ON THE TESSERAL-HARMONICS RESONANCE IN ARTIFICIAL-SATELLITE THEORY

PART II

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(NASA-CR-146854) ON THE TESSERAL-HARMONICS RESONANCE PROBLEM IN ARTIFICIAL-SATELLITE THEORY, PART 2 (Smithsonian Astrophysical Observatory) 32 p HC $4.00 CSCL 22A

N76-21237

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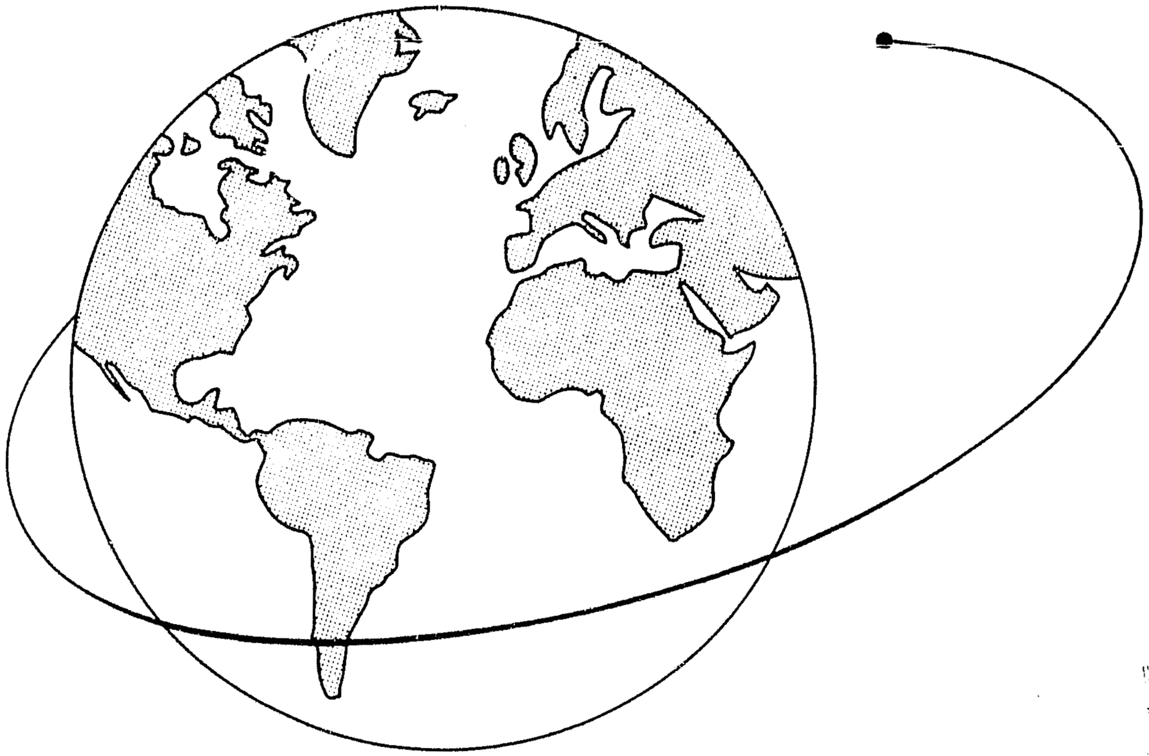

Smithsonian Astrophysical Observatory

SPECIAL REPORT 373

Research in Space Science
SAO Special Report No. 373

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Part II

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March 10, 1976

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ABSTRACT

In Paper I, we derived equations for the perturbations on an artificial satellite when the motion of the satellite is commensurable with that of the earth. This was done by first selecting the tesseral harmonics that contribute the most to the perturbations and then by applying Hori's method by use of Lie series. Here, we introduce some modifications to the perturbations, which now result in better agreement with numerical integration.

ON THE TESSERAL-HARMONICS RESONANCE PROBLEM IN ARTIFICIAL-SATELLITE THEORY*

Part II

Barbara A. Romanowicz†

1. INTRODUCTION

If the gravitational potential of the earth is expanded in terms of Legendre polynomials and functions, then in order to obtain a good approximation in the determination of the orbit of an artificial satellite, it is usually sufficient to consider the zonal, longitude-free terms of the expansion. However, the influence of the tesseral terms becomes important when the mean motion of the satellite and the rate of rotation of the earth around its axis are in a simple ratio; this is called tesseral-harmonics resonance.

In Paper I (Romanowicz, 1975), a theory was derived to compute analytically the perturbations on the motion of an artificial satellite in the case of tesseral-harmonics resonance. In a comparison with a numerical integration, however, some discrepancies appeared in the analytical method. After some modifications were made to the perturbation expressions, which we derive herein, the theory has proved to agree much better with numerical integration.

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*This work was supported in part by Grant NGR 09-015-002 from the National Aeronautics and Space Administration.

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2. NEW EXPRESSIONS OF THE PERTURBATIONS

2.1 Perturbing Function and Equations of Motion

The gravitational potential of the earth at exterior points can be expressed as follows [cf. equations (1) and (2) of Paper I]:

$$U(r,\theta,\lambda) = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\cos\theta) + \sum_{n=2}^{\infty}\sum_{m=1}^{n} J_{n,m}\left(\frac{R}{r}\right)^n P_{n,m}(\cos\theta)\cos m(\lambda-\lambda_{n,m})\right], \tag{1}$$

where (r,θ,λ) are spherical polar coordinates relative to the center of mass of the earth, the axis of rotation being the pole of coordinates; μ is the gravitational constant G times the mass of the earth; R is the mean equatorial radius of the earth; $P_n(z)$ is the n^{th} Legendre polynomial; $P_{n,m}(z)$ are associated Legendre functions:

$$P_{n,m}(z) = (1-z^2)^{m/2}\,\frac{d^n P_n(z)}{dz^n} \quad ;$$

and J_n and $J_{n,m}$ are dimensionless coefficients related to the normalized coefficients $\overline{C_{n,m}}$, $\overline{S_{n,m}}$ by

$$J_{n,m} = \overline{J_{n,m}}\ \sqrt{2(2n+1)\,(n-m)!/(n+m)!}\ , \qquad m \neq 0\ ,$$

$$-J_n = C_{n,0} = \sqrt{2n+1}\ \overline{C_{n,0}}\ , \tag{2}$$

$$\overline{J_{n,m}}\cos m(\lambda-\lambda_{n,m}) = \overline{C_{n,m}}\cos m\lambda + \overline{S_{n,m}}\sin m\lambda\ .$$

In choosing the perturbing function, we take the leading zonal harmonic (i.e., the one containing J_2) and we select those tesseral harmonics that, in the particular resonance being studied, give rise to long-period effects with the largest amplitude.

The general tesseral harmonic can be developed in terms of the osculating elliptic elements $(a, e, I, \Omega, \omega, M)$, referred to the equator of the earth, in the form (Kaula, 1966):

$$V_{n,m} = \left(\frac{\mu}{R}\right)\left(\frac{R}{a}\right)^{n+1} J_{n,m} \left\{ \sum_{p=0}^{p=n} F_{n,m,p}(I) \sum_{q} G_{n,p,q}(e) \cos\left[(n-2p)\omega + (n-2p+q)M \right.\right.$$
$$\left.\left. + m(\Omega - \nu t - \lambda_{n,m})\right] \right\} \quad , \tag{3}$$

where ν is the angular velocity of the rotation of the earth, t is the time, and $F_{n,m,p}(I)$ and $G_{n,p,q}(e)$ are, respectively, the inclination and eccentricity functions as defined by Kaula (1966). The functions $G_{n,p,q}(e)$ are of order $|q|$ in eccentricity.

Resonance occurs when a pair (α, β) of mutually prime integers exists such that the satellite performs β nodal periods while the earth rotates α times relative to the precessing satellite's orbit plane. This can be expressed by

$$\alpha(\dot{\omega} + \dot{M}) = \beta(\nu - \dot{\Omega}) \quad , \tag{4}$$

where $\dot{\omega}$, $\dot{M}$, and $\dot{\Omega}$ are the rates of change with time of ω, M, and Ω, respectively. The corresponding slowly varying arguments are of the form

$$k_1 \Phi_{\alpha,\beta} + \text{const} \quad ,$$

where

$$\Phi_{\alpha,\beta} = \alpha(\omega + M) + \beta(\Omega - \nu t)$$

is called the resonant variable and $k_1 = 1, 2, 3, \ldots$.

Considering that $\dot{\omega}$ is a small quantity and that the general argument in a tesseral harmonic is

$$\phi = (n-2p)\omega + (n-2p+q)M + m(\Omega - \nu t - \lambda_{n,m}) \quad ,$$

we shall keep those tesseral harmonics that contain arguments such that

$$n - 2p + q = k_1\alpha \quad , \qquad m = k_1\beta \quad , \qquad k_1 = 1, 2, 3, \ldots \quad . \tag{5}$$

Since $n \geq m$ and since lower order tesseral harmonics are bound to have larger effects because of the factor $(R/r)^n$, where $R/r < 1$, it can be assumed that it is sufficient to consider only the cases $k_1 = 1, 2,$ and 3. In a general manner, the tesseral harmonics to be kept are

$$V_{k_1\beta+k_0, k_1\beta} \quad ,$$

where $k_1 = 1, 2, 3, \ldots$ and for each k_1, the index k_0 takes values $0, 1, 2, \ldots$. We then write each $V_{n,m}$ in the form

$$V_{n,m} = V'_{n,m} + R_{n,m} \quad ,$$

where $R_{n,m}$, the "resonant" part of the tesseral harmonic, consists of all the terms that contain the resonant arguments, obtained by solving equations (5) for p and q [cf. equations (6), Paper I]. Thus, the resonant part of the tesseral harmonic $V_{n,m}$ is

$$R_{n,m} = \mu \frac{R^n}{a^{n+1}} J_{n,m} \sum_{x=0}^{\beta k_1 + k_0} F_{n,m,p_x}(I)\, G_{n,p_x,q_x}(e) \cos(k_1\Phi_{\alpha,\beta} - q_x\omega - m\lambda_{n,m}) \quad , \tag{6}$$

where

$$q_x = k_1(\alpha - \beta) - k_0 + 2x$$

and

$$p_x = x \quad .$$

The residual part of $V_{n,m}$ is $V'_{n,m}$, whose effects are much smaller than those from $R_{n,m}$.

In Paper I, we then defined $D(k_1)$ and $\theta_1(k_1)$ by

$$R_{n,m} = \sum_{k_1} D(k_1) \cos 2\theta_1(k_1) \quad ,$$

where

$$D(k_1) \cos 2\theta_1(k_1) = \sum_{k_0(k_1)} R_{k_1\beta + k_0, k_1\beta} \quad ,$$

the sum over $k_0(k_1)$ meaning that we have taken into account all the values of k_0 when k_1 has a given value. In order to obtain $D(k_1)$ and $\theta_1(k_1)$, we can write

$$S(k_1, k_0) = \frac{\mu R^n J_{n,m}}{a^{n+1}} \quad , \qquad n = k_1\beta + k_0 \quad , \qquad m = k_1\beta \quad ,$$

$$A(k_1, k_0, x) = F_{n,m,x}(I)\, G_{n,x,q_x}(e) \quad , \tag{7}$$

$$D(k_1) \exp 2i\theta_1(k_1) = \sum_{k_0(k_1)} S(k_1, k_0) \sum_{x=0}^{x=n} A(k_1, k_0, x) \exp i(k_1\Phi_{\alpha,\beta} - q\omega - k_1\beta\lambda_{n,m}) \quad .$$

If we write

$$\Lambda_{k_1,k_0} = k_1\beta\, \lambda_{k_1\beta + k_0, k_1\beta}$$

and

$$2\theta_1(k_1) = k_1 \Phi_{\alpha,\beta} - k_1(\alpha - \beta)\omega + \psi(k_1) \quad ,$$

then (see Appendix A)

$$D(k_1) \exp i\psi(k_1) = \sum_{k_0(k_1)} S(k_1, k_0) \sum_{x=0}^{x=n} A(k_1, k_0, x) \exp i\left[(k_0 - 2x)\omega - \Lambda_{k_1, k_0}\right] \quad . \tag{8}$$

Finally, the potential in which the satellite moves is

$$V = \frac{\mu}{r} - \mu J_2 \frac{R^2}{r^3} P_2(\cos\theta) + \sum R_{n,m} + \sum V'_{n,m} \quad , \tag{9}$$

the sums being taken over all the tesseral harmonics selected.

If we consider the canonical set of Delaunay variables,

$$\begin{aligned} L_D &= \sqrt{\mu a} \quad , & G_D &= \sqrt{\mu a(1 - e^2)} \quad , & H_D &= G_D \cos i \quad , \\ \ell_D &= M \quad , & g_D &= \omega \quad , & h_D &= \Omega \quad , \end{aligned} \tag{10}$$

then the Hamiltonian of the problem is

$$F_D = \frac{\mu^2}{2L_D^2} + F_1 + F_2 \quad , \tag{11}$$

where

$$F_1 = -J_2 \mu \frac{R^2}{r^3} P_2 (\cos\theta) \quad ,$$

$$F_2 = \sum V_{n,m} \quad .$$

To avoid the dependence on time of the Hamiltonian, we perform a canonical transformation, so that the new variables [cf. equations (12) of Paper I] are

$$
\begin{aligned}
L &= L_D , & \ell &= \ell_D , \\
G &= G_D , & g &= g_D , \\
H &= H_D , & h &= h_D - \nu t ,
\end{aligned}
$$

and the new Hamiltonian is

$$F = F_D + \nu H \quad .$$

The equations of motion are then

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{\partial F}{\partial \ell} , & \frac{dG}{dt} &= \frac{\partial F}{\partial g} , & \frac{dH}{dt} &= \frac{\partial F}{\partial h} , \\
\frac{d\ell}{dt} &= -\frac{\partial F}{\partial L} , & \frac{dg}{dt} &= -\frac{\partial F}{\partial G} , & \frac{dh}{dt} &= -\frac{\partial F}{\partial H} ,
\end{aligned}
\tag{12}
$$

with $F = F_0 + F_1 + F_2$:

$$F_0 = \frac{\mu^2}{2L^2} + \nu H \quad ,$$

$$F_1 = -J_2 \mu \frac{R^2}{r^3} \left[\frac{1}{4} \frac{H^2}{G^2} - \frac{3}{4} \left(1 - \frac{H^2}{G^2} \right) \cos 2u \right] , \tag{13}$$

$$F_2 = \sum V_{n,m} \quad ;$$

in the above, $u = g + f$, and f is the true anomaly.

The Hamiltonian is now expanded in terms of powers of the small parameter J_2:

$$F_1 = O(J_2) \quad ,$$

$$F_2 = O(J_2^2) \quad .$$

2.2 First-Order Perturbations

To simplify the writing, we shall now consider only one value of k_1, as defined in Section 2.1, and drop the index k_1 in expressions such as $D(k_1)$ and $\theta_1(k_1)$. The Hamiltonian is then such that

$$F_2 = D \cos 2\theta_1 \quad .$$

We define θ by

$$\theta = \frac{\pi}{2} - \theta_1 \quad .$$

To derive the complete perturbations, we shall sum the individual expressions obtained for each value of k_1.

In the process of removing short-period terms from the Hamiltonian in Paper I, we applied Hori's method by Lie series (Hori, 1966) and considered a canonical transformation

$$\begin{aligned} (L, G, H, \ell, g, h) &\longrightarrow (L', G', H', \ell', g', h') \\ F &\longrightarrow F' \end{aligned}$$

such that

$$L = L' + \frac{\partial S}{\partial \ell'} + \frac{1}{2}\left\{\frac{\partial S}{\partial \ell'}, S\right\} + \frac{1}{3!}\left\{\left\{\frac{\partial S}{\partial \ell'}, S\right\}, S\right\} + \dots \quad ,$$

$$G = G' + \frac{\partial S}{\partial g'} + \frac{1}{2}\left\{\frac{\partial S}{\partial g'}, S\right\} + \dots \quad ,$$

$$
\begin{aligned}
H &= H' + \frac{\partial S}{\partial h'} + \frac{1}{2}\left\{\frac{\partial S}{\partial h'}, S\right\} + \dots \quad , \\
\ell &= \ell' - \frac{\partial S}{\partial L'} - \frac{1}{2}\left\{\frac{\partial S}{\partial L'}, S\right\} - \dots \quad , \\
g &= g' - \frac{\partial S}{\partial G'} - \frac{1}{2}\left\{\frac{\partial S}{\partial G'}, S\right\} - \dots \quad , \\
h &= h' - \frac{\partial S}{\partial H'} - \frac{1}{2}\left\{\frac{\partial S}{\partial H'}, S\right\} - \dots \quad ,
\end{aligned}
\tag{14}
$$

as described by equations (39) in Paper I.

The transformation involves a generating function

$$S = S_{1/2} + SJ + SR \quad ,$$

where

$$SR = Z_1 + Z_2 \quad ,$$

$S_{1/2}$ and Z_1 are the perturbations due only to resonant harmonics, SJ the perturbations due only to J_2 (or to the zonals, if other zonals are also considered), and Z_2 the perturbations due to the interaction of J_2 with the resonant tesserals.

In order to avoid small divisors, S is developed in terms of powers of the square root of the small parameter J_2, as is the custom in resonance problems (Paper I). We reintroduce the characteristic small parameter of the problem:

$$\gamma = \nu\left(\frac{\beta}{\alpha}\right) - \frac{\mu^2}{L'^3} \quad .$$

To find $S_{1/2}$, Z_1, and Z_2, we gather terms of the same order relative to J_2 from (Hori, 1966)

$$F' = F + \{F, S\} + \frac{1}{2}\left\{\{F, S\}, S\right\} + \dots \quad , \tag{15}$$

where F' is the new Hamiltonian, and γ is assumed to be of order 1/2.

In comparing this theory with numerical integration, we found good agreement for the perturbations of highest order due only to the resonant tesserals. The relevant part of the determining function is $S_{1/2}$. In Appendix A, we repeat the derivation of $S_{1/2}$, expressions of the derivatives of this function with respect to the modified Delaunay variables, and definitions of the quantities A, I_2, I_0, k, and Δ.

Some problems developed, however, when the J_2 tesseral-interaction terms were considered. It turns out (cf. Paper I) that the determining function Z_2 cannot be defined by

$$\gamma\Delta\frac{\partial Z_2}{\partial \ell'} = -\left(b_1\frac{\partial S_{1/2}}{\partial \ell'} + b_2\frac{\partial S_{1/2}}{\partial g'} + b_3\frac{\partial S_{1/2}}{\partial h'}\right) \quad ,$$

where

$$b_1 = \frac{\partial F_1}{\partial L'} \quad , \qquad b_2 = \frac{\partial F_1}{\partial G'} \quad , \qquad b_3 = \frac{\partial F_1}{\partial H'} \quad ,$$

since here we have neglected the contribution of the secular rate to the variable g.

To the highest order, then, let us consider instead the following equation as a definition of Z_2:

$$\{F_0 + F_1, Z_2\} + \{F_1, S_{1/2}\} = 0 \quad . \tag{16}$$

We assume that Z_2 is a function only of L′, G′, H′, and θ, where we recall

$$2\theta = \alpha\ell' + \beta(g'+h') + \psi(g') - \pi \quad .$$

Then we obtain

$$\frac{\partial Z_2}{\partial \ell'} = \frac{a}{2} \frac{\partial Z_2}{\partial \theta} \quad ,$$

$$\frac{\partial Z_2}{\partial g'} = \frac{d}{2} \frac{\partial Z_2}{\partial \theta} \quad , \tag{17}$$

$$\frac{\partial Z_2}{\partial h'} = \frac{\beta}{2} \frac{\partial Z_2}{\partial \theta} \quad ,$$

with

$$d = \beta + \frac{\partial \psi}{\partial g'} \quad ,$$

which yields

$$\frac{a\gamma + \sigma}{2} \frac{\partial Z_2}{\partial \theta} = -\{F_1, S_{1/2}\} = -\sigma c_2 \gamma (\Delta - 1) - b_2 a_4 c_2 \gamma (I_2 - I_0) \quad , \tag{18}$$

where

$$\sigma = b_1 a + b_2 d + b_3 \beta$$

and

$$a_4 = \frac{1}{D} \frac{\partial D}{\partial g}$$

[cf. equation (44) of Paper I]. Hence,

$$Z_2 = B(I_2 - \theta) \quad , \tag{19}$$

where

$$B = -\frac{2\sigma c_2 \gamma}{a\gamma + \sigma} \quad .$$

The derivatives of Z_2 with respect to the modified Delaunay variables are as follows. If X is a modified Delaunay variable, then

$$\frac{\partial Z_2}{\partial X} = \frac{\partial B}{\partial X}(I_2 - \theta) + B\left(\frac{\partial I_2}{\partial X} - \frac{\partial \theta}{\partial X}\right) , \tag{20}$$

where $\partial I_2/\partial X$ and $\partial \theta/\partial X$ are given in Appendix A, and

$$\frac{\partial B}{\partial X} = -\frac{2c_2\gamma}{\alpha\gamma+\sigma}\frac{\partial \sigma}{\partial X}\left(1 - \frac{\sigma}{\alpha\gamma+\sigma}\right) + \varepsilon\left(-\frac{2\sigma}{\alpha\gamma+\sigma}\right)\left(4c_1\gamma + \frac{1}{\alpha} - \frac{\gamma}{\alpha\gamma+\sigma}\right) ,$$

in which

$$\varepsilon = 1 \quad \text{if} \quad X = L' ,$$

$$\varepsilon = 0 \quad \text{if} \quad X \neq L' ,$$

and

$$\frac{\partial \sigma}{\partial X} = \alpha\frac{\partial b_1}{\partial X} + d\frac{\partial b_2}{\partial X} + \beta\frac{\partial b_3}{\partial X} + b_2\frac{\partial^2 \psi}{\partial g\,\partial X} ,$$

The relevant expressions are as in equations (44) and (65b) and Appendix A of Paper I; they are repeated in Appendix B herein.

2.3 Second-Order Perturbations

Instead of considering the second-order perturbations due to the resonant tesserals separately from those due to the interaction terms, let us define a determining function Z_3 by

$$\{F_0+F_1, Z_3\} + \frac{1}{2}\left\{\{F_0+F_1, S_{1/2}+Z_2\},\ S_{1/2}+Z_2\right\} = 0 . \tag{21}$$

This yields, assuming $Z_3 = Z_3(L', G', H', \theta)$,

$$\frac{\alpha\gamma+\sigma}{2}\,\frac{\partial Z_3}{\partial\theta} = -\frac{1}{2}\left\{\{F_0+F_1, S_{1/2}+Z_2\}, S_{1/2}+Z_2\right\} \quad , \tag{22}$$

which, after some algebra, gives

$$\frac{\partial Z_3}{\partial\theta} = R_1\left(\Delta-\frac{1}{\Delta}\right)(\Delta-1) + R_2\,(I_2-I_0)\frac{k^2\sin\theta\cos\theta}{\Delta} + R_3\,(I_2-\theta)\frac{k^2\sin\theta\cos\theta}{\Delta}$$

$$+ R_4\,(\Delta-1)^2 \quad , \tag{23}$$

where

$$R_1 = R_2 = -\left(\alpha c_2\gamma^2\right)^2 x \quad ,$$

$$R_3 = -\alpha c_2\gamma^2\left(\frac{B_p}{2}+4\alpha c_1\gamma+1\right) \quad , \tag{24}$$

$$R_4 = -\frac{\alpha^2 c_2\gamma^3}{\alpha\gamma+\sigma}\,(3\alpha\gamma+2\sigma) \quad .$$

In equations (24), terms of higher order were neglected and the following were used:

$$x = \frac{\alpha a_1 + d a_2 + \beta a_3 - p_2 a_4}{2} \quad ,$$

$$c_1 = \frac{L^3}{3\mu^2\alpha} \quad , \qquad c_2 = \frac{L^4}{3\mu^2\alpha} \quad ,$$

$$a_1 = \frac{1}{D}\frac{\partial D}{\partial L'} \quad , \qquad a_2 = \frac{1}{D}\frac{\partial D}{\partial G'} \quad , \qquad a_3 = \frac{1}{D}\frac{\partial D}{\partial H'} \quad ,$$

$$B_p = \alpha\frac{\partial B}{\partial L'} + d\frac{\partial B}{\partial G'} + \beta\frac{\partial B}{\partial H'} \quad .$$

After some algebra, then, we obtain

$$Z_3 = P_1 \left[\left(1 - \frac{k^2}{2}\right) \theta + \frac{k^2}{4} \sin 2\theta\right] - P_2 I_2 + P_3 \theta + P_4 I_0 \quad , \tag{25}$$

where

$$P_1 = 2R_1 + R_4 + R_3 \quad ,$$

$$P_2 = R_1 + 2R_4 + R_3 + \Delta(R_1 + R_3) \quad ,$$

$$P_3 = R_4 - 2R_1 + \Delta R_3 \quad ,$$

$$P_4 = R_1(1 + \Delta) \quad .$$

The derivatives of Z_3 with respect to the modified Delaunay variables are as follows:

$$\frac{\partial Z_3}{\partial X} = \frac{\partial P_1}{\partial X} \left[\left(1 - \frac{k^2}{2}\right)\theta + \frac{k^2}{4} \sin 2\theta\right] - \frac{\partial P_2}{\partial X} I_2 + \frac{\partial P_3}{\partial X} \theta + \frac{\partial P_4}{\partial X} I_0$$

$$- P_2 \frac{\partial I_2}{\partial X} + P_3 \frac{\partial \theta}{\partial X} + P_4 \frac{\partial I_0}{\partial X} + P_1 Q \quad , \tag{26}$$

where

$$Q = \frac{\partial \theta}{\partial X} \left(1 - \frac{k^2}{2} + \frac{k^2}{2} \cos 2\theta\right) - \frac{k^2}{2} a_x \left(\theta - \frac{\sin 2\theta}{2}\right) \quad .$$

In equation (26), the derivatives $\partial I_2/\partial X$, $\partial \theta/\partial X$, and $\partial I_0/\partial X$ are given in Appendix A. Now we get

$$\frac{\partial R_1}{\partial X} = - (\alpha c_2 \gamma^2)^2 \frac{\partial x}{\partial X} - \varepsilon x (4\alpha c_2 \gamma^3) (1 + 4\alpha c_1 \gamma) \quad ,$$

$$\frac{\partial R_3}{\partial X} = \varepsilon \left[-2\gamma (2\alpha c_1 \gamma + 1) \left(\frac{B_p}{2} + 4\alpha c_1 \gamma + 1\right) - 4\alpha c_2 \gamma^2 \left(\frac{1}{L'} + \frac{\gamma L'}{\mu^2}\right)\right] \quad ,$$

$$\frac{\partial R_4}{\partial X} = \frac{a^3 c_2 \gamma^4}{(a\gamma+\sigma)^2}\frac{\partial\sigma}{\partial X} + \varepsilon\left|\left(-\frac{a^2\gamma^2}{a\gamma+\sigma}\right)\left(12ac_1\gamma^2 + 9\gamma + 8c_1\gamma\sigma + \frac{6\sigma}{a} + \frac{\sigma\gamma}{a\gamma+\sigma}\right)\right| \quad ,$$

where

$$\varepsilon = 1 \quad \text{if} \quad X = L' \quad ,$$

$$\varepsilon = 0 \quad \text{if} \quad X \neq L' \quad .$$

In conclusion, the determining function for the canonical transformation used to remove short-period terms from the Hamiltonian is

$$S = S_{1/2} + Z_2 + Z_3 \quad ,$$

and the new Hamiltonian is

$$F' = F'_0 + F'_1 + F'_2 \quad , \tag{27}$$

where

$$F'_0(L', H') = F_0(L', H') \quad ,$$

$$F'_1(L', G', H') = F_1(L', G', H') \quad ,$$

$$F'_2(L', G', H', g') = -\sum_{k_1} D(k_1)\ (L', G', H', g') \quad .$$

The new equations of motion are then

$$\begin{aligned}
\frac{dL'}{dt'} &= \frac{\partial F'}{\partial \ell'} = 0 \quad , & \frac{d\ell'}{dt'} &= -\frac{\partial F'}{\partial L'} \quad , \\
\frac{dG'}{dt'} &= \frac{\partial F'}{\partial g'} \quad , & \frac{dg'}{dt'} &= -\frac{\partial F'}{\partial G'} \quad , \\
\frac{dH'}{dt'} &= \frac{\partial F'}{\partial h'} = 0 \quad , & \frac{dh'}{dt'} &= -\frac{\partial F'}{\partial H'} \quad ,
\end{aligned} \tag{28}$$

and the long-period variable g' remains to be removed, following the procedure given in Section 5 of Paper I.

3. COMPARISON OF THE THEORY WITH A NUMERICAL INTEGRATION

The theoretical computations have been compared with a numerical integration provided by the routine DVDQ. The satellite considered was Geos 1, which passes through the $\alpha = 1$, $\beta = 12$ resonance with the following orbital elements:

semimajor axis	$a_0 = 8.074 \times 10^6$ m ,
eccentricity	$e_0 = 0.073$,
inclination	$I_0 = 59°$.

We take the constants of the problem to be

$$
\begin{aligned}
L_0'' &= \sqrt{\mu a_0} \quad , \\
G_0'' &= L_0'' (1 - e_0^2) \quad , \\
H_0'' &= G_0'' \cos I_0 \quad , \\
\ell_0'' &= 0 \quad , \\
g_0'' &= 0 \quad , \\
h_0'' &= 0 \quad .
\end{aligned}
\tag{29}
$$

In this case, k^2 is of the order of 10^{-4}, so the motion of the satellite is of the circulation type. The period of the short-term perturbations is approximately 7 days.

After removing the short- and long-period perturbations, we obtain errors in the Delaunay variables L, G, and H for a period of 8 days, with computations performed every 0.5 day. The results are shown in Table 1.

Table 1. Errors in the action variables as computed over a period of 8 days.

Variable	Amplitude of perturbation $\times 2^*$	Error*	Relative error (%)
L	48×10^{-5}	9.75×10^{-7}	0.2
G	67×10^{-5}	8.86×10^{-5}	13
H	57×10^{-3}	1.14×10^{-5}	0.2

*The semimajor axis is expressed in units of 10^6 m.

We obtain the following errors in the Delaunay angular variables:

ℓ = 7.79×10^{-7} rad in 8 days, corresponding to ~ 0.8 m per day in the ascending node of the orbit ($a = 8.074 \times 10^6$ m),

g = 7.05×10^{-7} rad in 8 days, corresponding to 0.75 m per day in the argument of perigee ($a = 8.074 \times 10^6$ m),

h = 8.7×10^{-9} rad in 8 days, corresponding to 0.002 m per day in the satellite's position in the orbit ($a = 8.074 \times 10^6$ m).

Moreover, the errors in ℓ and g have opposite signs, so that the error in ℓ + g is of the order of 8×10^{-8} rad in 8 days, or 10^{-8} rad per day (see Figure 1).

We conclude that there is satisfactory agreement in all variables except G. Adding or removing the contributions of second-order perturbations (the generating function Z_3) does not affect the results. We suggest a tentative explanation for the large error in G. In the process of removing the variables ℓ and h from the Hamiltonian, we have written

$$G = G' + \frac{\partial S}{\partial g'} \quad ,$$

where

$$S = S_{1/2} + Z_2 + Z_3 \quad .$$

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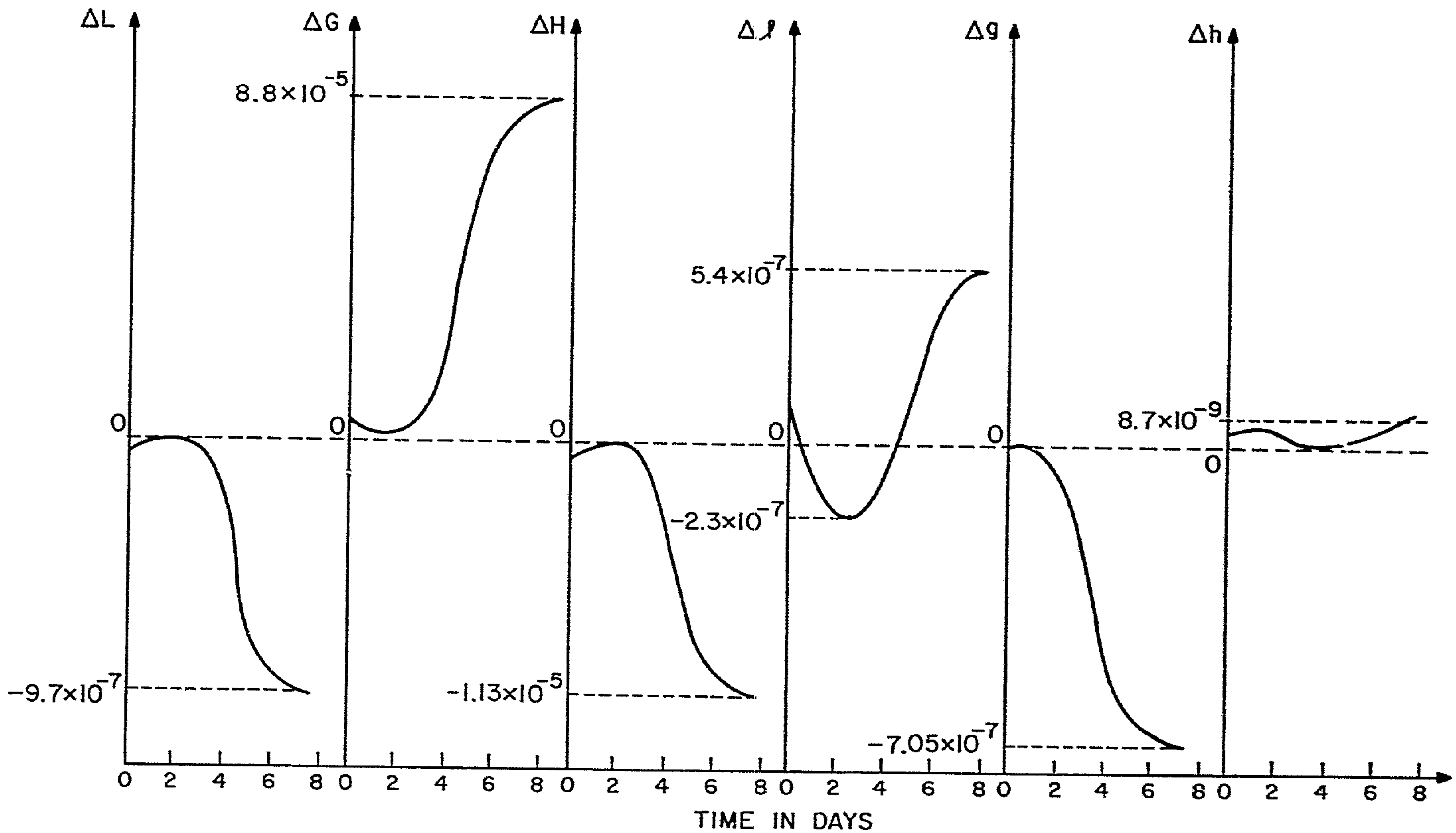

Figure 1. Variations of the errors over one oscillation.

We took Z_2 from equation (4), where we assumed that Z_2 depends on g' only through the resonant variable θ. However, after obtaining Z_2 and its derivatives, we get [see equation (20)]

$$\frac{\partial Z_2}{\partial g'} = \frac{\partial B}{\partial g'} (I_2 - \theta) + B \left[\frac{d}{2} (\Delta - 1) + \frac{a_4}{2} (I_2 - I_0) \right] \quad , \tag{30}$$

where $d = \beta + \psi(g')$ and $a_4 = (1/k^2)\,(\partial k^2/\partial g')$ are functions of L', G', H', and g', and both have significant derivatives with respect to g'. Hence, there is an inaccuracy in the dependence of Z_2 on g'. The correct equation for Z_2 is

$$\left(\frac{\alpha\gamma + \sigma}{2}\right) \frac{\partial Z_2}{\partial \theta} + \frac{\partial F_1}{\partial G'} \frac{\partial Z_2}{\partial g'} = - \{F_1, S_{1/2}\} \quad , \tag{31}$$

where

$$Z_2 = Z_2(L, G, H, \theta, g)$$

and

$$\frac{dZ_2}{dg'} = \frac{\partial Z_2}{\partial \theta} \frac{\partial \theta}{\partial g'} + \frac{\partial Z_2}{\partial g'} \quad .$$

Another factor in support of this explanation is that, when J_2 is set equal to zero (and, hence, $\partial F_1/\partial G' = 0$), the error in the variable G is of the order of 0.2%, as it is for the variables L and H.

It seems that, to obtain better agreement in the variable G, we must take into account the intricate dependence of Z_2 on g' and solve equation (31) more accurately.

4. ACKNOWLEDGMENT

I would like to thank Drs. K. Aksnes and E. M. Gaposchkin for the encouragement they gave me for this work.

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APPENDIX A

EXPRESSIONS OF $S_{1/2}$ AND ITS DERIVATIVES WITH RESPECT TO THE MODIFIED DELAUNAY VARIABLES

The function $S_{1/2}$ is obtained by solving the resonance equation

$$\{F_0, S_{1/2}\} + \frac{1}{2}\frac{\partial \gamma}{\partial L'}\left(\frac{\partial S_{1/2}}{\partial \ell'}\right)^2 + 2D \sin^2 \theta = 0 \quad , \qquad \text{(A-1)}$$

where

$$\gamma = \frac{\beta}{\alpha}\, \nu - \frac{\mu^2}{L'^3}$$

and

$$\{F_0, S_{1/2}\} = \gamma \frac{\partial S_{1/2}}{\partial \ell'} \quad .$$

If

$$A = \frac{2\gamma L'^4}{3\mu^2 \alpha k_1} \quad ,$$

$$k^2 = \frac{12 D \mu^2}{L'^4 \gamma^2} \quad ,$$

and

$$\Delta = \sqrt{1 - k^2 \sin^2 \theta} \quad ,$$

then, solving equation (A-1) and applying some continuity arguments yield

$$\frac{\partial S_{1/2}}{\partial \theta} = A(\Delta - 1)$$

and

$$S_{1/2} = A(I_2 - \theta) \quad ,$$

where we define

$$I_2 = E(\theta, k) \quad .$$

Let us also define

$$I_0 = F(\theta, k) \quad .$$

Here $F(\theta, k)$ and $E(\theta, k)$ are the elliptic integrals of the first and second kind, respectively. The derivatives of $S_{1/2}$ are then

$$\frac{\partial S_{1/2}}{\partial \ell'} = \alpha c_2 \gamma \, (\Delta - 1) \quad ,$$

$$\frac{\partial S_{1/2}}{\partial g'} = d c_2 \gamma \, (\Delta - 1) + c_2 \gamma a_4 \, (I_2 - I_0) \quad ,$$

$$\frac{\partial S_{1/2}}{\partial h'} = \beta c_2 \gamma \, (\Delta - 1) \quad ,$$

$$\frac{\partial S_{1/2}}{\partial L'} = \frac{2}{\alpha} \, (I_0 - \theta) + 8 c_1 \gamma \, (I_2 - \theta) + c_2 p_1 \gamma (\Delta - 1) + (c_2 \gamma a_1 - 4 c_1 \gamma) \, (I_2 - I_0) \quad , \tag{A-2}$$

$$\frac{\partial S_{1/2}}{\partial G'} = c_2 \gamma p_2 (\Delta - 1) + c_2 \gamma a_2 (I_2 - I_0) \quad ,$$

$$\frac{\partial S_{1/2}}{\partial H'} = \gamma c_2 p_3 (\Delta - 1) + c_2 \gamma a_3 (I_2 - I_0) \quad ,$$

where c_1, c_2, a_1, a_2, a_3 are as given in equations (24),

$$a_4 = \frac{1}{D}\frac{\partial D}{\partial g'} \quad ,$$

$$p_1 = \frac{\partial\psi}{\partial L'} \quad , \qquad p_2 = \frac{\partial\psi}{\partial G'} \quad , \qquad p_3 = \frac{\partial\psi}{\partial H'} \quad , \qquad p_4 = \frac{\partial\psi}{\partial g'} \quad .$$

We also have

$$\begin{aligned}
\frac{\partial\theta}{\partial L'} &= \frac{p_1}{2} \quad , & \frac{\partial\theta}{\partial \ell'} &= \frac{\alpha}{2} \quad , \\
\frac{\partial\theta}{\partial G'} &= \frac{p_2}{2} \quad , & \frac{\partial\theta}{\partial g'} &= \frac{p_4}{2} \quad , \\
\frac{\partial\theta}{\partial H'} &= \frac{p_3}{2} \quad , & \frac{\partial\theta}{\partial h'} &= \frac{\beta}{2} \quad ;
\end{aligned} \tag{A-3}$$

and, if X is a modified Delaunay variable,

$$\begin{aligned}
\frac{\partial\Delta}{\partial X} &= \frac{a_x}{2}\left(\Delta - \frac{1}{\Delta}\right) - \frac{k^2 \sin\theta\cos\theta}{\Delta}\,\frac{\partial\theta}{\partial X} \quad , \\
\frac{\partial I_2}{\partial X} &= \frac{\partial\theta}{\partial X}\,\Delta + \frac{a_x}{2}\,(I_2 - I_0) \quad , \\
\frac{\partial I_0}{\partial X} &= \frac{1}{\Delta}\,\frac{\partial\theta}{\partial X} + \frac{a_x}{2}\,(I_{-2} - I_0) \quad ,
\end{aligned} \tag{A-4}$$

where

$$I_{-2}(\theta) = \int_0^\theta \frac{dx}{\Delta^3} = \frac{1}{1-k^2}\left(I_2 - \frac{k^2 \sin\theta\cos\theta}{\Delta}\right) \quad .$$

In the above,

$$a_X = \begin{cases} 0 & \text{if} \quad X = \ell' \text{ or } h' \\ a_1 - \dfrac{4}{L'} - \dfrac{2}{\alpha c_2 \gamma} & \text{if} \quad X = L' \\ a_2 & \text{if} \quad X = G' \\ a_3 & \text{if} \quad X = H' \\ a_4 & \text{if} \quad X = g' \end{cases} .$$

APPENDIX B

DERIVATIVES OF b_1, b_2, b_3, AND $\psi(g')$, AND SECOND DERIVATIVES OF $\psi(g')$

We recall the definitions of D and ψ, introducing the quantities B and C:

$$D^2 = B^2 + C^2 \quad ,$$

$$D \cos \psi = B \quad ,$$

$$D \sin \psi = C \quad .$$

Then, if X is any of the variables L′, G′, H′, or g′, we get

$$D \frac{\partial D}{\partial X} = B \frac{\partial B}{\partial X} + C \frac{\partial C}{\partial X} \quad . \tag{B-1}$$

If $X \neq g'$, we have

$$\frac{\partial B}{\partial X} = \sum_{k_0} \sum_{x=0}^{k_1\beta+k_0} \frac{\partial}{\partial X} \left[S(k_1, k_0)\, A(k_1, k_0, x)\right] \cos \left[(k_0 - 2x)g - \Lambda_{k_1, k_0}\right] \quad ,$$

$$\frac{\partial C}{\partial X} = \sum_{k_0} \sum_{x=0}^{k_1\beta+k_0} \frac{\partial}{\partial X} \left[S(k_1, k_0)\, A(k_1, k_0, x)\right] \sin \left[(k_0 - 2x)g - \Lambda_{k_1, k_0}\right] \quad ,$$

(B-2)

$$\frac{\partial A(k_1, k_0, x)}{\partial X} = \frac{\partial F_{n, m, p}(I)}{\partial I} \frac{\partial I}{\partial X} G(e) + F_{n, m, p}(I) \frac{\partial G_{n, p, q}(e)}{\partial e} \frac{\partial e}{\partial X} \quad ,$$

$$\frac{\partial S(k_1 k_0)}{\partial X} = \begin{cases} 0 & \text{if} \quad X \neq L \\ -\dfrac{2\ (\beta k_1 + k_0 + 1)}{L}\, S(k_1, k_0) & \text{if} \quad X = L \quad ; \end{cases}$$

and if $X = g'$, we get

$$\frac{\partial B}{\partial g'} = -\sum_{k_0} \sum_{x=0}^{\beta k_1 + k_0} S(k_1, k_0)\, A(k_1, k_0, x)\, (k_0 - 2x) \sin\left[(k_0 - 2x)g' - \Lambda_{k_1, k_0}\right] ,$$

(B-3)

$$\frac{\partial C}{\partial g'} = \sum_{k_0} \sum_{x=0}^{\beta k_1 + k_0} S(k_1, k_0)\, A(k_1, k_0, x)\, (k_0 - 2x) \cos\left[(k_0 - 2x)g' - \Lambda_{k_1, k_0}\right] .$$

For the derivatives of ψ, we have

$$C = D \sin \psi ,$$

so that

$$\frac{\partial C}{\partial X} = D \frac{\partial \psi}{\partial X} \cos \psi + \frac{\partial D}{\partial X} \sin \psi \qquad \text{(B-4a)}$$

and

$$\frac{\partial \psi}{\partial X} = \frac{1}{B}\left(\frac{\partial C}{\partial X} - \frac{1}{D}\frac{\partial D}{\partial X} C\right) . \qquad \text{(B-4b)}$$

We also have

$$\frac{\partial b_1}{\partial X} = \frac{\partial b_2}{\partial X} = \frac{\partial b_3}{\partial X} = 0 \quad \text{if} \quad X = \ell' \text{ or } h' \text{ or } g' ,$$

$$\frac{\partial b_1}{\partial L'} = -\frac{4b_1}{L'} ,$$

$$\frac{\partial b_2}{\partial L'} = -\frac{3b_2}{L'} ,$$

$$\frac{\partial b_3}{\partial L'} = -\frac{3b_3}{L'} ,$$

(B-5)

$$\frac{\partial b_1}{\partial G'} = -\frac{3b_1}{G'} + \frac{9}{2}\frac{J_2 \mu^4 R^2 H'^2}{L'^4 G'^6} ,$$

$$\frac{\partial b_2}{\partial G'} = -\frac{4b_1}{G'} + \frac{15}{2}\frac{J_2 \mu^4 R^2 H'^2}{L'^3 G'^7} \quad ,$$

$$\frac{\partial b_3}{\partial G'} = -\frac{5b_3}{G'} \quad ,$$

$$\frac{\partial b_1}{\partial H'} = -\frac{9}{2}\frac{J_2 \mu^4 R^2 H'}{L'^4 G'^5} \quad ,$$

$$\frac{\partial b_2}{\partial H'} = -\frac{15}{2}\frac{J_2 \mu^4 R^2 H'}{L'^3 G'^6} \quad ,$$

$$\frac{\partial b_3}{\partial H'} = \frac{3}{2}\frac{J_2 \mu^4 R^2}{L'^3 G'^5} \quad .$$