TECHNICAL NOTE 2249

THE SPANWISE DISTRIBUTION OF LIFT FOR MINIMUM INDUCED DRAG OF WINGS HAVING A GIVEN LIFT AND A GIVEN BENDING MOMENT

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SUMMARY

The problem of the minimum induced drag of wings having a given lift and a given span is extended to include cases in which the bending moment to be supported by the wing is also given. As in the classical problem of induced drag, the theory is limited to lifting surfaces traveling at subsonic speeds. It is found that the required shape of the downwash distribution can be obtained in an elementary way which is applicable to a variety of such problems. Expressions for the minimum drag and the corresponding spanwise load distributions are also given for the case in which the lift and the bending moment about the wing root are fixed while the span is allowed to vary. The results show a 15-percent reduction of the induced drag with a 15-percent increase in span as compared with results for an elliptically loaded wing having the same total lift and bending moment.

INTRODUCTION

In the problem of minimum induced drag as originally treated by Munk (references 1 and 2) the span of the wing and the total lift were supposed to be given and the distribution of lift over the span resulting in a minimum of drag was sought. The solution of this problem thus provided a convenient lower bound for the induced drag of a wing of given dimensions.

In the practical design of wings the requirements for low induced drag and the requirements for structural strength are opposed. Here the bending moment developed by the lift becomes an important consideration - more important in many cases than the actual spanwise dimension of the wing. Such considerations lead to the problem of determining the minimum drag with limitations imposed on the bending moment as well as on the total lift. It is the purpose of the present paper to show how the methods of the earlier analysis can be extended in a very simple way to the solution of problems involving the bending moment of the load distribution.

A complete list of symbols employed in the analysis will be found in the appendix.

GENERAL FORMULAS FOR LIFT, DRAG, AND BENDING MOMENT

Reference may be made to the original papers of Prandtl and Munk (references 1 and 2), or to any of the standard text books on aerodynamics, for the fundamental developments of wing theory which form the basis for the calculations of induced drag. In these developments the over-all lift is given by

$$L = \rho V \int_{-s}^{+s} \Gamma \, dy \tag{1}$$

and the drag is given by

$$D_i = \rho \int_{-s}^{+s} w_i \, \Gamma \, dy \tag{2}$$

In these formulas the wing span is supposed to extend along the y axis between $-s$ and $+s$, Γ is the local circulation or vortex strength, and V is the constant velocity of flight. The induced downwash velocity w_i is variable along the span and is connected with the vortex distribution $\Gamma(y)$ through the relation

$$w_i(y) = \frac{1}{4\pi} \int_{-s}^{+s} \frac{d\Gamma/d\eta}{y-\eta} \, d\eta \tag{3}$$

With this value for w_i the expression for the drag may be converted to a double integral involving the spanwise distribution of lift as represented by the circulation strength Γ

$$D_i = \frac{\rho}{4\pi} \int_{-s}^{+s} \int_{-s}^{+s} \frac{\Gamma(y)\Gamma'(\eta)}{y-\eta} \, dy \, d\eta \tag{4}$$

This integral may be reduced to a more symmetric form if it is integrated by parts on the supposition that Γ falls to zero at the wing tips. Thus[1]

$$D_i = \frac{\rho}{4\pi} \int_{-s}^{+s} \int_{-s}^{+s} \frac{\Gamma(y)\Gamma(\eta)}{(y-\eta)^2} \, dy \, d\eta \tag{5}$$

[1]The validity of equations (3), (4), and (5) can be demonstrated by referring to the limiting values of complex integrals taken along a path a short distance above the singular point on the real axis. In the case of equations (3) and (4) this process yields the Cauchy principal value.

In mathematical terms the problem is to minimize the double integral, equation (5), while holding fixed values of

$$L = \rho V \int_{-s}^{+s} \Gamma \, dy \tag{6}$$

and

$$B = \rho V \int_{s_o}^{s} (y-s_o) \, \Gamma \, dy \tag{7}$$

where B is the bending moment specified about the point s_o. For the time being s_o will be taken as the origin, or wing root ($s_o = 0$), although later another example will appear.

Although the derivation of the formulas for induced drag makes use of the concept of the lifting line, it is important to note that the results are not actually restricted to this approximation. According to Munk's well-known stagger theorem the induced drag of a lifting surface will be equal to that of a lifting line if the spanwise load distributions are the same.

It should be noted further that the induced drag of a wing having a given lift and a given spanwise load distribution is not affected by the compressibility of the air at subsonic speeds. At supersonic speeds an additional drag associated with the formation of waves arises and the induced drag, which is associated with the vortex wake, becomes only a part of the total pressure drag.

THE DISTRIBUTION OF DOWNWASH FOR MINIMUM DRAG

In general, if the drag is to be a minimum, a small variation in the shape of the curve of spanwise loading will produce no first-order change in the drag. The variation in shape may take the form of a small addition to the original loading; it is then necessary to find conditions under which the drag added by a small additional loading is zero.

The solution of this latter problem is rendered especially simple by the mutual drag theorem (reference 1), which arises from the evident symmetry of the integral to be minimized (equation (5)). The theorem states that if the lift distribution (represented by Γ (y)) is the sum of two distributions Γ_1, and Γ_2, the drag of Γ_1 arising from the downwash field of Γ_2 is exactly equal to the drag of Γ_2 arising from the downwash of Γ_1.

Consider now an initial distribution designed to achieve minimum drag. (See fig. 1.) The drag added by a small additional loading will be composed of three parts, namely:

1. The drag of the additional lift acting alone
2. The drag of the original loading arising from the downwash field of the additional loading
3. The drag of the additional loading induced by the downwash field of the original loading

Item 1 is of second order in terms of the magnitude of the added lift for smooth distributions, that is, so-called "weak variations." (The fact that this second-order term is invariably positive insures that the drag will be a minimum and not a maximum.) Items 2 and 3 are equal by the mutual drag theorem. The first-order variation in drag can then be computed by considering only the drag of the small additional lift acting in the induced downwash field $w_i(y)$ of the original lift.

The conditions of fixed bending moment and fixed total lift are met by allowing only those curves of lift variation that produce no change in these quantities, that is, curves having zero area and zero moment. It can be seen that such curves of variation must have at least three elements to meet the conditions of zero area and zero moment. Furthermore, any curve meeting these conditions can be subdivided into groups of three elements so that the individual groups also satisfy the conditions. Hence, as the representative of such restricted curves of variation we may adopt three small elements having areas l_1, l_2, and l_3 (fig. 1). These elements, together with their positions y_1, y_2, and y_3 and the local values of the downwash w_{i_1}, etc., due to the original loading must satisfy the following three equations:

$$\left.\begin{array}{lll} \text{for} & \delta L = 0, & l_1 + l_2 + l_3 = 0 \\ \text{for} & \delta B = 0, & l_1 y_1 + l_2 y_2 + l_3 y_3 = 0 \\ \text{for} & \delta D_i = 0, & l_1 w_{i_1} + l_2 w_{i_2} + l_3 w_{i_3} = 0 \end{array}\right\} \tag{8}$$

It can be seen that these equations will be consistent if $w_{i_1} \sim a+by_1$, $w_{i_2} \sim a+by_2$ and $w_{i_3} \sim a+by_3$, where a and b are constants to be determined from the given conditions. Since such equations must be satisfied for all positions y_1, y_2, etc., it is concluded that, in general,

$$w_i \sim a + by \tag{9}$$

Hence, for a minimum induced drag with a given total lift and a given bending moment the downwash must show a linear variation along the span.[2] (See fig. 2.)

The foregoing method may be readily extended to a more general class of problems involving bending moments or rolling moments. Suppose, for example, a braced wing is considered, as in the dotted outline of figure 3. In this case the bending moment developed by that portion of the lift acting inboard of the point of bracing attachment may be of no concern, but it may be desired to limit the bending moment developed by that portion of the spanwise load curve extending between this point and the tip. In this case s_o will not be zero. At least three elements are required to preserve stationary values of the lift and bending moment, and it is evident that at least two of the elements must lie to the right of the point s_o. The three simultaneous equations are (see fig. 2):

$$\left.\begin{aligned} l_1 + l_2 + l_3 &= 0 \\ l_2(y_2-s_o) + l_3(y_3-s_o) &= 0 \\ l_1 w_{i_1} + l_2 w_{i_2} + l_3 w_{i_3} &= 0 \end{aligned}\right\} \qquad (10)$$

Here y_2 and y_3 are to the right of the point s_o and y_1 lies to the left of this point. For these equations[3] to be consistent w_i must have the form

$$w_{i_1} \sim a; \quad w_{i_2} \sim a+b(y_2-s_o); \quad w_{i_3} \sim a+b(y_3-s_o)$$

Hence, in general, the downwash will be a constant over the portion of the span for which the moment is not specified, as illustrated in figure 3. If no restriction whatever is placed on the moment there is obtained the solution of Munk's original problem, namely, that the downwash should be constant over the entire span.

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[2]It may be noticed at this point that, whereas the discussion has emphasized the idea of minimizing the drag, the analysis actually makes no distinction between the lift, bending moment, or drag, in that stationary values of all three are demanded. Thus equation (9) may be considered a necessary condition for the solution of the following problems: (1) given the total lift and the induced drag to find the distribution of lift over the span that will result in a minimum bending moment, and (2) given the bending moment and the induced drag to find the distribution resulting in the maximum total lift.

[3]See reference 3 for a discussion of solutions of such equations.

Determination of Span Loading and Induced Drag From the Downwash Distribution

The case of bilateral symmetry with moment specified about the root section will serve as an example of the calculation of the actual span loading and induced drag. It will be evident from the foregoing that the downwash distribution will consist of two straight-line segments with a reversal of slope at the plane of symmetry. It is then necessary to compute the spanwise variation of Γ corresponding to such a curve of downwash.

To perform this calculation by standard methods of airfoil theory, use is made of the idea that at a great distance behind the wing the vortex sheet forms a two-dimensional field of motion, with the discontinuity in the lateral velocity across the sheet given by $d\Gamma/dy$, and the downwash w given by twice the value of the induced downwash w_i at the wing. Hence, the quantity $1/2\ (d\Gamma/dy) - 2iw_i$ can be evaluated by means of the familiar complex velocity function $v - iw$ of the two-dimensional potential theory using for v its value just above the vortex sheet. In this theory if the vertical component of velocity w is given along the line representing the trace of the span, then the velocity vector at any other point in the field $\xi = y + iz$ may be obtained from the relation (reference 4)

$$v - iw = \frac{1}{\pi} \frac{1}{\sqrt{s^2 - \xi^2}} \int_{-s}^{+s} \frac{w(\eta) \sqrt{s^2-\eta^2}}{\eta-\xi} d\eta \tag{11}$$

As noted above,

$$\frac{d\Gamma}{dy} = v(y + oi) - v(y - oi) = 2v(y + oi) \tag{12}$$

Introducing $w = a + by$ for $y > 0$ and $w = a - by$ for $y < 0$ into equation (11) yields, after integration,

$$\frac{d\Gamma}{dy} = -2a \frac{y}{\sqrt{s^2-y^2}} + \frac{4s}{\pi} b \left(\frac{y}{s} \cosh^{-1} \frac{s}{|y|} - \frac{y}{\sqrt{s^2-y^2}} \right) \tag{13}$$

and hence

$$\Gamma = 2 \left(a + \frac{bs}{\pi} \right) \sqrt{s^2-y^2} + \frac{2b}{\pi} y^2 \cosh^{-1} \frac{s}{|y|}$$

The spanwise loading thus contains the elliptical distribution as one component.

Equation (13) for the spanwise distribution of circulation enables the determination of the over-all lift, bending moment, and drag in terms of the unassigned constants a and b. The use of equations (2), (6), and (7), together with the wing semispan s, yields the following values:

$$\left.\begin{aligned} L &= \rho V s^2 \left(\pi a + \frac{4}{3} bs \right) \\ B &= \rho V s^3 \left(\frac{2}{3} a + \frac{1}{\pi} bs \right) \\ D_i &= \frac{a}{2V} L + \frac{b}{V} B \end{aligned}\right\} \tag{14}$$

It is convenient to specify the bending moment of the lift in terms of the lateral position of the centroid, or center of pressure, of the load curve. The lateral centroid as a fraction of the semispan s may be denoted by y' (i.e., $y' = 2B/Ls$). Then, solving for a and b,

$$\left.\begin{aligned} a &= \frac{L}{\rho V s^2} 9 \left(\frac{1}{\pi} - \frac{2}{3} y' \right) \\ bs &= \frac{L}{\rho V s^2} 9 \left(\frac{\pi}{2} y' - \frac{2}{3} \right) \end{aligned}\right\} \tag{15}$$

The expression for induced drag in terms of the lift and the lateral center of pressure becomes

$$D_i = \frac{L^2}{\pi \frac{\rho}{2} V^2 (2s)^2} \left(\frac{9}{2} \pi^2 y'^2 - 12\pi y' + 9 \right) \tag{16}$$

This equation yields the minimum drag for the given position of y'. If the lateral center of pressure is specified so as to coincide with that for an elliptical loading (i.e., $b = 0$; $y' = 4/3\pi$), then the above formula reduces to

$$D_i = \frac{L^2}{\pi \frac{\rho}{2} V^2 (2s)^2} \tag{17}$$

The optimum distribution of loading for a given position of the centroid y' may be obtained from equation (13) with the aid of equations (15). The result is

$$2s\,\frac{\Gamma\rho V}{L} = \left(\frac{12}{\pi} - 6y'\right)\frac{\sqrt{s^2-y^2}}{s} + \left(18y' - \frac{24}{\pi}\right)\frac{y^2}{s^2}\cosh^{-1}\frac{s}{|y|} \qquad (18)$$

Drag for a Given Bending Moment with Unrestricted Span

The foregoing calculations show, as was to be expected, that the elliptic loading yields a smaller drag than any of the others within a restricted span. However, if the restriction on the span is removed, still lower values of the induced drag can be obtained without any increase in the bending moment at the wing root. The lower values are obtained by permitting the span to increase and at the same time adopting a more tapered form of the loading curve.

Equation (16) which contains the three variables lift, span, and center of pressure can be easily rearranged to show the variation of drag with span when the bending moment and the lift are held at fixed values. In this case, the lateral position of the center of pressure y's will be fixed, while the form and extent s of the load curve will vary. In order to provide a convenient basis for comparison the span and shape of the load curves will be related to the elliptic loading. If s/s_e denotes the ratio of the semispan of the wing to that of an elliptically loaded wing having the same total lift and bending moment, then equation (16) can be rewritten:

$$D_i = \frac{L^2}{\pi\frac{\rho}{2}V^2(2s_e)^2}\left[8\left(\frac{s_e}{s}\right)^4 - 16\left(\frac{s_e}{s}\right)^3 + 9\left(\frac{s_e}{s}\right)^2\right] \qquad (19)$$

The quantity in the bracket is the ratio of the induced drag to that of the corresponding elliptically loaded wing. This ratio is plotted in figure 4 to show the decrease of drag possible by increase of the span. The forms of load curve required for the minimum drag at various values of s/s_e are shown in figure 5.

It will be noted that a 15-percent reduction of the induced drag below that for elliptic loading can be achieved with a 15-percent increase in span. Further increases of span between 15 percent and 50 percent (s = 1.15 to 1.50) yield no significant reductions, however. At still larger values of s the drag becomes lower, and approaches zero at an infinite value of s. For extreme values of s/s_e the curves begin to show negative loadings at the tips and eventually the bending moment at certain points along the span will exceed that at the wing root.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics,
Moffett Field, Calif., Sept. 25, 1950.

APPENDIX

DEFINITIONS OF SYMBOLS

L	total lift
l	element of lift
D_i	induced drag
B	bending moment
ρ	air density
Γ	circulation
w_i	induced downwash velocity at wing
w	downwash velocity, at infinity ($w - 2w_i$)
v	lateral velocity
V	velocity of flight
y,η	distances along wing semispan
s_o	point of origin for bending moment
s	length of wing semispan
y'	lateral position of load centroid as a fraction of s
a,b	constants

REFERENCES

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4. Munk, Max M.: Elements of the Wing Section Theory and of the Wing Theory. NACA Rep. 191, 1924.

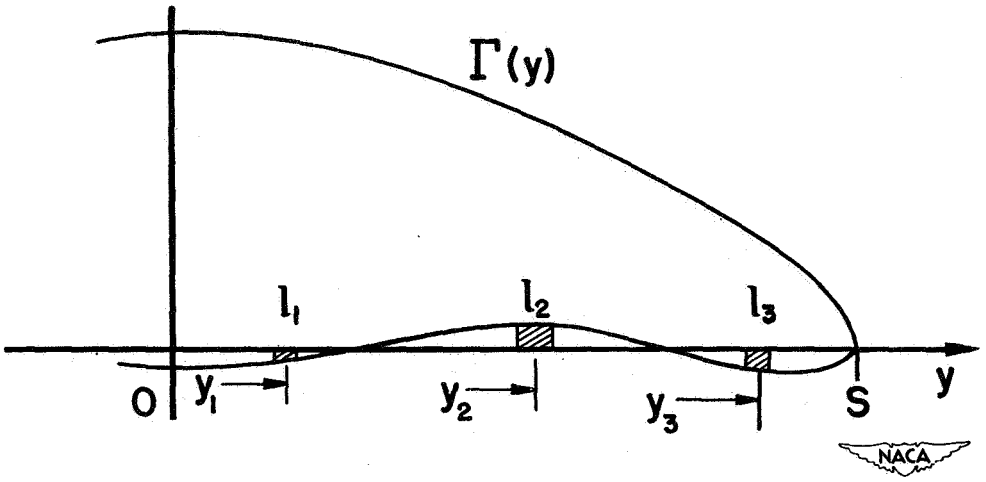

FIGURE 1.- SPANWISE LOAD CURVE WITH THREE ELEMENTS OF VARIATION

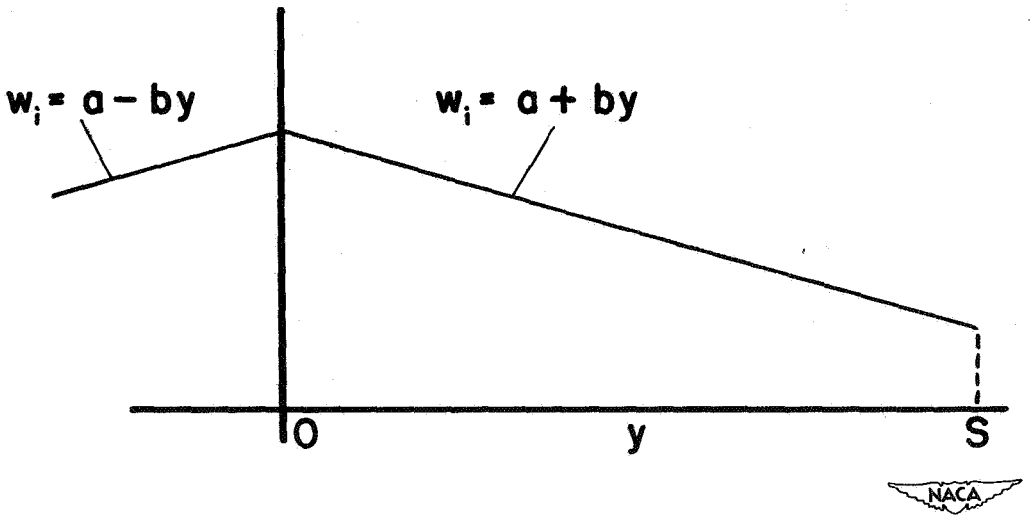

FIGURE 2.- DOWNWASH DISTRIBUTION FOR MINIMUM DRAG WITH RESTRICTED BENDING MOMENT ABOUT CENTER SECTION

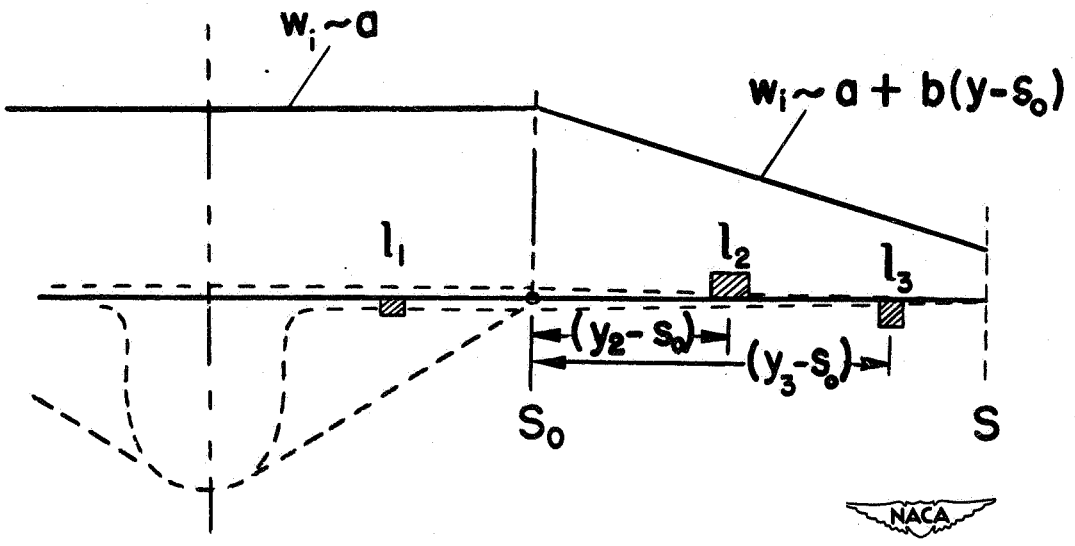

FIGURE 3.- DOWNWASH DISTRIBUTION FOR MINIMUM DRAG WITH RESTRICTION ON BENDING MOMENT OF THE OUTER PORTION OF THE LOAD CURVE

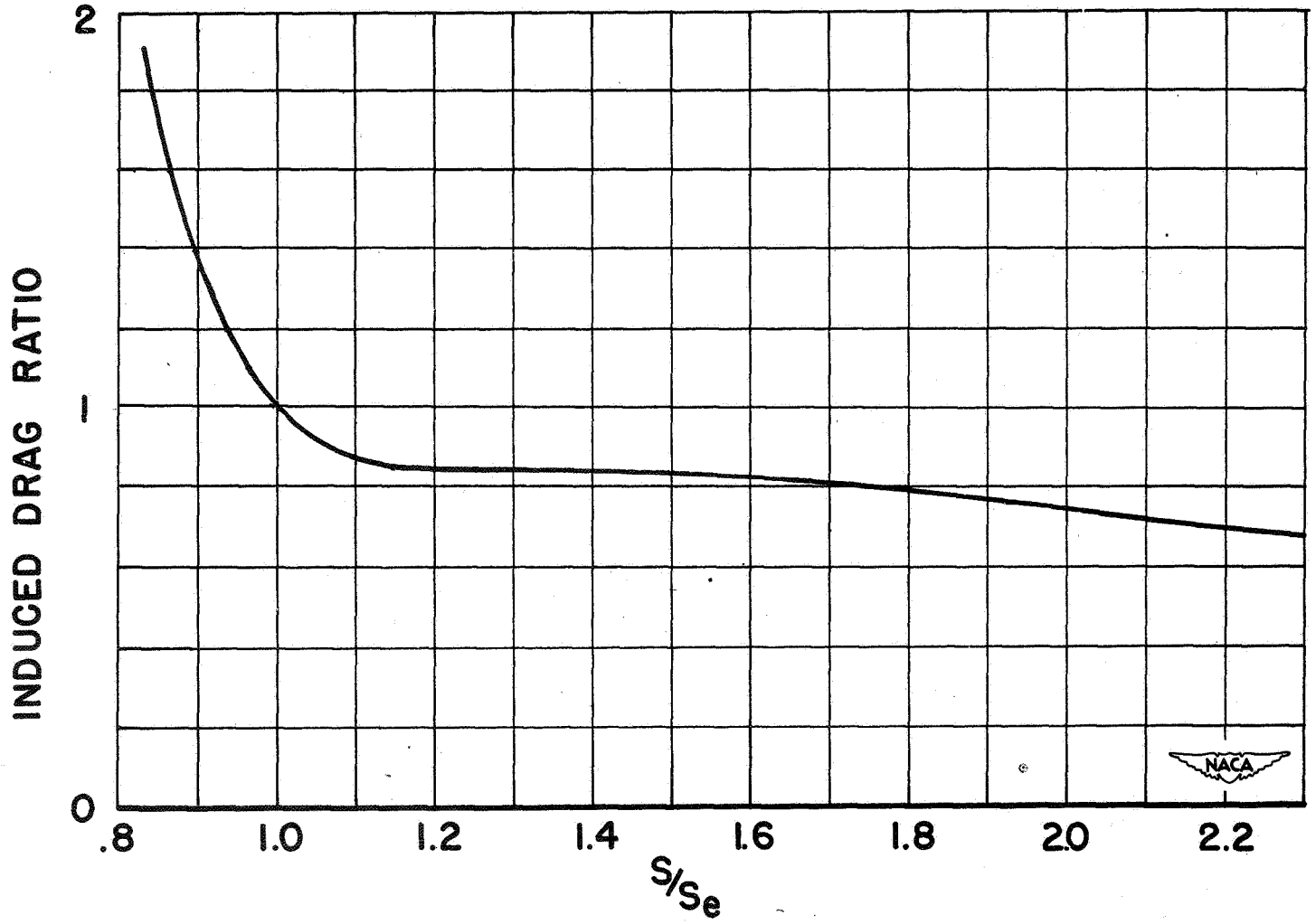

FIGURE 4.- VARIATION OF INDUCED DRAG WITH SEMISPAN RATIO S/S_e, FOR WINGS HAVING A FIXED TOTAL LIFT AND A FIXED BENDING MOMENT.

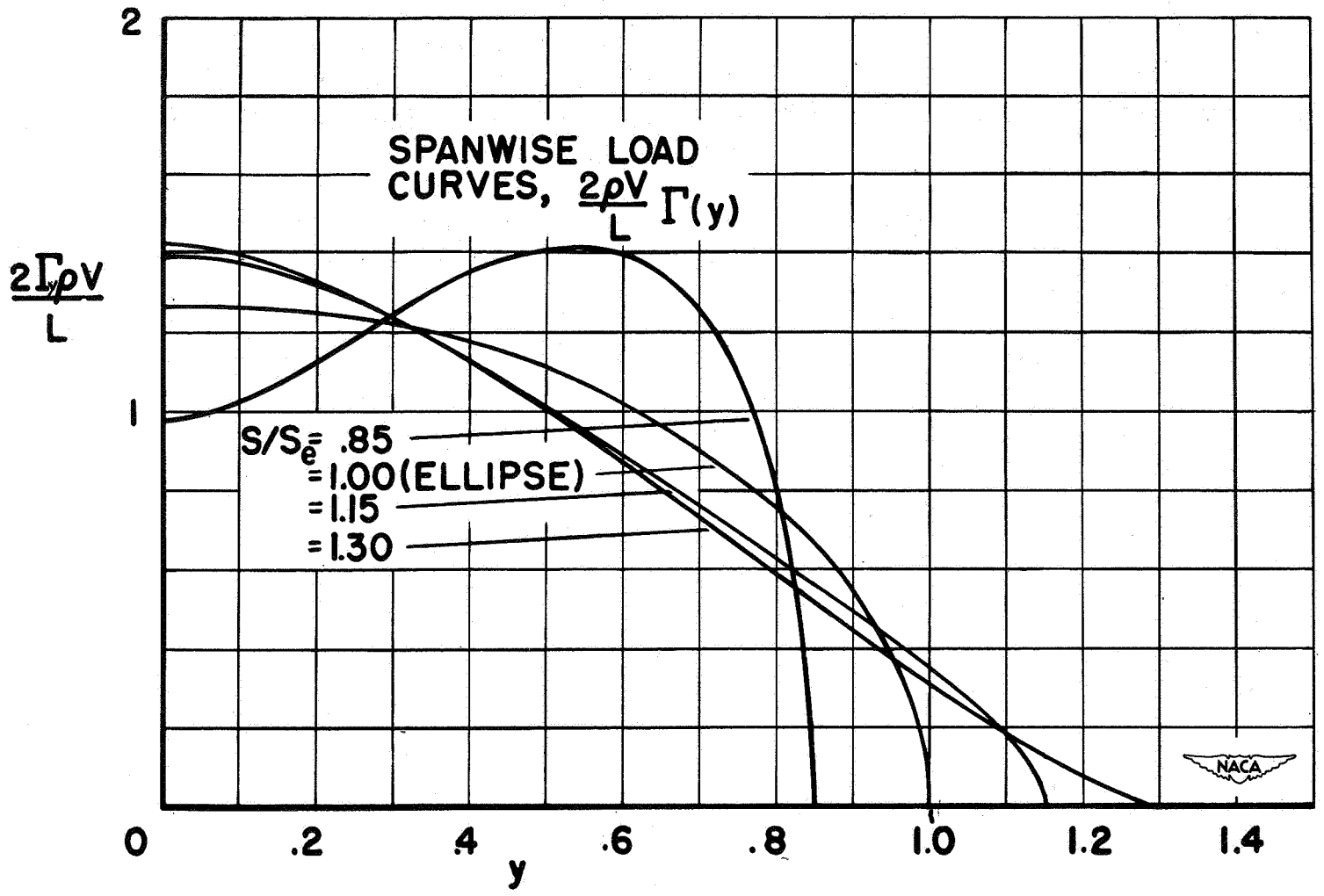

FIGURE 5.- VARIATION OF SHAPE OF THE SPANWISE LOADING CURVE WITH SEMISPAN RATIO S/S_e, FOR WINGS HAVING A FIXED TOTAL LIFT AND A FIXED BENDING MOMENT