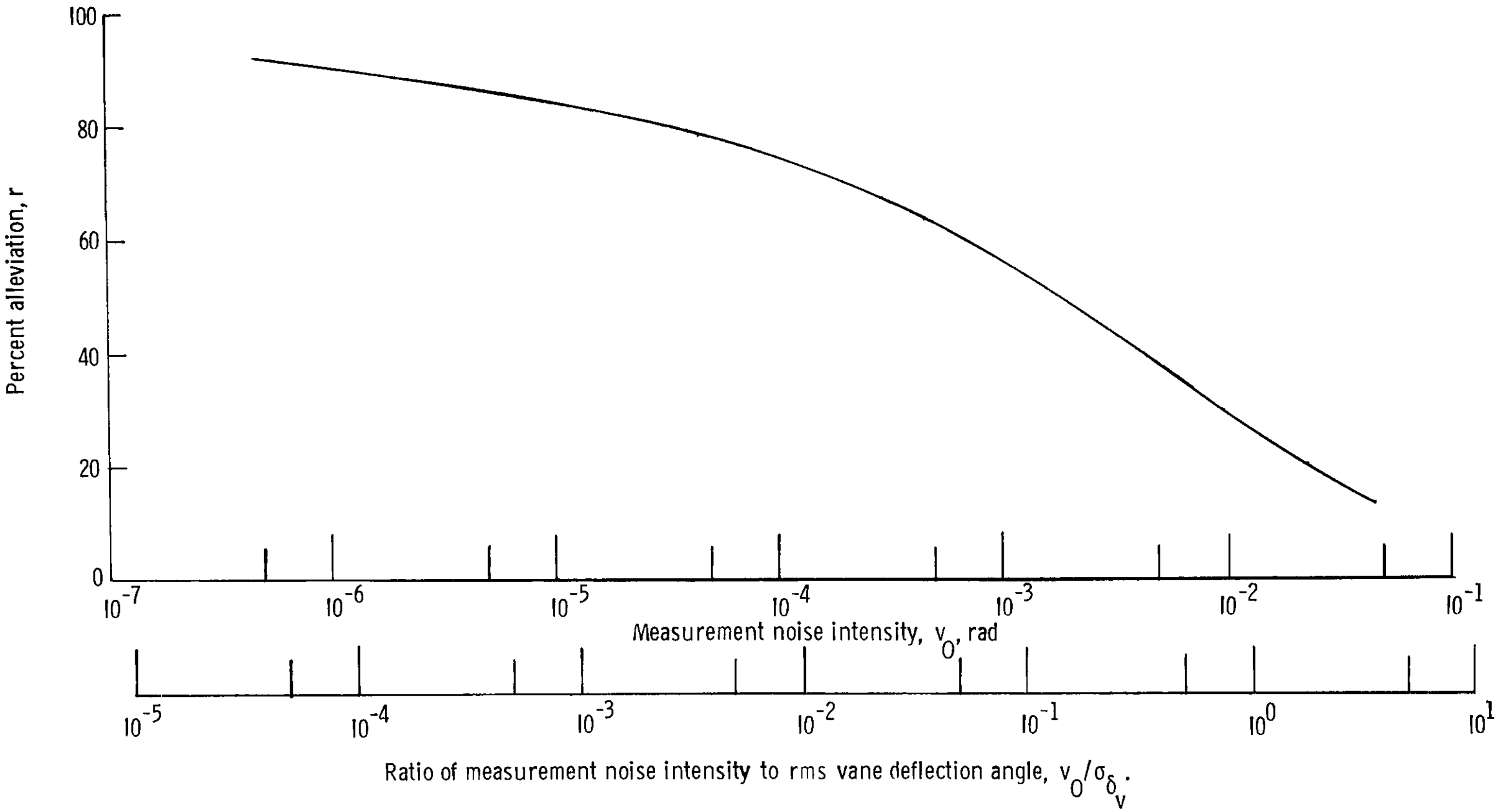


Figure 5.- Effect of measurement noise intensity on gust alleviation; $\beta = 3/\text{rad}^2$.

NASA-Langley, 1976 L-10361

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