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CONJUGATE QUASILINEAR DIRICHLET AND NEUMANN PROBLEMS AND A POSTERIORI ERROR BOUNDS

John E. Lavery

Goddard Space Flight Center
Greenbelt, Md. 20771

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INTRODUCTION

Interest in solving quasilinear Dirichlet and Neumann problems of the type

$$-\frac{\partial}{\partial x}\left[p\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},x,y\right)\frac{\partial u}{\partial x}\right]-\frac{\partial}{\partial y}\left[q\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},x,y\right)\frac{\partial u}{\partial y}\right]=0$$

arising in such areas as torsion, elasticity, and filtration has been sharpened in the past few decades by the development of numerical methods for efficiently solving these problems and the possibility of programming these methods on computers. Of great importance for the application of these numerical methods are computable a posteriori error bounds for the norm of the difference of an approximate solution and the unknown exact solution. In this document, the concept of conjugate quasilinear Dirichlet and Neumann problems is introduced with the goal of obtaining two-sided bounds for the energies of the solutions of the Dirichlet and Neumann problems. These two-sided bounds for the energies are subsequently used to obtain a posteriori error bounds for the norm of the difference of the approximate and exact solutions.*

PRELIMINARY REMARKS AND AUXILIARY RESULTS

All variables and constants will be exclusively real-valued. Lower case c's, with or without numerical subscripts, will denote finite constants. Derivatives will be denoted by literal subscripts for functions of two variables, for example, $u_{xy} = \partial^2 u/\partial y \partial x\ (x,y)$, and by numerical subscripts for functions of more than two variables, for example, $p_{32}(X,Y,x,y) = \partial^2 p/\partial Y \partial x\ (X,Y,x,y)$.

The letter D will denote the rectangular domain (-a,a) x (-b,b) in R^2. The area differential in D or in any domain in R^2 will be denoted by dA. The boundary $D^\bullet$ of D will be parametrized in a counterclockwise direction starting from the lower left corner by a parameter s. The four open straight line segments of $D^\bullet$ will be denoted by $D_i^\bullet$, i = 1,2,3,4 (in any order).

Differential will mean Gateaux differential. Let B and B_2 denote Banach spaces. Let B_0 denote a subspace of B and B_1 denote the set $\{\zeta \epsilon B: \zeta = \zeta_0 + \zeta_1, \zeta_0 \epsilon B_0\}$, where ζ_1 is some fixed element of B. For operators with differentials, the following generalizations of the

*Results presented in this document have been presented in more detail in Chapters I and II of the informal publication John E. Lavery, "Solution of a Class of Quasilinear Dirichlet and Neumann Problems by the Method of Moments with A Posteriori Error Bounds," GSFC X-814-73-29, Greenbelt, Maryland, January 1973.

mean-value theorem and Taylor theorem of classical analysis hold (reference 1, pp. 37-41). In these lemmas, A denotes an operator from B_1 into B_2.

Lemma 1. If the operator A possesses a differential A′ at every point of B_1, then, for any fixed ζ in B_1, any fixed η in B_0, and any fixed e in the *conjugate* space B_2^* of B_2, there exists a constant τ, $0 < \tau < 1$, such that

$$(A(\zeta + \eta) - A(\zeta), e) = (A'(\zeta + \tau\eta, \eta), e),$$

where τ depends on ζ, η, and e.

Lemma 2. If the functional A possesses a second differential A″ at every point of B_1, and $A''(\zeta + \tau\eta, \hat{\eta}, \tilde{\eta})$ is continuous as a function of the real variable τ for all fixed ζ in B_1 and all fixed η, $\hat{\eta}$, and $\tilde{\eta}$ in B_0, then, for any fixed ζ in B_1 and any fixed η in B_0, there exists a constant τ, $0 < \tau < 1$, such that

$$A(\zeta + \eta) - A(\zeta) = A'(\zeta, \eta) + \frac{1}{2} A''(\zeta + \tau\eta, \eta, \eta).$$

A functional A with a second differential will be called convex if there exists a positive constant c such that $A''(\zeta, \eta, \eta) \geqslant c \, \|\eta\|^2_{B_0}$. (This definition is more restrictive than usual, but is sufficient for our purposes.)

Lemma 3. Let the Banach space B_0 be a Hilbert space. Let A be a convex functional on B_1, and let A satisfy the hypotheses of lemma 2. Then there exists a unique point $\hat{\zeta}$ in B_1 at which A has an absolute minimum. The point $\hat{\zeta}$ is the unique stationary point of A, that is, the unique point at which $A'(\hat{\zeta}, \eta) = 0$ for all η in B_0.

The Sobolev space $W_2^m(D)$ will be denoted by $H^m(D)$, with inner product

$$(\phi, \hat{\phi})_{H^m(D)} = \sum_{|i|=0}^{m} \iint_D \phi^{(i)} \hat{\phi}^{(i)} \, dA.$$

$L_2^1(D)$ will denote the Hilbert space of equivalence classes of elements of $H^1(D)$ which differ by a constant, with the inner product

$$(\phi, \hat{\phi})_{L_2^1(D)} = \iint_D (\phi_x \hat{\phi}_x + \phi_y \hat{\phi}_y) dA.$$

$H_0^1(D)$ will denote the Hilbert space which is the closure of the set of infinitely differentiable functions with compact support in D in the norm $\| \cdot \|_{H^1(D)}$.

The following lemma is a generalization of a result of classical analysis. (See reference 2, p. 227, theorem 4.)

Lemma 4. Let ϕ and $\hat{\phi}$ be in $H^0(D)$ and let

$$\iint_D (\phi h_x - \hat{\phi} h_y)\, dA = 0$$

for all h in $H_0^1(D)$. Then there exists a unique Φ in $L_2^1(D)$ such that

$$\frac{\partial \Phi}{\partial x} = \hat{\phi} \text{ and } \frac{\partial \Phi}{\partial y} = \phi.$$

The standard Friedrichs inequality is stated in lemma 5.

Lemma 5. For all u in $H_0^1(D)$,

$$\| u \|^2_{L_2^1(D)} \geqslant \frac{1}{[4 \min (a^2, b^2) + 1]} \| u \|^2_{H^1(D)}.$$

$H^{1/2}(D^{\bullet})$ will denote the Banach space consisting of the traces on $D^{\bullet}$ of the elements of $H^1(D)$ (see reference 3). It is well known that $H^{1/2}(D^{\bullet}) \subset L_2(D^{\bullet})$. For any $g \in H^{1/2}(D^{\bullet})$, $H_g^1(D)$ will denote the set of elements $u \in H^1(D)$ such that $u|_{D^{\bullet}} = g$ ("the trace of u on $D^{\bullet}$ is g"). Consider the functional M defined for all u in $H_g^1(D)$ by

$$M(u) = \frac{1}{2} \iint_D (u_x^2 + u_y^2)\, d\,A.$$

This functional satisfies the hypotheses of lemma 3. Therefore, there exists a unique element $\hat{u}$ that minimizes M over $H_g^1(D)$. The norm $\|\cdot\|_{1/2}$ on the Banach space $H^{1/2}(D^{\bullet})$ will be given by

$$\| g \|_{1/2} = \sqrt{2M(\hat{u}) + \left(\int_{D^{\bullet}} g\, d\,s\right)^2} = \sqrt{\| \hat{u} \|^2_{L_2^1(D)} + \left(\int_{D^{\bullet}} g\, d\,s\right)^2}.$$

Let two elements of $H^{1/2}(D^{\bullet})$ be defined to be equivalent if they differ by a constant. Let $L^{1/2}(D^{\bullet})$ be the Banach space of equivalence classes of $H^{1/2}(D^{\bullet})$ defined by this equivalence relation, equipped with the norm

$$\| g \|_{L^{1/2}(D^{\bullet})} = \| \hat{u} \|_{L_2^1(D)},$$

where $\hat{u}$ is the element that minimizes M(u) over $H_{\hat{g}}^1(D)$, $\hat{g}$ being any member of the equivalence class g. To each element v in $L_2^1(D)$ there corresponds a unique element $g = v|_{D^{\bullet}}$ in $L^{1/2}(D^{\bullet})$.

$H^{-1/2}(D^{\bullet})$ will denote the linear space of linear functionals on $H^{1/2}(D^{\bullet})$. The value of the element $\hat{g} \in H^{-1/2}(D^{\bullet})$ at $g \in H^{1/2}(D^{\bullet})$ will be denoted by $(\hat{g}, g)$. $L_2(D^{\bullet})$ is a subspace of $H^{-1/2}(D^{\bullet})$

since, to every element $\tilde{g}$ of $L_2(D^\bullet)$, there corresponds the functional $\hat{g}$ of $H^{-1/2}(D^\bullet)$ defined by

$$(\hat{g}, g) = \int_{D^\bullet} \tilde{g} g \, ds.$$

Define $\overset{\circ}{H}{}^{-1/2}(D^\bullet) = \{\hat{g} \in H^{-1/2}(D^\bullet) : (\hat{g}, 1) = 0\}$. Let $L^{-1/2}(D^\bullet)$ be the linear space of linear functionals on $L^{1/2}(D^\bullet)$. The value of $\hat{g} \in L^{-1/2}(D^\bullet)$ at $g \in L^{1/2}(D^\bullet)$ will be denoted by $(\hat{g},g)$. It can be shown that $\overset{\circ}{H}{}^{-1/2}(D^\bullet)$ is isomorphic to $L^{-1/2}(D^\bullet)$. For a given $\hat{g}$ in $\overset{\circ}{H}{}^{-1/2}(D^\bullet)$, define the functional $N_{\hat{g}}$ on $L_2^1(D)$ by

$$N_{\hat{g}}(v) = \frac{1}{2} \iint_D \left(v_x^2 + v_y^2\right) dA - (\hat{g}, v|_{D^\bullet}).$$

This functional satisfies the hypotheses of lemma 3. Therefore, there exists a unique element $\hat{v}$ that minimizes $N_{\hat{g}}$ over $L_2^1(D)$. Define the norm $\|\bullet\|_{-1/2}$ on $\overset{\circ}{H}{}^{-1/2}(D^\bullet)$ by

$$\|\hat{g}\|_{-1/2} = \|\hat{v}\|_{L_2^1(D)}.$$

Under this norm, $\overset{\circ}{H}{}^{-1/2}(D^\bullet)$ is a Banach space.

The following lemma will permit defining the the derivative of an element of $H^{1/2}(D^\bullet)$.

Lemma 6. Let g and $\tilde{g}$ be any two fixed elements of $H^{1/2}(D^\bullet)$. Then the bilinear functional

$$\Phi(u, h) = \iint_D (u_y h_x - u_x h_y) \, dA$$

has the same value for all (u,h) in $H_g^1(D) \times H_{\tilde{g}}^1(D)$.

Let $\hat{u}$ be the unique element that minimizes M over $H_g^1(D)$. By lemma 6, $\Phi(\hat{u},h)$ is a linear functional on $h|_{D^\bullet}$ in $H^{1/2}(D^\bullet)$. Thus, the element g' of $H^{-1/2}(D^\bullet)$ is well-defined by the equality

$$(g', h|_{D^\bullet}) = \Phi(\hat{u}, h)$$

for all h in $H^1(D)$. Moreover, for all h such that $h|_{D^\bullet} = 1$,

$$(g', 1) = \iint_D [\hat{u}_y (1)_x - \hat{u}_x (1)_y] \, dA = 0.$$

Hence g' is in $\overset{\circ}{H}{}^{-1/2}(D^\bullet)$. The element g' of $\overset{\circ}{H}{}^{-1/2}(D^\bullet)$ will be called the derivative of the element g of $H^{1/2}(D^\bullet)$. Note that $g = c$ implies that $\hat{u} \equiv c$ and hence that $c' = 0$.

Conversely, the concept of a primitive of an element $\hat{g}$ of $\overset{\circ}{H}^{-1/2}(D^\bullet)$ can be defined as follows. Let $\hat{v}$ minimize $N_{\hat{g}}$ over $L_2^1(D)$. By lemma 3, $N'_{\hat{g}}(\hat{v},h) = 0$ for all h in $L_2^1(D)$ or, equivalently, for all h in $H^1(D)$. Hence, for all h in $H_0^1(D)$,

$$\iint_D (\hat{v}_x h_x + \hat{v}_y h_y) \, d A = N'_{\hat{g}}(\hat{v}, h) = 0.$$

Lemma 4 then implies that there exists a unique element $\hat{u}$ of $L_2^1(D)$ such that $\hat{u}_x = -\hat{v}_y$ and $\hat{u}_y = \hat{v}_x$. Define $g = \hat{u}|_{D^\bullet} \in L^{1/2}(D^\bullet)$. A primitive of $\hat{g}$ in $\overset{\circ}{H}^{-1/2}(D^\bullet)$ will then be any fixed element $\int \hat{g} \in H^{1/2}(D^\bullet)$ out of the equivalence class g. Note that any two primitives of $\hat{g}$ differ only by a constant.

The derivatives and primitives defined here can be multiplied by constants and added in the same manner as conventional derivatives and primitives, that is, $g' + c\tilde{g}' = (g + c\tilde{g})'$ and $\int\hat{g} + c\int\hat{\hat{g}} = c_0 + \int(\hat{g} + c\hat{\hat{g}})$. The following two lemmas also hold.

Lemma 7. Let g be in $H^{1/2}(D^\bullet)$. Then

$$\int (g') = g + c.$$

Lemma 8. Let $\hat{g}$ be in $\overset{\circ}{H}^{-1/2}(D^\bullet)$. Then

$$[(\int \hat{g}) + c]' = \hat{g}.$$

Using lemmas 6 and 7 the following result can be proved.

Lemma 9. Let g be in $H^{1/2}(D^\bullet)$. Then the following two statements are equivalent: $\tilde{u}$ is in $H^1_{g+c}(D)$, and $\tilde{u}$ is in $H^1(D)$ and is such that

$$\iint_D (\tilde{u}_y h_x - \tilde{u}_x h_y) \, d A = (g', h|_{D^\bullet})$$

for all h in $H^1(D)$.

It is of importance that, in all cases of computational interest, derivatives of elements of $H^{1/2}(D^\bullet)$ and primitives of elements of $\overset{\circ}{H}^{-1/2}(D^\bullet)$ can be calculated by conventional differentiation and integration along $D^\bullet$ rather than by solving the linear Dirichlet and Neumann problems of finding the $\hat{u}$ and $\hat{v}$ such that $M'(\hat{u}, h) = 0$ and $N'_g(\hat{v}, h) = 0$, that is, of minimizing $M(u)$ and $N_g(v)$. These results are stated in the following two lemmas.

Lemma 10. Let $g \equiv g(s)$ be in $\{g \in C^0(D^\bullet)\colon g|_{D_i^\bullet} \in W_2^1(D_i^\bullet),\ i = 1,2,3,4\}$. By dg/ds, denote the element of $L_2(D^\bullet)$ that equals the distributional derivative (counterclockwise direction) of $g|_{D^\bullet}$ on $D_i^\bullet$, $i = 1,2,3,4$. Then $g' = dg/ds$.

Proof. By lemma 9 and the density of the set of infinitely differentiable functions on D in $H^1(D)$.

Lemma 11. Let $\hat{g} \equiv \hat{g}(s)$ be in $L_2(D^\bullet) \cap \overset{\circ}{H}{}^{-1/2}(D^\bullet)$. Let $g \equiv g(s) = \int_0^s \hat{g}(\hat{s})\, d\hat{s}$. Then $\int \hat{g} = g + c$.

Proof. By lemmas 7 and 10.

Since the trace on $D^\bullet$ of any element of $H^1(D)$ is in $L_2(D^\bullet)$, the definition of the Hilbert space ${}_0H^1(D) = \{v \in H^1(D): \int_{D_1^\bullet} (v|_{D_1^\bullet})\, ds = 0\}$ with inner product $(\cdot,\cdot)_{H^1(D)}$ can be made. For this space, the following lemma holds.

Lemma 12. For all v in ${}_0H^1(D)$,

$$\|v\|^2_{L_2^1(D)} \geqslant \frac{1}{8\max(a^2, b^2) + 1} \|v\|^2_{H^1(D)}.$$

To conclude this section, a lemma concerning matrices will be stated.

Lemma 13. Let $\phi(x,y)$, $\hat{\phi}(x,y)$ and $\tilde{\phi}(x,y)$ be elements of $L_\infty(D)$, and let there exist positive constants $\hat{c}$, c, and $\tilde{c}$ such that

$$\hat{c} \leqslant \phi(x,y) \leqslant c,$$

$$\hat{c} \leqslant \hat{\phi}(x,y) \leqslant c,$$

and

$$\tilde{c} \leqslant \phi(x,y)\,\hat{\phi}(x,y) - [\tilde{\phi}(x,y)]^2$$

for almost all (x,y) in D. Then, for all (X,Y) in R^2,

$$(X\ Y) \begin{pmatrix} \phi(x,y) & \tilde{\phi}(x,y) \\ \tilde{\phi}(x,y) & \hat{\phi}(x,y) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \geqslant \frac{\tilde{c}}{2c}(X^2 + Y^2)$$

at almost all (x,y) in D.

CONJUGATE QUASILINEAR DIRICHLET AND NEUMANN PROBLEMS

Let there be given functions p(X,Y,x,y) and q(X,Y,x,y) satisfying the following conditions (1a)-(1e):

$$\left.\begin{array}{l} p(X,Y,x,y) \text{ and } q(X,Y,x,y) \text{ are defined and finite for all } (X,Y,x,y) \text{ in } R^2 \times D; \\ \text{for each fixed } (x,y) \text{ in } D,\ p(X,Y,x,y) \text{ and } q(X,Y,x,y) \text{ are in } C^1(R^2); \\ \text{for each fixed } (X,Y) \text{ in } R^2,\ p_i(X,Y,x,y) \text{ and } q_i(X,Y,x,y) \text{ are in } L_\infty(D),\ i = 1,2. \end{array}\right\} (1a)$$

There exist positive constants m_1, M_1, m_2, M_2, and m_3 such that for all (X,Y,x,y) in $R^2 \times D$

$$\left.\begin{aligned} m_1 \leq p(X,Y,x,y) \leq M_1, \\ m_1 \leq q(X,Y,x,y) \leq M_1, \end{aligned}\right\} \tag{1b}$$

$$\left.\begin{aligned} m_2 \leq p(X,Y,x,y) + p_1(X,Y,x,y)\,X \leq M_2, \\ m_2 \leq q(X,Y,x,y) + q_2(X,Y,x,y)\,Y \leq M_2, \end{aligned}\right\} \tag{1c}$$

$$\begin{aligned} m_3 \leq {} & [p(X,Y,x,y) + p_1(X,Y,x,y)\,X]\;[q(X,Y,x,y) + q_2(X,Y,x,y)\,Y] \\ & - [p_2(X,Y,x,y)\,X]\;[q_1(X,Y,x,y)\,Y]. \end{aligned} \tag{1d}$$

(Condition (1d) guarantees the uniform ellipticity of the Dirichlet and Neumann problems and is used to show that the energy functionals for these problems are convex.) For all (X,Y,x,y) in $R^2 \times D$,

$$p_2(X,Y,x,y)\,X = q_1(X,Y,x,y)\,Y. \tag{1e}$$

(This condition is the so-called potentialness condition, which guarantees the existence of potentials or energy functionals, the minima of which occur at the solutions of the Dirichlet and Neumann problems. See chapter II of reference 1 for general theorems on the existence of energy functionals.) Note that conditions (1c), (1d), and (1e) together imply that for all (X,Y,x,y) in $R^2 \times D$,

$$m_3 \leq \begin{matrix} [p(X,Y,x,y) + p_1(X,Y,x,y)\,X]\;[q(X,Y,x,y) + q_2(X,Y,x,y)\,Y] \\ - [p_2(X,Y,x,y)\,X] \cdot [q_1(X,Y,x,y)\,Y] \end{matrix} \leq M_2^2. \tag{1d'}$$

Inequalities (1c) and (1d′) and equality (1e) imply that

$$\left.\begin{aligned} |p_2(X,Y,x,y)\,X| \leq \sqrt{M_2^2 - m_3} \\ |q_1(X,Y,x,y)\,Y| \leq \sqrt{M_2^2 - m_3} \end{aligned}\right\} \tag{1f}$$

uniformly for all (X,Y,x,y) in $R^2 \times D$.

Let there be given a function g in $H^{1/2}(D^{\bullet})$. The Dirichlet problem consists in finding a function $U \in H_g^1(D)$ such that

$$\iint_D [p(U_x, U_y, x, y)\,U_x\,h_x + q(U_x, U_y, x, y)\,U_y\,h_y]\,dA = 0 \tag{2}$$

for all h in $H_0^1(D)$. (This is the variational formulation of the Dirichlet problem stated in the introduction.)

Define the energy functional of the Dirichlet problem to be

$$E(u) = \int_0^1 \iint_D [tp(tu_x, tu_y, x, y)\, u_x^2 + tq(tu_x, tu_y, x, y)\, u_y^2]\, dA\, dt$$

on $H_g^1(D)$. Using conditions (1) it can be shown that the first differential of E is

$$E'(u,h) = \iint_D [p(u_x, u_y, x, y)\, u_x h_x + q(u_x, u_y, x, y)\, u_y h_y]\, dA$$

and that the second differential of E is

$$E''(u,h,\hat{h}) = \iint_D \left\{ \begin{array}{l} [p(u_x, u_y, x, y) + p_1(u_x, u_y, x, y)\, u_x]\, h_x \hat{h}_x + [p_2(u_x, u_y, x, y)\, u_x]\, h_x \hat{h}_y \\ \\ +[q_1(u_x, u_y, x, y)\, u_y]\, h_y \hat{h}_x + [q(u_x, u_y, x, y) + q_2(u_x, u_y, x, y)\, u_y]\, h_y \hat{h}_y \end{array} \right\} dA,$$

where h and $\hat{h}$ are any elements of $H_0^1(D)$. The properties of E are used to prove the following two theorems.

Theorem 1. There exists a unique solution of the Dirichlet problem.

Proof. Define the 2-by-2 matrix function $P = P(x,y)$ by

$$P = \begin{pmatrix} p(u_x, u_y, x, y) + p_1(u_x, u_y, x, y)\, u_x & p_2(u_x, u_y, x, y)\, u_x \\ \\ q_1(u_x, u_y, x, y)\, u_y & q(u_x, u_y, x, y) + q_2(u_x, u_y, x, y)\, u_y \end{pmatrix}.$$

From conditions (1c) and (1d) and lemma 13, one obtains that

$$\begin{aligned} E''(u,h,h) &= \iint_D (h_x, h_y)\cdot P \cdot \begin{pmatrix} h_x \\ \\ h_y \end{pmatrix} dA \\ &\geqslant \frac{m_3}{2M_2} \iint_D (h_x^2 + h_y^2)\, dA \end{aligned} \tag{3}$$

uniformly for all u in $H_g^1(D)$ and all h in $H_0^1(D)$. Lemma 5 further implies that

$$E''(u,h,h) \geqslant \frac{m_3}{2M_2\,[4\min(a^2,b^2)+1]}\,\|h\|^2_{H^1(D)}. \tag{4}$$

Hence, E is convex over $H^1_g(D)$. Lemma 3 then implies that there exists a unique function U in $H^1_g(D)$ such that

$$E'(U,h) = 0 \tag{5}$$

for all h in $H^1_0(D)$. This equality coincides with equality (2). Hence, U is the unique solution of the Dirichlet problem.

Theorem 2. If $\{{}_nu\}_{n=1}^{\infty} \subset H^1_g(D)$ is a minimizing sequence for E(u), that is, if

$$E({}_nu) \to \inf_{u \in H^1_g(D)} E(u)$$

as $n \to \infty$, then ${}_nu \to U$ in $H^1(D)$.

Proof. By lemma 3, the unique absolute minimum of E(u) occurs at the solution U of the Dirichlet problem, that is,

$$\inf_{u \in H^1_g(D)} E(u) = E(U).$$

By lemma 2, for each $n \geqslant 1$ there exists a constant τ_n, $0 < \tau_n < 1$, such that

$$\begin{aligned} E({}_nu) - E(U) &= E'(U, {}_nu - U) + \frac{1}{2} E''(U + \tau_n({}_nu - U), {}_nu - U, {}_nu - U) \\ &= \frac{1}{2} E''(U + \tau_n({}_nu - U), {}_nu - U, {}_nu - U), \end{aligned}$$

the second equality following from equality (5). By inequality (4), therefore,

$$E({}_nu) - E(U) \geqslant \frac{m_3}{4M_2\,[4\min(a^2,b^2)+1]}\,\|{}_nu - U\|^2_{H^1(D)}, \tag{6}$$

which implies that ${}_nu \to U$ in $H^1(D)$.

Let there be given a function g in $\overset{\circ}{H}{}^{-1/2}(D^{\bullet})$. The Neumann problem consists in finding a function $V \in L^1_2(D)$ such that

$$\iint_D [p(V_x, V_y, x, y)\,V_x h_x + q(V_x, V_y, x, y)\,V_y h_y]\,dA - (g, h|_{D^{\bullet}}) = 0 \tag{7}$$

for all h in $L_2^1(D)$. (This is the variational formulation of the Neumann problem stated in the introduction.)

Define the energy functional of the Neumann problem to be

$$F(v) = \int_0^1 \iint_D [tp(tv_x, tv_y, x, y) v_x^2 + tq(tv_x, tv_y, x, y) v_y^2] \, dAdt - (g, v|_{D^{\cdot}})$$

on $L_2^1(D)$. Using conditions (1), it can be shown that the first differential of F is

$$F'(v, h) = \iint_D [p(v_x, v_y, x, y) v_x h_x + q(v_x, v_y, x, y) v_y h_y] \, dA - (g, h|_{D^{\cdot}})$$

and that the second differential of F is

$$F''(v, h, \hat{h}) = \iint_D \left\{ \begin{array}{l} [p(v_x, v_y, x, y) + p_1(v_x, v_y, x, y) v_x] h_x \hat{h}_x + [p_2(v_x, v_y, x, y) v_x] h_x \hat{h}_y \\ \\ + [q_1(v_x, v_y, x, y) v_y] h_y \hat{h}_x + [q(v_x, v_y, x, y) + q_2(v_x, v_y, x, y) v_y] h_y \hat{h}_y \end{array} \right\} dA,$$

where h and $\hat{h}$ are any elements of $L_2^1(D)$. The properties of F are used to prove the following two theorems.

Theorem 3. There exists a unique solution of the Neumann problem.

Proof. By the same reasoning as that used to prove inequality (3), one obtains that

$$F''(v, h, h) \geqslant \frac{m_3}{2M_2} \|h\|^2_{L_2^1(D)} \tag{8}$$

uniformly for all v and h in $L_2^1(D)$. Hence, F is convex over $L_2^1(D)$. Lemma 3 then implies that there exists a unique function V in $L_2^1(D)$ such that

$$F'(V, h) = 0 \tag{9}$$

for all h in $L_2^1(D)$. Let V denote also the unique element in $H^1(D)$ out of the equivalence class V which is in ${}_0H^1(D)$. Equality (9) then implies that V satisfies equality (7) for all h in $L_2^1(D)$ or, equivalently, for all h in $H^1(D)$. Hence, V is the unique solution of the Neumann problem.

Theorem 4. If $\{{}_nv\}_{n=1}^{\infty} \subset L_2^1(D)$ is a minimizing sequence for F(v), that is, if

$$F({}_nv) \to \inf_{v \in L_2^1(D)} F(v).$$

as $n \to \infty$, then ${}_n v \to V$ in $L_2^1(D)$. If each element of the sequence is normalized so that $\{{}_n v\}_{n=1}^{\infty} \subset {}_0H^1(D)$, then ${}_n v \to V$ in ${}_0H^1(D)$.

Proof. By lemma 3, the unique absolute minimum of F(v) occurs at the solution V of the Neumann problem, that is,

$$\inf_{v \in L_2^1(D)} F(v) = F(V).$$

By lemma 2, for each $n \geqslant 1$, there exists a constant τ_n, $0 < \tau_n < 1$, such that

$$F({}_n v) - F(V) = F'(V, {}_n v - V) + \frac{1}{2} F''(V + \tau_n({}_n v - V), {}_n v - V, {}_n v - V)$$

$$= \frac{1}{2} F''(V + \tau_n({}_n v - V), {}_n v - V, {}_n v - V),$$

the second equality following from equality (9). By inequality (8), therefore,

$$F({}_n v) - F(V) \geqslant \frac{m_3}{4M_2} \|{}_n v - V\|^2_{L_2^1(D)}, \tag{10}$$

which implies that ${}_n v \to V$ in $L_2^1(D)$. If V and the ${}_n v$ are normalized to be in ${}_0H^1(D)$, then inequality (10) and lemma 12 imply that

$$F({}_n v) - F(V) \geqslant \frac{m_3}{4M_2\,[8\max(a^2, b^2) + 1]} \|{}_n v - V\|^2_{H^1(D)}, \tag{11}$$

which implies that ${}_n v \to V$ in ${}_0H^1(D)$.

The concept of conjugate Dirichlet and Neumann problems is based on the following transformation from R^2 to R^2. Let functions p(X,Y,x,y) and q(X,Y,x,y) satisfying conditions (1) be given. Let (x,y) be a fixed but arbitrary point in D. Consider the transformation $(X,Y) \to (Z,W)$ from R^2 into R^2 defined by

$$\left.\begin{aligned} Z &= q(X, Y, x, y)\,Y, \\ W &= -p(X, Y, x, y)\,X. \end{aligned}\right\} \tag{12}$$

This transformation is a special case of the Legendre dual transformation (see reference 4). The Jacobian matrix of the transformation is

$$J(X, Y, x, y) = \begin{pmatrix} q_1(X, Y, x, y)\,Y & q(X, Y, x, y) + q_2(X, Y, x, y)\,Y \\ -p(X, Y, x, y) - p_1(X, Y, x, y)\,X & -p_2(X, Y, x, y)\,X \end{pmatrix},$$

a matrix which is nonsingular for all (X,Y) in R^2, since, by condition (1d),

$$0 < m_3 \leqslant \det J(X, Y, x, y).$$

The invertibility of J implies that the range of transformation (12) is open. Using condition (1b), it can be shown that the range is also closed. Since the range is nonempty ((0,0) is in the range), then, transformation (12) maps R^2 onto R^2. Let (X,Y) and $(\hat{X},\hat{Y})$ both be preimages of some point (Z,W). Then, by lemma 1,

$$0 = \left(\begin{pmatrix} Z \\ W \end{pmatrix} - \begin{pmatrix} Z \\ W \end{pmatrix}, \begin{pmatrix} \hat{Y} - Y \\ -\hat{X} + X \end{pmatrix}\right)_{R^2} = \left(J(X, Y, x, y) \begin{pmatrix} \hat{X} - X \\ \hat{Y} - Y \end{pmatrix}, \begin{pmatrix} \hat{Y} - Y \\ -\hat{X} + X \end{pmatrix}\right)_{R^2}$$

$$= (\hat{X}-X, \hat{Y}-Y) \begin{pmatrix} p(\tilde{X}, \tilde{Y}, x, y) + p_1(\tilde{X}, \tilde{Y}, x, y)\tilde{X} & p_2(\tilde{X}, \tilde{Y}, x, y)\tilde{X} \\ q_1(\tilde{X}, \tilde{Y}, x, y)\tilde{Y} & q(\tilde{X}, \tilde{Y}, x, y) + q_2(\tilde{X}, \tilde{Y}, x, y)\tilde{Y} \end{pmatrix} \begin{pmatrix} \hat{X}-X \\ \hat{Y}-Y \end{pmatrix}$$

$$\geqslant c\,[(\hat{X} - X)^2 + (\hat{Y} - Y)^2]$$

for some positive constant c, where $\tilde{X} = X + t(\hat{X} - X)$ and $\tilde{Y} = Y + t(\hat{Y} - Y)$ for some t in (0,1). The inequality above follows from the positive definiteness of the matrix in the second-to-last line (see lemma 13) and implies that $\hat{X} = X$ and $\hat{Y} = Y$. Hence, transformation (12) is not only onto but also one-to-one, that is, it is an isomorphism of R^2 onto R^2. This permits one to define conjugate functions $\hat{p}(Z,W,x,y)$ and $\hat{q}(Z,W,x,y)$ for all (Z,W) in R^2 by the relations

$$\left.\begin{aligned} \hat{p}(Z, W, x, y) &= \frac{1}{q(X, Y, x, y)}, \\ \hat{q}(Z, W, x, y) &= \frac{1}{p(X, Y, x, y)}. \end{aligned}\right\} \tag{13}$$

From equalities (12) and (13), it is clear that the inverse transformation $(Z,W) \rightarrow (X,Y)$ is given by

$$\left.\begin{aligned} X &= -\hat{q}(Z, W, x, y)\,W, \\ Y &= \hat{p}(Z, W, x, y)\,Z. \end{aligned}\right\} \tag{14}$$

The functions $\hat{p}$ and $\hat{q}$ defined by equalities (13) satisfy the same conditions as p and q (the constants m_i and M_i may be different). Indeed, equalities (13) imply that $\hat{p}$ and $\hat{q}$ satisfy condition (1b) with m_1 and M_1 replaced by $\hat{m}_1 = 1/M_1$ and $\hat{M}_1 = 1/m_1$, respectively. Since the mapping $(X,Y) \rightarrow (Z,W)$ is continuously differentiable with nonsingular Jacobian matrix, the inverse mapping is also continuously differentiable. This fact in turn implies that $\hat{p}$ and $\hat{q}$ satisfy conditions (1a), since from equalities (13) one obtains, for example,

$$\hat{p}_1(Z,W,x,y) = \frac{-q_1(X,Y,x,y)\dfrac{\partial X}{\partial Z} - q_2(X,Y,x,y)\dfrac{\partial Y}{\partial Z}}{[q(X,Y,x,y)]^2}$$

For fixed (x,y) in D, consider the Jacobian matrix

$$[J(X,Y,x,y)]^{-1} = \hat{J}(Z,W,x,y) \equiv \begin{pmatrix} -\hat{q}_1(Z,W,x,y)W & -\hat{q}(Z,W,x,y) - \hat{q}_2(Z,W,x,y)W \\ \hat{p}(Z,W,x,y) + \hat{p}_1(Z,W,x,y)Z & \hat{p}_2(Z,W,x,y)Z \end{pmatrix}$$

of the inverse transformation (14). From condition (1d′) for p and q, one obtains condition (1d) for $\hat{p}$ and $\hat{q}$:

$$\begin{aligned} \hat{m}_3 \leqslant [\det J(X,Y,x,y)]^{-1} &= \det \hat{J}(Z,W,x,y) \\ &= [\hat{p}(Z,W,x,y) + \hat{p}_1(Z,W,x,y)Z]\ [\hat{q}(Z,W,x,y) + \hat{q}_2(Z,W,x,y)W] \\ &\quad - [\hat{p}_2(Z,W,x,y)Z]\ [\hat{q}_1(Z,W,x,y)W], \end{aligned}$$

where $\hat{m}_3 = 1/M_2^2$. Equating the off-diagonal elements of $\hat{J}$ and J^{-1} and using conditions (1c) and (1d′) for p and q yields condition (1c) for $\hat{p}$ and $\hat{q}$ with m_2 and M_2 replaced by $\hat{m}_2 = m_2/(M_2)^2$ and $\hat{M}_2 = M_2/m_3$, respectively. Finally, equating the diagonal elements of $\hat{J}$ and J^{-1} and using condition (1e) for p and q imply that $\hat{p}$ and $\hat{q}$ satisfy condition (1e). Thus, $\hat{p}$ and $\hat{q}$ satisfy all of conditions (1) (with different constants).

The relationship between the energies of the solutions of conjugate Dirichlet and Neumann problems is a consequence of the following lemma.

Lemma 14. For all (X,Y) in R^2,

$$\begin{aligned} &\hat{p}(Z,W,x,y)Z^2 + \hat{q}(Z,W,x,y)W^2 \\ &- \int_0^1 [t\,\hat{p}(tZ,tW,x,y)\,Z^2 + t\,\hat{q}(tZ,tW,x,y)W^2]\,dt \\ &= \int_0^1 [t\,p(tX,tY,x,y)X^2 + t\,q(tX,tY,x,y)Y^2]\,dt. \end{aligned} \tag{15}$$

Proof. Fix (x,y) in D. Consider X and Y to be functions of Z and W. Define $\phi(Z,W)$ to be the left side of equality (15) minus the right side. Using relations (12) and (14), it is easily shown that $\partial\phi/\partial Z = \partial\phi/\partial W = 0$. This, and the fact that $\phi(0,0) = 0$, imply that $\phi(Z,W) = 0$ for all (Z,W) in R^2, from which equality (15) follows.

Let there be given the Dirichlet problem (2). Define the functions $\hat{p}$ and $\hat{q}$ by relations (13). Define $\hat{g}$ in $\overset{o}{H}{}^{-1/2}(D^{\bullet})$ by the equality

$$\hat{g} = g'.$$

The definition of g' involves solving a linear Dirichlet problem with coefficients $p = q \equiv 1$, but if g is in $\{ g \in C^0(D^{\bullet}) \colon g|_{D_i^{\bullet}} \in W_2^1(D_i^{\bullet}),\ i = 1,2,3,4 \}$, then, by lemma 10, g' is the conventional distributional derivative of g. The problem of finding $V \in {}_0H^1(D)$ such that

$$\iint_D [\hat{p}(V_x, V_y, x, y)\, V_x h_x + \hat{q}(V_x, V_y, x, y)\, V_y h_y]\, dA - (\hat{g}, h|_{D^{\bullet}}) = 0 \tag{16}$$

for all h in $H^1(D)$ is a well-posed Neumann problem with a unique solution. This problem will be called the Neumann problem conjugate to the given Dirichlet problem (2).

Let U be the solution of the Dirichlet problem. Since equality (2) holds for all h in $H_0^1(D)$, lemma 4 implies that there exists a unique $\hat{v}$ in $L_2^1(D)$ such that

$$\left.\begin{aligned} \hat{v}_x &= q(U_x, U_y, x, y)\, U_y, \\ \hat{v}_y &= -p(U_x, U_y, x, y)\, U_x. \end{aligned}\right\} \tag{17}$$

Denote also by $\hat{v}$ the unique element of the equivalence class $\hat{v}$ which is in ${}_0H^1(D)$. Equalities (17) imply that

$$U_x = -\hat{q}(\hat{v}_x, \hat{v}_y, x, y)\, \hat{v}_y,$$

$$U_y = \hat{p}(\hat{v}_x, \hat{v}_y, x, y)\, \hat{v}_x.$$

Therefore, by lemma 9,

$$\iint_D [\hat{p}(\hat{v}_x, \hat{v}_y, x, y)\, \hat{v}_x h_x + \hat{q}(\hat{v}_x, \hat{v}_y, x, y)\, \hat{v}_y h_y]\, dA = \iint_D (U_y h_x - U_x h_y)\, dA$$

$$= (g', h|_{D^{\bullet}})$$

for all h in $H^1(D)$, that is, $\hat{v} = V$, the solution of the Neumann problem (16). Hence, the relations

$$\left.\begin{aligned} V_x &= q(U_x, U_y, x, y)\, U_y, \\ V_y &= -p(U_x, U_y, x, y)\, U_x \end{aligned}\right\} \tag{18}$$

hold between the solution U of the given Dirichlet problem (2) and the solution V of the conjugate Neumann problem.

Let there be given the Neumann problem (16). Define p and q by relations (13). Define g in $H^{1/2}(D^\bullet)$ by the equality

$$g = (\int \hat{g}) + \hat{c}, \tag{19}$$

where $\hat{c}$ is a constant which can be adjusted according to external criteria. The definition of $\int \hat{g}$ involves solving a linear Neumann problem with coefficients $\hat{p} = \hat{q} \equiv 1$, but if $\hat{g}$ is $L_2(D^\bullet)$, then, by lemma 11, g coincides with the distributional primitive $g(s) = \int_0^s \hat{g}(\hat{s})\, d\hat{s} + c$ of $\hat{g}(s)$. The Dirichlet problem of finding $U \in H_g^1(D)$ such that equality (2) is satisfied, where p, q, and g are defined by equalities (13) and (19), will be called the Dirichlet problem conjugate to the given Neumann problem (16).

Equality (16) and lemma 4 imply that there exists a unique $\hat{u}$ in $L_2^1(D)$ such that

$$\left.\begin{aligned} \hat{u}_x &= -\hat{q}(V_x, V_y, x, y)\, V_y, \\ \hat{u}_y &= \hat{p}(V_x, V_y, x, y)\, V_x. \end{aligned}\right\} \tag{20}$$

Denote also by $\hat{u} \in H^1(D)$ the unique element of the equivalence class $\hat{u}$ which is such that $\int_{D^\bullet} \hat{u}|_{D^\bullet}\, ds = \int_{D^\bullet} g\, ds$. Then equalities (20) imply that

$$\iint_D (\hat{u}_y h_x - \hat{u}_x h_y)\, dA = \iint_D [\hat{p}(V_x, V_y, x, y)\, V_x h_x + \hat{q}(V_x, V_y, x, y)\, V_y h_y]\, dA$$

$$= (\hat{g}, h|_{D^\bullet})$$

for all h in $H^1(D)$. Hence, by equality (19) and lemmas 8 and 9, $\hat{u}|_{D^\bullet} = g + c$ for some constant c. But since $\int_{D^\bullet} \hat{u}|_{D^\bullet}\, ds = \int_{D^\bullet} g\, ds$ ($\hat{u}$ was chosen so that this condition is satisfied), c must be zero, that is, $\hat{u} \in H_g^1(D)$. Equalities (20) imply that

$$V_x = q(\hat{u}_x, \hat{u}_y, x, y)\, \hat{u}_y,$$

and

$$V_y = -p(\hat{u}_x, \hat{u}_y, x, y)\, \hat{u}_x.$$

Therefore, by lemma 9, for all h in $H_0^1(D)$,

$$\iint_D [p(\hat{u}_x, \hat{u}_y, x, y)\, \hat{u}_x h_x + q(\hat{u}_x, \hat{u}_y, x, y)\, \hat{u}_y h_y]\, dA = -\iint_D (V_y h_x - V_x h_y)\, dA$$

$$= -\left((V|_{D^\bullet})', 0\right)$$

$$= 0,$$

that is, $\hat{u} = U$, the solution of the Dirichlet problem (2). Hence, relations (18) hold between the solution V of the given Neumann problem (16) and the solution U of the conjugate Dirichlet problem.

An important property of the conjugation procedures described above is that the problem resulting from two successive conjugations is again the original problem. Specifically, if one constructs the Neumann problem conjugate to a given Dirichlet problem and constructs the Dirichlet problem conjugate to that Neumann problem, then it is clear from the symmetry of relations (13) and from lemma 7 that one obtains the original Dirichlet problem back again (by adjusting the constant $\hat{c}$ in equality (19) properly). If one constructs the Dirichlet problem conjugate to a given Neumann problem and constructs the Neumann problem conjugate to that Dirichlet problem, then it is clear from the symmetry of relations (13) and from lemma 8 that one obtains the original Neumann problem back again.

Of greatest importance is the property of conjugate Dirichlet and Neumann problems stated in theorem 5.

Theorem 5. Let the Dirichlet problem (2) and the Neumann problem (16) be conjugate to each other and have energy functionals E and F and solutions U and V, respectively. Then

$$-F(V) = E(U). \tag{21}$$

Remark. Sewell (reference 4) presents a similar result based on the Legendre dual transformation without discussion of the existence of the potential function of the conjugate problem or of the differentiation or integration of the boundary function.

Proof. Since $F'(V,h) = 0$ for all h in $H^1(D)$, clearly $F'(V,V) = 0$ or, equivalently,

$$(\hat{g}, V|_{D^\cdot}) = \iint_D [\hat{p}(V_x, V_y, x, y)\, V_x^2 + \hat{q}(V_x, V_y, x, y)\, V_y^2]\, dA.$$

Therefore,

$$-F(V) = \iint_D [\hat{p}(V_x, V_y, x, y)\, V_x^2 + \hat{q}(V_x, V_y, x, y)\, V_y^2]\, dA$$

$$- \int_0^1 \iint_D [t\, \hat{p}(t\, V_x, t\, V_y, x, y)\, V_x^2 + t\, \hat{q}(t\, V_x, t\, V_y, x, y)\, V_y^2]\, dAdt.$$

Relations (18) and lemma 14 (and Fubini's theorem, which permits interchanging integrations over D and over t) then imply that equality (21) holds.

A POSTERIORI ERROR BOUNDS

Let $\{{}_n u\}_{n=1}^{\infty} \subset H_g^1(D)$ and $\{{}_n v\}_{n=1}^{\infty} \subset {}_0H^1(D)$ be minimizing sequences for the energy functionals of the conjugate Dirichlet and Neumann problems. Then, by theorem 5, the inequalities

$$-F({}_n v) \leqslant E(U) \leqslant E({}_n u) \tag{22a}$$

and

$$-E({}_n u) \leqslant F(V) \leqslant F({}_n v) \tag{22b}$$

hold for all n. Moreover, since $E({}_n u) \to E(U)$ and $F({}_n v) \to F(V)$, the two-sided error bounds in inequalities (22) converge to E(U) and F(V), respectively, as $n \to \infty$. For the Dirichlet solution U, inequalities (6) and (22a) yield the a posteriori error bound

$$\frac{4 M_2 \, [4 \min(a^2, b^2) + 1]}{m_3} [E({}_n u) + F({}_n v)] \geqslant \| {}_n u - U \|^2_{H^1(D)}. \tag{23a}$$

In certain cases, the a posteriori error bound,

$$\frac{4 M_2}{m_3} [E({}_n u) + F({}_n v)] \geqslant \| {}_n u - U \|^2_{L_2^1(D)}, \tag{23b}$$

which is derived from inequality (3) in the same manner in which inequality (23a) was derived from inequality (4), is more meaningful physically than inequality (23a). For the Neumann problem, inequalities (10), (11), and (22b) yield the a posteriori error bounds.

$$\frac{4 \hat{M}_2}{\hat{m}_3} [F({}_n v) + E({}_n u)] \geqslant \| {}_n v - V \|^2_{L_2^1(D)} \tag{24a}$$

and

$$\frac{4 \hat{M}_2 \, [8 \max(a^2, b^2) + 1]}{\hat{m}_3} [F({}_n v) + E({}_n u)] \geqslant \| {}_n v - V \|^2_{H^1(D)}. \tag{24b}$$

Note that, by theorem 5, the bounds in inequalities (23) and (24) approach zero, since

$$E({}_n u) + F({}_n v) \to E(U) - E(U) = 0.$$

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