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A ONE-EQUATION MODEL OF TURBULENCE FOR USE WITH THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Moffett Field, California

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16. Abstract

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SYMBOLS

A	van Driest damping coefficient
C	modeling constant in term representing turbulence dissipation, equations (2) and (14)
c_p	specific heat at constant pressure
D	turbulence function, equation (8)
e	specific kinetic energy of turbulence in incompressible flow
H	function of turbulence Reynolds number, equation (10)
$\tilde{H}$	mass-averaged stagnation enthalpy of fluid
h''	mass-averaged fluctuating static enthalpy
K	Kármán constant in mixing length formulation
K_1	modeling constant in term representing the redistribution between Reynolds stresses
$\bar{k}$	mass-averaged specific turbulence kinetic energy, $\overline{\rho u_i'' u_i''}/\bar{\rho}$
L	length scale of turbulence
ℓ	mixing length
M	turbulence function, equation (7)
$\bar{p}$	mean local static pressure
p'	fluctuating local static pressure
q_i	instantaneous heat flux in ith direction by molecular means
r	Reynolds number of turbulence
r_o	empirical modeling constant, equation (13)
S	correlation length used by Glushko
S_{ij}	instantaneous rate of strain tensor
$\tilde{S}_{iu}$	mass-averaged mean rate of strain tensor
S''_{ij}	mass-averaged fluctuating rate of strain tensor
T	instantaneous temperature

$\tilde{T}$	mass-averaged mean temperature
T''	fluctuating temperature in mass-averaged coordinates
$\bar{u}$, $\bar{u}_1$	mean velocity in streamwise direction
$\tilde{u}_i$	mass-averaged mean velocity in ith direction
u'	fluctuating velocity in streamwise direction
u_i'	fluctuating velocity in ith direction
u_i''	mass-averaged fluctuating velocity in ith direction
$\bar{v}$, $\bar{u}_2$	mean velocity normal to a surface
v'	fluctuating velocity normal to a surface
x, y / x_1, x_2	coordinates in directions along and normal to a surface
α	empirical modeling constant, equation (12)
Γ	modeling constant related to a turbulent Prandtl number
δ	viscous layer thickness
δ_{ik}	Kronecker delta, $i = k$, $\delta_{ii} = 1$; $i \neq k$, $\delta_{ik} = 0$
ε	ratio of eddy diffusivity to kinematic viscosity
κ	modeling constant in term representing turbulence dissipation, equation (2)
λ	empirical modeling constant, equations (8) and (15)
ν	kinematic viscosity on incompressible fluid
ν_t	eddy diffusivity
$\bar{\rho}$	mean density in a compressible fluid
$\underline{\rho}$	density of an incompressible fluid
τ_{ij}	instantaneous stress caused by molecular motions (viscosity)
ζ	modeling constant in compressibility term, equation (66)
$(\ )_{,k}$	partial differentiation in kth direction
$(\bar{\ })$	time-averaged quantity

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SUMMARY

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INTRODUCTION

Recent advances in both computer technology and numerical analysis permit the evaluation of compressible flow fields by numerically solving the complete Navier-Stokes equations, including the energy equation for evaluating variable fluid properties (e.g., ref. 1). Thus, flow fields containing separated regions, which hitherto had to be treated by iterative procedures that coupled essentially viscous- and inviscid-flow regions, can be treated as a single field and solved with reasonable computer economy (refs. 2-5). A difficulty arises, however, when these techniques are applied to turbulent flows through the Reynolds stress formalism. For separated flows caused by adverse pressure gradients, the comparatively large pressure gradients, both along and normal to body surfaces, and the presence of the separated-flow region in close proximity to a surface are factors outside the bounds of previous turbulence modeling experience. Nonetheless, mixing length models, expressed by empirical formulas evaluated from experiments on attached turbulent boundary layers in equilibrium in the Clauser sense (ref. 6), have been tried in the numerical solutions and the results have been compared with experimental data on turbulent boundary layers separated by either incident shock waves or deflected surfaces such as ramps (refs. 3 and 4). Although these computations yield results for the flow-field quantities — surface skin friction and heat transfer that have the general character of the data — their quantitative differences are rather large and in certain regions there are significant qualitative differences as well. To improve the computations, intuitive relaxation models were tried that use empirical formulas for mixing length and/or eddy viscosity modified by a rate equation based on the degree of departure from local equilibrium. Although the predictions of the pressure distributions were improved for some experimental cases, the resulting skin-friction values generally were not significantly better than the original models (ref. 7). An early version of a two-equation model of turbulence was also tried (ref. 3) and the results were not any better than those resulting from the empirical models.

The failure of the early two-equation turbulence model to improve the calculations over the results from the empirical algebraic models illustrates that more generality and complexity in turbulence modeling do not necessarily assure greater accuracy in a particular application. It was decided therefore to try a one-equation model for use with the compressible Navier-Stokes equations, but one that contains more of the physics of turbulence than the relaxation models and that is not as complex as a two-equation model. The incompressible fluid model of Glushko (ref. 8) was adopted for this purpose because of its rather direct formulation, its lack of arbitrary damping functions, and its rather good representation of flat-plate boundary-layer flows. Also, this model has been carefully compared with a well-documented incompressible boundary-layer flow in an adverse pressure gradient (ref. 9); modeling modifications based on the data are indicated there. The same authors also show (ref. 10), that data obtained using the Glushko model agree fairly well with other boundary-layer data obtained under other arbitrary pressure distributions.

The present report suggests the modeling modifications necessary to extend the Glushko model to compressible flow within the full Navier-Stokes equations. Implied in this extension is the assumption of the adequacy of the empirical length scale inherent in the model to represent separated flows near the surface. This report first presents an outline of the Glushko model for incompressible boundary-layer flow. Then the compressible analog of the model is derived in the absence of the boundary-layer approximation. Compressibility introduces new terms modeled in two different ways to show the arbitrariness inherent in the modeling process. The tentative model, subject to future test against experiment, is then summarized.

GLUSHKO MODEL

The Glushko turbulence model for an incompressible, turbulent boundary-layer flow is used to guide the development of a one-equation model of turbulence applicable to the Navier-Stokes equations for compressible flows. Glushko derived a model for the Reynolds stress, $\overline{u'v'}$, in the x-direction momentum equation for an incompressible turbulent boundary-layer flow. The Reynolds stress was shown to be expressible by the product of an eddy viscosity and the strain of the mean flow, $\partial\bar{u}/\partial y$, in what is usually called the logarithmic region of the boundary layer. This eddy viscosity simplification was not assumed a priori, but resulted directly from the Reynolds stress equations in an analysis (ref. 11) using the boundary-layer assumptions and neglecting the convection and diffusion of the Reynolds stresses. The principal turbulence mechanisms used were the production, dissipation, and redistribution of the individual components of the Reynolds stresses. For the later mechanisms, Glushko used Rotta's models (ref. 12), namely,

$$\overline{p'(u'_{i,k} + u'_{k,i})} = -K_1 \frac{\sqrt{e}}{L} \left(\underline{\rho} \overline{u'_i u'_k} - \frac{2}{3} \delta_{ik} \underline{\rho} e \right) \tag{1}$$

for the redistribution of turbulence between the components of the Reynolds stress and

$$\nu \overline{u'_{i,j} u'_{k,j}} = \nu C \left(\frac{\overline{u'_i u'_k}}{2L^2} + \delta_{ik} \kappa \frac{\sqrt{e} L}{\nu} \frac{e}{3L^2} \right) \tag{2}$$

for the dissipation.

In addition to establishing an eddy viscosity form for the Reynolds shear stress, Glushko's analysis also demonstrated that the eddy viscosity was expressible in terms of a local turbulence Reynolds number defined as

$$r = \frac{\sqrt{e} L}{\nu} \tag{3}$$

Glushko identified the length scale, L, with half the distance, S, where the correlation $u_1(x_1, x_2) u_1(x_1, x_2 + S)$ vanishes. Although this definition of L is quite arbitrary in the modeling equations, L always occurs in a product with an empirical constant so that there is no loss of generality. Finally, Glushko did not use the relationship between ε and r found directly from the analysis of the Reynolds stress equations, but used it to justify the use of an empirically developed relationship between these quantities that presumably corrects for the previous neglect of the convection and diffusion of the individual components of the Reynolds stresses.

The Glusko model for a two-dimensional incompressible, turbulent boundary layer is

Continuity of mass:

$$\frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2} = 0 \tag{4}$$

Momentum balance in x_1 direction:

$$\bar{u}_1 \frac{\partial \bar{u}_1}{\partial x_1} + \bar{u}_2 \frac{\partial \bar{u}_1}{\partial x_2} = - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_1} + \frac{\partial}{\partial x_2} \left[\nu M \frac{\partial u_1}{\partial x_2} \right] \tag{5}$$

Kinetic energy of turbulence:

$$\bar{u}_1 \frac{\partial e}{\partial x_1} + \bar{u}_2 \frac{\partial e}{\partial x_2} = \nu (M - 1) \left(\frac{\partial u_1}{\partial x_2} \right)^2 - \nu C D \frac{e}{L^2} + \frac{\partial}{\partial x_2} \left(\nu D \frac{\partial e}{\partial x_2} \right) \tag{6}$$

where

$$M = 1 + \varepsilon(r) \tag{7}$$

and

$$D = 1 + \varepsilon(\lambda r) \tag{8}$$

Note that the argument of ε, the ratio of eddy diffusivity to kinematic viscosity, in the definition of D has been altered by the empirical constant λ to account for differences in the transport of momentum and kinetic energy. This can be considered as equivalent to introducing a Prandtl number for the transport of turbulence kinetic energy. The ratio of eddy diffusivity to kinematic viscosity is expressed as

$$\varepsilon = H(r)\alpha r \tag{9}$$

where $H(r)$ is expressed empirically as

$$H(r) = \begin{cases} r/r_o , & 0 \le r/r_o < 0.75 \\ r/r_o - (r/r_o - 0.75)^2 , & 0.75 \le r/r_o \le 1.25 \\ 1 , & 1.25 \le r/r_o < \infty \end{cases} \tag{10}$$

The length scale is also given by an empirical expression based on flat-plate, boundary-layer correlation length data, namely,

$$L/\delta = \begin{cases} x_2/\delta , & 0 \le x_2/\delta \le 0.23 \\ (x_2/\delta + 0.37)/2.61 , & 0.23 \le x_2/\delta \le 0.57 \\ (1.48 - x_2/\delta)/2.52 , & 0.57 \le x_2/\delta \le 1.48 \end{cases} \tag{11}$$

With the empirical constants,

$$\alpha = 0.2 \tag{12}$$

$$r_o = 110 \tag{13}$$

$$C = 3.93 \tag{14}$$

$$\lambda = 0.4 \tag{15}$$

Equations (3) through (11) form a closed system that can be used to calculate incompressible turbulent boundary-layer flow. It is emphasized again that e, the specific turbulence kinetic energy, is found from a differential equation involving the production, dissipation, and diffusion of turbulent kinetic energy so that the turbulence need not be in equilibrium with the mean flow. This dynamic characteristic is introduced back into the mean-flow equations through the Reynolds number, r.

COMPRESSIBLE ANALOG TO GLUSHKO MODEL

To extend the foregoing model of turbulence to the compressible Navier-Stokes equations, the use of mass-averaged dependent variables (ref. 13) is adopted. The mean field equations, written in abbreviated differential form, are

Continuity of mass:

$$\bar{\rho}_{,t} + (\bar{\rho}\tilde{u}_j)_{,j} = 0 \tag{16}$$

Conservation of momentum in ith direction:

$$(\bar{\rho}\tilde{u}_i)_{,t} + [\bar{\rho}\tilde{u}_j\tilde{u}_i + \delta_{ij}\bar{p} - (\bar{\tau}_{ij} - \overline{\rho u_i'' u_j''})]_{,j} = 0 \tag{17}$$

Conservation of thermal energy:

$$(\bar{\rho}\tilde{H} - \bar{p})_{,t} + \left[\bar{\rho}\tilde{u}_j\tilde{H} + \bar{q}_j + \overline{\rho u_j'' h''} - \tilde{u}_i(\bar{\tau}_{ij} - \overline{\rho u_i'' u_j''}) - u_i''\left(\tau_{ij} - \frac{\overline{\rho u_i'' u_j''}}{2}\right)\right]_{,j} = 0 \tag{18}$$

The last term in equation (18) is usually neglected since it is smaller than the thermal dissipation term to its left. The terms that must be evaluated then to close these equations are $\overline{\rho u_j'' h''}$ and $\overline{\rho u_i'' u_j''}$.

Consistent with the Glushko model, an eddy viscosity concept is adopted. For nonboundary-layer flows, however, an additional assumption that the eddy viscosity is a scalar quantity is necessary. With these assumptions, the components of the Reynolds stress tensor are written:

$$-\overline{\rho u_i'' u_j''} = \tilde{\nu}\varepsilon(\tilde{u}_{i,j} + \tilde{u}_{j,i} - \frac{2}{3}\delta_{ij}\tilde{u}_{k,k}) - \frac{2}{3}\delta_{ij}\bar{\rho}\bar{k} \tag{19}$$

in the notation of reference 13. The heat flux vector can be expressed as

$$\overline{\rho u_j'' h''} = -\frac{\tilde{\mu}\varepsilon}{Pr_t}\tilde{h}_{,j} \tag{20}$$

or, in keeping with the way Glushko handled the transport of turbulence kinetic energy,

$$\overline{\rho u_j'' h''} = -\tilde{\mu}\varepsilon(\Gamma\bar{r})\tilde{h}_{,j} \tag{21}$$

where Γ is a new universal constant related to the turbulent Prandtl number, Pr_t and $\bar{r}$ is defined later in equation (35).

From equation (19), the effective total shear stress, laminar plus turbulent, can be written as

$$\bar{\tau}_{ij} - \overline{\rho u_i'' u_j''} = \tilde{\mu}(1 + \varepsilon)\tilde{S}_{ij} - \frac{2}{3}\delta_{ij}\bar{\rho}\bar{k} \tag{22}$$

where the mean strain of the flow is

$$\tilde{S}_{ij} = \tilde{u}_{i,j} + \tilde{u}_{j,i} - \frac{2}{3}\delta_{ij}\tilde{u}_{k,k} \tag{23}$$

With equations (19) through (23), equations (16) through (18) are closed once ε is defined.

To define ε (the ratio of eddy diffusivity to kinematic viscosity) when the fluid is compressible, one must do the following:

(1) Show that past experience makes it reasonable to use equation (11) for the length scale in compressible flow.

(2) Show that ε is a function of a turbulence Reynolds number in compressible flow and show what form this Reynolds number takes.

(3) Model the turbulence kinetic energy equation to account for effects of compressibility.

Length Scales in Compressible Flow

For equilibrium turbulent boundary layers, it has been found that compressibility has little direct effect on the length scales of turbulence. For example, the early mixing length theories for a boundary layer on a flat plate (e.g., refs. 14 and 15) and the more recent computer codes for calculating the boundary-layer equations on two-dimensional bodies with mild pressure gradients (ref. 16) have shown good agreement with experimental data up to Mach numbers of 6 while treating the mixing length as being essentially independent of density. This independence can be illustrated quite clearly through a consideration of the turbulence model used in reference 16 where the eddy diffusivity is expressed in terms of an inner and outer part of the boundary layer. In either part of the boundary layer, reference 16 uses

$$\nu_t = \ell^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \qquad (24)$$

In the inner part, the mixing length scale is

$$\ell_{inner} = Ky[1 - \exp(-y/A)] \qquad (25)$$

with

$$A = \frac{26\ \mu_r}{\sqrt{\tau_w \rho_r}} \qquad (26)$$

on a flat plate with zero surface mass transfer and

$$K = 0.4 \qquad (27)$$

the usual Kármán constant value. In reference 16, local properties are used as the reference fluid properties in the definition of A in equation (27). This causes the value of A to depend somewhat on the density or compressibility of the fluid within the sublayer adjacent to the surface. Outside the sublayer, however, $y \geq A$ so that the exponential term in equation (25) becomes small and effects of compressibility tend to vanish from ℓ_{inner}. Thus, over most of the inner region of the boundary layer, the length scale is independent of the fluid density.

In the outer part of the boundary layer, reference 16 uses the Clauser form of eddy diffusivity:

$$\nu_{t,outer} = \left[0.0168 \int_0^\infty (u_e - u)\,dy\right] \Big/ \left[1 + 5.5\left(\frac{y}{\delta}\right)^6\right] \tag{28}$$

Note that the density does not appear explicitly in this expression. Since the eddy diffusivity in the outer part of the boundary layer is independent of density, then, together with equation (24), the suggestion is that the length scale in the outer part of the boundary layer is also independent of the density of the fluid, other than through the effect of compressibility on the boundary-layer thickness δ. This is seen more directly in the model used in reference 17, where outside the sublayer the mixing length is expressed as

$$\frac{\ell_{outer}}{\delta} = f\left(\frac{y}{\delta}\right) \tag{29}$$

so as to be independent of density.

From the above discussion of mixing length used in programs that accurately predict near-equilibrium turbulent boundary-layer behavior, it can be concluded that the length scale L in equation (11) will apply to compressible flows as well. Its application to separated flows is less clear and modifications may be needed under those conditions.

Turbulence Reynolds Number in Compressible Flow

In a compressible turbulent boundary layer, the total shear stress expressed in equation (22) reduces to

$$\tau = \bar{\tau}_{12} - \overline{\rho u_1'' u_2''} = \tilde{\mu}(1 + \varepsilon)\,\frac{\partial \tilde{u}}{\partial y} = \bar{\rho}\tilde{\nu}(1 + \varepsilon)\,\frac{\partial \tilde{u}}{\partial y} \tag{30}$$

The mixing length expression, equation (24), can be written in terms of the ratio of eddy diffusivity to the kinematic viscosity as

$$\nu_t = \tilde{\nu}\varepsilon = \ell^2\,\frac{\partial \bar{u}}{\partial y} \tag{31}$$

Equations (30) and (31) combine to yield

$$\varepsilon(1 + \varepsilon) = \frac{(\tau/\bar{\rho})\ell^2}{\tilde{\nu}^2} \tag{32}$$

In the fully turbulent portion of the boundary layer, $\varepsilon \gg 1$ and

$$\frac{\tau}{\bar{\rho}} = -\frac{\overline{\rho u''v''}}{\bar{\rho}} \tag{33}$$

so that under these conditions

$$\varepsilon = \frac{\sqrt{-(\overline{\rho u''v''}/\bar{\rho})}\,\ell}{\tilde{\nu}} \tag{34}$$

Thus, ε identifies with a turbulent Reynolds number based on local fluid properties and the local correlation of the fluctuating velocities. This suggests, by analogy, that the Glushko model can be extended to compressible flows by use of equation (9), but with the turbulence Reynolds number $\bar{r}$ redefined as

$$\bar{r} = \frac{\sqrt{\bar{k}}L}{\tilde{\nu}} \tag{35}$$

where $\bar{k}$ is the mass-averaged kinetic energy of turbulence and $\tilde{\nu}$ is the local kinematic viscosity.

Second-Order Modeling of Turbulence in Kinetic Energy Equation

The local mass-averaged specific kinetic energy of turbulence is represented in reference 13 by

$$(\bar{\rho}\bar{k})_{,t} + (\bar{\rho}\tilde{u}_j\bar{k})_{,j} = -(\overline{\rho u_i''u_j''})\tilde{u}_{i,j} - (\overline{\rho u_j''k})_{,j} - (\overline{u_i''p})_{,i} + \overline{pu''_{i,i}} + (\overline{u_i''\tau_{ij}})_{,j} - \overline{\tau_{ij}u''_{i,j}} \tag{36}$$

The first term on the right in equation (36) can be related through equation (19) to the mean velocity field and the compressible extension of equation (9), namely, $\varepsilon = H(\bar{r})\alpha\bar{r}$. The second term, having divergence form, represents a diffusion of the instantaneous turbulence kinetic energy by the turbulence itself. The third and fourth terms representing pressure and velocity correlations and the fifth and sixth terms containing molecular-shear and velocity correlations must be modeled to close equation (36).

Models of pressure and velocity correlations.- When the pressure in the abovementioned terms is represented as the sum of a mean value plus a fluctuating quantity, $p = \bar{p} + p'$, the pressure and velocity correlation terms become

$$\overline{PV} = -(\overline{u_i''}\bar{p})_{,i} - (\overline{u_i''p'})_{,i} + \bar{p}\overline{u''_{i,i}} + \overline{p'u''_{i,i}} \tag{37}$$

For incompressible flow, both $\overline{u_i''} = 0$ and $u''_{i,i} = 0$ so that only the second term on the right remains. This term, having a divergence form, is identified with the diffusion of turbulent pressure fluctuations by the turbulence and is usually grouped with the second term on the right in equation (36). The other terms in equation (37) require models that converge to zero when the flow is incompressible. Thus, grouping these terms yields

$$PV + (\overline{u_i''p'})_{,i} = -\overline{u_i''}\bar{p}_{,i} + \overline{p'u_{i,i}''} \tag{38}$$

To evaluate the quantity $\overline{u_i''}$ in the first term on the right in equation (38), it can be assumed first that the total temperature of the fluid is constant within a field of turbulent eddies:

$$T_t = T + \frac{u_k u_k}{2c_p} = \text{const} \tag{39}$$

When T is replaced by a mean and fluctuating mass-averaged quantity (ref. 13),

$$T = \tilde{T} + T'' \tag{40}$$

and

$$u_k = \tilde{u}_k + u_k'' \tag{41}$$

Then

$$\tilde{T} + T'' + \frac{\tilde{u}_k\tilde{u}_k}{2c_p} + \frac{\tilde{u}_k u_k''}{c_p} + \frac{u_k''u_k''}{2c_p} = T_t \tag{42}$$

On averaging, equation (42) becomes

$$\tilde{T} + \overline{T''} + \frac{\tilde{u}_k\tilde{u}_k}{2c_p} + \frac{\tilde{u}_k \overline{u_k''}}{c_p} + \frac{\overline{u_k''u_k''}}{2c_p} = T_t \tag{43}$$

Equation (43) subtracted from (42) yields

$$T'' - \overline{T''} + \frac{\tilde{u}_k}{c_p}(u_k'' - \overline{u_k''}) + \frac{u_k''u_k''}{2c_p} - \frac{\overline{u_k''u_k''}}{2c_p} = 0 \tag{44}$$

which is satisfied by

$$T'' + \frac{\tilde{u}_k}{c_p}u_k'' + \frac{u_k''u_k''}{2c_p} = 0 \tag{45}$$

Since the mean velocity in at least one direction will be much larger than the fluctuating velocity in that direction, the above equation can be well represented by its leading terms:

$$T'' = -\frac{\tilde{u}_k}{c_p}u_k'' \tag{46}$$

At this point, it is usual to argue that pressure fluctuations are of a smaller order than either density or temperature fluctuations, and this assumption, together with the perfect gas equation, permits one to identify the temperature fluctuation with a density fluctuation in equation (46), thus relating density and velocity fluctuations. Since this assumption is suspect at high Mach numbers, an alternative assumption will be proposed, namely, that the gas behaves in a polytropic manner. Then

$$\frac{p'}{\bar{p}} = n\,\frac{\rho'}{\bar{\rho}} = \frac{n}{(n-1)}\,\frac{T''}{\tilde{T}} \tag{47}$$

where n is the polytropic coefficient ($n = 0$, isobaric; $n = 1$, isothermal; $n = c_p/c_v$, isentropic, etc.).

From equations (46) and (47), the density fluctuation is related to the velocity fluctuation as

$$\rho' = -\frac{\bar{\rho}}{(n-1)c_p\tilde{T}}\,\tilde{u}_k u_k'' \tag{48}$$

With equation (48),

$$\overline{u_i''} = -\frac{\overline{\rho' u_i''}}{\bar{\rho}} = \frac{1}{(n-1)c_p\tilde{T}}\,\tilde{u}_k\overline{u_k'' u_i''} \tag{49}$$

If the third-order correlations between the density and velocity are neglected,

$$\overline{u_i''} = -\frac{1}{(n-1)c_p\tilde{T}}\,\tilde{u}_k\left(\tilde{\nu}\varepsilon\tilde{S}_{ik} - \frac{2}{3}\,\delta_{ik}\bar{k}\right) \tag{50}$$

Since the value of n is not known, it, in effect, becomes a modeling coefficient and equation (50) can be used to define the first term in equation (38). Note that $n \to \infty$ for incompressible flow (see eq. (47)) and $\overline{u_i''} \to 0$ as required. Under boundary-layer assumptions, the only appropriate value for the index is $i = 1$ and, consequently, equation (49) reduces to

$$\overline{u_1''} = \frac{\tilde{u}_1\overline{u_1'' u_1''}}{(n-1)c_p\tilde{T}} = \text{const}\,\frac{\tilde{u}_1\bar{k}}{(n-1)c_p\tilde{T}} = \text{const}\,\frac{(\gamma-1)}{(n-1)}\,\frac{\tilde{u}_1\bar{k}}{\tilde{a}^2} \tag{51}$$

Since $\bar{k}/a^2$ is a measure of the square of the turbulence Mach number, $\overline{u_1''}$ is expected to be small if n takes on a value of about 1.2 as expected in nonisentropic flows. This latter form is identical with the Alber and Wilcox representation (ref. 18), except that the n must be less than unity to have the same sign as in their work.

An alternative representation of $\overline{u_i''}$ that does not require the assumption of constant total temperature within the turbulence is

$$\overline{u_i''} = -\frac{\overline{\rho' u_i''}}{\bar{\rho}} = -\frac{1}{(n-1)\tilde{T}}\,\overline{T'' u_i''} \tag{52}$$

If density fluctuations are small compared to the mean density, then equation (52) can be rewritten as

$$\overline{u_i''} = - \frac{1}{(n-1)c_p\tilde{T}\bar{\rho}} \overline{\rho c_p T'' u_i''} = \frac{1}{(n-1)\bar{p}} \frac{(\gamma - 1)}{\gamma} \tilde{\mu}\varepsilon(\Gamma\bar{r})\tilde{h}_{,i} \tag{53}$$

or

$$\overline{u_i''} = \frac{(\gamma - 1)}{(n-1)} \frac{\tilde{\nu}\varepsilon(\Gamma\bar{r})}{\bar{a}^2} \tilde{h}_{,i} \tag{54}$$

In the absence of an application of either equation (50) or (54) to predict several sets of experimental data, it is not clear at this point which is the better model, although equation (54) has the advantage of being simpler and not requiring the assumption of a constant total temperature within an eddy. This latter assumption turns out to be rather significant. For example, in boundary-layer flow and with, say, $n > 1$, equation (51) would assign a positive value to $\overline{u_1''}$. On the other hand, for the same conditions, equation (54) can assign either positive or negative values, depending on whether the flow was over a cooled or a heated surface. This illustrates how tenuous modeling assumptions are until established by comparison with data.

With the assumption that state variables within an eddy behave in a polytropic fashion, the third term on the right in equation (38) is expressed as

$$\overline{p' u_{i,i}''} = \frac{n\bar{p}}{\bar{\rho}} \overline{\rho' u_{i,i}''} \tag{55}$$

The instantaneous continuity equation

$$\rho_{,t} + (\rho u_j)_{,j} = 0 \tag{56}$$

when written in terms of mean and fluctuating quantities is

$$\bar{\rho}_{,t} + \rho'_{,t} + (\bar{\rho}\tilde{u}_j + \bar{\rho}u_j'' + \rho'\tilde{u}_j + \rho' u_j'')_{,j} = 0 \tag{57}$$

When equation (16) is subtracted from this equation, the instantaneous fluctuations of density and velocity are related by

$$\rho'_{,t} + (\bar{\rho}u_j'' + \rho'\tilde{u}_j + \rho' u_j'')_{,j} = 0 \tag{58}$$

If the mean flow varies slowly over the scale of an eddy, and only the linear terms in the fluctuations are retained, equation (58) reduces to its highest order terms:

$$\rho'_{,t} + \tilde{u}_j\rho'_{,j} + \bar{\rho}u_{j,j}'' = 0 \tag{59}$$

With equation (59) multiplied by ρ' and then time-averaged,

$$\left(\frac{\overline{\rho'^2}}{2}\right)_{,t} + \tilde{u}_j\left(\frac{\overline{\rho'^2}}{2}\right)_{,j} + \bar{\rho}\overline{\rho' u''_j}_{,j} = 0 \tag{60}$$

The time-averaged quantities now scale with the dimensions of the mean flow field and can be compared with the mean continuity equation in similar form:

$$\bar{\rho}_{,t} + \tilde{u}_j\bar{\rho}_{,j} + \bar{\rho}\tilde{u}_{j,j} = 0 \tag{61}$$

If the rms density fluctuation intensity is called β:

$$\left\langle\frac{\rho'}{\bar{\rho}}\right\rangle = \beta \tag{62}$$

then equation (60) becomes

$$\beta^2(\bar{\rho}_{,t} + \tilde{u}_j\bar{\rho}_{,j}) + \bar{\rho}\beta(\beta_{,t} + \tilde{u}_j\beta_{,j}) + \overline{\rho' u''_j}_{,j} = 0 \tag{63}$$

It is now assumed that the intensity of turbulence changes relatively slowly along the streamlines in contrast to across the streamlines, so that the second term in equation (63) can be neglected. With equation (61), the resulting equation reduces to

$$\overline{\rho' u''_j}_{,j} = \beta^2\bar{\rho}\tilde{u}_{j,j} \tag{64}$$

The variation in turbulence intensity β from streamline to streamline, can be accounted for by setting β^2 proportional to the local kinetic energy of the turbulence. Usually, $\tilde{u}_1 > \tilde{u}_2$, even in separated flows, so that equation (48) permits one to write

$$\beta^2 = \text{const}\ \frac{\tilde{u}_1{}^2}{\tilde{a}^2} \cdot \frac{\bar{k}}{\tilde{a}^2} \tag{65}$$

which, together with equations (55) and (64), gives

$$\overline{p' u''_i}_{,i} = \zeta\bar{p}\ \frac{\tilde{u}_1{}^2}{\tilde{a}^2}\ \frac{\bar{k}}{\tilde{a}^2}\ \tilde{u}_{j,j} \tag{66}$$

where ζ is a modeling coefficient that includes the polytropic coefficient.

Models of molecular-shear and velocity correlations.- The molecular-shear and velocity correlation terms in equation (36) provide for the dissipation and diffusion, by molecular processes, of the specific turbulence kinetic energy. To demonstrate the assumptions underlying the modeling of some elements of these terms, one must first expand the form of the terms to reveal their components.

At an instant of time, the local molecular shear is expressed as

$$\tau_{ij} = \mu S_{ij} \tag{67}$$

where μ and S_{ij} are the instantaneous viscosity and strain. Specifically,

$$S_{ij} = u_{i,j} + u_{j,i} - \frac{2}{3}\delta_{ij}u_{k,k} \tag{68}$$

When each variable is expressed as the sum of a mass-averaged mean quantity plus a fluctuation, equation (67) becomes

$$\tau_{ij} = \tilde{\mu}\tilde{S}_{ij} + \tilde{\mu}S''_{ij} + \mu''\tilde{S}_{ij} + \mu''S''_{ij} \tag{69}$$

With equation (69), the molecular-shear and velocity correlation terms in equation (36) can be written as

$$\begin{aligned}\overline{MSV} &= (\overline{u''_i\tau''_{ij}})_{,j} - \overline{\tau_{ij}u''_{i,j}} \\ &= \overline{u''_i(\tilde{\mu}\tilde{S}_{ij})_{,j}} + \overline{u''_i(\tilde{\mu}S''_{ij})_{,j}} + \overline{u''_i(\mu''\tilde{S}_{ij})_{,j}} + \overline{u''_i(\mu''S''_{ij})_{,j}}\end{aligned} \tag{70}$$

The terms containing μ'' are neglected. The justification for this is that the third term should be negligibly small compared to the first because $\mu'' < \tilde{\mu}$ and the correlation coefficient between u''_i and μ''_{ij} is much less than unity. By similar reasoning, the fourth term is small compared to the second. When the remaining terms are expanded,

$$\begin{aligned}\overline{MSV} &= \overline{u''_i}(\tilde{\mu}\tilde{S}_{ij})_{,j} + (\tilde{\mu}\overline{u''_iS''_{ij}})_{,j} - \tilde{\mu}\overline{S''_{ij}u''_{i,j}} \\ &= \overline{u''_i}(\tilde{\mu}\tilde{S}_{ij})_{,j} + (\tilde{\mu}\tilde{k}_{,j})_{,j} - \tilde{\mu}\overline{u''_{i,j}u''_{i,j}} + \overline{u''_i(\tilde{\mu}u''_{j,i})_{,j}} - \frac{2}{3}\delta_{ij}[\overline{u''_i(\tilde{\mu}u''_{k,k})_{,j}}]\end{aligned} \tag{71}$$

The first, fourth, and fifth terms then vanish in incompressible flow since $\overline{u''_i} = 0$ and $u''_{k,k} = 0$. For compressible flow, the following arguments can be used to justify neglecting the fourth and fifth terms. Since they contain the second derivatives of fluctuating quantities before averaging, the smallest eddies should contribute most to these quantities. The smallest eddies, however, tend to be isotropic (ref. 19), and, if the variation of $\tilde{\mu}$ over the dimensions of these small eddies is neglected, slight extensions of the methods in reference 20 (ch. 3), show that $\overline{u''_iu''_{j,i,j}} = \overline{u''_iu''_{j,j,i}}$ vanish for isotropic turbulence in a compressible fluid.

The final expression then for the molecular-shear and velocity correlation term is

$$\overline{MSV} = u''_i(\tilde{\mu}\tilde{S}_{ij})_{,j} + (\tilde{\mu}\tilde{k}_{,j})_{,j} - \tilde{\mu}\overline{u''_{i,j}u''_{i,j}} \tag{72}$$

In keeping with the Glushko model, the third term representing the dissipation of the specific kinetic energy of the turbulence is modeled as

$$\tilde{\mu}\overline{u''_{i,j}u''_{i,j}} = \tilde{\mu}C\frac{\bar{k}}{L^2}[1 + \varepsilon(\lambda\bar{r})] \tag{73}$$

The $\overline{u''_i}$ in the first term can be taken from equation (54).

Model of third-order correlation term.- The divergence terms on the right in equation (36) are grouped and modeled as follows:

$$\begin{aligned}(\overline{\rho u''_j k})_{,j} + (\overline{u''_i p'})_{,i} &= [\overline{u''_j(p' + \rho u''_\ell u''_\ell)}]_{,j} \\ &= -\tilde{\mu}\varepsilon(\lambda\bar{r})\bar{k}_{,j}\end{aligned} \tag{74}$$

SUMMARY OF ONE-EQUATION TURBULENCE MODEL IN COMPRESSIBLE FLOW

The system of equations that represents a one-equation model of compressible turbulent flow is as follows:

Continuity of mass (see eq. (16)):

$$\bar{\rho}_{,t} + (\bar{\rho}\tilde{u}_j)_{,j} = 0 \tag{75}$$

Conservation of momentum in ith direction (see eqs. (17), (22), and (23)):

$$(\bar{\rho}\tilde{u}_i)_{,t} + [\bar{\rho}\tilde{u}_i\tilde{u}_j + \delta_{ij}\bar{p} - (\bar{\tau}_{ij} - \overline{\rho u''_i u''_j})]_{,j} = 0 \tag{76}$$

where

$$\bar{\tau}_{ij} - \overline{\rho u''_i u''_j} = \tilde{\mu}[1 + \varepsilon(\bar{r})]\tilde{S}_{ij} - \frac{2}{3}\delta_{ij}\bar{\rho}\bar{k} \tag{77}$$

and

$$\tilde{S}_{ij} = \tilde{u}_{i,j} + \tilde{u}_{j,i} - \frac{2}{3}\delta_{ij}\tilde{u}_{k,k} \tag{78}$$

Conservation of thermal energy (see eqs. (18) and (21)):

$$(\bar{\rho}\tilde{H} - \bar{p})_{,t} + [\bar{\rho}\tilde{u}_j\tilde{H} + \bar{q}_j + \overline{\rho u''_j h''} - \tilde{u}_i(\bar{\tau}_{ij} - \overline{\rho u''_i u''_j})]_{,j} = 0 \tag{79}$$

where

$$\bar{q}_j + \overline{\rho u''_j h''} = -\tilde{\mu}\left[\frac{1}{Pr} + \varepsilon(\Gamma\bar{r})\right]\tilde{h}_{,j} \tag{80}$$

Specific turbulence kinetic-energy equation (see eqs. (36), (39), (54), (74), and (77)):

$$(\bar{\rho}\bar{k})_{,t} + (\bar{\rho}\tilde{u}_j\bar{k})_{,j} = -\overline{\rho u''_i u''_j}\tilde{u}_{i,j} - (\overline{\rho u_j k})_{,j} - (\overline{u''_j p'})_{,j} + [\overline{PV} + \overline{MSV} + (\overline{u''_j p'})_{,j}] \tag{81}$$

$$\overline{(u_j''p' + \rho u_j''k)}_{,j} = -\tilde{\mu}\varepsilon(\lambda\bar{r})\bar{k}_{,j} \tag{82}$$

$$PV + \overline{(u_j''p')}_{,j} = -\overline{u_i''}\bar{p}_{,i} + \overline{p'u''_{i,i}} \tag{83}$$

$$\overline{u_i''} = \frac{(\gamma - 1)}{(n - 1)} \frac{\tilde{\nu}\varepsilon(\Gamma\bar{r})}{\bar{a}^2} \tilde{h}_{,i} \tag{84}$$

$$\overline{p'u''_{i,i}} = \zeta\bar{p} \frac{\tilde{u}_1{}^2}{a^2} \frac{\bar{k}}{a^2} \tilde{u}_{j,j} \tag{85}$$

$$\overline{MSV} = \overline{u_i''}(\tilde{\mu}\tilde{S}_{ij})_{,j} + (\tilde{\mu}\bar{k}_{,j})_{,j} - \tilde{\mu}C \frac{\bar{k}}{L^2} [1 + \varepsilon(\lambda\bar{r})] \tag{86}$$

These equations combine to yield

$$(\bar{\rho}\bar{k})_{,t} + (\bar{\rho}\tilde{u}_j\bar{k})_{,j} = \tilde{\mu}\varepsilon(\bar{r})\,\tilde{S}_{ij}\tilde{u}_{i,j} - \frac{2}{3}\,\delta_{ij}\bar{\rho}\bar{k}\tilde{u}_{i,j} + \{\tilde{\mu}[1 + \varepsilon(\lambda\bar{r})]\bar{k}_{,j}\}_{,j}$$

$$- \overline{u_i''}\bar{p}_{,i} + \zeta \frac{\bar{p}\bar{k}}{V^2} \tilde{u}_{j,j} + \overline{u_i''}(\tilde{\mu}\tilde{S}_{ij})_{,j} - \tilde{\mu}C \frac{\bar{k}}{L^2} [1 + \varepsilon(\lambda\bar{r})] \tag{87}$$

where n, ζ, and Γ are new modeling coefficients. Additional modeling quantities are

$$\bar{r} = \frac{\sqrt{\bar{k}}L}{\tilde{\nu}} \tag{88}$$

$$\varepsilon = H(\bar{r})\alpha\bar{r} \tag{89}$$

$$H(\bar{r}) = \begin{cases} \bar{r}/\bar{r}_o\ , & 0 \le \bar{r}/\bar{r}_o < 0.75 \\ \bar{r}/\bar{r}_o - (\bar{r}/\bar{r}_o - 0.75)^2\ , & 0.75 \le \bar{r}/\bar{r}_o < 1.25 \\ 1\ , & 1.25 \le \bar{r}/\bar{r}_o < \infty \end{cases} \tag{90}$$

and

$$L/\delta = \begin{cases} x_2/\delta\ , & 0 \le x_2/\delta < 0.23 \\ (x_2/\delta + 0.37)/2.61\ , & 0.23 \le x_2/\delta < 0.57 \\ (1.48 - x_2/\delta)/2.52\ , & 0.57 \le x_2/\delta < 1.48 \end{cases} \tag{91}$$

where

$$\left.\begin{aligned} \alpha &= 0.2 \\ \bar{r}_o &= 110 \\ C &= 3.93 \\ \lambda &= 0.4 \end{aligned}\right\} \tag{92}$$

CONCLUDING REMARKS

The set of equations summarized in the preceding section is closed when the modeling parameters n, ζ, and Γ are established. To do this, one must eventually compare the predictions given by the model with the data from a series of experiments that covers a sizable range of Mach and Reynolds numbers. At this point, all that can be done is to offer suggestions for the first set of trial values of these parameters. For example, in nonisentropic inviscid flows, values of $n = 1.2$ seem to fit much of the data. Consistent with this value of n, $\gamma = 1.4$ and for the relationship between $\overline{u''^2}$ and $\bar{k}$ that occurs in an equilibrium boundary layer, that is, $\overline{u''^2} \cong 8/9\ \bar{k}$, $\zeta = 8/11$. The quantity Γ is identified with the turbulent Prandtl number, $\Gamma = 1/Pr_t$, for which $\Gamma = 1.1$ is a reasonable first trial value.

REFERENCES

1. MacCormack, R. W.: Numerical Solution of the Interaction of a Shock Wave With a Laminar Boundary Layer. Lecture Notes in Physics, vol. 8, 1971, p. 151.

2. Hung, M.; and MacCormack, R. W.: Supersonic and Hypersonic Laminar Flows Over a Two-Dimensional Compression Corner. AIAA Paper 75-2, Jan. 1975.

3. Baldwin, B. S.; and MacCormack, R. W.: Numerical Solution of Interaction of a Strong Shock Wave with a Hypersonic Turbulent Boundary Layer. AIAA Paper 74-558, June 1974.

4. Marvin, J. G.; Horstman, C. C.; Rubesin, M. W.; Coakley, T. C.; and Kussoy, M. I.: An Experimental and Numerical Investigation of Shock-Wave-Induced Turbulent Boundary-Layer Separation at Hypersonic Speeds. AGARD Conference on Flow Separation, Göttingen, 27-30 May 1975, AGARD CP-168.

5. Mateer, G. G.; Brosh, A.; and Viegas, J. R.: A Normal Shock-Wave Turbulent Boundary-Layer Interaction at Transonic Speeds. AIAA Paper 76-161, Jan. 1976.

6. Clauser, F. H.: The Turbulent Boundary Layer, in Advances in Applied Mechanics, vol. 4, Academic Press, Inc., New York, 1956.

7. Baldwin, B. S.; and Rose, W. C.: Calculation of Shock-Separated Turbulent Boundary Layers. NASA SP-347, 1975.

8. Glushko, G. S.: Turbulent Boundary Layer on a Flat Plate in an Incompressible Fluid. Bull. Acad. Sci. USSR, Mech. Ser. 4, 1965, pp. 13-23.

9. Beckwith, I. E.; and Bushnell, D. M.: Detailed Description and Results of a Method for Computing Mean and Fluctuating Quantities in Turbulent Boundary Layers. NASA TN D-4815, 1968.

10. Beckwith, I. W.; and Bushnell, D. M.: Calculation of Mean and Fluctuating Properties of the Incompressible Turbulent Boundary Layer. Proceedings Computation of Turbulent Boundary Layers — 1968, AFOSR-IFP-Stanford Conference, vol. 1, Kline et al., Ed., p. 275.

11. Townsend, A. A.: The Structure of Turbulent Shear Flow, Cambridge Univ. Press, 1956.

12. Rotta, J.: Statistical Theory of Inhomogeneous Turbulence, Part 1, Z. Phys., vol. 129, 1951, pp. 547-572 (NASA TT F-14560). Part 2, Z. Phys., vol. 131, 1951, pp. 51-57 (NASA TT TT-11696).

13. Rubesin, M. W.; and Rose, W. C.: The Turbulent Mean-Flow, Reynolds Stress, and Heat Flux Equations in Mass-Averaged Dependent Variables. NASA TM X-62,248, 1973.

14. van Driest, E. R.: Turbulent Boundary Layer in Compressible Fluids. J. Aero. Sci., vol. 18, no. 3, March 1951, pp. 145-160.

15. Rubesin, M. W.: A Modified Reynolds Analogy for the Compressible Turbulent Boundary Layer on a Flat Plate. NACA TN-2917, 1953.

16. Cebeci, T.: Calculation of Compressible Turbulent Boundary Layers with Heat and Mass Transfer, AIAA J., vol. 9, no. 6, June 1971, pp. 1091-1097.

17. Price, J. M.; and Harris, J. E.: Computer Program for Solving Compressible Nonsimilar-Boundary-Layer Equations for Laminar, Transitional, or Turbulent Flows of a Perfect Gas. NASA TM X-2458, 1972.

18. Wilcox, D. C.; and Alber, I. E.: A Turbulence Model for High-Speed Flows, Proceedings of the 1972 Heat Transfer and Fluid Mechanics Institute, Stanford University Press, 1972, pp. 231-252.

19. Bradshaw, P.; and Ferriss, D. H.: Calculation of Boundary-Layer Development Using the Turbulent Energy Equation: Compressible Flow on Adiabatic Flows. J. Fluid Mech., Vol. 46, Pt. 1, 1971, pp. 83-110.

20. Hinze, J. O.: Turbulence, McGraw-Hill Book Co., Inc., 1959.