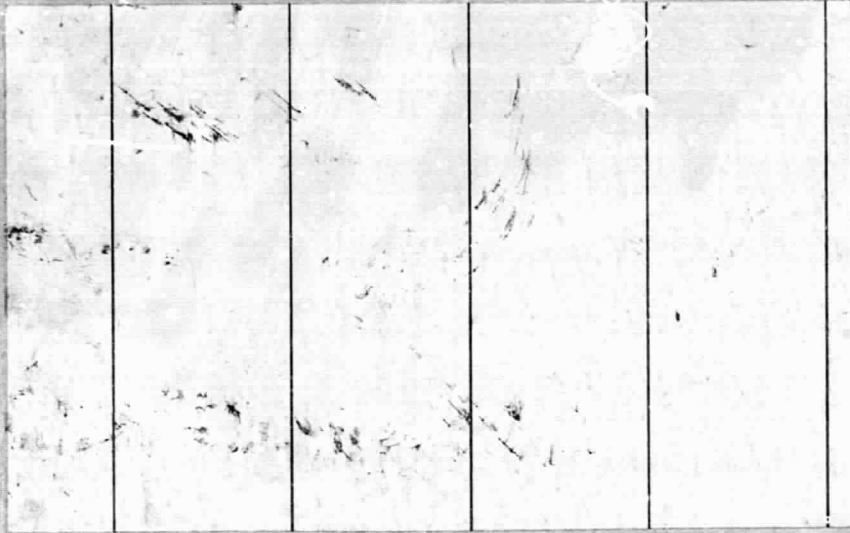# General Disclaimer

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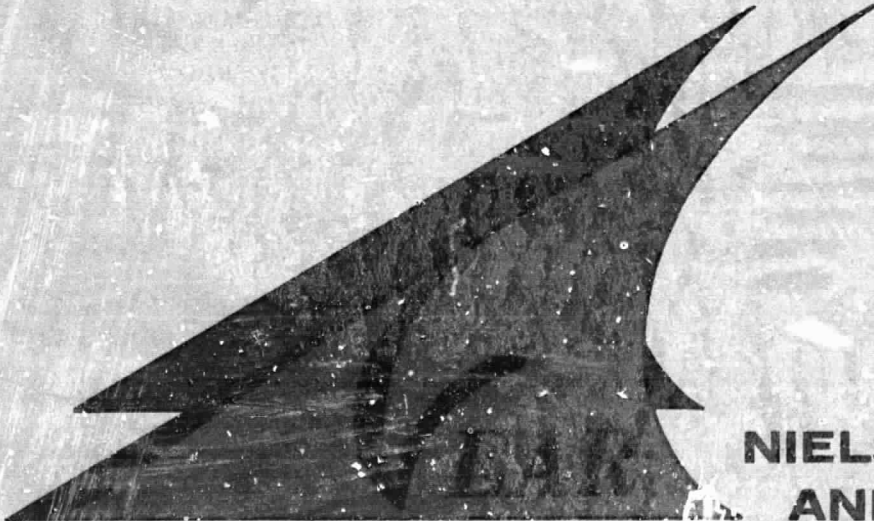Produced by the NASA Center for Aerospace Information (CASI)

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FORMULATION AND CLOSURE OF COMPRESSIBLE
TURBULENCE EQUATIONS IN THE LIGHT
OF KINETIC THEORY

by

Shunichi Tsugé and Kenshi Sagara

NEAR TR 109

February 1976

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by Shunichi Tsugé and Kenshi Sagara[*]
Nielsen Engineering & Research, Inc.

SUMMARY

On the basis of a kinetic hierarchy system developed previously by
the authors, fluid-dynamic moment equations are derived governing the inter-
action between turbulent and thermal fluctuations. Truncation of the hier-
archy under the hypothesis of "ternary" molecular chaos provides a unique
closure condition on the set of equations. The (multipoint) kinetic theory
is shown to reduce the inherent complexity of the conventional formalism
of compressible turbulence theory and to minimize arbitrariness in formu-
lating the closure condition. This assertion is based on two factors
characteristic of kinetic theory: First, all the turbulent correlation
terms appearing in the averaged equations of viscous compressible flow are
expressed as expansion coefficients of a single function, viz., the two-
particle correlation in the 2μ-space. Second, the closure condition is,
then, equivalent to deriving and solving the equation governing the two-
particle correlation. Actually, the first feature reflects favorably on the
fact that turbulence corrections to the gasdynamic equations representing
the Reynolds stress and the turbulent heat flux, are each formed by single
terms. Each of the expressions remain invariant in form whether the flow
is compressible or not; such invariance is not to be expected in the conven-
tional formalism of turbulent correlations. A superficial similarity of our
formalism to a formalism based on the concept of mass-averaging breaks down
at the second point at issue where our closure equations have a unique form
without recourse to modeling. An á posteriori way of reproducing the whole
set of equations (except for thermal-agitation terms) on a proper reinter-
pretation of the conventional fluid-dynamic equations is indicated. In
fact, a bilinear transformation of fluctuating quantities shows that the
two formalisms are identical with regard to terms of double correlations.
The essential difference between the two systems lies in the appearance of
higher order correlations; turbulence corrections by the kinetic theory

[*]Department of Physics, University of Tokyo, Tokyo, 113 Japan.

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include no such terms in principle, in contrast to the triple and the quadruple correlations intervening in conventional expressions of Reynolds stress and turbulent heat flux, respectively.

1. INTRODUCTION

1.1 Macroscale and Microscale Fluctuations in Nonequilibrium Gases

Due to its enormous number of degrees of freedom in motion at the molecular level, a macroscopic body of a gas exhibits various kinds of fluctuations around each "mean" value, viz., each thermodynamic variable of the gas, such as the density, the pressure, the fluid-dynamic velocity, etc. Roughly speaking, we can subdivide the whole fluctuation into two classes; micorscopic and macroscopic fluctuations. The microscopic fluctuation, for example, the thermal agitation, is characterized by the fact that its relative intensity has the form of $\alpha(\delta N)^{-1/2}$, where δN is the number of molecules in the body of a gas under consideration, and α is a certain function of the thermodynamic quantities. This quantity is, then, vanishingly small in the continuum limit where thermodynamics and fluid mechanics are spoken of except under anomalous conditions (e.g., the entropy or the specific volume fluctuations at phase equilibrium; $\alpha \rightarrow \infty$). On the other hand, a feature of the macroscopic fluctuation is that its intensity is independent of δN. Therefore, in the continuum limit, this fluctuation alone is detectable by means of macroscopic measurements. Another feature of the macroscopic fluctuation is that its existence is limited to the nonequilibrium state, whereas thermal agitations prevail in the equilibrium state as well as the nonequilibrium state. Among well-known fluctuations of the macroscopic type are fluid-dynamic turbulence, unstable phenomena in plasmas, and the mild (low temperature) ignition of a hydrogen/oxygen system.

In the present work we confine ourselves to a study of one of the interactions between microscopic and macroscopic fluctuations; namely, that between microscopic thermal fluctuations and the macroscopic fluctuations due to fluid-dynamic turbulence. Our objective is to derive equations governing the interaction of the basic turbulent field with the two fluctuations in the fluid-dynamic space for the case of a classical ideal gas in first-order translational nonequilibrium.[*]

1.2 Differences in Generation and Propagation
of Fluctuation Correlations

The microscale and macroscale fluctuations differ not only in their dependence on the scale of the system but also in the ways they generate and propagate correlations. The thermal fluctuation is generated by a spontaneous deviation from the mean value charaterizing the equilibrium state and is dispersed by relaxation. Microscopically, the generation and propagation of the fluctuation is via molecular collision, and the correlation formed by this mechanism relaxes as[1] $\exp(-c^2 t/\epsilon)$, where ϵ is a small parameter proportional to the mean free path, and c is the isothermal speed of sound. Then the characteristic time for the relaxation is $t_I \sim \epsilon/c^2$, a time proportional to the mean free path.

Only recently, the existence of a second mode of propagating correlation has been revealed in addition to the one via direct molecular collisions: Correlation generated and propagated by collective motion of macroscopic bodies of the gas. Grad[2] and Sastri[3] state that in contrast to the direct correlation propagation, which persists only for a few mean free times, the second mode of correlation must propagate at least over a hydrodynamic time, judging from the fact that the correlations are shown to obey certain "conservation" equations of two-point fluid mechanics. In fact, decay of the second mode of velocity-velocity correlation in a gas at rest obeys a diffusion equation[4] and therefore behaves like $\exp(-x^2/\epsilon t)$. This second mode, then, has the time constant $t_{II} \sim L^2/\epsilon$ (where L is a hydrodynamic length) and therefore decays much more slowly than the first mode with $t_I \sim \epsilon/c^2$. This implies a striking feature of the second mode: It must be detectable by macroscopic (fluid-dynamic) measurements. Furthermore, it has been shown[4] that the correlation of the second mode is even amplified if the flows under consideration are unstable with respect to the classical criteria of hydrodynamical stability. Fluctuation evolution in a Blasius boundary layer, calculated along the line of two-point kinetic theory, has exhibited agreement with the experimental results of Schubauer-Klebanoff.[5] Hence, we conclude that the second mode of correlation has a certain implication as a turbulent correlation.

1.3 Description in Terms of
Microscopic Density

A crucial factor of a kinetic theory in dealing with fluctuating phenomena is the quality of statistical processing that has to be invoked to make the problem tractable. For example, it has been shown that instantaneous turbulent fluctuations are wiped out[4] in defining the Boltzmann function as an expectation value for the number density. Also, the thermal fluctuations are missed[1] even in the most general formulation of the BBGKY formalism. To avert this difficulty, it is advisable to start with an exact distribution function where no probability concepts have yet been invoked and where therefore every fluctuation is self-contained. To this end, we shall start with a distribution which has been proposed in plasma kinetic theory,[6] and has become known as the microscopic density. It is defined by

$$\overset{o}{f}(z,t) = m \sum_{s=1}^{N} \delta\left[z - z^{(s)}(t)\right] \tag{1.1}$$

where m is the mass of a particle, and $z = (x,v)$ denotes a phase-space point, $z^{(s)}(t)$ gives the locus of the sth particle in the phase space, δ is Dirac's delta function, and the summation is over all particles N under consideration. The microscopic density has a favorable feature in comparison with the Liouville density function which has heretofore formed the basis for the BBGKY hierarchy: The microscopic density is defined in the (six-dimensional) phase space, in contrast to 6N space of the Liouville density, and has a definite physical meaning such that the expression

$$\frac{1}{\Delta x \, \Delta v} \int_{\Delta x \, \Delta v} \overset{o}{f} \, dx \, dv$$

gives the "exact" mass density at an instant t in the phase space z. This is an immediate consequence of the fact that each integrated delta function (1.1) is unity or zero depending on whether the given particle is located inside or outside, respectively, the volume $\Delta x \, \Delta v$.

In terms of this distribution, macroscopic variables are defined such that

$$\overset{o}{\rho} = \int \overset{o}{f} \, d\vec{v} = m \sum_{s} \delta\left[\vec{x} - \vec{x}^{(s)}(t)\right] \tag{1.2}$$

$$\overset{o}{m_i} = \int v_i \overset{o}{f} \, d\vec{v} = m \sum_{s} v_i^{(s)} \delta\left[\vec{x} - x^{(s)}(t)\right] \tag{1.3}$$

give, respectively, the density and the mass flux density which are subject to thermal as well as turbulent fluctuations. However, the presence of the delta function and the unknowable nature of initial values of $\vec{x}^{(s)}$ and $\vec{v}^{(s)}$ make it difficult to solve for $\overset{o}{\rho}$ and $\overset{o}{m_i}$ with instantaneous fluctuations restored (except by numerical experiments, e.g., the Monte Carlo method). The simplest way to make the formulation tractable is to introduce a smoothing or averaging procedure whose operation on the microscopic density yields the Boltzmann function, or more exactly, the one-particle distribution function of the BBGKY hierarchy

$$f = \overline{\overset{o}{f}} \tag{1.4}$$

In other words, f is the mean mass density in the phase space. Since, as noted above, neither turbulent nor thermal fluctuations survive the average, all the instantaneous fluctuations must be included in

$$\Delta f \equiv \overset{o}{f} - f \tag{1.5}$$

Accordingly, the density and the mass-flux density fluctuations are given, respectively, by

$$\Delta\rho = \overset{o}{\rho} - \rho = \int \Delta f \, d\vec{v} \tag{1.6}$$

$$\Delta m_i = \overset{o}{m_i} - m_i = \int v_i \, \Delta f \, d\vec{v} \tag{1.7}$$

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where ρ and m_i are the average density and the average mass-flux density obeying the (Boltzmann) formalism of the classical kinetic theory:

$$\rho = \int f \, \overrightarrow{dv} \qquad (1.8)$$

$$m_i = \int v_i f \, \overrightarrow{dv} \qquad (1.9)$$

Statistical behavior of the fluctuations is spoken of only through their correlations; for example, a tensor defined by

$$\overline{\Delta m_i \, \Delta \hat{m}_j} = \int v_i \hat{v}_j \, \overline{\Delta f \, \Delta \hat{f}} \, \overrightarrow{dv} \, \overrightarrow{d\hat{v}} \qquad (1.10)$$

describes the average characteristics of the fluctuation in the mass-flux density at a seven-dimensional point $(\overrightarrow{x}, \overrightarrow{\hat{x}}, t)$. In order to calculate the factor

$$\overline{\Delta f \, \Delta \hat{f}} = \overline{\overset{\circ\hat{\circ}}{ff}} - \overline{f}\,\hat{f} \qquad (1.11)$$

appearing in (1.10), a simple mathematical identity

$$\sum_{s=1}^{N} \delta\left[z - z^{(s)}(t)\right] \sum_{\hat{s}=1}^{N} \delta\left[\hat{z} - z^{(\hat{s})}(t)\right]$$

$$= \sum_{s \neq \hat{s}}^{N} \delta\left[z - {}^{(s)}(t)\right] \delta\left[\hat{z} - z^{(\hat{s})}(t)\right]$$

$$+ \delta(z - \hat{z}) \sum_{s=1}^{N} \delta\left[z - z^{(s)}(t)\right]$$

needs to be noted.

Let the average, defined by (1.4), be taken over the identity; then along with definition (1.1) the following equation results:

$$\overline{\overset{O}{f}(z)\,\overset{O}{f}(\hat{z})} = f_{II}(z,\hat{z}) + m\delta(z - \hat{z})f(z) \tag{1.12}$$

where f_{II} is defined by

$$f_{II}(z,\hat{z}) = m^2 \overline{\sum_{s \neq \hat{s}}^{N} \delta\left[z - z^{(s)}(t)\right] \delta\left[\hat{z} - z^{(\hat{s})}(t)\right]} \tag{1.13}$$

is called the two-particle distribution function in the BBGKY hierarchy. Customarily, the function f_{II} is decomposed further as

$$f_{II}(z,\hat{z}) = f(z)\,f(\hat{z}) + \psi(z,\hat{z}) \tag{1.14}$$

where ψ is called the two-particle correlation. Then equation (1.11) reads, in view of (1.12) and (1.13),

$$\overline{\Delta f(z)\,\Delta f(\hat{z})} = \psi(z,\hat{z}) + m\delta(z - \hat{z})f(z) \tag{1.15}$$

The physical meaning of each term on the right-hand side can be seen from (1.10) with (1.15). Assuming f to be a Maxwellian distribution, we have

$$\left. \begin{aligned} \overline{\Delta m_i \Delta \hat{m}_\ell} &= \int v_i \hat{v}_\ell \; \overline{\Delta f \, \Delta \hat{f}} \; d\vec{v}\, d\hat{\vec{v}} \\[2mm] &= \int v_i \hat{v}_\ell \psi(z,\hat{z}) \; d\vec{v}\, d\hat{\vec{v}} \\[2mm] &\quad + \frac{1}{n}\, \delta(\hat{\vec{x}} - \vec{x})\,(\rho a)^2 \delta_{i\ell} \end{aligned} \right\} \tag{1.16}$$

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where n is the mean number density and $a = (RT)^{1/2}$ is the isothermal
speed of sound. Then the second term on the right-hand side of equation
(1.16) turns out to be the thermal agitation,[7] whereas the first term
represents the macroscopic fluctuation correlation in nonequilibrium situ-
ations that is directly connected with the turbulent correlation.[4] Thus,
the macroscopic and microscopic fluctuations are uncoupled in (1.16), but
they will be shown to interact with each other through the equations of
fluid dynamics.

    In Section 2 we summarize our methodological basis for what follows;
a hierarchy system which is based on the master Boltzmann equation instead
of Liouville's master equation of the BBGKY hierarchy. Specifically, we
discuss the one-point and two-point equations of the hierarchy which, to-
gether with the assumption of "ternary" molecular chaos, form a closed set.
In Section 3, factors causing a crucial difference from the conventional
turbulence formalism are extracted and discussed. Fluid-dynamic moment
equations out of the one-particle equation are described in Section 4 with
special reference to the coexistence of compressibility and turbulence.
Differences in several nontrivial aspects from the Chapman-Enskog's or Grad's
classical schemes are pointed out. In Section 5, the connection of these
equations with those derived on a macroscopic basis is discussed: The two
formalisms are shown to yield the same equations in the lowest order
correlations. In Section 6, the two-particle equation which plays a central
role in the closure problem is detailed in its equivalent form of moment
equations of two-point fluid dynamics. These equations are shown to be
reconstructed except for thermal agitation terms on the basis of phenome-
nological gasdynamic equations if dependent variables are identified with
those defined in terms of microscopic density(Section 7).

2.   A HIERARCHY SYSTEM ON THE "MASTER"
BOLTZMANN EQUATION

2.1 Master Boltzmann Equation

    As is well-known, Liouville's equation, the master equation of the
BBGKY hierarchy, has an implication as the equation of continuity in the
(6N-dimensional) Γ space. A parallel procedure of generating a hierarchy
system in the (6-dimensional) μ space turns out to be practicable, starting
with the equation of continuity in this space. In reference 1 such an

equation for classical ideal gases has been shown to have the form of the Boltzman equation in which the Boltzmann function f is replaced by the microscopic density $\overset{\circ}{f}$ defined in (1.1);

$$B(\overset{\circ}{f}) \equiv \left(\frac{\partial}{\partial t} + \vec{v}\,\frac{\partial}{\partial \vec{x}}\right)\overset{\circ}{f} - J\left[\hat{z}\,\middle|\,z\right]\left[\overset{\circ}{f}(\hat{z})\,\overset{\circ}{f}(z)\right] = 0 \tag{2.1}$$

where J is the conventional Boltzmann operator for the short-range inter-molecular force

$$\left|\vec{F}\right| \sim \left|\vec{x} - \vec{\hat{x}}\right|^{-\sigma}, \quad \sigma > 3 \tag{2.2}$$

and where $\hat{z}$ stands for a field particle on which the integration is effected. Equation (2.1) may be called the "master" Boltzmann equation as distinguished from its averaged version (eq. 2.3 below) governing the Boltzmann function.

A salient feature of the master Boltzmann equation (2.1) is that no statistical processing such as the molecular chaos assumption nor even the probability concept (e.g., the distribution function as an expectation value for number density) has had to be invoked in deriving the equation.[1] It is simply an exact continuity equation, in the μ space at each instant, of fluctuating number

$$\frac{1}{\delta z}\int_{\delta z} m^{-1}\overset{\circ}{f}\ dz$$

of molecules in a specified volume δz. The temporal resolvability of this equation is limited by the nature of the operator J that is unable to resolve temporal evolution within the collision duration time $\tau = r/a$, where r is the characteristic radius of curvature of a field particle at molecular encounter. Therefore, τ is interpreted as the time-resolvability limit of the equation. It is vanishingly small for a rigid spherical molecule ($\sigma \to \infty$), and increases as σ is decreased (softer collision). We can show that the thermal agitations survive this fluctuation-filtering action of the operator J on condition that τ be smaller than the mean collision time. In fact, spontaneous stress fluctuations caused by thermal motion

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of a gas are shown to relax as[1] $\exp(-pt/\mu)$, (p, the pressure; μ, the viscosity coefficient), so that the characteristic time τ^* of thermal agitations, $\tau^* \sim \mu/p \sim \epsilon/a$, where ϵ denotes the mean free path, is seen to be of the order of the mean collision time. Then the condition $\tau^* \gg \tau$, or equivalently $\ell \gg r$ (the necessary and sufficient condition for binary molecular collision), ensures that thermal agitation will be re·····ed by equation (2.1) for a certain class of "hard" molecular encounters including Maxwellian molecules ($\sigma = 5$).

2.2 A Hierarchy System

In this section we present one-particle and two-particle kinetic equations of a new hierarchy system on the basis of the master Boltzmann equation (2.1), with emphasis on the contrast with their BBGKY counterparts. (We do not duplicate here a derivation of the equations, which is carried out in some detail in reference 1.) The equation of the lowest order, viz., the one-particle equation, is obtained by averaging equation (2.1);

$$\left(\frac{\partial}{\partial t} + \vec{v}\,\frac{\partial}{\partial \vec{x}}\right) f = J\left[\hat{z}\,|\,z\right]\left[f\hat{f} + \psi(z,\hat{z})\right] \qquad (2.3)$$

where (1.14) and (1.12) are incorporated, and where, on the right-hand side of (2.3), we have utilized the fact that $J\left[\delta(z - \hat{z})f\right] = 0$, implying that no effects due to thermal agitations are observed in the averaged equation. In fact, this equation is exactly the same as the one-particle equation of the BBGKY hierarchy, and has the form of the Boltzmann equation with no molecular chaos assumption invoked. Evidently this equation differs from the classical Boltzmann equation by the presence of the term ψ in the collision integral so that the equation is not closed by itself. We need, therefore, to address ourselves to the next equation of the hierarchy that governs the function ψ. This inherent difficulty of indeterminacy is not peculiar to the one-point equation. The two-particle equation includes higher-order terms as well; the three-particle distribution function f_{III} intervenes in the equation in an unavoidable fashion. Then a question arises: What is the simplest possible form of truncation rule which retains the essential features of turbulence ($\psi \neq 0$)? Two points need to be noted here. First, Sastri[3] has pointed out that a truncation rule

$$f_{III} = G(f, f_{II}) \qquad (2.4)$$

cannot be chosen arbitrarily, but should obey

$$\int L_{II}(f_{II}, G)\, dz = C L_I(f, f_{II}) \qquad (2.5)$$

where C is a constant and

$$L_N(f_N, f_{N+1}) = 0 \quad (N: \; I, II, III, \ldots) \qquad (2.6)$$

is the kinetic equation of Nth BBGKY hierarchy that is <u>exact</u> in the sense
that the Liouville equation is so. Condition (2.5) limits the choice of
G to an extent that the arbitrariness introduced in assumption (2.4),
thereby intervening in the two-particle equation does not "contaminate" the
exactness of the one-particle kinetic equation. The second point at issue
is that every G so far obtained and eligible in the sense above leads to
an identical form when truncated at the three-particle level, whereas they
differ at the four- (or more) particle level in effecting truncation. In
fact, among the truncations meeting condition (2.5) are the well-known
cumulant truncation rule[8] and the hypothesis of N-particle molecular chaos

$$\psi_N \equiv \overline{\prod_{n=1}^{N} \Delta f(z_n)} = 0 \qquad (2.7)$$

where Δf has been defined by (1.5) and where terms of macroscopic corre-
lation alone are taken into account. Both rules as well as some others[3]
are shown to yield an identical truncation rule at the three-particle stage

$$f_{III}(z, \hat{z}, \tilde{z}) - f\hat{f}\tilde{f} - f\psi(\hat{z}, \tilde{z}) - \hat{f}\psi(\tilde{z}, z) - \tilde{f}\psi(z, \hat{z}) = 0 \qquad (2.8)$$

Thus, we have seen that the truncation at this stage and of this form is
fairly general, although not perfectly so.

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Once the principle of the closure has been established, constructing an equation governing ψ is rather straightforward. The two-particle equation is provided by

$$\overline{\Delta f(z) B\left[\overset{o}{f}(\hat{z})\right]} + \Delta f(\hat{z}) B\left[\overset{o}{f}(z)\right] = 0 \qquad (2.9)$$

which, in view of (2.1), (2.3), and (2.8), is written in an alternative form[1]

$$\left[\frac{\partial}{\partial t} + \vec{v}\frac{\partial}{\partial \vec{x}} + \vec{\hat{v}}\frac{\partial}{\partial \vec{\hat{x}}}\right]\psi(z,\hat{z}) - K\left[z^+\Big|z\right]\left[f(z)\psi(z^+,\hat{z})\right]$$

$$- K\left[\hat{z}^+\Big|\hat{z}\right]\left[f(\hat{z})\psi(\hat{z}^+,z)\right] = K\left[z^+\Big|z\right]\left[\delta(z - \hat{z})f_{II}(z,z^+)\right]$$

$$+ K\left[\hat{z}^+\Big|\hat{z}\right]\left[\delta(z - \hat{z})f_{II}(\hat{z},\hat{z}^+)\right] - \delta(z - \hat{z})J\left[z^+\Big|z\right]\left[f_{II}(z,z^+)\right] \qquad (2.10)$$

with the operator K defined by

$$K(z^+\Big|z)g(z,z^+,\ldots) \equiv J(z^+\Big|z)\left[g(z,z^+,\ldots) + g(z^+,z,\ldots)\right] \qquad (2.11)$$

where z^+ indicates a field particle on which integration operates. If the terms on the right-hand side of equation (2.10) are set to zero, the equation reduces to the two-particle equation of the BBGKY hierarchy that is homogeneous in ψ. In the case of a non-zero right-hand side, note that the inhomogeneous terms have a common factor $\delta(x - \hat{x})$, which implies that they stand for effects due to thermal agitations. It should be stressed that equation (2.10) is exact with regard to the terms arising from thermal agitations: The closure approximation has been incorporated only in the macroscopic part of the three-particle distribution function (see eq. 2.8).

Equations (2.3) and (2.10) constitute a closed set of governing equations for f and ψ where the effects due to thermal agitations, missing in the BBGKY formalism, are taken into consideration. The remaining part of the present work is devoted to a derivation of the fluid-dynamic moment

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equations which are correct to the first-order in Δ^2 (double correlation), the mean free path ϵ, viz., the order of magnitude of transport coefficients, and n^{-1} (n: the mean number density) representing the first-order effect of thermal agitations.

3. EFFECTS OF TURBULENCE IN THE KINETIC THEORY FORMALISM

3.1 Expansion in Hermite Polynomials

In order to accomplish transition from the kinetic to the fluid-dynamic equations we expand f and ψ in a single and a double series of Hermite polynomial $\mathcal{H}$, respectively, and discuss equations governing their expansion coefficients. A systematic method of moment expansion has its origin in the so-called 13-moment method of Grad.[9] Since the expansion is around the (local) Maxwellian distribution (eq. 3.3 below) expansion in three-dimensional Hermite polynomials is considered to be most relevant. The 13-moment expansion[9] for f and its two-particle counterpart[4] for ψ are the following:

$$f(z) = f_0(z)\left[1 + \frac{P_{jk}}{2p}\mathcal{H}_{jk}^{(2)} + \frac{Q_j}{5pc}\mathcal{H}_j^{(3)}\right] \tag{3.1}$$

$$\psi(z,\hat{z}) = (\rho\hat{\rho})^{-1}f_0(z)f_0(\hat{z}) \sum_{(J,K) = (0,0)}^{(3,3)} \frac{R_{j\ldots,k\ldots}^{(J,K)}}{c^J\hat{c}^K J!K!}\mathcal{H}_{j\ldots}^{(J)}\hat{\mathcal{H}}_{k\ldots}^{(k)} \tag{3.2}$$

with

$$f_0(z) \equiv \rho(2\pi c^2)^{-3/2}\exp(-\frac{\vec{\xi}^2}{2}) \tag{3.3}$$

$$\vec{\xi} = c^{-1}(\vec{v} - \vec{w}) \tag{3.4}$$

where $\mathcal{H}$ is the Hermite polynomial defined in terms of the non-dimensional peculiar velocity $\vec{\xi}$; also P_{jk}, Q_j, ρ, p, c and $\vec{w}$ are the stress deviator tensor, the heat flux density, the density, the pressure, and the two reference velocities to be determined later. Each of these quantities except ρ is to be generalized properly to include effects due to turbulence.

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Two remarks in connection with expression (3.1) are in order at this point. First, we postulate the turbulent mean pressure p to have the same mathematical expression as it does in the laminar case regardless of its physical relevancy,

$$p = \frac{1}{3} \int c^2 \left(\mathcal{H}^{(2)} + 3 \right) f \, d\vec{v} \tag{3.5}$$

where $\mathcal{H}^{(2)} = \mathcal{H}_{ii}^{(2)} = \xi_i^2 - 3$ is the contracted Hermite polynomial of the second order. Second, we postulate also that

$$p_{ij} = 0 \tag{3.6}$$

be retained together with the condition of vanishing first-order terms. (See 3.1) Postulate (3.5) in which (3.1) is substituted for f determines the parameter c as

$$c^2 = \frac{p}{\rho} \tag{3.7}$$

Also, expansion (3.1) substituted in (1.9) determines the parameter w of (3.4) as

$$w_i = \frac{m_i}{\rho} \tag{3.8}$$

With regard to expansion (3.2), there are no such degrees of freedom remaining, so that the summation in (3.2) should span over all possible terms including nonvanishing first-order as well as contracted second-order terms.

All of the expansion coefficients in (3.1) and (3.2) are solved in view of the orthonormality property of the Hermite polynomials as

$$p_{ij} = \int c^2 \mathcal{H}_{ij}^{(2)} f \, d\vec{v} \tag{3.9}$$

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$$Q_i = \int \frac{1}{2} c^3 \mathcal{H}_i^{(3)} f \, d\vec{v} \tag{3.10}$$

where

$$\mathcal{H}_i^{(3)} = \mathcal{H}_{ijj}^{(3)} = \xi_i(\xi_j^2 - 5)$$

and

$$R_{j..,k..}^{(J,K)}(\vec{x},\vec{\hat{x}}) = \int c^J \mathcal{H}_{j..}^{(J)} \hat{c}^K \mathcal{H}_{k..}^{(K)} \psi \, d\vec{v} \, d\vec{\hat{v}} \tag{3.11}$$

3.2 Instantaneous Fluctuations

Expressions (1.8), (1.9), (3.5), (3.9) and (3.8) for ρ, m_i, p, P_{ij}, Q_i exhaust the set of 13 mean variables necessary for describing a fluid state. To each of these variables corresponds an instantaneous fluctuation some of which have been already given by (1.6) and (1.7). The whole set of the instantaneous fluctuations is given by

$$\begin{bmatrix} \Delta\rho \\ \Delta m_i \\ \Delta p \\ \Delta P_{ij} \\ \Delta Q_i \end{bmatrix} = \int \begin{bmatrix} 1 \\ w_i + c\,\mathcal{H}_i^{(1)} \\ c^2 + \frac{1}{3} c^2 \mathcal{H}^{(2)} \\ c^2 \mathcal{H}_{ij}^{(2)} \\ \frac{1}{2} c^3 \mathcal{H}_i^{(3)} \end{bmatrix} \Delta f \, d\vec{v} \tag{3.12}$$

whose form should be self-explanatory in view of the corresponding expressions for the mean values.

It should be noted that all the variables appearing in equation (3.12) have in common the characteristic of being proportional to number density. In fluid mechanics, however, there is another class of quantities whose members are <u>independent</u> of number density, for example, the velocity or the temperature. We have learned that quantities proportional to the density are more fundamental than those which are density-independent in the sense that fluctuations of the former quantities are directly defined in terms of Δf.

It is sometimes required, however, to express the fluctuation formulas with respect to the density-independent quantities. For example, laser-doppler velocimetry picks up the velocity fluctuation rather than that of the mass-flux density. Similarly, some measuring devices perceive the temperature fluctuation instead of the pressure fluctuation.

The density-independent quantities can be defined only indirectly. We shall show this for the velocity and the temperature. The instantaneous velocity $\overset{o}{u}_i$ is defined as

$$\overset{o}{u}_i = \frac{\overset{o}{m}_i}{\overset{o}{\rho}} \tag{3.13}$$

Separating this expression into average and fluctuating parts, we have

$$m_i = \rho u_i + \overline{\Delta\rho\ \Delta u_i} \tag{3.14}$$

$$\Delta m_i = \rho\ \Delta u_i + u_i\ \Delta\rho \tag{3.15}$$

As for the temperature, it would be most natural to define this quantity as equi-partitioned kinetic energy per molecule due to purely thermal motion, namely, that excluding the turbulent energy. Then we have

$$\frac{3}{2} k \overset{o}{T} = \frac{1}{\overset{o}{n}} \int \frac{1}{2} (v_i - \overset{o}{u}_i)^2 \overset{o}{f}\ d\vec{v} \tag{3.16}$$

where k is Boltzmann's constant and $\overset{o}{n}$ is the fluctuating number density. Splitting this expression into average and fluctuating parts leads to

$$\rho RT + R\ \overline{\Delta\rho\ \Delta T} = p - \frac{1}{3}\ \rho\ \overline{(\Delta u_i)^2} \tag{3.17}$$

$$RT\ \Delta\rho + R\rho\ \Delta T = \Delta p \tag{3.18}$$

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where

$$(R \equiv \tfrac{k}{m})$$

and where formulas (3.5) and (3.12) have been used.

3.3 Physical Meanings of $R^{(J,K)}_{j..k..}$

With these preliminaries on the instantaneous fluctuations, we can show a more direct physical meaning for $R^{(J,K)}$ than is implied in (3.11). As is seen from (1.15), the R's are linearly related to

$$\overline{\Delta A \; \Delta \hat{B}} = \int a(z) b(\hat{z}) \overline{\Delta f \; \Delta \hat{f}} \; d\vec{v} \; d\hat{\vec{v}} \tag{3.19}$$

namely, the fluctuation correlation between two thermodynamic quantities A and B at physical-space points $\vec{x}$ and $\hat{\vec{x}}$. Formula (3.19) is an immediate consequence of averaged product of expression (3.12) at the different points z and $\hat{z}$. Quantities ΔA and ΔB stand for fluctuations of macroscopic quantities standing on the left-hand side of (3.12), whereas a and b stand for those counterparts appearing within parentheses on the right-hand side. A straight-forward calculation of (3.19) in which $\overline{\Delta f \; \Delta \hat{f}}$ is replaced with the right-hand side of (1.15) yields

$$
\left.
\begin{aligned}
\overline{\Delta \rho \; \Delta \hat{\rho}} &= R^{(0,0)} + m\rho \; \delta(\vec{x} - \hat{\vec{x}}) \\[8pt]
\overline{\Delta \rho \; \Delta \hat{m}_\ell} &= R^{(0,1)}_\ell + \hat{\rho}^{-1}\hat{m}_\ell R^{(0,0)} + m m_\ell \delta(\vec{x} - \hat{\vec{x}}) \\[8pt]
\overline{\Delta m_i \; \Delta \hat{m}_\ell} &= R^{(1,1)}_{i,\ell} + m_i \rho^{-1} R^{(0,1)}_\ell + \hat{\rho}^{-1}\hat{m}_\ell R^{(1,0)}_i \\[4pt]
&\quad + (\rho\hat{\rho})^{-1} m_i \hat{m}_\ell R^{(0,0)} + m(\rho^{-1} m_{i\,\ell} + \rho\delta_{i\ell})\delta(\vec{x} - \hat{\vec{x}}) \\[8pt]
\overline{\Delta p \; \Delta \hat{\rho}} &= \tfrac{1}{3} R^{(2,0)}_{rr} + c^2 R^{(0,0)} + m p \delta(\vec{x} - \hat{\vec{x}})
\end{aligned}
\right\} \tag{3.20}
$$

and

$$\Delta m_i \Delta \hat{p} = \frac{1}{3} R_{i,rr}^{(1,2)} + \frac{1}{3} \rho^{-1} m_i R_{rr}^{(0,2)} + \hat{c}^2 R_i^{(1,0)} + c^2 \rho^{-1} m_i R^{(0,0)}$$

$$+ m_i c^2 \delta (\vec{x} - \vec{\hat{x}})$$

The above formulas show that fluctuation correlations, in general, are com-
posed of two factors: thermodynamic fluctuations, which are characterized
by the presence of the delta function, and nonthermal, nonequilibrium fluc-
tuations R, that are connected with a nonchaotic situation ($\psi \neq 0$)
(c.f. 1.16). On the other hand, terms on the left-hand side of (3.20) are
rewritten in terms of Δu_i and ΔT by the use of (3.15) and (3.18). Term-
by-term comparison of these expressions with those on the right-hand side of
(3.20), where the thermal agitation terms are neglected as of higher order,
leads to the following relationships:

$$R^{(0,0)} = \overline{\Delta \rho \ \Delta \hat{\rho}}$$

$$R_i^{(1,0)} = \rho \overline{\Delta u_i \Delta \hat{\rho}}$$

$$R_{i,\ell}^{(1,1)} = \rho \hat{\rho} \overline{\Delta u_i \Delta \hat{u}_\ell}$$

$$R^{(2,0)} = 3 \rho R \overline{\Delta T \ \Delta \hat{\rho}}$$

$$R_i^{(1,2)} = 3 \rho \hat{\rho} R \overline{\Delta u_i \Delta \hat{T}}$$

$$R^{(2,2)} = 9 \rho \hat{\rho} R^2 \overline{\Delta T \ \Delta \hat{T}}$$

$$\left.\right\} \quad (3.21)$$

3.4 Notes on the Pressure Concept in the Presence of Turbulence

We have introduced the average pressure p in the form of (3.5) for
the sake of conformity with the expansion (3.1) which refers to the pecu-
liar velocity ξ defined by (3.4). In fact, from the formal point of view,
this seems to be the most natural definition of the pressure extended so as

to allow for turbulence. According to this definition, however, p includes effects due to turbulence as well as the thermal motion of gas molecules, so that we can show

$$p \neq \overset{\overline{\circ}}{p}$$

provided that $\overset{\circ}{p}$ is to represent the instantaneous pressure including thermal effects alone. In order to obtain the relationship between p and

$$\pi = \overset{\overline{\circ}}{p} \tag{3.22}$$

we proceed as follows. A key to the correct form of $\overset{\circ}{p}$ is provided by (3.18) which is exactly the fluctuating part of the ideal gas relation

$$\overset{\circ}{p} = R\overset{\circ\circ}{\rho T} = k\overset{\circ\circ}{nT} \tag{3.23}$$

Since we have the definition (3.16) of the fluctuating temperature at hand, equation (3.23) leads to the definition of $\overset{\circ}{p}$ as

$$\overset{\circ}{p} = \int \frac{1}{3}(v_i - \overset{\circ}{u_i})^2 \overset{\circ}{f}\, d\vec{v} \tag{3.24}$$

Averaging this expression and utilizing equations (3.22), (3.21), (3.4), and (3.5), we have

$$p = \pi + \frac{1}{3\rho}\left[R^{(1,1)}_{i,i}\right]_{x = \hat{x}} \tag{3.25}$$

This relationship tells us that the quantity p defined by (3.5) includes effects due to turbulence in addition to the "genuine" pressure π which is defined by (3.22) and includes thermal effects alone. With this relationship, equation (3.17) is interpreted favorably: In fact, the equation reads

$$\pi = \rho RT + R \, \overline{\Delta\rho \; \Delta T}$$

or

$$\pi = \rho RT + \frac{1}{3\rho} \left[R^{(2,0)} \right]_{x \, = \, \hat{x}}$$

<div align="right">(3.26)</div>

which is nothing but equation (3.23) averaged. The difference between π and p, which is ascribed to random motion due to turbulence, is of $O(M^2\Delta^2)$ in comparison with the pressure (where M is the characteristic Mach number). Therefore, the difference is negligibly small for incompressible flows, but can reach an appreciable magnitude in certain hypersonic situations.

Correction formulas (3.25) and (3.14) tell us that the local Maxwellian, when written in terms of the average velocity and temperature, takes the form

$$f_o \sim \exp - \frac{\left[v_j - u_j - \rho^{-2} R_j^{(1,0)} \right]^2}{2 \left[RT + \frac{1}{3\rho^2} R^{(2,0)} + \frac{1}{3\rho^2} R_{j,j}^{(1,1)} \right]}$$

All the correction factors appearing in the expression disappear for incompressible flows because of non-fluctuating density $\left[R^{(J,0)} = 0 \right]$ and of vanishing Mach number $R_{j,j}^{(1,1)}/\rho^2 RT \sim O(M^2) \ll 1$.

<h4 align="center">4. TURBULENT GASDYNAMICS ON THE BASIS OF
THE CORRECTED BOLTZMANN EQUATION</h4>

<h4 align="center">4.1 Moment Equations</h4>

The only difference between the modified Boltzmann equation (2.3) and the conventional one is the presence of the term ψ in the collision integral. Thus, no difference is observed in the three moment equations (of conservation) corresponding to three summational invariants as moment functions. This situation is readily seen from the following expression of the general moment equation for equation (2.3) with the moment function $\alpha(z)$:

$$\frac{\partial}{\partial t} \int \alpha f \, d\vec{v} + \frac{\partial}{\partial x_r} \int \alpha v_r f \, d\vec{v} - \int \left(\frac{\partial \alpha}{\partial t} + v_r \frac{\partial \alpha}{\partial x_r} \right) f \, d\vec{v} = \frac{1}{2} \int \left[\!\left[\alpha\right]\!\right] \left[f(z) f(\hat{z}) \right.$$

$$\left. + \psi(z, \hat{z}) \right] d\vec{K} \, d\vec{v} \, d\hat{\vec{v}} \tag{4.1}$$

where the notation

$$\left[\!\left[\alpha\right]\!\right] = \alpha(z') + \alpha(\hat{z}') - \alpha(z) - \alpha(\hat{z}) \tag{4.2}$$

has been introduced. In deriving expression (4.1) we have assumed that quantities appearing in the collision integral of equation (2.3) are symmetrical with respect to z and $\hat{z}$. This condition is met with ψ since we have put $\vec{x} = \hat{\vec{x}}$ to secure collision encounter. The three conservation equations corresponding to the following choices

$$\alpha = 1, \quad v_i, \quad v^2$$

for which we have

$$\left[\!\left[\alpha\right]\!\right] = 0$$

are shown to have the form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial m_j}{\partial x_j} = 0 \tag{4.3}$$

$$\frac{\partial m_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho^{-1} m_i m_j + p \delta_{ij} + P_{ij}) = 0 \tag{4.4}$$

$$\frac{\partial}{\partial t} \left[\frac{3}{2} p + \frac{1}{2} \rho^{-1} m_j^2 \right] + \frac{\partial}{\partial x_j} \left[\frac{1}{2} \rho^{-2} m_j m_i^2 + \frac{5}{2} p \rho^{-1} m_j + \rho^{-1} m_i P_{ij} + Q_j \right] = 0 \tag{4.5}$$

These equations are formally identical to their classical counterparts[9] so far as the actual forms of p_{ij} and Q_j have been left unspecified.

As has been seen above, the way in which nonvanishing ψ affects the fluid dynamic equations (4.3) through (4.5) is via p_{ij} and Q_j, whose actual expressions are to be determined from moment equations with moment functions $\mathcal{H}_{ij}^{(2)}$ and $\mathcal{H}_j^{(3)}$, respectively. Since these moment functions are not summational invariants of collision, the moment equations necessarily include effects due to ψ. After some manipulation, we have moment equations, namely, equation (4.1) for $\alpha = \mathcal{H}_{ij}^{(2)}$ and $\alpha = \mathcal{H}_j^{(3)}$, respectively:

$$\frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}\left[\frac{m_r}{\rho} p_{ij}\right] + \frac{2}{5}\left[\frac{\partial Q_i}{\partial x_j} + \frac{\partial Q_i}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial Q_r}{\partial x_r}\right] + p_{ir}\frac{\partial}{\partial x_r}\left[\frac{m_j}{\rho}\right]$$

$$+ p_{jr}\frac{\partial}{\partial x_r}\left[\frac{m_i}{\rho}\right] - \frac{2}{3}\delta_{ij}p_{rs}\frac{\partial}{\partial x_s}\left[\frac{m_r}{\rho}\right] + p\left\{\frac{\partial}{\partial x_j}\left[\frac{m_i}{\rho}\right] + \frac{\partial}{\partial x_i}\left[\frac{m_j}{\rho}\right]\right.$$

$$\left. - \frac{2}{3}\delta_{ij}\frac{\partial}{\partial x_r}\left[\frac{m_r}{\rho}\right]\right\} = -\frac{6\rho B}{m}\left\{p_{ij} + \frac{1}{\rho}\left[R_{ij}^{(2,0)} - R_{i,j}^{(1,1)}\right.\right.$$

$$\left.\left. - \frac{1}{3}\delta_{ij}R^{(2,0)} + \frac{1}{3}\delta_{ij}R_{r,r}^{(1,1)}\right]\right\} \qquad (4.6)$$

and

$$\frac{\partial Q_i}{\partial t} + \frac{\partial}{\partial x_r}\left[\frac{m_r}{\rho}Q_i\right] + \frac{7}{5}q_r\frac{\partial}{\partial x_r}\left[\frac{m_i}{\rho}\right] + \frac{2}{5}q_r\frac{\partial}{\partial x_i}\left[\frac{m_r}{\rho}\right] + \frac{2}{5}q_i\frac{\partial}{\partial x_r}\left[\frac{m_r}{\rho}\right]$$

$$+ RT\frac{\partial p_{ir}}{\partial x_r} + \frac{7}{2}p_{ir}\frac{\partial}{\partial x_r}\left[\frac{p}{\rho}\right] - \frac{p_{ir}}{p}\frac{\partial}{\partial x_s}(p\delta_{sr} + p_{sr}) + \frac{5}{2}p\frac{\partial}{\partial x_i}\left[\frac{p}{\rho}\right]$$

$$= -\frac{4\rho B}{m}\left\{Q_i + \frac{1}{\rho}\left[R_i^{(3,0)} - 2R_i^{(2,1)} + R_{ir,r}^{(2,1)}\right]\right\} \qquad (4.7)$$

where B is related to the viscosity coefficient μ and the heat conductivity coefficient λ as

$$\mu = \frac{mRT}{6B}$$

and

(4.8)

$$\lambda = \frac{5mR^2T}{8B}$$

All R's in equations (4.6) and (4.7) represent the correlations at the same point $(x = \hat{x})$; therefore, the relation $R^{(J,K)} = R^{(K,J)}$ holds.

4.2 Generalized Navier-Stokes and Fourier Laws

Equations (4.6) and (4.7) have elucidated, at the level of the 13-moment approximation, the manner in which the stress and heat flux are affected by the presence of ψ, namely, the two-particle correlation. Neglecting the quantities of the order of transport coefficients in the left-hand sides of equations (4.6) and (4.7), that is, reducing the level of description to that of fluid dynamics, we obtain the following expressions for the stress and heat flux:

$$P_{ij} = (p_{ij})_{NS} + \frac{1}{\rho}\left[-R_{ij}^{(2,0)} + R_{i,j}^{(1,1)} + \frac{1}{3}\delta_{ij}R^{(2,0)} - \frac{1}{3}\delta_{ij}R_{r,r}^{(1,1)} \right] \quad (4.9)$$

and

$$Q_i = (Q_i)_F + \frac{1}{2\rho}\left[-R_i^{(3,0)} + 2R_i^{(2,1)} - R_{ir,r}^{(2,1)} \right] \quad (4.10)$$

where $(p_{ij})_{NS}$ and $(Q_i)_F$ are given by the classical Navier-Stokes and Fourier laws, respectively, as

$$(p_{ij})_{NS} \equiv -\mu\left[\frac{\partial}{\partial x_j}\left(\frac{m_i}{\rho}\right) + \frac{\partial}{\partial x_i}\left(\frac{m_j}{\rho}\right) - \frac{2}{3}\delta_{ij}\frac{\partial}{\partial x_r}\left(\frac{m_r}{\rho}\right) \right]$$

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and

$$(Q_i)_F \equiv -\lambda \frac{\partial}{\partial x_i} \left[\frac{p}{\rho R} \right]$$

Further simplification of expressions (4.10) and (4.11) is feasible if we make use of the fact[4] that the correlations between quantities containing a nonsummational invariant are smaller than those between summational invariants by a factor proportional to transport coefficients ϵ. We have, for example,

$$R_{ij}^{(2,0)} = \frac{1}{3} \delta_{ij} R^{(2,0)} + r_{ij}^{(2,0)}$$

and

$$\frac{r_{ij}^{(2,0)}}{R^{(2,0)}} \sim 0(\epsilon)$$

With this decomposition rule and the simplifying approximation, equations (4.10) and (4.11) read

$$p_{ij} = (p_{ij})_{NS} + \frac{1}{\rho} \left[R_{i,j}^{(1,1)} - \frac{1}{3} \delta_{ij} R_{r,r}^{(1,1)} \right] \tag{4.12}$$

$$Q_i = (Q_i)_F + \frac{5}{6\rho} R_i^{(2,1)} \tag{4.13}$$

Substituting equation (4.12) into equation (4.4) with the relationship (3.25) between π and p in mind, we readily see that the term $\rho^{-1} R_{i,j}^{(1,1)}$ behaves just like the Reynolds stress appearing in the equation of incompressible turbulent flows. The "turbulent" Navier-Stokes relation (4.12) might look deceptively simple, however, when we are reminded that the relation holds as well for compressible flows. In fact, we have imposed no restriction on the compressibility in the course of the derivation. Also, the fact that the turbulent heat transfer (the second term of 4.13) is expressed by a single term is not to be expected in the conventional formalism. Actually, in the conventional approach, the averaged Navier-

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Stokes and the energy equations, in which mean and fluctuating parts are separated, are written in the forms:

$$\frac{\partial}{\partial t} \rho u_i + \frac{\partial}{\partial x_j} (\rho u_i u_j + \pi \delta_{ij} + p_{ij}^{\dagger}) = 0$$

and

$$\frac{\partial}{\partial t} \left[\frac{1}{2} \rho u_i^2 + \rho h - \pi \right] + \frac{\partial}{\partial x_j} \left\{ \rho u_j \left[\frac{u_i^2}{2} + h \right] + u_i p_{ij}^{\dagger} + Q_j \right\} = 0$$

with $p_{ij}^{\dagger}$ and $Q_i^{\dagger}$ given by

$$p_{ij}^{\dagger} = (p_{ij})_{NS} + \rho \overline{\Delta u_i \Delta u_j} + u_i \overline{\Delta \rho \, \Delta u_j} + u_j \overline{\Delta \rho \, \Delta u_i} + \overline{\Delta \rho \, \Delta u_i \Delta u_j} \qquad (4.14)$$

and

$$Q_i^{\dagger} = (Q_i)_F + \frac{1}{2} \rho u_i \overline{(\Delta u_r)^2} - \frac{1}{2} u_r^2 \overline{\Delta \rho \, \Delta u_i} + u_i \overline{\Delta h \, \Delta \rho} + \rho \overline{\Delta u_i \Delta h}$$

$$+ h \overline{\Delta \rho \, \Delta u_i} + \overline{\Delta \rho \, \Delta h \, \Delta u_i} + \frac{1}{2} \overline{\rho (\Delta u_j)^2 \Delta u_i} + \frac{1}{2} \rho u_i \overline{\Delta \rho (\Delta u_j)^2}$$

$$+ \frac{1}{2} \overline{\Delta \rho \, \Delta u_i (\Delta u_j)^2} \qquad (4.15)$$

where h is the specific enthalpy. These expressions for $p_{ij}^{\dagger}$ and $Q_i^{\dagger}$ are the exact versions of those appearing in the textbooks on turbulence.[10,11] Note that the Reynolds stress includes triple correlation terms and that the turbulent heat transfer includes terms up to quadruple correlation. It is natural, then, to ask whether the two pairs of transport relations (4.12), (4.13), (4.14), and (4.15) are essentially or only seemingly different. This question is investigated in some detail in the following sections.

5. RELATION BETWEEN KINETIC AND MACROSCOPIC
FORMALISMS OF TURBULENCE EQUATIONS

5.1 Comparison of the Two Formalisms

The system of equations derived above, viz., equations (4.3) through (4.5) supplemented by (4.12) and (4.13), is shown to have an intuitive interpretation in terms of á priori gasdynamic equations whose dependent variables are the Klimontovich variables, $\overset{o}{\rho}$, $\overset{o}{u}_j (= \overset{o}{m}_j/\overset{o}{\rho})$, $\overset{o}{p}$, defined in previous sections by

$$L_o \equiv \frac{\partial \overset{o}{\rho}}{\partial t} + \frac{\partial \overset{o}{m}_j}{x_j} = 0 \tag{5.1}$$

$$L_j \equiv \frac{\partial \overset{o}{m}_j}{\partial t} + \frac{\partial}{\partial x_k}\left[\frac{\overset{o}{m}_j \overset{o}{m}_k}{\overset{o}{\rho}} + \overset{o}{p}\, \delta_{jk} + (\overset{o}{p}_{jk})_{NS} \right] = 0 \tag{5.2}$$

$$L_4 \equiv \frac{\partial}{\partial t}\left[\frac{1}{\gamma-1}\, \overset{o}{p} + \frac{1}{2}\frac{\overset{o}{m}_j^2}{\overset{o}{\rho}} \right] + \frac{\partial}{\partial x_k}\left[\frac{1}{2}\frac{\overset{o}{m}_k \overset{o}{m}_j^2}{\overset{o}{\rho}^2} + \frac{\gamma}{\gamma-1}\frac{\overset{oo}{pm}_k}{\overset{o}{\rho}} \right.$$

$$\left. + \frac{\overset{o}{m}_j (\overset{o}{p}_{jk})_{NS}}{\overset{o}{\rho}} + (\overset{o}{Q}_k)_F \right] = 0 \tag{5.3}$$

In the above, γ is the adiabatic index and takes the value 5/3 for monatomic gases now under consideration, and $(\overset{o}{P}_{jk})_{NS}$ and $(\overset{o}{Q}_k)_F$ are defined, respectively, by

$$(\overset{o}{P}_{jk})_{NS} = -\mu\left[\frac{\partial \overset{o}{u}_k}{\partial x_j} + \frac{\partial \overset{o}{u}_j}{\partial x_k} - \frac{2}{3}\, \delta_{jk}\, \frac{\partial \overset{o}{u}_r}{\partial x_r} \right] \tag{5.4}$$

$$(\overset{o}{Q}_k)_F = -\lambda\left[\frac{\partial}{\partial x_k} \right]\, \overset{o}{T} \tag{5.5}$$

We shall show that our kinetic turbulence equations are equivalent to the averaged version of equations (5.1) through (5.3) if, in the course of

averaging, triple or higher-order correlations are disregarded and the transport coefficients (μ and λ) which depend on the temperature are interpreted as material constants, and are therefore not subject to fluctuations.

No explanation is needed for the continuity equation: Simple averaging of (5.1) gives (4.3). With regard to the Navier-Stokes equation, we proceed as follows. By the turbulent Navier-Stokes (TNS) equation we mean hereafter momentum-conservation equation (4.4) in which p_{ij} is replaced with the generalized Navier-Stokes relation (4.12), namely,

$$\frac{\partial m_i}{\partial t} + \frac{\partial}{\partial x_j} \left[\frac{m_i m_j}{\rho} + \pi \delta_{ij} + (p_{ij})_{NS} + \frac{1}{\rho} R_{i,j}^{(1,1)} \right] = 0 \qquad (5.6)$$

First, we note that the limitation as to the equivalence of (5.6) and (5.2) is not existent when the flow is incompressible ($\overset{o}{\rho} = \rho$: a material constant), in which case averaged equation (5.2) leads exactly to equation (5.6) in view of relation (3.14) with $\Delta \rho = 0$. For compressible flows, however, the identity is reached only conditionally because of the density variation. Then a straightforward averaging of (5.2), written in terms of m_j instead of u_j yields

$$\text{l.h.s. of (5.6)} + O(\epsilon \, \Delta^2, \Delta^3) = 0 \qquad (5.7)$$

where Δ stands for fluctuating quantities. It is easily checked that the terms Δ^2 and Δ^3 include a common factor $\Delta \rho$, so that equation (5.6) is exactly retrieved for incompressible flow. For compressible flows, identity of the two formalisms is fulfilled only within the approximation

$$\left. \begin{array}{c} \epsilon \, \Delta^2 = 0 \\[2em] \Delta^3 = 0 \end{array} \right\} \qquad (5.8)$$

If, in the course of deriving (5.7), the variable u_i is used instead of m_i/ρ, we obtain equation (4.14). Apparently the form of (5.6) is more convenient than that of (4.14) in its invariance with respect to compressibility.

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The same method is shown to be applicable to correlate the á priori energy equation with the energy equation of the present formalism. The latter equation, which is equation (4.5) with turbulent Navier-Stokes and Fourier laws (4.12) and (4.13) incorporated, is written as

$$
\frac{\partial}{\partial t}\left[\frac{3}{2}p + \frac{1}{2}\frac{m_j^2}{\rho}\right] + \frac{\partial}{\partial x_j}\left[\frac{m_j m_i^2}{2\rho^2} + \frac{5}{2}p\frac{m_j}{\rho} + \frac{m_i}{\rho}\left\{(P_{ij})_{NS} + \frac{1}{\rho}\left[R_{i,j}^{(1,1)}\right.\right.\right.
$$

$$
\left.\left.\left. - \frac{1}{3}\delta_{ij}R_{r,r}^{(1,1)}\right]\right\} + (Q_j)_F + \frac{5}{6\rho}R_j^{(1,2)}\right] = 0 \qquad (5.9)
$$

The former equation, with average taken and relationships (3.20) incorporated, reads under approximation (5.8),

$$
\frac{\partial}{\partial t}\left[\frac{1}{\gamma - 1}p + \frac{1}{2\rho}m_j^2\right] + \frac{\partial}{\partial x_j}\left[\frac{1}{2\rho^2}m_j m_i^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}m_j + \frac{m_i}{\rho}\left\{(P_{ij})_{NS}\right.\right.
$$

$$
\left.\left. + \frac{1}{\rho}\left[R_{i,j}^{(1,1)} - \frac{1}{3}\delta_{ij}R_{r,r}^{(1,1)}\right]\right\} + (Q_j)_F + \frac{\gamma}{3(\gamma - 1)}\frac{1}{\rho}R_j^{(1,2)}\right]
$$

$$
+ \frac{(3\gamma - 5)}{6(\gamma - 1)}\left\{\frac{\partial}{\partial t}\left[\frac{1}{\rho}R_{r,r}^{(1,1)}\right] + \frac{\partial}{\partial x_j}\left[\frac{m_j}{\rho^2}R_{r,r}^{(1,1)} - \frac{1}{\rho}R_j^{(1,2)}\right]\right\} = 0 \qquad (5.10)
$$

where, in deriving the equation, $\pi = \overline{p}$ has been eliminated through the use of (3.25). This equation tells us that for a monatomic gas ($\gamma = 5/3$) the equation reduces to equation (5.9) and further, that the factor 5/6 appearing in the generalized Fourier law (4.13) is to be interpreted as $\gamma/3(\gamma - 1)$ in the case where the gas is not monatomic.

It should be remarked that equation (5.9) (and eq. 5.10 as well) is written in terms of the "pseudo" pressure p, so that it is not in perfect conformity with the turbulent Navier-Stokes equation (5.6) that is described in terms of the "genuine" pressure π and has a form easier to understand in connection with laminar cases. The parallel expression for the energy is feasible by introducing the internal energy per unit of volume:

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$$E = \frac{3}{2} p = \frac{3}{2} \pi + \frac{1}{2\rho} R_{i,i}^{(1,1)} \tag{5.11}$$

Then equation (5.9) takes the form

$$\frac{\partial}{\partial t} \left[\frac{1}{2} \frac{m_j^2}{\rho} + E \right] + \frac{\partial}{\partial x_j} \left\{ \frac{m_i^2 m_j}{2\rho^2} + \frac{m_j}{\rho} (E + \pi) + \frac{m_i}{\rho} \left[(p_{ij})_{NS} + \frac{1}{\rho} R_{i,j}^{(1,1)} \right] \right.$$

$$\left. + (Q_j)_F + \frac{5}{6\rho} R_j^{(1,2)} \right\} = 0 \tag{5.12}$$

This expression is apparently consistent with that of (5.6) and also with the laminar case: It consists of physically identifiable terms provided that we interpret $E + \pi$ as the enthalpy per unit of volume. This observation allows the following interpretation for the turbulent gas. Turbulent flow of an ideal gas, in its average description, can be considered as an equivalent imperfect gas with thermal and caloric equations of state given by (3.26) and (5.11), respectively.

5.2 Essential Difference Between the Two Formalisms

As we have seen above, the kinetic and macroscopic methods have led to the same equations of turbulence so far as the lowest order in correlation terms is concerned. However, it seems to be equally important to stress the difference between the two formalisms with regard to the appearance of higher-order correlations, because this point proves to be the essential feature of the kinetic-theory formalism that is not to be attained by the conventional formalism.

First of all, let us recall that the classical turbulent corrections to transport relations (4.14) and (4.15), which are exact within their regimes, include triple and quadruple correlations, respectively. These have their origin in the terms $\rho \overset{oo}{u_i} \overset{o}{u_j}$ and $\rho \overset{oo}{u_i} \overset{o}{u_j^2}$ in the Navier-Stokes and the energy equations, respectively. On the other hand, TNS equation (5.6) and turbulent energy equation (5.9), both of which are consequences of the kinetic theory, include only double correlations $R_{i,j}^{(1,1)}$ and $R_i^{(2,1)}$ through generalized Navier-Stokes and Fourier laws (4.12) and (4.13), respectively. It can be shown easily that the one-particle equations,

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namely, the Navier-Stokes and the energy equations of fluid mechanics, should not include correlations higher than double correlations, in principle. In other words, this is true independently of the multi-particle stage at which truncation is effected. To prove this, only two facts need be remarked. First, a general property of the BBGKY hierarchy is that each hierarchy equation concerns only two distribution functions of consecutive hierarchies. In fact, the equation of each hierarchy has the form

$$Df_n = Jf_{n+1} \tag{5.13}$$

where D and J are certain streaming and collision operators, respectively. For $n = 1$, the equation reduces to equation (2.3). The other fact which is necessary is that triple correlation Δ^3 is directly connected with a nonreducible part of the three-particle distribution function

$$\psi_{III}(z,\hat{z},\hat{\hat{z}}) \equiv \overline{\left[\overset{O}{f}(z) - f(z)\right]\left[\overset{O}{f}(\hat{z}) - f(\hat{z})\right]\left[\overset{O}{f}(\hat{\hat{z}}) - f(\hat{\hat{z}})\right]}$$

in such a manner that

$$\overline{\Delta A \; \Delta\hat{B} \; \Delta\hat{\hat{C}}} = \int a(z)b(\hat{z})c(\hat{\hat{z}})\psi_{III}\vec{dv}\;\vec{d\hat{v}}\;\vec{d\hat{\hat{v}}}$$

therefore, nonvanishing terms of Δ^2 are directly connected with nonvanishing ψ_{III}, and do not appear in equation (5.13) for $n = 1$. Thus, we conclude, as a general rule, that terms of order Δ^3 do not appear in any of the moment equations stemming from equation (2.3). Actually, formulas (4.12) and (4.13) do not include such terms although no approximation to rule out higher-order correlations has been effected. These retain a level of accuracy comparable with formulas (4.14) and (4.15) of the conventional expressions where triple and quadruple correlations intervene in unavoidable fashions.

The above statement does not imply, however, that formulas (4.12) and (4.13) are not at all affected by the form of the function ψ_{III} (assumed or solved). It does affect them _implicitly_ through the double correlations

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$R_{i,j}^{(1,1)}$ and $R_i^{(1,2)}$ whose actual forms are to be solved from equation (5.13) with $n = 2$, where the term ψ_{III} appears on the right-hand side of the equation.

6. CLOSURE EQUATIONS

6.1 Comparison with Conventional Closure

M. Rubesin[12] first pointed out the close similarity between the system of equations (5.6) and (5.12) and a system involving mass averaging.[13] Mass averaging is an artifice which formally eliminates density fluctuations from the momentum and the energy equations, thereby enabling application of the modeling conventionally used in incompressible turbulent flows. In the present formalism an equivalent artifice is forbidden by the structure of the theory. Here, all the fluctuation terms appearing in the one-particle system emerge from a single function ψ as coefficients of its expansion, and the governing equation for ψ is already at hand (eq. 2.10). Therefore, invoking a special kind of averaging instead of the straightforward one we have employed would make equations (3.25) and (3.14) over-determined. For the same reason, it would be redundant to introduce special heuristic arguments concerning the Reynolds stress $R_{i,j}^{(1,1)}$ of (4.12) and the turbulent heat flux $R_i^{(1,2)}$ of (4.13), such as, for example, the mixing length concept and a turbulent Prandtl number of unity. The evolution of these quantities hinges totally upon solution of the partial differential equations in $(\vec{x},\vec{\hat{x}},t)$ space derived from kinetic equation (2.10) by the moment expansion.

In the case of incompressible flow, these equations have been worked out in reference 4. There, the equations are shown to be separable into two groups of variables $(\vec{x},t)$ and $(\vec{\hat{x}},t)$, and thereafter reducible to two Orr-Sommerfeld equations with different physical implications. The generalization now at issue is two-fold; inclusion of compressibility which allows for a nonvanishing $R^{(J,0)}$ and inclusion of thermal-fluctuation effects which give rise to inhomogeneous terms in the correlation equations. The two-particle moment equations including these generalizations are developed in what follows.

6.2 Two-Particle Moment Equations

A general expression for the moment equations of equation (2.10) has been derived by the authors (eq. (65) of ref. 1). The kinetic equation, after being multiplied by a moment function $\alpha(z)\beta(\hat{z})$ and integrated with respect to $\vec{v}$ and $\hat{\vec{v}}$, is written in the form

$$\left\langle\!\!\left\langle \frac{\alpha\beta\psi}{\mathcal{D}t} + \frac{\partial}{\partial x_k}\,\alpha\beta c\xi_k\psi + \frac{\partial}{\partial \hat{x}_k}\,\alpha\beta\hat{c}\hat{\xi}_k\psi - \left(\alpha\frac{D\beta}{Dt} + \beta\frac{D\alpha}{Dt}\right)\psi - c\xi_k\frac{\partial\alpha}{\partial x_k}\,\beta\psi \right.\right.$$

$$\left.\left. - \hat{c}\hat{\xi}_k\alpha\frac{\partial\beta}{\partial \hat{x}_k}\,\psi \right\rangle\!\!\right\rangle_{v\hat{v}} - \left\langle\!\!\left\langle \left[\!\left[\alpha\right]\!\right]\beta f(z)\psi(\tilde{z},\hat{z}) \right\rangle\!\!\right\rangle^{\dagger}_{v\tilde{v}}\,\hat{v}$$

$$- \left\langle\!\!\left\langle \left[\!\left[\beta\right]\!\right]\alpha f(\hat{z})\psi(z,\tilde{z}) \right\rangle\!\!\right\rangle^{\dagger}_{\hat{v}\tilde{v}}\,v = \delta\!\left[\vec{x} - \hat{\vec{x}}\right]G \qquad (6.1)$$

with

$$G \equiv \frac{1}{2}\left\langle\!\!\left\langle \left\{ -\left[\!\left[\alpha(z)\beta(z)\right]\!\right] \;\dot{\div}\; \left[\alpha(z) + \alpha(\hat{z})\right]\left[\!\left[\beta(z)\right]\!\right] \right.\right.\right.$$

$$\left.\left.\left. + \left[\beta(z) + \beta(\hat{z})\right]\left[\!\left[\alpha(z)\right]\!\right]\right\} f_{II}(z,\hat{z}) \right\rangle\!\!\right\rangle^{\dagger}_{v\hat{v}} \qquad (6.2)$$

where the following abbreviations are employed:

$$\left\langle Z \right\rangle_v = \int Z\,d\vec{v}, \qquad \left\langle\!\!\left\langle Z \right\rangle\!\!\right\rangle^{\dagger}_{v\hat{v}} = \int Z\,d\vec{K}(z,\hat{z})d\vec{v}\,d\hat{\vec{v}}, \left.\vphantom{\int}\right\}$$

$$\frac{\mathcal{D}Z}{\mathcal{D}t} = \frac{\partial Z}{\partial t} + \frac{\partial(m_k/\rho)Z}{\partial x_k} + \frac{\partial(\hat{m}_k/\hat{\rho})Z}{\partial \hat{x}_k}, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{m_k}{\rho}\frac{\partial}{\partial x_k} \qquad (6.3)$$

It may be relevant to point out here that term G on the right-hand side of equation (6.1) gives the effects due to thermal agitations and that the term is nonvanishing unless the product $\alpha\beta$ is a summational invariant as

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well as α and β themselves. For example, the two-point fluid-dynamic equation describing momentum-momentum correlation is subject to an inhomogeneous term ascribed to thermal agitations. These situations are actually the case in the following set of equations which exhausts the whole range of possib. choices of α and β when both are summational invariants:

$$\frac{\mathcal{D}R^{(0,0)}}{\mathcal{D}t} + \frac{\partial R_r^{(1,0)}}{\partial x_r} + \frac{\partial R_r^{(0,1)}}{\partial \hat{x}_r} = 0 \tag{6.4}$$

$$\frac{\mathcal{D}R_i^{(1,0)}}{\mathcal{D}t} + \frac{\partial}{\partial x_i}\left[\frac{1}{3}R^{(2,0)} + c^2 R^{(0,0)}\right] + \frac{\partial r_{is}^{(2,0)}}{\partial x_s} + \frac{\partial}{\partial \hat{x}_s} R_{i,s}^{(1,1)}$$

$$+ R_s^{(1,0)}\frac{\partial}{\partial x_s}\left(\frac{m_i}{\rho}\right) + R^{(0,0)} \frac{D}{Dt}\left(\frac{m_i}{\rho}\right) = 0 \tag{6.5}$$

$$\frac{\mathcal{D}R_{i,\ell}^{(1,1)}}{\mathcal{D}t} + \frac{\partial}{\partial x_i}\left[\frac{1}{3}R_\ell^{(2,1)} + c^2 R_\ell^{(0,1)}\right] + \frac{\partial}{\partial x_s}r_{is,\ell}^{(2,1)} + \frac{\partial}{\partial \hat{x}_\ell}\left[\frac{1}{3}R_i^{(1,2)}\right.$$

$$\left. + c^2 R_i^{(1,0)}\right] + \frac{\partial}{\partial \hat{x}_s}r_{i,s\ell}^{(1,2)} + R_{r,\ell}^{(1,1)}\frac{\partial}{\partial x_r}\left(\frac{m_i}{\rho}\right) + R_{i,r}^{(1,1)}\frac{\partial}{\partial \hat{x}_r}\left(\frac{\hat{m}_\ell}{\hat{\rho}}\right)$$

$$+ R_\ell^{(0,1)} \frac{D}{Dt}\left(\frac{m_i}{\rho}\right) + R_i^{(1,0)} \frac{D}{Dt}\left(\frac{\hat{m}_\ell}{\hat{\rho}}\right) = 6B\, \delta(\vec{x} - \vec{\hat{x}})\rho(p_{i\ell})_{NS} \tag{6.6}$$

$$\frac{\mathcal{D}R^{(2,0)}}{\mathcal{D}t} + \frac{\partial}{\partial x_s}\left[r_s^{(3,0)} + 5c^2 R_s^{(1,0)}\right] + \frac{\partial}{\partial \hat{x}_s} R_s^{(2,1)} - \frac{2R^{(0,0)}}{\rho}\frac{\partial \mathcal{Q}_s}{\partial x_s}$$

$$+ 2\left[\frac{\delta_{ts}}{3} R^{(2,0)} + r_{ts}^{(2,0)} - \frac{R^{(0,0)}}{\rho}p_{ts}\right]\frac{\partial}{\partial x_t}\left(\frac{m_s}{\rho}\right) + 2R_r^{(1,0)}\rho\frac{D}{Dt}\left(\frac{m_r}{\rho}\right)$$

$$- 3c^2 \frac{\partial}{\partial x_r} R_r^{(1,0)} = 0 \tag{6.7}$$

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$$\frac{\mathcal{D}R^{(2,1)}_i}{\mathcal{D}t} + \frac{\partial}{\partial x_s}\left[r^{(3,1)}_{s,i} + 5c^2 R^{(1,1)}_{s,i}\right] + \frac{\partial}{\partial \hat{x}_i}\left[\frac{1}{3}R^{(2,2)} + c^2 R^{(2,0)}\right]$$

$$+ \frac{\partial}{\partial \hat{x}_s}r^{(2,2)}_{,is} - \frac{2R^{(2,1)}_i}{\rho}\frac{\partial Q_r}{\partial x_r} + 2\left(R^{(2,1)}_{rs,i} - \frac{R^{(0,1)}_i}{\rho}\cdot P_{rs}\right)\frac{\partial}{\partial x_r}\left(\frac{m_s}{\rho}\right)$$

$$+ 2R^{(1,1)}_{r,i}\frac{D}{Dt}\left(\frac{m_r}{\rho}\right) - 3c^2\frac{\partial}{\partial x_r}R^{(1,1)}_{r,i} + R^{(2,1)}_r\frac{\partial}{\partial \hat{x}_r}\left(\frac{\hat{m}_i}{\hat{\rho}}\right)$$

$$+ R^{(2,0)}\frac{D}{Dt}\left(\frac{\hat{m}_i}{\hat{\rho}}\right) = -8B\ \delta(\vec{x} - \vec{\hat{x}})\ \rho(Q_i)_F \tag{6.8}$$

$$\frac{\mathcal{D}R^{(2,2)}}{\mathcal{D}t} + \frac{\partial}{\partial x_s}\left[r^{(3,2)}_s + 5c^2 R^{(1,2)}_s\right] + \frac{\partial}{\partial \hat{x}_s}\left[r^{(2,3)}_s + 5c^2 R^{(2,1)}_s\right]$$

$$- \frac{R^{(2,2)}}{\rho}\frac{\partial Q_s}{\partial x_s} - \frac{R^{(2,2)}}{\hat{\rho}}\frac{\partial \hat{Q}_s}{\partial \hat{x}_s} + 2\left[\frac{\delta_{st}}{3}R^{(2,2)} + r^{(2,2)}_{st},\right.$$

$$\left.- \frac{R^{(0,2)}}{\rho}P_{st}\right]\frac{\partial}{\partial x_{st}}\left(\frac{m_s}{\rho}\right) + 2\left[\frac{\delta_{st}}{3}R^{(2,2)} + r^{(2,2)}_{,st}\right.$$

$$\left.- \frac{R^{(2,0)}}{\hat{\rho}}\hat{P}_{rs}\right]\frac{\partial}{\partial \hat{x}_r}\left(\frac{\hat{m}_s}{\hat{\rho}}\right) + 2R^{(1,2)}_r\rho\frac{D}{Dt}\left(\frac{m_r}{\rho}\right) + 2R^{(2,1)}_r\frac{D}{Dt}\left(\frac{\hat{m}_r}{\hat{\rho}}\right)$$

$$- 3c^2\frac{\partial}{\partial x_r}R^{(1,2)}_r - 3\hat{c}^2\frac{\partial}{\partial \hat{x}_r}R^{(2,1)}_r = -4B\ \delta(\vec{x} - \vec{\hat{x}})\frac{5}{3}R^{(2,2)} \tag{6.9}$$

with

$$R^{(J,K)} = R^{(J,K)}(x,\hat{x}) \tag{6.10}$$

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In the above equations we have also employed the following express:ons:

$$r_{ij,}^{(2,K)} \equiv R_{ij,}^{(2,K)} - \frac{1}{3} \delta_{ij} R^{(2,K)} \qquad (6.11)$$

$$r_i^{(3,K)} \equiv R_i^{(3,K)} \qquad (6.12)$$

where the moment $\hat{\mathcal{R}}^{(K)}$ corresponding to the second variable is a summa-
tional invariant. Equations (6.4) through (6.9) are regarded as the evo-
lution equations for correlations $R^{(J,K)}$ between summational invariants.
As the actual expressions show, these are not closed by themselves, but
include correlation terms involving the r's associated with nonsummational
invariants. In deriving terms on the right-hand side, viz., terms of ther-
mal agitations, only linear terms in R or ϵ have been retained.

In order to have a closed set of equations we need to construct equa-
tions for the r's; that is, the fluctuation correlations between the stress
tensor/heat flux vector and summational invariants. To extract the essen-
tial feature of equations governing the r's, an inductive approach seems
to be more palatable than a deductive one. For this purpose we derive
equations governing two correlations of (4.11) and (4.12) for $K = 0$:

$$\frac{\mathcal{D} R_{ij,}^{(2,0)}}{\mathcal{D}t} + \frac{\partial}{\partial x_s} \left\{ r_{ijs,}^{(3,0)} - \frac{\delta_{ij}}{3} r_s^{(3,0)} + c^2 \left[R_i^{(1,0)} \delta_{js} + R_j^{(1,0)} \delta_{is} \right. \right.$$

$$\left. \left. - \frac{2}{3} \delta_{ij} R_s^{(1,0)} \right] \right\} + \frac{\partial}{\partial \hat{x}_s} r_{ij,s}^{(2,1)} + \left[c^2 R^{(0,0)} + \frac{1}{3} R^{(2,0)} \right] \left[\frac{\partial}{\partial x_j} \left(\frac{m_i}{\rho} \right) \right.$$

$$\left. + \frac{\partial}{\partial x_i} \left(\frac{m_j}{\rho} \right) - \frac{2}{3} \delta_{ij} \frac{\partial}{\partial x_s} \left(\frac{m_s}{\rho} \right) \right] + r_{is}^{(2,0)} \frac{\partial}{\partial x_s} \left(\frac{m_j}{\rho} \right) + r_{js}^{(2,0)} \frac{\partial}{\partial x_s} \left(\frac{m_i}{\rho} \right)$$

$$- \frac{2}{3} \delta_{ij} r_{st}^{(2,0)} \frac{\partial}{\partial x_s} \left(\frac{m_t}{\rho} \right) + R_i^{(1,0)} \frac{D}{Dt} \left(\frac{m_j}{\rho} \right) + R_j^{(1,0)} \frac{D}{Dt} \left(\frac{m_i}{\rho} \right)$$

(Continued on next page)

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REPRODUCIBILITY OF THE

$$- \frac{2}{3} \delta_{ij} R_s^{(1,0)} \frac{D}{Dt}\left(\frac{m_s}{\rho}\right) + \frac{6B}{m}\left[R^{(0,0)} P_{ij} + \rho r_{ij}^{(2,0)}\right]$$

$$= -6B \; \delta(\vec{x} - \hat{\vec{x}})\left[\rho P_{ij} + r_{ij}^{(2,0)} - R_{i,j}^{(1,1)} + \frac{1}{3}\delta_{ij}R_{s,s}^{(1,1)}\right] \tag{6.13}$$

<div align="right">(Concluded)</div>

$$\frac{\mathscr{D} r_i^{(3,0)}}{\mathscr{D} t} + \frac{\partial}{\partial x_s}\left[7c^2 R_{is}^{(2,0)} + c^2 R^{(2,0)}\delta_{is}\right] + \frac{\partial}{\partial \hat{x}_s} r_{i,s}^{(3,1)}$$

$$+ 2c^2 R_s^{(1,0)}\left[\frac{\partial}{\partial x_s}\left(\frac{m_i}{\rho}\right) + \frac{\partial}{\partial x_i}\left(\frac{m_s}{\rho}\right) - \frac{2}{3}\delta_{is}\frac{\partial}{\partial x_t}\left(\frac{m_t}{\rho}\right)\right] + r_s^{(3,0)}\frac{\partial}{\partial x_s}\left(\frac{m_s}{\rho}\right)$$

$$+ 2r_{ist,}^{(3,0)}\frac{\partial}{\partial x_t}\left(\frac{m_s}{\rho}\right) + \left[R^{(2,0)}\delta_{is} + 2R_{is}^{(2,0)}\right]\frac{D}{Dt}\left(\frac{m_s}{\rho}\right)$$

$$- \frac{10}{3\rho}R_i^{(1,0)}\left[P_{st}\frac{\partial}{\partial x_s}\left(\frac{m_t}{\rho}\right) + \frac{\partial Q_s}{\partial x_s}\right] + 5c^2 R^{(0,0)}\frac{\partial c^2}{\partial x_i} - 5c^2\frac{\partial}{\partial x_s}R_{is}^{(2,0)}$$

$$+ \frac{4B}{m}\left[\rho r_i^{(3,0)} + 2R^{(0,0)}q_i + R_s^{(1,0)}P_{is}\right] = -4B \; \delta(\vec{x}-\hat{\vec{x}})\left[2\rho Q_i R_i^{(3,0)}\right.$$

$$\left. - 2R_i^{(2,1)} + R_{is,s}^{(2,1)}\right] \tag{6.14}$$

These equations are simplified by imposing an order-of-magnitude estimate

$$r \sim O(R^2, R\epsilon) \tag{6.15}$$

which will be proved á posteriori. In fact, retaining only terms of
$O(1,R/\epsilon)$ and leading terms of delta function in the equations, we have

$$r_{ij}^{(2,0)} = -\mu\left[\frac{\partial}{\partial x_j}\left(\frac{R_i^{(1,0)}}{\rho}\right) + \frac{\partial}{\partial x_i}\left(\frac{R_j^{(1,0)}}{\rho}\right) - \frac{2}{3}\delta_{ij}\frac{\partial}{\partial x_s}\left(\frac{R_s^{(1,0)}}{\rho}\right)\right]$$

$$- \frac{R^{(0,0)}}{\rho}\left[P_{ij} - (P_{ij})_{NS}\right] + \frac{R^{(2,0)}}{3\rho}(P_{ij})_{NS} - \frac{m}{\rho}\delta(\vec{x} - \overset{\times}{x})\left[\rho P_{ij}\right.$$

$$\left. - R_{i,j}^{(1,1)} + \frac{1}{3}\delta_{ij}R_{s,s}^{(1,1)}\right] - m\,\delta(\vec{x} - \overset{\times}{x})(P_{ij})_{NS} \tag{6.16}$$

$$r_i^{(3,0)} = - \frac{\lambda}{(3/2)Rc^2}\frac{\partial}{\partial x_i}\left(\frac{c^2 R^{(2,0)}}{\rho}\right) - \frac{2R^{(0,0)}}{\rho}\left[Q_i - (Q_i)_F\right]$$

$$+ \frac{R_s^{(1,0)}}{\rho}\left[3(P_{is})_{NS} - P_{is}\right] - \frac{m}{\rho}\delta(\vec{x} - \overset{\times}{x})(2\rho Q_i)_F \tag{6.17}$$

These are the two-particle versions of the Navier-Stokes and the Fourier laws generalized to include effects due to thermal agitations. These results also assure that postulate (6.15) actually holds. It should be remarked, however, that not all terms in equations (6.16) and (6.17) are equally important. For example, when (6.16) is substituted for $r_{ij}^{(2,0)}$ in equation (6.5), all the terms except the first on the right-hand side of (6.16) find respective terms that are larger by the factor $O(\epsilon^{-1})$ or $O(R^{-1})$. On the other hand, there are no terms to compare with regard to the first term of (6.16), therefore, this term alone should be retained. This is also the case with equation (6.17), as well as with the expressions for general $r_{ij}^{(2,K)}$ and $r_i^{(3,K)}$. Having carried out the simplification, we are led to the following equations:

$$r_{ij}^{(2,K)} = -\mu\left[\frac{\partial}{\partial x_j}\left(\frac{R_i^{(1,K)}}{\rho}\right) + \frac{\partial}{\partial x_i}\left(\frac{R_j^{(1,K)}}{\rho}\right) - \frac{2}{3}\delta_{ij}\frac{\partial}{\partial x_s}\left(\frac{R_s^{(1,K)}}{\rho}\right)\right] \tag{6.18}$$

$$r_i^{(3,K)} = - \frac{\lambda}{(3/2)Rc^2}\frac{\partial}{\partial x_i}\left(\frac{c^2}{\rho}R^{(2,K)}\right) \tag{6.19}$$

The connection of these expressions with the Navier-Stokes and the Fourier laws of transport process should be self-explanatory.

6.3 Interaction Equations

Equations (6.18) and (6.19) express fluid-dynamic correlations between nonsummational quantities in terms of those between summational invariants; therefore, together with equations (6.4) through (6.9), they form a closed set. For incompressible flows (ρ, a material constant), terms including the density fluctuation disappear $\left[R^{(J,0)} = 0 \right]$, and the system reduces to a closed set of equations (6.5), (6.6), and (6.8) for $K = 1$. Eliminating $r^{(1,2)}$ from the latter equations, we have equations governing the interaction of turbulence with thermal agitations in incompressible flows;

$$\frac{\partial R_{i,\ell}^{(1,1)}}{\partial x_i} = 0 \tag{6.20}$$

$$\left[\frac{\mathcal{D}}{\mathcal{D}t} - \nu(\nabla^2 + \hat{\nabla}^2) \right] R_{i,\ell}^{(1,1)} + \frac{\partial u_i}{\partial x_k} R_{k,\ell}^{(1,1)} + \frac{\partial \hat{u}_\ell}{\partial \hat{x}_k} R_{i,k}^{(1,1)}$$

$$+ \frac{1}{3} \left[\frac{\partial}{\partial x_i} R_\ell^{(2,1)} + \frac{\partial}{\partial \hat{x}_\ell} R_i^{(1,2)} \right] = \frac{p}{\nu} \frac{1}{n} \delta(\vec{x} - \vec{\hat{x}}) (p_{i\ell})_{NS} \tag{6.21}$$

where u_i is the mean velocity and is identified with $\rho^{-1} m_i$ in this situation. Equations (6.20) and (6.21) differ from the correlation equations of the previous theory[4] by the presence of the term with the delta function on the right-hand side of (6.21) This term, lacking in the BBGKY formalism, provides the only mechanism for generating correlation in a perfectly quiescent shear flow. Also, this term, which is ascribed to the effect of thermal agitations, is seen to be very small because of the presence of the factor n^{-1}, which is of $O(10^{-10})$ under normal temperature and pressure condition. Under realistic conditions, therefore, this effect in initiating turbulence is masked by other macroscopic mechanisms, such as noise or free-stream turbulence. Taking the thermal motion into account makes sense only in evaluating the maximum transition Reynolds number in a shear flow. This evaluation has been attempted for the Blasius flow only qualitatively in reference 4, because the form of thermal agitation effects

. s not known at that stage. In this regard, equations (6.6) and (6.8)
a.. of particular interest. These equations governing $R_{i,l}^{(1,1)}(x,\hat{x})$ and
$R^{(1,2)}(x,\hat{x})$ which reduce for $x = \hat{x}$ to the Reynolds stress and the turbu-
lent heat flux, respectively, quantitatively describe their nature as
mechanisms driving the shear stress and the heat flow at the molecular level.
In principle, these equations allow us to determine evolution of the
Reynolds stress and the turbulent heat transfer beginning with fluctuations
at the molecular level and ending with fully developed turbulence.

7. MACROSCOPIC DERIVATION OF CORRELATION EQUATIONS

A we have shown in Section 5, the one-particle equations for a com-
pressible turbulent flow which were originally derived on the basis of
kinetic theory, could be derived also from the phenomenological equations
of fluid dynamics. Therefore, it might well be expected that our two-point
equations, viz., equations (6.4) through (6.9), (6.8), and (6.19), should
similarly be interpreted in terms of fluid-dynamic equations (5.1) through
(5.3) governing "fluctuating" quantities $\overset{o}{\rho}$, $\overset{o}{u}$, and $\overset{o}{p}$. We note that these
equations are c...dered as deterministic only in the sense that they have
as many different solutions as number of different choices of the initial
values $\left[\vec{x}^{(s)}, \vec{v}^{(s)}\right]$, (s: 1,..,N) at the molecular level. Therefore it
would be more realistic to categorize the equations as <u>stochastic</u> because
specifying 6N initial values is not practicable. Then, transition to the
deterministic scheme of gasdynamics is effected through averaging at various
hierarchy levels, the lowest of which has led to the one-particle equations
(5.6) and (5.9). In order to construct the two-particle hierarchy out of
the stochastic fluid-dynamic equations we proceed as follows: let all the
fluid-dynamic correlations be decomposed in the form

$$R_{ij..,\ell m..}^{(J,K)} = \int q_{ij..}^{(J)}(\vec{x},\lambda)\hat{q}_{,m..}^{(K)}(\vec{\hat{x}},\lambda)d\lambda \qquad (7.1)$$

This expression being substituted in equation (6.6) we have, except for the
terms of thermal agitations,

$$\int d\lambda \left\{ q_i^{(1)} \left[\frac{D\hat{q}_\ell^{(1)}}{\hat{Dt}} + \frac{\partial}{\partial \hat{x}_\ell} \left(\frac{1}{3} \hat{q}^{(2)} + c^2 \hat{q}^{(0)} \right) + \frac{\partial}{\partial \hat{x}_s} q_{s\ell}^{(2)} \right. \right.$$

$$\left. + q_s^{(1)} \frac{\partial}{\partial \hat{x}_s} \left(\frac{\hat{m}_\ell}{\hat{\rho}} \right) + \hat{q}^{(0)} \frac{D}{Dt} \left(\frac{\hat{m}_\ell}{\hat{\rho}} \right) \right] + (\text{terms with} \quad x \leftrightarrow \hat{x}$$

$$\left. \text{and} \quad i \leftrightarrow \ell \quad \text{interchanged}) \right\} = 0 \qquad (7.2)$$

First we note, in view of equations (7.1), (3.2) and (1.15), that

$$q_{ij..}^{(J)}(\vec{x}, \lambda) = \int c^J \not{p}_{ij..}^{(J)} \, \Delta f \, d\vec{v} \qquad (7.3)$$

we have, then, from (3.12),

$$q^{(0)} = \Delta \rho$$

$$q_i^{(1)} = \Delta m_i - \rho^{-1} m_i \, \Delta \rho \qquad (7.4)$$

$$\frac{1}{3} q^{(2)} = \Delta p - c^2 \, \Delta \rho$$

With these linear relations for fluctuations, equation (7.2) is written in an equivalent form

$$\int \left(q_i^{(1)} \hat{L}_\ell + \hat{q}_\ell^{(1)} L_i \right) \, d\lambda = 0 \qquad (7.5)$$

where L_i has been defined by (5.2). By a similar procedure we can show that the whole system of correlation equations (6.4) through (6.9) is simply equivalent to

$$S_{\alpha\beta} \equiv \overline{\Delta w_\alpha \hat{L}_\beta + \Delta \hat{w}_\beta L_\alpha} = 0 \qquad (\alpha,\beta; \; 0 \text{ through } 4) \qquad (7.6)$$

with

$$\Delta w_o = \Delta\rho$$

$$\Delta w_i = \Delta m_i - \rho^{-1} m_i \, \Delta\rho \qquad \left.\right\} \qquad (7.7)$$

$$\Delta w_4 = \Delta p - c^2 \, \Delta\rho$$

A set of equations $\overline{L}_\alpha = 0$ and $S_{\alpha\beta} = 0$, consisting of 5 and 15 independent equations, respectively, forms a complete set to determine 20 unknowns, namely, ρ, m_i or w_i, p, and the lowest 15 components of the tensor $R^{(J,K)}$. For incompressible flows the equations with component $\alpha = 4$ (the energy equation) are deleted. On the other hand, ρ (= const.) and $R^{(0,K)}$ (= 0) are not considered as dependent variables; accordingly we have $4 + 10$ unknowns and the same number of equations.[4] If, further, the flow is homogeneous and isotropic, the first set of equations turns out to be trivial, whereas the second set is equivalent to the Karman-Howarth equation describing the final stages of decay.[14]

Summarizing, the closure problem of the compressible, shear turbulence equations reduces, under the assumption of ternary molecular chaos, to solving equations (7.6) in conjunction with (4.3) through (4.5) supplemented by (4.12) and (4.13). These two systems are coupled through only two variables $R_{j,k}^{(1,1)}$ and $R_j^{(1,2)}$ which intervene in the latter groups of equations. Despite their seeming complexity, the closure equations may be solved by the method of separation of variables in $(\vec{x},t)$ and $(\vec{\hat{x}},t)$ as is inferred from their rearranged form (7.6). The practicability of this method has been ensured in the case of incompressible flows.[4] Obtaining solutions, however, is beyond the scope of the present report.

Nielsen Engineering & Research, Inc.
    Mountain View, California
        February 1976

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TABLE I

PHYSICAL IMPLICATIONS OF $R^{(J,K)}$ FOR $(J,K) \leq (2,2)$

Expansion Coefficients	Incompressible Flows	Physical Meaning
$R^{(0,0)}$	0	Density-density correlation
$R_i^{(1,0)}$	0	Turbulent correction to the mean velocity as distinguished from the mean mass-flux density divided by the density (cf. eq. (35))
$\rho^{-1}R_{i,j}^{(1,1)}$	Nonvanishing	Reynolds stress (cf. eq. (54))
$R^{(2,0)}$	0	Turbulent correction to the mean temperature as distinguished from the pressure divided by the density (cf. eq. (36))
$\rho^{-1}R_i^{(2,1)}$	Nonvanishing*	Turbulent heat flux density (cf. eq. (57))
$(3\rho R)^{-2}R^{(2,2)}$	Nonvanishing*	Temperature-temperature correlation

*But has no practical meaning because the temperature is a redundant variable in incompressible flow equations.