A GENERAL GEOMETRIC THEORY OF ATTITUDE DETERMINATION FROM DIRECTIONAL SENSING

BY

Bertrand T. Fang

February 18, 1976

(NASA-CR-144744) A GENERAL GEOMETRIC THEORY OF ATTITUDE DETERMINATION FROM DIRECTIONAL SENSING (Wolf Research and Development Corp.) 36 p HC $4.00 CSCL 22C

N76-20183

Unclas
G3/17 19946

Backup Document for AIAA Synoptic Scheduled for Publication in the Journal of Spacecraft and Rockets, June 1976

Washington Analytical Services Center Inc.
Wolf Research and Development Group
6801 Keniworth Avenue - P. O. Box 398
Riverdale, Maryland 20840

SYNOPTIC BACKUP DOCUMENT

This document is made publicly available through the NASA scientific and technical information system as a service to readers of the corresponding "Synoptic" which is scheduled for publication in the following (checked) technical journal of the American Institute of Aeronautics and Astronautics.

- [ ] AIAA Journal
- [ ] Journal of Aircraft
- [X] Journal of Spacecraft & Rockets, June 1976
- [ ] Journal of Hydronautics

A Synoptic is a brief journal article that presents the key results of an investigation in text, tabular, and graphical form. It is neither a long abstract nor a condensation of a full length paper, but is written by the authors with the specific purpose of presenting essential information in an easily assimilated manner. It is editorially and technically reviewed for publication just as is any manuscript submission. The author must, however, also submit a full backup paper to aid the editors and reviewers in their evaluation of the synoptic. The backup paper, which may be an original manuscript or a research report, is not required to conform to AIAA manuscript rules.

For the benefit of readers of the Synoptic who may wish to refer to this backup document, it is made available in this microfiche (or facsimile) form without editorial or makeup changes.

A GENERAL GEOMETRIC THEORY OF ATTITUDE DETERMINATION FROM DIRECTIONAL SENSING

Bertrand T. Fang*
EG&G/Washington Analytical Services Center, Inc.

ABSTRACT

Spacecraft attitude determination is generally based on the output of on-board direction sensors which measure external reference directions relative to the spacecraft. A general geometric theory of attitude determination from such sensors is presented. Outputs of different sensors are reduced to two kinds of basic directional measurements. Errors in these measurement equations are studied in detail. The partial derivatives of measurements with respect to the spacecraft orbit, the spacecraft attitude, and the error parameters form the basis for all orbit and attitude determination schemes and error analysis programs and are presented in a series of tables. The question of attitude observability is studied with the introduction of a graphical construction which provides a great deal of physical insight. The result is applied to the attitude observability of the IMP-8 spacecraft.

This work was sponsored by NASA Goddard Space Flight Center under Contract NAS 5-20098

*Senior Scientific Specialist, Wolf Research and Development Group

NOMENCLATURE

$[\ ]$ = matrix

$[\ ]^T$ = matrix transpose

$[\tilde{P}][Q] = [P]x[Q] = \overline{P}x\overline{Q}$, cross product in matrix notation

$[\tilde{P}] = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}$ = antisymmetric matrix associated with the vector $[P] = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$

$[P^I] = \begin{bmatrix} P_1^I \\ P_2^I \\ P_3^I \end{bmatrix}$ = the vector $\overline{P}$ expressed in a set of X_j^I-axes

$[A_{I/B}] = [a_{ij} = \cos(\overline{X}_i^I , \overline{X}_j^B)]$, direction cosine matrix relating a set of spacecraft axes $\overline{X}_i^B$ to another set of axes $\overline{X}_j^I$, usually the inertial axes.

$\overline{I}, \overline{J}, \overline{K}$ or [I], [J], [K] = a set of spacecraft-fixed orthogonal unit vectors

$\overline{D}$ or [D] = a spacecraft-fixed unit vector representing a sensitivity axis of direction sensor

$\overline{R}$ or [R] = a unit vector representing a reference direction

$\overline{\rho}$ or $[\rho]$ = spacecraft position vector

$\overline{\omega}$ or $[\omega]$ = spacecraft angular velocity vector

y_1, y_2, y_3 = standard measurements as defined in Equations (3), (4), (5)

t = time

$\Delta(\ )$ = error in ( )

θ = attitude parameter or Euler's angle

$q_0, q_1, q_2 q_3$ = components of the unit rotation quaternion

α, β = parameters specifying the error in the direction of a unit vector as defined in Equation (11)

$\varepsilon_1, \varepsilon_2, \varepsilon_3$ = instrument package mounting error vector representing small rotations about $-X_1^B$, $-X_2^B$ and $-X_3^B$ axes.

INTRODUCTION

Spacecraft attitude determination is generally based on the output of on-board directional sensors which measure external reference directions relative to the spacecraft. Examples of such sensors include magnetometers, interferometers, horizon scanners, star trackers, and etc. The reference direction sensed may be an ambient field vector such as the geomagnetic field vector. More frequently, it is a spacecraft to object vector as traced by an electromagnetic radiation path. If the object is a distant star, the vector is a fixed direction in inertial space. If the object is a landmark on Earth, the vector depends on the position or orbit of the spacecraft. In that case the directional measurements are related to both the spacecraft attitude and orbit and would be useful in both attitude and orbit determination.

A great deal of interest has arisen recently concerning: the need to develop a standardized attitude determination software package for use with the diversity of attitude sensors used on different spacecraft; and the possible use of attitude sensors to gain information about the orbit.

With these as motivations this paper presents the fundamental geometric aspect of attitude sensors with a view toward the development of a systematic and general theory independent of particular sensor types.

THE TWO BASIC DIRECTIONAL MEASUREMENTS

In order to develop a general theory independent of particular sensor types, it is necessary to categorize the available measurements into their most fundamental ingredients. The fundamental building blocks for most sensors are: 1. the angle between two lines, one of which is the reference direction and the other a spacecraft-fixed direction; and 2. the angle between two planes, one of which contains the reference direction and a spacecraft-fixed direction, the other is spacecraft-fixed and contains the same spacecraft-fixed direction.

It is convenient to use unit vectors to represent directions. Let $\bar{R}$ be the reference unit vector and $\bar{K}$ a spacecraft-fixed unit vector. Then the above two classes of directional information are essentially those represented by the two vector products; i.e., $\bar{R}\cdot\bar{K}$ in the first case and $\frac{\bar{R}x\bar{K}}{|\bar{R}x\bar{K}|}$ in the second case. Since $\bar{R}x\bar{K}$ is a vector in a plane orthogonal to $\bar{K}$, and $|\bar{R}x\bar{K}| = \sqrt{1-(\bar{R}\cdot\bar{K})^2}$,

$$\frac{\bar{R}x\bar{K}}{|\bar{R}x\bar{K}|} = \frac{\bar{R}\cdot\bar{J}}{\sqrt{1-(\bar{R}\cdot\bar{K})^2}}\,\bar{I} - \frac{\bar{R}\cdot\bar{I}}{\sqrt{1-(\bar{R}\cdot\bar{K})^2}}\,\bar{J} \qquad (1)$$

where $\bar{I}$, $\bar{J}$ and $\bar{K}$ form a set of spacecraft-fixed right-handed orthogonal unit vectors. Obviously this second kind of measurement contains a little more information than the first kind of measurement, but it is related in a more complicated non-linear manner to the reference direction $\bar{R}$.

THE DIRECTION COSINE MATRIX AND THE OBSERVATION EQUATIONS

A convenient way to describe the spacecraft attitude is to specify the direction cosine matrix

$$[A_{I/B}] = [a_{ij} = \cos (X_i^I , X_j^B)] , (i,j = 1,2,3) \qquad (2)$$

relating a set of spacecraft axes X_i^B to another set of axes X_j^I of known orientation, which for the convenience of argument may be taken to be a set of inertial axes.

It is now possible to rewrite the basic directional measurements in terms of the reference direction in inertial coordinates $[R^I]$, a known spacecraft fixed direction $[D^B]$, three known spacecraft fixed orthogonal directions, say, $[I^B]$, $[J^B]$, $[K^B]$, the direction cosine matrix $[A_{B/I}]$ and the equivalent instrument outputs y's as follows:

$$y_1 = \bar{R}\cdot\bar{D} = [R^I]^T [D^I] = [R^I]^T [A_{I/B}][D^B] \qquad (3)$$

$$y_2 = \sin(\bar{R}\text{-}\bar{K},\bar{I}\text{-}\bar{K}) = \frac{\bar{R}\cdot\bar{J}}{\sqrt{1-(\bar{R}\cdot\bar{K})^2}} = \frac{[R^I]^T [A_{I/B}][J^B]}{\sqrt{1-([R^I]^T [A_{I/B}][K^B])^2}} \qquad (4)$$

$$y_3 = \cos(\bar{R}\text{-}\bar{K},\bar{I}\text{-}\bar{K}) = \frac{\bar{R}\cdot\bar{I}}{\sqrt{1-(R\cdot K)^2}} = \frac{[R^I]^T [A_{I/B}][I^B]}{\sqrt{1-([R^I]^T [A_{I/B}][K^B])^2}} \qquad (5)$$

with $y_2^2+y_3^2 = 1$.

In the above forms, various quantities of interest are clearly delineated. The left-hand sides of these equations, the y's, are either direct sensor outputs or are quantities easily computed from the sensor outputs. The right-hand sides are what "the sensors are supposed to sense." The spacecraft-fixed directions generally depend on the instrument alignment and when expressed in spacecraft coordinates, are independent of the spacecraft motion. These measurements are not related to the rate variables. Any information concerning the spacecraft orbit must enter through $[R^I]$ and that concerning the attitude through $[A_{I/B}]$. In the usual attitude determination problem $[R^I]$ is assumed known and $[A_{I/B}]$ is the unknown to be determined. If the attitude $[A_{I/B}]$ is known and $[R^I]$ is an unknown spacecraft to object vector, these equations can be used to determine orbit. In a combined attitude/orbit determination problem, both $[A_{I/B}]$ and $[R^I]$ are treated as unknowns.

ERRORS IN MEASUREMENT EQUATIONS

In reality measurements involve errors. Although different instruments have different errors, as long as attitude estimates are derived from the basic measurement equations, instrument errors must enter, or transform into errors in these equations. For the purpose of illustration, let us consider the observation Equation (3), which is written as

$$y_1(t) = [R^I(t)]^T \; [A_{I/B}(t)] \; [D^B] \tag{6}$$

with the explicit dependence on time shown. In the attitude determination problem y_1, $[R^I]$ and $[D^B]$ enter this equation as given quantities and $[A_{I/B}]$ as the unknown to be solved for. In general there are the following two kinds of errors:

1. Measurement process modeling error; i.e., Equation (5) may not represent the true functional relation between the measurement y_1 and the other quantities appearing on the right-hand side of that equation. This kind of error is somewhat rare for most sensor systems. Usually the only case of importance is the "triggering time" error; i.e., although the measurement is thought to be made at time t, the actual measurement is made at time $t+\Delta t$.

2. Errors in the "given quantities"; for instance, the reference direction $[R^I]$ may not be known exactly, the instrument output y_1 may be read with a bias, etc.

When these errors are considered, the basic observation equations appear in the following forms with the Δ's indicating the errors.

$$y_1(t+\Delta t) = [R(t+\Delta t)^I+\Delta R^I]^T [A_{I/B}(t+\Delta t)][D^B+\Delta D^B] + \Delta_1 \quad (7)$$

$$y_2(t+\Delta t) = \frac{[R(t+\Delta t)^I+\Delta R^I]^T [A_{I/B}(t+\Delta t)][J^B+\Delta J^B]}{\sqrt{1-([R(t+\Delta t)^I][A_{I/B}(t+\Delta t)][K^B+\Delta K^B])^2}} + \Delta_2 \quad (8)$$

$$y_3(t+\Delta t) = \frac{[R(t+\Delta t)^I+\Delta R^I]^T [A_{I/B}(t+\Delta t)][I^B+\Delta I^B]}{\sqrt{1-([R(t+\Delta t)^I+\Delta R^I][A_{I/B}(t+\Delta t)][K^B+\Delta K^B])^2}} + \Delta_3 \quad (9)$$

It is seen only the following errors enter into these equations:

1. Instrument reading errors, Δ_1, Δ_2, Δ_3.

2. Reference direction error $[\Delta R^I]$.

3. Instrument alignment errors, $[\Delta D^B],[\Delta I^B][\Delta J^B],[\Delta K^B]$.

4. Timing error, Δt.

Let us investigate these errors in a little more detail. The instrument reading errors may be biases and/or random fluctuations. Since the y's may be converted rather than original measurements, Δ_1, Δ_2, Δ_3 may be equivalent biases, etc. But they are readily calculated once the corresponding errors in the original measurements are given. Most of the time the reference direction $\bar{R}$ is the line of sight from the spacecraft to a "spacemark", and may be written as

$$\bar{R} = \frac{\bar{\rho}-\bar{T}}{|\bar{\rho}-\bar{T}|}$$

where $\bar{\rho}$ is the orbital position vector and $\pm\bar{T}$ is the position vector of the spacemark i[illegible]pendent of the orbital position. Therefore one may write the reference direction error as

$$[\Delta R^I] = \frac{[\rho^I]-[T^I]+[\Delta\rho^I]-[\Delta T^I]}{|[\rho^I]-[T^I]+[\Delta\rho^I]-[\Delta T^I]|} - \frac{[\rho^I]-[T^I]}{|[\rho^I]-[T^I]|}$$

$$\approx \frac{1}{|[\rho^I]-[T^I]|}\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - [R^I][R^I]^T\right)([\Delta\rho^I]-[\Delta T^I]) \quad (10)$$

which exhibits explicitly the effect of orbital error and spacemark position error on the error in the reference direction.

Sometimes one does not know the error sources, but has some idea about the accuracy that may be expected of the reference direction. In that case it is more convenient to express the reference direction error as

$$[\Delta R^I] = \alpha[P^I] + \beta[Q^I] \tag{11}$$

where $[R^I]$-$[P^I]$-$[Q^I]$ form a set of right-handed orthogonal vectors. Equation (11) expresses the fact that since $[R^I]$ is a unit vector, $[\Delta R^I]$ must be orthogonal to $[R^I]$ and depends on only two parameters α and β.

If the sensor has a single sensitivity axis, the alignment error of this axis depends on two parameters just as in the case of reference direction errors. On the other hand, for a carefully aligned instrument package with several sensitivity axes, it may often be assumed that alignment errors are instrument package mounting errors. The small mounting errors may be represented by a misalignment vector $[\varepsilon^B] = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3]^T$ such that

$$[\Delta(\ )^B] = [(\ )^B] \times [\varepsilon^B] \tag{12}$$

where ( ) = D,I,J,K, etc., as the case may be, and ε_1, ε_2 and ε_3 represent small misaligning rotations of the instrument package about the three spacecraft body axes. In this case there will only be a maximum of three misalignment error parameters regardless of how many instruments there are in the instrument package.

As seen in Equations (7) through (9), the timing error enters into both the reference direction and the direction cosine matrix. Since the attitude motion generally has a shorter time constant than that of the orbital motion, one may disregard the error in reference direction due to timing error. If the timing error is small,

$$[A_{I/B}(t+\Delta t)] \cong [A_{I/B}(t)] + \frac{d}{dt} [A_{I/B}(t) \Delta t$$

$$= [A_{I/B}(t)] + [A_{I/B}(t)] [\tilde{\Omega}^B] \Delta t .$$

where $[\tilde{\Omega}^B]$ is the anti-symmetric angular velocity tensor.

So far our primary concern has been with attitude determination. If the measurements are to be used for orbit determination, the direction cosine matrix $[A_{I/B}]$ will be considered as known quantities, and errors in $[A_{I/B}]$ will have to be considered; i.e., $[A_{I/B}]$ in Equations (7) through (9) should be replaced by $[A_{I/B}] + [\Delta A_{I/B}]$. The elements of the direction cosine error matrix $[\Delta A_{I/B}]$ are not all independent. Their expressions in terms of the computationally convenient quaternions are given in the next section. For other commonly used attitude parameters, see Ref. 1.

PARTIAL DERIVATIVES OF MEASUREMENTS WITH RESPECT TO ORBIT, ATTITUDE, AND ERROR PARAMETERS

The partial derivatives of measurements with respect to the orbit, the spacecraft attitude and the error parameters gives the linear first order relations among these quantities and are the basis for all orbit and attitude determination schemes and error analysis programs. These partial derivatives may be determined from the equations of the preceding section and are presented in a series of tables. In these tables y_1, y_2 and y_3 denote the standard measurements defined in Equations (3), (4) and (5). The partial derivatives of the measurement y_3 are obtainable from the relation

$$\frac{\partial y_3}{\partial(\ )} = -\frac{y_2}{y_3}\frac{\partial y_2}{\partial(\ )} \tag{14}$$

and are not listed separately.

Table 1 gives the partial derivatives of the standard measurements with respect to error parameters discussed in the preceding section. Since the actual measurements may not be the standard measurements, Table 2 presents a compilation of common sensors, their actual measurements and the equivalent standard measurements. Generally, the directional measurements are not directly related to the spacecraft orbital or angular velocities. The partial derivatives of the standard measurements with respect to the spacecraft position vector are given in Table 3. Formulas for the partial derivatives of the standard measurements with respect to the spacecraft attitude are given in Table 4. It is seen that the dependence on attitude enters only through the direction cosine matrix $[A_{I/B}]$. A complication arises because different investigators tend to use different sets of attitude parameters and some of the parameters are not independent. For the case the direction cosine matrix [A] is expressed in terms of the quaternion parameters q_0, q_1, q_2, q_3 as

$$[A] = (2q_0^2 - 1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} q_1^2 & q_1 q_2 & q_1 q_3 \\ q_1 q_2 & q_2^2 & q_2 q_3 \\ q_1 q_3 & q_2 q_3 & q_3^2 \end{bmatrix}$$

$$-2q_0\begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} , \qquad (15)$$

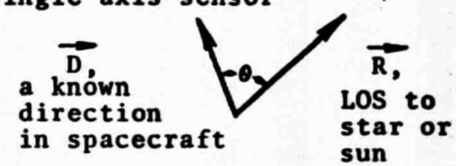
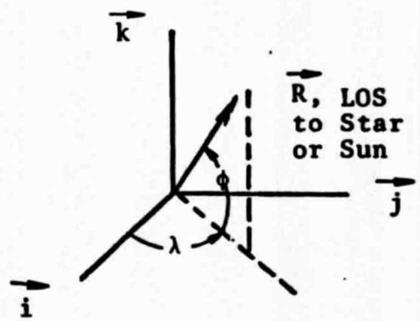
the results are given in Table 5. For other parametric representations, see Ref. 1.

Table 1. Partial Derivatives of Standard Measurements y_1 and y_2 with Respect to Error Parameters

Error Parameter	Partial Derivatives of Standard Measurements with respect to error parameter
Instrument Reading Errors: Δ_1, Δ_2	$\frac{\partial y_1}{\partial \Delta_1} = 1$ $\frac{\partial y_2}{\partial \Delta_2} = 1$
Timing Error, Δt	$\frac{\partial y_1}{\partial \Delta t} = [R^I]^T [A_{I/B}][\tilde{\Omega}^B][D^B]$ $\frac{\partial y_2}{\partial \Delta_t} = \frac{[R^I]^T [A_{I/B}][\tilde{\Omega}^B][J^B]}{\sqrt{1-([R^I]^T [A_{I/B}][K^B])^2}} + \frac{y_2 [R^I]^T [A_{I/B}][\dot{K}^B]}{1-([R^I]^T [A_{I/B}][K^B])^2} [R^I]^T [A_{I/B}][\tilde{\Omega}^B][K^B]$
Instrument Alignment Error, $[\Delta D^B] = \gamma [F^B] + \delta [G^B]$, $[F^B] = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}$ $[D^B]$, $[F^B]$, $[G^B]$ forming a set of right-handed orthogonal vectors	$\frac{\partial y_1}{\partial \gamma} = [R^I]^T [A_{I/B}][F^B]$ $\frac{\partial y_1}{\partial \delta} = [R^I]^T [A_{I/B}][G^B]$

Instrument Mounting Error, $[\Delta(\ )^B] = [\tilde{(\ )}^B]\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix}$	$\frac{\partial y_1}{\partial[\varepsilon]} \triangleq \left[\frac{\partial y_1}{\partial \varepsilon_1} \vdots \frac{\partial y_1}{\partial \varepsilon_2} \vdots \frac{\partial y_1}{\partial \varepsilon_3}\right] = \underset{1x3}{[R^I]^T}\ \underset{3x3}{[A_{I/B}]}\ \underset{3x3}{[\tilde{(\ )}^B]}$ -- $\frac{\partial y_2}{\partial[\varepsilon]} \triangleq \left[\frac{\partial y_2}{\partial \varepsilon_1} \vdots \frac{\partial y_2}{\partial \varepsilon_2} \vdots \frac{\partial y_2}{\partial \varepsilon_3}\right]$ $= \frac{[R^I]^T[A_{I/B}][\tilde{J}^B]}{\sqrt{1-([R^I]^T[A_{I/B}][K^B])^2}} + \frac{y_2[R^I]^T[A_{I/B}][K^B]}{1-([R^I]^T[A_{I/B}][K^B])^2}[R^I]^T[A_{I/B}][\tilde{K}^B]$
Reference Direction Error, $[\Delta R^I] = \alpha[P^I] + \beta[Q^I]$ $[P^I] = \begin{bmatrix}p_1\\ p_2\\ 0\end{bmatrix}$ $[R^I]$, $[P^I]$, $[Q^I]$ forming a set of right-handed orthogonal vectors	$\frac{\partial y_1}{\partial \alpha} = [P^I]^T[A_{I/B}][D^B]$ -- $\frac{\partial y_2}{\partial \alpha} = \frac{[P^I]^T[A_{I/B}][J^B]}{\sqrt{1-([R^I]^T[A_{I/B}][K^B])^2}} + \frac{y_2[R^I]^T[A_{I/B}][K^B]}{1-([R^I]^T[A_{I/B}][K^B])^2}[P^I]^T[A_{I/B}][K^B]$ $\frac{\partial y_1}{\partial \beta}$ and $\frac{\partial y_2}{\partial \beta}$ obtainable from the above equations by replacing $[P^I]$ by $[Q^I]$

Table 2. Common Sensors, Their Actual Measurements and Equivalent Standard Measurements

Sensor	Reference Direction	Possible Measurement	Equivalent Standard Measurements	Partial Derivatives of Equivalent Measurements w.r.t. original measurements
	$[R^I] = \frac{[\rho^I]-[T^I]}{\lvert [\rho^I]-[T^I] \rvert}$ $[\rho^I]$ = orbital radius vector $[T^I]$ = independent of orbit			
Star and Sun Sensor	$-[T^I]$ is position vector of star or sun	θ as shown below for a single-axis sensor 	$y_1=\cos\theta=[R^I]^T[A_{I/B}][D^B]$	$\frac{\partial y_1}{\partial\theta} = -\sin\theta$
		λ and ϕ shown below for a two-axes sensor  $\bar{i},\bar{j},\bar{k}$ are known orthogonal directions in spacecraft	Either, $y_1=\sin\phi=[R^I]^T[A_{I/B}][K^B]$ $y_2=\sin\lambda=\frac{[R^I]^T[A_{I/B}][J^B]}{\sqrt{1-([R^I]^T[A_{I/B}][K^B])^2}}$ $y_3=\cos\lambda=\frac{[R^I]^T[A_{I/B}][I^B]}{\sqrt{1-([R^I]^T[A_{I/B}][K^B])^2}}$ or, $y_1=\sin\phi=[R^I]^T[A_{I/B}][K^B]$ $y_1'=\cos\phi\cos\lambda=[R^I]^T[A_{I/B}][I^B]$ $y_1''=\cos\phi\sin\lambda=[R^I]^T[A_{I/B}][J^B]$	$\frac{\partial y_1}{\partial\phi} = \cos\phi$ $\frac{\partial y_2}{\partial\lambda} = \cos\lambda$ $\frac{\partial y_3}{\partial\lambda} = -\sin\lambda$ $\frac{\partial y_1}{\partial\phi} = \cos\phi$ $\frac{\partial y_1'}{\partial\phi} = -\sin\phi\cos\lambda$ $\frac{\partial y_1'}{\partial\lambda} = -\cos\phi\sin\lambda$ $\frac{\partial y_1''}{\partial\phi} = -\sin\phi\sin\lambda$ $\frac{\partial y_1''}{\partial\lambda} = \cos\phi\cos\lambda$

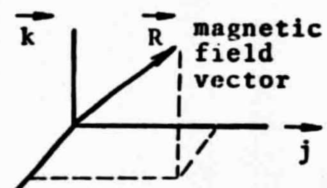
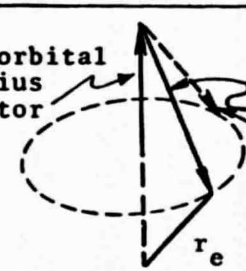
		Star or Sun Sight Time τ	Same as Two-Axes Sensor	See timing error in Table 1
Interferometer	$[T^I]$ is position vector of transmitting station	$\cos\theta = (\frac{\lambda}{\alpha})\ (\frac{\Delta\phi}{\pi})$  	$y_1 = \cos\theta = [R^I]^T [A_{I/B}][D^B]$	Actual measurements are standard measurements
Magnetometer	Exception. $[R^I]$ = magnetic field vector, which depends on the orbital position, and is also not a unit vector	$\bar{R}\cdot\bar{i}$, $\bar{R}\cdot\bar{j}$, $\bar{R}\cdot\bar{k}$  $\vec{I},\vec{J},\vec{K}^I$ are known orthogonal directions in spacecraft	$y_1 = [R^I]^T [A_{I/B}][I^B]$ $y_1' = [R^I]^T [A_{I/B}][J^B]$ $y_1'' = [R^I]^T [A_{I/B}][K^B]$	Actual measurements are standard measurements
Horizon Scanner	$[T^I] = [0]$	 Horizon entry time t and earth width Δt	$y_1 = \sqrt{1-\left(\frac{r_e}{\rho}\right)^2} = -[R^I]^T [A_{I/B}][D(t)^B]$ $y_1' = 0 = [R^I]^T [A_{I/B}]\left([D(t+\Delta t)^B] - [D(t)^B]\right)$	See timing error in Table 1

Table 3. Partial Derivatives of Standard Measurements y_1 and y_2 with Respect to Spacecraft Position Vector $[\rho^I]$

$$\frac{\partial y_1}{\partial[\rho^I]} = \frac{\partial[R^I]}{\partial[\rho^I]}[A_{I/B}][D^B]$$

$$\frac{\partial y_2}{\partial[\rho^I]} = \frac{1}{\sqrt{1-([R^I]^T[A_{I/B}][K^B])^2}} \frac{\partial[R^I]}{\partial[\rho^I]}[A_{I/B}]$$

$$\left\{[J^B]+[K^B] \quad \frac{y_2[R^I]^T[A_{I/B}][K^B]}{\sqrt{1-([R^I]^T[A_{I/B}][K^B])^2}}\right\}$$

with

$$\frac{\partial[R^I]}{\partial|[\rho^I]|} = \frac{1}{|[\rho^I]-[T^I]|}([E]-[R^I][R^I]^T)$$

$[E]$ = 3x3 identity matrix

$[R^I]$ = $[\rho^I]-[T^I]$

$\pm[T^I]$ = position vector of "spacemark"

Table 4. Partial Derivatives of Standard Measurements y_1 and y_2 with Respect to an Attitude Parameter θ

$$\frac{\partial y_1}{\partial \theta} = [R^I]^T \frac{\partial [A_{I/B}]}{\partial \theta} [D^B]$$

$$\frac{\partial y_2}{\partial \theta} = \frac{[R^I]^T \frac{\partial [A_{I/B}]}{\partial \theta} [J^B]}{\sqrt{1-([R^I]^T [A_{I/B}][K^B])^2}} + \frac{y_2 \, ([R^I]^T [A_{I/B}][K^B])}{1-([R^I]^T [A_{I/B}][K^B])^2} [R^I]^T \frac{\partial [A_{I/B}]}{\partial \theta} [K^B]$$

Table 5. Partial Derivatives of Direction Cosine Matrix [A] with Respect to the Quaternion Parameters q_0, q_1, q_2 and q_3 .

$$\delta[A] = \frac{\partial[A]}{\partial q_0}\delta q_0 + \frac{\partial[A]}{\partial q_1}\delta q_1 + \frac{\partial[A]}{\partial q_2}\delta q_2 + \frac{\partial[A]}{\partial q_3}\delta q_3$$

$$\frac{\partial[A]}{\partial q_0} = 4q_0\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - 2\begin{bmatrix}0 & -q_3 & q_2\\ q_3 & 0 & -q_1\\ -q_2 & q_1 & 0\end{bmatrix}$$

$$\frac{\partial[A]}{\partial q_1} = 2\begin{bmatrix}2q_1 & q_2 & q_3\\ q_2 & 0 & -q_0\\ q_3 & -q_0 & 0\end{bmatrix}$$

$$\frac{\partial[A]}{\partial q_2} = 2\begin{bmatrix}0 & q_1 & -q_0\\ q_1 & 2q_2 & q_3\\ q_0 & q_3 & 0\end{bmatrix}$$

$$\frac{\partial[A]}{\partial q_3} = 2\begin{bmatrix}0 & q_0 & q_1\\ -q_0 & 0 & q_2\\ q_1 & q_2 & 2q_3\end{bmatrix}$$

and the δq's satisfy the constraint equation

$$q_0\delta q_0 + q_1\delta q_1 + q_2\delta q_2 + q_3\delta q_3 = 0$$

ATTITUDE OBSERVABILITY

The basic observation equations are given in the preceding section. If measurement noise is not considered, any meaningful observation corresponds to some attitude. The question is, "What combinations of these basic measurements are required to resolve the attitude unambiguously?"

It is well-known and also easily understood that three non-colinear points of a rigid-body, or equivalently, two non-colinear body-fixed vectors determine the body attitude completely. The two vectors with six components represent three instead of six independent pieces of information because the lengths of the vectors and the angle between the vectors must be independent of the attitude. The fact that two vectors are required to describe the attitude accounts for the comparative complexity in rotation than translation and for the necessity of using more complicated tools of matrices and tensors.

Most spacecraft attitude sensors measure space-fixed directions relative to spacecraft. Attitude is a relative notion. Knowledge of two space-fixed directions relative to the spacecraft body determines the attitude of space relative to the body, or, the attitude of body in space. That one space-fixed direction is not sufficient is also obvious from the fact that "roll" about that direction cannot be distinguished.

One may conclude from the above that:

1. Attitude observability is equivalent to the observability of two spacecraft-fixed or space-fixed directions.

2. At least three independent measurements related to two reference directions are required to determine the spacecraft attitude.

Measurements are generally non-linear functions of attitude parameters. The question of observability is mathematically equivalent to that of the uniqueness of the real solution of a set of non-linear algebraic equations, and is not easy to answer. In the following we shall introduce a graphical construction, which provides a great deal of insight to the problem.

Let us represent direction by a unit vector, or the terminus of that vector on a unit sphere. The two classes of directional information discussed before may be called "small circle" and "half great circle" measurements because of their interpretations on the unit sphere as follows:

1. The measurement of the component of a reference direction $\bar{e}$ along a spacecraft-fixed direction $\bar{r}$ say

   $$d = \frac{\bar{e} \cdot \bar{r}}{|\bar{r}|}$$ restricts the terminus of $\bar{e}$ to lie

   on a small circle which is the intersection of the unit sphere with a plane perpendicular to $\bar{r}$ and at a distance d from the center of the sphere (Figure 1a). The diameter of the circle is $2\sqrt{1-d^2}$.

2. The measurement of the "meridianal" angle ϕ restricts the terminus of $\bar{e}$ to lie on a "half great circle" (Figure 1b).

Obviously, the intersection of a half great circle with any other great circle, determines $\bar{e}$ uniquely. On the other hand, two "small circle" measurements merely determine $\bar{e}$ as one of two possible directions which are represented by the two points of intersection of the small circles (Figure 2a).

Any other non-trivial measurement serves to remove the ambiguity. Sometimes one has some idea about the general location of $\bar{e}$ and that information may enable one to discriminate the true direction from the false direction. Depending on geometry, a half great circle and a small circle measurement may or may not determine $\bar{e}$ uniquely. For instance, if the component of $\bar{e}$ along a direction in a plane perpendicular to the great circle is measured, the corresponding small circle could only intersect the half great circle at one point and $\bar{e}$ is determined uniquely (Figure 3b). On the other hand, if the component of $\bar{e}$ along a direction near the intersection of the great circle plane and the "equatorial" plane is measured, there are likely to be two points of intersection, and ambiguity about $\bar{e}$ exists (Figure 3c).

The above discussion describes the combinations of independent measurements required to determine a direction. Whether measurements are independent or not also becomes very clear from the geometrical construction. Parallel (coincident) circles obviously carry no new information. Only components along linearly independent directions are independent measurements. Nearly parallel circles cannot resolve their points of intersection accurately. For this reason one generally prefers orthogonal measurements or orthogonal circles.

To illustrate the results of this section, consider the spin-stabilized IMP-8 spacecraft. During a complete spin, measurements shown in Figure 4 are made. The spin is determined as $\omega = \frac{2\pi}{T}$, T being the period between successive sun sightings. When ω is known, the Earth-in time and Earth-width time may be converted to angles.

Since measurements are made on-board, it is convenient to take the point of view of a person on-board the spacecraft. The measurements are represented in Figure 5 on a unit sphere relative to spacecraft-fixed directions. One sees immediately that

1. The solar elevation measurements restrict the Sun vector to lie on a small circle orthogonal to the spin-axis.

2. The horizon entry vector has a known direction on-board. The Earth-in rotation angle restricts the Sun vector to lie on a half great circle. This half great circle and the small circle specifying the solar elevation has only one intersection. The point of intersection determines a unique Sun vector.

3. The angle between the Sun vector and Earth vector known from the orbital information restricts the Earth vector to lie on a small circle orthogonal to the now determined Sun vector.

4. The Earth vector must also lie on two small circles of identical radius about the known (known on-board) horizon entry and exit vectors. These small circles are of course the horizon as seen by the spacecraft at known distance from the Earth.

5. Therefore the Earth vector must be at the intersection of the three small circles about the Sun, horizon entry and exit vectors. If these three vectors are linearly independent (non-coplanar), there can only be a single point of intersection. The Earth vector is determined uniquely and the IMP-J attitude is also determined. Generally this will be the case, although exceptions may occur when either

   a. Earth width becomes zero, or,

b. The Sun vector is colinear with one of the horizon crossing vectors. With the given instrument angle and the orbit, this may occur only if the following occur simultaneously

(1) The solar elevation $\approx \frac{\pi}{2}$.

(2) The Earth-in rotation angle $\approx \pi$ or π-Earth width.

6. Some redundancy exists about Earth vector for data smoothing.

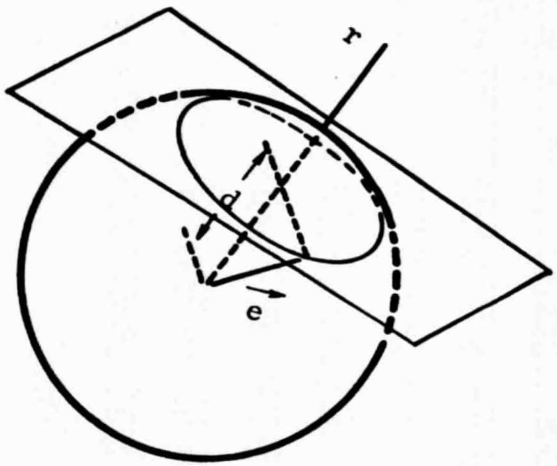
It is somewhat unfortunate that during an important part of the mission of IMP-8, the situation of "5b" nearly occurred, which makes the attitude determination difficult in the presence of the inevitable measurement noises, even though a large number of measurements are available and a rather sophisticated statistical attitude estimation scheme was used (Ref. 2).

REFERENCES

1. Fang, B.T., "Mathematical Specifications for an Attitude/Orbit Error Analysis System," Planetary Sciences Department Report No. 009-74, October 1974, Wolf Research and Development Corporation, Riverdale, Maryland.

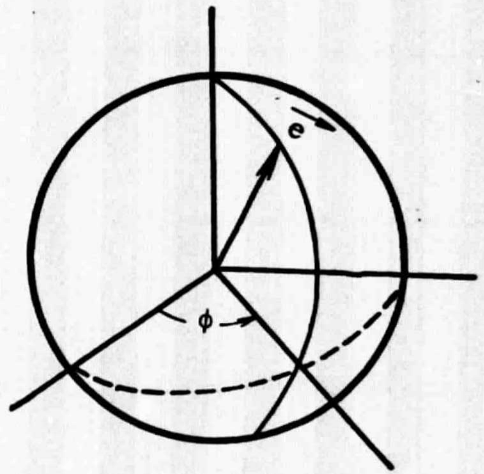
2. Gibbs, B.P., Haley, D.R., and Fang, B.T., "IMP-8 Attitude Study," Feb. 1975, Wolf Research and Development Corporation, Riverdale, Maryland.

CAPTIONS OF FIGURES



Small circle
representing all
possible $\bar{e}$ which
has component d
along $\bar{r}$

Figure 1a. Measurement of First Kind as A Small Circle on a Unit Sphere



Half great circle
representing all
possible directions
with the same
meridianal angle ϕ

Figure 1b. Measurement of Second Kind as A Half Great Circle on a Unit Sphere

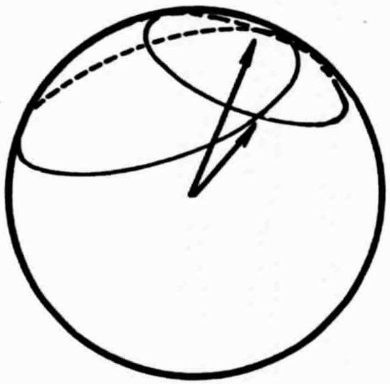


Figure 2a. Intersection of Two Small Circles Gives Two Possible Directions

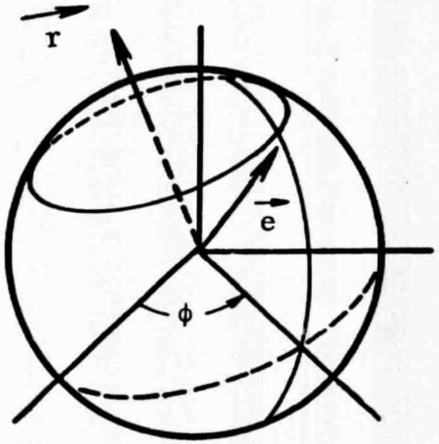


Figure 2b. With Favorable Geometry Intersection of a Half Great Circle and a Small Circle Determines a Unique Direction

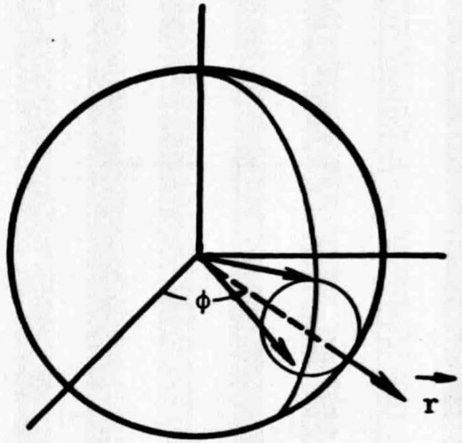


Figure 2c. With Unfavorable Geometry Intersection of a Half Great Circle and a Small Circle Gives Two Possible Directions

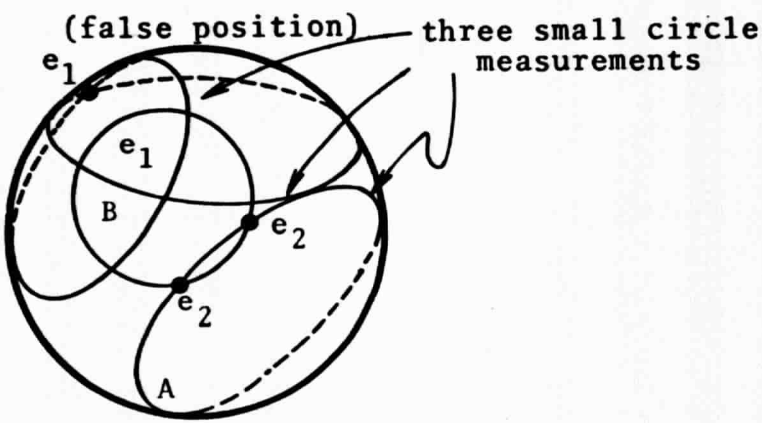


Figure 3a. Three Small Circle Measurements About Two Reference Directions $\overline{e}_1$ and $\overline{e}_2$

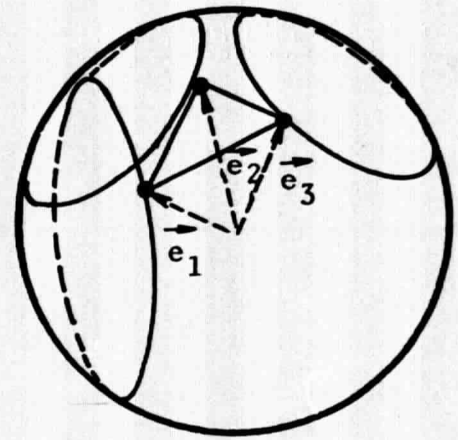
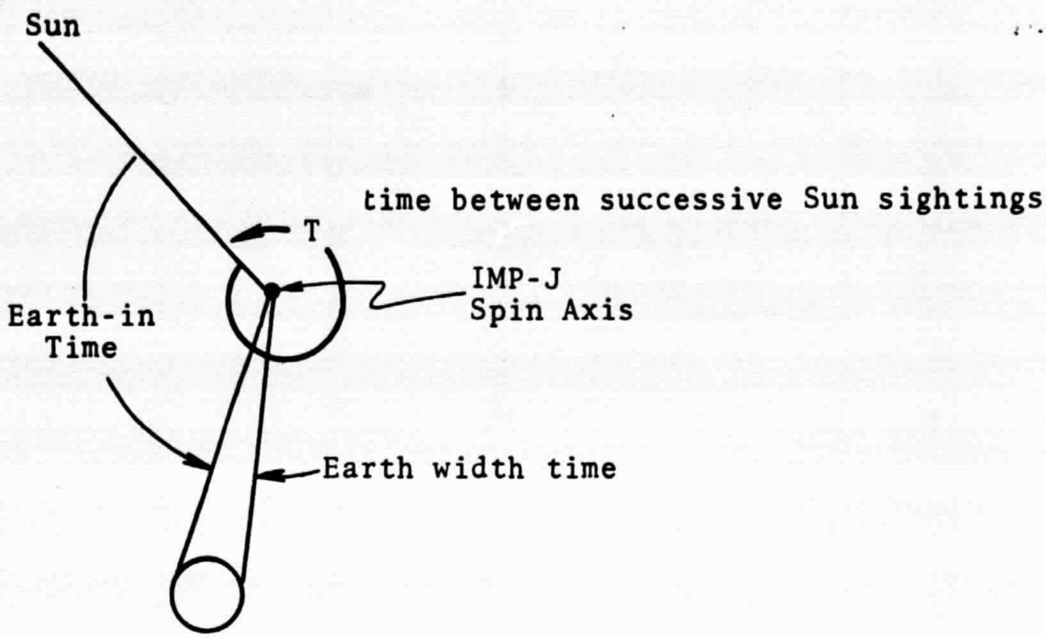


Figure 3b. Three Small Circle Measurements About Three Reference Directions $\overline{e}_1$, $\overline{e}_2$ and $\overline{e}_3$



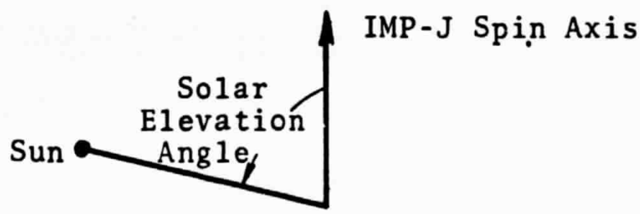


Figure 4. IMP-J Spacecraft Attitude Measurements

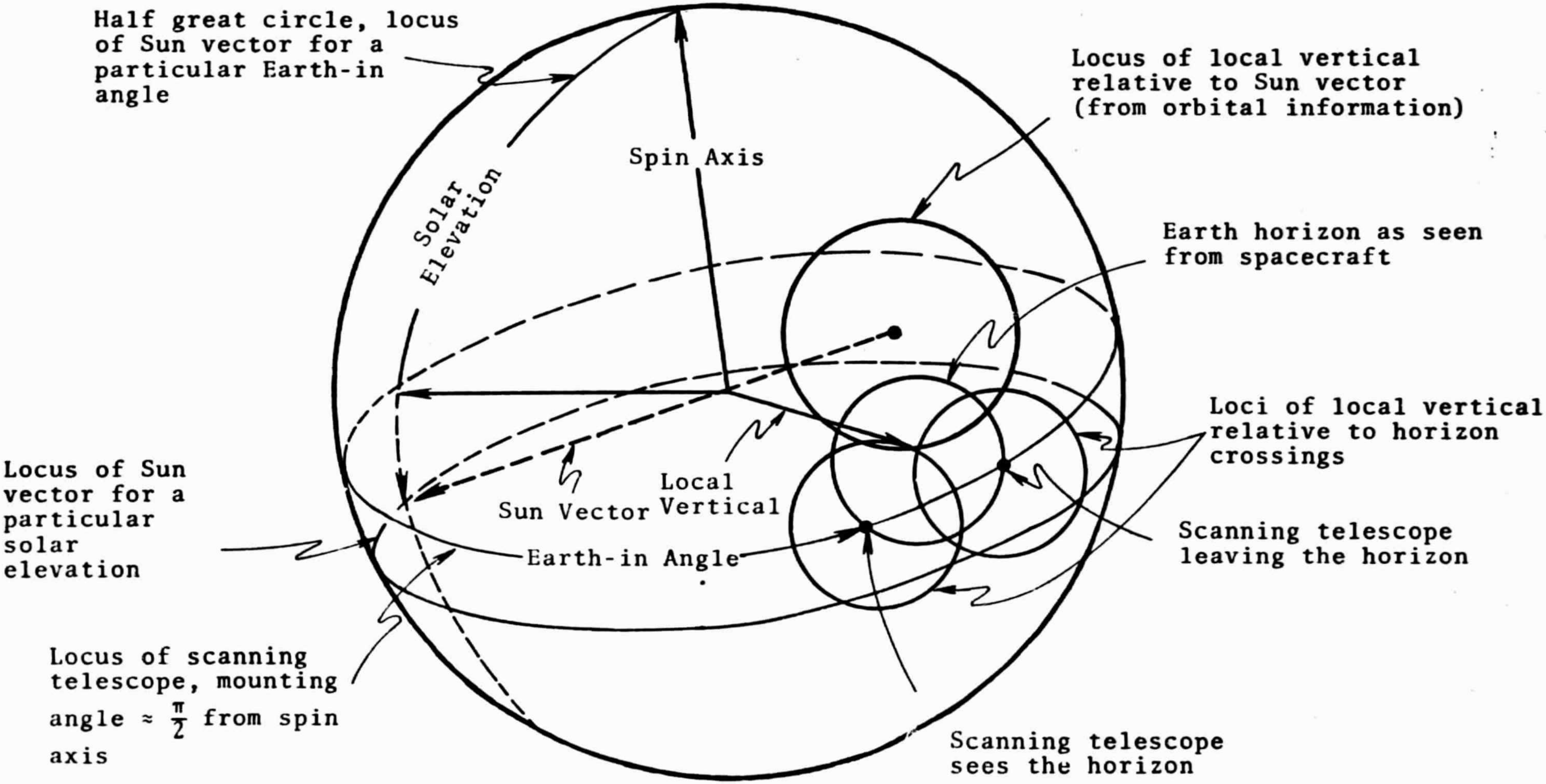


Figure 5. Unit Sphere Representation of IMP-J Measurements as Viewed On-board