STABILITY AND RELATIVE STABILITY OF LINEAR SYSTEMS WITH MANY CONSTANT TIME DELAYS

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ABSTRACT

A method of determining the stability of linear systems with many constant time delays is developed. This technique, an extension of the τ-decomposition method, is used to examine not only the stability but also the relative stability of retarded systems with many delays and of a class of neutral equations with one delay.

Analytical equations are derived for partitioning the delay space of a retarded system with two time delays. The stability of the system in each of the regions defined by the partitioning curves in the parameter plane is determined using the extended τ-decomposition method. In addition, relative stability boundaries are defined using the extended τ-decomposition method in association with parameter plane techniques.

Several applications of the extended τ-decomposition method are presented and compared with stability results obtained from other analyses. In all cases the results obtained using the method outlined herein coincide with and extends those of previous investigations. The extended τ-decomposition method applied to systems with time delays requires less computational effort and yields more complete stability analyses than previous techniques.

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SYMBOLS

A, A_ℓ	N X N matrices of real constants
a,b,c,d a_j,b_j,c_j,d_j $a_{11},a_{12},a_{21},a_{22}$ $b_{11},b_{12},b_{21},b_{22}$	real constants
b	wing span in equation (4.25), ft
c_{Knj},b_{Knj},d_{Knj}	real constants (see equations (1.4)and (1.5))
$C_{n_{\delta_r}}$	change in yawing moment coefficient with rudder deflection, per radian
e_2,e_1,e_0	constants defined by equations (4.25), (4.26), and (4.27), respectively
$G(s)$	modified characteristic quasi-polynomial in equation (1.23)
$g(z)$	function of z in equation (1.25)
$H_K(s)$	function of s in equation (1.3)
$h(z)$	function of z in equation (1.25)
I	N X N identity matrix
I_z	moment of inertia about principal axis, slug-ft^2
i	imaginary unit, $\sqrt{-1}$
$J_K(s)$	function of s in equation (1.3)
j,ℓ,n	integers
K	refers to θ_K
K_j	integer associated with θ_j
$L(s),L(s, \theta_j)$	characteristic quasi-polynomial

$L_0(s)$, $L(s, 0)$	resulting polynomial when delays are made zero in $L(s)$
M	number of θ_j delays in system
N	dimension of system
$N(\tau)$	number of roots of $W(z)$ with positive real parts
$N(\theta_K, \bar{\sigma})$	number of roots of $L(s)$ with $\sigma > \bar{\sigma}$ at θ_K for fixed θ_j, $j \neq K$
$P(s), Q(s), R(s)$	polynomials in s with real constant coefficients
Q	dynamic pressure in equation (4.25), lb/ft^2
p	integer
$\hat{p}$	real number defined in equation (3.23)
q, r	angles defined in figure 2
S	wing area, ft^2
s	complex variable, $\sigma + i\omega$
$\lvert s \rvert_m$	an upper bound on magnitude of s which satisfies $L(s) = 0$, where $s = \bar{\sigma} + i\omega$
s^*	root of $L(s)$
T	number of time delays τ_ℓ in system
t	time, sec
$\tilde{t}$	nondimensional time
u, v	real and imaginary parts, respectively, of $H_K(s)/J_K(s)$
V_j	integer associated with θ_j
$W(z), W_1(z), W_2(z)$	functions of z defined in equations (1.26), (1.27), and (1.28)
W_K, $W_K(\bar{\sigma}, \omega)$	testing function defined in equation (1.24)
$x(t)$	scalar function of time
$x_1(\tilde{t})$	scalar function of nondimensional time

$\vec{x}(t)$	N X 1 vector function of time
y_1, y_2	defined by equations (2.13) and (2.14), respectively
z	complex variable
α	argument of $Q(i\omega)\, e^{-i\omega\theta_1}$ in figure 2
α_1, α_2	real numbers in theorem 5
β	argument of $P(i\omega)$ in figure 2
β_p	used to denote values of θ_K at intersection points
γ	argument of $R(i\omega)\, e^{-i\omega\theta_2}$ in figure 2
γ_p	used to denote values of θ_K at intersection points
δ	small positive number
δ_r	rudder deflection, radians
ϵ	small positive number; also denotes "is an element of"
ζ	damping parameter
η	gain in equation (4.1)
θ_j	constant real delays
θ_K	particular delay being varied
$\tilde{\theta}_j$	fixed value of θ_j
$\bar{\theta}_j$	final desired value of θ_j or maximum value of θ_j to be considered
λ	radius of circular contour containing all N roots of $L_0(s)$
μ_1, μ_2	slopes of lines in figure 5
ξ	real gain constant
ρ	radius of small circular contour around any root of L(s)
σ	real part of complex variable s

$\bar{\sigma}$	specified value of real part of complex variabl s
σ_K	real part of large rodulus roots of L(s)(see equation (1.41))
$\tau, \tau_1, \tau_2, \tau_\ell$	constant real time delays
τ^*	particular value of τ at an intersection point of root locus curve with imaginary axis
φ	identification of an intersection point in theorem 4
$\Omega_0, \Omega_{1K}, \Omega_{2K}, \Omega_{3K}, \Omega_{4K}$	sets defined in equation (1.8)
$\psi(t)$	yaw angle, radians
ω	imaginary part of complex variable s
ω_m	an upper bound on ω in $L(s) = 0$, where $s = \bar{\sigma} + i\omega$
ω_n	natural frequency or constant parameter
ω^*	particular value of ω at an intersection point of a root-locus curve with the imaginary axis

Mathematical notations:

$\| \ \|$	absolute value or magnitude
$[\alpha_1, \omega_m]$	closed interval $\alpha_1 \leqq \omega \leqq \omega_m$
$(0, \omega_m]$	semiclosed interval $0 < \omega \leqq \omega_m$
arg	argument
ϵ	is an element of
det	determinant
$x^{(j)}(t)$	jth derivative of x(t) with respect to t
$\{s : B\}$	set of all s having property B
Δ	small incremental value
$\bigcup$	union of sets
0^+	arbitrarily small positive values

Dots over a symbol denote derivatives with respect to time.

INTRODUCTION

Time delays in the mathematical description of a physical system occur whenever the system is affected not only by conditions at the present time, but also by conditions which have occurred in the past. In general, every physical system contains a certain amount of time delay in its operation or functioning. If the transmission time of an action cannot be neglected, then the mathematical description of the system must include this time delay. This leads to so-called differential-difference equations (reference 1).

Early forms of differential-difference equations appeared in 1750 in the works of Euler (reference 2) and in 1759 in the works of Lagrange (reference 3). By 1894, it was recognized that delayed actions were one of the primary causes of the hunting of governed engines (reference 2). Systems with a time delay subsequently appeared in various research areas, such as: automatic control, theory of epidemics, combustion of rocket engines, and simulation of pilot tracking tasks. (See references 1 to 14, for example.)

The differential-difference equations considered in the present analysis are linear and time-invariant; that is, the coefficients and time delays are constants. There are three basic types of differential-difference equations; namely: advanced, retarded, and neutral (references 1 and 15). The advanced systems are always unstable (reference 16); whereas the retarded and neutral equations may be either stable or unstable. The stability analysis, however, is complicated by the fact that the associated characteristic equation has an infinite number of roots. The most effective method at the present time for examining the stability of a retarded system with one time delay appears to be the τ-decomposition method (reference 17). The research of this thesis is to extend the τ-decomposition method to examine the stability, and also the relative stability (reference 18) of retarded systems with many time delays and also to a special class of scalar neutral equations with one delay.

CHAPTER I

EXTENDED τ-DECOMPOSITION STABILITY METHOD

The Mikhailov criteria (reference 19), which is similar to the Nyquist criteria (reference 20) is used to examine the stability of linear dynamical systems described by linear ordinary differential equations with constant coefficients. A. A. Sokolov (reference 21) showed the possibility of extending the Mikhailov criteria to linear systems with a transport time delay. Later, as mentioned by Popov (reference 19), Miasnikov worked out in detail the application of the extended Mikhailov criteria; and following Miasnikov, Popov considers some individual cases. Krall (reference 22) also develops some useful theorems related to the criteria.

It was subsequently recognized by Kashiwagi and Flügge-Lotz (reference 23) and Lee and Hsu (reference 24) that the extended Mikhailov criteria, as discussed by Popov, had limited validity. However, by extending the work of Krall and modifying the earlier work of Sokolov and Miasnikov, Lee and Hsu (reference 24) were able to rigorously refine the criteria and proceeded to call it the τ-decomposition method of stability analysis for retarded dynamic systems.

The τ-decomposition method, a powerful and convenient tool, has been applied previously to retarded systems with only one constant time delay (references 14 and 17). The extension of this method to examine the stability and relative stability of systems with many constant delays is presented in this chapter.

A. Retarded Systems

A linear homogeneous differential-difference equation of the retarded type with constant coefficients and delays can be expressed as

$$\dot{\vec{x}}(t) = A\vec{x}(t) + \sum_{\ell=1}^{T} A_{\ell}\, \vec{x}(t - \tau_{\ell}) \qquad (1.1)$$

where $\vec{x}(t)$ is an N X 1 vector, A and A_{ℓ} are N X N constant matrices, and $\tau_{\ell} \geq 0$ are constant time delays. For brevity, equation (1.1) is referred to as a retarded equation, or retarded system. Mathematical treatments of retarded systems can be found in references 1, 9, 15, and 25. The nature of the solution of equation (1.1) as well as the solution itself is of great importance in many physical applications.

1. Stable systems

The characteristic quasi-polynomial associated with equation (1.1) is

$$L(s) = \det\left(sI - A - \sum_{\ell=1}^{T} A_{\ell}\, e^{-\tau_{\ell} s}\right) \qquad (1.2)$$

It is known (references 1, 15, and 26) that the solution of equation (1.1) is

(1) unstable if any root of L(s) has a positive real part, or if any purely imaginary root of L(s) is a multiple root;

(2) stable (bounded) if all purely imaginary roots are simple (not multiple), and the remaining roots have negative real parts; and

(3) asymptotically stable (exponentially) if all roots of $L(s)$ have negative real parts.

The difficulty is that there are an infinite number of roots of $L(s)$. For one delay, the τ-decomposition method provides a convenient means of determining the number of roots with positive real parts. This method is extended in this chapter to accomplish this objective when there are many delays.

2. Expansion of characteristic quasi-polynomial

The determinant in equation (1.2) can be expanded and written in the form

$$L(s) = H_K(s) - J_K(s)e^{-\theta_K s} \tag{1.3}$$

where

$$H_K(s) = \sum_{n=0}^{N} a_n s^n + \sum_{n=0}^{N-1} \sum_{\substack{j=1 \\ j\neq K}}^{M} b_{Knj} s^n e^{-\theta_j s} \tag{1.4}$$

and

$$J_K(s) = \sum_{n=0}^{N-1} c_{Knj} s^n + \sum_{n=0}^{N-1} \sum_{\substack{j=1 \\ j\neq K}}^{M} d_{Knj} s^n e^{-\theta_j s} \tag{1.5}$$

for each $K = 1, 2, \ldots, M$. All τ_ℓ in the expansion of the determinant are absorbed in θ_K, $K = 1, 2, \ldots, M$. A subscript K appears in equations (1.4) and (1.5) to denote that the specific form of these equations changes with the choice of θ_K in equation (1.3), although $H_K(s)$ and $J_K(s)$ do not depend explicitly on θ_K itself. (See appendix A for examples of equation (1.3).)

Equation (1.3) will be used to examine the change in the relative stability of a system with variations in particular values of θ_K, while the remaining delays are held fixed.

The following three basic theorems can be used to justify subsequent statements about the roots of L(s).

Theorem 1: The roots of the quasi-polynomial L(s) are continuous functions of θ_j, j = 1, 2, ..., M.

Theorem 2: Let $L_0(s)$ be the Nth-order polynomial obtained by setting all the delays in the quasi-polynomial L(s) equal to zero. Then, for sufficiently small values of the delays θ_j, L(s) has N roots which are arbitrarily close to the N roots of $L_0(s)$. In addition, any other roots of L(s) have arbitrarily large moduli for θ_j sufficiently small.

Theorem 3: The arbitrarily large modulus roots of the quasi-polynomial L(s) of a retarded system have negative real parts.

Theorems 1, 2, and 3 are proved in appendix B.

It follows readily by taking the complex conjugate of equation (1.3) that all characteristic roots occur in complex conjugate pairs. Hence, hereafter, only those roots in the upper half of the s-plane are of interest; that is, $s = \sigma + i\omega$ with $\omega \geq 0$.

3. Relative stability

A system can be stable and yet be unsatisfactory because it is too lightly damped. This leads to the concept of relative stability (references 18 and 27), which is, in a sense, a measure of the degree of stability. Relative stability is examined herein by computing the number of roots on and to the right of a specified line parallel to the imaginary axis in the s-plane. This line is identified by $s = \bar{\sigma} + i\omega$, where $\bar{\sigma}$ is a fixed real number. Stability is a special case of relative stability, for if there are no roots $s = \sigma + i\omega$ of $L(s)$ with $\sigma \geq \bar{\sigma} = 0$, then the retarded system is stable.

The procedure for finding the relative stability of a retarded system for a set of constant delays, say $\bar{\theta}_j$, is to: (1) find the relative stability of the system for a particular set of delays, say $\tilde{\theta}_j$, and; (2) examine the relative stability change as the delays θ_j are varied from $\tilde{\theta}_j$ to $\bar{\theta}_j$. If $\tilde{\theta}_j = 0$ $(j = 1, 2, \ldots, M)$, the initial relative stability is obtained by solving for the roots of $L(s)$ in equation (1.3) with zero delays, that is, solving for the roots of the equation

$$L_0(s) = H_K(s) - J_K(s) = 0 \tag{1.6}$$

The number of roots $s = \sigma + i\omega$ of equation (1.6) with $\omega \geq 0$ and $\sigma \geq \bar{\sigma}$ is determined. In order for this number of roots to change as the delays are varied in a continuous manner to a final set of delays $\bar{\theta}_j$, a root-locus curve must intersect the $(\bar{\sigma}, \omega)$ line in the s-plane. The intersection points, which occur as a delay θ_K is varied, are roots of equation (1.3). Hence, at an intersection point of the $(\bar{\sigma}, \omega)$ line and a root-locus curve

$$L(s) = H_K(s) - J_K(s)e^{-\theta_K s} = 0 \tag{1.7}$$

This relation is called the characteristic equation of the retarded system.

4. Intersection points

The intersection points, which occur on the line $(\bar{\sigma}, \omega)$ as θ_K varies with the remaining delays fixed, are contained in the following mutually exclusive (or disjoint) sets:

$$\left.\begin{aligned}
\Omega_0 &= \{s: \; s = 0\} \\
\Omega_{1K} &= \{s: \; H_K(s) = J_K(s) = 0, \; s \neq 0\} \\
\Omega_{2K} &= \{s: \; Re(s) \neq 0, \; Im(s) = 0\} - \Omega_{1K} \\
\Omega_{3K} &= \{s: \; Re(s) = 0, \; Im(s) \neq 0\} - \Omega_{1K} \\
\Omega_{4K} &= \{s: \; Re(s) \neq 0, \; Im(s) \neq 0\} - \Omega_{1K}
\end{aligned}\right\} \tag{1.8}$$

where $Re(s)$ and $Im(s)$ denote the real and imaginary parts of s, respectively. The intersection points are now discussed relative to these sets.

If $s = 0$ is a root of $L(s)$, then it belongs to the set Ω_0. These roots are easily obtained by solving equation (1.6).

In order to consider the intersection points $s \epsilon \Omega_{1K}$, it is useful to note that $L(s) = 0$ in equation (1.3) implies

$$H_K(s) = J_K(s) = 0 \tag{1.9}$$

or

$$H_K(s) \neq 0, \; J_K(s) \neq 0 \tag{1.10}$$

for finite values of θ_K and $\bar{\sigma}$. If s satisfies equation (1.9), then $s \epsilon \Omega_{1K}$ and is actually invariant with changes in θ_K. It is not necessary to explicitly calculate these roots for the stability method to be presented later.

The intersection points $s \epsilon \Omega_{2K}$ are now determined. First, with $s = \sigma$ (or $\omega = 0$), equation (1.7) can be solved for θ_K as

$$\theta_K = -\frac{1}{\sigma} \ln \left[\frac{H_K(\sigma)}{J_K(\sigma)} \right] \tag{1.11}$$

In particular, for intersection points $s \epsilon \Omega_{2K}$, equation (1.11) becomes

$$\theta_K = -\frac{1}{\bar{\sigma}} \ln \left[\frac{H_K(\bar{\sigma})}{J_K(\bar{\sigma})} \right] \tag{1.12}$$

Notice that the real numbers $H_K(\bar{\sigma})$ and $J_K(\bar{\sigma})$ in equation (1.12) must be of the same algebraic sign in order for real θ_K to exist. Furthermore, for non-negative delays ($\theta_K \geq 0$) and a stable system ($\bar{\sigma} < 0$)

$$\left| \frac{H_K(\bar{\sigma})}{J_K(\bar{\sigma})} \right| \geq 1 \tag{1.13}$$

Finally, for $s \epsilon \Omega_{3K} \cup \Omega_{4K}$, equation (1.7) evaluated at the intersection points is equivalent to the following two equations

$$\left| H_K(\bar{\sigma}, \omega) \right| = \left| J_K(\bar{\sigma}, \omega) \right| e^{-\theta_K \bar{\sigma}} \tag{1.14}$$

and

$$\theta_K = -\frac{1}{\omega} \left\{ \arg \left[\frac{H_K(\bar{\sigma}, \omega)}{J_K(\bar{\sigma}, \omega)} \right] - 2p\pi \right\} \tag{1.15}$$

$$(p = 0, \pm 1, \pm 2, \ldots)$$

where $H_K(\bar{\sigma}, \omega) = H_K(\bar{\sigma} + i\omega)$, $J_K(\bar{\sigma}, \omega) = J_K(\bar{\sigma} + i\omega)$. and

$$- \pi < \arg \left[\frac{H_K(\bar{\sigma}, \omega)}{J_K(\bar{\sigma}, \omega)} \right] \leqq \pi \tag{1.16}$$

Equations (1.14) and (1.15) yield the only combination values of ω and θ_K which result in an intersection point $s \epsilon \Omega_{3K} \cup \Omega_{4K}$ of a root-locus curve with the line $(\bar{\sigma}, \omega)$ for θ_j fixed, $j \neq K$.

Substituting equation (1.15) into equation (1.14) results in an equation for ω in the form

$$\left|H_K(\bar{\sigma}, \omega)\right| = \left|J_K(\bar{\sigma}, \omega)\right| \exp\left\{\frac{\bar{\sigma}}{\omega}\left[\arg\left(\frac{H_K(\bar{\sigma}, \omega)}{J_K(\bar{\sigma}, \omega)}\right) - 2p\pi\right]\right\} \tag{1.17}$$

In general, equation (1.17) cannot be solved explicitly for $\omega \geq 0$, so that a graphical or iteration process is required to locate the roots. For this purpose, a useful upper bound ω_m on ω is derived in appendix C. Also, an upper bound on the integer p in equation (1.15) is obtained as

$$|p| \leqq \frac{1}{2} + \frac{\omega_m \bar{\theta}_K}{2\pi} \tag{1.18}$$

where $\theta_K \leqq \bar{\theta}_K$ and $\omega \leqq \omega_m$.

For $\bar{\sigma} = 0$ $(s \epsilon \Omega_{3K})$, equations (1.14) and (1.15) become

$$\left|H_K(0, \omega)\right| = \left|J_K(0, \omega)\right| \tag{1.19}$$

and

$$\theta_K = -\frac{1}{\omega}\left\{\arg\left[\frac{H_K(0, \omega)}{J_K(0, \omega)}\right] - 2p\pi\right\} \tag{1.20}$$

Equation (1.19) is independent of the delay θ_K; and for each ω solution of equation (1.19), the corresponding delays in equation (1.20) are shifted by the factor $\frac{2p\pi}{\omega}$.

An alternate form of equation (1.19) is

$$\left|H_K(0, \omega)\right|^2 = \left|J_K(0, \omega)\right|^2 \tag{1.21}$$

which, for only one delay θ_K, is a polynomial equation in ω of degree N^2. The real solutions of equation (1.19) and (1.21) are the same.

5. Effect on relative stability of changes in the delays

A zero root $s\epsilon\Omega_0$ of $L(s)$ is invariant with respect to changes in any of the delays; and, an intersection point $s\epsilon\Omega_{1K}$ is invariant with respect to changes in θ_K. It is now of interest to find out how the relative stability changes across the intersection points $s\epsilon\Omega_{2K}$ and $s\epsilon\Omega_{3K}\bigcup\Omega_{4K}$.

Let $s\epsilon\Omega_{2K}$ be an intersection point. Then, equation (1.11) gives the relationship between θ_K and σ on the root-locus curve. In this equation, ρ_K is a single-valued function of σ (θ_K can, of course, assume the same value for different values of σ). Thus, σ must either increase or decrease as θ_K increases beyond the value given by equation (1.12) for the intersection point. The variation of σ (in relation to $\bar{\sigma}$) with θ_K may be determined by calculating $\frac{d\sigma}{d\theta_K}$ from equation (1.11) and evaluating this derivative at $\sigma = \bar{\sigma}$, or by calculating values of θ_K on either side of $\sigma = \bar{\sigma}$. The retarded system becomes relatively more stable in the direction of decreasing σ.

The change in the relative stability at the intersection points $s = \bar{\sigma} + i\omega \in \Omega_{3K} \cup \Omega_{4K}$ is now discussed. For this purpose, equation (1.3) is written as

$$L(s) = - H_K(s) e^{-\theta_K(s - \bar{\sigma})} \left[- e^{\theta_K(s - \bar{\sigma})} + \frac{J_K(s)}{H_K(s)} e^{-\theta_K \bar{\sigma}} \right] \tag{1.22}$$

The roots of equation (1.22) for $H_K(s) \neq 0$ are the same as the roots of the modified quasi-polynomial

$$G(s) = W_K(s) - e^{\theta_K(s - \bar{\sigma})} \tag{1.23}$$

where

$$W_K(s) = \frac{J_K(s)}{H_K(s)} e^{-\theta_K \bar{\sigma}} \tag{1.24}$$

Now, let

$$\left.\begin{aligned} &\theta_K = \tau \\ &s = z + \bar{\sigma} \\ &J_K(s)e^{-\theta_K \bar{\sigma}} = J_K(z + \bar{\sigma})e^{-\theta_K \bar{\sigma}} = - h(z) \\ &H_K(s) = H_K(z + \bar{\sigma}) = g(z) \\ &G(s) = G(z + \bar{\sigma}) = W(z) \end{aligned}\right\} \tag{1.25}$$

Then, equation (1.23) can be rewritten using the notation of Lee and Hsu (reference 24) in the form

$$W(z) = W_2(z) - W_1(z) \tag{1.26}$$

where

$$W_1(z) = e^{\tau z} \tag{1.27}$$

and

$$W_2(z) = - \frac{h(z)}{g(z)} \tag{1.28}$$

The following theorem is attributed to Lee and Hsu (reference 24) where $N(\tau)$ denotes the number of roots of $W(z)$ with positive real parts.

<u>Theorem 4</u>: Let $z = i\omega^*$, where $\omega^* > 0$, be a purely imaginary root of $W(z)$ with corresponding delay $\tau^* \geq 0$. Let φ be an intersection point of the testing function $W_2(i\omega)$ with the unit circle $W_1(i\omega)$.

(1) $N(\tau)$ increases by 1 as τ increases across τ^* if the testing path enters the unit circle at φ as ω increases across ω^* for $\omega > 0$;

(2) $N(\tau)$ decreases by 1 as τ increases across τ^* if the testing path leaves the unit circle as ω increases across ω^* for $\omega > 0$; and

(3) $N(\tau)$ remains the same as τ increases across τ^* if the testing function remains on the same side of the unit circle as ω increases across ω^* for $\omega > 0$.

Lee and Hsu use an equation similar in form to equation (1.26) in proving the results of theorem 4 to examine the stability ($\bar{\sigma} = 0$) of a retarded dynamical system with one constant time delay τ. The development of theorem 4 is sufficiently general, however, such that the only restriction required on the $W_2(z)$ function in equation (1.28) is that it be analytic (or regular) at the intersection points being considered. This is certainly true for the intersection points

$s \epsilon \Omega_{3K} \cup \Omega_{4K}$, for which $H_K(s) \neq 0$, $J_K(s) \neq 0$, and $\omega \neq 0$. Therefore, the following theorem is deduced in Appendix B using equations (1.24), (1.25), (1.28) and theorem 4. In the theorem, $N(\theta_K, \bar{\sigma})$ denotes the number of roots of $L(s)$ with $\sigma > \bar{\sigma}$ at θ_K for fixed θ_j, $j \neq K$.

Theorem 5: Let $s = \bar{\sigma} + i\omega \epsilon \Omega_{3K} \cup \Omega_{4K}$ with $\omega > 0$ be an intersection point with corresponding delay θ_K; or, equivalently, let ω and θ_K be simultaneous solutions of equations (1.14) and (1.15). Let $\alpha_1 < \omega$ and $\alpha_2 > \omega$ be real numbers for which $W_K(\bar{\sigma}, \alpha_1)$ and $W_K(\bar{\sigma}, \alpha_2)$ are defined; and such that there are no other intersection points with imaginary parts which lie on the intervals $[\alpha_1, \omega]$ and $[\omega, \alpha_2]$. Now, for ϵ an arbitrarily small positive number

(1) $N(\theta_K + \epsilon, \bar{\sigma}) = N(\theta_K, \bar{\sigma}) + 1$

if $|W_K(\bar{\sigma}, \alpha_1)| > 1$ and $|W_K(\bar{\sigma}, \alpha_2)| < 1$;

(2) $N(\theta_K + \epsilon, \bar{\sigma}) = N(\theta_K, \bar{\sigma}) - 1$

if $|W_K(\bar{\sigma}, \alpha_1)| < 1$ and $|W_K(\bar{\sigma}, \alpha_2| > 1$; and

(3) $N(\theta_K + \epsilon, \bar{\sigma}) = N(\theta_K, \bar{\sigma})$

if both $|W_K(\bar{\sigma}, \alpha_1)|$ and $|W_K(\bar{\sigma}, \alpha_2)|$ are greater than 1, or both less than 1.

Geometrically, an intersection point between the root-locus curve and the $s = \bar{\sigma}$ line occurs whenever the testing function $W_K(\bar{\sigma}, \omega)$ intersects the unit circle. These intersection points are illustrated

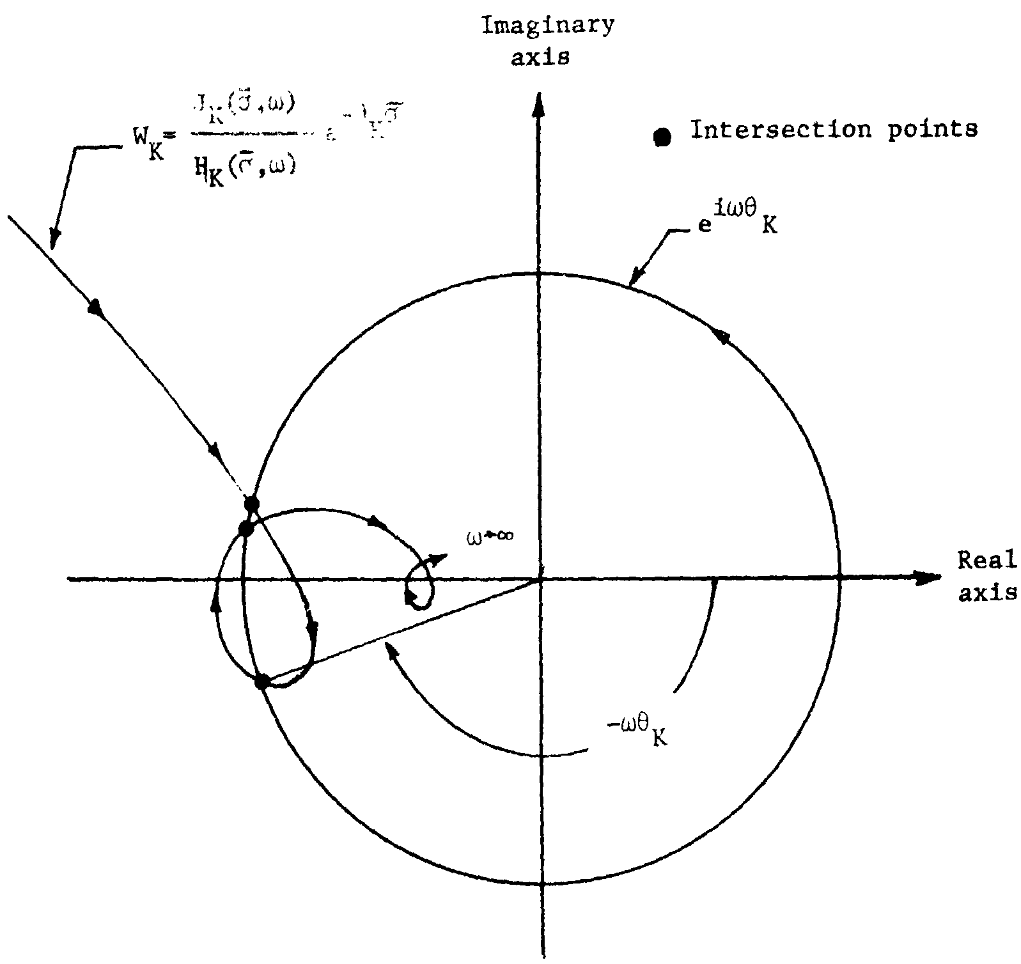


Figure 1.- Illustration of intersection points with respect to testing function W_K and unit circle.

in figure 1. Items 1, 2, and 3 of theorem 5 correspond, respectively, to the function $W_K(\bar{\sigma}, \omega)$ entering, leaving, and being tangent to the unit circle.

A significant feature of theorem 5 is that it is not necessary to be concerned with finding the first nonzero derivative of the real part of root-locus curves with respect to θ_K as is needed in the method of reference 28, for example; or whether the intersection points are simple.

Theorem 5 is used to determine the change in the relative stability of the system. Each delay is varied, one at a time, in some continuous manner, so that ultimately all delays attain their final desired values. As each delay is varied, it becomes θ_K in theorem 5.

6. Relative stability for sequential variation of delays

The method for examining the change in the relative stability of a system with a particular delay θ_K while the remaining delays are fixed is now developed.

The characteristic quasi-polynomial below is obtained after varying the delays $\theta_1, \ldots, \theta_{K-1}$ to their final desired values, allowing θ_K to vary, and fixing the values of the remaining delays $\theta_{K+1}, \ldots, \theta_M$.

$$L(s;\bar{\theta}_1,\ldots,\bar{\theta}_{K-1}, \theta_K, \widetilde{\theta}_{K+1},\ldots\widetilde{\theta}_M)$$

$$= H_K(s;\bar{\theta}_1,\ldots,\bar{\theta}_{K-1},\widetilde{\theta}_{K+1},\ldots,\widetilde{\theta}_M) - J_K(s;\bar{\theta}_1,\ldots,\bar{\theta}_{K-1},\widetilde{\theta}_{K+1},\ldots\widetilde{\theta}_M)e^{-\theta_K s} \tag{1.29}$$

where $\bar{\theta}_1, \ldots \bar{\theta}_{K-1}$ are the final desired values of the delays and $\tilde{\theta}_{K+1}, \ldots, \tilde{\theta}_M$ are fixed delays.

For $s \in \Omega_{2K}$, the change in the relative stability with θ_K is determined by using equation (1.11). For the remaining intersection points ($s \in \Omega_{3K} \cup \Omega_{4K}$) a digital computer can be used to increment $\omega \epsilon (0, \omega_m]$ in the following equations, for different values of p and a specified value of $\bar{\sigma}$.

$$s = \bar{\sigma} + i\omega \tag{1.30}$$

$$H_K(s) = H_K(s; \bar{\theta}_1, \ldots, \bar{\theta}_{K-1}, \tilde{\theta}_{K+1}, \ldots \tilde{\theta}_M) \tag{1.31}$$

$$J_K(s) = J_K(s; \bar{\theta}_1, \ldots, \bar{\theta}_{K-1}, \tilde{\theta}_{K+1}, \ldots \tilde{\theta}_M) \tag{1.32}$$

$$\frac{H_K(s)}{J_K(s)} = u + iv \tag{1.33}$$

$$\theta_K = \cdot \frac{1}{\omega} \left[\tan^{-1} \left(\frac{v}{u} \right) - 2p\pi \right] \tag{1.34}$$

$$F = |H_K| - |J_K| \, e^{\bar{\theta}_K \bar{\sigma}} \tag{1.35}$$

$$|W_K| = \left| \frac{J_K}{H_K} \right| e^{-\theta_K \bar{\sigma}} \tag{1.36}$$

An intersection point occurs whenever $F = 0$ or $|W_K| = 1$. In general, there will be an algebraic sign change in F in the vicinity of the intersection point. At $F = 0$, the delay θ_K is calculated. Then, $N(\theta_K + \epsilon)$ in theorem 5 is determined by comparing values of W_K on both sides of the intersection point. This procedure is repeated for all values of p for which $\theta_K \leq \bar{\theta}_K$, where $\bar{\theta}_K$ is the desired value of the K^{th} delay. Finally, the values of θ_K at all the intersections are ordered by increasing magnitudes to obtain the change in the relative stability as θ_K increases to $\tilde{\theta}_K$. This procedure is then applied for all values of K to determine the relative stability of the system as each of the delays is varied from zero to its final value.

If, during an iteration on ω, either $H_K(s)$ or $J_K(s)$ equals zero, then this indicates one of the following possibilities. First, this value of s is not an intersection point since equation (1.9) is not satisfied; or second, s is an intersection point belonging to the set Ω_{1K} and will remain invariant with changes in θ_K Thus, except for a computational inconvenience, this value of s is of no particular consequence. This being the case, the computation is continued with the next iteration on ω.

B. Scalar Neutral Systems With One Delay

The extended τ-decomposition stability method may be applied to a special class of scalar neutral equations with one constant time delay of the form (reference 1).

$$\sum_{j=0}^{N}\left[a_j\, x^{(j)}(t) + b_j\, x^{(j)}(t - \tau)\right] = 0 \qquad (1.37)$$

$$(a_N \neq 0,\ b_N \neq 0)$$

The characteristic quasi-polynomial associated with equation (1.37) can be expressed as

$$L(s) = H_K(s) - J_K(s)e^{-\theta_K s} \qquad (1.38)$$

where $\theta_K = \tau$,

$$H_K(s) = \sum_{j=0}^{N} a_j \, s^j \tag{1.39}$$

and

$$J_K(s) = - \sum_{j=0}^{N} b_j \, s^j \tag{1.40}$$

Equation (1.37) is asymptotically stable if and only if all the roots of $L(s)$ have negative real parts (references 1 and 26).

The basic problem in applying the extended τ-decomposition method to a neutral equation is that theorem 3 no longer can be applied. However, the real parts of the large modulus roots of equation (1.38) approach (reference 15)

$$\sigma_K = - \frac{1}{\theta_K} \ln \left| \frac{a_N}{b_N} \right| \tag{1.41}$$

The extended τ-decomposition is used in conjunction with equation (1.41) to examine the stability or relative stability of the solution of equation (1.37). An upper bound on the finite roots $s = \bar{\sigma} + i\omega$ which satisfy $L(s) = 0$ is determined in appendix C. For stability calculations ($\bar{\sigma} = 0$), the intersection points of the root locus curves with the imaginary axis can be computed using equation (1.21).

As long as $\sigma_K < \bar{\sigma}$, the extended τ-decomposition method is applied in the usual manner.

CHAPTER II

PARAMETER-PLANE METHODS

Parameter-plane methods deal with the plane of two system parameters in which lines separating the regions of stability or relative stability are plotted. These regions then give immediate information concerning the acceptable combination values of the parameters. Early use of this basic idea can be found in the works of Vishnegradskii (reference 19), Sokolov (reference 21), and Neimark (reference 22). Subsequent investigators include, for example: Mitrović and Šiljak (reference 18), Eisenberg (reference 27), and Krall (reference 22).

A. Stability Boundaries in Two-Delay Space for Retarded Systems

It appears that practical applications of systems with time delays have been limited thus far to one and two delays; for example, see references 5, 6, 8, 11, 12, and 14. If there is only one delay in the system, then the τ-decomposition method conveniently yields the intervals of the delay for which the system is stable and unstable. If there are two delays, then the extended τ-decomposition method can be used to construct stability boundaries in the plane of the two delays; however, this process involves the repeated application of the extended τ-decomposition method to obtain enough points to define the stability boundaries, and also involves the solution of transcendental equations for points other than those lying along either delay axis.

The purpose of this section is to develop explicit analytical equations for partitioning the delay space for retarded systems with two constant delays into different regions, each of which is either stable or unstable. Afterwards, the extended τ-decomposition method is used to identify the stable and unstable regions.

1. Class of retarded systems

Stability boundaries are examined for the class of retarded systems which has a characteristic quasi-polynomial of the form:

$$L(s) = P(s) + Q(s)\, e^{-\theta_1 s} + R(s)\, e^{-\theta_2 s} \tag{2.1}$$

where

$$P(s) = \sum_{j=0}^{N} a_j s^j \qquad (a_N \neq 0) \tag{2.2}$$

$$Q(s) = \sum_{j=0}^{N-1} b_j s^j \tag{2.3}$$

$$R(s) = \sum_{j=0}^{N-1} c_j s^j \tag{2.4}$$

The stability condition (stable or unstable) of a retarded system with characteristic quasi-polynomial equation (2.1) is the same as that for the same system with zero delays ($\theta_1 = \theta_2 = 0$), if the values of θ_1 and θ_2 in equation (2.1) are sufficiently small (theorem 2).

As the delays are varied in some continuous manner from essentially zero, the roots of the quasi-polynomial equation (2.1) move continuously and generate an infinite number of continuous root-locus curves in the complex root plane; that is, in the $\sigma\omega$-plane, $s = \sigma + i\omega$ satisfies $L(s) = 0$. Clearly, it is impossible to plot all the root-locus curves of equation (2.1) for a given continuous variation of the delays. It is possible, however, to determine the number of roots with positive real parts as the delays are varied by examining the behavior of the root-locus curves on the imaginary axis.

2. Partitioning delay space into different regions

Root-locus curves are generated by the roots of equation (2.1) as the delays are varied. If a root-locus curve comes into contact with, or crosses, the imaginary axis at ω (intersection point), then by definition, $s = i\omega$ satisfies

$$L(i\omega) = P(i\omega) + Q(i\omega)e^{-i\omega\theta_1} + R(i\omega)\ e^{-i\omega\theta_2} = 0 \qquad (2.5)$$

Since the root-locus curves are symmetrical about the real axis, only $\omega \geqq 0$ values are considered.

Equation (2.5) can be considered as two equations (real and imaginary parts) in two unknowns. The two unknowns which are chosen vary with the different methods which have been used to examine the stability of systems with delays. For example, in reference 23, $\theta_2 = 0$ and the two unknowns are ω and θ_1; in Neimark's method (reference 22), the delays are held fixed, and the two coefficients are chosen as the unknowns with ω as a coordinating parameter; in reference 10; the unknowns are a delay and a coefficient or gain. In the present case, the two unknowns are θ_1 and θ_2 with ω as a coordinating parameter.

A vector representation of the complex quantities appearing in equation (2.1) is shown in figure 2. The two distinct solution sets to equation (2.1) are shown graphically in figure 3. From the geometric properties of a triangle, it follows that a solution to equation (2.5) exists if and only if the following three relationships simultaneously hold:

$$|P(i\omega)| + |Q(i\omega)| \geqq |R(i\omega)| \tag{2.6}$$

$$|P(i\omega)| + |R(i\omega)| \geqq |Q(i\omega)| \tag{2.7}$$

$$|R(i\omega)| + |Q(i\omega)| \geqq |P(i\omega)| \tag{2.8}$$

These relations express the fact that the sum of the lengths of any two sides of a triangle must be greater than or equal to the length of the remaining side. An equality sign in either of equations (2.6), (2.7) or (2.8) corresponds to collinear vectors.

It follows by using figure 2 that the angles r and q in figure 3(a) are given by

$$r = \pi - \beta + \alpha$$

$$r = \pi - \arg P(i\omega) + \arg Q(i\omega) - \omega\theta_1 \tag{2.9}$$

and

$$q = \pi - \beta + \gamma$$

$$q = \pi - \arg P(i\omega) + \arg R(i\omega) - \omega\theta_2 \tag{2.10}$$

Now, solving equation (2.9) for θ_1 and equation (2.10) for θ_2 gives

$$\theta_1 = \frac{1}{\omega}\left[\pi - \arg P\ (i\omega) + \arg Q(i\omega) - r\right] \tag{2.11}$$

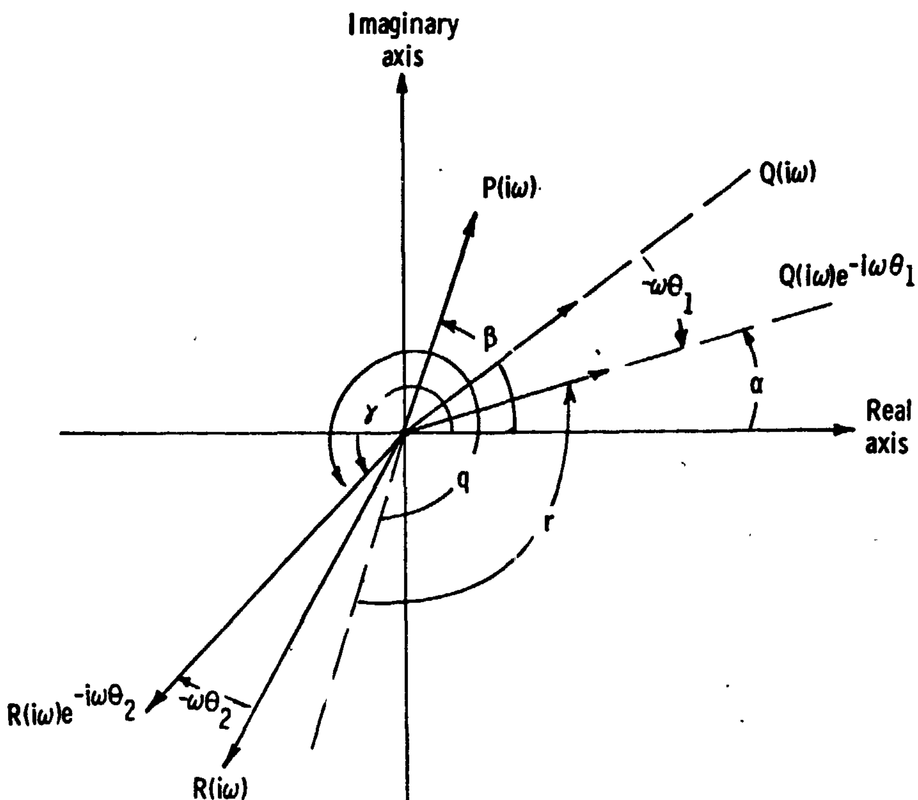
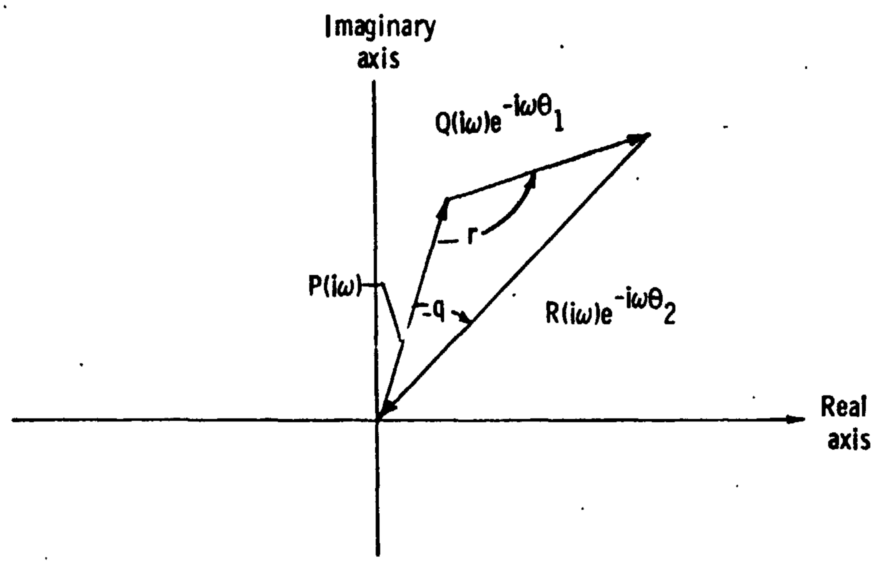
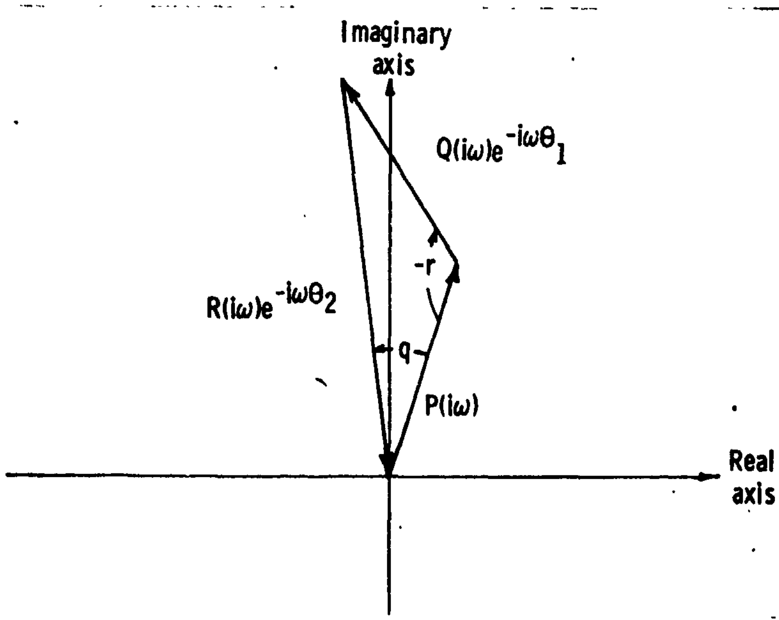


Figure 2.- Complex quantities and angles. Angles are measured positive in the counterclockwise direction.



(a) $r \leqq \pi$.



(b) $r \geqq \pi$.

Figure 3.- Graphical representation of two distinct solution sets for equation (2.5).

and

$$\theta_2 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg R(i\omega) - q\right] \tag{2.12}$$

The angles r and q are obtained from figure 3(a) by applying the law of cosines as

$$\cos r = \frac{|P(i\omega)|^2 + |Q(i\omega)|^2 - |R(i\omega)|^2}{2|P(i\omega)||Q(i\omega)|} \equiv y_1 \tag{2.13}$$

and

$$\cos(-q) = \frac{|P(i\omega)|^2 + |R(i\omega)|^2 - |Q(i\omega)|^2}{2|P(i\omega)||R(i\omega)|} \equiv y_2 \tag{2.14}$$

Choosing $0 \leqq \cos^{-1} y_j \leqq \pi$ and using the geometry in figure 3(a) gives

$$\theta_1 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg Q(i\omega) - \cos^{-1} y_1 + 2\pi K_1\right] \tag{2.15}$$

and

$$\theta_2 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg R(i\omega) + \cos^{-1} y_2 + 2\pi K_2\right] \tag{2.16}$$

where K_1 and K_2 are integers.

Using the geometry in figure 3(b) gives

$$\theta_1 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg Q(i\omega) + \cos^{-1} y_1 + 2\pi V_1\right] \tag{2.17}$$

and

$$\theta_2 = \frac{1}{\omega}\left[\pi - \arg P(i\omega) + \arg R(i\omega) - \cos^{-1} y_2 + 2\pi V_2\right] \tag{2.18}$$

where V_1 and V_2 are integers. A solution to equation (2.1) exists if and only if θ_1 and θ_2 satisfy the pair of equations (2.15) and (2.16) or the pair of equations (2.17) and (2.18).

Combination values of θ_1 and θ_2 for which a root-locus curve touches the imaginary axis can be determined by plotting θ_1 and θ_2 against ω. Alternately, θ_2 can be plotted directly against θ_1 for corresponding values of ω. This latter type of figure is actually a partitioning of the delay space into regions of stability (stable or unstable). It should be noted that Neimark's D-partition methods (reference 22) can be used to construct regions of stability in the plane of two real parameters (gains or coefficients) which occur linearly in the characteristic quasi-polynomial for fixed delays. Other methods, developed within the last decade, are discussed briefly in reference 29.

The stability region for the smallest values of delays may be all that is required to show that the delays are not large enough to make the system unstable or to indicate how much the delays can be increased before the system becomes unstable.

3. Special values of ω

There are certain values of ω which are useful in evaluating the equations for θ_1 and θ_2.

Upper bound on ω.- An upper bound on ω can be computed by using equation (2.5). The dominant power on ω occurs, by definition, in $P(i\omega)$; hence, let $\omega = \omega_m$ be the largest positive real root of the equation

$$\left|a_N\right|\omega^N - \sum_{n=0}^{N-1}\left(\left|a_n\right| + \left|b_n\right| + \left|c_n\right|\right)\omega^n = 0 \qquad (2.19)$$

Then, it follows that $\omega \leq \omega_m$ in equation (2.5) and in the pertinent pair of equations for θ_1 and θ_2.

Border values of ω.- Partitioning curves are defined only for those values of ω which satisfy equations (2.6) to (2.8). These meaningful values of ω are determined by using border values of ω. A border value of ω is defined as a non-negative real value of ω which satisfies any of the equality relations in equations (2.6) to (2.8), which are:

$$|P(i\omega)| + |Q(i\omega)| = |R(i\omega)| \tag{2.20}$$

$$|P(i\omega)| + |R(i\omega)| = |Q(i\omega)| \tag{2.21}$$

$$|R(i\omega)| + |Q(i\omega)| = |P(i\omega)| \tag{2.22}$$

These equality relationships also follow by setting $\cos r = \pm 1$ or $\cos(-q) = \pm 1$ in equations (2.13) or (2.14), respectively.

The finite number of border values separates the ω space into different intervals. To determine if a partitioning curve is defined for values of ω in an interval, $\cos r$ or $\cos(-q)$ may be evaluated at some value of ω within the interval. If the magnitude of $\cos r$ or $\cos(-q)$ determined from equations (2.13) and (2.14), respectively, is less than or equal to unity, then the partitioning curves exist for all values of ω in that interval.

4. Stable and unstable regions

The delay space is partitioned into different regions which are either stable or unstable. The stability character of these regions now must be identified. Toward this end equation (2.1) is written as

$$L(s) = H_K(s) - J_K(s)\, e^{-\theta_K s} \qquad (K = 1, 2) \tag{2.23}$$

where

$$H_1(s) = P(s) + R(s)\, e^{-\tilde{\theta}_2 s} \tag{2.24}$$

$$J_1(s) = -\, Q(s) \tag{2.25}$$

$$H_2(s) = P(s) + Q(s)\, e^{-\tilde{\theta}_1 s} \tag{2.26}$$

$$J_2(s) = -\, R(s) \tag{2.27}$$

and where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are fixed values of θ_1 and θ_2. The extended τ-decomposition method is used to identify the stable and unstable regions in the delay space. The initial stability is determined by using equation (1.6), which actually is equation (2.1) equated to zero with $\theta_1 = \theta_2 = 0$.

The procedure used to partition the delay space into different regions and to identify the different regions as stable or unstable is summarized in the following steps:

(1) The initial stability of the system is computed using equation (2.1) with $\theta_1 = \theta_2 = 0$.

(2) The range of acceptable values of ω are computed using equations (2.20) to (2.22).

(3) Partitioning curves in the delay space are plotted using θ_1 and θ_2 equations as ω varies over the predetermined range of acceptable values.

(4) If the delays do not cross a partitioning curve as they are varied in combination from zero to their final desired values, then the retarded system maintains its initial stability. However, if these delays cannot be varied to their final values without crossing a partitioning curve, the extended τ-decomposition method is applied using equations (2.23) to (2.27).

The stability on each side of a partitioning curve is determined by counting the number of roots with positive real parts as the curves are crossed. Stability boundaries are those partitioning curves which divide the delay space into stable or unstable regions.

The procedure presented in this section can be applied to retarded systems with more than two delays if the remaining delays are held fixed. A particular example of the parameter-plane method in conjunction with the extended τ-decomposition method is presented in a subsequent chapter.

B. Relative Stability Boundaries in the Gain-Delay Space

1. Retarded system

Let the characteristic quasi-polynomial of a retarded system be expressed as (reference 10)

$$L(s) = P(s) + \xi\, Q(s)\, e^{-\theta_K s} \qquad (2.28)$$

where the gain ξ and delay θ_K are the two system parameters of interest, and where P(s) and Q(s) are defined by equations (2.2) and (2.3), respectively.

After encountering combination values of ξ and θ_K which satisfy

$$P(s) + \xi\, Q(s)\, e^{-\theta_K s} = 0 \tag{2.29}$$

for $s = \bar{\sigma} + i\omega$, the number of roots with $\sigma > \bar{\sigma}$ may increase or decrease. Each of these combination values corresponds to a root-locus curve intersecting the line $(\bar{\sigma}, \omega)$ in the s-plane. Hence, if all these combination values can be determined and plotted in the plane of ξ and θ_K, then regions can be established, throughout which the relative stability of the system remains unchanged.

If Q(s) = 0 in equation (2.29), then the relative stability of the system is not influenced by the gain ξ or the delay θ_K and is determined completely by the roots of P(s). If $Q(s) \neq 0$ and $\omega \neq 0$, then ξ and θ_K are obtained from equation (2.29) as

$$|\xi| = \left|\frac{P(s)}{Q(s)}\right| e^{-\bar{\sigma}\theta_K} \tag{2.30}$$

and

$$\theta_K = -\frac{1}{\omega}\left[\arg \frac{P(s)}{Q(s)} - 2p\pi\right] \tag{2.31}$$

$$(p = 0, \pm 1, \ldots)$$

where $s = \bar{\sigma} + i\omega$. Since ξ does not appear in equation (2.31), the partitioning lines generated by using equations (2.30) and (2.31) will be symmetrical about the $\xi = 0$ line.

The complex quantities in equation (2.29) are illustrated geometrically in figure 4. For a given value of s, the vectors P(s) and Q(s) are fixed. Multiplication of Q(s) by $e^{-\theta_K s}$ rotates and changes the magnitude (if $\bar{\sigma} \neq 0$) of Q(s) as θ_K changes. The value of θ_K in equation (2.31) make Q(s) $e^{-\sigma_K s}$ collinear with P(s). Then, ξ in equation (2.30) makes the magnitude of the vectors equal. A negative value of ξ means that Q(s) $e^{-\theta_K s}$ is in the same direction as P(s); whereas, a positive value of ξ means that these vectors are in opposite directions.

If $Q(s) \neq 0$, $\omega = 0$, and $\bar{\sigma} \neq 0$, then

$$P(\bar{\sigma}) + \xi\, Q(\bar{\sigma})\, e^{-\theta_K \bar{\sigma}} = 0 \qquad (2.32)$$

from which another partitioning line

$$\theta_K = -\frac{1}{\bar{\sigma}} \ln \left[\frac{-P(\bar{\sigma})}{\xi Q(\bar{\sigma})} \right] \qquad (2.33)$$

is obtained.

After the plane of ξ and θ_K is partitioned into different regions, the extended τ-decomposition method with

$$H_K(s) = P(s) \qquad (2.34)$$

and

$$J_K(s) = -\xi\, Q(s) \qquad (2.35)$$

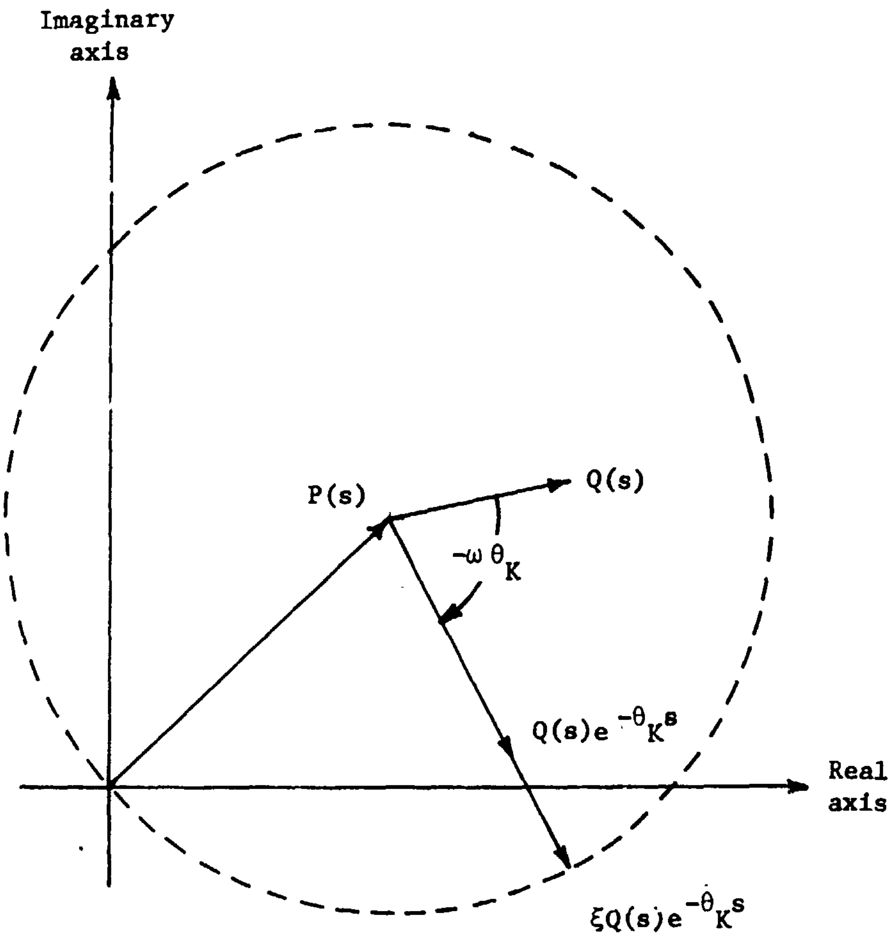


Figure 4.- Illustration of complex quantities in quasi-polynomial with gain ξ and delay θ_K.

is used to calculate the number of the resulting partitioned regions with $\sigma > \bar{\sigma}$.

It is expedient to note that the relative stability of a region which contains a segment of the ξ-axis ($\theta_K = 0$) may be determined by using the characteristic equation for zero delay, equation (1.6), with ξ equal to any value on this segment.

2. Neutral system

Parameter-plane techniques may be used also for the neutral system previously considered. The differential-difference equation (1.37) may be written as

$$\sum_{j=0}^{N} \left[a_j \, x^{(j)}(t) + \xi \, c_j \, x^{(j)}(t - \tau) \right] = 0 \qquad (2.36)$$

where b_j has been replaced by $\xi \, c_j$.

The characteristic equation may be expressed as

$$P(s) + \xi \, Q(s) \, e^{-\theta_K s} = 0 \qquad (2.37)$$

where $\theta_K = \tau$,

$$P(s) = \sum_{j=0}^{N} a_j \, s^j \qquad (2.38)$$

and

$$Q(s) = \sum_{j=0}^{N} c_j \, s^j$$

The partitioning equations for equation (2.37) are analogous to those for equation (2.28) and are not repeated here. To examine the relative stability of the different regions, the characteristic quasi-polynomial is written as

$$L(s) = H_K(s) - J_K(s) \tag{2.40}$$

where

$$H_K(s) = P(s) \tag{2.41}$$

$$J_K(s) = -\xi Q(s) \tag{2.42}$$

Now, for $\theta_K \to 0^+$, L(s) has N roots arbitrarily close to the N roots of equation (1.6); whereas, the remaining roots have moduli which are arbitrarily large ($|s| \to \infty$). From equation (1.41), the real part of these large modulus roots is approximately

$$\sigma_K = -\frac{1}{\theta_K} \ln \left| \frac{a_N}{\xi c_N} \right| \tag{2.43}$$

Hence, for $\theta_K \to 0^+$ and

$$\left| \frac{a_N}{\xi c_N} \right| > 1 \tag{2.44}$$

$\sigma_k \to -\infty$; whereby, $\sigma_k < \bar{\sigma}$. Therefore, for any point (ξ, θ_K), arbitrarily close to the ξ-axis in the plane of ξ and θ_K, the number of roots with $\sigma > \bar{\sigma}$ is given by equation (1.6). In moving from a region which contains a segment of the ξ-axis to other subsequent adjoining regions, which do not, equation (2.43) and the extended τ-decomposition method are used to determine the relative stability change.

CHAPTER III

APPLICATIONS FOR RETARDED SYSTEMS

Several applications of the extended τ-decomposition stability analysis technique for retarded systems will be considered in this chapter. Stability of solutions of versions of the equation

$$\frac{d^2}{d\tilde{t}^2} x_1(\tilde{t}) + 2\zeta \frac{d}{d\tilde{t}} x_1(\tilde{t} - \theta_1) + x_1(\tilde{t} - \theta_2) = 0 \tag{3.1}$$

which is a normalized form (reference 28) of the equation

$$\ddot{x}(t) + 2\zeta\, \omega_n\, \dot{x}\,(t - \tau_1) + \omega_n^2\, x(t - \tau_2) = 0 \tag{3.2}$$

where $\tilde{t} = \omega_n t$, $\theta_1 = \omega_n \tau_1$, and $\theta_2 = \omega_n \tau_2$

will be determined.

The stability character of the solutions in some cases is evaluated using the parameter-plane methods.

A. Second-Order Differential Equation With One Delay

1. Stability for delay in velocity term

The asymptotic stability for the retarded system

$$\frac{d^2}{d\tilde{t}^2} x_1(\tilde{t}) + 2\zeta \frac{d}{d\tilde{t}} x_1(\tilde{t} - \theta_K) + x_1(\tilde{t}) = 0 \tag{3.3}$$

with $\zeta > 0$ and $\theta_K \geq 0$ is now considered.

The characteristic quasi-polynomial of equation (3.3) is

$$L(s) = H_K(s) + J_K(s)\, e^{-\theta_K s} \tag{3.4}$$

where

$$H_K(s) = s^2 + 1 \qquad (3.5)$$

$$J_K(s) = -2\zeta s \qquad (3.6)$$

The intersection points of the root-locus curves and the imaginary axis are obtained by using equation (1.21), which for this example becomes

$$(\omega^2 - 1)^2 - (2\zeta\omega)^2 = 0 \qquad (3.7)$$

The only two non-negative roots of equation (3.7) are denoted by

$$\omega_1 = -\zeta + \sqrt{\zeta^2 + 1} \qquad (3.8)$$

$$\omega_2 = \zeta + \sqrt{\zeta^2 + 1} \qquad (3.9)$$

For convenience in the following development, the θ_K delays corresponding to ω_1 are represented by

$$\beta_p = -\frac{1}{\omega_1}\left[\arg\left(\frac{\omega_1^2 - 1}{2\zeta\omega_1 i}\right) - 2p\pi\right] \qquad (3.10)$$

or

$$\beta_p = \frac{1}{\omega_1}(4p - 1)\frac{\pi}{2} \qquad (3.11)$$

$$(p = 0, \pm 1, \ldots)$$

The delays corresponding to ω_2 are denoted by

$$\gamma_p = -\frac{1}{\omega_2}\left[\arg\left(\frac{\omega_2^2 - 1}{2\zeta\omega_2 i}\right) - 2p\pi\right] \qquad (3.12)$$

or

$$\gamma_p = \frac{1}{\omega_2} (4p + 1) \frac{\pi}{2} \qquad (p = 0, \pm 1, \ldots) \tag{3.13}$$

The testing function for use in the extended τ-decomposition method is obtained from equation (1.24) with $\bar{\sigma} = 0$ in the form

$$W_K(0, \omega) = \frac{2\zeta\, \omega i}{-\omega^2 + 1} \tag{3.14}$$

Therefore,

$$\left|W_K(0, \omega)\right| = \left|\frac{2\zeta\omega}{\omega^2 - 1}\right| \tag{3.15}$$

Some relationships used in the application of theorem 5 are

$$0 < \omega_1 < \omega_1 + \epsilon < \omega_2 < \omega_2 + \epsilon \tag{3.16}$$

$$\left|W_K(0, 0)\right| = 0 < 1 \tag{3.17}$$

$$\left|W_K(0, \omega_1 + \epsilon)\right| \approx \left|\frac{1 + \frac{\epsilon}{\omega_1}}{1 - \frac{\epsilon}{\zeta}}\right| > 1 \tag{3.18}$$

$$W_K(0, \omega_2 + \epsilon) \approx \left|\frac{1 + \frac{\epsilon}{\omega_2}}{1 + \frac{\epsilon}{\zeta}}\right| < 1 \tag{3.19}$$

where ϵ is an arbitrarily small positive number. Then application of theorem 5 yields

$$N(\beta_p + \epsilon) = N(\beta_p) - 1 \tag{3.20}$$

$$N(\gamma_p + \epsilon) = N(\gamma_p) + 1 \tag{3.21}$$

Intersection points of the root-locus curves and the imaginary axis occur when $\theta_K = \beta_p$ and $\omega = \omega_1$ and when $\theta_K = \gamma_p$ and $\omega = \omega_2$. With $\omega = \omega_1$ or $\omega = \omega_2$ understood, these intersection points are discussed relative to the delays β_p and γ_p.

The discrete points given by equations (3.11) and (3.13) lie along the dashed straight lines illustrated in figure 5. For $\zeta > 0$, the system is stable when $\theta_K = 0$; that is, N(0) = 0. Since the system does not have a root with positive real part, it cannot lose such a root; therefore, the first intersection point for a positive delay must occur on the dashed line for γ_p, as indicated by relations (3.20) and (3.21). This also follows directly from equations (3.11) and (3.13) since the smallest positive delay is $\gamma_0 = \pi/2\omega_2$. Thus, the system remains stable for

$$0 \leqq \theta_K < \frac{\pi}{2\omega_2} \tag{3.22}$$

For $\theta_K = \gamma_0 + \epsilon$, the system is unstable with one root with positive real part. From equations (3.11) and (3.13), $\beta_p > 0$ and $\gamma_p > 0$ for $p > 0$. The next intersection point depends on the location of the crossing point

$$\hat{p} = \frac{\sqrt{\zeta^2 + 1}}{4\zeta} \tag{3.23}$$

of the two dashed lines in figure 5 with respect to the integer values of p.

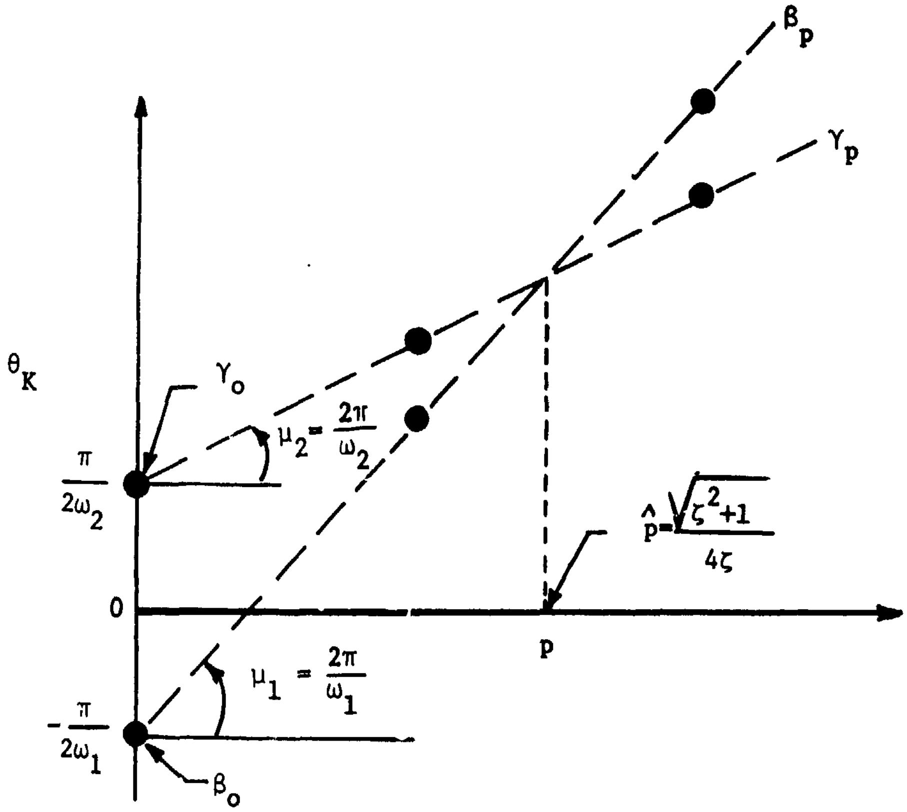


Figure 5.- Illustration of β_p and γ_p intersection points.

For $p = 1 < \hat{p}$, the second intersection point must occur at β_1. Thereafter, the system loses the root with positive real part and becomes stable again. The third intersection point cannot lie on the dashed line for β_p, since the system is stable. Thus, the third intersection point must be γ_1. This means the system is stable for

$$\beta_1 < \theta_K < \gamma_1 \tag{3.24}$$

For $\theta_K = \gamma_1 + \epsilon$, the system again becomes unstable with one root with positive real part. A continuation of this reasoning shows the system stable for

$$\beta_p < \theta_K < \gamma_p \tag{3.25}$$

and unstable for

$$\gamma_p < \theta_K < \beta_{p+1} \tag{3.26}$$

whenever $p < \hat{p}$.

Suppose $\hat{p}$ is an integer; so that it corresponds to an intersection point on both of the dashed lines in figure 6. Thus, at $\hat{p}$, the system loses one root with positive real part and simultaneously gains one root with positive real part. Thereafter, the stability condition of the system remains the same as for the previous intersection point $p < \hat{p}$.

For $p > \hat{p}$, the dashed line for β_p is located above the dashed line for γ_p. The first intersection point for $p > \hat{p}$ must occur on the γ_p line. After this intersection point, the system gains a second root with positive real part. This second root is actually one of the roots which had an arbitrarily large negative real part for θ_K sufficiently small.

It is now shown that two consecutive intersection points on the dashed line for β_p cannot happen; and, therefore, the system can never be stable again for $p > \hat{p}$. A proof by contradiction is used. Let two intersection points on the dashed line for β_p be denoted by β_{p_1} and β_{p_1+1}, as shown in figure 6. Assume that there are no intersection points on the dashed line for γ_p between these two points. This means that the dashed line for γ_p does not cross either of the segments 1 or 2 in figure 6. Since the exterior angle of a triangle exceeds either of the remote interior angles,

$$\mu_2 > \mu_1 \qquad (3.27)$$

But, $\mu_2 = \frac{2\pi}{\omega_2}$ and $\mu_1 = \frac{2\pi}{\omega_1}$ in figure 5, and $\omega_2 > \omega_1$ in equations (3.8) and (3.9). Hence, equation (3.27) is a contradiction. The following theorem results:

<u>Theorem 6:</u> For $\zeta > 0$ and $\theta_K \geqq 0$, the solution of equation (3.3) is asymptotically stable if and only if θ_K lies on an interval

$$\frac{2\pi p - \frac{\pi}{2}}{-\zeta + \sqrt{\zeta^2 + 1}} < \theta_K < \frac{2\pi p + \frac{\pi}{2}}{\zeta + \sqrt{\zeta^2 + 1}} \qquad (3.28)$$

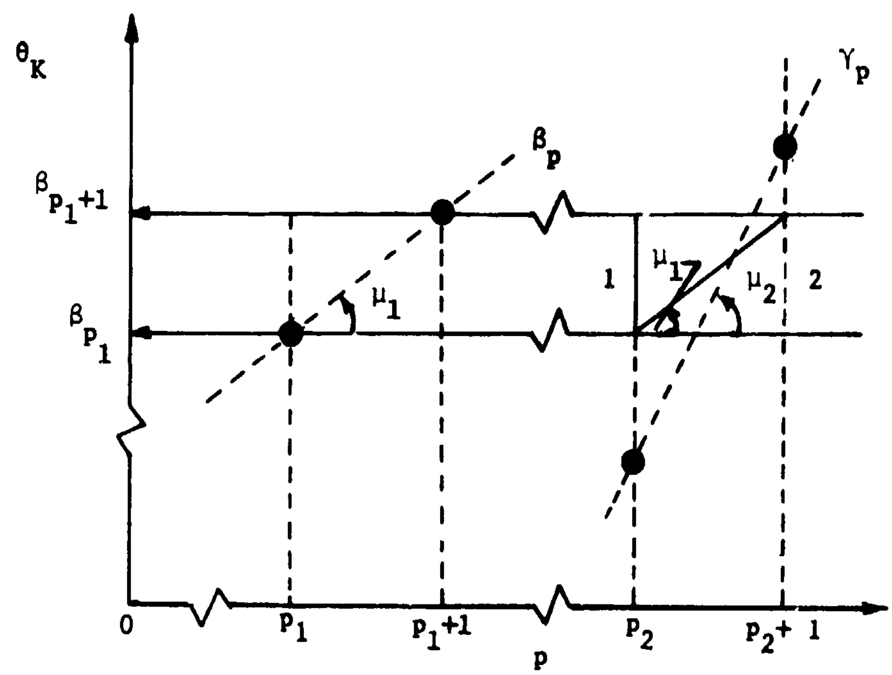


Figure 6.- Illustration of consecutive β_p intersection points.

for values of $p = 1, 2, \ldots,$ where

$$p < \frac{\sqrt{\zeta^2 + 1}}{4\zeta} \tag{3.29}$$

An immediate consequence of theorem 6 is that the solution of equation (3.3) is asymptotically stable only on the interval

$$0 \leqq \theta_K < \frac{\pi/2}{\zeta + \sqrt{\zeta^2 + 1}} \tag{3.30}$$

whenever

$$\zeta > \frac{1}{\sqrt{15}} \tag{3.31}$$

This result follows by selecting ζ in equation (3.29) so that $p < 1$.

The regions of stability described by theorem 6 agree with results obtained by Kashiwagi (reference 30), who used a different method which involves evaluating partial derivatives and the assumption of simple roots.

2. Stability for delay in displacement term

The asymptotic stability of the equation

$$\frac{d^2}{d\tilde{t}^2} x_1(\tilde{t}) + 2\zeta \frac{d}{d\tilde{t}} x_1(\tilde{t}) + x_1(\tilde{t} - \theta_K) = 0 \tag{3.32}$$

with $\zeta > 0$ and $\theta_K \geq 0$ is examined in this section.

The characteristic quasi-polynomial is

$$L(s) = H_K(s) - J_K(s)\, e^{-\theta_K s} \tag{3.33}$$

where

$$H_K(s) = s(s + 2\zeta) \tag{3.34}$$

$$J_K(s) = -1 \tag{3.35}$$

Equations (1.21) and (1.20) may be used to obtain, respectively,

$$\omega^4 + 4\zeta^2\,\omega^2 - 1 = 0 \tag{3.36}$$

and

$$\theta_K = -\frac{1}{\omega}\left[\tan^{-1}\left(\frac{-2\zeta}{\omega}\right) - 2p\pi\right] \quad (p = 0, \pm 1, \ldots) \tag{3.37}$$

The only positive real root of equation (3.36) is

$$\omega = \sqrt{-2\zeta^2 + \sqrt{4\zeta^2 + 1}} \tag{3.38}$$

Using equation (1.24) with $\bar{\sigma} = 0$ results in

$$W_K(0, \omega) = \frac{-1}{i\omega(i\omega + 2\zeta)} \tag{3.39}$$

Taking the square of the magnitude of equation (3.39) gives

$$\left|W_K(0, \omega)\right|^2 = \frac{1}{\omega^2(\omega^2 + 4\zeta^2)} \tag{3.40}$$

The following relations hold:

$$\omega - \epsilon < \omega < \omega + \epsilon \tag{3.41}$$

$$\left|W_K(0, \omega - \epsilon)\right| > 1 \tag{3.42}$$

$$\left|W_K(0, \omega + \epsilon)\right| < 1 \tag{3.43}$$

where ϵ is an arbitrarily small positive number. Application of theorem 5 produces

$$N(\theta_K + \epsilon) = N(\theta_K) + 1 \tag{3.44}$$

Since $\zeta > 0$, the solution of equation (3.3?) is asymptotically stable for $\theta_K = 0$ and remains stable as θ_K increases until the first intersection point is encountered. Thereafter, the solution is unstable. The following theorem results:

Theorem 7: For $\zeta > 0$ and $\theta_K \geqq 0$, the solution of equation (3.32) is asymptotically stable if and only if

$$0 \leqq \theta_K < \frac{1}{\omega} \tan^{-1}\left(\frac{2\zeta}{\omega}\right) \tag{3.45}$$

where ω is given by equation (3.38).

This result was also obtained by Kashiwagi (reference 30) using the method mentioned previously.

B. Second-Order Differential Equation With Two Delays

1. Stability boundaries

Stability boundaries are constructed in the θ_1, θ_2 plane for the equation

$$\frac{d^2}{d\tilde{t}^2} x_1(\tilde{t}) + 2\zeta \frac{d}{d\tilde{t}} x_1(\tilde{t} - \theta_1) + x_1(\tilde{t} - \theta_2) = 0 \tag{3.46}$$

where

$\zeta > 0$, $\theta_1 \geqq 0$, and $\theta_2 \geqq 0$.

The characteristic quasi-polynomial associated with equation (3.46) can be expressed in the form of equation (2.1); namely:

$$L(s) = P(s) + Q(s)\, e^{-\theta_1 s} + R(s)\, e^{-\theta_2 s} \tag{3.47}$$

where

$$P(s) = s^2 \tag{3.48}$$

$$Q(s) = 2\zeta\, s \tag{3.49}$$

$$R(s) = 1 \tag{3.50}$$

Since $\zeta > 0$, the solution of equation (3.46) is asymptotically stable for zero delays (initial stability).

Setting $s = i\omega$ in equations (3.48) to (3.50) gives

$$P(i\omega) = -\omega^2 \tag{3.51}$$

$$Q(i\omega) = 2\zeta\omega i \tag{3.52}$$

$$R(i\omega) = 1 \tag{3.53}$$

The first pair of equations for the delays θ_1 and θ_2 (eqs. (2.15) and (2.16)) becomes

$$\theta_1 = \frac{1}{\omega}\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{\omega^4 + 4\zeta^2\omega^2 - 1}{4\zeta\omega^3}\right) + 2\pi K_1\right] \tag{3.54}$$

and

$$\theta_2 = \frac{1}{\omega}\left[\cos^{-1}\left(\frac{\omega^4 - 4\zeta^2\omega^2 + 1}{2\omega^2}\right) + 2\pi K_2\right] \tag{3.55}$$

The second pair of equations for θ_1 and θ_2 (eqs. (2.17) and (2.18) becomes

$$\theta_1 = \frac{1}{\omega}\left[\frac{\pi}{2} + \cos^{-1}\left(\frac{\omega^4 + 4\zeta^2\omega^2 - 1}{4\zeta\omega^3}\right) + 2\pi V_1\right] \tag{3.56}$$

and

$$\theta_2 = \frac{1}{\omega}\left[-\cos^{-1}\left(\frac{\omega^4 - 4\zeta^2\omega^2 + 1}{2\omega^2}\right) + 2\pi V_2\right] \tag{3.57}$$

An upper bound ω_m.- The upper bound of ω is obtained by using equation (2.19) which becomes

$$\omega^2 - 2\zeta\omega - 1 = 0 \tag{3.58}$$

The largest real non-negative value of ω which satisfies equation (3.58) is denoted by ω_m. In this case,

$$\omega_m = \zeta + \sqrt{\zeta^2 + 1} \tag{3.59}$$

Border values of ω.- The border values of ω are obtained by using the relations in equations (2.20), (2.21), and (2.22), which become

$$\omega^2 + 2\zeta\omega - 1 = 0 \tag{3.60}$$

$$\omega^2 - 2\zeta\omega + 1 = 0 \tag{3.61}$$

$$\omega^2 - 2\zeta\omega - 1 = 0 \tag{3.62}$$

Notice that equation (3.62) is the same as equation (3.58), so that ω_m is a border value. Other border values are obtained from the solutions of equations (3.60) and (3.61) which are, respectively,

$$\omega = -\zeta \pm \sqrt{\zeta^2 + 1} \tag{3.63}$$

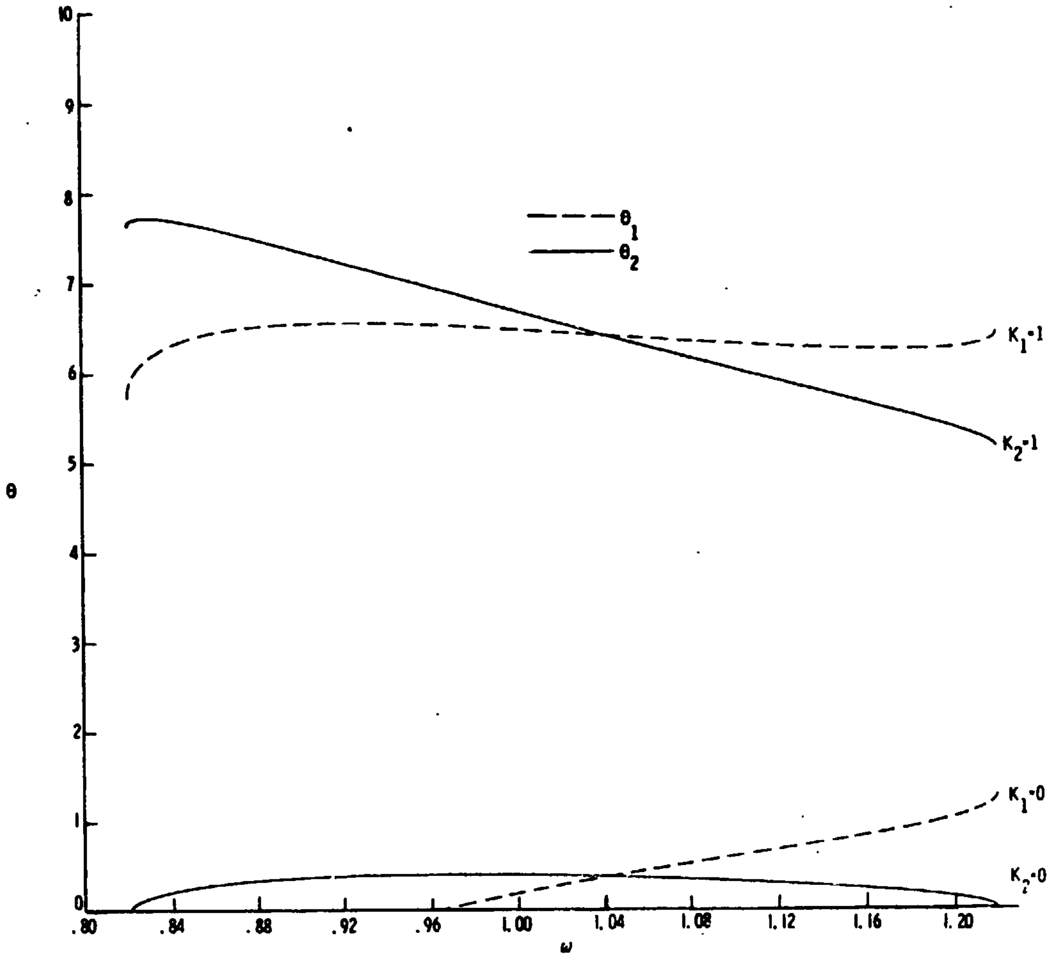
$$\omega = \zeta \pm \sqrt{\zeta^2 - 1} \tag{3.64}$$

Only the positive radical in equation (3.63) is of interest since $\omega \geqq 0$. Equation (3.64) is only of interest when $\zeta \geqq 1$.

Specific Calculations.- The border values and upper bound of ω are shown in table I for damping parameter values ζ of 0.2, 0.5, and 1.

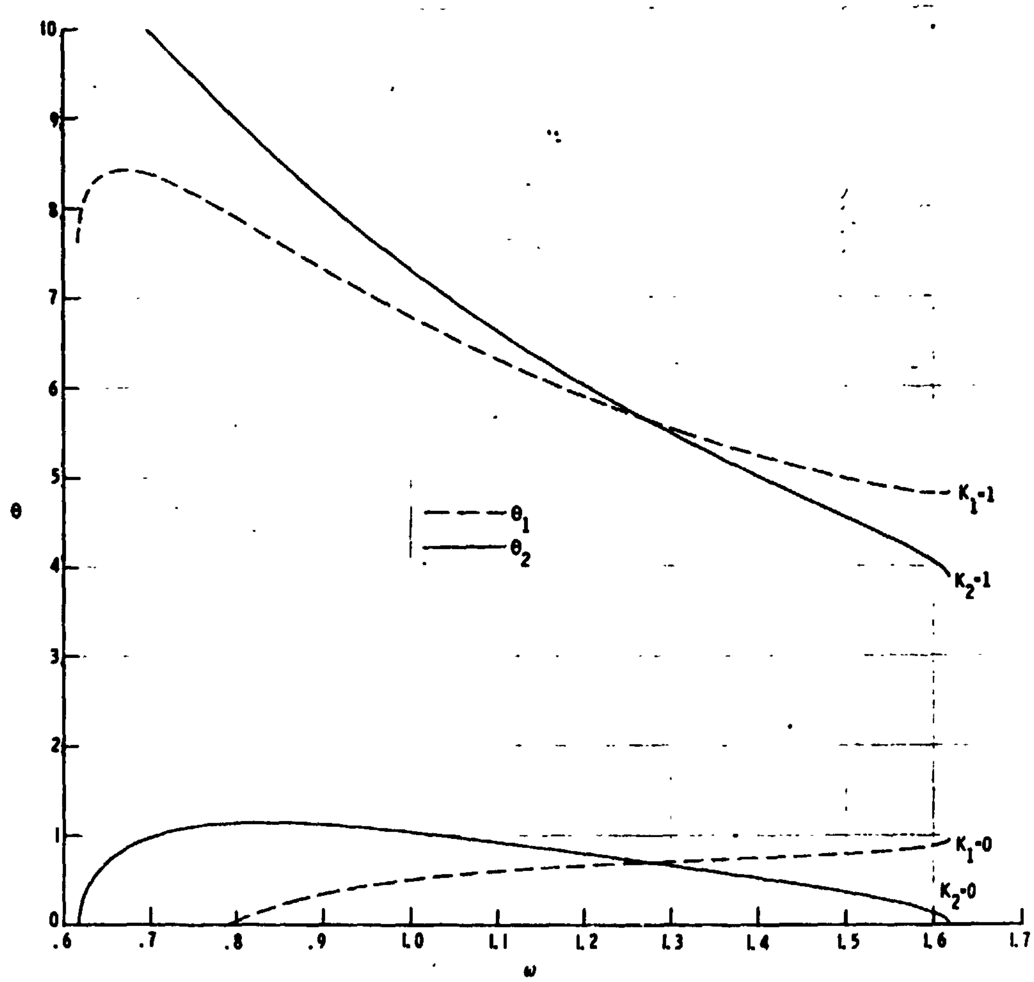
TABLE I.- BORDER VALUES AND UPPER BOUND OF ω

Damping parameter, ζ	Border values of ω	Upper bound, ω_m
0.2	0.82 1.22	1.22
.5	.62 1.62	1.62
1.0	.41 1.00 2.41	2.41



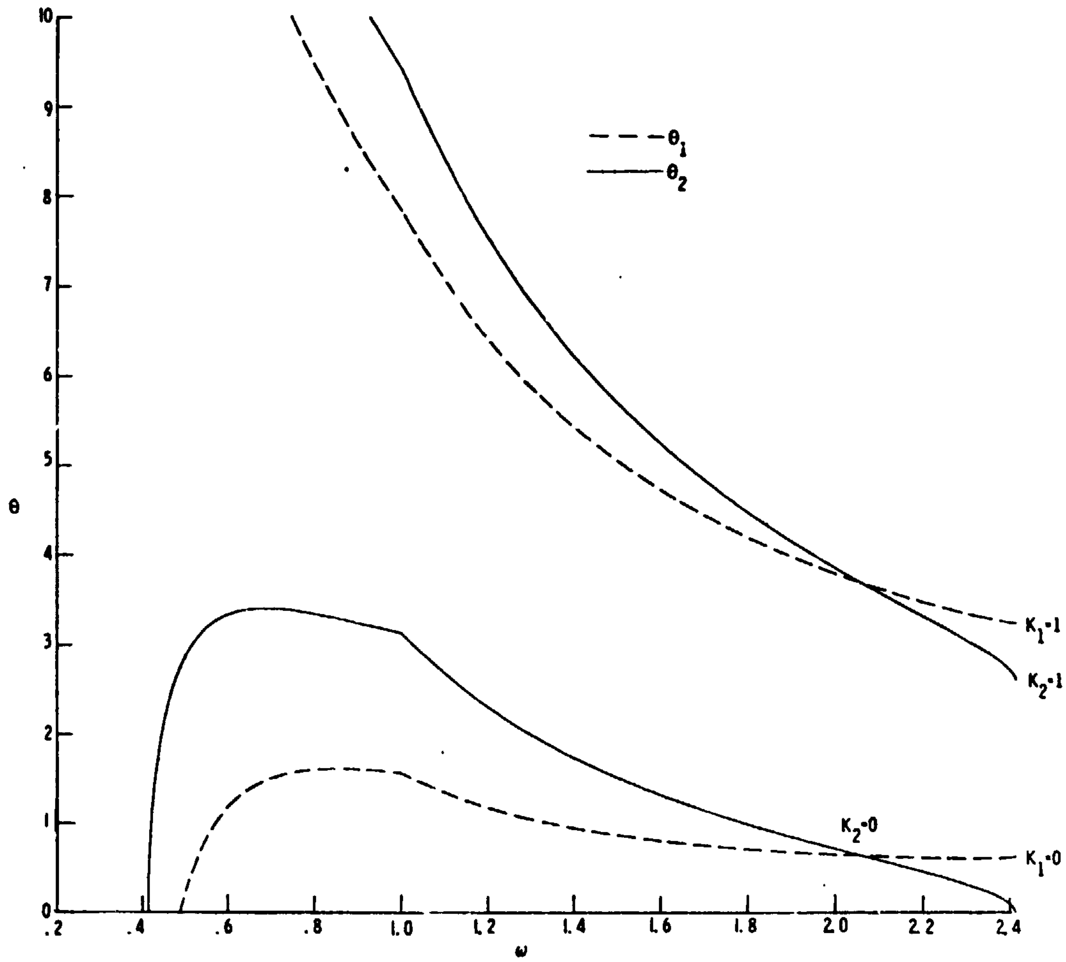
(a) $\zeta = 0.2$.

Figure 7.- Pairs of values of the delays θ_1 and θ_2 which result in an intersection point for values of K_1 and K_2.



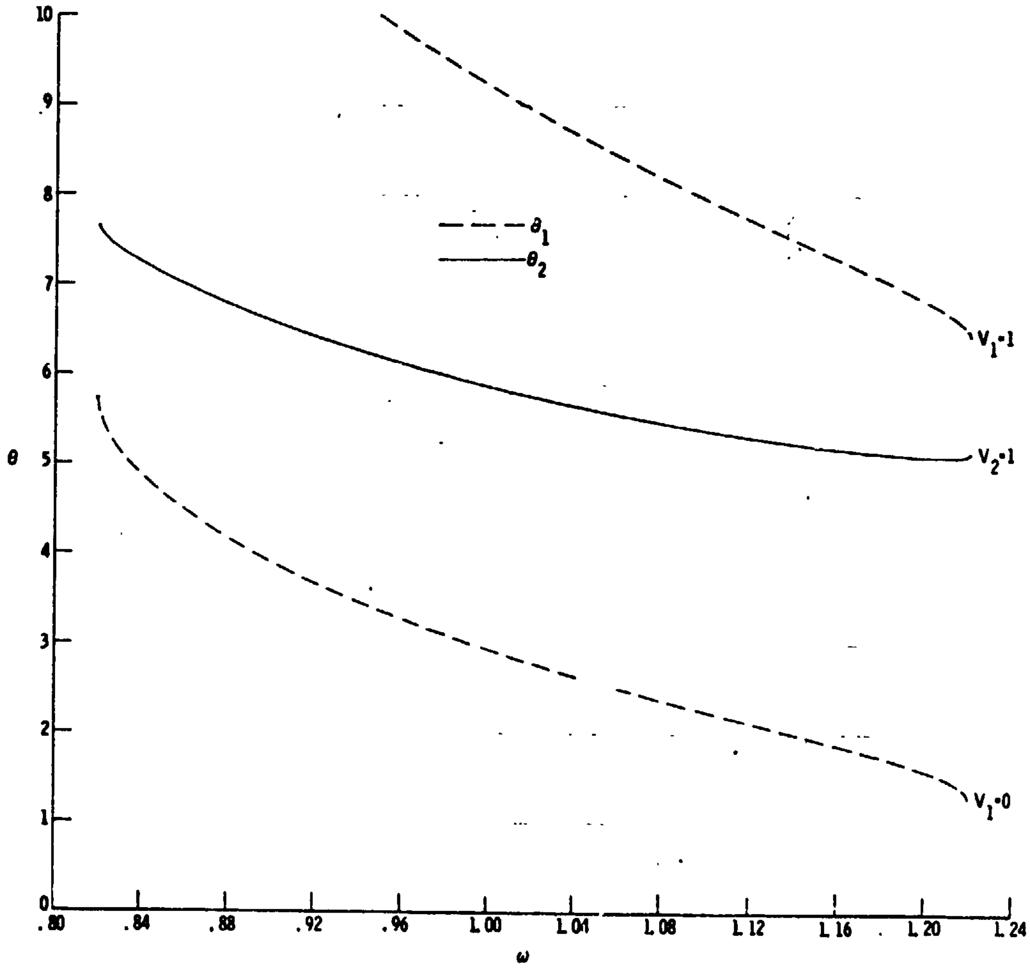
(b) $\zeta = 0.5$.

Figure 7.- Continued.



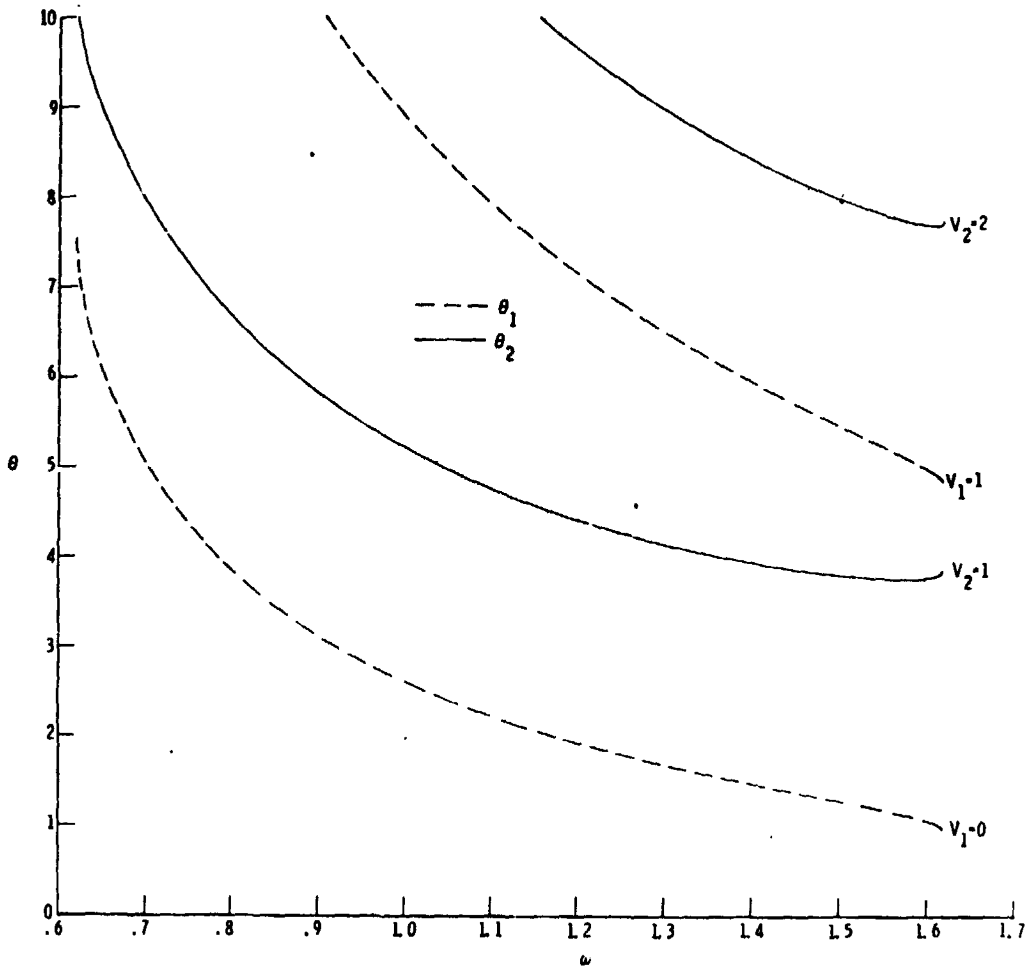
(c) $\zeta = 1$.

Figure 7.- Concluded.



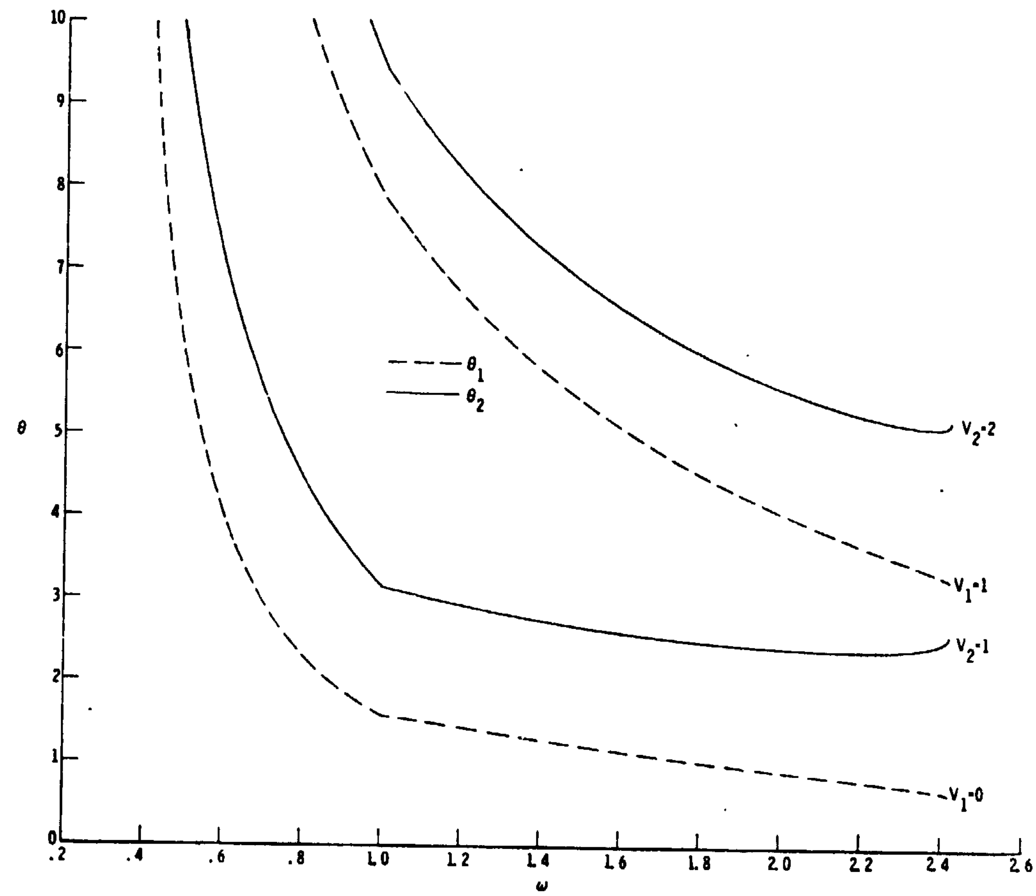
(a) $\zeta = 0.2$.

Figure 8.- Pairs of values of delays θ_1 and θ_2 which result in an intersection point for various values of V_1 and V_2.



(b) ζ = 0.5.

Figure 8.- Continued.



(c) $\zeta = 1.$

Figure 8.- Concluded

Figures 7 and 8 show the pairs of the delays θ_1 and θ_2 which result in a root-locus curve coming into contact with, or crossing, the imaginary axis in the complex root plane at ω (intersection point) for various values of (K_1, K_2) and (V_1, V_2), respectively. Results are presented for $\zeta = 0.2$, 0.5, and 1.

The consecutive K_j curves in figure 7 and the consecutive V_j curves in figure 8 both differ by $2\pi/\omega$, as shown by equations (3.54) to (3.57). Thus, the $V_2 = 0$ curve falls off the scale in figure 8.

The terminal points of the curves in figures 7 and 8 correspond to a zero delay or to a border value in table I. All border values, however, do not necessarily specify a terminal point on the curves. The value $\omega = 1$ is a border value that occurs along the curves in figures 7(c) and 8(c) and is identified by a singularity in the slope of the curves at this value.

Figures 7 and 8 can be used to obtain the points ω which occur as the delays are varied in some continuous manner from zero to their final constant values. These points clearly depend on the manner in which the delays are varied.

A partitioning of the delay space results when θ_2 is plotted directly against θ_1 with ω as a coordinating parameter, as shown in figure 9. The solid curves correspond to various values of (K_1, K_2) and are generated by using the pair of equations (3.54) and (3.55). The dashed curves correspond to various values of (V_1, V_2) and are generated by using the pair of equations (3.56) and (3.57). The totality of curves for (K_1, K_2) and (V_1, V_2) partition the delay

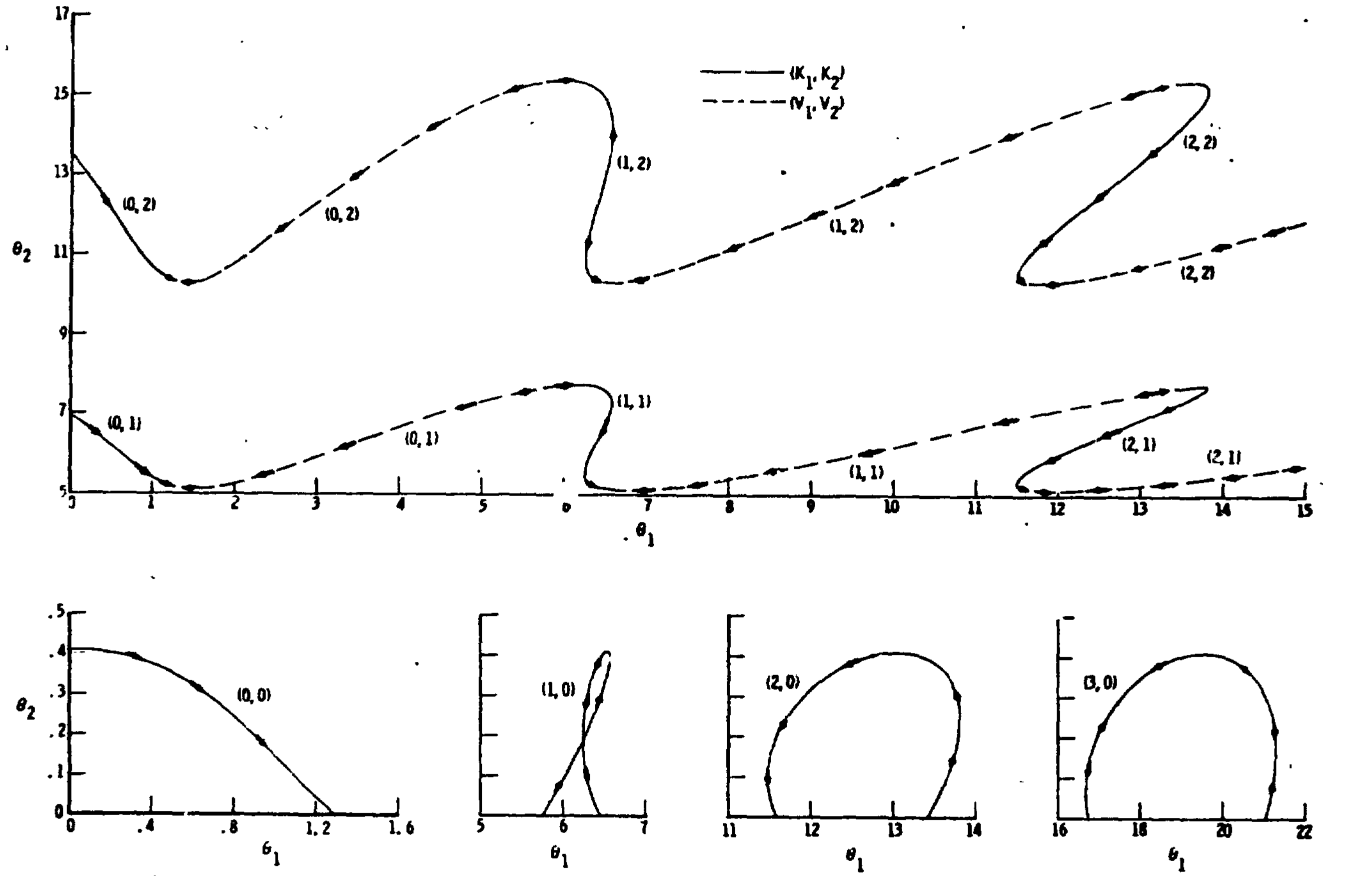


Figure 9.- A partitioning of the delay space by curves generated in the delay space for $\zeta = 0.2$ and various values of (K_1,K_2) and (V_1,V_2). Arrows indicate direction of increasing ω.

space into regions. The arrows on the curves denote the direction of increasing ω.

Since the system is initially stable, it will remain stable until the the curve $(K_1, K_2) = (0,0)$ in figure 9 is touched. To examine the stability beyond this point, the extended τ-decomposition method is used.

The number of roots with positive real parts in each region containing a point of the θ_1-axis is determined by using equations (3.11), (3.13), (3.20), and (3.21). The number of roots with positive real parts in any region containing a point on the θ_2-axis is determined by using equation (3.44), which holds at all intersection points.

Each point on the partitioning curves in figure 9 corresponds to an intersection point. Proceeding off the $(K_1, K_2) = (0, 0)$ curve along the θ_1-axis in figure 9 results in the gain of a root with positive real part. The system, therefore, becomes unstable. Entering the lower triangular region of the $(K_1, K_2) = (1, 0)$ curve from the left results in the loss of a root with positive real part, so that the system becomes stable again inside the triangular region. Upon leaving the triangular region on the right along the θ_1-axis, however, a root with positive real part is gained, and the system becomes unstable. There are no other stable regions along the θ_1-axis.

Proceeding off the $(K_1, K_2) = (0, 0)$ curve along the θ_2-axis results in the gain of a root with positive real part. A second root with positive real part is gained after crossing the $(K_1, K_2) = (0, 1)$

curve. The system accumulates an additional root with positive real part following each succeeding crossing along the θ_2-axis.

The remaining region to be examined is the upper looped portion of the curve $(K_1, K_2) = (1, 0)$. There are no intersection points for $\theta_1 = 0$ as θ_2 increases from zero to 0.3, since the curve $(K_1, K_2) = (0, 0)$ is not crossed. Hence, the system is stable for $(\theta_1, \theta_2) = (0, 0.3)$. Now, with $\theta_2 = 0.3$, let θ_1 increase from zero to some point inside the looped region. An intersection point occurs upon touching the looped region from the left. Theorem 5 is used to determine the change in the number of roots with positive real parts at this intersection point.

If $\theta_2 = 0.3$ is associated with an intersection point, then ω is either one of two values on the $K_2 = 0$ curve in figure 7. It is not necessary to know these exact values. It is sufficient to note that the maximum value $\theta_2 = 0.42$ occurs on the $K_2 = 0$ curve (figure 7(a)) when $\omega = 0.96$. Thus, the two values of ω occur on opposite sides of $\omega = 0.96$. The direction of the arrows on the $(K_1, K_2) = (1, 0)$ curve show that the value of ω at the intersection point on the left of the looped region is the one which exceeds 0.96. Hence, let $\alpha_1 = 0.96$ and $\alpha_2 = \infty$ in theorem 5 for this intersection point. The square of the magnitude of the testing function is

$$\left|W_K(0, \bar{\sigma})\right|^2 = \frac{4\zeta^2\omega^2}{\omega^4 - 2\omega^2 \cos \omega \tilde{\theta}_2 + 1} \qquad (3.65)$$

where $\tilde{\theta}_2 = 0.3$ and $\zeta = 0.2$. Note that

$$\left|W_K(0,\ 0.96)\right| > 1 \tag{3.66}$$

and

$$\left|W_K(0,\ \infty)\right| < 1 \tag{3.67}$$

Therefore, by theorem 5, the system gains a root with positive real part upon entering the looped region from the left. So, inside the looped boundary, the system has two roots with positive real part, and is unstable.

The value of ω at the exit point on the right of the looped region satisfies $0 < \omega < 0.96$. But

$$\left|W_K(0,\ 0)\right| = 0 \tag{3.68}$$

Hence, using equations (3.68). (3.66), and theorem 5 shows that leaving the looped region on the right results in the loss of a root with positive real part. However, the system had two such roots inside the looped boundary, and is, therefore, still unstable.

Thus, it has been shown that there are only two stable regions in figure 9. These are the initial stability region bounded by the curve $(K_1, K_2) = (0, 0)$ and the lower triangular region of the $(K_1, K_2) = (1, 0)$ curve.

Figure 10 shows the stability boundaries for the retarded system represented by equation (3.46) when $\zeta = 0.2$, 0.5, and 1. The hatched lines indicate the stable side of the boundaries. These analytically determined results are in agreement with those of

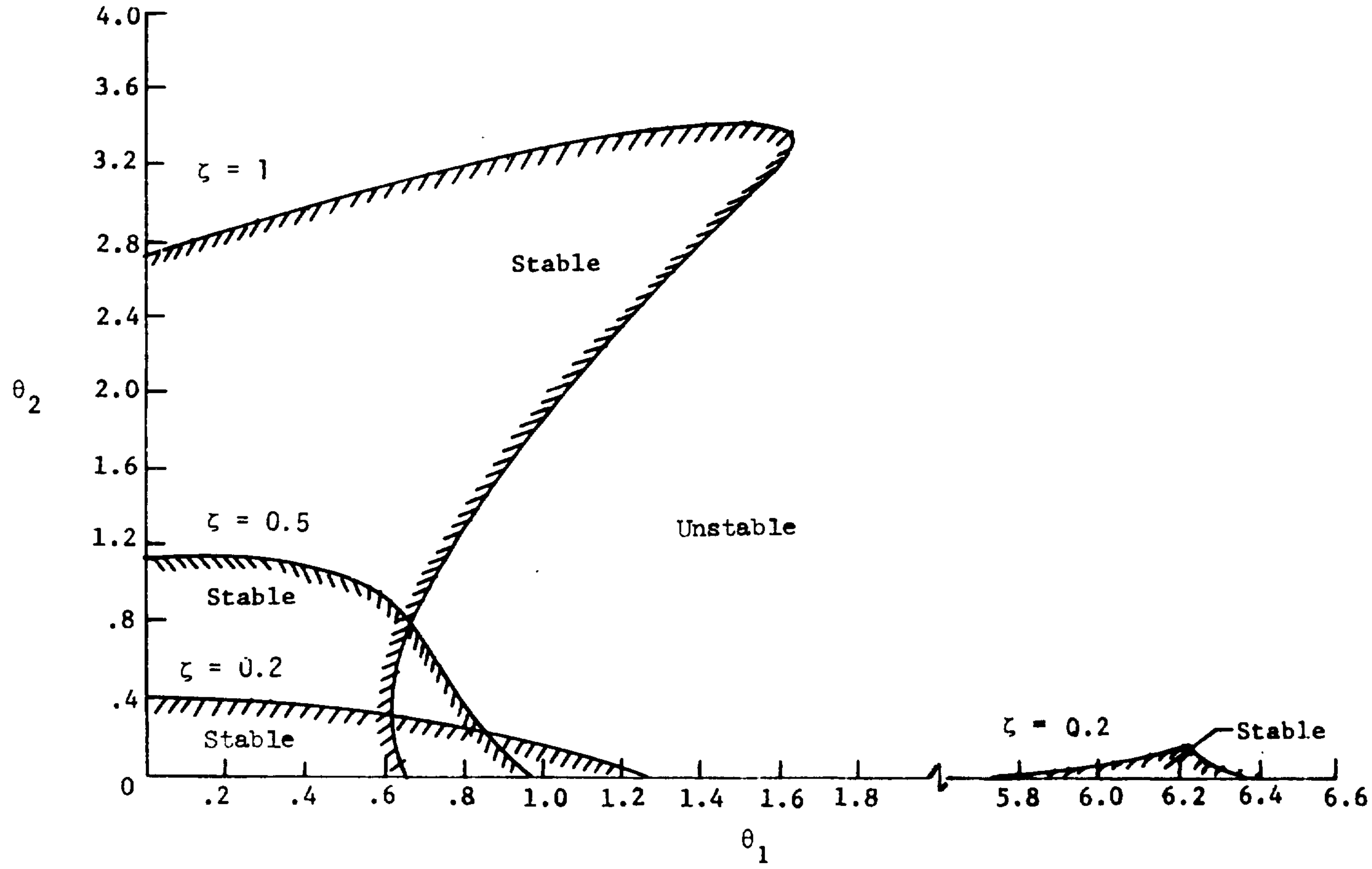


Figure 10.- Stability boundaries for $\frac{d^2}{d\tilde{t}^2} x_1(\tilde{t}) + 2\zeta \frac{d}{d\tilde{t}} x_1(\tilde{t} - \theta_1) + x_1(\tilde{t} - \theta_2) = 0$

for ζ = 0.2, 0.5, and 1.

reference 28 which were obtained by repeatedly solving transcendental equations and evaluating partial derivatives. Another advantage of the extended τ-decomposition method over the method of reference 28 is that it is not necessary to know the exact intersection points.

It is well known that the solution of equation (3.46) becomes more highly damped as ζ increases if $\theta_1 = \theta_2 = 0$. If $\theta_1 \neq 0$, this may not be the case, as shown in figure 10. For example, let $\theta_2 = 0$ and $\theta_1 = 0.8$. Then, the solution of equation (3.46) is stable for $\zeta = 0.2$ and unstable for $\zeta = 1$.

2. Relative stability boundaries

The stability boundaries ($\bar{\sigma} = 0$) in figure 10 for $\zeta = 0.2$ are shown again in figure 11, along with $\bar{\sigma} = -0.05$ relative stability boundaries. Explicit analytical equations are not available for directly generating the latter boundaries. These boundaries were computed using equations (1.30) to (1.36) and theorem 5. The delay θ_1 was held constant while $\theta_K = \theta_2$ was allowed to vary which generated points on the relative stability boundaries for values of $\bar{\sigma}$. The procedure was repeated with other values of θ_1 until enough points were obtained to clearly define the boundaries.

The hatching convention in figure 11 is as follows. Passing from the hatched (unhatched) side of a boundary line corresponding to a particular value of $\bar{\sigma}$ to the unhatched (hatched) side of the boundary results in the gain (loss) of exactly one root with $\sigma > \bar{\sigma}$.

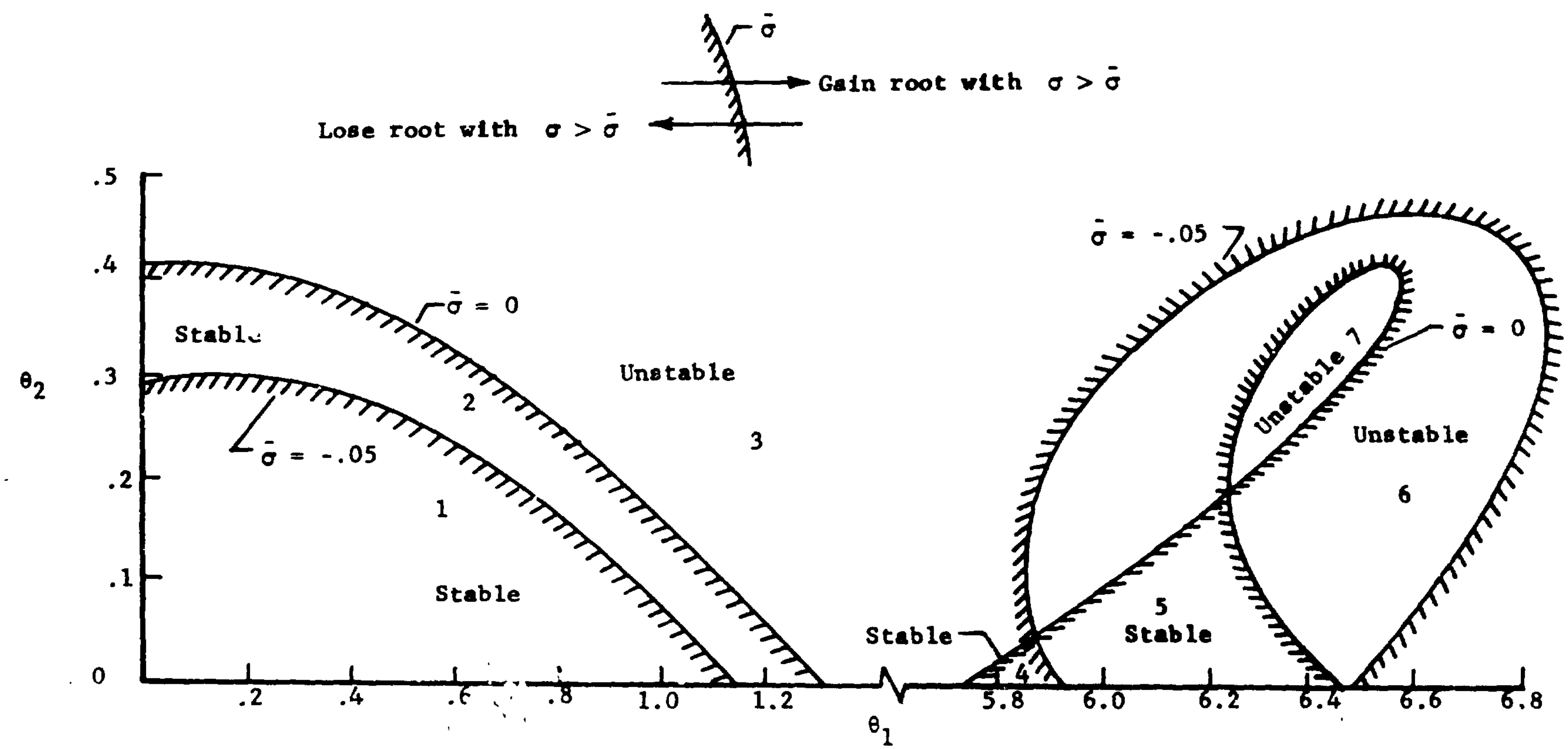


Figure 11.- Relative stability boundaries in θ_1, θ_2-plane.

The hatching convention on the boundary lines in figure 11 can be used to obtain information about the roots in each of the different regions. These results are presented in table II.

TABLE II.- ROOTS IN RELATIVE STABILITY REGIONS

Region	Number of roots with -		Stability condition
	$\sigma > 0$	$\sigma > -0.05$	
1	0	0	Stable
2	0	1	Stable
3	1	1	Unstable
4	0	1	Stable
5	0	2	Stable
6	1	2	Unstable
7	2	2	Unstable

The remaining roots not referred to in table II have $\sigma < -0.05$.

Every point on the curves in figure 11 represents an intersection point $s = \bar{\sigma} + i\omega \in \Omega_{3K} \cup \Omega_{4K}$. However, for $\bar{\sigma} \neq 0$, intersection points $s = \bar{\sigma} \in \Omega_{2K}$ may also occur according to equation (1.12). In the present example, these intersection points only occur for very large delays and, therefore, do not influence the results shown in figure 11.

CHAPTER IV

APPLICATIONS FOR NEUTRAL SYSTEMS

The techniques of Chapters I and II are applied now to two specific problems described by neutral systems.

A. Longitudinal Oscillations of a Launch Vehicle

The longitudinal oscillation of a vehicle in powered flight can lead to an oscillating thrust. The time delay between the thrust oscillation and the structural oscillation can influence the stability of the system. In connection with this problem, Glaser (reference 11) examines the stability of a neutral equation which, except for notational differences, is

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) + \eta \ddot{x}(t - \theta_K) = 0 \qquad (4.1)$$

The extended τ-decomposition method is used now to examine the asymptotic stability of equation (4.1); and advantage of this method is noted.

The characteristic quasi-polynomial associated with equation (4.1) is

$$L(s) = H_K(s) - J_K(s)\, e^{-\theta_K s} \qquad (4.2)$$

where

$$H_K(s) = (s + 2\zeta\, \omega_n)\, s + \omega_n^2 \qquad (4.3)$$

and

$$J_K(s) = -\, \eta\, s^2 \qquad (4.4)$$

Equation (4.1) is a special case of equation (1.37); hence, from equation (1.41), the real parts of the large modulus roots ($|s| \to \infty$) of L(s) approach

$$\sigma_K = -\frac{1}{\theta_K} \ln \left|\frac{1}{\eta}\right| \tag{4.5}$$

The initial stability of equation (4.1) is determined by using equation (4.5) and equation (1.6), which becomes

$$L_0(s) = s^2(1 + \eta) + 2\zeta\ \omega_n\ s + \omega_n^2 = 0 \tag{4.6}$$

For $\theta_K > 0$ and $\left|\frac{1}{\eta}\right| \leqq 1$ in equation (4.5), $\sigma_K \geqq 0$; therefore, the system is not asymptotically stable when

$$|\eta| \geqq 1 \tag{4.7}$$

For $|\eta| < 1$, the roots of equation (4.6) have negative real parts so that the system is asymptotically stable for θ_K sufficiently small ($\theta_K \to 0^+$).

The only intersection points that need be considered in this example are $s \in \Omega_{3K}$. Hence, equations (1.20), (1.21), and (1.24) and theorem 5 are used to examine the change in stability as θ_K increases. These equations, for this application, are:

$$\theta_K = -\frac{1}{\omega}\left[\tan^{-1}\left(\frac{2\zeta\ \omega_n\ \omega}{\omega_n^2 - \omega^2}\right) - 2p\pi\right]_{(p = 0, \pm 1, \ldots)} \tag{4.8}$$

$$(\omega_n^2 - \omega^2)^2 + (2\zeta\ \omega_n\ \omega)^2 = \omega^4\ \eta^2 \tag{4.9}$$

$$W_K(0, \omega) = \frac{J_K(i\omega)}{H_K(i\omega)} = \frac{\eta\omega^2}{-\omega^2 + 2\zeta\,\omega_n\,\omega i + \omega_n^2} \tag{4.10}$$

The square of the magnitude of the testing function $W_K(0, \omega)$ which is needed in theorem 5 is

$$\left|W_K(0, \omega)\right|^2 = \frac{\eta^2\,\omega^4}{(\omega_n^2 - \omega^2)^2 + (2\zeta\,\omega_n\,\omega)^2}$$

which can also be written as

$$\left|W_K(0, \omega)\right|^2 = \frac{\eta^2}{\left[\left(\frac{\omega_n}{\omega}\right)^2 - 1\right]^2 + \left[2\zeta\left(\frac{\omega_n}{\omega}\right)\right]^2} \tag{4.11}$$

Equation (4.9) can be solved for ω^2 to obtain

$$\omega^2 = \frac{-(2\zeta^2\omega_n^2 - \omega_n^2) \pm \sqrt{(2\zeta^2\omega_n^2 - \omega_n^2)^2 + \omega_n^4(1 - \eta^2)}}{1 - \eta^2} \tag{4.12}$$

or,

$$\omega^2 = \frac{-\omega_n^2(2\zeta^2 - 1) \pm \omega_n^2\sqrt{4\zeta^4 - 4\zeta^2 + \eta^2}}{1 - \eta^2} \tag{4.13}$$

There are no real values of ω which satisfy equation (4.13) when the radicand is negative. Hence, the system continues to be asymptotically stable for all $\theta_K > 0$ as long as

$$|\eta| < 2\zeta\sqrt{1 - \zeta^2} < 1 \tag{4.14}$$

The remaining values of η of interest satisfy

$$2\zeta\sqrt{\zeta^2 - 1} \leq |\eta| < 1 \tag{4.15}$$

There is exactly one finite positive real value of ω which satisfies equation (4.13) when the equality holds, that is, when

$$2\zeta\sqrt{\zeta^2 - 1} = |\eta| < 1 \tag{4.16}$$

Intersection points of root-locus curves then occur with the imaginary axis at this value of ω for any corresponding value of θ_K in equation (4.8). But, since $0 < \omega < \infty$, $|W_K(0, 0)| = 0$, and $|W_K(0, \infty)| = |\eta| < 1$, it follows by condition 3 of theorem 5 that the testing function W_K in figure 1 is tangent to the unit circle; or, equivalently, that the root-locus curves are tangent to the imagainary axis at the intersection points. Thus, for equation (4.16), the system is asymptotically stable, except at the values of θ_K which result in an intersection point.

There are two distinct positive real values of ω which satisfy equation (4.13) whenever

$$2\zeta\sqrt{\zeta^2 - 1} < |\eta| < 1 \tag{4.17}$$

These values are denoted by ω_1 and ω_2, where $\omega_1 < \omega_2$. The values of θ_K corresponding to ω_1 are denoted by

$$\theta_p = -\frac{1}{\omega_1}\left[\tan^{-1}\left(\frac{2\zeta\omega_n\ \omega_1}{\omega_n^2 - \omega_1^2}\right) - 2p\pi\right] \tag{4.18}$$

The values of θ_K corresponding to ω_2 are denoted by

$$\gamma_p = -\frac{1}{\omega_2}\left[\tan^{-1}\left(\frac{2\zeta\omega_n\,\omega_2}{\omega_n^2 - \omega_2^2}\right) - 2p\pi\right] \tag{4.19}$$

Only positive values of β_p and γ_p are of interest.

The delays β_p and γ_p are ordered in an increasing sequence. Then, the change in the stability of the system as θ_K increases across this sequence is determined by using theorem 5.

It is expedient to note that

$$0 < \omega_1 < d < \omega_2 < \infty \tag{4.20}$$

$$\left|W_K(0,\ 0)\right| = 0 \tag{4.21}$$

$$\left|W_K(0,\ d)\right| > 1 \tag{4.22}$$

$$\left|W_K(0,\ \infty)\right| = \eta < 1 \tag{4.23}$$

Hence, by theorem 5,

$$N(\beta_p + \epsilon) = N(\beta_p) - 1 \tag{4.23}$$

and

$$N(\gamma_p + \epsilon) = N(\gamma_p) + 1 \tag{4.24}$$

Therefore, the system loses a root with positive real part at the intersection points corresponding to β_p and gains such a root at intersection points corresponding to γ_p. Since the system is initially asymptotically stable $[N(0) = 0]$, it does not have a root with positive real part to lose. This means that, as θ_K increases from 0^+, the first intersection point must occur for a value of γ_p by equations (4.23) and (4.24).

The stability of the system is completely determined in an algebraic manner once particular values of ζ and ω_n are specified by using equations (4.18), (4.19), (4.23), and (4.24).

To summarize:

(1) The system is unstable for

$$|\eta| \geqq 1$$

(2) The system is asymptotically stable for

$$|\eta| < 2\zeta\sqrt{1 - \zeta^2} < 1$$

(3) The system is asymptotically stable for

$$2\zeta\sqrt{\zeta^2 - 1} = |\eta| < 1,$$

except for values of θ_K at intersection points.

(4) The system's stability condition (stable or unstable) is determined algebraically using equations (4.18) and (4.19) along with the results shown in equations (4.23) and (4.24) for

$$2\zeta\sqrt{\zeta^2 - 1} < |\eta| < 1$$

Using the Nyquist criteria, Glaser (reference 11) obtained conditions (1) and (2) for positive values of η. The stability condition for other values of η required the calculation of the roots of a transcendental equation for particular values of ζ and

ω_n; whereas, conditions (3) and (4) are completely algebraic.

4.2 Airplane yaw damper control system

The yaw damper control system examined by Beckhardt (reference 12) requires that the airplane rudder deflection be proportional to the yawing acceleration which has occurred in the recent past. The equations used by Beckhardt, except for some notational changes, are:

$$e_2 \ddot{\psi}(t) + e_1 \dot{\psi}(t) + e_0 \psi(t) = C_{n_{\delta_r}} \delta_r(t) \tag{4.23}$$

$$\delta_r(t) = \xi \ddot{\psi}(t - \theta_K) \tag{4.24}$$

$$e_2 = \frac{I_z}{QSb} \tag{4.25}$$

$$e_1 = - C_{n_r} \tag{4.26}$$

$$e_0 = C_{n_\beta} \tag{4.27}$$

where θ_K is a constant time delay, and ξ is a constant gain. Equation (4.24) can be substituted into equation (4.23) to obtain

$$e_2 \ddot{\psi}(t) + e_1 \dot{\psi}(t) + e_0 \psi(t) - C_{n_{\delta_r}} \xi \ddot{\psi} (t - \theta_K) = 0 \tag{4.28}$$

Equation (4.28) is the same as equation (4.1) with the substitutions:

$$2\zeta \omega_n = \frac{e_1}{e_2} \tag{4.29}$$

$$\omega_n^2 = \frac{e_0}{e_2} \tag{4.30}$$

$$\eta = \frac{-C_{n_{\delta_r}}}{e_2} \tag{4.31}$$

Hence, the discussion in Section 4.1 also applies to equation (4.28). Stability ($\bar{\sigma} = 0$) and relative stability ($\bar{\sigma} \neq 0$) boundaries are generated in the plane of ξ and θ_K by means of the extended τ-decomposition method in conjunction with the parameter-plane method.

The characteristic quasi-polynomial for equation (4.28) can be written as

$$L(s) = P(s) + \xi\, Q(s)\, e^{-\theta_K s} \tag{4.32}$$

where

$$P(s) = e_2 s^2 + e_1 s + e_0 \tag{4.33}$$

and

$$Q(s) = -C_{n_{\delta_r}} s^2 \tag{4.34}$$

or, as

$$L(s) = H_K(s) - J_K(s)\, e^{-\theta_K s} \tag{4.35}$$

where

$$H_K(s) = e_2 s^2 + e_1 s + e_0 \tag{4.36}$$

and

$$J_K(s) = \xi\, C_{n_{\delta_r}} s^2 \tag{4.37}$$

Equations (4.33) and (4.34) are used in equations (2.30) and (2.31) to partition the ξ, θ_K- plane into different regions; whereas, equations (4.36) and (4.37) are used in equations (1.34) and (1.35) in applying the extended τ-decomposition method. The equations were solved on a digital computer by incrementing ω in $s = \bar{\sigma} + i\omega$, with $\bar{\sigma}$ specified.

The initial stability of equation (4.28) along the ξ-axis ($\theta_K \to 0^+$) is evaluated by using equations (2.43) and (1.6), which become:

$$\sigma_K = -\frac{1}{\theta_K} \ln \left| \frac{e_2}{\xi\, C_{n_{\delta_r}}} \right| \tag{4.38}$$

and

$$L_0(s) = e_2\, s^2 + \epsilon_1\, s + e_0 - C_{n_{\delta_r}}\, \xi\, s^2 = 0 \tag{4.39}$$

Specific Calculations.- Constants used by Beckhardt are: $e_2 = 0.01024$, $e_1 = 0.00704$, $e_0 = 0.250$, and $C_{n_{\delta_r}} = -0.163$. With these constants, equations (4.38) and (4.39) become:

$$\sigma_K = -\frac{1}{\theta_K} \ln \left| \frac{0.01024}{0.163\, \xi} \right| \tag{4.40}$$

and

$$L_0(s) = (0.01024 + 0.163\, \xi) s^2 + 0.00704\, s + 0.250 = 0 \tag{4.41}$$

The solid curves in figure 12 are partitioning curves. For constant ξ, the extended τ-decomposition method was used to determine which side of the curves should be hatched. Each point on a partitioning curve corresponds to an intersection point.

The number of roots in each region of figure 12 with $\sigma > \bar{\sigma}$ is determined by using the hatching on the curves and equations (4.40) and (4.41). For example, let $\xi = 0.04$. Then, by equation (4.40) and

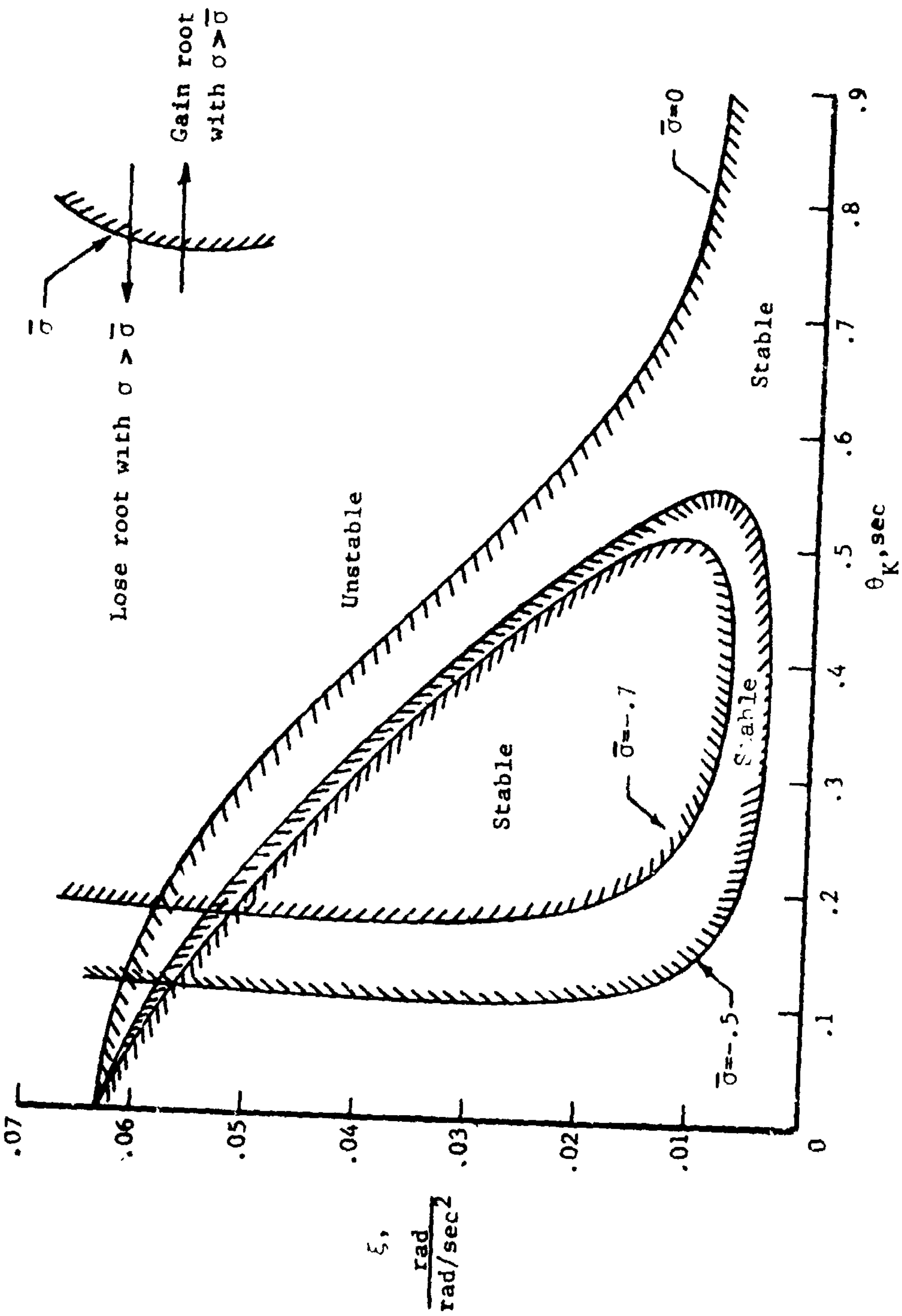


Figure 12.- Relative stability boundaries in ξ, θ_K-plane.

the roots of equation (4.41), the system has one root with $\sigma = -0.21$ when $\theta_K \to 0^+$, and the remaining roots have arbitrarily large negative real parts. Now, as θ_K increases with $\xi = 0.04$ fixed, the boundary line for $\bar{\sigma} = -0.5$ is crossed, and the system loses a root with $\sigma > -0.5$. (This is the root which originally had $\sigma = -0.21$.) At the point $(\xi, \theta_K) = (0.4, 0.2)$, the system has no roots with $\sigma > -0.7$. The value of σ_K in equation (4.40) at this point is $\sigma_K = -2.257$.

Every point on the partitioning curves in figure 12 corresponds to an intersection point $s \in \Omega_{3K} \cup \Omega_{4K}$. The intersection points $s = \bar{\sigma} \in \Omega_{2K}$ need not be considered since real values of θ_K in equation (2.33) do not exist for the present application.

Beckhardt (reference 12) uses cross plotting to construct a region in the gain-delay space. The region constructed corresponds to values of gain and delay which satisfy a criterion for satisfactory damping of short-period lateral oscillations. This region is essentially that of the $\bar{\sigma} = -0.5$ curve in figure 12. No discussion of stability is given by Beckhardt.

A more thorough analysis of the control gearing and time delay necessary for determining a specified damping of the motions of an aircraft equipped with an autopilot having constant-time-lag characteristics is presented in reference 31. However, the gain and loss of roots as the boundary lines in the gain-delay space are crossed is not discussed. The extended τ-decomposition method can be easily applied for this purpose.

CHAPTER V

CONCLUDING REMARKS

Differential-difference equations are used to describe the dynamic behavior of physical systems with time delays. One of the major problems involved in dealing with differential-difference equations is the stability analysis. The reason for this is that the characteristic equation of the system is transcendental and, therefore, has an infinite number of roots.

A method of determining the stability of linear systems with many constant time delays has been developed. The τ-decomposition method of stability analysis has been extended and used to examine not only the stability but also the relative stability of retarded systems with many delays and of a class of neutral equations with one delay.

Analytical equations have been derived for partitioning the delay space of a retarded system with two time delays. The stability of the system in each of the regions defined by the partitioning curves in the parameter plane has been determined using the extended τ-decomposition method. In addition, relative stability boundaries of the retarded system have been defined using the extended τ-decomposition method in association with parameter plane techniques.

Several applications of the extended τ-decomposition method have been presented and compared with stability results obtained from other analyses. In all cases the stability analyses obtained using the method outlined herein have been shown to coincide with and extend those of previous investigations.

The extended τ-decomposition method applied to systems with time delays has been shown to require less computational effort and yield more complete stability analyses than previous techniques.

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VITA

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APPENDIX A

EXAMPLES OF EXPANSIONS OF CHARACTERISTIC QUASI-POLYNOMIAL

Example 1: Consider the following scalar differential equation with two constant real time delays $\tau_1 \geqq 0$ and $\tau_2 \geqq 0$:

$$\sum_{n=0}^{N} a_n\, x^{(n)}(t) + \sum_{n=0}^{N-1} b_n\, x^{(n)}(t - \tau_1) + \sum_{n=0}^{N-1} c_n\, x^{(n)}(t - \tau_2) = 0 \quad (B1)$$

where $x^{(n)}(t)$ denotes the nth derivative of $x(t)$. The coefficients a_n, b_n, and c_n are real constants. Equation (B1) can be written in the form of equation (1); however, this is unnecessary in obtaining the characteristic quasi-polynomial, which is

$$L(s) = \sum_{n=0}^{N} a_n\, s^n + \left(\sum_{n=0}^{N-1} b_n\, s^n\right) e^{-\tau_1 s} + \left(\sum_{n=0}^{N-1} c_n\, s^n\right) e^{-\tau_2 s} \quad (B2)$$

Let

$$\theta_1 = \tau_1$$

$$\theta_2 = \tau_2$$

Then, equation (B2) can be expressed as

$$L(s) = H_K(s) - J_K(s)\, e^{-\theta_K s}$$

$$(K = 1,\ 2)$$

where

$$H_1(s) = \sum_{n=0}^{N} a_n s^n + \left(\sum_{n=0}^{N-1} c_n s^n \right) e^{-\theta_2 s}$$

$$J_1(s) = - \sum_{n=0}^{N-1} b_n s^n$$

$$H_2(s) = \sum_{n=0}^{N} a_n s^n + \left(\sum_{n=0}^{N-1} b_n s^n \right) e^{-\theta_1 s}$$

$$J_2(s) = - \sum_{n=0}^{N-1} c_n s^n$$

Example 2: Let the system be described by two coupled linear equations with one constant time delay $\tau \geqq 0$ as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_1(t - \tau) \\ x_2(t - \tau) \end{bmatrix} \tag{B3}$$

The characteristic quasi-polynomial associated with equation (B3) can be written as

$$L(s) = P(s) + Q(s)\, e^{-\tau s} + R(s)\, e^{-2\tau s} \tag{B4}$$

where

$$P(s) = s^2 - (a_{11} + a_{22})s + a_{11} a_{22} - a_{21} a_{12}$$

$$Q(s) = - (b_{11} + b_{22})s + a_{11} b_{22} + a_{22} b_{11} - a_{21} b_{12} - a_{12} b_{21}$$

$$R(s) = b_{11} b_{22} - b_{21} b_{12}$$

Now, let

$$\theta_1 = \tau$$

$$\theta_2 = 2\tau$$

Then, equation (B4) can be written as

$$L(s) = H_K(s) - J_K(s)\, e^{-\theta_K s} \qquad (K = 1,\ 2)$$

where

$$H_1(s) = P(s) + R(s)\, e^{-\theta_2 s}$$

$$J_1(s) = -\ Q(s)$$

$$H_2(s) = P(s) + Q(s)\, e^{-\theta_1 s}$$

$$J_2(s) = -\ R(s)$$

Note that although there is only one time delay τ in equation (B3), there can be two delays θ_1 and θ_2 in the associated quasi-polynomial.

A special case of equation (B3) occurs in examining airplane stability for the controls-fixed case of reference 6 which is modeled in reference 13. Here, τ accounts for the fact that there is a time delay between the occurrence of a given angle of attack of the wing and the occurrence of the associated downwash at the tail (reference 33). It can be shown that the characteristic quasi-polynomial for the controls-free case has the form of equation (B4), except that the polynomials P(s), Q(s), and R(s) are of higher degrees.

<u>Example 3:</u> Let the differential-difference equation be

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{x}(t - \tau_1) + C\vec{x}(t - \tau_2) \tag{B5}$$

where

$$A = -\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and $\tau_1 \geqq 0$ and $\tau_2 \geqq 0$ are constant time delays.

The characteristic quasi-polynomial associated with equation (B5) can be expressed as

$$L(s) = (s + 1)^2 - (1 - e^{-\tau_1 s})(1 - e^{-\tau_2 s}) \tag{B6}$$

or

$$L(s) = (s + 1)^2 - 1 + e^{-\tau_1 s} + e^{-\tau_2 s} + e^{-(\tau_1 + \tau_2)s} \tag{B7}$$

Setting $\theta_1 = \tau_1$, $\theta_2 = \tau_2$, and $\theta_3 = \tau_1 + \tau_2$ in equation (B7) results in three delays; whereas, setting $\theta_1 = \tau_1$ and $\theta_2 = \tau_2$ in equation (B6) only results in two delays. The latter is the better choice, because the number of computations necessary to determine the stability of a system with delays increases with the total number of θ_j delays. Equation (B6) can be expressed as

$$L(s) = H_K(s) - J_K(s)\, e^{-\theta_K s}$$

$$(K = 1, 2)$$

where

$$H_1(s) = s^2 + 2s + e^{-\tau_2 s}$$

$$J_1(s) = e^{-\tau_2 s} - 1$$

$$H_2(s) = s^2 + 2s + e^{-\tau_1 s}$$

$$J_2(s) = e^{-\tau_1 s} - 1$$

APPENDIX B

PROOF OF THEOREMS

<u>Theorem 1:</u> The finite roots of the quasi-polynomial $L(s)$ are continuous functions of the delays θ_j, where $j = 1, 2, \ldots, M$.

<u>Proof:</u> The quasi-polynomial $L(s)$ is written as $L(s;\ \theta_j)$ to indicate the dependence of $L(s)$ on θ_j. The quasi-polynomial $L(s;\ \theta_j)$ is an entire function, so that its roots in the finite s-plane are isolated. Let s^* be any one of the roots of $L(s;\ \theta_j)$. Let ρ be a positive number such that the only root of $L(s;\ \theta_j)$ contained inside the circle $|s - s^*| = \rho$ is s^* and there are no roots of $L(s;\ \theta_j)$ and $L(s;\ \theta_j + \Delta\theta_j)$ on this circle. A Taylor series expansion of the exponential term $e^{-\theta_j s}$ can be used to show that for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$\left|L(s;\ \theta_j) - L(s;\ \theta_j + \Delta\theta_j)\right| < \epsilon$$

for $|s - s^*| \leqq \rho < \epsilon$ whenever $|\Delta\theta_j| < \delta(\epsilon)$. Let $\epsilon < \min\left|L(s;\ \theta_j)\right|$ on $|s - s^*| = \rho$, then

$$\left|L(s;\ \theta_j) - L(s;\ \theta_j + \Delta\theta_j)\right| < \epsilon < \left|L(s;\ \theta_j)\right|$$

for $|\Delta\theta_j|$ sufficiently small and $|s - s^*| = \rho$. By Rouche's theorem (reference 32), $L(s;\ \theta_j)$ and $L(s;\ \theta_j + \Delta\theta_j)$ have the same number of roots inside the circle $|s - s^*| = \rho < \epsilon$. Thus, if s_1 is a root of $L(s;\ \theta_j + \Delta\theta_j)$ inside the circle, then $|s_1 - s^*| < \epsilon$ whenever $|\Delta\theta_j| < \delta(\epsilon)$. Therefore, the roots of $L(s;\ \theta_j)$ move continuously with θ_j.

<u>Theorem 2:</u> Let $L_0(s)$ be the Nth-order polynomial obtained by setting all the delays in the quasi-polynomial $L(s)$ equal to zero. Then, for sufficiently small values of the delays θ_j, $L(s;\ \theta_j)$ has N roots which are arbitrarily close to the N roots of $L_0(s)$. In addition, any other roots of $L(s;\ \theta_j)$ have arbitrarily large moduli for θ_j sufficiently small.

<u>Proof:</u> In the proof, $L(s) = L(s;\ \theta_j)$ and $L_0(s) = L(s,\ 0)$. Choose a positive constant λ such that the N roots of $L(s,\ 0)$ lie inside the circle $|s| = \lambda$ and $L(s;\ \theta_j) \neq 0$ on this circle. Taylor series expansions of the $e^{-\theta_j s}$ terms show that for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\left|L(s,\ 0) - L(s,\ \theta_j)\right| < \epsilon$$

for $|s| \leq \lambda$ and $|\theta_j| < \delta(\epsilon)$, for every j. Let $\epsilon < \min \left|L(s,\ 0)\right|$ on $|s| = \lambda$. Then

$$\left|L(s;\ 0) - L(s;\ \theta_j)\right| < \epsilon < \left|L(s,\ 0)\right|$$

for $|s| = \lambda$ and all θ_j sufficiently small. By Rouche's theorem (reference 32), $L(s,\ 0)$ and $L(s,\ \theta_j)$ have the same number of roots inside the circle $|s| = \lambda$.

Let s^* be any one of the roots of $L(s,\ 0)$ inside $|s| = \lambda$, and let ρ be a positive number such that the only root of $L(s,\ 0)$ contained inside or on the contour $|s - s^*| = \rho$ is s^*. Now, as before, it can be shown that there exists a $\delta(\epsilon) > 0$ such that whenever $|\theta_j| < \delta(\epsilon)$, for every j,

$$\left|L(s, 0) - L(s, \theta_j)\right| < \left|L(s, 0)\right|$$

for $|s - s^*| = \rho$. Thus, by Rouche's theorem, $L(s, \theta_j)$ and $L(s, 0)$ have the same number of roots inside the contour $|s - s^*| = \rho$.

Following the same procedure for each root of $L(s, 0)$ shows that $L(s, \theta_j)$ has N roots which are as close as desired to the N roots of $L(s, 0)$ for θ_j sufficiently small.

All other roots of $L(s, \theta_j)$ lie outside the contour $|s| = \lambda$, and λ can be made as large as desired by making θ_j sufficiently small.

Theorem 3: The arbitrarily large modulus roots of the quasi-polynomial $L(s)$ of a retarded system have negative real parts.

Proof: It is more expedient to use equation (1.2) than equation (1.3) in this proof. Thus, if $s = \sigma + i\omega$ is a root of $L(s)$, then

$$\det\left(sI - A - \sum_{\ell=1}^{T} A_\ell e^{-\tau_\ell s}\right) = 0 \tag{A1}$$

Divide equation (A1) by $s \neq 0$ to get

$$\det\left(I - \frac{A}{s} - \sum_{\ell=1}^{T} A_\ell \frac{e^{-\tau_\ell s}}{s}\right) = 0 \tag{A2}$$

Now, as $|s| \to \infty$, there are three possibilities for the behavior of σ. These are:

Case 1: $\sigma \to \sigma^*$ (a finite number)

Case 2: $\sigma \to \infty$

Case 3: $\sigma \to -\infty$

For case 1, useful relationships are

$$\lim_{|s|\to\infty}\left|\frac{1}{s}\right| = 0 \qquad \text{(A3)}$$

and

$$\lim_{\substack{|s|\to\infty\\ \sigma\to\sigma^*}}\left|\frac{e^{-\tau_\ell s}}{s}\right| = \lim_{\sigma\to\sigma^*} e^{-\tau_\ell \sigma} \cdot \lim_{|s|\to\infty}\left|\frac{1}{s}\right| = e^{-\tau_\ell \sigma^*} \cdot 0 = 0$$

Thus, det (I) → 0 in equation (A2). However, this is a contradiction, since det (I) = 1.

For case 2, equation (A3) holds and, since $\tau_\ell \geqq 0$,

$$\lim_{\substack{|s|\to\infty\\ \sigma\to\infty}}\left|\frac{e^{-\tau_\ell s}}{s}\right| = \lim_{\sigma\to\infty} e^{-\tau_\ell \sigma} \cdot \lim_{|s|\to\infty}\left|\frac{1}{s}\right| = 0 \cdot 0 = 0$$

As for case 1, these relationships imply the contradictory result that det (I) → 0.

The only remaining possibility is case 3, so that $\sigma \to -\infty$ as $|s| \to \infty$.

A consequence of Rouche's theorem is Hurwitz's theorem. Krall (reference 22) considers a quasi-polynomial with one time delay and a system gain and states that from Hurwitz's theorem the points on the root locus curve are continuous functions of the gain. Rouche's theorem is used directly in theorem 1 to show that the finite roots of L(s) are continuous functions of the delays. The method of proof in theorem 2 was suggested by El'sgolt's (reference 16), and theorem 3 is a generalization of a theorem by Shaughnessy and Kashawagi (reference 28).

Theorem 5: Let $s = \bar{\sigma} + i\omega \in \Omega_{3K} \cup \Omega_{4K}$ with $\omega > 0$ be an intersection point with corresponding delay θ_K; or, equivalently, let ω and θ_K be simultaneous solutions of equations (1.14) and (1.15). Let $\alpha_1 < \omega$ and $\alpha_2 > \omega$ be real numbers for which $W_K(\bar{\sigma}, \alpha_1)$ and $W_K(\bar{\sigma}, \alpha_2)$ are defined; and such that there are no other intersection points with imaginary parts which lie on the intervals $[\alpha_1, \omega]$ and $[\omega, \alpha_2]$. Now, for ϵ an arbitrarily small positive number

(1) $N(\theta_K + \epsilon, \bar{\sigma}) = N(\theta_K, \bar{\sigma}) + 1$

if $|W_K(\bar{\sigma}, \alpha_1)| > 1$ and $|W_K(\bar{\sigma}, \alpha_2)| < 1$;

(2) $N(\theta_K + \epsilon, \bar{\sigma}) = N(\theta_K, \bar{\sigma}) - 1$

if $|W_K(\bar{\sigma}, \alpha_1)| < 1$ and $|W_K(\bar{\sigma}, \alpha_2| > 1$; and

(3) $N(\theta_K + \epsilon, \bar{\sigma}) = N(\theta_K, \bar{\sigma})$

if both $|W_K(\bar{\sigma}, \alpha_1)|$ and $|W_K(\bar{\sigma}, \alpha_2)|$ are greater than 1, or both less than 1.

Proof: Let $s = \bar{\sigma} + i\omega^* \in \Omega_{3K} \cup \Omega_{4K}$ with $\omega^* > 0$ be a particular intersection point with corresponding delay θ_K^*; or equivalently, let $\omega = \omega^*$ and $\theta_K = \theta_K^*$ be simultaneous solutions of equations (1.14) and (1.15). Let $\alpha_1 < \omega^*$ and $\alpha_2 > \omega^*$ be real numbers for which $W_K(\bar{\sigma}, \alpha_1)$ and $W_K(\bar{\sigma}, \alpha_2)$ are defined; and such that there are no other intersection points with imaginary parts which lie on the intervals $[\alpha_1, \omega^*]$ and $[\omega^*, \alpha_2]$.

If $|W_K(\bar{\sigma}, \alpha_1)| > 1$ and $|W_K(\bar{\sigma}, \alpha_2)| < 1$, then the testing function $W_K(\bar{\sigma}, \omega) = W_K(\bar{\sigma} + i\omega)$ enters the unit circle $e^{i\omega\theta_K}$ as ω increases across ω^*. From equations (1.24), (1.25), and (1.28), it can be seen that $W_K(\bar{\sigma}, \omega) = W_2(i\omega)$ and $\theta_K = \tau$. This implies that $W_2(i\omega)$ enters the unit circle $e^{i\omega\tau} = e^{i\omega\theta_K}$ as ω increases across ω^*. But, by theorem 4, this means $N(\tau)$ increases by 1 as τ increases across τ^*. However, since $s = z + \bar{\sigma}$, this means $N(\theta_K, \bar{\sigma})$ increases by 1 as θ_K increases across θ_K^*; that is, $N(\theta_K^* + \epsilon, \bar{\sigma}) = N(\theta_K^*, \bar{\sigma}) + 1$, where ϵ is an arbitrarily small positive number. If θ_K^* is considered to be θ_K from equation (1.15) and ω^* is considered to be ω from equation (1.14) [or equation (1.17)], then the asterisk can be removed from θ_K^* and ω^*. Therefore, item 1 of the theorem has been established. Items 2 and 3 follow in a similar manner.

APPENDIX C

UPPER BOUND ω_m

In general, the values of ω which correspond to an intersection point must be found by an iteration process. In this respect, it is useful to restrict the possible values of ω to a finite interval $[0, \omega_m]$, where ω_m is an upper bound on ω. The purpose of this appendix is to derive a polynomial equation, from which ω_m can be computed.

A. Retarded Systems

Let $s = \bar{\sigma} + i\omega$, where $\bar{\sigma} \leqq 0$ is a specified constant. Then, equation (1.3), (1.4), and (1.5) can be used to show that

$$|a_N||s|^N \leqq \sum_{n=0}^{N-1} \left[|a_n| + |c_{Knj}| e^{-\bar{\theta}_K \bar{\sigma}} + \sum_{\substack{j=1 \\ j \neq K}}^{M} |b_{Knj}| e^{-\bar{\theta}_j \bar{\sigma}} + |d_{Knj}| e^{-(\bar{\theta}_j + \bar{\theta}_K)\bar{\sigma}} \right] |s|^j \tag{C1}$$

where $\theta_j \leqq \bar{\theta}_j$ $(j = 1, 2, \ldots, M)$. The largest value of $|s|$ which can satisfy equation (C1) is the largest non-negative real root $|s|$ of the polynomial equation

$$|a_N| \; |s|^N - \sum_{n=0}^{N-1} \left[|a_n| + |c_{Knj}| e^{-\bar{\theta}_K \bar{\sigma}} + \sum_{\substack{j=1 \\ j \neq K}}^{M} |b_{Knj}| e^{-\bar{\theta}_j \bar{\sigma}} + |d_{Knj}| e^{-(\bar{\theta}_j + \bar{\theta}_K)\bar{\sigma}} \right] |s|^j = 0 \tag{C2}$$

Denote this largest root by $|s|_m$. Then, since $|s|^2 = (\bar{\sigma})^2 + \omega^2 \leqq |s|_m^2 = (\bar{\sigma})^2 + \omega_m^2$, it follows that

$$\omega \leqq \omega_m = \sqrt{|s|_m^2 - (\bar{\sigma})^2} \tag{C3}$$

B. Neutral System

The neutral system under consideration is governed by equation (1.37). The characteristic quasi-polynomial is given by equation (1.38), which can be used to write

$$\left| |a_N| - |b_N| e^{-\bar{\sigma}\theta_K} \right| |s|^N \leqq \sum_{j=0}^{N-1} \left(|a_j| + |b_j| e^{-\bar{\sigma}\theta_K} \right) |s|^j \tag{C4}$$

where $s = \bar{\sigma} + i\omega$. For specified values of $\bar{\sigma}$ and θ_K, the largest value of $|s|$ which can satisfy the inequality in equation (C4) is the largest non-negative real root (if it exists) of the polynomial equation

$$\left| |a_N| - |b_N| e^{-\bar{\sigma}\theta_K} \right| |s|^N - \sum_{j=0}^{N-1} \left(|a_j| + |b_j| e^{-\bar{\sigma}\theta_K} \right) |s|^j = 0 \tag{C5}$$

Call this largest real root $|s|_m$, then

$$\omega \leqq \omega_m \leqq \sqrt{|s|_m^2 - (\bar{\sigma})^2} \tag{C6}$$

Equation (C5) can be used also to obtain an upper bound on ω for a range of parameters. As an example, let $0 \leqq \theta_K \leqq \bar{\theta}_K$, and $c \leqq b_N \leqq d$, $\bar{\sigma} \leqq 0$, and $|a_N| > |d| e^{-\bar{\sigma}\bar{\theta}_K}$. Then, for $\bar{\sigma} \leqq 0$, it is not difficult to see that the maximum value of $|s|$ which satisfies equation (C5) is certainly no larger than the maximum value of $|s|$ which satisfies the polynomial equation:

$$\left(\left|a_N\right| - \left|d\right| e^{-\bar{\sigma}\,\bar{\theta}_K}\right)\left|s\right|^N - \sum_{j=0}^{N-1}\left(\left|a_j\right| + \left|b_j\right| e^{-\bar{\sigma}\,\bar{\theta}_K}\right)\left|s\right|^j = 0 \qquad \text{(C7)}$$

Hence, for the range of parameters mentioned, an upper bound on ω can be obtained by setting $|s|_m$ in equation (C6) equal to the largest root $|s| \geqq 0$ of equation (C7).